Deriving Egalitarian and Proportional Principles from Individual Monotonicity*

Yukihiko Funaki† Yukio Koriyama‡

May 20, 2024

Abstract

The problem of efficient allocation of the grand coalition worth in transferable-utility games boils down to specifying how the surplus is distributed among individuals, in the situation where the individual share is well-defined. We show that the Individual Monotonicity axiom for Equal Surplus, together with Efficiency and Equal Treatment, implies Egalitarian Surplus Sharing, while the same axiom for Equal Ratio implies Proportional Division. The results thus illustrate the common structure in deriving two principles of surplus distribution, egalitarian and proportional, from the Individual Monotonicity axioms. We further show that relaxation of Equal Treatment leads to Weighted Surplus Sharing and Shifted Proportional Division, highlighting the common structure in which Individual Monotonicity characterizes the allocations that can incorporate social objectives of a redistributive nature.

JEL classification: C71, D63

Keywords: TU-games, monotonicity, egalitarian surplus sharing, proportional division, redistribution.

1 Introduction

Monotonicity plays a crucial role in characterizing solution concepts in cooperative games. One of the primary examples is that the strong monotonicity, together with the axioms of efficiency and symmetry, characterizes the Shapley

*Declarations of interest: none. We are grateful to Satoshi Nakada for fruitful discussions and helpful comments. Financial support by Investissements d'Avenir, ANR-11-IDEX-0003/Labex Ecodec/ANR-11-LABX-0047, JSPS Core-to-Core Program (JPJSCCA20200001), and JSPS KAKENHI (17H02503, 18KK004, 22H00829) is gratefully acknowledged.

†School of Political Science and Economics, Waseda University. E-mail: `funaki@waseda.jp`.

‡CREST, Ecole Polytechnique, Institut Polytechnique de Paris. E-mail: `yukio.koriyama@polytechnique.edu`.

value (Young, 1985).[1] A variety of characterizations using monotonicity conditions have also been obtained in the literature (Casajus and Yokote, 2019). Weak monotonicity, for instance, leads to the egalitarian Shapley value (Joosten, 1996; van den Brink et al., 2013; Casajus and Huettner, 2014b), while grand coalition monotonicity implies the equal division value (Casajus and Huettner, 2014b). Based on linear algebraic arguments, Yokote and Funaki (2017) provides a unified approach, in which various combinations of monotonicity conditions imply linear combinations of the corresponding solution concepts. In particular, they show that surplus+individual monotonicity, together with efficiency and symmetry, characterizes the Center of Imputation Set (CIS), introduced by Driessen and Funaki (1991).

In addition to the above result, several characterizations of the CIS value have been explored in the literature. Two kinds of characterizations are provided in van den Brink and Funaki (2009), one with consistency and standardness, and another with efficiency, symmetry, linearity and weak individual rationality. The characterization given by Casajus and Huettner (2014a) is based on the axiom of coalition surplus monotonicity, which requires that if all zero-normalized worths increase in coalitions that include a player, then the excess amount the player receives over the individual worth also increases.

Our characterization is simple and constructive. *Individual Monotonicity for Equal Surplus* (IMES) requires that, if a player's individual worth increases, the payoff of the player also increases, given that the grand coalition surplus remains the same. Combined with the Efficiency and Equal Treatment axioms, we obtain the CIS value. Given the simplicity of our proof technique, the characterization is extended in several directions. First, we extend it to more general forms of *individual share*, which can be considered as a legitimate claim of each individual in the society. For instance, if the individual share is the separable contribution, i.e., the increase in worth when she joins a coalition and forms the grand coalition (Driessen and Funaki, 1991), then we obtain a characterization of the egalitarian non-separable contribution (ENSC) value. More generally, we allow the individual share to be defined by a symmetric and affine function. The surplus is then the remainder of the grand coalition worth net of the sum of individual shares. We show that the only solution which satisfies the IMES, Efficiency and Equal Treatment axioms is the Egalitarian Surplus Sharing (ESS) value. The above CIS and ENSC characterizations are special cases when the individual share has a specific form. A remarkable feature of our characterization results is that none of them relies on the linearity axiom. This line of research is in line with the recent work by Nakada (2024) which provides an explanation how linearity is derived from monotonicity, by using decision-theoretic tools.

Furthermore, we show that the same proof technique can be applied to the characterization of the Proportional Division by the axiom of *Individual Monotonicity for Equal Ratio* (IMER). The proof is analogous. Instead of considering

[1] In the original work by Shapley (1953), the Shapley value is characterized by the axioms of efficiency, symmetry, linearity and the null player property. The linearity axiom plays an essential role in the proof.

the remainder after subtracting the total individual shares from the grand coalition worth, we consider the ratio after dividing the latter by the former. We then show that the surplus should be divided proportionally. This line of characterization of the Proportional Division value is novel and different from those developed in recent papers (Zou et al., 2021, 2022; van den Brink et al., 2023). What is remarkable in our results is the common structure in which the principles of egalitarian and proportional surplus sharing are derived by the Individual Monotonicity axioms.

In Section 5, we provide further characterizations by dropping the Equal Treatment axiom and requiring Homogeneity instead. By applying the same technique again, we show that IMES characterizes the Weighted Surplus Sharing (WSS), which includes Egalitarian Surplus Sharing as a special equal-weight case. Since the WSS is written as a class of allocations obtained by a zero-sum redistribution based on the ESS, our characterization shows that dropping symmetry corresponds to unequal treatment of individuals, which allows us to incorporate social objectives of an asymmetric nature, such as minority protection, support for the disabled, consideration of seniority, and so on.

A novel finding is that the Proportional Division is also extended in an analogous way. By dropping Equal Treatment, IMER characterizes the Shifted Proportional Division, a class of allocations obtained as a zero-sum redistribution from the Proportional Division. Given that the redistribution terms are written as proportional to the Equal Surplus and Equal Ratio respectively, our results again highlight the central role of the Individual Monotonicity axioms and the common structure of the characterization.

The rest of the paper is organized as follows. The characterization results of the CIS and the ENSC, are presented in Section 2. We extend the axiom to the individual share defined by an arbitrary affine function, and obtain characterization of the Egalitarian Surplus Sharing in Section 3. Our axiom is further extended to the Individual Monotonicity for Equal Ratio, and a characterization of Proportional Division is obtained in Section 4. We relax the Equal Treatment axiom and characterize Weighted Surplus Sharing and Shifted Proportional Division in Section 5. The proofs of the lemmas are relegated to the Appendix.

2 Individual Monotonicity for Equal Surplus

2.1 Preliminary

Let $N = \{1, 2, \cdots, n\}$ be the set of the players. Let $\mathcal{V}^N = \left\{v : 2^N \to \mathbb{R} | v(\emptyset) = 0\right\}$ denote the set of all cooperative transferable utility games (TU-games) on N. For $S \subseteq N$, $v(S)$ is called the *worth* of coalition S. Let $\varphi : \mathcal{V}^N \to \mathbb{R}^n$ be a *solution* where $\varphi_i(v)$ is the *value* assigned to player $i \in N$. Together with the following axiom of Efficiency, $\varphi_i(v)$ is also called the *payoff* of player i in game v, which describes the portion of the grand coalition worth $v(N)$ that is allocated to player i.

Axiom 1 (Efficiency) *For any* $v \in \mathcal{V}^N$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Axiom 2 (Equal Treatment) *For any $v \in \mathcal{V}^N$, and for any players $i, j \in N$ with $i \neq j$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, then $\varphi_i(v) = \varphi_j(v)$.*

Throughout the paper, the Efficiency axiom is required. The Equal Treatment axiom is required except in Section 5, in which we examine its relaxation.

2.2 A characterization of CIS

We consider the following axiom of Individual Monotonicity for Equal Surplus. The axiom requires that, if the surplus defined as the grand coalition worth net of the individual worths is equal, the solution value is monotonic in the individual worth.

Axiom 3 (Individual Monotonicity for Equal Surplus: IMES) *For any $v, w \in \mathcal{V}^N$, if*

$$v(N) - \sum_{k \in N} v(\{k\}) = w(N) - \sum_{k \in N} w(\{k\})$$

and $v(\{i\}) \geq w(\{i\})$, then $\varphi_i(v) \geq \varphi_i(w)$.

Our first result is that the IMES axiom, together with Efficiency and Equal Treatment, characterizes the Center of the Imputation Set (CIS), which is defined as follows (Driessen and Funaki, 1991).

Definition 4 (CIS) *The Center of the Imputation Set is:*

$$CIS_i(v) = v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{k \in N} v(\{k\})\right), \ \forall i \in N.$$

The following theorem provides a characterization of the CIS. Notice that it does not hinge on the linearity axiom, which is often used in characterization of the solution concepts in TU-games.

Theorem 5 *A solution satisfies (Efficiency), (Equal Treatment) and (IMES), if and only if it is the CIS value.*

Before proving Theorem 5, we introduce the following axiom, Individualistic Property for Equal Surplus (IES), which requires the solution to depend solely on the individualistic stand-alone worth under equal surplus.

Axiom 6 (Individualistic Property for Equal Surplus: IES) *For any $v, w \in \mathcal{V}^N$, if $v(N) - \sum_{k \in N} v(\{k\}) = w(N) - \sum_{k \in N} w(\{k\})$ and $v(\{i\}) = w(\{i\})$, then $\varphi_i(v) = \varphi_i(w)$.*

It is straightforward to see that (IMES) implies (IES). Suppose $v(\{i\}) = w(\{i\})$. Then, obviously $v(\{i\}) \geq w(\{i\})$ and $v(\{i\}) \leq w(\{i\})$. Applying (IMES) to both inequalities, we obtain $\varphi_i(v) = \varphi_i(w)$.

Now, we prove Theorem 5.

Proof of Theorem 5. It is obvious that (CIS) satisfies (Efficiency), (Equal Treatment) and (IMES). To show the opposite, fix a game $v \in \mathcal{V}^N$. For each $k = 0, \cdots, n$, construct a game v^k as follows:

$$v^k(\{i\}) = \begin{cases} v(\{i\}) & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases} \tag{1}$$

for the stand-alone coalitions,

$$v^k(S) = v(S) \tag{2}$$

for all $S \subsetneq N$ with $2 \leq |S| < n$, and

$$v^k(N) = v(N) - \sum_{i=k+1}^{n} v(\{i\}) \tag{3}$$

for the grand coalition. Notice that for $k = n$, v^n coincides with v. On the other hand, v^0 satisfies $v^0(\{i\}) = 0$ for all i.

Now, let u^N be the N-unanimity game multiplied by a constant $v^0(N) = v(N) - \sum_{i \in N} v(\{i\})$, that is, $u^N(S) = 0$ for all $S \subsetneq N$ and $u^N(N) = v^0(N)$. Then, (Equal Treatment) implies that $\varphi_i(u^N) = \varphi_j(u^N)$ for any $i, j \in N$. By (Efficiency), $\varphi_i(u^N) = v^0(N)/n$ for all $i \in N$. We can apply (IES) to u^N and v^0, and we have $\varphi_i(v^0) = v^0(N)/n$ for all $i \in N$. By letting $k = 0$ in (3), we have

$$\varphi_i(v^0) = \frac{1}{n}\left(v(N) - \sum_{k \in N} v(\{k\})\right). \tag{4}$$

Now, for any $k = 0, 1, \cdots, n$, (1) and (3) imply that

$$v^k(N) - \sum_{i \in N} v^k(\{i\}) = v(N) - \sum_{i \in N} v(\{i\}).$$

Note that the sequence of games $(v^k)_{k=0}^n$ is constructed so that they all have the same surplus as v. In particular, fix any $k \in N$ and we have

$$v^k(N) - \sum_{i \in N} v^k(\{i\}) = v^{k-1}(N) - \sum_{i \in N} v^{k-1}(\{i\}).$$

Moreover, (1) implies that $v^k(\{i\}) = v^{k-1}(\{i\}) = 0$ for any $i > k$, and $v^k(\{i\}) = v^{k-1}(\{i\}) = v\{i\}$ for any $i < k$. Therefore, by applying (IES), we have:

$$\varphi_i(v^k) = \varphi_i(v^{k-1}) \text{ for any } i \neq k. \tag{5}$$

Hence,

$$v^k(N) - \sum_{i \neq k} \varphi_i(v^k) = v^k(N) - \sum_{i \neq k} \varphi_i(v^{k-1}). \tag{6}$$

On the other hand, the sequence $(v^k)_{k=0}^n$ is constructed so that the grand coalition worth increases by the individual worth $v(\{k\})$. More precisely, using (3) for k and $k-1$, we obtain

$$v^{k-1}(N) = v(N) - \sum_{i=k}^{n} v(\{i\}) = v^k(N) - v(\{k\}). \tag{7}$$

Hence, by (6) and (7),

$$v^k(N) - \sum_{i \neq k} \varphi_i\left(v^k\right) = v(\{k\}) + v^{k-1}(N) - \sum_{i \neq k} \varphi_i\left(v^{k-1}\right).$$

Then, (Efficiency) implies that

$$\varphi_k\left(v^k\right) = v(\{k\}) + \varphi_k\left(v^{k-1}\right), \forall k \in N. \tag{8}$$

Now, for any $i \in N$, by (5) and (8),

$$\varphi_i(v^n) = \varphi_i\left(v^i\right) = v(\{i\}) + \varphi_i\left(v^{i-1}\right) = v(\{i\}) + \varphi_i\left(v^0\right). \tag{9}$$

Remember $v^n = v$. By (4), we have

$$\begin{aligned} \varphi_i(v) &= v(\{i\}) + \varphi_i\left(v^0\right) \\ &= v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{k \in N} v(\{k\})\right). \end{aligned}$$

This is exactly (CIS). ■

2.3 An analogous characterization of ENSC

The dual concept of the CIS is the ENSC, the Egalitarian Non-Separable Contribution (Driessen and Funaki, 1991). In the ENSC, the non-separable contribution, defined as the remaining part of the grand coalition value net of the total separable contributions of all players, is distributed equally among all players.

Definition 7 (SC, NSC) *The separable contribution of player i in game v is:*

$$SC_i(v) = v(N) - v(N \backslash \{i\}).$$

The non-separable contribution of game v is:

$$NSC(v) = v(N) - \sum_{i \in N} SC_i(v).$$

Definition 8 (ENSC) *The Egalitarian Non-Separable Contribution is defined by:*

$$ENSC_i(v) = SC_i(v) + \frac{1}{n} NSC(v), \ \forall i \in N.$$

A characterization of ENSC, analogous to Theorem 5, can be provided by the Individual Monotonicity axiom defined over v^*, the dual of v.

Definition 9 (Dual) *The dual v^* of game v is defined by $v^*(S) = v(N) - v(N \backslash S)$, for any $S \subseteq N$.*

In particular, $v^*(\{i\}) = v(N) - v(N \backslash \{i\}), \forall i$. In the following axiom, the surplus to be shared among the players is defined by the dual of v.

Axiom 10 (IMES*) *For any $v, w \in \mathcal{V}^N$, if*

$$v(N) - \sum_{k \in N} v^*(\{k\}) = w(N) - \sum_{k \in N} w^*(\{k\})$$

and $v^*(\{i\}) \geq w^*(\{i\})$, then $\varphi_i(v) \geq \varphi_i(w)$.

We have the following characterization.

Theorem 11 *A solution φ satisfies (Efficiency), (Equal Treatment) and (IMES*), if and only if it is the ENSC value.*

Proof. The "if" part is obvious. To show the "only if" part, notice that it is straightforward by definition that ENSC is the dual of CIS:

$$ENSC_i(v) = CIS_i(v^*), \forall i, \forall v.$$

If (IMES*) is satisfied for game v, (IMES) is satisfied for game v^*. By applying Theorem 5 to v^*, we obtain $\varphi_i(v) = CIS_i(v^*), \forall i, \forall v$. Hence, $\varphi_i(v) = ENSC_i(v)$. ■

3 Generalized individual share

In the previous section, we considered two types of individual share: the stand-alone individual worth and the separable contribution. The surplus was defined as the remainder after subtracting the sum of individual shares from the grand coalition worth. Both cases are extreme in that the individual share only represents the contribution of the stand-alone coalition in the former and that to the grand coalition in the latter.

In this section, we extend the definition of individual share using a function f that can provide more general structures in what is considered to be an individual claim. Such a function f may represent the social consensus according to which how much the legitimate claim of each individual should reflect the factors such as individual contribution to the coalitions of diverse sizes. The stand-alone coalition worth $f_i(v) = v(\{i\})$ and the separable contribution $f_i(v) = SC_i(v)$ are the special examples. Then, the Individual Monotonicity axiom can be extended to the one requiring monotonic relationship of the payoff with respect to the individual share defined by f.

Let $f : \mathcal{V}^N \to \mathbb{R}^n$ be a function. Efficiency is not required on f.

Axiom 12 (f-Individual Monotonicity for Equal Surplus: f-IMES) *For any $v, w \in \mathcal{V}^N$, if*

$$v(N) - \sum_{k \in N} f_k(v) = w(N) - \sum_{k \in N} f_k(w)$$

and $f_i(v) \geq f_i(w)$, then $\varphi_i(v) \geq \varphi_i(w)$.

Axiom 13 (f-Individualistic Property for Equal Surplus: f-IES) *For any $v, w \in \mathcal{V}^N$, if $v(N) - \sum_{k \in N} f_k(v) = w(N) - \sum_{k \in N} f_k(w)$ and $f_i(v) = f_i(w)$, then $\varphi_i(v) = \varphi_i(w)$.*

It is straightforward to see that (f-IMES) implies (f-IES), by applying the inequality in the definition of (f-IMES) to both directions.

The corresponding Egalitarian Surplus Sharing value with respect to f is defined as follows:

Definition 14 (Egalitarian Surplus Sharing: ESS) *We say that the solution φ is the Egalitarian Surplus Sharing value with respect to f, if*

$$\varphi_i(v) = f_i(v) + \frac{1}{n}\left(v(N) - \sum_{k \in N} f_k(v)\right), \forall i \in N. \qquad (f\text{-ESS})$$

The CIS characterization (Theorem 5) and the ENSC characterization (Theorem 11) of the previous section are special cases of the f-ESS characterization where f_i takes a specific function of v, $f_i = v(\{i\})$ and $f_i = v(N) - v(N \setminus \{i\})$, respectively. In this section, we pursue further generalization.

3.1 Symmetric and linear f

We first define symmetry and linearity on f. Let π be a permutation on N, defined as a bijection from N to itself. For any $S \subseteq N$, define $\pi S := \{\pi(i)\}_{i \in S}$ and $\pi v(\pi S) := v(S)$.

Axiom 15 (Symmetry) *$f_{\pi(i)}(\pi v) = f_i(v)$, for any permutation π on N, individual $i \in N$ and game $v \in \mathcal{V}^N$.*

Axiom 16 (Linearity) *$f(v + w) = f(v) + f(w)$ and $f(av) = af(v)$ for any $v, w \in \mathcal{V}^N$ and $a \in \mathbb{R}$.*

Theorem 17 *Suppose that f satisfies (Symmetry) and (Linearity). A solution φ satisfies (Efficiency), (Equal Treatment) and (f-IMES), if and only if it is the ESS value.*

We use the following lemmas in the proof.

Lemma 18 *If f satisfies (Symmetry) and (Linearity), then there exist constants $\left(\alpha^k\right)_{k=1}^n \in \mathbb{R}^n$ and $\left(\beta^k\right)_{k=1}^{n-1} \in \mathbb{R}^{n-1}$ such that*

$$f_i\left(v\right) = \sum_{S \ni i} \alpha^{|S|} v\left(S\right) + \sum_{S \not\ni i} \beta^{|S|} v\left(S\right). \tag{10}$$

Lemma 19 *Let $i \in N$ and $S \subseteq N$. Define a matrix as follows:*

$$A = \left(a_{iS}\right)_{i \in N, 1 \leq |S| < n}$$

where

$$a_{iS} = \begin{cases} \alpha^{|S|} & \text{if } i \in S, \\ \beta^{|S|} & \text{if } i \notin S. \end{cases}$$

If there exists $k \in \{1, 2, \cdots, n-1\}$ such that $\alpha^k \neq \beta^k$, then the matrix A has a full rank n.

The proofs of Lemmas 18 and 19 are relegated to the Appendix.

Proof of Theorem 17. The "if" part is obvious. We show the "only if" part. By Lemma 18, we have

$$f_i\left(v\right) = \sum_{S \ni i} \alpha^{|S|} v\left(S\right) + \sum_{S \not\ni i} \beta^{|S|} v\left(S\right). \tag{11}$$

First, suppose $\alpha^j = \beta^j$ for all $j \in \{1, 2, \cdots, n-1\}$. Then, (11) becomes

$$f_i\left(v\right) = \sum_{S \subseteq N} \alpha^{|S|} v\left(S\right),$$

and thus all $\left(f_i\right)_{i \in N}$ are identical. Then, by (f-IES), all $\left(\varphi_i\right)_{i \in N}$ are equal. Only the equal division is the solution, which trivially satisfies (f-ESS). In the following, suppose $\exists j \in \{1, 2, \cdots, n-1\}$ such that $\alpha^j \neq \beta^j$.

For $i, k \in N$, let

$$\hat{f}_{i,k}\left(v\right) = f_i\left(v\right) \cdot \mathbf{1}\left(i \leq k\right), \tag{12}$$

where $\mathbf{1}\left(\cdot\right)$ is the characteristic function, which takes the value 1 if the inside of the bracket is true and 0 otherwise. We construct a sequence of games $v^0, v^1, \cdots, v^n$ such that, for $k = 0, 1, \cdots, n$,

$$f_i\left(v^k\right) = \hat{f}_{i,k}\left(v\right), \ \forall i, \text{ and} \tag{13}$$

$$v^k\left(N\right) = v\left(N\right) - \sum_{i > k} f_i\left(v\right). \tag{14}$$

For $k = n$, (13) and (14) are satisfied by letting $v^n = v$. Now, fix $k \in \{0, 1, \cdots, n-1\}$. Here, the challenge is that we need to show the existence of a game v^k which satisfies (13) and (14). Since (14) can be trivially satisfied by seeing it as a definition of the grand coalition value of game v^k, the variables to be determined are $\left(v^k(S)\right)_{1\le|S|<n}$, and there are 2^n-2 of them. Since (11) should hold for all i, the values of $\left(v^k(S)\right)_{1\le|S|<n}$ should satisfy:

$$\sum_{S\ni i}\alpha^{|S|}v^k(S)+\sum_{S\not\ni i}\beta^{|S|}v^k(S)=\hat{f}_{i,k}(v)\,,\forall i. \tag{15}$$

There are n equations (for each $i\in N$) in (15), which is equivalent to:

$$\sum_{S\ni i,S\subsetneq N}\alpha^{|S|}v^k(S)+\sum_{S\not\ni i}\beta^{|S|}v^k(S)=\hat{f}_{i,k}(v)-\alpha^n v^k(N)\,,\forall i. \tag{16}$$

By regarding it as a system of linear equations, define a matrix as follows:

$$A=\left(a_{iS}\right)_{i\in N,1\le|S|<n}$$

where

$$a_{iS}=\begin{cases}\alpha^{|S|} & \text{if } i\in S,\\ \beta^{|S|} & \text{if } i\notin S.\end{cases}$$

Letting $\mathbf{v}=\left(v^k(S)\right)_{1\le|S|<n}$ and $\mathbf{b}=\left(\hat{f}_{i,k}(v)-\alpha^n v^k(N)\right)_{i\in N}$, (16) is equivalent to:

$$A\mathbf{v}=\mathbf{b}. \tag{17}$$

Notice that $\mathbf{b}$ is fully determined by v and f, since $v^k(N)$ is fixed by (14). Therefore, a sufficient condition for the existence of a solution $\mathbf{v}$ in (17) is that the matrix A of size $n\times(2^n-2)$ has a full rank n.

Remember that $\exists j\in\{1,2,\cdots,n-1\}$ such that $\alpha^j\neq\beta^j$. By Lemma 19, A has full rank n. Then, for any $\mathbf{b}$, there exists at least one solution $\mathbf{v}$ in (17), which guarantees that there exists a game v^k which satisfies both (13) and (14), for each $k\in\{0,1,\cdots,n-1\}$.

By (12), (13) and (14),

$$v^k(N)-\sum_{i\in N}f_i\left(v^k\right)=\left(v(N)-\sum_{i>k}f_i(v)\right)-\sum_{i\le k}f_i(v)=v(N)-\sum_{i\in N}f_i(v)\,.$$

Notice that the last part is independent of k. Hence, for $k=1,2,\cdots,n$, we have:

$$v^k(N)-\sum_{i\in N}f_i\left(v^k\right)=v^{k-1}(N)-\sum_{i\in N}f_i\left(v^{k-1}\right).$$

Also by (13), we have $f_i\left(v^k\right)=f_i\left(v^{k-1}\right)$ for all $i\neq k$. Hence, we can apply (f-IES) to v^k and v^{k-1}, and obtain

$$\varphi_i\left(v^k\right)=\varphi_i\left(v^{k-1}\right)\ \text{ for all } i\neq k. \tag{18}$$

Then, for $i\in N$,

$$\begin{cases}\varphi_i\left(v^n\right)=\varphi_i\left(v^{n-1}\right)=\cdots=\varphi_i\left(v^i\right),\\ \varphi_i\left(v^{i-1}\right)=\varphi_i\left(v^{i-2}\right)=\cdots=\varphi_i\left(v^0\right).\end{cases} \tag{19}$$

Now, consider v^0. By (12) and (13), we have $f_i\left(v^0\right) = \hat{f}_{i,0}\left(v\right) = 0$, for $i \in N$. By (f-IES), (Equal Treatment) and (Efficiency), the unique solution of v^0 is the equal division. Therefore, by (14),

$$\varphi_i\left(v^0\right) = \frac{1}{n}\left(v\left(N\right) - \sum_{k\in N} f_k\left(v\right)\right), \forall i. \tag{20}$$

Now, consider any $k \in N$. By (Efficiency),

$$\begin{aligned} v^k\left(N\right) &= \sum_{i\in N}\varphi_i\left(v^k\right) = \varphi_k\left(v^k\right) + \sum_{i\neq k}\varphi_i\left(v^k\right), \\ v^{k-1}\left(N\right) &= \sum_{i\in N}\varphi_i\left(v^{k-1}\right) = \varphi_k\left(v^{k-1}\right) + \sum_{i\neq k}\varphi_i\left(v^{k-1}\right). \end{aligned}$$

By (18) and (19),

$$v^k\left(N\right) - v^{k-1}\left(N\right) = \varphi_k\left(v^k\right) - \varphi_k\left(v^{k-1}\right) = \varphi_k\left(v^n\right) - \varphi_k\left(v^0\right).$$

By (14),

$$v^k\left(N\right) = v^{k-1}\left(N\right) + f_k\left(v\right).$$

Hence,

$$f_k\left(v\right) = \varphi_k\left(v^n\right) - \varphi_k\left(v^0\right).$$

Since $v^n = v$,

$$\varphi_k\left(v\right) = f_k\left(v\right) + \varphi_k\left(v^0\right).$$

By (20), we obtain:

$$\varphi_k\left(v\right) = f_k\left(v\right) + \frac{1}{n}\left(v\left(N\right) - \sum_{i\in N} f_i\left(v\right)\right),$$

which is exactly (f-ESS). ■

Remark 20 *Suppose* $\hat{\alpha}^S = \mathbf{1}\left(|S| = 1\right)$, *and* $\hat{\beta}^S = 0, \forall S$. *Then,* $f_i\left(v\right) = v\left(\{i\}\right), \forall i, v$. *This corresponds to the (CIS) characterization in Theorem 5. Suppose* $\hat{\alpha}^S = \mathbf{1}\left(S = N\right)$, *and* $\hat{\beta}^S = -\mathbf{1}\left(|S| = n-1\right)$. *Then,* $f_i\left(v\right) = v^*\left(\{i\}\right) = v\left(N\right) - v\left(N\backslash\{i\}\right), \forall i, v$. *This corresponds to the (ENSC) characterization in Theorem 11. Moreover, by letting* f_i *be any linear combination of* $v\left(\{i\}\right)$ *and* $v^*\left(\{i\}\right)$, *we obtain a characterization of the linear combination of the CIS and the ENSC.*

In addition, when the linear requirement for f is replaced by affinity, we can show that, the solution which satisfies Efficiency and Individual Monotonicity is invariant with respect to the constant term of the affinity condition.

Axiom 21 (Affinity) *For any i, $f_i(v)$ is an affine function of $\{v(S)\}_{S\subseteq N}$, that is, there exist constants $\left(\hat{\alpha}^S,\hat{\beta}^S\right)_{S\subseteq N}$ and γ such that*

$$f_i(v)=\sum_{S\ni i}\hat{\alpha}^S v(S)+\sum_{S\not\ni i}\hat{\beta}^S v(S)+\gamma. \qquad \text{(Aff)}$$

Proposition 22 *Suppose that f satisfies (Symmetry) and (Affinity). A solution φ satisfies (Efficiency), (Equal Treatment) and (f-IMES), if and only if it is the ESS value. Moreover, the solution is invariant with respect to γ, the constant term in the Affinity axiom.*

Proofs of the propositions in this section are relegated to the Appendix.

3.2 Characterization of the f-ESS family

In the previous subsection, we have first fixed a specific individual share represented by a linear and symmetric function f, and then provided a characterization of the f-ESS solution. Here, we provide a full characterization of the *f-ESS family*, the set of all solutions that can be obtained as a result of the egalitarian surplus sharing from *some* individual share f.

First, if no restriction is imposed on f, the answer becomes trivial. Any efficient solution φ can be written as an f-ESS by regarding φ itself as f. On the other hand, any f-ESS solution is efficient by definition. Therefore, the set of solutions which can be written as an f-ESS by any f coincides with the set of all efficient values.

Second, if f is restricted to be linear, the answer is straightforward: a solution φ is the f-ESS for some linear f, if and only if φ is efficient and linear. This follows from linearity of (f-ESS) in Definition 14 (a formal statement and its proof are relegated to the Appendix).

Now, suppose that f is linear and symmetric. Then, the set of f-ESS solutions turns out to include known values such as CIS, ENSC and the Equal Division. To provide a full description of the result, define a sequence of solutions $\left(\psi^k\right)_{k=1}^n$ as follows:

Definition 23 *For each k, define $\psi^k:\mathcal{V}^N\to\mathbb{R}^n$ by:*

$$\psi_i^k(v)=\left(1-\frac{k}{n}\right)\left(\sum_{S:|S|=k,S\ni i}v(S)\right)-\frac{k}{n}\left(\sum_{S:|S|=k,S\not\ni i}v(S)\right)+\frac{1}{n}v(N). \qquad (21)$$

In particular, note that ψ^1 coincides with the CIS, ψ^{n-1} coincides with the ENSC, and ψ^n coincides with the Equal Division: $ED_i(v)=v(N)/n$, $\forall i,\forall v$. Note also that $\psi^k(v)$ is efficient for each k, which can be shown by simple algebra.

Proposition 24 *There exists a linear and symmetric function* $f : \mathcal{V}^N \to \mathbb{R}^n$ *such that a solution* φ *is* f*-ESS, if and only if* φ *is written as an affine combination of* $\left(\psi^k(v)\right)_{k=1}^n$*, that is, there exist coefficients* $\left(\lambda^k\right)_{k=1}^n$ *such that* $\sum_{k=1}^n \lambda^k = 1$ *and*

$$\varphi(v) = \sum_{k=1}^n \lambda^k \psi^k(v). \tag{22}$$

The set of solutions described by (22) coincides with the set of all efficient, linear and symmetric solutions (Ruiz et al. (1998), Lemma 9).

As can be seen from these propositions, the process of deriving an efficient solution φ by ESS from an arbitrary individual share f can be viewed as an efficient extension operator using the principle of egalitarian surplus sharing. Further discussion on the characterization of extension operators is beyond the scope of the current paper and readers are invited to refer to Funaki et al. (2024).

4 Individual Monotonicity for Equal Ratio

4.1 A characterization of Proportional Division

The IMES axiom can be extended to the one which requires monotonicity with respect to the equal *ratio*, rather than the equal *surplus*. We then obtain a characterization of the Proportional Division value. To see that, consider the following class of games with positive individual worths:

$$\mathcal{V}_+^N := \left\{ v \in \mathcal{V}^N \,\middle|\, \sum_{k \in N} v(\{k\}) > 0 \right\}.$$

Axiom 25 (Individual Monotonicity for Equal Ratio: IMER) *For any* $v, w \in \mathcal{V}_+^N$*, if*

$$\frac{v(N)}{\sum_{k\in N} v(\{k\})} = \frac{w(N)}{\sum_{k\in N} w(\{k\})}$$

and $v(\{i\}) \geq w(\{i\})$*, then* $\varphi_i(v) \geq \varphi_i(w)$.

It is straightforward to see that if φ satisfies (IMER), then it also satisfies the Individualistic Property for Equal Ratio (IER).

Axiom 26 (Individualistic Property for Equal Ratio: IER) *For any* $v, w \in \mathcal{V}_+^N$*, if*

$$\frac{v(N)}{\sum_{k\in N} v(\{k\})} = \frac{w(N)}{\sum_{k\in N} w(\{k\})}$$

and $v(\{i\}) = w(\{i\})$*, then* $\varphi_i(v) = \varphi_i(w)$.

The Proportional Division is defined as follows.

Definition 27 (PD) *The Proportional Division value is defined as:*

$$\varphi_i(v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N), \ \forall i \in N. \tag{PD}$$

Theorem 28 *A solution φ satisfies (Efficiency), (Equal Treatment) and (IMER), if and only if it is the PD value.*

Proof. It is obvious that (PD) satisfies (Efficiency), (Equal Treatment) and (IMER). To show the opposite, fix a game $v \in \mathcal{V}_+^N$.

For each $k = 0, \cdots, n$, construct a game v^k as follows:

$$v^k(\{i\}) = \begin{cases} v(\{i\}) & \text{if } i \le k \\ 1 & \text{if } i > k \end{cases} \tag{23}$$

for the stand-alone coalitions,

$$v^k(S) = v(S) \tag{24}$$

for all $S \subsetneq N$ with $2 \le |S| < n$, and

$$v^k(N) = \frac{\sum_{i\le k} v(\{i\}) + n - k}{\sum_{i\in N} v(\{i\})} v(N) \tag{25}$$

for the grand coalition. Notice that $v^n = v$. On the other hand, v^0 satisfies $v^0(\{i\}) = 1$ for all i.

Now, define a game u^N by $u^N(S) = 1$ for all $S \subsetneq N$ and $u^N(N) = v^0(N)$. Then, (Equal Treatment) implies that $\varphi_i(u^N) = \varphi_j(u^N)$ for any $i, j \in N$. By (Efficiency), $\varphi_i(u^N) = v^0(N)/n$ for all $i \in N$. We can apply (IES) to u^N and v^0, and we have $\varphi_i(v^0) = v^0(N)/n$ for all $i \in N$. By letting $k = 0$ in (25), we have

$$\varphi_i(v^0) = \frac{v(N)}{\sum_{k\in N} v(\{k\})}. \tag{26}$$

Now, notice that, for any $k = 0, 1, \cdots n$, (23) and (25) imply that

$$\frac{v^k(N)}{\sum_{i\in N} v^k(\{i\})} = \frac{v^k(N)}{\sum_{i\le k} v(\{i\}) + n - k} = \frac{v(N)}{\sum_{i\in N} v(\{i\})}.$$

Therefore, fix any $k \in N$ and we have

$$\frac{v^k(N)}{\sum_{i\in N} v^k(\{i\})} = \frac{v^{k-1}(N)}{\sum_{i\in N} v^{k-1}(\{i\})}.$$

Moreover, (23) implies that $v^k(\{i\}) = v^{k-1}(\{i\}) = 1$ for any $i > k$, and $v^k(\{i\}) = v^{k-1}(\{i\}) = v(\{i\})$ for any $i < k$. Therefore, by applying (IER), we have:

$$\varphi_i(v^k) = \varphi_i(v^{k-1}) \text{ for any } i \ne k. \tag{27}$$

Therefore,

$$v^k(N) - \sum_{i \neq k} \varphi_i\left(v^k\right) = v^k(N) - \sum_{i \neq k} \varphi_i\left(v^{k-1}\right). \tag{28}$$

By (25), we have

$$\begin{aligned} v^k(N) &= \frac{\sum_{i \leq k} v(\{i\}) + n - k}{\sum_{i \in N} v(\{i\})} v(N), \\ v^{k-1}(N) &= \frac{\sum_{i \leq k-1} v(\{i\}) + n - (k-1)}{\sum_{i \in N} v(\{i\})} v(N). \end{aligned}$$

Hence,

$$v^k(N) - v^{k-1}(N) = \frac{v(\{k\}) - 1}{\sum_{i \in N} v(\{i\})} v(N),$$

that is,

$$v^k(N) = v^{k-1}(N) + \frac{v(\{k\}) - 1}{\sum_{i \in N} v(\{i\})} v(N) \tag{29}$$

Plugging in (29) into the right-hand side of (28), we obtain

$$v^k(N) - \sum_{i \neq k} \varphi_i\left(v^k\right) = v^{k-1}(N) + \frac{v(\{k\}) - 1}{\sum_{i \in N} v(\{i\})} v(N) - \sum_{i \neq k} \varphi_i\left(v^{k-1}\right).$$

By (Efficiency), we have $v^k(N) - \sum_{i \neq k} \varphi_i\left(v^k\right) = \varphi_k\left(v^k\right)$ and $v^{k-1}(N) - \sum_{i \neq k} \varphi_i\left(v^{k-1}\right) = \varphi_k\left(v^{k-1}\right)$. Hence,

$$\varphi_k\left(v^k\right) = \frac{v(\{k\}) - 1}{\sum_{i \in N} v(\{i\})} v(N) + \varphi_k\left(v^{k-1}\right). \tag{30}$$

Now, for any $i \in N$, by (27) and (30),

$$\varphi_i(v^n) = \varphi_i\left(v^i\right) = \frac{v(\{i\}) - 1}{\sum_{k \in N} v(\{k\})} v(N) + \varphi_i\left(v^{i-1}\right) = \frac{v(\{i\}) - 1}{\sum_{k \in N} v(\{k\})} v(N) + \varphi_i\left(v^0\right).$$

Remember $v^n = v$. By (26), we have:

$$\begin{aligned} \varphi_i(v) &= \frac{v(\{i\}) - 1}{\sum_{k \in N} v(\{k\})} v(N) + \varphi_i\left(v^0\right) \\ &= \frac{v(\{i\}) - 1}{\sum_{k \in N} v(\{k\})} v(N) + \frac{v(N)}{\sum_{k \in N} v(\{k\})} \\ &= \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} v(N). \end{aligned}$$

This is exactly (PD). ■

Readers may notice that the structure of the proof is analogous to that of Theorem 17 in the previous section, where Egalitarian Surplus Sharing is characterized by the IMES axiom. The resulting allocation becomes the PD value, if the axiom is replaced by that of IMER. Our proofs illustrate that the difference stems from the way in which the surplus is distributed. The difference of requiring IMES and IMER is reflected to the way whether the surplus is distributed additively or proportionally.

4.2 IMER with respect to f

We generalize the axiom of Individual Monotonicity for Equal Ratio with respect to the individual share defined by an arbitrary log-linear function f. For this purpose, we consider the following class of games with positive coalition worths:

$$\mathcal{V}_{++}^{N} := \left\{ v \in \mathcal{V}^{N} | v(S) > 0, \, \forall S \subseteq N, S \neq \emptyset \right\}.$$

We consider the set of positive individual shares $\mathcal{F}_{+} := \left\{ f : \mathcal{V}_{++}^{N} \to \mathbb{R}_{+}^{n} \right\}$.

Axiom 29 (f-Individual Monotonicity for Equal Ratio: f-IMER) *Fix $f \in \mathcal{F}_{+}$. For any $v, w \in \mathcal{V}_{++}^{N}$, if*

$$\frac{v(N)}{\sum_{k \in N} f_k(v)} = \frac{w(N)}{\sum_{k \in N} f_k(w)}$$

and $f_i(v) \geq f_i(w)$, then $\varphi_i(v) \geq \varphi_i(w)$.

Suppose that the function f satisfies symmetry (Axiom 15) and the following log-linearity.

Axiom 30 (Log-linearity) *For any i, $f_i(v)$ is a log-linear function of $\{v(S)\}_{S \subseteq N}$, that is, there exist constants $\left(\hat{\alpha}^S, \hat{\beta}^S\right)_{S \subseteq N, S \neq \emptyset}$ such that*

$$f_i(v) = \sum_{S \ni i} \hat{\alpha}^S \log v(S) + \sum_{S \not\ni i, S \neq \emptyset} \hat{\beta}^S \log v(S).$$

We first show the following lemma.

Lemma 31 *If f satisfies (Symmetry) and (Log-linearity), then there exist constants $\left(\alpha^k\right)_{k=1}^{n} \in \mathbb{R}^n$ and $\left(\beta^k\right)_{k=1}^{n-1} \in \mathbb{R}^{n-1}$ such that*

$$f_i(v) = \sum_{S \ni i} \alpha^{|S|} \log v(S) + \sum_{S \not\ni i, S \neq \emptyset} \beta^{|S|} \log v(S). \tag{31}$$

Proof. The proof is entirely analogous to that of Lemma 18. We obtain the result by replacing $v(S)$ with $\log v(S)$. ■

The Proportional Division value with respect to f is defined as follows:

Definition 32 (f-PD) *The Proportional Division value with respect to f is:*

$$\varphi_i(v) = \frac{f_i(v)}{\sum_{k \in N} f_k(v)} v(N). \tag{f-PD}$$

Theorem 33 *Suppose that $f \in \mathcal{F}_+$ satisfies (Symmetry) and (Log-linearity). A solution φ satisfies (Efficiency), (Equal Treatment) and (f-IMER), if and only if it is the f-PD value.*

Proof. The "if" part is obvious. We prove the "only if" part. By Lemma 31, f satisfies (31). Then, as in the proof of Theorem 17, we can construct a sequence of games $v^0, v^1, \cdots, v^n = v$ such that

$$f_i\left(v^k\right) = \begin{cases} f_i(v) & \text{if } i \leq k \\ 1 & \text{if } i > k \end{cases} \tag{32}$$

and

$$v^k(N) = \frac{\sum_{i \leq k} f_i(v) + n - k}{\sum_{i \in N} f_i(v)} v(N) \tag{33}$$

for $k = 0, 1, \cdots, n$.

Then, for $k = 1, \cdots, n$, we can apply (f-IMER) to v^k and v^{k-1}, and obtain that

$$\varphi_i\left(v^k\right) = \varphi_i\left(v^{k-1}\right) \text{ for any } i \neq k.$$

By (Efficiency),

$$\varphi_k\left(v^k\right) = \varphi_k\left(v^{k-1}\right) + \left(v^k(N) - v^{k-1}(N)\right).$$

Therefore,

$$\varphi_k(v) = \varphi_k\left(v^0\right) + \left(v^k(N) - v^{k-1}(N)\right).$$

As in the proof of Theorem 28, apply (Efficiency) and (Equal Treatment) to v^0 and we obtain $\varphi_i\left(v^0\right) = v^0(N)/n$ for all $i \in N$. Letting $k = 0$ in (33), we have:

$$\varphi_k\left(v^0\right) = \frac{v(N)}{\sum_{i \in N} f_i(v)}.$$

Also by (33),

$$v^k(N) - v^{k-1}(N) = \frac{f_k(v) - 1}{\sum_{i \in N} f_i(v)} v(N).$$

Therefore, we obtain:

$$\begin{aligned} \varphi_k(v) &= \frac{v(N)}{\sum_{i \in N} f_i(v)} + \frac{f_k(v) - 1}{\sum_{i \in N} f_i(v)} v(N) \\ &= \frac{f_k(v)}{\sum_{i \in N} f_i(v)} v(N), \end{aligned}$$

which is exactly equal to (f-PD). ■

By Definition 32, it is straightforward to see that the f-PD value is written as the sum of the following two terms:

$$\varphi_i(v) = f_i(v) + \frac{f_i(v)}{\sum_{k \in N} f_k(v)} \left(v(N) - \sum_{k \in N} f_k(v) \right). \tag{34}$$

This means that the f-PD allocation can be seen as the proportional distribution of the surplus, defined as the grand coalition worth net of the sum of individual shares specified by f. Each individual i first receives the individual share f_i and the remaining surplus is shared proportionally to f_i.

5 Characterization without Equal Treatment

We have required Equal Treatment as an axiom that brings symmetry among individuals in all the characterization results obtained in the previous sections. Without the requirement of Equal Treatment of individuals in society, however, the solution concept can incorporate social objectives of an asymmetric nature. In this section, we consider characterization of the solutions without Equal Treatment.

5.1 Weighted Surplus Sharing

We first drop the Equal Treatment axiom used in the characterization of the Egalitarian Surplus Sharing. We consider the Homogeneity axiom instead.

Axiom 34 (Homogeneity: H) *For any $\lambda \in \mathbb{R}$ and any game $v \in \mathcal{V}^N$, $\varphi(\lambda v) = \lambda \varphi(v)$, where λv is the game in which all coalition worths of v are multiplied by λ.*

Recall that the surplus is shared equally among all individuals in the ESS. Instead, we consider Weighted Surplus Sharing (WSS), in which the surplus is shared in proportion to a constant weight which sums up to one (Kongo, 2019; Yang et al., 2019). More precisely, WSS is defined as follows:

Definition 35 (Weighted Surplus Sharing: WSS) *A solution φ is a Weighted Surplus Sharing value, if there exists a constant vector $a = (a_i)_{i \in N}$ satisfying $\sum_{i \in N} a_i = 1$, such that*

$$\varphi_i(v) = v(\{i\}) + a_i \left(v(N) - \sum_{k \in N} v(\{k\}) \right), \ \forall i \in N, \forall v \in \mathcal{V}^N \tag{WSS}$$

The ESS value is a special case of WSS in which $a_i = 1/n$ for all i. Also, notice that we do not impose the restriction of $a_i \geq 0, \forall i \in N$, although the weight is often assumed to be non-negative in the literature (Kongo, 2019; Yang et al., 2019). The reason is that we do not exclude general forms of surplus sharing, in which certain individuals are taxed in order to achieve a desirable redistribution. The following theorem provides a characterization of WSS.

Theorem 36 *The solution φ satisfies (Efficiency), (IMES) and (H) if and only if it is a WSS value.*

Proof. Suppose φ is a WSS. Then, (Efficiency), (IMES) and (H) are obviously satisfied.

To show the opposite, fix a game $v \in \mathcal{V}^N$. Define a sequence of games $\left(v^k\right)_{k=0}^n$ as in (1), (2) and (3). Then, $v^0(\{i\}) = 0$ for all $i \in N$.

Now, let u^N be the standard N-unanimity game, that is, $u^N(S) = 1$ if $S = N$, and $u^N(S) = 0$ otherwise. Define a game $w := v^0(N)\, u^N$. Then, $w(N) = v^0(N)$ and $w(\{i\}) = 0$ for all $i \in N$. Hence, we have:

$$v^0(N) - \sum_{k \in N} v^0(\{k\}) = w(N) - \sum_{k \in N} w(\{k\})$$

and $v^0(\{k\}) = w(\{k\}) = 0$ for all $k \in N$. Therefore, we can apply (IMES) to obtain that

$$\varphi_k\left(v^0\right) = \varphi_k(w), \forall k \in N.$$

By (H), $\varphi_k(w) = v^0(N)\, \varphi_k\left(u^N\right)$. Letting $a_k := \varphi_k\left(u^N\right)$, we have:

$$\varphi_k\left(v^0\right) = v^0(N)\, a_k, \forall k \in N. \tag{35}$$

On the other hand, exactly in the same way as in the proof of Theorem 5, (1), (2) and (3), together with (Efficiency) and (IMES) imply (9), that is,

$$\varphi_i(v^n) = v(\{i\}) + \varphi_i\left(v^0\right), \ \forall i \in N.$$

Remember $v^n = v$. By (3) and (35),

$$\begin{aligned} \varphi_i(v) &= v(\{i\}) + v^0(N)\, a_i \\ &= v(\{i\}) + a_i \left(v(N) - \sum_{k \in N} v(\{k\}) \right). \end{aligned}$$

This is exactly (WSS) with $a_i = \varphi_i\left(u^N\right)$. ■

In Theorem 36, we have extended the set of characterized allocations from ESS to WSS by dropping Equal Treatment and requiring Homogeneity instead. Note that Homogeneity is independent of Equal Treatment. Readers may have noticed that the Homogeneity axiom is applied only to the unanimity game in the proof, and thus the requirement of Homogeneity is more than necessary. Such an intuition is correct. However, we have stated Theorem 36 using the Homogeneity axiom, because we believe it is sensible for an axiomatization theory to deliver axioms which are natural and easy to interpret.

On the other hand, if one aims to understand more precisely the mathematical boundary up to which one can relax the axioms in the characterization of ESS and WSS, further investigation would be useful.

The following axiom requires homogeneity of the solution only among the unanimity games.

Axiom 37 (Weak Homogeneity: WH) *For any* $\lambda \in \mathbb{R}$, $\varphi\left(\lambda u^N\right) = \lambda\varphi\left(u^N\right)$.

By definition, if a solution satisfies Homogeneity, it also satisfies WH. It is also obvious that Homogeneity is satisfied by both ESS and WSS. Moreover, Equal Treatment implies Weak Homogeneity, under Efficiency. Therefore, characterization of WSS is obtained if (H) is replaced by (WH) in Theorem 36, which is a slightly more powerful statement.

In the same vein, we can consider the following axiom between Equal Treatment and Weak Homogeneity (van den Brink, 2007):

Axiom 38 (Weak Symmetry: WS) *For every* $v \in \mathcal{V}^N$*, if* $v(S \cup \{i\}) = v(S \cup \{j\})$*,* $\forall S \subseteq N \backslash \{i, j\}$*,* $\forall i, j \in N$ *with* $i \neq j$*, then there exists a constant* $c \in \mathbb{R}$ *such that* $\varphi_i(v) = c$ *for all* $i \in N$*.*

By definition, it is straightforward to see that the requirement of Weak Symmetry is weaker than that of Equal Treatment, i.e., if a solution satisfies Equal Treatment, then it also satisfies WS. Since Equal Treatment is applied only to the unanimity games in the proofs of Theorems 5, 11, 17 and Proposition 22, we obtain all the characterizations in these claims, by weakening Equal Treatment to Weak Symmetry.

In turn, under the assumption of Efficiency, Weak Symmetry implies Weak Homogeneity, i.e., if a solution satisfies Efficiency and WS, then it also satisfies WH. Therefore, the proof of Theorem 36 also indicates that a characterization boundary between ESS and WSS lies between the requirement of Weak Symmetry and that of Weak Homogeneity. The relationship is summarized in Figure 1.

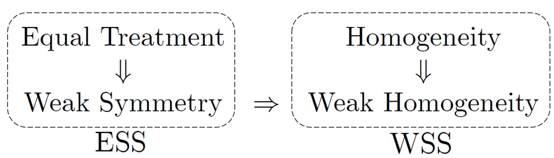

Figure 1: Relaxation of Equal Treatment (Efficiency is assumed).

5.2 Shifted Proportional Division

An analogous extension can be applied to the characterization of the Proportional Division value. Instead of Equal Treatment, we require Grand Coalition Homogeneity. For any game $v \in \mathcal{V}_+^N$, let v^λ be the game in which the grand coalition worth is multiplied by $\lambda \in \mathbb{R}^+$, while the worth $v(S)$ remains the same for all other coalitions $S \subsetneq N$.

Axiom 39 (Grand Coalition Homogeneity: GCH) *For any game* $v \in \mathcal{V}_+^N$ *and any* $\lambda \in \mathbb{R}^+$, $\varphi\left(v^\lambda\right) = \lambda\varphi(v)$.

Definition 40 (Shifted Proportional Division: SPD) *We say that φ is a Shifted Proportional Division value, if $\exists (b_i)_{i\in N}$ such that $\sum_i b_i = 0$ and*

$$\varphi_i(v) = \frac{v(\{i\}) + b_i}{\sum_{k\in N} v(\{k\})} v(N), \ \forall i \in N. \tag{SPD}$$

Theorem 41 *A solution satisfies (Efficiency), (IMER) and (GCH), if and only if it is a SPD value.*

Proof. The "if" part is obvious. We prove the "only if" part.

Define a sequence of games $(v^k)_{k=0}^n$ as follows. For each $k = 0, \cdots, n$, let

$$\begin{cases} v^k(N) = \frac{\sum_{j\leq k} v(\{j\}) + n - k}{\sum_{j\in N} v(\{j\})} v(N), \\ v^k(\{i\}) = \begin{cases} v(\{i\}) & \text{if } i \leq k \\ 1 & \text{if } i > k \end{cases}, \\ v^k(S) = v(S) \ \text{if } 1 < |S| < n. \end{cases} \tag{36}$$

In particular for $k = 0$, we have:

$$\begin{cases} v^0(N) = \frac{n}{\sum_{j\in N} v(\{j\})} v(N), \\ v^0(\{i\}) = 1, \forall i \in N. \end{cases}$$

Note also that $v^n = v$. Furthermore, the sequence is constructed so that

$$\frac{v^k(N)}{\sum_{j\in N} v^k(\{j\})} = \frac{v(N)}{\sum_{j\in N} v(\{j\})} \ \text{for all } k = 0, \cdots, n.$$

and

$$v^k(\{i\}) = v^{k-1}(\{i\}) \ \text{for all } k = 1, \cdots, n \text{ and } i \neq k.$$

Therefore, we can apply (IMER) to v^k and v^{k-1}, and obtain:

$$\varphi_i(v^k) = \varphi_i(v^{k-1}) \ \text{for all } k = 1, \cdots, n \text{ and } i \neq k. \tag{37}$$

In particular, for each $k = 1, \cdots, n$, we have:

$$\varphi_k(v^k) = \varphi_k(v^n) \ \text{and } \varphi_k(v^{k-1}) = \varphi_k(v^0). \tag{38}$$

Now, let w defined by $w(S) = 1$ for all nonempty $S \subseteq N$, and let $a_i = \varphi_i(w)$ for all $i \in N$. By (Efficiency), $\sum_{i\in N} a_i = 1$. Consider the game w^λ with $\lambda = v^0(N)$. Then, by (GCH), $\varphi_i(w^\lambda) = \lambda\varphi_i(w) = \lambda a_i = a_i v^0(N), \forall i \in N$. Since $v^0(N) = w^\lambda(N)$ and $v^0(\{i\}) = w^\lambda(\{i\}) = 1, \forall i \in N$, we can apply (IMER) to v^0 and w^λ and obtain $\varphi_i(v^0) = \varphi_i(w^\lambda), \forall i \in N$. Hence,

$$\varphi_i(v^0) = \varphi_i(w^\lambda) = a_i v^0(N) = \frac{a_i n}{\sum_{j\in N} v(\{j\})} v(N), \forall i \in N. \tag{39}$$

By (Efficiency),

$$\begin{aligned} v^k(N) &= \sum_{i\in N}\varphi_i\left(v^k\right)=\varphi_k\left(v^k\right)+\sum_{i\neq k}\varphi_i\left(v^k\right), \\ v^{k-1}(N) &= \sum_{i\in N}\varphi_i\left(v^{k-1}\right)=\varphi_k\left(v^{k-1}\right)+\sum_{i\neq k}\varphi_i\left(v^{k-1}\right). \end{aligned}$$

By (37), the last terms of the above equations are equal. Hence,

$$v^k(N)-v^{k-1}(N)=\varphi_k\left(v^k\right)-\varphi_k\left(v^{k-1}\right).$$

By (38),

$$v^k(N)-v^{k-1}(N)=\varphi_k\left(v^n\right)-\varphi_k\left(v^0\right).$$

Remember $v^n=v$ and by (39),

$$\begin{aligned} \varphi_k(v) &= v^k(N)-v^{k-1}(N)+\varphi_k\left(v^0\right) \\ &= v^k(N)-v^{k-1}(N)+\frac{a_k n}{\sum_{j\in N}v(\{j\})}v(N). \end{aligned}$$

On the other hand, by construction (36),

$$v^k(N)-v^{k-1}(N)=\frac{v(k)-1}{\sum_{j\in N}v(\{j\})}v(N).$$

Hence,

$$\varphi_k(v)=\frac{v(k)-1+a_k n}{\sum_{j\in N}v(\{j\})}v(N).$$

By letting $b_k=a_k n-1$, we have $\sum_{k\in N}b_k=n\sum_{k\in N}a_k-n=0$, and we obtain the result. ■

Grand Coalition Homogeneity is independent of the Equal Treatment axiom. Indeed, the requirement of GCH is stronger than what is necessary for characterizing the SPD. To see how far we can relax Equal Treatment, let us consider the following axiom analogous to the Weak Homogeneity defined above. For $\lambda\in\mathbb{R}^+$, let $\tilde{u}^\lambda$ be the game such that:

$$\tilde{u}^\lambda(S)=\begin{cases}1 & \text{if } S\neq N,\emptyset \\ \lambda & \text{if } S=N\end{cases}.$$

Axiom 42 *(Weak Grand Coalition Homogeneity: WGCH) For $\lambda\in\mathbb{R}^+$, $\varphi\left(\tilde{u}^\lambda\right)=\lambda\varphi\left(\tilde{u}^1\right)$.*

If a solution satisfies GCH, then it also satisfies WGCH. Since GCH is applied only to the game w in the proof of Theorem 41, we immediately have the following characterization:

Proposition 43 *A solution satisfies (Efficiency), (WGCH) and (IMER), if and only if it is a SPD value.*

Note that if a solution satisfies Equal Treatment and Efficiency, it also satisfies WGCH. Our results therefore indicate how far the Equal Treatment axiom can be relaxed so that the set of characterized solutions expands from PD to SPD.

5.3 Interpretation of the weakening of ET

We have seen above that the set of characterized solutions is expanded from the ESS to the WSS when the Equal Treatment axiom is relaxed. It is worth emphasizing that the WSS can be written as a shifted allocation based on the ESS. To see that, let $b_i := a_i - 1/n$, $\forall i \in N$ in (WSS). Then, we have:

$$\begin{aligned} WSS_i(v) &= v(\{i\}) + \left(\frac{1}{n} + b_i\right)\left(v(N) - \sum_{k\in N} v(\{k\})\right) \\ &= ESS_i(v) + b_i \left(v(N) - \sum_{k\in N} v(\{k\})\right). \end{aligned}$$

Similarly, the Shifted Proportional Division can be written as:

$$\begin{aligned} SPD_i(v) &= \frac{v(\{i\}) + b_i}{\sum_{k\in N} v(\{k\})} v(N). \\ &= PD_i(v) + b_i \frac{v(N)}{\sum_{k\in N} v(\{k\})}. \end{aligned}$$

In both cases, the vector of coefficients $b = (b_i)_{i\in N}$ satisfies $\sum_i b_i = 0$, and $b = 0$ is the special case in which the Equal Treatment axiom is satisfied. Therefore, relaxing the Equal Treatment axiom corresponds to an adjustment by a zero-sum transfer proportional to the vector b, which is fixed and applied to all games v.

Our results thus imply that the extended sets of allocations can incorporate social objectives of an asymmetric nature, such as redistribution, minority protection, support for the disabled, consideration of seniority, and so on. The coefficient vector b is fixed exogeneously in each society, but the same b is applied to all games v. As seen from the expressions above, the resulting allocation is written as a redistribution based on the Egalitarian Surplus Sharing or the Proportional Division, which represents the egalitarian or proportional principle, respectively. What is common in both cases is the structure in which relaxation of the Equal Treatment axiom leads to the redistribution term in the above expressions. Notice that the term multiplied by b_i corresponds to the equal surplus and the equal ratio, specified in the Individual Monotonicity axiom, respectively. Our characterization results thus highlight the common structure in the characterization of ESS and PD, and the central role played by the Individual Monotonicity axioms.

6 Conclusional remarks

In this paper, we provide a characterization of the Center of the Imputation Set value using the axioms of Individual Monotonicity for Equal Surplus, Efficiency and Equal Treatment. We show that the characterization is extended to the

Egalitarian Surplus Sharing, by defining surplus as the remainder of the grand coalition worth after subtracting the sum of the individual shares specified by any symmetric and affine function. Our characterization demonstrates that the three axioms lead to the *egalitarian allocation principle*, according to which each individual receives the sum of the two terms, the individual share and the egalitarian share of the surplus.

When the Individual Monotonicity axiom is required for Equal Ratio, again combined with the Efficiency and Equal Treatment, we obtain the *proportional principle*, according to which each individual receives the payoff proportional to the individual share. The main structure is the same: what each individual receives is the sum of the two terms, the individual share itself and the portion of the surplus distributed proportionally to the individual share. Our characterization thus highlights the essential role of the Individual Monotonicity axioms played in the characterization of two allocational principles.

We then relax the Equal Treatment axiom and show that the set of characterized allocations is extended to the Weighted Surplus Sharing and the Shifted Proportional Division, respectively. These allocations can be written as the result of a zero-sum redistribution based on Egalitarian Surplus Sharing and the Proportional Division respectively. Our result therefore explicitly demonstrate how the relaxation of Equal Treatment corresponds to the redistribution term in the resulting allocation. It turns out that redistribution is proportional to the equal surplus and equal ratio specified in the Individual Monotonicity axioms. Consequently, our results suggest that integrating social objectives of an asymmetric nature boils down to how to redistribute the equal surplus and equal ratio, while the remainder is allocated according to the two allocational principles also induced by the Individual Monotonicity axioms.

Our results are applicable to the discussion on the efficient allocation where there is a social agreement concerning the individual share which does not necessarily satisfy efficiency. For example, Banzhaf index is a semivalue which reflects individual's influence on the social outcome, and does not satisfy efficiency in general. In the commonly used normalization, the surplus is distributed proportionally to the individual share. While characterizations of the normalized Banzhaf value are available in the literature (van den Brink and van der Laan, 1998), our characterization provides a common ground for the analysis of proportional and egalitarian surplus sharing. Although the direct comparison of two types of normalization based on the common feature of Individual Monotonicity axioms is intriguing, further analysis is beyond the scope of the current paper and we leave it for future research.

References

Casajus, A. and F. Huettner (2014a): "Null, nullifying, or dummifying players: The difference between the Shapley value, the equal division value, and the equal surplus division value," *Economics Letters*, 122, 167–169.

——— (2014b): "Weakly monotonic solutions for cooperative games," *Journal of Economic Theory*, 154, 162–172.

Casajus, A. and K. Yokote (2019): "Weakly differentially monotonic solutions for cooperative games," *International Journal of Game Theory*, 48, 979–997.

Driessen, T. and Y. Funaki (1991): "Coincidence of and collinearity between game theoretic solutions," *OR Spektrum*, 13, 15–30.

Funaki, Y., Y. Koriyama, and S. Nakada (2024): "A characterization of the f-ESS solutions," Mimeo.

Joosten, R. A. M. G. (1996): "Dynamics, equilibria, and values," Ph.D. thesis, Maastricht University.

Kongo, T. (2019): "Players' nullification and the weighted (surplus) division values," *Economics Letters*, 183, 108539.

Nakada, S. (2024): "Shapley Meets Debreu: A Decision-theoretic Foundation for Monotonic Solutions of TU-games," Mimeo.

Ruiz, L. M., F. Valenciano, and J. M. Zarzuelo (1998): "The Family of Least Square Values for Transferable Utility Games," *Games and Economic Behavior*, 24, 109–130.

Shapley, L. S. (1953): *A Value for n-Person Games*, Princeton: Princeton University Press, 307–318.

van den Brink, R. (2007): "Null or nullifying players: The difference between the Shapley value and equal division solutions," *Journal of Economic Theory*, 136, 767–775.

van den Brink, R., Y. Chun, Y. Funaki, and Z. Zou (2023): "Balanced externalities and the proportional allocation of nonseparable contributions," *European Journal of Operational Research*, 307, 975–983.

van den Brink, R. and Y. Funaki (2009): "Axiomatizations of a class of equal surplus sharing solutions for TU-games," *Theory and Decision*, 67, 303–340.

van den Brink, R., Y. Funaki, and Y. Ju (2013): "Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values," *Social Choice and Welfare*, 40, 693–714.

van den Brink, R. and G. van der Laan (1998): "Axiomatizations of the normalized Banzhaf value and the Shapley value," *Social Choice and Welfare*, 15, 567–582.

Weber, R. J. (1988): *Probabilistic values for games*, Cambridge University Press, 101–120.

Yang, H., W. Wang, and Z. Ding (2019): "The Weighted Surplus Division Value for Cooperative Games," *Symmetry*, 11.

Yokote, K. and Y. Funaki (2017): "Monotonicity implies linearity: characterizations of convex combinations of solutions to cooperative games," *Social Choice and Welfare*, 49, 171–203.

Young, H. (1985): "Monotonic solutions of cooperative games," *International Journal of Game Theory*, 14, 65–72.

Zou, Z., R. van den Brink, Y. Chun, and Y. Funaki (2021): "Axiomatizations of the proportional division value," *Social Choice and Welfare*, 57, 35–62.

Zou, Z., R. van den Brink, and Y. Funaki (2022): "Sharing the surplus and proportional values," *Theory and Decision*, 93, 185–217.

A Appendix

A.1 Proof of Lemmas

Proof of Lemma 18. By Theorem 1 of Weber (1988), linearity of f implies that there exists a set of constants $\left(\alpha_i^S\right)_{i\in N, S\subseteq N}$ such that:

$$f_i(v) = \sum_{S\subseteq N} \alpha_i^S v(S), \forall i \in N, \forall v \in \mathcal{V}^N. \tag{40}$$

We first show the following claim.

Claim 44 *Suppose that f satisfies (Symmetry) and (Linearity). Then, there exists a set of constants $\left(\hat{\alpha}^S, \hat{\beta}^S\right)_{S\subseteq N}$ such that:*

$$f_i(v) = \sum_{S\ni i} \hat{\alpha}^S v(S) + \sum_{S\not\ni i} \hat{\beta}^S v(S), \forall i \in N, \forall v \in \mathcal{V}^N. \tag{41}$$

Proof of Claim 44. (Symmetry) implies the following (Equal Treatment) on f: if $v(S\cup\{i\}) = v(S\cup\{j\}), \forall S \subseteq N\setminus\{i,j\}$, then $f_i(v) = f_j(v)$.

Take an arbitrary $S \subseteq N$. Consider a game v such that $v(S) \neq 0$ and $v(T) = 0$ for all $T \neq S$, $T \subseteq N$. Then, by (40), $f_i(v) = \alpha_i^S v(S)$ and $f_j(v) = \alpha_j^S v(S)$. Now, suppose that there exist two players i and $j\,(\neq i)$ such that $i, j \in S$. By applying (Equal Treatment) to v and (i,j), we obtain $f_i(v) = f_j(v)$, which implies $\alpha_i^S = \alpha_j^S$. This holds for any $i, j \in S$. Similarly, suppose that there exist two players i and $j\,(\neq i)$ such that $i, j \notin S$. Again, we apply (Equal Treatment) to v and (i,j), and obtain $f_i(v) = f_j(v)$, which implies $\alpha_i^S = \alpha_j^S$. This holds for any $i, j \notin S$.

Now, for each $S \subseteq N$, replace α_i^S by $\hat{\alpha}^S$ for any i such that $i \in S$, and by $\hat{\beta}^S$ for any i such that $i \notin S$. From (40), we obtain (41). ■

In order to proceed with the proof of Lemma 18, fix a permutation π. Take any $i, j \in N$ such that $\pi(i) = j$. Then, by (Symmetry),

$$f_i(v) = f_{\pi(i)}(\pi v) = f_j(\pi v), \forall v. \tag{42}$$

Apply (Linearity) to $f_j(\pi v)$, and we have:

$$f_j(\pi v) = \sum_{S \ni j} \hat{\alpha}^S \pi v(S) + \sum_{S \not\ni j} \hat{\beta}^S \pi v(S). \tag{43}$$

Since $\pi v(\pi S) = v(S)$ by definition, (43) is equal to:

$$\sum_{S \ni j} \hat{\alpha}^S v\left(\pi^{-1}(S)\right) + \sum_{S \not\ni j} \hat{\beta}^S v\left(\pi^{-1}(S)\right). \tag{44}$$

For each S such that $S \ni j$, let $S' = \pi^{-1}(S \setminus \{j\}) \cup \{i\}$. Since $\pi(i) = j$, this induces a bijection from $\{S | S \ni j\}$ to $\{S' | S' \ni i\}$. Moreover, $\pi(S') = S$. Therefore, the first term of (44) becomes:

$$\sum_{S \ni j} \hat{\alpha}^S v\left(\pi^{-1}(S)\right) = \sum_{S' \ni i} \hat{\alpha}^{\pi(S')} v(S').$$

Similarly, by setting $S'' = \pi^{-1}(S)$, the second term of (44) becomes:

$$\sum_{S \not\ni j} \hat{\beta}^S v\left(\pi^{-1}(S)\right) = \sum_{S'' \not\ni i} \hat{\beta}^{\pi(S'')} v(S'').$$

Therefore, by (43), we have:

$$f_j(\pi v) = \sum_{S' \ni i} \hat{\alpha}^{\pi(S')} v(S') + \sum_{S'' \not\ni i} \hat{\beta}^{\pi(S'')} v(S'').$$

By (Linearity) and (42), the following equality should hold for any v:

$$\sum_{S \ni i} \hat{\alpha}^S v(S) + \sum_{S \not\ni i} \hat{\beta}^S v(S) = \sum_{S' \ni i} \hat{\alpha}^{\pi(S')} v(S') + \sum_{S'' \not\ni i} \hat{\beta}^{\pi(S'')} v(S'').$$

This is an identity with respect to $\{v(S)\}_{S \subseteq N}$. By comparing the coefficients of $v(S)$ on both sides for any S such that $S \ni i$, we have $\hat{\alpha}^S = \hat{\alpha}^{\pi(S)}$. Since this should hold for any $i, j \in N$ such that $\pi(i) = j$ and $i \in S$, there exists a constant α^k such that $\alpha^k = \hat{\alpha}^S$ for any $S \ni i$ such that $|S| = k$. Similarly, by comparing the coefficients of $v(S)$ for any S such that $S \not\ni i$, we obtain $\hat{\beta}^S = \hat{\beta}^{\pi(S)}$. Hence, there exists a constant β^k such that $\beta^k = \hat{\beta}^S$ for any $S \not\ni i$ such that $|S| = k$. Finally, we obtain (10). ■

Proof of Lemma 19. Suppose A has a rank less than n. Then, there exists a linear combination of n row vectors which sum up to the zero vector, that is, there exists a non-zero vector $t = (t_i)_{i=1}^n \in \mathbb{R}^n$ such that

$$\sum_i t_i a_{iS} = 0, \forall S \subsetneq N. \tag{45}$$

Then, for any $S \subsetneq N$ such that $|S| = k$,

$$\alpha^k \left(\sum_{i \in S} t_i \right) + \beta^k \left(\sum_{i \notin S} t_i \right) = 0.$$

Take any $i, j \in N$ such that $i \neq j$, and $S' \subseteq N \setminus \{i, j\}$ such that $|S'| = k - 1$. Let $S_1 = S' \cup \{i\}$, $S_2 = S' \cup \{j\}$, and we have:

$$\begin{aligned} \alpha^k \left(\sum_{i' \in S_1} t_{i'} \right) + \beta^k \left(\sum_{i' \notin S_1} t_{i'} \right) &= 0, \\ \alpha^k \left(\sum_{i' \in S_2} t_{i'} \right) + \beta^k \left(\sum_{i' \notin S_2} t_{i'} \right) &= 0, \end{aligned}$$

By subtracting one from the other, we obtain

$$\alpha^k (t_i - t_j) - \beta^k (t_i - t_j) = 0.$$

Since $\alpha^k \neq \beta^k$, we have $t_i = t_j$. Since the choice of i and j was arbitrary, we have $t_1 = t_2 = \cdots = t_n$. Together with (45), we have $t_i = 0, \forall i$, which is a contradiction. ■

A.2 Proof of Propositions

Proof of Proposition 22. Since f satisfies affinity, $\exists \gamma \in \mathbb{R}$ and f' such that

$$f_i(v) = f'_i(v) + \gamma, \forall i \tag{46}$$

and f' satisfies linearity. Whenever

$$v(N) - \sum_{k \in N} f_k(v) = w(N) - \sum_{k \in N} f_k(w)$$

and $f_i(v) \geq f_i(w)$, we have

$$v(N) - \sum_{k \in N} f'_k(v) = w(N) - \sum_{k \in N} f'_k(w)$$

and $f'_i(v) \geq f'_i(w)$. Therefore, whenever we can apply (f-IMES), we can also apply (f'-IMES). We can thus apply Theorem 17 to f', and we obtain:

$$\varphi_i(v) = f'_i(v) + \frac{1}{n} \left(v(N) - \sum_{k \in N} f'_k(v) \right), \forall i \in N. \tag{f'-ESS}$$

Plugging (46) in it, we have

$$\begin{aligned}\varphi_i(v) &= f_i(v) - \gamma + \frac{1}{n}\left(v(N) - \sum_{k\in N}(f_k(v) - \gamma)\right) \\ &= f_i(v) - \gamma + \frac{1}{n}\left(v(N) - \sum_{k\in N} f_k(v) + n\gamma\right) \\ &= f_i(v) + \frac{1}{n}\left(v(N) - \sum_{k\in N} f_k(v)\right).\end{aligned}$$

The solution is exactly the same as (f-ESS), implying invariance of the solution with respect to γ. ■

Proof of Proposition 24. We start with the "only if" part. Suppose there exists a linear and symmetric function $f : \mathcal{V}^N \to \mathbb{R}^n$. Then, by Lemma 18, there exist constants $\left(\alpha^k\right)_{k=1}^n \in \mathbb{R}^n$ and $\left(\beta^k\right)_{k=1}^n \in \mathbb{R}^n$ such that

$$f_i(v) = \sum_{S\ni i} \alpha^{|S|} v(S) + \sum_{S\not\ni i} \beta^{|S|} v(S).$$

Then, we have:

$$\begin{aligned}\sum_{i\in N} f_i(v) &= \sum_{i\in N}\left(\sum_{S\ni i} \alpha^{|S|} v(S) + \sum_{S\not\ni i} \beta^{|S|} v(S)\right) \\ &= \sum_{S\subseteq N}\left\{|S|\,\alpha^{|S|} + (n - |S|)\,\beta^{|S|}\right\} v(S).\end{aligned}$$

Since φ is the f-ESS solution,

$$\begin{aligned}\varphi_i(v) &= f_i(v) + \frac{1}{n}\left(v(N) - \sum_{j\in N} f_j(v)\right) \\ &= \left(\sum_{S\ni i} \alpha^{|S|} v(S) + \sum_{S\not\ni i} \beta^{|S|} v(S)\right) + \frac{1}{n}\left(v(N) - \sum_{j\in N} f_j(v)\right) \\ &= \left(\begin{array}{l}\sum_{S\ni i}\left\{\alpha^{|S|} - \frac{|S|\alpha^{|S|} + (n-|S|)\beta^{|S|}}{n}\right\} v(S) \\ +\sum_{S\not\ni i}\left\{\beta^{|S|} - \frac{|S|\alpha^{|S|} + (n-|S|)\beta^{|S|}}{n}\right\} v(S)\end{array}\right) + \frac{1}{n} v(N) \\ &= \left(\begin{array}{l}\sum_{S\ni i}\left\{\frac{n-|S|}{n}\left(\alpha^{|S|} - \beta^{|S|}\right)\right\} v(S) \\ +\sum_{S\not\ni i}\left\{\frac{|S|}{n}\left(\beta^{|S|} - \alpha^{|S|}\right)\right\} v(S)\end{array}\right) + \frac{1}{n} v(N).\end{aligned}$$

Let $\gamma^k := \alpha^k - \beta^k$ for $k = 1, \cdots, n-1$. We thus obtain:

$$\varphi_i(v) = \sum_{S\ni i}\left(1 - \frac{|S|}{n}\right)\gamma^{|S|} v(S) - \sum_{S\not\ni i} \frac{|S|}{n}\gamma^{|S|} v(S) + \frac{1}{n} v(N). \tag{47}$$

Now, let $\bar{\varphi} := \varphi - ED$ and $\bar{\psi}^k := \psi^k - ED$. Then, (47) implies that $\bar{\varphi}$ is written as a linear combination of $\left(\bar{\psi}^k(v)\right)_{k=1}^{n-1}$ as follows:

$$\bar{\varphi}(v) = \sum_{k=1}^{n-1} \gamma^k \bar{\psi}^k(v).$$

Therefore, φ is written as an affine combination of $\left(\psi^k(v)\right)_{k=1}^{n-1}$ and ED.

Now we show the "if" part. Suppose that φ is written as an affine combination of $\left(\psi^k(v)\right)_{k=1}^{n-1}$ and ED as in (22). Let

$$f_i(v) = \sum_{S \ni i} \lambda^{|S|} v(S).$$

Then, f is linear and symmetric. We show that the induced f-ESS solution coincides with φ.

First, notice that when $(f_j)_{j \in N}$ are summed up, each $S \subseteq N$ is counted exactly $|S|$ times. Hence, we have

$$\begin{aligned} \sum_{j \in N} f_j(v) &= \sum_{S \subseteq N} |S| \lambda^{|S|} v(S) \\ &= \sum_{S \ni i} |S| \lambda^{|S|} v(S) + \sum_{S \not\ni i} |S| \lambda^{|S|} v(S). \end{aligned}$$

The f-ESS solution is then:

$$\begin{aligned} & f_i(v) + \frac{1}{n} \left\{ v(N) - \sum_{j \in N} f_j(v) \right\} \\ = & \sum_{S \ni i} \lambda^{|S|} v(S) + \frac{1}{n} \left\{ v(N) - \left(\sum_{S \ni i} |S| \lambda^{|S|} v(S) + \sum_{S \not\ni i} |S| \lambda^{|S|} v(S) \right) \right\} \\ = & \sum_{S \ni i} \left(1 - \frac{|S|}{n}\right) \lambda^{|S|} v(S) - \sum_{S \not\ni i} \frac{|S|}{n} \lambda^{|S|} v(S) + \frac{1}{n} v(N). \end{aligned} \tag{48}$$

By (21), (48) is equal to:

$$\begin{aligned} & \sum_{k=1}^{n-1} \lambda^k \psi_i^k(v) + \left(1 - \sum_{k=1}^{n-1} \lambda^k\right) \frac{1}{n} v(N) \\ = & \sum_{k=1}^{n-1} \lambda^k \psi_i^k(v) + \lambda^n \frac{1}{n} v(N), \end{aligned}$$

which is equal to $\varphi_i(v)$ in (22). ■

The following are a formal statement and its proof for the case where f is efficient and linear.

Proposition 45 *A solution φ is the f-ESS value for some linear f, if and only if φ is efficient and linear.*

Proof of Proposition 45. To show the only if part, suppose that there exists a linear f such that φ is the f-ESS solution, as in Definition 14. Then, it is obvious by definition that φ is efficient: $\sum_{i \in N} \varphi_i(v) = v(N), \forall v$. Moreover, for any $v, w \in \mathcal{V}^N$,

$$\begin{aligned}
\varphi_i(v+w) &= f_i(v+w) + \frac{1}{n}\left((v+w)(N) - \sum_{k \in N} f_k(v+w)\right) \\
&= f_i(v) + f_i(w) + \frac{1}{n}\left(v(N) + w(N) - \sum_{k \in N} f_k(v) - \sum_{k \in N} f_k(w)\right) \\
&= \varphi_i(v) + \varphi_i(w).
\end{aligned}$$

The first and the third equalities are by definition of φ, and the second is by linearity of f. Hence, φ is also linear.

If part is obvious: if φ is efficient and linear, then by setting φ itself as f, φ is the f-ESS. ∎