Price Discovery in Waiting Lists:
A Connection to Stochastic Gradient Descent

Itai Ashlagi[*]   Jacob D. Leshno[†]   Pengyu Qian[‡]   Amin Saberi[§]

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Abstract

Waiting lists offer agents a choice among types of items and associated non-monetary prices given by required waiting times. These non-monetary prices are endogenously determined by a simple tâtonnement-like price discovery process: an item's price increases when an agent queues for it, and decreases when an item arrives and a queuing agent is assigned. By drawing a connection between price adjustments in waiting lists and the stochastic gradient descent optimization algorithm, we show that the waiting list mechanism achieves the optimal allocative efficiency minus a loss due to price fluctuations that is bounded by the granularity of price changes. We further consider a price discovery process inspired by the waiting list mechanism and show that this simple price discovery process performs well if the granularity of price changes is chosen to appropriately trade-off the speed of price adaptation and loss from price fluctuations.

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[*]Stanford University, iashlagi@stanford.edu.

[†]University of Chicago, Booth, jacob.Leshno@ChicagoBooth.edu. This work is supported by the Robert H. Topel Faculty Research Fund at the University of Chicago Booth School of Business.

[‡]Purdue University, pqian20@gmail.com.

[§]Stanford University, saberi@stanford.edu.

1 Introduction

Public goods are commonly assigned via waiting lists, which offer agents a choice among items and associated waiting times. The required waiting cost for an item can be interpreted as the item's price, as waiting times serve a similar role to monetary prices in guiding the allocation and rationing items. For example, the New York City Housing Authority (NYCHA) holds a separate queue for each public housing development, and applicants can join one development's queue.[1] Allowing agents to choose developments allows NYCHA to better match applicants to developments, and the mechanism rations desirable projects by requiring higher waiting times for desirable projects.

A key feature of waiting lists is that waiting times are endogenously determined and adjust with every change to the queue lengths. Waiting times adjustments are similar to a tâtonnement process: the wait for an item increases when an agent joins its queue, and the wait decreases when an item is supplied to an agent that leaves the queue. In this paper we view this adjustment process as a price discovery process, and analyze its allocative efficiency.[2]

We consider an economy in which agents and items arrive over time. The utility of an agent is quasi-linear in waiting costs, allowing us to consider expected waiting costs as prices. Agents have unit demand and their values for items are drawn independently from an arbitrary joint value distribution. There are finitely many types of items, and the waiting list mechanism maintains a queue for each item type. Arriving agents can join a single queue and wait until being assigned the respective item. The agent observes the queue lengths and chooses the item that maximizes her value minus the price.[3]

Because the length of the queues randomly changes as agents and items arrive, the price (waiting cost) for an item fluctuates over time. To illustrate the inefficiency caused by such fluctuations, we present a simple example.

Example 1. *A single kind of item is allocated by queue. Items arrives according to a Poisson process with rate* 1. *Agents arrive according to a Poisson process with rate* 2. *Agents have heterogeneous values for the item distributed* $v \sim U[0,1]$ *and a cost of waiting* $c(w) = 0.02 \cdot w$. *Arriving agents choose whether to join the queue or leave unassigned after observing the length of the queue.*

Intuitively, only half the agents in Example 1 can be assigned items, and items will be assigned to the highest value agents if all agents face the market-clearing price of $1/2$. This

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occurs when agents observe a queue length of 24.[4] Figure 1a presents the probability an arriving agent will observe any possible queue length in steady state. It shows that queue lengths are likely to be close to 24, but fluctuate considerably. These fluctuations reduce the allocative efficiency, as shown in Figure 1b. For example, agents with a value of $0.4, 0.6$ are assigned the item with probabilities 18% and 81%, respectively.

(a) Queue length distribution
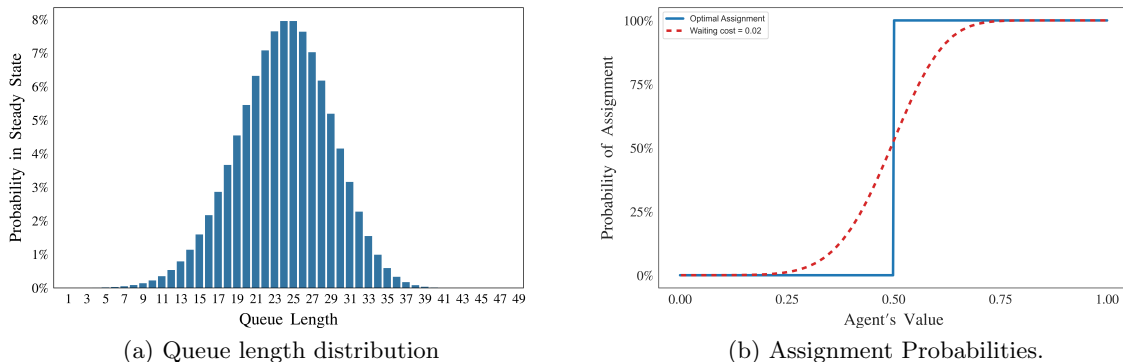
(b) Assignment Probabilities.

Figure 1: Distribution of queue lengths and implied assignment probabilities for Example 1.

The stochastic evolution of prices poses a challenge to the analysis. As illustrated in the example, prices depend on the lengths of the queues, and thus fluctuate over time. A natural approach for analyzing this random process is to calculate the stationary distribution and the implied distribution of prices. This approach is not tractable when there are more than two item types because the stationary distribution cannot be tractably calculated.[5] Instead, this paper takes a different technical approach.

Our main result bounds the allocative efficiency of the waiting list mechanism using the granularity of price adjustments. Define the adjustment size Δ to be the maximal change in an item's price (waiting costs) due to the addition of one agent to the item's queue. We show that the allocative efficiency of the waiting lists mechanism is at least as high as the maximal possible allocative efficiency minus an efficiency loss that is roughly equal to the adjustment size Δ.

As an illustration, suppose waiting costs are linear. The adjustment size is the maximal cost of waiting for one item's arrival. Thus, the allocative efficiency of the waiting list mechanism is roughly at least as high as optimally assigning all agents but making all agents wait for an additional arrival. To illustrate why the efficiency loss is affected by the adjustment size, consider a type of apartments whose market clearing price is equivalent

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[4]An agent who observes a queue length of 24 will wait until 25 copies of the item arrive, as the first 24 copies will be assigned to the agents ahead of him.

[5]For this reason, several papers were limited to considering economies with at most two kinds of items (see, e.g., Baccara et al. (2018); Leshno (2017); Bloch and Cantala (2017)).

to 6 months of waiting. If an apartment arrives each month, a single agent that joins the queue changes the apartment's price significantly and these large fluctuations can cause significant efficiency losses. If an apartment arrives daily, an additional agent that joins the queue does not change the apartment's price by much, and the efficiency loss is relatively low.

Because the adjustment size Δ decreases as agents become more patient, our main result implies that allocative efficiency under the waiting list mechanism approaches the optimal allocative efficiency as agents become more patient. In addition, we show that the waiting list mechanism generates prices that are close to market prices with high probability. Thus, when agents become patient, the outcome of the waiting list mechanism approximates the competitive market outcome.

Additional results identify conditions for when the efficiency loss is small or large. To show that our bound is essentially tight, we construct an economy in which the allocative efficiency under the waiting list mechanism is lower than the optimal allocative efficiency by approximately Δ. We also identify economies in which the efficiency loss is small. We show that in generic economies with finitely many agent types, the loss decreases exponentially with the adjustment size Δ.

Our results are enabled by a connection between the waiting list mechanism and the stochastic gradient descent (SGD) algorithm. The SGD algorithm is an iterative optimization method that takes random steps that improve its objective in expectation.[6] When used as an optimization algorithm, the step size is reduced to zero as the SGD algorithm approaches the optimal value. We show that the run of the waiting list mechanism is equivalent to a run of the SGD algorithm that optimizes allocative efficiency, but with step size that remains constant. The step size is exogenously given by the item arrival rates and the agent waiting costs, and is bounded by the adjustment size. Thus, prices under the waiting list mechanism tend to adjust towards market clearing prices, but never converge due to the constant step size. Intuitively, the fluctuations and the efficiency loss are smaller if the adjustment size is smaller.

The relation to SGD allows us to employ technical tools that are commonly used for analyzing such algorithms. The proof leverages Lyapunov theory to obtain tractable bounds for allocative efficiency. By using a Lyapunov function, we are able to obtain a bound that does not require calculation of the stationary distribution.

Finally, we consider a price discovery process inspired by the waiting list mechanism. We show that an appropriately chosen adjustment size produces a pricing heuristic that performs well. Consider a market with finite horizon in which items and agents arrive over time. A planner wishes to set monetary prices to maximize welfare, but has no knowledge

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[6]This algorithm is commonly used with great success and gained much attention in recent years because of its usefulness for training neural networks (see e.g. LeCun et al. (2012)).

about the distribution of agent preferences or the market clearing prices. The waiting list mechanism suggests a simple pricing heuristic: increasing an item's price by a fixed adjustment size when it is demanded and decreasing an item's price by the adjustment size when it is supplied. We show that this simple heuristic is asymptotically optimal when the adjustment size is chosen according to the market's time horizon.

This simple heuristic exhibits a form of price rigidity. In markets with finite horizon, the optimal adjustment size needs to trade off two different sources of loss. Roughly speaking, if prices are far from optimal, a large adjustment size helps the heuristic approach optimal prices faster. But when prices are close to optimal prices, a smaller adjustment size reduces price fluctuations by preventing the heuristic from reacting to noise. Knowledge of the market's time horizon allows the planner to choose an adjustment size that balances this trade off. The optimal adjustment size is flexible enough to allow the heuristic to learn and adapt, but rigid enough to prevent fluctuations.

Related Literature This paper contributes to the growing literature on waiting lists as dynamic assignment mechanisms. Two important motivating applications are the assignment of public housing (Kaplan, 1984, 1988) and organ allocation (Zenios, 1999; Su and Zenios, 2004). The works of Bloch and Cantala (2017); Leshno (2017); Arnosti and Shi (2020); Baccara et al. (2018); Che and Tercieux (2021) develop tractable theoretical analysis of dynamic assignment problems. These papers restrict attention to markets with at most two kinds of items because the analysis in these papers relies on calculating the stationary distribution of the stochastic process, which is not possible when there are three or more kinds of items. A growing empirical literature studies dynamic assignment of public housing apartments (Waldinger, 2021; van Dijk, 2019), transplant organs (Agarwal et al., 2021), and hunting licences (Verdier and Reeling, 2021).

Several papers are concerned with the stochasticity in dynamic two-sided trading markets in order to optimize clearing timing decisions Mendelson (1982); Kelly and Yudovina (2018); Loertscher et al. (2018). These papers also restrict attention to either a single asset or a binary type space. This paper takes a different technical approach, allowing us to consider economies with any number of items.

The literature on waiting lists as dynamic assignment mechanisms builds upon the large literature going back to Naor (1969) that studies queues with strategic agents (see Hassin and Haviv (2003); Hassin (2016) for a review). This literature highlights the inefficiencies caused by fluctuations of a queue's length. A growing literature studies matching in queues under non-strategic stochastic arrivals (see Weiss (2021) for a review). Adan et al. (2018) give a Markovian characterization for stochastic matching models, but conjectures that calculating the stationary distribution is computationally hard.

This paper also relates to papers that study the convergence of tâtonnement processes

using gradient descent. Numerous paper analyze these processes in markets with multiple goods (Uzawa, 1960; Cole and Fleischer, 2008; Cheung et al., 2018, 2019) and in congestion or transportation settings (Powell and Sheffi, 1982; Correa and Stier-Moses, 2010). Bubeck et al. (2019) and Roth et al. (2020) leverage SGD type algorithms to develop dynamic pricing algorithms that efficiently learn. These papers consider the problem of a planner that needs to learn prices and identify processes that converge to optimal prices. In contrast, under the waiting list mechanism prices never converge. A key distinction is that under the waiting list mechanism the adjustment size is fixed, while processes that converge to optimal prices must shrink the step size over time.

Organization of the Paper Section 2 provides the model. The main results are stated in Section 3. Section 4 provides the intuition and proofs for our main results for linear waiting costs. The proof for general waiting costs is in Appendix A. Section 5 analyzes an SGD pricing heuristic that mimics the price adjustments of the waiting list mechanism and shows that such heuristics perform well in finite horizon economies if the adjustment size is appropriately set. Section 6 concludes with a discussion of our modeling choices.

Appendix B contains all other omitted proofs. Appendix C provides additional results: calculating the adjustment size under non-linear waiting costs, and providing explicit convergence rates.

2 Model

We study an infinite horizon economy, in which agents and items arrive randomly over time. We describe the economy, set benchmarks for allocative efficiency, and describe the waiting list mechanism.

Economy. We consider a market in which items and unit-demand agents arrive over time. Agents arrive according to a Poisson process with rate λ. Each agent has a type θ drawn independently according to a probability distribution F over the set of types Θ. We assume that Θ is a compact subset of an Euclidean space, and allow for both finitely many agent types as well as a continuum of agents.

Items arrive according to a Poisson process with total rate μ normalized to 1. The agent and item arrival processes are independent. Each arriving item is of a type $j \in \mathcal{J} = \{1, 2, \ldots, J\}$. An item is of type j with probability $\mu_j > 0$, where $\sum_{j \in \mathcal{J}} \mu_j = \mu = 1$. Denote by $\mu_{\min} \triangleq \min_{j \in \mathcal{J}} \mu_j > 0$. We define an auxiliary item type $\emptyset$, which denotes being unassigned and use $\mathcal{J}_\emptyset \triangleq \mathcal{J} \cup \{\emptyset\}$.

The value an agent of type $\theta \in \Theta$ obtains from getting assigned to an item of type $j \in \mathcal{J}_\emptyset$ is given by $v(\theta, j)$, where we normalize $v(\theta, \emptyset) = 0$. Agents' utilities are quasi-linear

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in waiting costs; an agent of type θ that is assigned to item of type j after waiting w units of time receives a utility of

$$u_\theta(j, w) \triangleq v(\theta, j) - c(w) \,,$$

where $c(w)$ is the cost of waiting w units of time.

A market is specified by $\left[\Theta, \mathcal{J}, F, \lambda, \{\mu_j\}_{j \in \mathcal{J}}, v(\cdot, \cdot), c(\cdot)\right]$. We say that there are finitely many agent types if F corresponds to finitely many atoms. A market with finitely many agent types can be specified by $\left[\Theta, \mathcal{J}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{v}, c(\cdot)\right]$ with $\boldsymbol{\lambda} \in \mathbb{R}_{++}^{|\Theta|}$, $\boldsymbol{\mu} \in \mathbb{R}_{++}^{|\mathcal{J}|}$, and $\mathbf{v} \in \mathbb{R}^{|\Theta| \times |\mathcal{J}|}$.

We make the following technical assumptions. We assume that for each $j \in \mathcal{J}$, $v(\theta, j)$ is continuous in θ and bounded from above by $v_{\max} \in \mathbb{R}$. We assume that the waiting cost function is smooth, strictly increasing, weakly concave,[7] and $c(0) = 0$, $\lim_{w \to \infty} c(w) = \infty$.

To simplify notation, we consider an equivalent discrete time process[8] indexed by t which records the sequence of arrivals. For each arrival epoch t, the indicator ξ_t equals one if the t-th arrival is an agent, and equals zero if the t-th arrival is an item. If $\xi_t = 1$, let θ_t denote the type of the agent arriving at t. If $\xi_t = 0$, let $j_t \in \mathcal{J}$ denote the type of item arriving at t. To simplify notation, we also let $\theta_t = \emptyset$ and $v(\theta_t, \cdot) = 0$ for $\xi_t = 0$.

Assignments and Allocative Efficiency. An *assignment* η assigns each agent with one item, and each arriving item to at most one agent. The allocative efficiency of a matching is defined as the average item's value to its assigned agent. Formally, given assignment η, for each epoch t such that $\xi_t = 1$, let $\eta_t \in \mathcal{J}_\emptyset$ be the kind of item assigned under η to the agent of type θ_t that arrived in epoch t. Let $A(T) = \{t \leq T \mid \xi_t = 1\}$ be the set of epochs up to epoch T in which agents arrived. Allocative efficiency under η is defined as

$$W(\eta) = \liminf_{T \to \infty} \frac{1}{|A(T)|} \sum_{t \in A(T)} v(\theta_t, \eta_t) \,. \tag{1}$$

We restrict attention to assignments that satisfy a *no-Ponzi condition*. Loosely speaking, this condition ensures the assignment is approximately valid if the market terminates at some large finite time.[9] Formally, let $\mathcal{R}_T(\eta)$ denote the number of agents and items that arrived by time T and are waiting to be assigned at time T.[10] The assignment η satisfies

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[7] Our results also extend to convex $c(w)$ such that both $c'(w)$ and $c''(w)$ are subexponential, i.e., there exists α such that $c'(w), c''(w) \leq e^{\alpha w}$ for all $w \geq 0$.

[8] The equivalence is due to the Arrival Theorem of Poisson-driven processes (see, e.g., Wolff, 1982).

[9] For example, in the market of Example 1 there exists an assignment that assigns all agents to items by assigning agents to items that are further and further into the future (allowing the queue to grow arbitrarily large). However, if the market terminates at any finite time, only approximately half the agents can be assigned items. The no-Ponzi condition rules out such problematic assignments.

[10] In other words, $\mathcal{R}_T(\eta)$ counts the number of agents who arrived before time T and are assigned under

the no-Ponzi condition if there exists a finite $M \in \mathbb{R}$ such that $\mathcal{R}_T(\eta) < M$ for all T.

Define optimal allocative efficiency to be

$$W^{\text{OPT}} = \mathbb{E}\left[\sup_{\eta \in H} W(\eta)\right] ,$$

where H is the set of no-Ponzi assignments and the expectation is taken over all possible realizations.

Assignment by a Waiting List. Our main interest is to analyze the allocative efficiency of the assignment generated by a standard waiting list mechanism. The mechanism holds a separate First-Come-First-Served queue for each item. An arriving agent observes the length of the queue for each item and chooses to join the end of one of the queues, or take the auxiliary item immediately (i.e., balk). An agent who joins a queue will wait in that queue until receiving an item. When an item arrives, it is assigned to the agent at the head of its queue, if there is any; if the item's queue is empty, the item is discarded.

To formally describe the mechanism, let $\mathbf{q} = (q_1, \ldots, q_J) \in \mathbb{Z}_+^J$ denote the state where there are q_j agents in the queue for item j. An arriving agent of type θ who observes $\mathbf{q}$ and chooses to join the queue for item j will wait a random amount of time w_j before receiving item $j \in \mathcal{J}_\emptyset$, and will receive an expected utility of $v(\theta, j) - \mathbb{E}[c(w_j)|q_j]$. Thus, the agent will choose to join the queue for item $a(\theta, \mathbf{q}) \in \mathcal{J}_\emptyset$ given by

$$a(\theta, \mathbf{q}) \in \operatorname*{argmax}_{j \in \mathcal{J}_\emptyset} \left\{ v(\theta, j) - p_j(q_j) \right\}, \tag{2}$$

where we define $p_j(q_j) \triangleq \mathbb{E}[c(w_j)|q_j]$. We refer to p_j as the *price* of item j. We allow agents to leave without joining any queue, and simplify notation by setting $p_\emptyset(\cdot) \equiv 0$. For notation simplicity, denote $\mathbf{p}(\mathbf{q}) \triangleq [p_1(q_1), \cdots, p_J(q_J)]$. Denote the queue lengths just before the t-th arrival by $\mathbf{q}_t$. That is, an agent that arrives at epoch t will face prices $\mathbf{p}_t = \mathbf{p}(\mathbf{q}_t)$, which depend on the current state of the queues $\mathbf{q}_t$. No queue length can ever exceed $q_{\max} \triangleq \max_{j \in \mathcal{J}} p_j^{-1}(v_{\max})$.

Given a realization, let η^{WL} denote the assignment induced by the waiting list. Under our assumptions, η^{WL} satisfies the no-Ponzi condition.[11] We denote the expected allocative efficiency of the waiting list mechanism by

$$W^{\text{WL}} \triangleq \mathbb{E}\left[W(\eta^{\text{WL}})\right] .$$

η to an item that arrives after time T, plus the number of items that arrive before time T and are assigned to agents that arrive after time T.

[11]Because no queue length can ever exceed $q_{\max} \triangleq \max_{j \in \mathcal{J}} p_j^{-1}(v_{\max})$ we have that $\mathcal{R}_t(\eta^{\text{WL}}) \leq |\mathcal{J}| \cdot q_{\max}$.

We refer to $W^{\text{OPT}} - W^{\text{WL}}$ as the *allocative efficiency loss*, or loss for short.

2.1 The static assignment problem and prices

It will be helpful to consider a natural corresponding static assignment problem in which all items and agents are simultaneously assigned. To transform the dynamic assignment problem into a static assignment problem, consider a planner that accumulates all agents and items that arrive up to some large time T. At time T, the planner assigns all agents and items to maximize allocative efficiency. In expectation, the number of items that arrive up to time T is proportional to its arrival rate and likewise for agents.[12]

We denote the maximal value per agent generated by the static assignment by W^* and refer to it as the optimal static allocative efficiency. The static assignment problem and W^* are given by problem (3).

$$
\begin{aligned}
W^* = \max_{\{x_{\theta j}\}_{\theta \in \Theta, j \in \mathcal{J}}} \quad & \sum_{j \in \mathcal{J}} \int_{\Theta} x_{\theta j} \, v(\theta, j) \, dF(\theta) \\
\text{subject to} \quad & \sum_{j \in \mathcal{J}} x_{\theta j} \leq 1, \ x_{\theta j} \in [0, 1] \qquad\qquad \forall \theta \in \Theta \qquad (3) \\
& \int_{\Theta} \lambda \, x_{\theta j} \, dF(\theta) \leq \mu_j \qquad\qquad\qquad \forall j \in \mathcal{J}
\end{aligned}
$$

In problem (3), $x_{\theta j}$ is the share of agents of type θ that are assigned item j. The problem's objective is the average agent's value of assigned items. The first constraint requires that the shares $x_{\theta j}$ are well defined. The second constraint is the resource constraint; it requires that the expected number of j items arriving per unit time should be at least as large as the expected number of agents that arrive per unit time and are assigned to item j. The following proposition shows that optimal allocative efficiency is identical under the static and dynamic problems.

Proposition 1. *The optimal allocative efficiency for the dynamic assignment problem is*

$$
W^{\text{OPT}} = W^* \, ,
$$

where W^* is the optimal static allocative efficiency.

To gain intuition for Proposition 1, observe that $W^{\text{OPT}} \leq W^*$ because the static problem allows the planner to ignore constraints due to arrival order.[13] To see that $W^{\text{OPT}} \geq W^*$, let

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[12]For any finite time T the realized number of arrivals is stochastic, but the stochastic error becomes negligible as $T \to \infty$ by the central limit theorem.

[13]The no-Ponzi condition ensures that long-run allocative efficiency is approximated by the allocative efficiency up to finite time T for $T \to \infty$.

$\mathbf{x}^*$ be an optimal solution for the static assignment problem (3) and consider the dynamic assignment η that assigns an arriving agent θ to the next unclaimed arrival of a j item with probability $x_{\theta j}^*$.[14] The proof of Proposition 1 is in Appendix B.1.

The static assignment problem is also helpful for providing us with market clearing prices. We use $\mathcal{P}^*$ to denote the set of market clearing prices for the static assignment problem. Standard results (e.g., Bertsekas (1981); Demange et al. (1986)) imply the existence of market clearing prices $\mathbf{p}^* \in \mathcal{P}^*$ that induce an optimal static assignment $\mathbf{x}^*$. Such market clearing prices $\mathbf{p}^*$ can be used to maximize allocative efficiency in the dynamic problem: we can generate an optimal dynamic assignment by asking each arriving agent to join the queue for an item $j \in \mathcal{J}_\emptyset$ that maximizes $v(\theta, j) - p_j^*$.

3   Efficiency of the Price Discovery Process

Prices under the waiting list mechanism are not set by a planner or a market clearing condition. Instead, prices change over time according to a tâtonnement like process: the price (waiting cost) of item j increases when an agent chooses to join queue j, and the price of item j decreases when item j arrives and one agent is removed from queue j. As Example 1 illustrates, this price discovery process does not necessarily converge to a market clearing prices $\mathbf{p}^*$, or to any single price. Agents arriving to the waiting list may observe prices that differ from equilibrium prices, possibly resulting in lower allocative efficiency.

A natural approach to analyze the waiting list mechanism is to calculate the distribution of prices agents face by calculating the stationary distribution of waiting list. However, this stationary distribution is not tractable when $|\mathcal{J}| \geq 3$ (i.e., there are more than 3 types of items).[15]

We therefore take a different approach that allows us to analyze general markets with any number of items, a general (possibly continuous) distribution of agent types, and nonlinear waiting costs. Our analysis shows that the following attribute plays an central role in determining allocative efficiency:

Definition 1. *The* adjustment size Δ *is the maximal change in price due to a single arrival, and is given by*

$$\Delta \triangleq \max_{j \in \mathcal{J}} \max_{1 \leq q \leq q_{\max}} \big(p_j(q) - p_j(q-1) \big).$$

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[14]If the resource constraint for item j holds with equality under $\mathbf{x}^*$, the number of agents waiting to be assigned item j behaves like the number of agents in a queue with arrival and departure rates equal to μ_j. Because this queue is critical, it violates the no-Ponzi condition. The proof shows that we can achieve approximately the same allocative efficiency with assignments that satisfy the no-Ponzi condition, for example by imposing a limit on the number of agents that can be waiting for item j.

[15]Because of this limitation, previous papers that relied on calculation of the stationary distribution were limited to models with at most 2 kinds of items (e.g., Leshno (2017); Bloch and Cantala (2017); Baccara et al. (2018)).

In other words, each arrival of an item j reduces the price of item j by at most Δ. Each arrival of an agent who joins the queue of item j increases the price of item j by at most Δ. If waiting costs are linear $c(t) = c \cdot t$, we have $\Delta = c/\mu_{\min}$, that is, Δ is the expected cost of waiting for a single arrival of the least frequent item.

Our main result provides a lower bound for the allocative efficiency under the waiting list. Theorem 1 bounds $W^{\mathrm{OPT}} - W^{\mathrm{WL}}$, which we refer to as the allocative efficiency loss under the waiting list mechanism. This bound holds for any distribution of agent preferences.

Theorem 1. *The allocative efficiency under the waiting list is*

$$W^{\mathrm{WL}} \geq W^{\mathrm{OPT}} - \frac{\lambda + 2}{2\lambda}\Delta. \tag{4}$$

Theorem 1 states that allocative efficiency under the waiting list mechanism is at least the optimal allocative efficiency minus a loss that depends on the adjustment size. If waiting costs are linear and the arrival rate of agents is twice as large as the arrival rate of items $(\lambda = 2, \mu = 1)$, then the allocative efficiency loss is bounded by the cost of waiting for a single arrival of the least frequent item times. The loss will be small if items arrival frequently and the cost of waiting for another arrival is small, but the loss can be large if items arrive infrequently.

As an illustration, consider the economy of Figure 1. Recall that the distribution of agent's value is $U[0,1]$ and half the agents can be assigned an item, implying that the market clearing price is $p^* = 0.5$. If $c(w) = 0.02w$, the market clearing is the expected wait for an agent who joins a queue with 24 agents, as $\mathbb{E}[c(w)|24] = 0.5$. Each arrival or departure of an agent change the price by $\Delta = 0.02$, which is 4% of p^*. In contrast, if the waiting cost is $\hat{c}(w) = 0.1w$, the market clearing price is the expected wait for an agent who joins a queue with 4 agents, as $\mathbb{E}[\hat{c}(w)|4] = 0.5$. In this case, each arrival or departure of an agent change the price by $\hat{\Delta} = 0.1$, which is 20% of p^*. Figure 2 presents the assignment probabilities under $c(\cdot)$ and $\hat{c}(\cdot)$, showing more misallocation under $\hat{c}(\cdot)$.

The proof of Theorem 1 in Section 4 shows that price adaptation in the waiting list mechanism is identical to the price adaptation of the stochastic gradient descent (SGD) optimization algorithm, which is an iterative optimization algorithm commonly used for training neural networks (see, e.g., LeCun et al., 2012). The SGD algorithm searches for an optimal solution by taking many iterative steps that each improve its objective in expectation. Each arrival moves prices in a random direction: a price increase if the arrival is an agent who chooses to join a queue, or a price decrease if the arrival is an item that removes an agent from a queue. But the expected change in an item's price is proportional to the difference between supply and demand for the item given current prices. In other words, in expectation each arrival moves prices towards market clearing prices. This makes
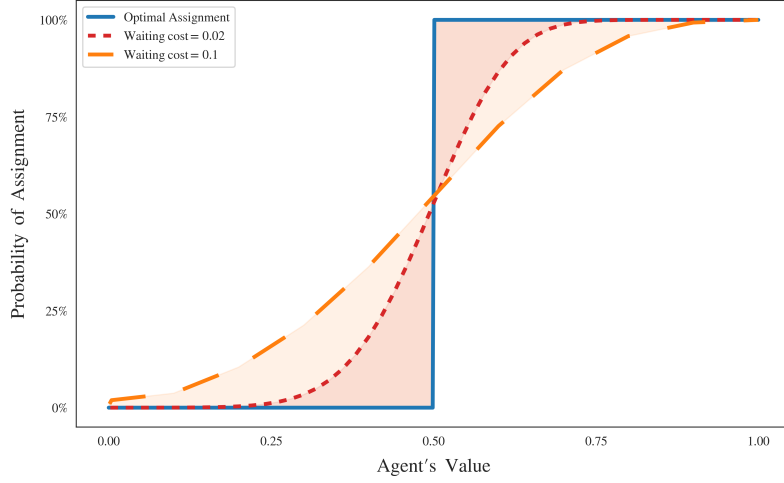
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Figure 2: Assignment probabilities under $c(w) = 0.02w$ and under $\hat{c}(w) = 0.1w$.

the price adaptation in the waiting list mechanism equivalent to a run of the SGD algorithm.

The adjustment size corresponds to the step size of the SGD algorithm. When used for optimization, the SGD algorithm is run with a step size that decreases to zero over time. In contrast, the adjustment size in the waiting list is determined exogenously by the item arrival rates and agent waiting costs. Because the adjustment size is fixed and bounded away from zero, the SGD algorithm never converges.

Theorem 1 helps identify situations for which public housing assignment through the waiting list mechanism may perform poorly. The adjustment size is determined by the agents' cost of waiting, as well as the frequency of apartment arrivals. Thus, the allocative efficiency loss from the fluctuating waiting times will be small when apartments arrive frequently or when agents are patient. In such situations, the queues for each item will typically hold many agents, and relative waiting cost fluctuations will be small. The loss may be large if apartments arrive infrequently, or if agents are impatient.

The following theorem provides a tighter bound under a few additional assumptions. It shows that if all market clearing prices are strictly positive, then the allocative efficiency loss is bounded by the adjustment size Δ plus an exponentially small loss.

Theorem 2. *Given an arbitrary distribution of agent types F and arbitrary item and agent arrival rates $\{\mu\}_{j \in \mathcal{J}}, \lambda$, consider market ℓ with linear waiting cost $c(w) = c_\ell \cdot w$. Let $W^{\mathrm{WL}}(c_\ell)$ denote the allocative efficiency of the waiting list mechanism for market ℓ, and let $\Delta_\ell = c_\ell / \mu_{\min}$ denote the adjustment size for market ℓ.*

Suppose that $\lambda \geq 1$ and $p_j^* > 0$ for any item $j \in \mathcal{J}$ under any market clearing price $\mathbf{p}^*$.

Then we have that

$$W^{\text{WL}}(c_\ell) \geq W^{\text{OPT}} - \Delta_\ell - \varepsilon$$

and there exist $\alpha, \beta, c_0 > 0$, *such that for any* $c_\ell < c_0$ *we have* $\varepsilon < \beta \cdot e^{-\alpha/\Delta_\ell}$.

The proof of Theorem 2 can be found in Appendix B.2. The remainder of this section provides additional results that further explore when the allocative efficiency loss is small or high.

3.1 Asymptotic optimality

The following corollaries of Theorem 1 show that the allocative efficiency under the waiting list mechanism approaches its optimal value as agents become more patient or item arrivals become more frequent.

Corollary 1. *Given an arbitrary distribution of agent types* F *and arbitrary item and agent arrival rates* $\{\mu\}_{j \in \mathcal{J}}, \lambda$, *consider market* ℓ *with linear waiting cost* $c(w) = c_\ell \cdot w$. *Let* $W^{\text{WL}}(c_\ell)$ *denote the allocative efficiency of the waiting list mechanism for market* ℓ. *We have that*

$$W^{\text{WL}}(c_\ell) \xrightarrow[c_\ell \to 0]{} W^{\text{OPT}}.$$

Observe that W^{OPT} and the market clearing prices $\mathbf{p}^*$ are independent of c_ℓ. As $c_\ell \to 0$, agents become more patient and the adjustment size $\Delta = c_\ell / \mu_{\min}$ tends to zero. Thus, by Theorem 1 the loss tends to zero. Intuitively, the length of queue j that is required to generate the optimal price is $q_j^* = p_j^*/c_\ell$, which is increasing as $c_\ell \to 0$. When queue j is longer, the price fluctuations due to agents joining and leaving the queue become relatively small.

Corollary 2. *Given an arbitrary distribution of agent types* F, *arbitrary item and agent arrival rates* $\{\mu\}_{j \in \mathcal{J}}, \lambda$, *and waiting cost* $c(\cdot)$, *consider market* n *in which the agent arrival rate is* $n \cdot \lambda$ *and the arrival rate of item* j *is* $n \cdot \mu_j$. *Let* $W^{\text{WL}}(n)$ *denote the allocative efficiency of the waiting list mechanism for market* n. *We have that*

$$W^{\text{WL}}(n) \xrightarrow[n \to \infty]{} W^{\text{OPT}}.$$

Intuitively, if the market thickens in the sense that arrivals of agents and items become more frequent, the expected cost of waiting for a single arrival becomes lower. For example, if $c(w) = c \cdot w$ and $\mu_{\min} \to \infty$, then $\Delta = c/\mu_{\min} \to 0$.

3.2 Lower bound for the allocative efficiency loss

A natural question is whether the bound given in Theorem 1 is tight. This section gives a lower bound for the allocative efficiency loss by constructing an economy in which the loss is approximately Δ.

Proposition 2. *For any $J \in \mathbb{N}, \Delta > 0$ and $\varepsilon > 0$, there exists a market with J items in which*

$$W^{\mathrm{WL}} \leq W^{\mathrm{OPT}} - \Delta + \varepsilon$$

The proof of Proposition 2 is in Appendix B.3. It shows the loss under the waiting list mechanism is large for the following market.

Example 2. *Consider a market in which $\Theta = \mathcal{J}$, that is, the set of items is $\mathcal{J} = \{1, 2, \ldots, J\}$ and there is a corresponding agent type for each item type. The distribution of agent types and item types are uniform, i.e., $\mathbb{P}(\theta = j) = \mu_j = 1/J, \forall j \in \mathcal{J}$. The total agent arrival rate is $\lambda = 1$, and the waiting cost is linear, i.e. $c(w) = c \cdot w$. The value of agent θ for item j is*

$$v(\theta, j) = \left\{ \begin{array}{ll} \gamma & \text{if } \theta = j \,, \\ 0 & \text{if } \theta \neq j \,. \end{array} \right.$$

In the market of Example 2, any price $\mathbf{p}$ where $0 \leq p_j \leq \gamma$ is a market clearing price. When the price exceeds γ agents leave without being assigned, causing a loss equal to γ. The length of queue j follows a reflected unbiased random walk over $0, 1, 2, \ldots, \lceil \gamma/\Delta \rceil$, and an arriving agent is equally likely to observe either of these queue lengths. Thus, the probability that agents observe a price equal to $\Delta \cdot \lceil \gamma/\Delta \rceil > \gamma$ is roughly Δ/γ and the expected loss is roughly $\gamma \cdot \Delta/\gamma = \Delta$.

The example of Proposition 2 can be changed slightly to show that the bound in Theorem 2 is tight up to exponentially small terms. Although the market of Example 2 does not satisfy the condition that $p_j^* > 0$ under any market price, we can create a similar market that satisfies this condition by slightly increasing the agent arrival rate to add agents that have a value Δ for all goods. The resulting market will have roughly the same allocative efficiency, and strictly positive prices for all items.

3.3 Small allocative efficiency loss with finitely many agents

The connection between the waiting list mechanism and the SGD optimization algorithm allows us to further explore the parameters of the economy that determine the magnitude of the allocative efficiency loss. In the market in Example 2, there is a wide range of market

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clearing prices. Under the waiting list mechanism, prices fluctuate widely (see the proof of Proposition 2), resulting in frequent misallocation. However, in markets with finitely many agent types, multiplicity of market clearing prices occurs only under a knife-edge set of parameters $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{v})$.

Lemma 1. *A market with finitely many agent types $\left[\Theta, \mathcal{J}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{v}, c(\cdot)\right]$ has unique market clearing prices for an open and dense set of vectors $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{v}) \in \mathbb{R}_{++}^{|\Theta|} \times \mathbb{R}_{++}^{|\mathcal{J}|} \times \mathbb{R}^{|\Theta| \cdot |\mathcal{J}|}$.*

The following results shows that the waiting list mechanism performs well in markets with finitely many agent types that have a unique market clearing price.

Theorem 3. *Given $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{v}) \in \mathbb{R}_{++}^{|\Theta|} \times \mathbb{R}_{++}^{|\mathcal{J}|} \times \mathbb{R}^{|\Theta| \cdot |\mathcal{J}|}$, consider market ℓ with linear waiting cost $c(w) = c_\ell \cdot w$. Assume that given $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{v})$ the market has a unique market clearing price. Let Δ_ℓ denote the adjustment size for market ℓ. Let $W^{\mathrm{WL}}(c_\ell)$ denote the allocative efficiency of the waiting list mechanism for market ℓ. Then there exists $\alpha, \beta, c_0 > 0$, such that for any $c_\ell < c_0$ we have that*

$$W^{\mathrm{WL}}(c_\ell) \geq W^{\mathrm{OPT}} - \beta \cdot e^{-\alpha/\Delta_\ell}$$

The intuition behind Theorem 3 is the following. If an economy with finitely many agent types has a unique market clearing price, the price is robust in the following sense: as long as prices are within some $\delta > 0$ distance from the unique market clearing price $\mathbf{p}^*$, agents choose items they can be assigned under an efficient assignment. In other words, loss can occur only when agents observe a price that is more than δ away from $\mathbf{p}^*$. In addition, any price different from $\mathbf{p}^*$ causes significant imbalance between supply and demand in the static assignment problem. This implies that the price adjustment under the waiting list mechanism follows a biased random walk towards $\mathbf{p}^*$, and agents are unlikely to observe prices that are far from $\mathbf{p}^*$. Together, these facts imply that prices that cause misallocation are unlikely to occur.

3.4 Price concentration

The following result shows that the waiting costs borne by agents under the waiting list mechanism are approximately equal to market clearing prices for the static assignment problem. We use $\mathrm{dist}(\mathbf{p}, \mathcal{P}^*) \triangleq \inf_{\mathbf{p}' \in \mathcal{P}^*} ||\mathbf{p} - \mathbf{p}'||_2$ to denote the minimal distance of a price to a market clearing price, and use $B_\epsilon(\mathcal{P}^*) = \{\mathbf{p} \in \mathbb{R}_+^{|\mathcal{J}|} : \mathrm{dist}(\mathbf{p}, \mathcal{P}^*) < \epsilon\}$ to denote the set of prices that are within ϵ of some market clearing price.

Proposition 3. *Given an arbitrary distribution of agent types F and arbitrary item and agent arrival rates $\{\mu\}_{j \in \mathcal{J}}, \lambda$, consider market ℓ with linear waiting cost $c(w) = c_\ell \cdot w$ and adjustment size Δ_ℓ. Let $\mathbf{p}(c_\ell)$ be a random variable generated by drawing the price observed*

by a randomly drawn agent arriving to the waiting list mechanism in market ℓ. Then for every $\epsilon > 0$ there exist $\alpha, \beta, c_0 > 0$ such that for any $c_\ell < c_0$ we have that

$$\mathbb{P}\Big(\mathbf{p}(c_\ell) \notin B_\epsilon(\mathcal{P}^*)\Big) < \beta \cdot e^{-\alpha/\Delta_\ell}.$$

Proposition 3 and Corollary 1 together imply that if agents are sufficiently patient, the waiting list mechanism approximates the competitive outcome of the static assignment problem. The proof of Proposition 3 is in Appendix B.5. Appendix C.2 gives explicit constants for the rate of convergence shown in Proposition 3.

An immediate implication of Proposition 3 is that items whose market clearing price is strictly positive are rarely wasted.

Corollary 3. *Given a distribution of agent types F and item and agent arrival rates $\{\mu\}_{j\in\mathcal{J}}, \lambda$, consider market ℓ with linear waiting cost $c(w) = c_\ell \cdot w$ and adjustment size Δ_ℓ. Suppose that for item j we have that $p_j > 0$ under any market clearing prices $\mathbf{p}$. Then there exist $\alpha, \beta, c_0 > 0$ such that for every market with $c_\ell < c_0$ the probability that an arriving item j is discarded is at most $\beta \cdot e^{-\alpha/\Delta_\ell}$.*

4 Intuition and Proofs

Before providing the proof of Theorem 1, we show the connection between the waiting list mechanism and the stochastic gradient descent (SGD) optimization algorithm. This connection provides the intuition for the Theorem 1 and inspired the technical approach in the proof. To formally show this connection, we use duality to formulate the static assignment problem as an optimization problem over possible prices. We then show that, in expectation, the run of the waiting list mechanism is a run of the SGD algorithm for this dual problem.

The Dual of the Static Assignment Problem   Consider the dual problem of the static assignment problem (3), which optimizes over possible prices. The following strong duality result is well-known.[16]

Lemma 2. *[Monge-Kantorovich duality] The optimal value W^* of assignment problem (3), coincides with the optimal value of the following dual optimization problem:*

$$\underset{\mathbf{p} \geq \mathbf{0}}{\text{minimize}} \ \ h(\mathbf{p})$$

---

[16]Problem (3) is known as the (unbalanced) optimal transport problem, which has the strong duality property stated in Lemma 2. For further details, see, e.g., Galichon (2018).

where (here we let $p_\phi \triangleq 0$)

$$h(\mathbf{p}) \triangleq \int_\Theta \max_{j \in \mathcal{J}_\emptyset} \left[v(\theta, j) - p_j \right] dF(\theta) + \frac{1}{\lambda} \sum_{j \in \mathcal{J}} \mu_j p_j \,. \tag{5}$$

Throughout the proof, we denote by $\mathbf{p}^*$ an optimal value for the dual problem, and also refer to $\mathbf{p}^*$ by optimal prices or market clearing prices.

Relation to Stochastic Gradient Descent. The SGD optimization algorithm is an iterative optimization algorithm. Each step of the SGD algorithm is random, but in expectation each step is a move in a subgradient direction of the algorithm's objective function. Under the waiting list mechanism, each arrival is a random draw of an agent or an item, but the expected arrival corresponds to supply and demand given current prices. In particular, the stochastic price adjustment from one arrival corresponds to a step in the SGD algorithm that seeks to minimize the dual objective $h(\mathbf{p})$. The next lemma formalizes this.

Lemma 3. *If the system is in state $\mathbf{q}_t$ such that $q_{j,t} \geq 1$ for all j, the expected change to the queue length from a single arrival $\mathbb{E}[\mathbf{q}_t - \mathbf{q}_{t+1}]$ equals $\frac{\lambda}{1+\lambda}$ times a subgradient of the dual objective $h(\mathbf{p}_t)$ at $\mathbf{p}_t = \mathbf{p}(\mathbf{q}_t)$.*

Indeed, observe that the expected adjustment to the length of queue j from a single arrival is

$$\mathbb{E}[q_{j,t+1} - q_{j,t}] = \mathbb{E}\left[\mathbb{1}_{\{\xi_t = 1, \, a(\theta_t, \mathbf{q}_t) = j\}} - \mathbb{1}_{\{\xi_t = 0, \, j_t = j\}} \right] \tag{6}$$

$$= \frac{\lambda}{1+\lambda} \int_\Theta \mathbb{1}_{\{j = \mathrm{argmax}_{j \in \mathcal{J}_\emptyset} \{v(\theta, j) - p_j(q_{j,t})\}\}} dF(\theta) - \frac{1}{1+\lambda} \mu_j \,,$$

and that $\frac{1+\lambda}{\lambda} \mathbb{E}[q_{j,t+1} - q_{j,t}]$ is a subgradient of $h(\mathbf{p}_t)$ at $\mathbf{p}_t = \mathbf{p}(\mathbf{q}_t)$.

So, on average, prices adjust in the right direction (but note the actual adjustments are random and depend on the realization of the arrival). For a gradient descent optimization algorithm to converge, the step size must shrink when approaching the optimal value. In the waiting list, the size of the adjustment is fixed and bounded by the step size Δ. Therefore, the price adjustment in the waiting list corresponds to the run of an SGD algorithm with a fixed step size and never converges.

4.1 Proof of Theorem 1

The proof is based on a Lyapunov analysis. A Lyapunov function will capture the "potential" in each state, which will allow us to decompose the expected value from the next arrival into a combination of the objective, change in potential, and a per-period loss.

We define the Lyapunov function to be the following quadratic function:

$$L(\mathbf{p}) = \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j p_j^2.$$

The following lemma is the key step in the proof.

Lemma 4. *The following inequality holds:*

$$\mathbb{E}[v(\theta_t, a(\theta_t, \mathbf{q}_t))|\mathbf{q}_t] \geq \frac{\lambda}{\lambda + 1} W^* - \underbrace{\frac{1}{\mu_{\min} \cdot \Delta} \left(L(\mathbf{p}_t) - \mathbb{E}[L(\mathbf{p}_{t+1})|\mathbf{q}_t] \right)}_{\text{(I) change in potential}} - \underbrace{\frac{2 + \lambda}{2(1 + \lambda)} \Delta}_{\text{(II) loss}}. \tag{7}$$

To interpret equation (7), observe that $\frac{\lambda}{\lambda+1} W^* = \frac{\lambda}{\lambda+1} W^{\text{OPT}}$ is the average per-arrival[17] value under the optimal assignment. Equation (7) shows that the waiting list achieves this value minus change in potential and a per-period loss. Summing over many periods, the change in potential (I) forms a telescoping series, and will therefore remain bounded. Therefore, as we average over many periods, we have that (I) tends to zero. The loss term (II) is uniformly bounded for any $\mathbf{p}_t$, which allows to bound the loss without calculating the stationary distribution.

Proof of Theorem 1. Let $W_T(\eta^{\text{WL}})$ be the total value of items assigned to agents that arrive before epoch T. That is

$$W_T(\eta^{\text{WL}}) = \sum_{t=1}^{T} v(\theta_t, a(\theta_t, \mathbf{q}_t)) ,$$

where we abuse notation by defining $v(\theta_t, a(\theta_t, \mathbf{q}_t)) \equiv 0$ when $\xi_t = 0$.[18] Therefore

$$W^{\text{WL}} = \frac{1 + \lambda}{\lambda} \mathbb{E} \left[\liminf_{T \to \infty} \frac{W_T(\eta^{\text{WL}})}{T} \right].$$

Observe that

$$\mathbb{E}\left[W_T(\eta^{\text{WL}})\right] = \mathbb{E}\left[\sum_{t=1}^{T} \xi_t \cdot v(\theta_t, a(\theta_t, \mathbf{q}_t)) \right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\sum_{t=1}^{T} \xi_t \cdot v(\theta_t, a(\theta_t, \mathbf{q}_t)) \mid \mathbf{q}_t \right] \right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}\left[\xi_t \cdot v(\theta_t, a(\theta_t, \mathbf{q}_t)) \mid \mathbf{q}_t \right] \right]$$

---

[17] Including both agent arrivals and item arrivals.
[18] That is, when an item arrives in epoch t .

$$\geq \mathbb{E}\left[\sum_{t=1}^{T} \frac{\lambda}{1+\lambda} W^* - \frac{1}{\mu_{\min} \cdot \Delta}\left(L(\mathbf{p}_t) - \mathbb{E}[L(\mathbf{p}_{t+1} \mid \mathbf{q}_t)]\right) - \frac{2+\lambda}{2(1+\lambda)}\Delta\right]$$

$$= T\frac{\lambda}{1+\lambda} W^* - \frac{1}{\mu_{\min} \cdot \Delta}\left(L(\mathbf{p}_1) - \mathbb{E}[L(\mathbf{p}_{T+1})]\right) - T\frac{2+\lambda}{2(1+\lambda)}\Delta \qquad (8)$$

where the inequality follows from Lemma 4.

Lemma 8, stated in Appendix B.2, uses standard argument to establish the following equality:

$$W^{\mathrm{WL}} = \frac{1+\lambda}{\lambda} \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[W_T(\eta^{\mathrm{WL}})\right] . \qquad (9)$$

Equality (9) allows us to translate the bound for $\mathbb{E}[W_T(\eta^{\mathrm{WL}})]$ to a bound for W^{WL}. Plugging in (8) to (9), we obtain

$$W^{\mathrm{WL}} \geq W^* - \frac{2+\lambda}{2\lambda}\Delta,$$

which completes the proof of Theorem 1. $\qquad\qquad \square$

Proof of Lemma 4: We begin with a few definitions. Let $\mathbf{a}_t$ and $\mathbf{g}_t$ be the vectors of indicators representing the arriving agent and items (goods) at time t, respectively:

$$\mathbf{a}_t \triangleq \mathbf{e}_{a(\theta_t,\mathbf{q}_t)}\xi_t , \qquad \mathbf{g}_t \triangleq \mathbf{e}_{j_t}(1-\xi_t) .$$

That is, $a_{j,t} = \mathbf{1}_{\{\xi_t=1,\, a(\theta_t,\mathbf{q}_t)=j\}}$ and $g_{j,t} = \mathbf{1}_{\{\xi_t=0,\, j_t=j\}}$. Let the indicator $l_{j,t} \triangleq \max\{0, g_{j,t} - q_{j,t} - a_{j,t}\}$ denote whether an item of type j arrived at time t and was discarded.[19] Using this notation, we can write the dynamics governing the evolution of the length of queue j by

$$q_{j,t+1} = [q_{j,t} + a_{j,t} - g_{j,t}]^+ = q_{j,t} + a_{j,t} - g_{j,t} + l_{j,t} , \quad \text{for each } j \in \mathcal{J}.$$

Step 1. We show that

$$\mathbb{E}\left[v(\theta_t, a(\theta_t, \mathbf{q}_t))\xi_t - \langle \mathbf{p}_t, \mathbf{a}_t - \mathbf{g}_t\rangle \mid \mathbf{q}_t\right] = \frac{\lambda}{1+\lambda}h(\mathbf{p}_t). \qquad (10)$$

Observe that

$$\mathbb{E}\left[v(\theta_t, a(\theta_t, \mathbf{q}_t))\xi_t - \langle \mathbf{p}_t, \mathbf{a}_t - \mathbf{g}_t\rangle \mid \mathbf{q}_t\right]$$

---

[19]Recall that under our definition of the waiting list mechanism, an item is discarded if the item finds its corresponding queue to be empty when it arrives.

$$= \mathbb{E}\Big[v(\theta_t, a(\theta_t, \mathbf{q}_t))\xi_t - \sum_{j \in \mathcal{J}} p_{j,t}(a_{j,t} - g_{j,t}) \mid \mathbf{q}_t\Big]$$

$$= \mathbb{E}\Big[\max_{j \in \mathcal{J}_\emptyset} \; [v(\theta_t, j) - p_{j,t}]\,\xi_t + \sum_{j \in \mathcal{J}} p_{j,t}g_{j,t} \mid \mathbf{q}_t\Big]$$

$$= \mathbb{E}[\xi_t]\mathbb{E}\Big[\max_{j \in \mathcal{J}_\emptyset} \; [v(\theta_t, j) - p_{j,t}] \mid \xi_t = 1, \mathbf{q}_t\Big] + \mathbb{E}\Big[\sum_{j \in \mathcal{J}} p_{j,t}g_{j,t} \mid \mathbf{q}_t\Big]$$

$$= \frac{\lambda}{1+\lambda} \int_\Theta \max_{j \in \mathcal{J}_\emptyset} \; [v(\theta_t, j) - p_{j,t}]\, dF(\theta) + \frac{1}{1+\lambda}\sum_{j \in \mathcal{J}} \mu_j p_{j,t}$$

$$= \frac{\lambda}{1+\lambda} h(\mathbf{p}_t)\,.$$

Step 2. To simplify the exposition, we focus here on the case in which the waiting cost is linear. The proof for general waiting costs is in Appendix A.

Assume that the waiting cost takes a linear form, where $c(w) = c \cdot w$ for some $c > 0$. Thus, $c = \Delta \cdot \mu_{\min}$.

We show that:

$$\mathbb{E}[v(\theta_t, a(\theta_t, \mathbf{q}_t))|\mathbf{q}_t] = \frac{\lambda}{1+\lambda}h(\mathbf{p}_t) - \frac{1}{c}\left(L(\mathbf{p}_t) - \mathbb{E}[L(\mathbf{p}_{t+1}) \mid \mathbf{q}_t]\right) \tag{11}$$

$$- \frac{1}{2}\sum_{j \in \mathcal{J}} \frac{c}{\mu_j}\mathbb{E}[(a_{j,t} - g_{j,t})^2 + l_{j,t}^2 \mid \mathbf{q}_t]$$

and

$$\frac{1}{2}\sum_{j \in \mathcal{J}} \frac{c}{\mu_j}\mathbb{E}[(a_{j,t} - g_{j,t})^2 + l_{j,t}^2 \mid \mathbf{q}_t] \leq \frac{2+\lambda}{2(1+\lambda)}\Delta\,. \tag{12}$$

We first show equality (11). We have that the drift of Lyapunov function $L(\mathbf{p})$ in one period is

$$L(\mathbf{p}_t) - L(\mathbf{p}_{t+1})$$

$$= \frac{1}{2}\sum_{j \in \mathcal{J}} \mu_j \left[p_{j,t}^2 - \left(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t} + l_{j,t})\right)^2\right]$$

$$= \frac{1}{2}\sum_{j \in \mathcal{J}} \mu_j \left[p_{j,t}^2 - \left(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t})\right)^2 - \frac{c^2}{\mu_j^2}l_{j,t}^2 - \frac{2c}{\mu_j}\left(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t})\right)l_{j,t}\right]$$

$$\overset{(a)}{=} \frac{1}{2}\sum_{j \in \mathcal{J}} \mu_j \left[p_{j,t}^2 - \left(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t})\right)^2 - \frac{c^2}{\mu_j^2}l_{j,t}^2\right]\,,$$

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where the equality (a) follows from the fact that $\left(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t})\right) l_{j,t} \equiv 0$ for all $j \in \mathcal{J}$.

We further simplify the Lyapunov drift as follows.

$$
\begin{aligned}
-\frac{1}{c}\left(L(\mathbf{p}_t) - L(\mathbf{p}_{t+1})\right) &= -\frac{1}{2c}\sum_{j \in \mathcal{J}} \mu_j \left[p_{j,t}^2 - \left(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t})\right)^2 \right] + \frac{1}{2}\sum_{j \in \mathcal{J}} \frac{c}{\mu_j} l_{j,t}^2 \\
&= \frac{1}{2}\sum_{j \in \mathcal{J}} \mu_j \left[\frac{2}{\mu_j}(a_{j,t} - g_{j,t})p_{j,t} + \frac{c}{\mu_j^2}(a_{j,t} - g_{j,t})^2 \right] + \frac{1}{2}\sum_{j \in \mathcal{J}} \frac{c}{\mu_j} l_{j,t}^2 \\
&= \sum_{j \in \mathcal{J}}(a_{j,t} - g_{j,t})p_{j,t} + \frac{1}{2}\sum_{j \in \mathcal{J}} \frac{c}{\mu_j}\left((a_{j,t} - g_{j,t})^2 + l_{j,t}^2\right) \\
&= \langle \mathbf{p}_t, \mathbf{a}_t - \mathbf{g}_t \rangle + \frac{1}{2}\sum_{j \in \mathcal{J}} \frac{c}{\mu_j}\left((a_{j,t} - g_{j,t})^2 + l_{j,t}^2\right) . \qquad (13)
\end{aligned}
$$

By taking the expectation conditional on $\mathbf{q}_t$ on both sides of equation (13), adding equation (10) and rearranging, this establishes equation (11).

We proceed to show inequality (12). Observe that

$$
\frac{c}{2\mu_j}(a_{j,t} - g_{j,t})^2 + \frac{c}{2\mu_j}l_{j,t}^2 = \begin{cases} \frac{c}{\mu_j} & \text{if } g_{j,t} = 1 \text{ and } q_j = 0, \\ \frac{c}{2\mu_j} & \text{otherwise} . \end{cases}
$$

That is, the term above is equal to $c/\mu_j \leq \Delta$ for arrivals that correspond to an item that is discarded because its queue is empty, and equal to $c/2\mu_j \leq \Delta/2$ for all other arrivals. Note that the probability that an item is discarded is at most $1/(1 + \lambda)$ (the probability that an item arrives). Therefore

$$
\begin{aligned}
&\frac{1}{2}\sum_{j \in \mathcal{J}} \frac{c}{\mu_j}\mathbb{E}[(a_{j,t} - g_{j,t})^2 + l_{j,t}^2 \mid \mathbf{q}_t] \\
&\leq \frac{\Delta}{2} + \frac{1}{1 + \lambda}\frac{\Delta}{2} \\
&= \frac{2 + \lambda}{2(1 + \lambda)}\Delta ,
\end{aligned}
$$

The lemma follows from inequalities (11) and (12) and the inequality $h(\mathbf{p}_t) \geq W^*$ which is implied by Lemma 2. $\qquad \square$

4.2 Proof of Theorem 3

Proof of Theorem 3. Consider a market market with finitely many agent types and linear waiting costs $\left[\Theta, \mathcal{J}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{v}, c(\cdot)\right]$ that has unique dual price $\mathbf{p}^*$. To simplify notation, we use

$\mathbf{p}_t$ to denote the random variable generated by drawing the price observed by a randomly drawn agent arriving to the waiting list mechanism in market. By the arrival theorem, the distribution of $\mathbf{p}_t$ is equal to the steady state distribution.

Step 1. We first establish that there exists a $\delta > 0$ such that the allocative efficiency loss is small if prices are within δ of $\mathbf{p}^*$ with high probability.

For each agent $\theta \in \Theta$, let $\mathcal{J}_\theta^* \triangleq \operatorname{argmax}_{j \in \mathcal{J}_\emptyset}\{v(\theta, j) - p_j^*\}$ be the set of items agent θ can optimally demand under prices $\mathbf{p}^*$. The following definition[20] of δ_1 ensures that for any $\mathbf{p}$ such that $\|\mathbf{p} - \mathbf{p}^*\|_\infty < \delta_1$ we have that $a(\theta, \mathbf{p}) \in \mathcal{J}_\theta^*$ for all $\theta \in \Theta$:

$$\delta_1 \triangleq \min_{\theta \in \Theta} \left\{ \max_{j \in \mathcal{J}_\emptyset}\{v(\theta, j) - p_j^*\} - \max_{j \in \mathcal{J}_\emptyset \setminus \mathcal{J}_\theta^*}\{v(\theta, j) - p_j^*\} \right\} .$$

Let $\delta_2 = \min_{j \in \mathcal{J}}\{p_j^* \mid p_j^* > 0\}$, and set $\delta = \frac{1}{2}\min\{\delta_1, \delta_2\}$.

Step 2. Let $\kappa = \mathbb{P}(\|\mathbf{p}_t - \mathbf{p}^*\|_\infty > \delta)$. By Proposition 3, there exist α, β, c_0 such that $\kappa < \beta e^{-\alpha/\Delta_\ell}$ for any $c < c_0$. We choose $c_0 > 0$ to be sufficiently small so that $\beta e^{-\alpha/\Delta_\ell} < 1/2$. In addition, we choose $c_0 > 0$ to be sufficiently small so that $c_0/\mu_{\min} < \delta/2$, which implies that if at time t we have that $\|\mathbf{p}_t - \mathbf{p}^*\|_\infty < \delta$ then for any j such that $p_j^* > 0$ the j-th queue is not empty at epoch t (that is, $q_{jt} > 0$).

Step 3. Let $\mathbf{x}$ denote the assignment probabilities of agents arriving to the waiting list mechanism in states where $\|\mathbf{p}_t - \mathbf{p}^*\|_\infty < \delta$. By the arrival theorem, $\mathbf{x}$ is given by

$$\mathbf{x}_{\theta j} = \mathbb{E}\left[\mathbb{1}_{\{a(\theta, \mathbf{p}_t)=j\}} \mid \|\mathbf{p}_t - \mathbf{p}^*\|_\infty < \delta\right] ,$$

let $\mathbf{y}$ denote the assignment probabilities of remaining agents

$$y_{\theta j} = \mathbb{E}\left[\mathbb{1}_{\{a(\theta, \mathbf{p}_t)=j\}} \mid \|\mathbf{p}_t - \mathbf{p}^*\|_\infty \geq \delta\right] ,$$

and denote $x_j = \sum_{\theta \in \Theta} \lambda_\theta x_{\theta j}$, $y_j = \sum_{\theta \in \Theta} \lambda_\theta y_{\theta j}$.

Because all agents are eventually assigned, we have that $(1-\kappa)x_j + \kappa y_j \leq \mu_j$ for every $j \in \mathcal{J}$. If an item j with $p_j^* > 0$ arrives in epoch t such that when $\|\mathbf{p}_t - \mathbf{p}^*\|_\infty < \delta$, then the item is assigned to a waiting agent. Therefore, at most $\kappa \cdot \mu_j$ of j items are wasted, and for every $j \in \mathcal{J}$ such that $p_j^* > 0$ we have $(1-\kappa)x_j + \kappa y_j \geq (1-\kappa)\mu_j$ or

$$x_j \geq \mu_j - \frac{\kappa}{1-\kappa}y_j \geq \mu_j - \frac{\kappa}{1-\kappa}\lambda . \tag{14}$$

---

[20]We use the convention that if $\mathcal{J}_\emptyset \setminus \mathcal{J}_\theta^* = \emptyset$ then $\max_{j \in \mathcal{J}_\emptyset \setminus \mathcal{J}_\theta^*}\{v(\theta, j) - p_j^*\} = -\infty$.

Observe that for an optimal assignment $\mathbf{x}^*$ we can write

$$
\begin{aligned}
W^* &= \sum_{\theta,j} \lambda_\theta x_{\theta j}^* v(\theta, j) \\
&= \sum_{\theta,j} \lambda_\theta x_{\theta j}^* (v(\theta, j) - p_j^*) + \sum_{\theta,j} \lambda_\theta x_{\theta j}^* p_j^* \\
&= \sum_{\theta,j} \lambda_\theta x_{\theta j}^* (v(\theta, j) - p_j^*) + \sum_{j} \left(\mu_j p_j^* + (x_j^* - \mu_j) p_j^* \right) \\
&= \sum_{\theta,j} \lambda_\theta x_{\theta j}^* (v(\theta, j) - p_j^*) + \sum_{j} \mu_j p_j^*
\end{aligned}
$$

where the last equality follows from complementary slackness. By the definition of δ, we have that for every θ, j such that $x_{\theta j} > 0$ we have that $j \in \mathcal{J}_\theta^*$, implying that

$$
\sum_{\theta,j} \lambda_\theta x_{\theta j}^* (v(\theta, j) - p_j^*) = \sum_{\theta,j} \lambda_\theta x_{\theta j} (v(\theta, j) - p_j^*) .
$$

Thus, we have

$$
\sum_{\theta,j} \lambda_\theta x_{\theta j} (v(\theta, j) - p_j^*) = W^* - \sum_{j} \mu_j p_j^* . \tag{15}
$$

Using (14),(15) we therefore have

$$
\begin{aligned}
\mathbb{E}\Big[v(\theta_t, a(\theta_t, \mathbf{p}_t)) \Big] &= (1-\kappa) \mathbb{E}\Big[v(\theta_t, a(\theta_t, \mathbf{p}_t)) \mid \|\mathbf{p}_t - \mathbf{p}^*\|_\infty < \delta \Big] \\
&\quad + \kappa \mathbb{E}\Big[v(\theta_t, a(\theta_t, \mathbf{p}_t)) \mid \|\mathbf{p}_t - \mathbf{p}^*\|_\infty \geq \delta \Big] \\
&\geq (1-\kappa) \sum_{\theta,j} \lambda_\theta x_{\theta j} v(\theta, j) \\
&= (1-\kappa) \left(\sum_{\theta,j} \lambda_\theta x_{\theta j} \big(v(\theta, j) - p_j^* \big) + \sum_{j} x_j p_j^* \right) \\
&= (1-\kappa) \left(W^* + \sum_{j} (x_j - \mu_j) p_j^* \right) \\
&\geq (1-\kappa) \left(W^* + \sum_{j} (\mu_j - \frac{\kappa}{1-\kappa} \lambda - \mu_j) p_j^* \right) \\
&\geq (1-\kappa) \left(W^* - J \frac{\kappa}{1-\kappa} \lambda\, v_{\max} \right) \\
&\geq W^* - \kappa W^* - \kappa J \lambda\, v_{\max}
\end{aligned}
$$

We chose c_0 such that $\kappa < \beta e^{-\alpha/\Delta_\ell}$, and therefore we have that

$$
\begin{aligned}
W^{\mathrm{WL}}(c_\ell) &= \mathbb{E}\Big[v(\theta_t, a(\theta_t, \mathbf{p}_t))\Big] \\
&\geq W^* - \kappa\left(W^* + J\lambda\, v_{\max}\right) \\
&\geq W^* - \beta\left(W^* + J\lambda\, v_{\max}\right) e^{-\alpha/\Delta_\ell}\,,
\end{aligned}
$$

which completes the proof. $\qquad\square$

5 On the Optimal Adjustment Size

In this section we consider a planner who can set prices, but does not know the distribution of agent preferences or the market clearing prices. The waiting list mechanism suggests a simple pricing heuristic: prices adjust over time, increasing an item's price when it is demanded and decreasing an item's price when it is supplied. We show that this simple pricing heuristic approximately maximizes allocative efficiency. This heuristic can be used by a planner that has minimal knowledge of the market: the planner only needs to set the adjustment size according to the market's time horizon.

Finite Horizon Economy. The economy is as defined as in Section 2, except that the world ends after T arrival epochs and that agents have no cost of waiting (i.e., $c(w) \equiv 0$). A feasible assignment η assigns each agent that arrives before time T to an item that arrives before T (or leaves the agent unassigned). For notational convenience, we define the allocative efficiency of allocation η as the total value of assigned items.

$$
W_T(\eta) = \sum_{t=1}^{T} \xi_t v(\theta_t, \eta_t)\,. \tag{16}
$$

Define optimal allocative efficiency to be

$$
W_T^{\mathrm{OPT}} = \mathbb{E}\left[\sup_\eta W_T(\eta)\right]\,,
$$

where we take the supremum over all possible assignments and the expectation is taken over all possible realizations. The following proposition relates the optimal allocative efficiency in the finite horizon market to the optimal allocative efficiency of the static assignment problem.

Proposition 4. *Let* $W_T^* \triangleq \frac{\lambda}{\lambda+1} T \cdot W^*$, *where* W^* *is the optimal static allocative efficiency.*

The optimal allocative efficiency in the finite horizon economy is bounded by

$$W_T^{\mathrm{OPT}} \leq W_T^* \,.$$

The proof of Proposition 4 is almost identical to the proof of Proposition 1 and is therefore omitted.

5.1 The SGD pricing algorithm.

We now formally define the SGD pricing algorithm, which is inspired by the waiting list mechanism. Instead of using waiting times to guide the assignment, the SGD pricing algorithm uses monetary prices. Prices adjust with each arrival, replicating the price adjustment under the waiting list except that the adjustment size can be specified by the planner (in contrast to the exogenously given adjustment size of the waiting list mechanism).

With slight abuse of notation, we use $\mathbf{p}_t$ to denote the monetary prices used by the planner in arrival epoch t. In an epoch t in which an agent arrives, (i.e., $\xi_t = 1$), the agent can choose any item $j \in \mathcal{J}_\emptyset$ at price $p_{j,t}$.[21] An agent of type θ_t chooses item type

$$a(\theta_t, \mathbf{p}_t) \in \underset{j \in \mathcal{J}_\emptyset}{\operatorname{argmax}} \left\{ v(\theta, j) - p_{j,t} \right\} \,.$$

If $a(\theta_t, \mathbf{p}_t) = j \in \mathcal{J}$ the agent joins the queue to wait for item j (recall that waiting is costless). In an epoch t in which an item arrives, (i.e., $\xi_t = 0$), denote the item's type by $j_t \in \mathcal{J}$. The item is assigned to some agent waiting in the queue for that type of item.[22] If there are no agents waiting for item j_t, the item is discarded.

Prices adjust with each arrival, as follows. Initialize the prices of all items to zero, i.e., $p_{1,j} = 0$, $\forall j \in \mathcal{J}$. Prices for epoch $t+1$ are adjusted according to the arrival in epoch t as follows:

$$p_{j,t+1} = \left[p_{j,t} + \Delta \cdot \mathbb{1}_{\{\xi_t = 1, j = a(\theta_t, \mathbf{p}_t)\}} - \Delta \cdot \mathbb{1}_{\{\xi_t = 0, j = j_t\}} \right]^+ \,. \tag{17}$$

That is, the price adjustment mimics the adjustment of waiting times under the waiting list mechanism. If an agent choose item j, the price of item j increases by a Δ increment. If item j arrived, the price of item j decreases by Δ (or stays at 0).

Let W_T^{WL} denote the allocative efficiency of the SGD pricing algorithm. Theorem 4 shows that this simple pricing algorithm performs well if the adjustment size Δ is set appropriately.

---

[21]The price of being unassigned is $p_{\emptyset,t} = 0$ for all $t \leq T$.

[22]For concreteness, items are assigned to agents in the queue according to a First-come First-Served order. Because waiting is costless, the choice of agent in the queue can be arbitrary.

Theorem 4. *The allocative efficiency of the SGD algorithm with step size $\Delta = 1/\sqrt{T}$ satisfies:*

$$W_T^{\mathrm{WL}} \geq W_T^* - \sqrt{T}\left(\frac{2+\lambda}{2+2\lambda} + J \cdot v_{\max}^2\right).$$

Setting the adjustment size $\Delta = 1/\sqrt{T}$, the pricing algorithm obtains allocative efficiency that is within $O(\sqrt{T})$ of W_T^*. Devanur et al. (2019) shows that the allocative efficiency under any online pricing algorithm is at most $W_T^* - \Omega(\sqrt{T})$, implying that the SGD pricing algorithm is asymptotically optimal.[23]

We note two immediate extensions of Theorem 4. First, regardless of the choice of initial prices $\mathbf{p}_1 \in [0, v_{\max}]^J$, the allocative efficiency of the SGD pricing algorithm is within $O(\sqrt{T})$ of the optimal W_T^*. Second, setting the adjustment size to $\Delta = v_{\max}/\sqrt{T}$ can improve the allocative efficiency bound. We state both of these as the following corollary.

Corollary 4. *The allocative efficiency of the SGD algorithm with arbitrary initial prices $\mathbf{p}_1 \in [0, v_{\max}]^J$ and adjustment size $\Delta = v_{\max}/\sqrt{T}$ satisfies:*

$$W_T^{\mathrm{WL}} \geq W_T^* - v_{\max}\sqrt{T}\left((1 + \mu_{\min}^{-1})J + \frac{2+\lambda}{2+2\lambda}\right).$$

Intuition and proof. Before giving the proof of Theorem 4, we give a stylized explanation of the result. Roughly speaking, we can divide the run of the SGD pricing algorithm into two phases. First, the algorithm gradually raises prices to "learn" the market clearing prices. Second, after some time the distribution of prices is close to the steady state distribution of prices. In the second phase the algorithm incurs losses from price fluctuations (as discussed in Section 3). In the first phase prices are too low, allowing agents with low valuations to join the queue. This is wasteful because of there is a limited supply of items that arrive by time T: although we always allow agents to join the queue for any item j, if too many agents choose item j, then at time T there will be agents waiting in the j queue, and these agents will not be assigned any item.

A lower adjustment size Δ reduces the loss from fluctuations in the second phase. A higher adjustment size Δ helps the algorithm approach market clearing prices faster and reduces the loss from the first phase. Setting the adjustment size to an intermediate value $\Delta = 1/\sqrt{T}$ allows the algorithm to balance the two sources of loss.

Proof. Let $\tilde{W}_T(\eta) \triangleq \sum_{t=1}^T v(\theta_t, \eta_t)$ denote the allocative efficiency of the SGD algorithm if

---

[23]Given arbitrary functions $f(\cdot)$, and a positive function $h(\cdot)$ we will say that $f(\rho) = O(h(\rho))$ if there exists positive constants α, ρ_0 such that $0 \leq f(\rho) \leq \alpha h(\rho)$ for any $\rho > \rho_0$. Similarly, we will say that $f(\rho) = g(\rho) - \Omega(h(\rho))$ if there exists positive constants α, ρ_0 such that $0 \leq \alpha h(\rho) \leq g(\rho) - f(\rho)$ for any $\rho > \rho_0$.

all agents who choose an item are assigned; that is, $\tilde{W}_T(\eta)$ is the allocative efficiency of the SGD algorithm if there were no supply constraints. We separately bound $W_T^* - \tilde{W}_T^{\mathrm{WL}}$ and $W_T^{\mathrm{WL}} - \tilde{W}_T^{\mathrm{WL}}$.

From the analysis of Theorem 1, equation (8), we have:

$$\mathbb{E}\left[\tilde{W}_T(\eta^{\mathrm{WL}})\right] \geq T\frac{\lambda}{1+\lambda}W^* - \frac{1}{\Delta\mu_{\min}}\left(L(\mathbf{p}_1) - \mathbb{E}[L(\mathbf{p}_{T+1})]\right) - T\frac{2+\lambda}{2(1+\lambda)}\Delta\,.$$

Therefore we have:

$$W_T^* - \tilde{W}_T^{\mathrm{WL}} \leq \frac{1}{\Delta\mu_{\min}}\left(L(\mathbf{p}_1) - \mathbb{E}[L(\mathbf{p}_{T+1})]\right) + T\frac{2+\lambda}{2(1+\lambda)}\Delta\,.$$

Setting $\Delta = 1/\sqrt{T}$, and observing that $L(\cdot) \geq 0$ and $L(\mathbf{p}_1) = L(0) = 0$, we have that

$$W_T^* - \tilde{W}_T^{\mathrm{WL}} \leq \sqrt{T}\frac{2+\lambda}{2(1+\lambda)}\,. \tag{18}$$

To bound $\tilde{W}_T^{\mathrm{WL}} - W_T^{\mathrm{WL}}$, observe that this difference is the total value of items assigned to agents that are waiting in the queue in epoch T when the economy ends. By the definition price adjustments, the number of agents waiting for item j at time T is $p_{j,T}/\Delta$. Since no agent will join queue j when $p_{j,t} > v_{\max}$, we have $p_{j,t} \leq v_{\max}$ for all $j \in \mathcal{J}$ and $t \leq T$. Therefore,

$$\tilde{W}_T^{\mathrm{WL}} - W_T^{\mathrm{WL}} \leq \sum_{j \in \mathcal{J}} v_{\max} \cdot \frac{p_{j,T}}{\Delta} \leq J \cdot v_{\max}^2 \cdot \sqrt{T}\,. \tag{19}$$

Combining (18) and (19), we conclude the proof. $\square$

Proof of Corollary 4. The corollary follows from the proof of Theorem 4 by plugging in $\Delta = v_{\max}/\sqrt{T}$ to obtain the following inequalities:

$$\begin{aligned}
W_T^* - \tilde{W}_T^{\mathrm{WL}} &\leq \frac{1}{\Delta\mu_{\min}}\left(L(\mathbf{p}_1) - \mathbb{E}[L(\mathbf{p}_{T+1})]\right) + T\frac{2+\lambda}{2(1+\lambda)}\Delta \\
&\leq \frac{\sqrt{T}}{\mu_{\min}v_{\max}}\sum_{j \in \mathcal{J}}\mu_j p_{j,1}^2 + T\frac{2+\lambda}{2(1+\lambda)}\frac{v_{\max}}{\sqrt{T}} \\
&\leq \mu_{\min}^{-1}Jv_{\max}\sqrt{T} + \frac{2+\lambda}{2+2\lambda}v_{\max}\sqrt{T}\,.
\end{aligned}$$

And,

$$\tilde{W}_T^{\mathrm{WL}} - W_T^{\mathrm{WL}} \leq \sum_{j \in \mathcal{J}} v_{\max} \cdot \frac{p_{j,T}}{\Delta} = \sum_{j \in \mathcal{J}} v_{\max} \cdot \frac{p_{j,T}}{v_{\max}/\sqrt{T}} \leq J \cdot v_{\max} \cdot \sqrt{T}\,.$$

Combining these inequalities and rearranging terms yields the desired result.

□

6 Conclusion

This paper analyzed the endogenous determination of waiting times under the waiting list mechanism. Waiting times continuously adjust according to a tâtonnement-like process, which is equivalent to running the SGD optimization algorithm with a fixed and exogenously given adjustment size. The resulting allocative efficiency is at least the optimal allocative efficiency minus a loss equal to the adjustment size, given by the marginal increase in expected waiting cost from having one more agent in the queue. Waiting list mechanisms approximately replicate the static competitive equilibrium outcome when the adjustment size is small, but can incur an additional loss from fluctuations when the adjustment size is large.

In practice, the adjustment size will be large in applications where agents are impatient or items arrive infrequently. In the context of public housing assignment, our results raise concerns for losses due to misallocation when applicants have urgent housing needs or when apartments arrive infrequently. We emphasize that such losses can occur even if there are many agents who seek housing. On the other hand, if housing projects are large and similar apartments arrive frequently, losses due to misallocation are small.

To focus our analysis on the endogenous determination of waiting times, we focused on the standard waiting list mechanism which holds a separate queue for each item. This mechanism is commonly used, and is particularly suitable for our analysis because agents face a single choice that is analogous to the choice agents make in competitive equilibrium models. Insights from the analysis apply to other mechanisms as well. For example, changes to the queueing policy (e.g., Leshno (2017)) can effectively reduce the adjustment size and thus reduce misallocation. Similarly, giving agents infrequent updates can be seen as another method for reducing the effective adjustment size.

Our analysis abstracts away from potential distinctions between waiting times and monetary price. If payments for items are transfers, competitive equilibria is welfare maximizing (in addition to maximizing allocative efficiency). In applications such as public housing waiting lists, applicants bear a cost while they wait for an apartment, e.g. higher rent payments. Potential applicants bear this waiting costs whether or not they participate in the waiting list mechanism. Each assigned apartment reduces the waiting cost of one agent. Thus, total waiting costs are constant under any assignment that assigns all available apartments, and waiting times are transfers that do not affect social welfare. Our analysis shows that if all items have a positive price, the waiting list mechanism assigns almost all available items. However, we note that allocative efficiency maximization is distinct from

welfare maximization in applications where agents have an outside option that reduces their waiting costs (for example, visitors to a theme park can choose to watch a show instead of queueing for a ride).

Finally, we consider a simple pricing heuristic inspired by the waiting list mechanism. We show that a simple SGD pricing heuristic with an appropriate adjustment size performs well in finite-horizon markets. A planner can implement this simple heuristic with almost no knowledge of the market; the planner only needs to know the market's time horizon to select an appropriate adjustment size.

This simple SGD heuristic gives rise to a form of price rigidity. The optimal adjustment size balances two potential losses. A larger adjustment size facilitates a faster adaptation towards market-clearing prices, but a smaller adjustment size creates rigidity that protects from overreaction to noise. Intuitively, the optimal adjustment size balances the two, and induces more rigidity when the time horizon T indicates a more stable environment. Because of this trade off, the SGD heuristic will seem too rigid and slow to adjust prices in initial periods.

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A   Proof of Theorem 1 for General Waiting Costs

In this section we complete the proof of Theorem 1 for the general case. The proof is identical to the proof in the main body except for the second step of Lemma 4. The following three lemmas establish inequalities that give versions of inequalities (11) and (12) for general waiting cost functions that satisfy the conditions in Section 2. Replacing Step 2 in the proof of Lemma 4 with these lemmas gives the general proof.

We use the same Lyapunov function in the analysis, where $L(\mathbf{q})$ is such that $\nabla L(\mathbf{q}) = \mathbf{p}(\mathbf{q})$. The analysis uses the Bregman divergence generated by $L(\mathbf{q})$ as the notion of proximity, which is defined as follows.

$$D_L(\mathbf{q}_1, \mathbf{q}_2) \triangleq L(\mathbf{q}_1) - L(\mathbf{q}_2) - \langle \nabla L(\mathbf{q}_2), \mathbf{q}_1 - \mathbf{q}_2 \rangle \,.$$

Lemma 5. *We have that*

$$L(\mathbf{q}_{t+1}) \leq L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t) + \frac{\Delta}{2} \cdot \mathbb{1}\{\xi_t = 0\} \,.$$

Proof. By definition of Bregman divergence, we have

$$
\begin{aligned}
D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_{t+1}) &= L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t) - L(\mathbf{q}_{t+1}) - \langle \mathbf{p}(\mathbf{q}_{t+1}), \mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t - \mathbf{q}_{t+1} \rangle \\
&= L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t) - L(\mathbf{q}_{t+1}) + \langle \mathbf{p}(\mathbf{q}_{t+1}), \mathbf{u}_t \rangle \,.
\end{aligned}
$$

Therefore

$$L(\mathbf{q}_{t+1}) = L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t) + \langle \mathbf{p}(\mathbf{q}_{t+1}), \mathbf{u}_t \rangle - D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_{t+1}) \,. \tag{20}$$

To bound the RHS of (20), we consider two cases:

Case 1. If $\exists j \in \mathcal{J}$ such that $d_{j,t} = 1$ and $q_{j,t} = 0$, we have $q_{j,t+1} = 0$ and $l_{j,t} = 1$. Note that in this case $\xi_t = 0$. Let $P_j(q)$ be an anti-derivative of $p_j(q)$, then $L(\mathbf{q}) = \sum_{j \in \mathcal{J}} P_j(q)$ is a Lyapunov function because it satisfies $\nabla L(\mathbf{q}) = \mathbf{p}(\mathbf{q})$. We have

$$
\begin{aligned}
&\langle \mathbf{p}(\mathbf{q}_{t+1}), \mathbf{u}_t \rangle - D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_{t+1}) \\
&= p_j(0) - (P_j(-1) - P_j(0) - p_j(0) \cdot (-1)) \\
&= P_j(0) - P_j(-1) \,.
\end{aligned}
\tag{21}
$$

Since $p_j(\cdot)$ is non-negative and Δ-Lipshitz, we have

$$P_j(0) \leq P_j(-1) + \int_0^1 \Delta \cdot x \, dx = P_j(-1) + \Delta/2 \,.$$

Plugging in the above equality to (21), we have

$$\langle \mathbf{p}(\mathbf{q}_{t+1}), \mathbf{u}_t \rangle - D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_{t+1}) \leq \frac{\Delta}{2} \cdot \mathbb{1}\{\xi_t = 0\}. \tag{22}$$

Case 2. If the condition in Case 1 does not hold, we have $\mathbf{u}_t = \mathbf{0}$ and $\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t = \mathbf{q}_{t+1}$, hence

$$\langle \mathbf{p}(\mathbf{q}_{t+1}), \mathbf{u}_t \rangle - D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_{t+1}) = 0.$$

Therefore, plugging in the above two cases to (20), we have

$$L(\mathbf{q}_{t+1}) \leq L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t) + \frac{\Delta}{2} \cdot \mathbb{1}\{\xi_t = 0\}.$$

$\square$

Lemma 6. *We have that*

$$\mathbb{E}[v(\theta_t, a(\theta_t, \mathbf{q}_t))|\mathbf{q}_t] \geq \frac{\lambda}{1+\lambda} h(\mathbf{p}_t) - (L(\mathbf{q}_t) - \mathbb{E}[L(\mathbf{q}_{t+1}) \mid \mathbf{q}_t]) - \frac{\Delta}{2(1+\lambda)}$$
$$- \mathbb{E}[D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_t) \mid \mathbf{q}_t].$$

Proof. We have that the drift of Lyapunov function $L(\mathbf{q})$ in one period is

$$L(\mathbf{q}_t) - L(\mathbf{q}_{t+1})$$
$$\geq L(\mathbf{q}_t) - L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t) - \frac{\Delta}{2} \cdot \mathbb{1}\{\xi_t = 0\}$$
$$= -\langle \mathbf{p}(\mathbf{q}_t), \mathbf{a}_t - \mathbf{g}_t \rangle - D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_t) - \frac{\Delta}{2} \cdot \mathbb{1}\{\xi_t = 0\}, \tag{23}$$

where the inequality follows from Lemma 5, and the equality comes from the definition of Bregman divergence.

Adding $v(\theta_t, a(\theta_t, \mathbf{q}_t))$ to both sides of equation (23), we have

$$v(\theta_t, a(\theta_t, \mathbf{q}_t)) + L(\mathbf{q}_t) - L(\mathbf{q}_{t+1})$$
$$\geq v(\theta_t, a(\theta_t, \mathbf{q}_t)) - \langle \mathbf{p}(\mathbf{q}_t), \mathbf{a}_t - \mathbf{g}_t \rangle - D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_t) - \frac{\Delta}{2} \cdot \mathbb{1}\{\xi_t = 0\}.$$

Taking expectation conditional on $\mathbf{q}_t$ and applying equation (10), we have

$$\mathbb{E}[v(\theta_t, a(\theta_t, \mathbf{q}_t))|\mathbf{q}_t] + (L(\mathbf{q}_t) - \mathbb{E}[L(\mathbf{q}_{t+1}) \mid \mathbf{q}_t])$$
$$\geq \frac{\lambda}{1+\lambda} h(\mathbf{p}_t) - \mathbb{E}[D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_t) \mid \mathbf{q}_t] - \frac{\Delta}{2(1+\lambda)}.$$

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Rearranging the terms, and we obtain the desired inequality. $\qquad\square$

Lemma 7. *We have that for any* $\mathbf{q}_t$,

$$\mathbb{E}[D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_t) \mid \mathbf{q}_t] \le \frac{\Delta}{2}\,.$$

Proof. Note that $L(\mathbf{q})$ is convex because its gradient $\nabla L(\mathbf{q}) = \mathbf{p}(\mathbf{q})$ is increasing in each coordinate. Also note that $L(\mathbf{q})$ has a Δ-Lipschitz gradient, because for queue lengths $\mathbf{q}_1, \mathbf{q}_2$,

$$||\nabla L(\mathbf{q}_1) - \nabla L(\mathbf{q}_2)|| = ||\mathbf{p}(\mathbf{q}_1) - \mathbf{p}(\mathbf{q}_2)|| \le \Delta ||\mathbf{q}_1 - \mathbf{q}_2||\,.$$

Equivalently, $L(\mathbf{q})$ is Δ-strongly smooth, i.e.,

$$L(\mathbf{q}_2) - L(\mathbf{q}_1) \le \langle \nabla L(\mathbf{q}_1), \mathbf{q}_2 - \mathbf{q}_1 \rangle + \frac{\Delta}{2} ||\mathbf{p}(\mathbf{q}_2) - \mathbf{p}(\mathbf{q}_1)||^2\,.$$

By definition of Bregman divergence, we have

$$
\begin{aligned}
& D_L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t, \mathbf{q}_t) \\
&= L(\mathbf{q}_t + \mathbf{a}_t - \mathbf{g}_t) - L(\mathbf{q}_t) - \langle \nabla L(\mathbf{q}_t), \mathbf{a}_t - \mathbf{g}_t \rangle \\
&\le \frac{\Delta}{2} ||\mathbf{a}_t - \mathbf{g}_t||^2 \\
&= \frac{\Delta}{2}\,.
\end{aligned}
$$

Here the second last inequality follows from the strong smoothness of $L(\mathbf{q})$. This concludes the proof. $\qquad\square$

B   Omitted Proofs

B.1   Proof of Proposition 1 from Section 2

In this section we prove Proposition 1, showing that $W^{\mathrm{OPT}} = W^*$.

Proof of Proposition 1. We first show that $W^{\mathrm{OPT}} \le W^*$. This part of the proof mostly consists of a careful treatment of expectations and limits.

Let $\eta \in H$ be any no-Ponzi allocation. Without loss of generality, assume that $\Theta = [0, 1]$ and that F is a CDF on $[0, 1]$. This allows us to rewrite problem (3) as

$$W^* = \max_{G_j(\theta),\, j \in \mathcal{J}} \quad \sum_{j \in \mathcal{J}} \int_\Theta v(\theta, j)\, dG_j(\theta)$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} G_j(\theta) \leq F(\theta) \qquad\qquad\qquad \forall \theta$$

$$G_j(0) = 0; \ G_j(1) \leq \mu_j/\lambda \qquad\qquad\qquad \forall j \qquad (24)$$

$$G_j(\cdot) \text{ is non-decreasing and right-continuous} \qquad \forall j$$

Let $A_T = |A(T)|$, and recall that $A(T)$ is set of epochs up to T in which an agent arrived. Recall that $\eta_t \in \mathcal{J}_\emptyset$ is the kind of item assigned under η to the agent that arrived at epoch t. For each $j \in \mathcal{J}$ and $\theta \in \Theta$, define $\hat{G}_j^T(\theta)$ as

$$\hat{G}_j^T(\theta) \triangleq \frac{1}{A_T} \sum_{t \leq T} \mathbb{1}_{\{\xi_t = 1, \theta_t \leq \theta, \eta_t = j\}} \,.$$

Therefore, $\hat{G}_j^T(\theta)$ is proportional to the empirical cumulative distribution function of the types of the agents in A_T who are assigned a type j item. When $A_T = 0$, we set $\hat{G}_j^T(\theta) = 0$ for all $j \in \mathcal{J}$ and $\theta \in \Theta$. By definition, the allocative efficiency under η is defined as

$$W(\eta) = \liminf_{T \to \infty} \frac{1}{A_T} \sum_{j \in \mathcal{J}} \sum_{t \leq T} \xi_t v(\theta_t, j) = \liminf_{T \to \infty} \sum_{j \in \mathcal{J}} \int_{\theta \in \Theta} v(\theta, j) d\hat{G}_j^T(\theta) \,. \qquad (25)$$

Note that for any T, $\hat{G}_j^T(\theta)$ satisfies

$$\sum_{j \in \mathcal{J}} \hat{G}_j^T(\theta) \leq \frac{1}{A_T} \sum_{t \leq T} \xi_t \mathbb{1}_{\{\theta_t \leq \theta\}} \,, \qquad\qquad \forall \theta \in \Theta \qquad (26)$$

$$\hat{G}_j^T(0) = 0 \,, \hat{G}_j^T(1) \leq \frac{1}{A_T} \left(\sum_{t \leq T} \mathbb{1}_{\{\eta_t = j\}} + M \right) \qquad \forall j \in \mathcal{J} \qquad (27)$$

$$\hat{G}_j^T(\theta) \text{ is non-decreasing and right-continuous.} \qquad \forall j \in \mathcal{J} \qquad (28)$$

for some $M \in \mathbb{R}$. Here (26) and (28) are trivial. (27) is satisfied by any no-Ponzi assignment for the following reason: The agents arriving in epochs in $A(T)$ who are assigned a type $j \in \mathcal{J}$ item are either assigned before the T-th epoch or after the T-th epoch. The number of those who are assigned before T cannot exceed the total number of type j items that arrive before T. The number of those who are assigned after T is bounded by some $M \in \mathbb{R}$ by the definition of no-Ponzi assignments.

Combining the above, we have

$$\mathbb{E}\left[\sum_{j \in \mathcal{J}} \int_{\theta \in \Theta} v(\theta, j) d\hat{G}_j^T(\theta) \right] \leq \mathbb{E}\left[\max_{\hat{G}_j(\theta) \text{ satisfying } (26)(27)(28)} \sum_{j \in \mathcal{J}} \int_{\theta \in \Theta} v(\theta, j) d\hat{G}_j(\theta) \right] \,.$$

It is easy to check that the optimal value of the inner maximization problem above is concave and non-decreasing in the RHS of (26) and (27). Note that

$$\text{Expectation of RHS of (26)} = F(\theta)\,, \qquad\qquad \forall \theta \in \Theta\,,$$

$$\text{Expectation of RHS of (27)} = \frac{\mu_j}{\lambda} + \frac{(1+\lambda)M}{\lambda T}\,, \qquad\qquad \forall j \in \mathcal{J}\,.$$

It follows from Fatou's Lemma that

$$\mathbb{E}[W(\eta)] = \mathbb{E}\left[\liminf_{T\to\infty} \sum_{j\in\mathcal{J}} \int_{\theta\in\Theta} v(\theta,j) d\hat{G}_j^T(\theta)\right] \le \liminf_{T\to\infty} \mathbb{E}\left[\sum_{j\in\mathcal{J}} \int_{\theta\in\Theta} v(\theta,j) d\hat{G}_j^T(\theta)\right]\,.$$

Applying Jensen's inequality, we have

$$\liminf_{T\to\infty} \mathbb{E}\left[\sum_{j\in\mathcal{J}} \int_{\theta\in\Theta} v(\theta,j) d\hat{G}_j^T(\theta)\right] \le W^*\,,$$

where W^* is defined in (3). Therefore $\mathbb{E}[W(\eta)] \le W^*$ for any $\eta \in H$. Since $W(\eta)$ is uniformly bounded above by $v_{\max}$, by the Bounded Convergence Theorem we have $W^{\text{OPT}} = \mathbb{E}[\sup_{\eta\in H} W(\eta)] = \sup_{\eta\in H} \mathbb{E}[W(\eta)] \le W^*$. This concludes the proof.

Next we prove that $W^{\text{OPT}} \ge W^*$. We explicitly construct a sequence of randomized policies that can achieve allocative efficiencies which are arbitrarily close to W^*. Note that the constructed policies are more of technical devices used to prove the desired bound, rather than practical policies.

Denote the optimal solution of the optimization problem (3) by $\mathbf{x}^*$. Consider the following randomized policy: Maintain a separate First-Come-First-Served queue for each item. An arriving agent will be assigned to one of the queues or rejected based on a coin-toss (to be specified later). An agent who joins a queue will wait in that queue until receiving an item. When an item arrives, it is assigned to the agent at the head of its queue; if the queue is empty, the item is discarded. The coin-toss is defined as follows: Fix $M \in \mathbb{Z}_+$. If the arriving agent is of type θ, it is assigned to a queue j with probability $x_{\theta j}^*$, or rejected from all queues with probability $1 - \sum_{j\in\mathcal{J}} x_{\theta j}^*$. If the length of the queue to which the agent is assigned exceeds M, the agent is also rejected.

Denote the match value collected by the randomized policy in epoch t by v_t^{RD}. Then by definition of the policy, we have

$$\mathbb{E}[v_t^{\text{RD}} | q_{j,t} < M, \forall j \in \mathcal{J}] = \frac{\lambda}{1+\lambda} \sum_{j\in\mathcal{J}} \int_\Theta x_{\theta j}^* \, v(\theta,j) dF(\theta) = \frac{\lambda}{1+\lambda} W^*\,.$$

It follows that

$$\mathbb{E}[v_t^{\mathrm{RD}}] \geq \frac{\lambda}{1+\lambda} W^* \cdot \mathbb{P}(q_{j,t} < M, \forall j \in \mathcal{J}).$$

Let W^{RD} be the allocative efficiency of the randomized policy, we therefore have

$$W^{\mathrm{RD}} \geq W^* \cdot \mathbb{P}(q_{j,\infty} < M, \forall j \in \mathcal{J}),$$

where $\mathbf{q}_\infty$ is the steady-state queue length distribution. The allocative efficiency loss of the randomized policy can be bounded as:

$$\begin{aligned}
W^* - W^{\mathrm{RD}} &\leq W^* - W^* \cdot \mathbb{P}(q_{j,\infty} < M, \forall j \in \mathcal{J}) \\
&\leq v_{\max} \cdot \mathbb{P}(q_{j,\infty} = M, \exists j \in \mathcal{J}) \\
&\leq v_{\max} \cdot \sum_{j \in \mathcal{J}} \mathbb{P}(q_{j,\infty} = M),
\end{aligned} \tag{29}$$

where the second inequality follows from the fact that $W^* \leq v_{\max}$, and the last inequality comes from the union bound. It remains to bound $\mathbb{P}(q_{j,\infty} = M)$ for any $j \in \mathcal{J}$ under the randomized policy. Fix $j \in \mathcal{J}$, then $q_{j,t}$ is a birth-death process on $\{0, 1, \cdots, M\}$ with death rate μ_j and birth rate

$$\lambda_j^* = \lambda \int_\Theta x_{\theta j}^* dF(\theta).$$

It follows from the constraint in (3) that $\lambda_j^* \leq \mu_j$. As a result, $\mathbb{P}(q_{j,\infty} = M) \leq \mathbb{P}(q_{j,\infty} = M - 1) \leq \cdots \leq \mathbb{P}(q_{j,\infty} = 0)$, hence $\mathbb{P}(q_{j,\infty} = M) \leq \frac{1}{M+1}$. Plugging in the bound on $\mathbb{P}(q_{j,\infty} = M)$ to (29), we have

$$W^* - W^{\mathrm{RD}} \leq \frac{v_{\max}|\mathcal{J}|}{M+1}.$$

Notice that by definition, $W^{\mathrm{RD}} \leq W^{\mathrm{OPT}}$, hence

$$W^* - W^{\mathrm{OPT}} \leq \frac{v_{\max}|\mathcal{J}|}{M+1}.$$

Since M can be chosen arbitrarily, it must be true that $W^* - W^{\mathrm{OPT}} \leq 0$. This concludes the proof. $\qquad\square$

B.2   Omitted proofs from Section 3

The following Lemma was used in the proof of Theorem 1. Its proof uses standard arguments, and is presented here for completeness.

Lemma 8. *The following equality holds:*

$$W^{\text{WL}} = \frac{1+\lambda}{\lambda} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[W_T(\eta^{\text{WL}})\right] . \tag{30}$$

Proof. We need to argue that the limiting operator and the expectation can be interchanged. This is done in the following two steps.

First, we show that the stochastic process $\{(\xi_t, v(\theta_t, a(\theta_t, \mathbf{q}_t)))\}_{t \geq 0}$ is ergodic. Note that $v(\theta_t, a(\theta_t, \mathbf{q}_t))$ only depends on $\mathbf{q}_t$ and independent variables θ_t. The finite state Markov chain $\{\mathbf{q}_t\}_{t \geq 0}$ is irreducible and aperiodic,[24] therefore it has a unique steady state distribution and $\{(\xi_t, v(\theta_t, a(\theta_t, \mathbf{q}_t)))\}_{t \geq 0}$ is ergodic.

Second, we exchange the order of limit and expectation. It follows from the Birkhoff's ergodic theorem that $\frac{W_T(\eta^{\text{WL}})}{A_T}$ converges almost surely to $\mathbb{E}[v_\infty | \xi_\infty = 1]$, where (ξ_∞, v_∞) is the steady state distribution of $(\xi_t, v(\theta_t, a(\theta_t, \mathbf{q}_t)))$. Since $\frac{W_T(\eta^{\text{WL}})}{A_T}$ is non-negative and uniformly bounded from above by $v_{\max}$ for all $T > 0$, we have

$$W^{\text{WL}} = \mathbb{E}\left[\lim_{T \to \infty} \frac{W_T(\eta^{\text{WL}})}{A_T}\right] = \lim_{T \to \infty} \mathbb{E}\left[\frac{W_T(\eta^{\text{WL}})}{A_T}\right] = \mathbb{E}[v_\infty | \xi_\infty = 1],$$

where we apply the Bounded Convergence Theorem in the second equality to exchange the limits; the last equality holds because the boundedness of $\frac{W_T(\eta^{\text{WL}})}{A_T}$ and its almost sure convergence implies L_1 convergence. Finally, observe that

$$\mathbb{E}[v_\infty] = \mathbb{E}[v_\infty | \xi_\infty = 1] \cdot \mathbb{P}(\xi_\infty = 1) + \mathbb{E}[v_\infty | \xi_\infty = 0] \cdot \mathbb{P}(\xi_\infty = 0)$$

$$= \mathbb{E}[v_\infty | \xi_\infty = 1] \cdot \frac{\lambda}{1+\lambda} + 0$$

$$= W^{\text{WL}} \cdot \frac{\lambda}{1+\lambda},$$

where the second equality follows from the fact that all the rewards are collected when agents arrive, i.e., $\xi_t = 1$. Note that

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[W_T(\eta^{\text{WL}})\right] = \mathbb{E}[v_\infty],$$

---

[24]Irreducibility follows from the fact that all states can go to **0** with positive probability. Aperiodicity comes from the fact that the state can stay at **0** for an arbitrary number of periods.

we have

$$W^{\text{WL}} = \frac{1+\lambda}{\lambda}\mathbb{E}[v_\infty] = \frac{1+\lambda}{\lambda}\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[W_T(\eta^{\text{WL}})\right].$$

$\square$

Proof of Theorem 2. The proof is identical to the proof of Theorem 1 presented in Section 4, except for the following changes. We replace inequality (12) with the following inequality

$$\frac{1}{2}\sum_{j\in\mathcal{J}}\frac{c}{\mu_j}\mathbb{E}[(a_{j,t}-g_{j,t})^2 + l_{j,t}^2 \mid \mathbf{q}_t] \leq \frac{1}{2}\Delta + \varepsilon. \tag{31}$$

and show there exist $\alpha,\beta,c_0 > 0$, such that for any $c_\ell < c_0$ we have $\varepsilon < \beta\cdot e^{-\alpha/\Delta_\ell}$.

To show inequality (31), observe, that

$$\frac{c}{2\mu_j}(a_{j,t}-g_{j,t})^2 + \frac{c}{2\mu_j}l_{j,t}^2 = \left\{\begin{array}{ll} \frac{c}{\mu_j} & \text{if } g_{j,t}=1 \text{ and } q_j = 0\,, \\ \frac{c}{2\mu_j} & \text{otherwise}\,. \end{array}\right.$$

That is, the term above is equal to $c/\mu_j \leq \Delta$ for arrivals that correspond to an item that is discarded because its queue is empty, and equal to $c/2\mu_j \leq \Delta/2$ for all other arrivals. By Corollary 3, there exist $\alpha,\beta,c_0 > 0$ such that for every market with $c_\ell < c_0$ the probability that an arriving item j is discarded is at most $\beta\cdot e^{-\alpha/\Delta_\ell}$. Thus,

$$\frac{1}{2}\sum_{j\in\mathcal{J}}\frac{c}{\mu_j}\mathbb{E}[(a_{j,t}-g_{j,t})^2 + l_{j,t}^2 \mid \mathbf{q}_t]$$
$$\leq \frac{\Delta}{2} + \beta\cdot e^{-\alpha/\Delta_\ell}\frac{\Delta}{2}.$$

Using (31), we can replace (8) in the proof of Theorem 1 with

$$\mathbb{E}\left[W_T(\eta^{\text{WL}})\right] = \mathbb{E}\left[\sum_{t=1}^T \xi_t\cdot v(\theta_t, a(\theta_t, \mathbf{q}_t))\right]$$
$$\geq \mathbb{E}\left[\sum_{t=1}^T \frac{\lambda}{1+\lambda}W^* - \frac{1}{\mu_{\min}\cdot\Delta}\left(L(\mathbf{p}_t) - \mathbb{E}[L(\mathbf{p}_{t+1}\mid\mathbf{q}_t)] - \frac{1}{2}\Delta - \varepsilon\right)\right]$$
$$= T\frac{\lambda}{1+\lambda}W^* - \frac{1}{\mu_{\min}\cdot\Delta}\left(L(\mathbf{p}_1) - \mathbb{E}[L(\mathbf{p}_{T+1})]\right) - T\frac{1}{2}\Delta - T\varepsilon \tag{32}$$

Following the same steps as in the proof of Theorem 1 and using the assumption that

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$\lambda \geq 1$ we have that

$$
\begin{aligned}
W^{\mathrm{WL}} &= \frac{1+\lambda}{\lambda} \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[W_T(\eta^{\mathrm{WL}})\right] \\
&\geq W^* - \frac{1+\lambda}{2\lambda}\Delta - \frac{1+\lambda}{\lambda}\varepsilon \\
&\geq W^* - \Delta - 2\varepsilon .
\end{aligned}
$$

completing the proof.

$\square$

B.3 Proofs for Section 3.2

Proof of Proposition 2. We prove the result by calculating the allocative efficiency loss in the market of Example 2. By Proposition 1, we have that $W^{\mathrm{OPT}} = \gamma$.

Under the waiting list, an agent of type θ will only join the queue for item $j = \theta$. An agent arriving in epoch t of type $\theta_t = j$ will choose to join queue j to receive a value of 1 only if

$$
\gamma \geq p_j(q_j) = \frac{c}{\mu_j}(1 + q_{t,j}) = \Delta(1 + q_{t,j}) ,
$$

or

$$
q_{t,j} \leq \frac{\gamma}{\Delta} - 1 .
$$

Therefore, the possible states of each queue $j \in \mathcal{J}$ are $0, 1, \dots, K$ with $K = \lfloor \gamma/\Delta \rfloor$. Let $\pi_j(k)_{0 \leq k \leq K}$ denote the steady distributions over the length of queue j. Because the length of the queue follows a reflected unbiased random walk, all states are equally likely and $\pi_j(k) = \frac{1}{K+1}$.[25]

The allocative efficiency loss under the waiting list is given by

$$
\begin{aligned}
W^{\mathrm{OPT}} - W^{\mathrm{WL}} &= \gamma - \sum_{j \in \Theta} F(j) \left(\pi_j(K) \cdot 0 + \sum_{k < K} \pi_j(k) \cdot \gamma \right) \\
&= \gamma - J\frac{1}{J}\left(\frac{1}{K+1} \cdot 0 + \frac{K}{K+1} \cdot \gamma \right) \\
&= \frac{1}{K+1} \cdot \gamma \\
&= \frac{1}{\lfloor \gamma/\Delta \rfloor + 1} \cdot \gamma
\end{aligned}
$$

---

[25]To see this directly, observe that equating probability flows across a cut gives for any $0 < k < K$ that $\pi_j(k)\lambda/J = \pi_j(k+1)\mu_j$, which implies that $\pi_j(0) = \pi_j(1) = \cdots = \pi_j(K)$.

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By choosing γ such that $\lfloor \gamma/\Delta \rfloor \approx \gamma/\Delta - 1$ we get that

$$W^{\text{OPT}} - W^{\text{WL}} \geq \frac{1}{\gamma/\Delta - 1 + 1} \cdot \gamma - \varepsilon$$
$$= \Delta - \varepsilon$$

□

B.4 Omitted proofs for Section 3.3

Proof of Lemma 1. A problem instance $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is said to satisfy *generalized imbalance (GI)* if there are no pair of nonempty subsets of agent types $\mathcal{I}' \subset \Theta$ and item types $\mathcal{J}' \subset \mathcal{J}$ such that the total arrival rate of agents with type in $\mathcal{I}'$ exactly matches the total arrival rate of items with types in $\mathcal{J}'$, that is

$$\sum_{j \in \mathcal{J}'} \mu_j \neq \sum_{\theta \in \mathcal{I}'} \lambda_\theta \qquad \forall \mathcal{J}' \subset \mathcal{J}, \, \mathcal{I}' \subset \Theta \, .$$

The set of $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ satisfying GI is open and dense in $\mathbb{R}_{++}^{|\mathcal{I}|} \times \mathbb{R}_{++}^{|\mathcal{J}|}$. The result follows because GI implies uniqueness of market clearing prices (see Proposition C.2 of Johari et al. (2016)).

□

B.5 Proofs for Section 3.4

Proof of Proposition 3. For ease of reading, the proof is divided into three steps. First, we provide a lower bound for the dual objective. Second, we use that lower bound to construct a lower bound on the Lyapunov drift. Third, we drive a concentration bound using the Lyapunov drift.

Step 1. We first establish a lower bound of the dual function $h(\mathbf{p})$, namely, for any $\epsilon > 0$, there exists $\gamma > 0$, such that $\forall \mathbf{p} \notin B_\epsilon(\mathcal{P}^*)$,

$$h(\mathbf{p}) - h(\mathbf{p}^*) \geq \gamma \cdot \text{dist}(\mathbf{p}, \mathcal{P}^*) \, . \tag{33}$$

To establish (33), we use the ϵ-level set of the dual function $h(\mathbf{p})$ which we denote by $\mathcal{P}_h(\epsilon) \triangleq \{\mathbf{p} \in \mathbb{R}_+^{|\mathcal{J}|} : h(\mathbf{p}) \leq h(\mathbf{p}^*) + \epsilon\}$ for $\epsilon > 0$, where $\mathbf{p}^* \in \mathcal{P}^*$ is some market clearing price.

Let there be $\epsilon > 0$. By continuity of $h(\mathbf{p})$ and boundedness of $\mathcal{P}_h(\epsilon)$ for any ϵ, there exists $\epsilon_1 > 0$ such that $\mathcal{P}_h(\epsilon_1) \subset B_\epsilon(\mathcal{P})$. Thus, for any $\mathbf{p} \notin B_\epsilon(\mathcal{P})$, we have that $\mathbf{p} \notin \mathcal{P}_h(\epsilon_1)$.

Since $h(\mathbf{p})$ is convex and $\mathcal{P}^*, \mathcal{P}_h(\epsilon)$ are bounded, Proposition 1 in Deng (1998) implies that for $\epsilon_1 > 0$, there exists $\gamma_1 > 0$, such that $\forall \mathbf{p} \notin \mathcal{P}_h(\epsilon_1)$,

$$h(\mathbf{p}) - h(\mathbf{p}^*) - \epsilon_1 \geq \gamma_1 \cdot \text{dist}(\mathbf{p}, \mathcal{P}_h(\epsilon_1)). \tag{34}$$

We will show that the desired inequality (33) holds for $\gamma \triangleq \min\{\gamma_1, \frac{\epsilon_1}{\text{dist}(\mathcal{P}^*, \mathcal{P}_h^c(\epsilon_1))}\} > 0$, where $\text{dist}(\mathcal{P}^*, \mathcal{P}_h^c(\epsilon_1)) \triangleq \inf_{\mathbf{p} \in \mathcal{P}^*, \mathbf{p}' \notin \mathcal{P}_h(\epsilon_1)} ||\mathbf{p} - \mathbf{p}'||_2$.

Note that $\text{dist}(\mathcal{P}^*, \mathcal{P}_h^c(\epsilon)) > 0$ for any $\epsilon > 0$. It follows from triangle inequality that

$$\text{dist}(\mathbf{p}, \mathcal{P}_h(\epsilon_1)) \geq \text{dist}(\mathbf{p}, \mathcal{P}^*) - \text{dist}(\mathcal{P}^*, \mathcal{P}_h^c(\epsilon_1)).$$

We thus have that for any given $\mathbf{p} \notin B_\epsilon(\mathcal{P})$,

$$\begin{aligned}
h(\mathbf{p}) - h(\mathbf{p}^*) &\geq \gamma_1 \cdot \text{dist}(\mathbf{p}, \mathcal{P}_h(\epsilon_1)) + \epsilon_1 \\
&\geq \gamma\Big(\text{dist}(\mathbf{p}, \mathcal{P}^*) - \text{dist}(\mathcal{P}^*, \mathcal{P}_h^c(\epsilon_1))\Big) + \epsilon_1 \\
&\geq \gamma \cdot \text{dist}(\mathbf{p}, \mathcal{P}^*) - \frac{\epsilon_1}{\text{dist}(\mathcal{P}^*, \mathcal{P}_h^c(\epsilon_1))} \text{dist}(\mathcal{P}^*, \mathcal{P}_h^c(\epsilon_1)) + \epsilon_1 \\
&\geq \gamma \cdot \text{dist}(\mathbf{p}, \mathcal{P}^*).
\end{aligned}$$

This concludes the proof of step 1.

Step 2. We now bound the Lyapunov drift for the Lyapunov functions

$$\tilde{L}(\mathbf{p}) = \inf_{\mathbf{p}' \in \mathcal{P}^*} \left\{ \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j (p_j - p_j')^2 \right\} \qquad \text{and} \qquad \tilde{V}(\mathbf{p}) \triangleq \sqrt{\tilde{L}(\mathbf{p})}.$$

showing that

$$\mathbb{E}[\tilde{L}(\mathbf{p}_{t+1}) | \mathbf{q}_t] - \tilde{L}(\mathbf{p}_t) \leq -c\left(h(\mathbf{p}_t) - h(\mathbf{p}^*)\right) + c\frac{2+\lambda}{2(1+\lambda)}\Delta. \tag{35}$$

Fix $\mathbf{p}_t$ and let $\mathbf{p}_t^* \triangleq \text{argmin}_{\mathbf{p}' \in \mathcal{P}^*} \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j (p_{j,t} - p_j')^2$ be the closest market clearing price to $\mathbf{p}_t$ in the weighted norm (note that the minimum is attainable since $\mathcal{P}^*$ is compact).

As an intermediate step, we first show that

$$\mathbb{E}[\bar{L}_t(\mathbf{p}_{t+1}) | \mathbf{q}_t] - \bar{L}_t(\mathbf{p}_t) \leq -c\left(h(\mathbf{p}_t) - h(\mathbf{p}^*)\right) + c\frac{2+\lambda}{2(1+\lambda)}\Delta, \tag{36}$$

for $\bar{L}_t(\mathbf{p}_t) \triangleq \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j (p_{j,t} - p_{j,t}^*)^2$.

To see that (36) holds, recall that $p_{j,t+1} = p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t} + l_{j,t})$. We have that

$$
\begin{aligned}
&\bar{L}(\mathbf{p}_{t+1}) - \bar{L}(\mathbf{p}_t) \\
&= \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j \left(\frac{c^2}{\mu_j^2}(a_{j,t} - g_{j,t} + l_{j,t})^2 + \frac{2c}{\mu_j}(p_{j,t} - p_{j,t}^*)(a_{j,t} - g_{j,t} + l_{j,t}) \right) \\
&= c\langle \mathbf{p}_t - \mathbf{p}_t^*, \mathbf{a}_t - \mathbf{g}_t \rangle + \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j \left(\frac{c^2}{\mu_j^2}(a_{j,t} - g_{j,t} + l_{j,t})^2 + \frac{2c}{\mu_j}(p_{j,t} - p_{j,t}^*)l_{j,t} \right) \\
&\overset{(a)}{\leq} c\langle \mathbf{p}_t - \mathbf{p}_t^*, \mathbf{a}_t - \mathbf{g}_t \rangle + \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j \left(\frac{c^2}{\mu_j^2}(a_{j,t} - g_{j,t} + l_{j,t})^2 + \frac{2c}{\mu_j}p_{j,t} \cdot l_{j,t} \right) \\
&= c\langle \mathbf{p}_t - \mathbf{p}_t^*, \mathbf{a}_t - \mathbf{g}_t \rangle \\
&\quad + \frac{1}{2} \sum_{j \in \mathcal{J}} \mu_j \left(\frac{c^2}{\mu_j^2}\left((a_{j,t} - g_{j,t})^2 + l_{j,t}^2\right) + \frac{2c}{\mu_j}\left(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t})\right)l_{j,t} \right).
\end{aligned}
$$

where inequality (a) follows because $p_{j,t}^* \geq 0$ and $l_{j,t} \geq 0$. Note that $(p_{j,t} + \frac{c}{\mu_j}(a_{j,t} - g_{j,t}))l_{j,t} \equiv 0$.

Recall equation (12), which establishes that

$$
\frac{1}{2} \sum_{j \in \mathcal{J}} \frac{c}{\mu_j} \mathbb{E}[(a_{j,t} - g_{j,t})^2 + l_{j,t}^2 | \mathbf{q}_t] \leq \frac{2 + \lambda}{2(1 + \lambda)} \Delta.
$$

By taking expectations and plugging in the above to the RHS, we have

$$
\mathbb{E}[\bar{L}(\mathbf{p}_{t+1}) | \mathbf{q}_t] - \bar{L}(\mathbf{p}_t) \leq c\mathbb{E}[\langle \mathbf{p}_t - \mathbf{p}^*, \mathbf{a}_t - \mathbf{g}_t \rangle | \mathbf{q}_t] + c\frac{2 + \lambda}{2(1 + \lambda)}\Delta.
$$

Use the fact that $\mathbb{E}[\mathbf{a}_t - \mathbf{g}_t | \mathbf{q}_t] \in -\partial h(\mathbf{p}_t)$, we have

$$
\mathbb{E}[\bar{L}(\mathbf{p}_{t+1}) | \mathbf{q}_t] - \bar{L}(\mathbf{p}_t) \leq -c\left(h(\mathbf{p}_t) - h(\mathbf{p}^*)\right) + c\frac{2 + \lambda}{2(1 + \lambda)}\Delta. \tag{37}
$$

To see the bound (36) for $\bar{L}$ implies the desired bound (35) for $\tilde{L}$, observe that $\bar{L}_t(\mathbf{p}_t) = \tilde{L}(\mathbf{p}_t)$ (by definition), and that $\tilde{L}(\mathbf{p}_{t+1}) \leq \bar{L}_t(\mathbf{p}_{t+1})$.

Step 3. Let $\epsilon > 0$. By the previous steps, there exists γ such that for all $\mathbf{p} \notin B_\epsilon(\mathcal{P}^*)$ we have that

$$
\begin{aligned}
\mathbb{E}[\tilde{L}(\mathbf{p}_{t+1}) | \mathbf{q}_t] - \tilde{L}(\mathbf{p}_t) &\leq -c\left(h(\mathbf{p}_t) - h(\mathbf{p}^*)\right) + c\frac{2 + \lambda}{2(1 + \lambda)}\Delta \\
&\leq -c\gamma\|\mathbf{p}_t - \mathbf{p}^*\|_2 + c\frac{2 + \lambda}{2(1 + \lambda)}\Delta.
\end{aligned}
$$

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Because $f(x) = \sqrt{x}$ is concave for $x \geq 0$, we have that for $y > x > 0$, $f(y) - f(x) \leq (y-x)f'(x) = \frac{y-x}{2\sqrt{x}}$. Therefore,

$$\tilde{V}(\mathbf{p}_{t+1}) - \tilde{V}(\mathbf{p}_t) \leq \frac{\tilde{L}(\mathbf{p}_{t+1}) - \tilde{L}(\mathbf{p}_t)}{2\tilde{V}(\mathbf{p}_t)} . \tag{38}$$

Taking conditional expectation given $\mathbf{q}_t$ on both sides of (38) and plug in (35), we have

$$\mathbb{E}[\tilde{V}(\mathbf{p}_{t+1})|\mathbf{q}_t] - \tilde{V}(\mathbf{p}_t) \leq \frac{\mathbb{E}[\tilde{L}(\mathbf{p}_{t+1})|\mathbf{q}_t] - \tilde{L}(\mathbf{p}_t)}{2\tilde{V}(\mathbf{p}_t)}$$

$$\leq -\frac{c\gamma}{2} + \frac{2+\lambda}{2(1+\lambda)}\frac{c^2}{2\mu_{\min}\tilde{V}(\mathbf{p}_t)} .$$

For $\mathbf{p}_t$ such that

$$\tilde{V}(\mathbf{p}_t) \geq \frac{c}{\mu_{\min}\gamma}\frac{2+\lambda}{1+\lambda} ,$$

we have

$$\mathbb{E}[\tilde{V}(\mathbf{p}_{t+1})|\mathbf{q}_t] - \tilde{V}(\mathbf{p}_t) \leq -\frac{c\gamma}{4} .$$

Now we use a concentration bound from Bertsimas et al. (2001) to prove the desired result. $\tilde{V}(\cdot)$ is a Lyapunov function with exception parameter $\frac{c}{\mu_{\min}\gamma}\frac{2+\lambda}{1+\lambda}$ and negative drift $\frac{c\gamma}{4}$. In each step, the Lyapunov function can increase by at most $\frac{c}{\mu_{\min}}$. Using Theorem 1 in Bertsimas et al. (2001), we have for any $r = 0, 1, \ldots,$

$$\mathbb{P}\left(\tilde{V}\left(\mathbf{p}(c_\ell)\right) > \frac{c}{\mu_{\min}\gamma}\frac{2+\lambda}{1+\lambda} + 2r\frac{c}{\mu_{\min}}\right) \leq \left(\frac{\frac{c}{\mu_{\min}}}{\frac{c}{\mu_{\min}} + \frac{c\gamma}{4}}\right)^{r+1} = \left(\frac{1}{1 + \frac{\gamma\mu_{\min}}{4}}\right)^{r+1} .$$

If $\mathbf{p} \notin B_\epsilon(\mathcal{P}^*)$ we have that $\tilde{V}(\mathbf{p}) \geq \frac{\mu_{\min}}{2}\epsilon$. Choosing $c_0 = \mu_{\min}^2\epsilon\gamma/8$ we have that $\Delta_\ell = c_\ell/\mu_{\min} \leq \mu_{\min}\epsilon\gamma\frac{1+\lambda}{4(2+\lambda)}$ for any $c_\ell < c_0$. Plugging in $r = \frac{\mu_{\min}\epsilon}{8\Delta_\ell}$ we have

$$\mathbb{P}(\mathbf{p}(c_\ell) \notin B_\epsilon(\mathcal{P})) \leq \mathbb{P}\left(\tilde{V}(\mathbf{p}(c_\ell)) > \frac{\mu_{\min}}{2}\epsilon\right)$$

$$\leq \mathbb{P}\left(\tilde{V}(\mathbf{p}(c_\ell)) > \frac{c}{\mu_{\min}\gamma}\frac{2+\lambda}{1+\lambda} + 2r\frac{c}{\mu_{\min}}\right)$$

$$\leq \left(\frac{1}{1 + \frac{\gamma\mu_{\min}}{4}}\right)^{\frac{\mu_{\min}\epsilon}{8\Delta_\ell}}$$

$$= \exp\left(-\log\left(1 + \frac{\gamma\mu_{\min}}{4}\right)\left(\frac{\mu_{\min}\epsilon}{8\Delta_\ell}\right)\right) .$$

This concludes the proof.

$\square$

Proof of Corollary 3. Let κ_j denote the probability that an arriving item j is discarded. Denote $\varepsilon = \frac{1}{2}\inf\{p_j \mid \mathbf{p} \in \mathcal{P}^*\}$. We have that if $\mathbf{p} \in B_\varepsilon(\mathcal{P}^*)$, then $p_j \geq \varepsilon > 0$.

For sufficiently small $c_1 > 0$ we have that $\mathbb{E}[c(w_j)|q_j = 0] = c/\mu_j < \varepsilon$ for every $c < c_1$. Thus, $\mathbf{p}_t \in B_\varepsilon(\mathcal{P}^*)$ and $c < c_1$ implies $q_{jt} > 0$, that is, the j-th queue is not empty at epoch t and an item arriving in epoch t is assigned to some agent. By Proposition 3 there exist $\alpha, \beta, c_0 > 0$ such that for any $c_\ell < \min\{c_0, c_1\}$ we have that

$$\kappa \leq \mathbb{P}\Big(\mathbf{p}(c_\ell) \notin B_\varepsilon(\mathcal{P}^*)\Big) < \beta \cdot e^{-\alpha/\Delta_\ell}\,.$$

$\square$

C   Auxiliary Results

This appendix provides auxiliary results to complement our analysis. Appendix C.1 bounds the adjustment size for nonlinear waiting cost functions. Appendix C.2 provides a lemma that tightens the bound of Proposition 3.

C.1   Adjustment size of nonlinear waiting cost functions

We stated our main result (Theorem 1) in terms of the adjustment size Δ. This section shows how Δ is related to the waiting cost function $c(w)$ for nonlinear waiting costs. We focus on the waiting costs that satisfy the assumption below.

Assumption 1. *We consider the following classes of waiting cost functions.*

- *Convex waiting costs. $c(w)$ is convex, twice-differentiable for $w \geq 0$, and that $c'(w)$ and $c''(w)$ are subexponential, i.e., there exists α such that $c'(w), c''(w) \leq e^{\alpha w}$ for all $w \geq 0$.*

- *Concave waiting costs. $c(w)$ is concave and twice-differentiable for $w \geq 0$.*

Proposition 5. *Consider the asymptotic regime in Corollary 2 and waiting cost functions satisfying Assumption 1. The following holds:*

1. *For convex $c(w)$, there exists $\ell_0 < \infty$ such that for $\ell \geq \ell_0$, $\Delta \leq \frac{2c'(c^{-1}(v_{\max}))}{\ell\mu_{\min}}$.*

2. *For concave $c(w)$, for any $\ell > 0$, $\Delta \leq \frac{c'(0)}{\ell\mu_{\min}}$.*

Proof of Proposition 5. Consider the system with index ℓ. Let X_t be the interarrival time between the t-th type j item and the $(t+1)$-th type j item. Hence, X_t is an exponential random variable with rate $\ell\mu_j$, and $\{X_t\}_{t=1}^{\infty}$ are i.i.d. Let $S_n \triangleq \sum_{t=1}^{n} X_t$.

Let $q_{\max,\ell}$ be the threshold queue length above which no arriving agent will join that queue. Then approximately

$$v_{\max} = p_j(q_{\max,\ell}) = \mathbb{E}[c(S_{q_{\max,\ell}})].$$

For convex cost function $c(w)$, by Jensen's inequality we have

$$\mathbb{E}[c(S_{q_{\max,\ell}})] \geq c\left(\mathbb{E}[S_{q_{\max,\ell}}]\right) = c\left(\frac{q_{\max,\ell}}{\ell\mu_j}\right).$$

Compare the above two inequalities, we have $q_{\max,\ell} \leq \ell\mu_j c^{-1}(v_{\max})$. Notice that

$$p_j(q_j+1) - p_j(q_j) = \mathbb{E}[c(S_{q_j} + X_{q_j+1}) - c(S_{q_j})] \leq \mathbb{E}[c'(S_{q_j} + X_{q_j+1}) \cdot X_{q_j+1}],$$

where the inequality follows from the convexity of $c(w)$. Take the supremum over all $0 \leq q_j \leq q_{\max,\ell}$ on both sides of the above inequality. Because $c(w)$ is convex, $c'(w)$ must be non-decreasing, hence

$$\Delta = \sup_{0 \leq q_j \leq q_{\max,\ell}} (p_j(q_j+1) - p_j(q_j)) \leq \mathbb{E}[c'(S_{q_{\max,\ell}} + X_{q_{\max,\ell}+1}) \cdot X_{q_{\max,\ell}+1}].$$

Using Holder's inequality, we have

$$\mathbb{E}[c'(S_{q_{\max,\ell}} + X_{q_{\max,\ell}+1}) \cdot X_{q_{\max,\ell}+1}] \leq \sqrt[\alpha]{\mathbb{E}[(c'(S_{q_{\max,\ell}+1}))^\alpha]} \cdot \sqrt[\beta]{\mathbb{E}[X_1^\beta]},$$

where $\alpha, \beta \in (1, \infty)$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Because $c(w)$ satisfies Assumption 1, for any $\alpha \in (1, \infty)$ we can apply Lemma 9 and it follows that

$$\lim_{\ell \to \infty} \sqrt[\alpha]{\mathbb{E}[(c'(S_{q_{\max,\ell}}))^\alpha]} = \sqrt[\alpha]{c'\left(\mathbb{E}[S_{q_{\max,\ell}}]\right)^\alpha} = c'\left(\frac{q_{\max,\ell}}{\ell\mu_j}\right).$$

Therefore

$$\lim_{\ell \to \infty} \mathbb{E}[c'(S_{q_{\max,\ell}} + X_{q_{\max,\ell}+1}) \cdot X_{q_{\max,\ell}+1}] \leq c'\left(\frac{q_{\max,\ell}}{\ell\mu_j}\right) \inf_{\beta>1} \sqrt[\beta]{\mathbb{E}[X_1^\beta]}$$

$$\leq c'\left(\frac{q_{\max,\ell}}{\ell\mu_j}\right) \mathbb{E}[X_1] = \frac{c'\left(\frac{q_{\max,\ell}}{\ell\mu_j}\right)}{\ell\mu_j}.$$

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As a result, there exists $\ell_0 > 0$ such that for $\ell \geq \ell_0$, it holds that

$$\Delta \leq \frac{2c'\left(\frac{q_{\max,\ell}}{\ell\mu_j}\right)}{\ell\mu_j} \leq \frac{2c'\left(c^{-1}(v_{\max})\right)}{\ell\mu_j} \,. \tag{39}$$

For concave $c(w)$, using its concavity we have for any $0 \leq q_j \leq q_{\max,\ell}$,

$$p_j(q_j+1) - p_j(q_j) = \mathbb{E}[c(S_{q_j} + X_{q_j+1}) - c(S_{q_j})] \leq \mathbb{E}[c'(0)X_1] = \frac{c'(0)}{\ell\mu_j} \,. \tag{40}$$

Combine (39) and (40), we conclude the proof. $\qquad\square$

Lemma 9. *Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. exponential random variables with rate 1, $\bar{X}_n \triangleq \frac{1}{n}\sum_{i=1}^{n} X_i$, and $\alpha \in (0,\infty)$. For any continuously differentiable function $f(x)$ defined on $\mathbb{R}^+$ such that there exists $C_1, C_2 \in (0,\infty)$ and $f(x) \leq C_1 e^{C_2 x}$, $f'(x) \leq C_1 e^{C_2 x}$ for all $x \in \mathbb{R}^+$, we have*

$$\lim_{N\to\infty} \mathbb{E}\left[f(\alpha \cdot \bar{X}_N)\right] = f(\alpha) \,.$$

Proof of Lemma 9. The result simply follows the proof of Theorem 1(c) in Kozakiewicz et al. (1947), therefore we omit the details. $\qquad\square$

C.2   The rate of convergence of waiting costs

This section provides additional results that provide explicit constants for the rate of convergence shown in Proposition 3. We show that in discrete economies with unique market clearing prices are very likely to be close to $\mathbf{p}^*$ by showing that prices that deviates from $\mathbf{p}^*$ quickly adjusts back. The rate of price adjustment is related to the "sharpness" of the dual objective (5).

The proof of shows that if for $\gamma(\mathbf{v},\boldsymbol{\lambda},\boldsymbol{\mu}) > 0$ it holds for any $\mathbf{p} \in \mathbb{R}^{|\mathcal{J}|}$ that

$$h(\mathbf{p}) - h(\mathbf{p}^*) \geq \gamma(\mathbf{v},\boldsymbol{\lambda},\boldsymbol{\mu})||\mathbf{p} - \mathbf{p}^*||_2 \,, \tag{41}$$

then the following bound holds:

$$\mathbb{P}(\mathbf{p}(c_\ell) \notin B_\epsilon(\mathcal{P})) \leq \exp\left(-\log\left(1 + \frac{\gamma\mu_{\min}}{4}\right)\left(\frac{\mu_{\min}\epsilon}{8\Delta_\ell}\right)\right) \,.$$

The following lemma provides an explicit lower bound for such γ.

Lemma 10 (Geometry of dual function)**.** *There exists $\gamma(\mathbf{v},\boldsymbol{\lambda},\boldsymbol{\mu}) > 0$ such that for any $\mathbf{p} \in \mathbb{R}^{|\mathcal{J}|}$, we have*

$$h(\mathbf{p}) - h(\mathbf{p}^*) \geq \gamma(\mathbf{v},\boldsymbol{\lambda},\boldsymbol{\mu})||\mathbf{p} - \mathbf{p}^*||_2 \,. \tag{42}$$

Moreover,

$$\gamma(\mathbf{v}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \geq \frac{1}{\lambda} \left\{ \min_{\mathcal{I} \subset \mathcal{J}} \frac{\lambda - \sum_{j \in \mathcal{I}} \mu_j}{\sqrt{|\mathcal{I}|}} , \min_{\{\mathcal{I} : \mathcal{I} \subset \mathcal{J}, \mathcal{I} \supset \cup_{\theta \in \Theta^*} \mathcal{J}_\theta^*\}} \frac{\sum_{j \in \mathcal{I}} \mu_j - \sum_{\theta \in \Theta^*} \lambda_\theta}{\sqrt{|\mathcal{I}|}} , \min_{j \in \mathcal{J}} \mu_j \right\}.$$

The proof shows that the rate of adjustment is positive for $\mathbf{p} \in \mathcal{P}$ and by convexity of $h(\mathbf{p}) - h(\mathbf{p}^*)$ this holds also for prices not in $\mathcal{P}$.

Proof of Lemma 10. We proceed in two steps.

Step 1. We first show that we can lower bound $h(\mathbf{p}) - h(\mathbf{p}^*)$ by a support function:

$$h(\mathbf{p}) - h(\mathbf{p}^*) \geq \sup_{s \in \mathcal{S}} \langle \mathbf{p}^* - \mathbf{p}, \mathbf{s} \rangle.$$

for some convex set $\mathcal{S}$.

For each agent type $\theta \in \Theta$, define

$$\Delta_\theta \triangleq \left\{ \mathbf{x} \in \mathbb{R}_+^{|\mathcal{J}|} : \sum_{j \in \mathcal{J}} x_j = 1 , x_j = 0 \text{ for } j \notin \mathcal{J}_\theta^* \right\},$$

$$\tilde{\Delta}_\theta \triangleq \left\{ \mathbf{x} \in \mathbb{R}_+^{|\mathcal{J}|} : \sum_{j \in \mathcal{J}} x_j \leq 1 , x_j = 0 \text{ for } j \notin \mathcal{J}_\theta^* \right\},$$

$$\Delta \triangleq \left\{ \mathbf{x} \in \mathbb{R}_+^{|\mathcal{J}|} : \sum_{j \in \mathcal{J}} x_j = 1 \right\}, \quad \tilde{\Delta} \triangleq \left\{ \mathbf{x} \in \mathbb{R}_+^{|\mathcal{J}|} : \sum_{j \in \mathcal{J}} x_j \leq 1 \right\}.$$

Using the definitions above, we can rewrite the dual function 5 as

$$h(\mathbf{p}) = \frac{1}{\lambda} \sum_{\theta \in \Theta} \lambda_\theta \left(\max_{\mathbf{x}_\theta \in \tilde{\Delta}} \sum_{j \in \mathcal{J}} (v(\theta, j) - p_j) x_{\theta, j} \right) + \frac{1}{\lambda} \sum_{j \in \mathcal{J}} \mu_j p_j.$$

Let $\mathbf{x}_\theta^*$ be a maximizer of the inner maximization problem above. Define $\mathbf{s} \in \mathbb{R}^{|\mathcal{J}|}$ where

$$s_j = \sum_{\theta \in \Theta} \lambda_\theta \cdot x_{\theta, j}^* - \mu_j,$$

then it is easy to see that $-\frac{1}{\lambda}\mathbf{s}$ is a subgradient of $h(\mathbf{p})$ at $\mathbf{p}$, denoted by $-\frac{1}{\lambda}\mathbf{s} \in \partial h(\mathbf{p})$.

For $\theta \in \Theta^*$, let

$$\mathbf{x}_\theta' \triangleq \operatorname{argmax}_{\mathbf{x}_\theta \in \Delta_\theta} \sum_{j \in \mathcal{J}} (p_j^* - p_j) x_{\theta, j},$$

for $\theta \in \Theta \backslash \Theta^*$, let

$$\mathbf{x}'_\theta \triangleq \mathrm{argmax}_{\mathbf{x}_\theta \in \tilde{\Delta}_\theta} \sum_{j \in \mathcal{J}} (p^*_j - p_j) x_{\theta,j} \,.$$

The interpretation of $\mathbf{x}'_\theta$ is as follows. Consider a type θ agent. If the current price is exactly $\mathbf{p}^*$, then the agent is indifferent between the items in $\mathcal{J}^*_\theta$, and strictly prefers these items to other items. If the price deviates a little from $\mathbf{p}^*$: (1) if $\theta \in \Theta^*$, the agent will prefer the item in $\mathcal{J}^*_\theta$ that is the cheapest; (2) if $\theta \notin \Theta^*$, the agent's optimal utility is zero, hence she will choose an item in $\mathcal{J}^*_\theta$ that is the cheapest only if the price is lower than the optimal price, otherwise she will not choose any item. $\mathbf{x}'_\theta$ characterizes the choice of an agent when $\mathbf{p}$ is sufficiently close to $\mathbf{p}^*$.

A key observation is that when $\mathbf{p} \in \mathcal{P}$, the above observation for "sufficiently close" $\mathbf{p}$ holds. Therefore, for $\mathbf{s}' \in \mathbb{R}^{|\mathcal{J}|}$ defined as

$$s'_j \triangleq \sum_{\theta \in \Theta} \lambda_\theta \cdot x'_{\theta,j} - \mu_j \,,$$

we have $-\frac{1}{\lambda} \mathbf{s}' \in \partial h(\mathbf{p})$, and that for $\mathbf{p} \in \mathcal{P}^*$,

$$\begin{aligned}
\lambda h(\mathbf{p}) &= \sum_{\theta \in \Theta} \lambda_\theta \sum_{j \in \mathcal{J}} (v(\theta,j) - p_j) x'_{\theta,j} + \sum_{j \in \mathcal{J}} \mu_j p_j \\
&= \sum_{\theta \in \Theta} \lambda_\theta \sum_{j \in \mathcal{J}} (p^*_j - p_j) x'_{\theta,j} + \sum_{j \in \mathcal{J}} \mu_j (p_j - p^*_j) \\
&\quad + \sum_{\theta \in \Theta} \lambda_\theta \sum_{j \in \mathcal{J}} (v(\theta,j) - p^*_j) x'_{\theta,j} + \sum_{j \in \mathcal{J}} \mu_j p^*_j
\end{aligned}$$

Note that the sum of the terms in the last row is exactly $\lambda h(\mathbf{p}^*)$. This is because for an agent of type $\theta \in \Theta^*$, under price $\mathbf{p}$ she must choose an item from $\mathcal{J}^*_\theta$, hence

$$\sum_{j \in \mathcal{J}} (v(\theta,j) - p^*_j) x'_{\theta,j} = \max_{j \in \mathcal{J}} \left(v(\theta,j) - p^*_j \right),$$

whereas for agent of type $\theta \notin \Theta^*$, she either chooses an item from $\mathcal{J}^*_\theta$, or she balks. Hence

$$\sum_{j \in \mathcal{J}} (v(\theta,j) - p^*_j) x'_{\theta,j} = 0 = \max_{j \in \mathcal{J}} \left(v(\theta,j) - p^*_j \right).$$

Thus we have

$$\lambda(h(\mathbf{p}) - h(\mathbf{p}^*)) = \sum_{\theta \in \Theta^*} \lambda_\theta \left(\max_{\mathbf{x}_\theta \in \Delta_\theta} \sum_{j \in \mathcal{J}} (p^*_j - p_j) x_{\theta,j} \right)$$

$$+ \sum_{\theta \in \Theta \setminus \Theta^*} \lambda_\theta \left(\max_{\mathbf{x}_\theta \in \tilde{\Delta}_\theta} \sum_{j \in \mathcal{J}} (p_j^* - p_j) x_{\theta,j} \right) + \sum_{j \in \mathcal{J}} \mu_j (p_j - p_j^*) . \qquad (43)$$

Define the *rate region* $\mathcal{S}$ as:

$$\mathcal{S} \triangleq \left\{ \sum_{\theta \in \Theta} \lambda_\theta \mathbf{x}_\theta - \boldsymbol{\mu} : \mathbf{x}_\theta \in \Delta_\theta \text{ for } \theta \in \Theta^* , \mathbf{x}_\theta \in \tilde{\Delta}_\theta \text{ for } \theta \in \Theta \setminus \Theta^* \right\} ,$$

which is the set of possible rates of change of dual prices when $\mathbf{p} \in \mathcal{P}$. Therefore we can rewrite the RHS of (43) as

$$\sup_{\mathbf{s} \in \mathcal{S}} \langle \mathbf{p}^* - \mathbf{p}, \mathbf{s} \rangle .$$

Using the fact that $h(\mathbf{p})$ is convex, we have for any $\mathbf{p}$,

$$h(\mathbf{p}) - h(\mathbf{p}^*) \geq \frac{1}{\lambda} \sup_{\mathbf{s} \in \mathcal{S}} \langle \mathbf{p}^* - \mathbf{p}, \mathbf{s} \rangle .$$

This concludes step 1.

Step 2. Characterizing the set $\mathcal{S}$. Note that $\mathcal{S}$ is the Minkowski sum of simplices shifted by $\boldsymbol{\mu}$, which is known as the generalized permutohedron (see, e.g., Postnikov, 2009). Using Proposition 6.3 from Postnikov (2009), we have the following defining inequalities of $\mathcal{S}$:

$$\sum_{j \in \mathcal{I}} s_j \leq \lambda - \sum_{j \in \mathcal{I}} \mu_j , \qquad\qquad \forall \mathcal{I} \subset \mathcal{J} ,$$

$$\sum_{j \in \mathcal{I}} s_j \geq \sum_{\theta \in \Theta^*} \lambda_\theta - \sum_{j \in \mathcal{I}} \mu_j , \qquad\qquad \forall \mathcal{I} : \mathcal{I} \subset \mathcal{J} , \mathcal{I} \supset \cup_{\theta \in \Theta^*} \mathcal{J}_\theta^* ,$$

$$s_j \geq -\mu_j , \qquad\qquad \forall j \in \mathcal{J} .$$

We first argue that there exists $\epsilon > 0$ such that the ball $B(\mathbf{0}, \epsilon)$ is contained in $\mathcal{S}$. This can be proved by contradiction: if it is not true, then using (43), we can show that the minimizer of $h(\mathbf{p})$ is non-unique, leading to contradiction with the assumption that the dual optimum is unique. Note that this already leads to a lower bound of $h(\mathbf{p}) - h(\mathbf{p}^*)$: we have $\epsilon \frac{\mathbf{p}^* - \mathbf{p}}{||\mathbf{p}^* - \mathbf{p}||_2} \subset \mathcal{S}$, hence $\lambda(h(\mathbf{p}) - h(\mathbf{p}^*)) \geq \epsilon ||\mathbf{p}^* - \mathbf{p}||_2$.

It remains to quantitatively characterize ϵ. To simplify the notation, we consider the centered version of $\mathbf{p}$, defined as $\tilde{\mathbf{p}} \triangleq \mathbf{p}^* - \mathbf{p}$; let $\tilde{h}(\tilde{\mathbf{p}}) \triangleq h(\mathbf{p}) - h(\mathbf{p}^*)$.

Since $\mathcal{S}$ is defined "locally" (i.e., for $\mathbf{p} \in \mathcal{P}$), all the arguments below assume that $\mathbf{p} \in \mathcal{P}$.

We have derived that $\tilde{h}(\tilde{\mathbf{p}}) = \sup_{\mathbf{s} \in \mathcal{S}} \langle \tilde{\mathbf{p}}, \mathbf{s} \rangle$. Define the level sets of $\tilde{h}(\tilde{\mathbf{p}})$:

$$\mathcal{L} \triangleq \left\{ \tilde{\mathbf{p}} \in \mathbb{R}^{|\mathcal{J}|} : \tilde{p}_j \leq 0 \text{ for } j \neq \mathcal{J}^*, \tilde{h}(\tilde{\mathbf{p}}) \leq 1 \right\} .$$

Here the constraints $\tilde{p}_j \leq 0$ for $j \neq \mathcal{J}^*$ come from the fact that $\mathbf{p} \geq \mathbf{0}$. Using the theory of polar duality (see, e.g., Gallier, 2008), since the ball $B(\mathbf{0}, \epsilon)$ is contained in $\mathcal{S}$, we have that

$$\left(\mathcal{S} \cap \{ \tilde{\mathbf{p}} : \tilde{p}_j \leq 0 \text{ for } j \neq \mathcal{J}^* \} \right)^* \subset \left(B(\mathbf{0}, \epsilon) \cap \{ \tilde{\mathbf{p}} : \tilde{p}_j \leq 0 \text{ for } j \neq \mathcal{J}^* \} \right)^* .$$

Here the asterisk outside stands for polar set. Denote the LHS set as $\mathcal{B}^*$, the RHS set as $\mathcal{L}^*$. We have

$$\mathcal{B}^* = \left\{ \mathbf{s} + \sum_{j \notin \mathcal{J}^*} \gamma_j \mathbf{e}_j : \|\mathbf{s}\|_2 \leq \epsilon , \gamma_j \geq 0 \right\} ,$$

$$\mathcal{L}^* = \left\{ \mathbf{s} + \sum_{j \notin \mathcal{J}^*} \gamma_j \mathbf{e}_j : \mathbf{s} \in \mathcal{S} , \gamma_j \geq 0 \right\} .$$

Because $\mathcal{B}^* \subset \mathcal{L}^*$, ϵ can take value up to the inradius of $\mathcal{S}$, which is larger than the minimum of the distances between $\mathbf{0}$ and the defining hyperplanes of $\mathcal{S}$. It follows that

$$\epsilon \geq \left\{ \min_{\mathcal{I} \subset \mathcal{J}} \frac{\lambda - \sum_{j \in \mathcal{I}} \mu_j}{\sqrt{|\mathcal{I}|}} , \min_{\{\mathcal{I}:\mathcal{I} \subset \mathcal{J}, \mathcal{I} \supset \cup_{\theta \in \Theta^*} \mathcal{J}_\theta^*\}} \frac{\sum_{j \in \mathcal{I}} \mu_j - \sum_{\theta \in \Theta^*} \lambda_\theta}{\sqrt{|\mathcal{I}|}} , \min_{j \in \mathcal{J}} \mu_j \right\} .$$

$\square$