Strategy-proofness and Efficiency with Nonquasi-linear Preferences: A Characterization of Minimum Price Walrasian Rule*

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Abstract

We consider the problems of allocating objects to a group of agents and how much agents should pay. Each agent receives at most one object and has nonquasi-linear preferences. Nonquasi-linear preferences describe environments where payments influence agents' abilities to utilize objects or derive benefits from them. The "minimum price Walrasian (MPW) rule" is the rule that assigns a minimum price Walrasian equilibrium allocation to each preference profile. We establish that the MPW rule is the unique rule satisfying *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy for losers*. Since the outcome of the MPW rule coincides with that of the simultaneous ascending (SA) auction, our result supports SA auctions adopted by many governments.

Keywords: minimum price Walrasian equilibrium, simultaneous ascending auction, strategy-proofness, efficiency, heterogeneous objects, nonquasi-linear preferences
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1 Introduction

1.1 Purpose

Since the 1990s, governments in numerous countries have conducted auctions to allocate a variety of objects or assets including spectrum rights, vehicle ownership licenses, and land, etc. Although auction revenues sometimes amount to government annual budgets, the announced goals of many government auctions are rather to allocate objects "efficiently", *i.e.*, to agents who benefit most from them.[1] Agents benefiting more are willing to pay higher prices for them, and thus would have more chances to win the objects in auctions. However, large-scale auction payments would influence agents' abilities to utilize objects or benefit from them, thereby complicating *efficient* allocations. This article analyzes rules that allocate auctioned objects *efficiently* even when payments are so large that it impairs agents' abilities to utilize them or realize their benefits. We investigate *what types of allocation rules can allocate objects efficiently in such environments.*

1.2 Main Result

An *allocation rule*, or simply a *rule*, is a function that assigns to each preference profile an *allocation*, which consists of an assignment of objects and agents' payments. Each agent receives one object at most. The *domain* of rules is the class of preference profiles. We assume that preferences satisfy monotonicity, continuity, and "finite compensation", which means that, given an assignment, any change of assigned object is compensated by a finite amount of money. We call such preferences "classical". It is well-known that in this model, there is a minimum price Walrasian equilibrium (MPWE),[2] and that the allocation associated with the MPWE coincides with the outcome of a certain type of auctions, called the "simultaneous ascending (SA) auction".[3] Under SA auctions, bids on all objects start simultaneously, and the sale of any object is not settled as long as new bids are made on some objects. We focus on the rule that assigns an MPWE allocation to each preference profile. We refer to this rule as the "*minimum price Walrasian (MPW) rule*".

The MPW rule satisfies four desirable properties. First is *(Pareto-)efficiency.* An allocation is *efficient* if no agent can be better off without either some other agent being made worse off or the government's revenue being reduced.[4] Note that *efficiency* is evaluated based on agents' preferences. Thus, an *efficient* allocation cannot be chosen without information about agents' preferences. Since preferences are private information, agents may have an incentive to behave strategically to influence the final outcome in their favor. *Strategy-proofness* is an incentive-compatibility property, which gives a strong incentive for each agent to reveal his true preferences. It says that for each preference profile, in the normal form game induced by the rule, it is a (weakly) dominant strategy for each agent to reveal his true preference. The

[1]For example, frequency auctions in the United States were introduced to promote "efficient and intensive use of the electromagnetic spectrum". See McAfee and McMillan (1996, p.160).

[2]See Demange and Gale (1985).

[3]For example, see Demange, Gale, and Sotomayor (1986).

[4]In our auction model, *efficiency* is defined by taking government revenue into account.

MPW rule satisfies *strategy-proofness*,[5] and chooses an *efficient* allocation corresponding to the revealed preferences.

The third property of the MPW rule is *individual rationality*, which requires that any agent should not be made worse off than he would be if he had received no object and paid nothing. This property induces agents's voluntary participation. Fourth is *no subsidy for losers.* Under the MPW rule, the governments never subsidize losers. This property prevents agents with little abilities from flocking to auctions to sponge subsidies.

The primary conclusion of this article is that *only the minimum price Walrasian rule satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers* (Theorem 4.1). Since the outcome of the MPW rule coincides with that of the SA auction (Proposition 5.1), the result supports SA auctions adopted by many governments.

1.3 Related Literature

Holmström (1979) establishes a fundamental result relating to our question that applies when agents' benefits from auctioned objects are not influenced by their payments, *i.e.*, agents have "quasi-linear" preferences. He assumes that preferences are quasi-linear, and shows that *only the Vickrey–Clarke–Groves type (VCG)*[6] *allocation rules satisfy strategy-proofness and efficiency.*[7] His result implies that *on the quasi-linear domain, only the Vickrey rule*[8] *satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers.*[9] In reality, preferences are approximately quasi-linear if payments are sufficiently low. However, quasi-linearity is not an appropriate assumption for large-scale auctions. Excessive payment for the auctioned objects may damage bidders' budgets and render effective use of the objects impossible. Or bidders may need to obtain loans to bid high amounts, and typically financial costs are nonlinear in borrowings, which makes bidders' preferences on objects and payments nonquasi-linear.[10] In spectrum license auctions and vehicle ownership license auctions, license prices often equal or exceed bidders' annual revenues. Thus, bidders' preferences are nonquasi-linear for such important auctions.[11] As contrasted with Holmström (1979), our result can be applied even to such environments.

Saitoh and Serizawa (2008) investigate a problem similar to ours in the case where the domain includes nonquasi-linear preferences, and there are multiple units of the same object. They generalize Vickrey rules by employing compensating valuations from no object and no payment, and characterize the generalized Vickrey rule by *strategy-proofness, efficiency, indi-*

[5]In addition, the MPW rule satisfies *group strategy-proofness*, *i.e.,* by misrepresenting their preferences, no group of agents should obtain assignments that they prefer.

[6]See Clarke (1971), Groves (1973), and Vickrey (1961).

[7]More precisely, Holmström (1979) studies public goods models. When agents have quasi-linear preferences, his result can be applied to the auction model.

[8]See Section 6 for the formal definition.

[9]Remember that the payment of an agent under the VCG rule is decomposed into two parts. The first part is what is called "Vickrey price", the social opportunity cost to allocate him an object; the second is the term which is independent of his preference. *Individual rationality* and *no subsidy for losers* imply that the second part is zero. See also Chew and Serizawa (2007).

[10]See Saitoh and Serizawa (2008) for numerical examples.

[11]Ausubel and Milgrom (2002) also discuss the importance of the analysis under nonquasi-linear preferences.

vidual rationality, and no subsidy.[12] We stress that when preferences are not quasi-linear, the heterogeneity of objects makes the MPW rule different from the generalized Vickrey rule.[13]

Although the assumption of quasi-linearity neglects the serious effects of large-scale auction payments in actual practice, it is difficult to investigate the above question without this assumption. Quasi-linearity simplifies the description of *efficient* allocations. More precisely, under quasi-linear preferences, an *efficient* allocation of objects can be achieved simply by maximizing the sum of realized benefits from objects (agents' net benefits), and hence, *efficient* allocations of objects are independent of how much agents pay. In this sense, Holmström (1979) characterizes only the payment part of *strategy-proof* and *efficient* rules. On the other hand, without quasi-linearity, *efficient* allocations of objects do depend on payments, and thus are complicated to identify. Furthermore, as mentioned earlier, on nonquasi-linear domains, the MPW rule is different from the generalized Vickrey rule, and the former outperforms the latter in terms of our desirable properties: *strategy-proofness, efficiency, individual rationality,* and *no subsidy for losers.* Needless to say, Holmström's (1979) results cannot be applied to prove our results on a nonquasi-linear domain. It is worthwhile to mention that most standard results of auction theory, such as the Revenue Equivalence Theorem, also depend on assuming quasi-linearity. In this article, we overcome that difficulty.

Since Hurwicz's (1972) seminal work, many authors have investigated *efficient* and *strategy-proof* rules in pure exchange economies.[14] In pure exchange economies, classical[15] preferences are standard, but no rule satisfies *strategy-proofness*, *efficiency*, and *individual rationality* on the classical domain. On the other hand, Demange and Gale (1985) show that, in the model studied in this article, the MPW rule satisfies *strategy-proofness*, *efficiency*, and *individual rationality* on the classical domain.[16] Miyake (1998) shows that only the MPW rule satisfies *strategy-proofness* among "Walrasian rules".[17] Note that the Walrasian rules are a small part of the class of allocation rules satisfying *efficiency*, *individual rationality*, and *no subsidy for losers.* By developing analytical tools different from Miyake's (1998), we extend his characterization in that we establish the uniqueness of the rules satisfying the desirable properties without confinement to Walrasian rules.

Many authors have analyzed SA auctions in quasi-linear settings (Ausubel, 2004, 2006; Ausubel and Milgrom, 2002; de Vries, Schummer, and Vohra, 2007; Gul and Stacchetti, 2000; and Mishra and Parkes, 2007; Andersson, Andersson and Talman, 2012, etc). In nonquasi-

[12]Sakai (2008) also obtains a result similar to theirs.

[13]In Section 6, we give a detailed discussion on this point by contrasting the MPW rule with the generalized Vickrey rule.

[14]For example, see Zhou (1991); Barberà and Jackson (1995); Schummer (1997); Serizawa (2002); and Serizawa and Weymark (2003).

[15]In pure exchange economies, where consumption spaces are some multidimensional Euclidean space, classical preferences are assumed to satisfy convexity in addition to continuity and monotonicity. Clearly, the class of such preferences contains nonquasi-linear preferences.

[16]More precisely, Demange and Gale (1985) study two-sided matching markets which contain our model as a special case, and show the rules selecting an optimal stable assignment for one side of the market are *group strategy-proof* for the agents on that side.

[17]A "Walrasian rule" is the rule that assigns a Walrasian equilibrium allocation to each preference profile.

linear settings, the MPW rules differ from the generalized Vickrey rules, and it is the MPWE allocation that coincides with the outcome of the SA auction.[18] Therefore, our result demonstrates that the SA auction analyzed by those works is more important in nonquasi-linear settings.

1.4 Organization

The article is organized as follows. Section 2 sets up the model and introduces basic concepts. Section 3 defines and characterizes minimum price Walrasian equilibrium. Section 4 provides our main result. Section 5 defines the SA auction, and shows that its outcome coincides with the MPWE allocation. Section 6 introduces generalized Vickrey rules, and contrasts them with the MPW rule. Section 7 concludes. All proofs appear in the Appendix.

2 The Model and Definitions

There are n agents and m objects, where $2 \leq n < \infty$ and $1 \leq m < \infty$. We denote the set of agents by $N \equiv \{1, \ldots, n\}$, and the set of objects by $M \equiv \{1, \ldots, m\}$. Let $L \equiv \{0\} \cup M$. Each agent consumes one object at most. We denote the object that agent $i \in N$ receives by $x_i \in L$. Object 0 is referred to as the "null object", and $x_i = 0$ means that agent i receives no object. We denote the money that agent i pays by $t_i \in \mathbb{R}$. For each $i \in N$, agent i's **consumption set** is $L \times \mathbb{R}$, and agent i's **(consumption) bundle** is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$. Let $\mathbf{0} \equiv (0, 0)$.

Each agent i has a complete and transitive preference relation R_i on $L \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relation associated with R_i, respectively. Given a preference R_i and a bundle $z_i \in L \times \mathbb{R}$, let the **upper contour set** and **lower contour set of $\boldsymbol{R_i}$ at $\boldsymbol{z_i}$** be $\boldsymbol{UC(R_i, z_i)} \equiv \{\hat{z}_i \in L \times \mathbb{R} : \hat{z}_i \, R_i \, z_i\}$ and $\boldsymbol{LC(R_i, z_i)} \equiv \{\hat{z}_i \in L \times \mathbb{R} : z_i \, R_i \, \hat{z}_i\}$, respectively. For each $i \in N$, agent i's preference R_i satisfies the following properties:

Money monotonicity: For each $x_i \in L$ and each $t_i, \hat{t}_i \in \mathbb{R}$, if $\hat{t}_i < t_i$, then, $(x_i, \hat{t}_i) \, P_i \, (x_i, t_i)$.

Finiteness: For each $t_i \in \mathbb{R}$ and each $x_i, \hat{x}_i \in L$, there exist $\hat{t}_i, \bar{t}_i \in \mathbb{R}$ such that $(\hat{x}_i, \hat{t}_i) \, R_i \, (x_i, t_i)$ and $(x_i, t_i) \, R_i \, (\hat{x}_i, \bar{t}_i)$.

Continuity: For each $z_i \in L \times \mathbb{R}$, $UC(R_i, z_i)$ and $LC(R_i, z_i)$ both are closed.[19]

Let $\mathcal{R}^E$ be the class of continuous, money monotonic, and finite preferences, which we call the "**extended domain**". Given $R_i \in \mathcal{R}^E$, $z_i \in L \times \mathbb{R}$, and $y_i \in L$, we define the **compensating valuation $\boldsymbol{CV_i(y_i; z_i)}$ of $\boldsymbol{y_i}$ from $\boldsymbol{z_i}$ for $\boldsymbol{R_i}$** by $(y_i, CV_i(y_i; z_i)) \, I_i \, z_i$. Note that by continuity and finiteness of preferences, $CV_i(y_i; z_i)$ exists, and by money monotonicity, $CV_i(y_i; z_i)$ is unique. The compensating valuation for $\hat{R}_i$ is denoted by $\widehat{CV}_i$.

We introduce another property of preferences.

Desirability of objects: For each $x_i \in M$ and each $t_i \in \mathbb{R}$, $(x_i, t_i) \, P_i \, (0, t_i)$.[20]

[18] Alaei, Jain, and Malekian (2013) construct an alternative algorithm computing MPWE in nonquasi-linear setting.

[19] Money monotonicity and finiteness imply continuity. All of results hold without assuming continuity.

[20] A preference R_i satisfies *weak desirability of objects* if for each $x_i \in M$, $(x_i, 0) \, P_i \, \mathbf{0}$. All the results in this article still hold if desirability of objects is replaced by weak desirability of objects.

Definition 2.1. A preference R_i is **classical** if it satisfies continuity, money monotonicity, finiteness, and desirability of objects.

Let $\mathcal{R}^C$ be the class of classical preferences, which we call the "**classical domain**". Note that $\mathcal{R}^C \subsetneq \mathcal{R}^E$.

Definition 2.2. A preference R_i is **quasi-linear** if there is a valuation function $v_i : L \longrightarrow \mathbb{R}_+$ such that (i) $v_i(0) = 0$, (ii) for each $x \in M$, $v_i(x) > 0$, and (iii) for each $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$, and each $\hat{z}_i \equiv (\hat{x}_i, \hat{t}_i) \in L \times \mathbb{R}$, $z_i \, R_i \, \hat{z}_i$ if and only if $v_i(x_i) - t_i \geq v_i(\hat{x}_i) - \hat{t}_i$.

We denote the class of quasi-linear preferences by $\mathcal{R}^Q$, which we call the "quasi-linear domain". Note that $\mathcal{R}^Q \subsetneq \mathcal{R}^C$.

An **object allocation** is an n-tuple $(x_1, \ldots, x_n) \in L^n$ such that for each $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$, that is, no two agents receive the same object except when both agents receive the null object. Let X be the set of object allocations. A **(feasible) allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((x_1, t_1), \ldots, (x_n, t_n)) \in [L \times \mathbb{R}]^n$ of bundles such that $(x_1, \ldots, x_n) \in X$. Let Z be the set of feasible allocations. We denote the object allocation and agents' payments under an allocation $\hat{z}$ by $\hat{x} \equiv (\hat{x}_1, \ldots, \hat{x}_n)$ and $\hat{t} \equiv (\hat{t}_1, \ldots, \hat{t}_n)$, respectively.

Let $\mathcal{R}$ be a class of preferences such that $\mathcal{R} \subseteq \mathcal{R}^E$. A **preference profile** is an n-tuple $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$. Given $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{-N'} \equiv (R_i)_{i \in N \setminus N'}$.

An allocation rule, or simply a **rule**, on $\mathcal{R}^n$ is a function f from $\mathcal{R}^n$ to Z. Given a rule f and a preference profile $R \in \mathcal{R}^n$, we denote agent i's assignment of objects under f at R by $f_i^x(R)$ and agent i's payment under f at R by $f_i^t(R)$, and we write

$$f_i(R) \equiv (f_i^x(R), f_i^t(R)), \; f(R) \equiv (f_1(R), \ldots, f_n(R)), \text{ and } f^x(R) \equiv (f_j^x(R))_{j \in N}.$$

We introduce basic properties of rules. The efficiency condition defined below takes the auctioneer's preference into account and assumes that he is only interested in his revenue. An allocation $\hat{z} \in Z$ is **(Pareto-)dominates $\boldsymbol{z} \in Z$ for $\boldsymbol{R} \in \mathcal{R}^n$** if

$$\text{(i)} \sum_{i \in N} \hat{t}^i \geq \sum_{i \in N} t^i, \text{ (ii) for each } i \in N, \hat{z}_i \, R_i \, z_i, \text{ and (iii) for some } j \in N, \hat{z}_j \, P_j \, z_j.$$

An allocation $z \in Z$ is **(Pareto-)efficient for $\boldsymbol{R} \in \mathcal{R}^n$** if there is no feasible allocation $\hat{z} \in Z$ that dominates z.

Efficiency: For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

Individual rationality below requires that a rule should never assign an allocation which makes some agent worse off than he would be if he had received no object and paid nothing. *No subsidy* requires that the payment of agents always should be nonnegative. *No subsidy for losers* requires that the payment of agents who do not obtain objects always should be nonnegative. *No subsidy* implies *no subsidy for losers.*

Individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \, R_i \, \mathbf{0}$.

No subsidy: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i^t(R) \geq 0$.

No subsidy for losers: For each $R \in \mathcal{R}^n$ and each $i \in N$, if $f_i^x(R) = 0$, then $f_i^t(R) \geq 0$.

The two properties below have to do with incentive-compatibility. The first says that by misrepresenting his preferences, no agent should obtain an assignment that he prefers.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}_i \in \mathcal{R}$, $f_i(R) \, R_i \, f_i(\hat{R}_i, R_{-i})$.

The second is a stronger property: by misrepresenting their preferences, no group of agents should obtain assignments that they prefer.

Group strategy-proofness: For each $R \in \mathcal{R}^n$ and each $\hat{N} \subseteq N$, there is no $\hat{R}_{\hat{N}} \in \mathcal{R}^{\#\hat{N}}$ such that for each $i \in \hat{N}$, $f_i(\hat{R}_{\hat{N}}, R_{-\hat{N}}) \, P_i \, f_i(R)$.[21]

3 Minimum Price Walrasian Equilibrium

3.1 Definition of Walrasian equilibria

We define "Walrasian equilibrium" and "minimum price Walrasian equilibrium" in this model. Let $\mathcal{R} \subseteq \mathcal{R}^E$ in this section. All results in this section also hold on the classical domain $\mathcal{R}^C$.

Given a price vector $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$, the **(common) budget set** is defined as $\boldsymbol{B(p)} \equiv \{(x, p^x) : x \in L\}$, where $p^x = 0$ if $x = 0$. Given $i \in N$, $R_i \in \mathcal{R}$ and $p \in \mathbb{R}^m_+$, agent i's **demand set** is defined as $\boldsymbol{D(R_i, p)} \equiv \{x \in L : \text{ for each } y \in L, \, (x, p^x) \, R_i \, (y, p^y)\}$.

Definition 3.1. Let $R \in \mathcal{R}^n$. A pair $(z, p) \in Z \times \mathbb{R}^m_+$ is a **Walrasian equilibrium for $\boldsymbol{R}$** if

$$\begin{aligned} &\text{(WE-i) for each } i \in N, \, x_i \in D(R_i, p) \text{ and } t_i = p^{x_i}, \text{ and} \\ &\text{(WE-ii) for each } x \in M, \text{ if for each } i \in N, x_i \neq x, \text{ then, } p^x = 0. \end{aligned}$$

Condition (WE-i) says that each agent receives the object he demands, and pays its price. Condition (WE-ii) says that an object's price is zero if it is not assigned.

Fact 3.1. *For each $R \in \mathcal{R}^n$, there is a Walrasian equilibrium for R.*

Fact 3.1 is already known.[22] The existence of Walrasian equilibrium is also guaranteed by Proposition 5.1 in Section 5 by using the SA auction.

Given $R \in \mathcal{R}^n$, let $\boldsymbol{W(R)}$ be the **set of Walrasian equilibria for $\boldsymbol{R}$**, and let $\boldsymbol{Z(R)}$ and $\boldsymbol{P(R)}$ be the **projections of $\boldsymbol{W(R)}$ onto $\boldsymbol{Z}$ and $\mathbb{R}^m_+$**, respectively, *i.e.*,

$$\begin{aligned} \boldsymbol{Z(R)} &\equiv \{z \in Z : \text{ for some } p \in \mathbb{R}^m_+, \, (z, p) \in W(R)\}, \text{ and} \\ \boldsymbol{P(R)} &\equiv \{p \in \mathbb{R}^m_+ : \text{ for some } z \in Z, \, (z, p) \in W(R)\}. \end{aligned}$$

That is, $Z(R)$ and $P(R)$ are the sets of Walrasian equilibrium allocations and prices for R, respectively. Fact 3.2 below is the so-called First Welfare Theorem.

[21] Let $\#A$ denote the cardinality of set A.

[22] For example, see Alkan and Gale (1990). Our model is a special case of their model.

Fact 3.2. *Let $R \in \mathcal{R}^n$ and $z \in Z(R)$. Then, z is efficient for R.*[23]

Fact 3.3 below says that for each preference profile R, there is the minimum price $p_{\min}(R)$ among Walrasian equilibrium prices. The minimum price Walrasian equilibrium (hereafter MPWE) is the Walrasian equilibria associated with the minimum price.

Fact 3.3 (Demange and Gale, 1985). *For each $R \in \mathcal{R}^n$, there is $p_{\min}(R) \in P(R)$ such that for each $p \in P(R)$ and each $x \in M$, $p^x_{\min}(R) \leq p^x$.*

Given $R \in \mathcal{R}^n$, let $\boldsymbol{W_{\min}(R)}$ be the **set of minimum price Walrasian equilibria for $\boldsymbol{R}$**, and let

$$\boldsymbol{Z_{\min}(R)} \equiv \{z \in Z : (z, p_{\min}(R)) \in W_{\min}(R)\}.$$

By Facts 3.1 and 3.3, for each $R \in \mathcal{R}^n$, the set $Z_{\min}(R)$ is nonempty. Although the correspondence $Z_{\min}$ is set valued, it is *essentially single-valued*, *i.e.,* for each $R \in \mathcal{R}^n$, each pair $z, z' \in Z_{\min}(R)$, and each $i \in N$, $z_i \, I_i \, z'_i$.[24]

3.2 Illustration of minimum price Walrasian equilibrium

We illustrate a MPWE in a simple case where there are three agents, and two objects, say A and B, by using Figure 1. In Figure 1, there are three horizontal lines. The lowest horizontal line corresponds to no object. The middle and highest ones correspond to objects A and B, respectively. The intersection of the vertical line and each horizontal line denotes the consumption bundle of the corresponding object and no payment. For example, the origin $\mathbf{0}$ denotes the consumption bundle of null object and no payment. For each point z_i on the one of three horizontal lines, the distance from z_i to the vertical line denotes payment. For example, z_1 denotes the consumption bundle of object A and payment p^A. Agents' preferences are described by indifference curves depicted on the three horizontal lines. That is, if two points on the horizontal lines are connected by a indifference curve of an agent, it means that the agent is indifferent between the two points.

Assume that agents' preferences R_i, $i = 1, 2, 3$, are ones depicted in Figure 1. The compensating valuations from the origin are ranked as $CV_1(A; \mathbf{0}) > CV_3(A; \mathbf{0}) > CV_2(A; \mathbf{0})$ and $CV_1(B; \mathbf{0}) > CV_2(B; \mathbf{0}) > CV_3(B; \mathbf{0})$.

The MPWE for the preference profile $R = (R_1, R_2, R_3)$ is as follows: Agent 1 receives object A and pays $CV_3(A; \mathbf{0})$, *i.e.*, the price p^A of object A is $CV_3(A; \mathbf{0})$. This agent 1's consumption point is depicted as z_1 in Figure 1. Agent 2 receives object B and pays $CV_1(B; z_1)$, *i.e.*, the price p^B of object B is $CV_1(B; z_1)$. This agent 2's consumption point is depicted as z_2 in Figure 1. Agent 3's consumption point is $\mathbf{0}$ and depicted as z_3.

[23]To see this, suppose that $z \equiv (z_1, \ldots, z_n)$ is not *efficient for R*. Then, there is $\hat{z} \equiv (\hat{z}_1, \ldots, \hat{z}_n)$ such that

$$\text{(i)} \sum_{i \in N} \hat{t}^i \geq \sum_{i \in N} t^i, \text{ (ii) for each } i \in N,\ \hat{z}_i \, R_i \, z_i, \text{ (iii) for some } j \in N,\ \hat{z}_j \, P_j \, z_j.$$

Since $z \in Z(R)$, there is a price vector $p \in \mathbb{R}^m_+$ such that $(z, p) \in W(R)$. Then, by (ii) and (WE-i), for each $i \in N$, $\hat{t}_i \leq p^{\hat{x}_i}$. By (iii) and (WE-i), $\hat{t}_j < p^{\hat{x}_j}$. Thus, $\sum_{i \in N} \hat{t}_i < \sum_{i \in N} p^{\hat{x}_i} = \sum_{i \in N} t_i$. This contradicts (i).

[24]An allocation $z' \in Z$ *is obtained by an indifferent permutation from* $z \in Z$ if there is a permutation π on N such that for each $i \in N$, $z'_i = z_{\pi(i)}$ and $z'_i \, I_i \, z_i$ (Tadenuma and Thomson, 1991). Note that for each pair $z, z' \in Z_{\min}(R)$, z' is obtained by an indifferent permutation from z.

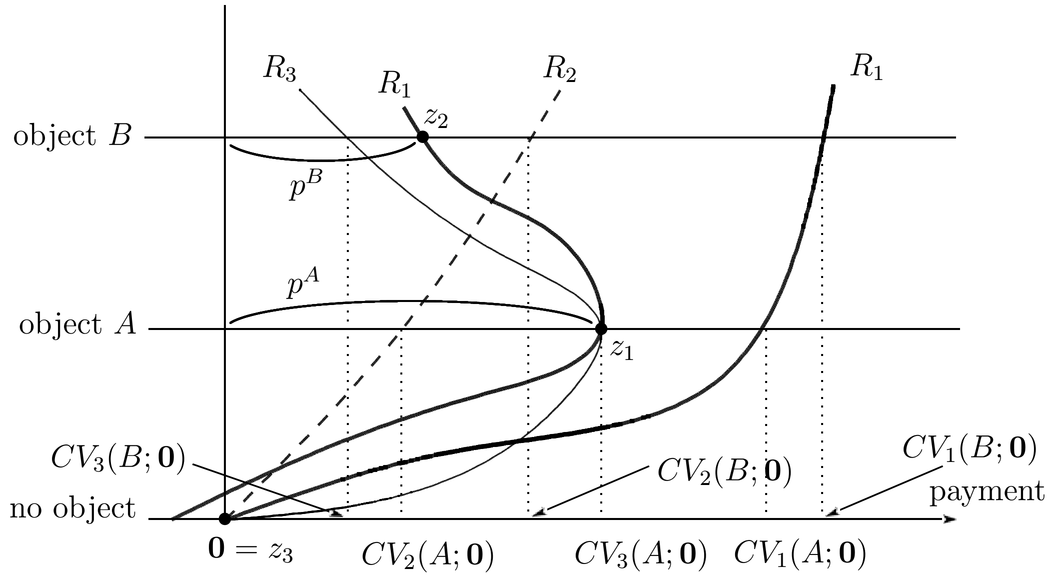

Figure 1: Illustration of nonquasi-linear preferences and the minimum Walrasian equilibrium

Let's see why the allocation $z \equiv (z_1, z_2, z_3)$ is the MPWE for R. First, note that for each agent $i = 1, 2, 3$, z_i is maximal for R_i in the budget set $\{\mathbf{0}, (A, p^A), (B, p^B)\}$. Thus, z is a Walrasian equilibrium.

Next, let $(\widetilde{p}^A, \widetilde{p}^B)$ be a Walrasian equilibrium price. We show $\widetilde{p}^A \geq p^A$ and $\widetilde{p}^B \geq p^B$. If $\widetilde{p}^A < p^A$ and $\widetilde{p}^B < p^B$, then, all agents prefer $(A, \widetilde{p}^A)$ or $(B, \widetilde{p}^B)$ to $\mathbf{0}$, that is, all three agents demand A or B or both. In that case, one agent cannot receive an object he demands, contradicting (WE-i). Therefore, $\widetilde{p}^A \geq p^A$ or $\widetilde{p}^B \geq p^B$. If $\widetilde{p}^A < p^A$, then $\widetilde{p}^B \geq p^B$, and so, both agents 1 and 3 prefer $(A, \widetilde{p}^A)$ to $\mathbf{0}$ and $(B, \widetilde{p}^B)$, that is, both demand only A. In that case, agent 1 or 3 cannot receive the object he demands, contradicting (WE-i). Therefore, $\widetilde{p}^A \geq p^A$. If $\widetilde{p}^B < p^B$, both agents 1 and 2 prefer $(B, \widetilde{p}^B)$ to $\mathbf{0}$ and $(A, \widetilde{p}^A)$, and so, agent 1 or 2 cannot receive the object he demands, contradicting (WE-i). Therefore, $\widetilde{p}^B \geq p^B$. Hence, (z, p) is the MPWE for R.

3.3 Overdemanded and underdemanded sets

We introduce the concepts of "overdemanded set" and "underdemanded set" (Mishra and Talman, 2010; etc.), and relate these concepts to Walrasian equilibria.

Definition 3.2. (i) A set $M' \subseteq M$ of objects is **(weakly) overdemanded at $\boldsymbol{p}$ for $\boldsymbol{R}$** if

$$\#\{i \in N : D(R_i, p) \subseteq M'\} \; (\geq) > \#M'.$$

(ii) A set $M' \subseteq M$ of objects is **(weakly) underdemanded at $\boldsymbol{p}$ for $\boldsymbol{R}$** if

$$[\forall x \in M', p^x > 0] \implies \#\{i \in N : D(R_i, p) \cap M' \neq \emptyset\} \; (\leq) < \#M'.$$

We illustrate the above concepts by using the preferences in Figure 1. Note that $\{i \in N : D(R_i, p) \subseteq \{A\}\} = \emptyset$, $\{i \in N : D(R_i, p) \subseteq \{B\}\} = \{2\}$, $\{i \in N : D(R_i, p) \subseteq \{A, B\}\} = \{1, 2\}$,

$\{i \in N : D(R_i, p) \cap \{A\} \neq \emptyset\} = \{1, 3\}$, $\{i \in N : D(R_i, p) \cap \{B\} \neq \emptyset\} = \{1, 2\}$, and $\{i \in N : D(R_i, p) \cap \{A, B\} \neq \emptyset\} = \{1, 2, 3\}$. Thus, no set of objects is overdemanded or weakly underdemanded in Figure 1.

Fact 3.4 below is a characterization of Walrasian equilibria by means of the concepts of overdemanded and underdemanded sets. This fact is established by Mishra and Talman (2010) for quasi-linear preferences. However, their proof also works for preferences in the extended domain.

Fact 3.4 (Mishra and Talman, 2010). *Let $R \in \mathcal{R}^n$. A price vector p is a Walrasian equilibrium price for R if and only if no set of objects is overdemanded and no set of objects is underdemanded at p for R.*

Theorem 3.1 below is a characterization of the *minimum* price Walrasian equilibrium by means of the concepts of overdemanded and *weakly* underdemanded sets. Mishra and Talman (2010) first obtain the same conclusion on the quasi-linear domain. We emphasize, in contrast to Fact 3.1, that Mishra and Talman's (2010) proof crucially depends on the quasi-linearity. It relies on the simple fact that when preferences are quasi-linear, if a set M' of objects is weakly underdemanded at a Walrasian equilibrium (z, p), then all the prices of M' can be slightly lowered by the same amount while maintaining the Walrasian equilibrium conditions (WE-i) and (WE-ii).[25] However, it is not true when preferences are not quasi-linear. Theorem 3.1 below is a novel result in that point.

Theorem 3.1 is the key to obtaining Theorem 4.1 in Section 4 and Proposition 5.1 in Section 5. Since we obtain the existence of Walrasian equilibrium as a by-product of Proposition 5.1, this theorem is also a key to the existence of Walrasian equilibrium.

Theorem 3.1. *Let $R \in \mathcal{R}^n$. A price vector p is a minimum Walrasian equilibrium price for R if and only if no set of objects is overdemanded and no set of objects is weakly underdemanded at p for R.*

Corollary 3.1 of Theorem 3.1 below says that if the number of objects is greater than or equal to the number of agents, the price of some objects is 0. It is useful to prove Fact 4.1 in Section 4. Corollary 3.2 says that each object bearing a positive price is connected by agents' demands to the null object or to an object with a price of 0. For example, in Figure 1, object B has a positive equilibrium price, agent 1's demand connects objects A and B, and agent 3's demand connects object A and null object.

Corollary 3.1 (Existence of Free Object). *Let $m \geq n$, $R \in \mathcal{R}^n$, and $z \in Z_{\min}(R)$. Then, there is $i \in N$ such that $p^{x_i}_{\min}(R) = 0$.*

Corollary 3.2 (Demand Connectedness).[26] *Let $R \in \mathcal{R}^n$ and (z, p) be a minimum Walrasian equilibrium price for R. For each $x \in M$ with $p^x > 0$, there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $x_{i_1} = 0$, or $x_{i_1} \neq 0$ and $p^{x_{i_1}} = 0$, (ii) $x_{i_K} = x$, and (iii) for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}, x_{i_{k+1}}\} \subseteq D(R_{i_k}, p)$.*

[25] For details, refer to the proof of Lemma 3 in Mishra and Talman (2010).

[26] This structure is discussed by Demange, Gale, and Sotomayor (1986) and Miyake (1998).

Here, we also introduce a concept of "d_i-truncation" of a preference, which is important to prove Theorem 3.1 above and Fact 4.1 in Section 4. It says that the welfare position of each consumption bundle $z_i \in M \times \mathbb{R}$ is lowered as much as d_i in terms of money, but their relative positions are kept.

Given $R_i \in \mathcal{R}$ and $d_i \in \mathbb{R}$, the $\boldsymbol{d_i}$**-truncation of** $\boldsymbol{R_i}$ is the preference $\hat{R}_i$ such that for each $z_i \in M \times \mathbb{R}$, $\widehat{CV}_i(0; z_i) = CV_i(0; z_i) + d_i$. Given $R \in \mathcal{R}^n$, the $\boldsymbol{d}$**-truncation of** $\boldsymbol{R}$ is the preference profile $\hat{R}$ such that for each $i \in N$, $\hat{R}_i$ is the d_i-truncation of R_i.

We have a remark and a fact below relating truncations.

Remark 3.1. Let $R_i \in \mathcal{R}$, $d_i \in \mathbb{R}$, and $\hat{R}_i$ be the d_i-truncation of R_i. Then, for each $z_i, \hat{z}_i \in M \times \mathbb{R}$, $z_i \, R_i \, \hat{z}_i$ if and only if $z_i \, \hat{R}_i \, \hat{z}_i$.

Fact 3.5 (Roth and Sotomayor, 1990). *Let $R \in \mathcal{R}^n$ and $\hat{R}$ be the d-truncation of R such that for each $i \in N$, $d_i \geq 0$. Then, $p_{\min}(\hat{R}) \leq p_{\min}(R)$.*

4 Main Results

In this section, we provide a characterization of the minimum price Walrasian equilibrium by means of the properties of rules. Let $\mathcal{R} \subseteq \mathcal{R}^E$.

Definition 4.1. A rule f on $\mathcal{R}^n$ is a **minimum price Walrasian rule** if for each $R \in \mathcal{R}^n$, $f(R) \in Z_{\min}(R)$.

4.1 Properties of the minimum price Walrasian rule

Let g be a minimum price Walrasian rule on $\mathcal{R}^n$. First, by Fact 3.2, for each $R \in \mathcal{R}^n$, $g(R)$ is *efficient for R*. Let $R \in \mathcal{R}^n$. Then, there is a price vector $p \equiv (p^1, \cdots, p^m) \in \mathbb{R}^m_+$ such that for each $i \in N$, (a) $g_i(R) \in B(p)$, and (b) for each $\hat{z}_i \in B(p)$, $g_i(R) \, R_i \, \hat{z}_i$. Let $i \in N$. Note that, for each $x \in M$, $p^x \geq 0$, and $B(p) = \{(0,0), (1,p^1), (2,p^2), \cdots, (m,p^m)\}$. Thus, by (a), $g^t_i(R) \geq 0$, and by (b), $g_i(R) \, R_i \, \mathbf{0}$. Therefore, the minimum price Walrasian rules satisfy *efficiency*, *individual rationality*, and *no subsidy*.

Fact 4.1 below is first shown by Demange and Gale (1985). By using Theorem 3.1, we show this fact more directly in Appendix B.

Fact 4.1 (Demange and Gale, 1985). *The minimum price Walrasian rules are group strategy-proof.*

4.2 Characterizations

In this subsection, we focus on the case where each agent has a classical preference and the number of agents exceeds the number of objects. Remember that all results established in Section 3 also hold in this case. Theorem 4.1 below is a main conclusion of this article, a characterization of the minimum price Walrasian rule.

Theorem 4.1. *Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. A rule f on $\mathcal{R}^n$ satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers if and only if it is a minimum price Walrasian rule: for each $R \in \mathcal{R}^n$, $f(R) \in Z_{\min}(R)$.*

Proof of Theorem 4.1 is in Appendix B. Since the minimum price Walrasian rules are *group strategy-proof*, Theorem 4.1 implies that only the minimum price Walrasian rules satisfies *group strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy for losers*. Since *no subsidy* implies *no subsidy for losers*, Theorem 4.1 also implies that only the minimum price Walrasian rules satisfies *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy*.

4.3 Indispensability of the axioms and assumptions

The *only if* part of Theorem 4.1 fails if we drop any of the four axioms, *strategy-proofness, efficiency, individual rationality,* and *no subsidy for losers.* The following examples establish the independence of the axioms in Theorem 4.1.

Example 4.1 (Dropping *strategy-proofness*). Let f be a rule that chooses a "maximum" price Walrasian equilibrium allocation for each preference profile. Then, the rule f satisfies *efficiency*, *individual rationality*, and *no subsidy for losers*, but not *strategy-proofness.*[27]

Example 4.2 (Dropping *efficiency*). Let f be the rule such that for each preference profile, each agent receives no object and pays nothing. Then, the rule f satisfies *strategy-proofness*, *individual rationality*, and *no subsidy for losers*, but not *efficiency.*

Next, we introduce variants of Walrasian equilibria, ones with "entry fees". A pair $(z, p) \in Z \times \mathbb{R}^m$ of a feasible allocation and a price vector is a *Walrasian equilibrium with entry fees for* $R \in \mathcal{R}^n$ if there is an entry fee *vector* $e = (e_1, \cdots, e_n) \in \mathbb{R}^m$, and

$$\text{(WE-i*) for each } i \in N,\ x_i \in D(R_i, p) \text{ and } t_i = p^{x_i} + e_i, \text{ and}$$
$$\text{(WE-ii) for each } x \in M, \text{ if for each } i \in N, x_i \neq x, \text{ then, } p^x = 0.$$

Note that, by Facts 3.1 and 3.3, for each preference profile $R \in \mathcal{R}^n$ and each $e = (e_1, \cdots, e_n) \in \mathbb{R}^m$, there is a minimum price Walrasian equilibrium with entry fees e, and it is *efficient.*

A rule f on $\mathcal{R}^n$ is a **minimum price Walrasian rule with entry fees** if there is a list $\{e_i(\cdot)\}_{i\in N}$ of entry fee functions defined on $\mathcal{R}^n$, and for each R, $f(R)$ is a minimum price Walrasian equilibrium with entry fees $\{e_i(R)\}_{i\in N}$. If the entry fee function $e_i(\cdot)$ of each agent i depends on only other agents' preferences, then the associated minimum price Walrasian rule with entry fees satisfies *strategy-proofness.* Thus, we assume that for each $i \in N$, his entry fee function $e_i(\cdot)$ is defined on the class of the other agents' preference profiles $\mathcal{R}^{n-1}$.

Example 4.3 (Dropping *individual rationality*). Let a list $\{e_i(\cdot)\}_{i\in N}$ of entry fee functions be such that for each i and each $R_{-i} \in \mathcal{R}^{n-1}$, $e_i(R_{-i}) > 0$. Then, the associated minimum price Walrasian rule with entry fees satisfies *strategy-proofness, efficiency*, and *no subsidy for losers*, but not *individual rationality.*

Example 4.4 (Dropping *no subsidy for losers*). Let a list $\{e_i(\cdot)\}_{i\in N}$ of entry fee functions be such that for each i and each $R_{-i} \in \mathcal{R}^{n-1}$, $e_i(R_{-i}) < 0$. Then, the associated minimum price

[27] Demange and Gale (1985) also show that for each preference profile, there is a maximum price Walrasian equilibrium. When there is only one object, the maximum price Walrasian equilibrium corresponds to the first price auction. It is well-know that the first price auction is not *strategy-proof.*

Walrasian rule with entry fees satisfies *strategy-proofness*, *efficiency*, and *individual rationality*, but not *no subsidy for losers*.

One might wonder if minimum price Walrasian rules with entry fees can be characterized by only *strategy-proofness* and *efficiency*. Our proof of Theorem 4.1 fails if *individual rationality* and *no transfer for losers* are dropped. However, we have not found an example of a rule that satisfies *strategy-proofness* and *efficiency*, but is not *a minimum price Walrasian rule with entry fees*. Therefore, it is an open question whether the class of minimum price Walrasian rules with entry fees can be characterized by only *strategy-proofness* and *efficiency* or not.

One might also wonder if the assumption that $n > m$ can be dropped in Theorem 4.1. Our proof of Theorem 4.1 also fails if $n \leq m$. However, we have not found an example of a rule that satisfies the four properties of Theorem 4.1, but is not *a minimum price Walrasian rule* even if $n > m$ is dropped. Therefore, this question is also open.

5 Simultaneous Ascending Auction

We define a class of simultaneous ascending auctions, and show that they achieve the minimum price Walrasian equilibrium. Let $\mathcal{R} \subseteq \mathcal{R}^E$, $R \in \mathcal{R}^n$, and $p \in \mathbb{R}^m_+$.

Definition 5.1. A set $M' \subseteq M$ of objects is a **minimal** overdemanded set at p for R if M' is overdemanded at p for R, and there is no $M'' \subsetneq M'$ such that M'' is overdemanded at p.

Under a (continuous time) "simultaneous ascending auction", there is a constant $d > 0$, and at each time, each bidder submits his demand at the current price vector, and the prices of the objects in a minimal overdemanded set are raised at a speed at least d.

Definition 5.2. A **simultaneous ascending (SA) auction** is a function $\hat{p}$ from $\mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathcal{R}^n$ to $\mathbb{R}^m_+$ such that
(i) for each $p \in \mathbb{R}^m_+$, each $R \in \mathcal{R}^n$, and each $x \in M$, $\hat{p}^x(0, p, R) \equiv 0$,
(ii) $\hat{p}$ is absolutely continuous with respect to t and p,
(iii) there is $d > 0$ such that for each $t \in \mathbb{R}_+$, each $p \in \mathbb{R}^m_+$, each $R \in \mathcal{R}^n$, and each $x \in M$,
(iii-a) if $\hat{p}^x$ is differentiable at (t, p), and x is in a minimal overdemanded, $d\hat{p}^x(t, p, R)/dt \geq d$, and
(iii-b) $d\hat{p}^x(t, p, R)/dt = 0$ otherwise.

Remark 5.1.[28] For each $R \in \mathcal{R}^n$, a SA auction $\hat{p}$ generates a **price path** $p(\cdot)$ such that for each $x \in M$ and each $t \in \mathbb{R}_+$,

$$p^x(t) = \int_0^t \frac{d\hat{p}^x(s, p(s), R)}{ds} ds.$$

Proposition 5.1. *For each preference profile, the price path generated by any simultaneous ascending auction converges to the minimum Walrasian equilibrium price in a finite time.*

[28]Condition (ii) of Definition 5.2 (absolute continuity) guarantees the intergrability of $d\hat{p}^x(t, p(t), R)/dt$ with respect to t, that is, the existence and uniqueness of the price path for each SA auction and each preference profile.

The proof is in Appendix C. Proposition 5.1 says that for each $R \in \mathcal{R}^n$, the price path $p(\cdot)$ generated by an SA auction has a **final time** T such that for each $t \geq T$, $p(t) = p(T) = p_{\min}(R)$, and at the final price $p(T)$, each agent receives an object from his demand. Moreover, this proposition implies the existence of Walrasian equilibrium.

6 Generalized Vickrey Rule

In this section, we introduce generalized Vickrey rules, and contrast them with the minimum price Walrasian rules.

6.1 Generalized Vickrey rule

Each quasi-linear preference R_i can be defined by means of valuation function $v_i : L \to \mathbb{R}_+$, and a preference profile R in the quasi-linear domain corresponds to a valuation profile $v(R) \equiv (v_1(R_1), \ldots, v_n(R_n))$. Given a valuation profile $v = (v_1, \ldots, v_n)$, let $(x_1^*(v), \ldots, x_n^*(v)) \in \arg\max_{(x_1,\ldots,x_n)\in X} \sum_i v_i(x_i)$, $\sigma_{-i}(v) \equiv \sum_{j\neq i} v_j(x_j^*(v))$, and $\overline{\sigma}_{-i}(v) \equiv \max_{(x_1,\ldots,x_n)\in X} \sum_{j\neq i} v_j(x_j)$. On the quasi-linear domain, the Vickrey rules are defined as follows.

Definition 6.1. A rule f on the quasi-linear domain is a **Vickrey rule** if for each valuation profile v, $f^x(v) \in \arg\max_{(x_1,\ldots,x_n)\in X} \sum_i v_i(x_i)$, and for each $i \in N$, $f_i^t(v) = \overline{\sigma}_{-i}(v) - \sigma_{-i}(v)$.

To generalize the above definition to the classical domain, we need to use some valuation function v_i for each classical preference R_i. Compensating valuation $CV_i(\cdot; \mathbf{0})$ from the origin $\mathbf{0}$ is defined for each classical preference R_i and a generalization of valuation function, and so a natural candidate. Given a classical preference R_i, let $v_i(\cdot; R_i)$ be a function defined as: for each $x \in L$, $v_i(x; R_i) \equiv CV_i(x; \mathbf{0})$. Given a classical preference profile R, let $\overline{v}(R) \equiv (v_1(\cdot; R_1), \ldots, v_n(\cdot; R_n))$.

Definition 6.2. A rule f on the classical domain is a **generalized Vickrey rule** if for each valuation profile $\overline{v}(R)$, $f^x(\overline{v}(R)) \in \arg\max_{(x_1,\ldots,x_n)\in X} \sum_i v_i(x_i; R_i)$, and for each $i \in N$, $f_i^t(\overline{v}(R)) = \overline{\sigma}_{-i}(\overline{v}(R)) - \sigma_{-i}(\overline{v}(R))$.

A classical preference R_i is **object-blind** if for each $x, y \in M$ and each $t \in \mathbb{R}$, $(x, t)\, I_i\, (y, t)$. If objects are homogeneous, agents have object-blind preferences. We call the class of object-blind preferences the "object-blind domain." On the object-blind domain, the generalized Vickrey rules give objects to agents with m highest compensating valuations from $\mathbf{0}$, and they pay $(m+1)$-th highest compensating valuation from $\mathbf{0}$. Saitoh and Serizawa (2008) and Sakai (2008) characterize the generalized Vickrey rules on this domain.

Fact 6.1 (Saitoh and Serizawa, 2008; Sakai, 2008). *A rule f on the object-blind domain satisfies strategy-proofness, efficiency, individual rationality, and no subsidy if and only if it is a generalized Vickrey rule.*[29]

One the quasi-linear domain, the classes of Vickrey rules, the generalized Vickrey rules, and the MPW rules all coincide. Fact 6.1 suggests that the generalized Vickrey rules are

[29] It is straightforward that on the object-blind domain, *strategy-proofness, efficiency, individual rationality,* and *no subsidy for losers* imply *no subsidy*.

natural generalizations of the Vickrey rules on the object-blind domain. On the object-blind domain, the classes of the generalized Vickrey rules and the MPW rules also coincide. However, Theorem 4.1 and Fact 6.1 are mathematically independent.

6.2 Generalized Vickrey rule vs. minimum price Walrasian rule

Notice that in Example of Section 3 (Figure 1), agent 2's payment p^B in the MPWE cannot be computed from the compensating valuations $v_i(\cdot; R_i), i = 1, 2, 3$, from the origin $\mathbf{0}$. Payments of the MPW rule are influenced by compensating valuations from various points. It is worthwhile to mention that for the preference profile in Figure 1, it is agent 1's preference R_1 that determined whether agent 2 or 3 receives an object in the MPWE allocation. In Figure 1, agent 1 prefers $(A, CV_3(A; \mathbf{0}))$ to $(B, CV_2(B; \mathbf{0}))$, and agent 2 receives an object. However, if agent 1 prefers $(B, CV_2(B; \mathbf{0}))$ to $(A, CV_3(A; \mathbf{0}))$, agent 3 instead receives an object. Object allocations of the MPW rule are also influenced by compensating valuations from various points. Thus, the outcome z of the MPW rule is not the one of the generalized Vickrey rule. Accordingly, the MPW rule does not coincide with the generalized Vickrey rule.

One can easily check that the generalized Vickrey rules are not *efficient* nor *strategy-proof* on the classical domain with heterogeneous objects. To check this fact, let $R_1 \in \mathcal{R}^C$, $R_2 \in \mathcal{R}^Q$, and $R_3 \in \mathcal{R}^Q$ be such that $CV_1(A; \mathbf{0}) = 9$, $CV_1(B; \mathbf{0}) = 10$, $(A, 6)\, P_1\, (B, 5)$, $CV_2(A; \mathbf{0}) = 3$, $CV_2(B; \mathbf{0}) = 5$, $CV_3(A; \mathbf{0}) = 6$, and $CV_3(B; \mathbf{0}) = 2$. Then, the outcome of the generalized Vickrey rule for R is $z \equiv ((B, 5), (0, 0), (A, 4))$. Let $\hat{z} \equiv ((A, 6), (B, 5), (0, -2))$. Then, $\hat{z}$ Pareto-dominates z, and so the generalized Vickrey rule is not *efficient*. Let $\hat{R}_1 \in \mathcal{R}^Q$ be such that $\widehat{CV}_1(A; \mathbf{0}) = 8$ and $\widehat{CV}_1(B; \mathbf{0}) = 5$. Then, under the generalized Vickrey rule, the consumption point agent 1 will obtains for $(\hat{R}_1, R_{-1})$ is $(A, 6)$. Since $(A, 6)\, P_1\, (B, 5)$, the generalized Vickrey rule is not *strategy-proof*.

The generalized Vickrey rule employs only a small part of the information about agents' preferences (*i.e.,* "compensating valuations from the origin"). On the other hand, the MPW rule employs other information (*i.e.,* "compensating valuations from various points"). As we stated in Section 4, only the MPW rule satisfies *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy for losers* on the domain including nonquasi-linear preferences. Thus, the information about compensating valuations from various points is necessary to design rules satisfying the above four properties on this domain. Proposition 5.1 states that the SA auction achieves the same outcome as the MPW rule.

7 Concluding Remarks

In this article, we mainly focus on the analysis of rules that allocate objects *efficiently*, and show that only the MPW rules are desirable based on the four properties, *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy for losers*. It would be also important to investigate rules that produce more revenue for the auctioneer. An interesting question relating to this issue is to examine whether there are *strategy-proof*, *efficient*, and *individually rational* rules that produce a revenue of more than the total payment under the MPW rule for each preference profile. We believe that the results and techniques developed in this article will be useful for the study of this research topic.

Appendix: Proofs

In this Appendix, we provide the proofs of all results of the article. In Section A, we prove Theorem 3.1 and Corollaries 3.1 and 3.2 in Section 3. In Section B, we give the proofs of the main results (Fact 4.1 and Theorem 4.1 in Section 4). Section C gives the proof of Proposition 5.1 in Section 5.

In Section D, we provide the proof of Fact 3.4 in Section 3. The proof is the same as Mishra and Talman's (2010), but we provide it for completeness of the article. Fact 3.2 is already shown by the authors such as Demange and Gale (1985) and Roth and Sotomayor (1990). We also give the proof of Fact 3.5 in Section E for completeness.

A Proofs for Section 3 (Theorem 3.1 and Corollaries 3.1 and 3.2)

Let $\mathcal{R} \subseteq \mathcal{R}^E$ in this section.

Lemma A.1. *Let $R \in \mathcal{R}^n$, $(z,p) \in W(R)$, and $\hat{R}$ be the d-truncation of R such that for each $i \in N$ with $x_i \neq 0$, $d_i \leq -CV_i(0;z_i)$, and for each $i \in N$ with $x_i = 0$, $d_i \geq 0$. Then, $(z,p) \in W(\hat{R})$.*

Proof of Lemma A.1. Since $(z,p) \in W(R)$, (z,p) satisfies (WE-i) and (WE-ii) for R. Since (WE-ii) is independent of preferences, we show only (WE-i) for $\hat{R}$, that is, that for each $i \in N$ and each $y \in L$, $(x_i, p^{x_i})\, \hat{R}_i\, (y, p^y)$. Let $i \in N$ and $y \in L$.

First, consider the case where $x_i \neq 0$. If $y \neq 0$, then by Remark A.1, $(x_i, p^{x_i})\, \hat{R}_i\, (y, p^y)$. If $y = 0$, then by $d_i \leq -CV_i(0; z_i)$, $(x_i, p^{x_i})\, \hat{R}_i\, \mathbf{0} = (y, p^y)$.

Next, consider the case where $x_i = 0$. If $y = 0$, then by $(y, p^y) = \mathbf{0} = (x_i, p^{x_i})$, $(x_i, p^{x_i})\, \hat{R}_i\, (y, p^y)$. If $y \neq 0$, then by $(x_i, p^{x_i})\, R_i\, (y, p^y)$ and $d_i \geq 0$, $(x_i, p^{x_i})\, \hat{R}_i\, (y, p^y)$. $\square$

Lemma A.2. *Let $i \in N$, $R_i \in \mathcal{R}$, $d_i \in \mathbb{R}$, and $\hat{R}_i \in \mathcal{R}$ be the d_i-truncation of R_i. Let $p, q \in \mathbb{R}^m_+$, $x \in M$, and $y \in L$ be such that $x \in D(R_i, p)$ and $y \in D(\hat{R}_i, q)$.*
(i) If $q^x < p^x$ and $y \in M$, then, $(y, q^y)\, P_i\, (x, p^x)$ and $q^y < p^y$.
(ii) If $q^x < p^x$ and $d_i \leq -CV_i(0; (x, p^x))$, then, $y \in M$, $(y, q^y)\, P_i\, (x, p^x)$, and $q^y < p^y$.

Proof of Lemma A.2.
Proof of (i). Let $q^x < p^x$ and $y \in M$. By $y \in D(\hat{R}_i, q)$, $(y, q^y)\, \hat{R}_i\, (x, q^x)$. Since $\hat{R}_i$ is the d_i-truncation of R_i, it follows from Remark 3.1 that $(y, q^y)\, R_i\, (x, q^x)$. Thus,

$$(y, q^y)\, R_i\, (x, q^x)\, P_i\, (x, p^x)\, R_i\, (y, p^y),$$

where the second preference relation follows from $q^x < p^x$, and the third from $x \in D(R_i, p)$. Thus, $(y, q^y)\, P_i\, (x, p^x)$. Also, $(y, q^y)\, P_i\, (y, p^y)$ implies that $q^y < p^y$.

Proof of (ii). Let $q^x < p^x$ and $d_i \leq -CV_i(0;(x,p^x))$. Then, $\widehat{CV}_i(0;(x,p^x)) \leq 0$, and so $(x,p^x)\,\hat{R}_i\,\mathbf{0}$. Thus,

$$(y,q^y)\,\hat{R}_i\,(x,q^x)\,\hat{P}_i\,(x,p^x)\,\hat{R}_i\,\mathbf{0},$$

where the first preference relation follows from $y \in D(\hat{R}_i, q)$, and the second from $q^x < p^x$. Then, $(y,q^y)\,\hat{P}_i\,\mathbf{0}$ implies $y \in M$. Thus, by (i) of Lemma A.2, $(y,q^y)\,P_i\,(x,p^x)$ and $q^y < p^y$. $\square$

We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1. We first show *if* part of Theorem 3.1. Then, we prove *only if* part.
Proof of "IF" part. Assume that no set of objects is overdemanded and no set of objects is weakly underdemanded at p for R. Then, by Fact 3.4, $p \in P(R)$. Suppose that there is $q \in P(R)$ such that $q \leq p$ and $q \neq p$. Without loss of generality, assume that for each $x \in M'$, $q^x < p^x$, and for each $x \notin M'$, $q^x = p^x$, where $M' \equiv \{1,\ldots,m'\}$ and $1 \leq m' \leq m$.

Since M' is not weakly underdemanded at p for R, there is $N' \subseteq N$ such that $\#N' > \#M'$ and for each $i \in N'$, $D(R_i,p) \cap M' \neq \emptyset$. For each $i \in N'$, let $y_i \in D(R_i,p) \cap M'$. Since for each $x \in M'$, $q^x < p^x$, and for each $x \notin M'$, $q^x = p^x$, it follows that for each $i \in N'$ and each $x \notin M'$, $(y_i, q^{y_i})\,P_i\,(y_i,p^{y_i})\,R_i\,(x,p^x) = (x,q^x)$. Thus, for each $i \in N'$, $D(R_i,q) \subseteq M'$. By $\#N' > \#M'$, this implies M' is overdemanded at q. Since $q \in P(R)$, by Fact A.1, this is a contradiction.

Proof of "ONLY IF" part. Let $p \equiv p_{\min}(R)$. Then, by Fact 3.4, no set of objects is overdemanded and no set of objects is underdemanded at p for R. We show that no set of objects is weakly underdemanded at p for R. Suppose that there is a set M' of objects that is weakly underdemanded at p for R, *i.e.,* for each $x \in M'$, $p^x > 0$, and $\#\{i \in N : D(R_i,p) \cap M' \neq \emptyset\} \leq \#M'$. Let $N' \equiv \{i \in N : D(R_i,p) \cap M' \neq \emptyset\}$. Without loss of generality, assume that M' is minimal among the weakly underdemanded sets at p for R, *i.e.,* no proper subset of M' is weakly underdemanded at p. Since $p \in P(R)$, there is an allocation $z \in Z$ such that for each $i \in N$, $x_i \in D(R_i,p)$ and $t_i = p^{x_i}$. Since no set of objects is underdemanded at p for R, $\#N' = \#M'$. Without loss of generality, let $M' \equiv \{1,\ldots,m'\}$ and $N' \equiv \{1,\ldots,m'\}$.

Step 1. *For each $i \in N'$, $x_i \in M'$.*

Proof of Step 1. Since for each $x \in M'$, $p^x > 0$, it follows from (WE-ii) that for each $x \in M'$, there is $i(x) \in N'$ such that $x_{i(x)} = x$. Then, by $\#N' = \#M'$, for each $i \in N'$, $x_i \in M'$. $\square$

For each $x \in M'$, let $q^x \equiv \max\{CV_j(x;z_j) : j \in N \setminus N'\} \cup \{0\}$. Then, for each $x \in M'$, $q^x < p^x$.[30] Let $\hat{R}_{m'+1} \in \mathcal{R}$ be such that for each $x \in M'$, if $q^x > 0$, $\widehat{CV}_{m'+1}(x;\mathbf{0}) = q^x$, and if $q^x = 0$, $\widehat{CV}_{m'+1}(x;\mathbf{0}) \in (0,p^x)$. Consider the economy E' with objects M', agents $N'' \equiv N' \cup \{m'+1\}$, and their preference profile $(R_{N'}, \hat{R}_{m'+1})$. Let $z_{m'+1} \equiv \mathbf{0}$ and $z_{N''} \equiv (z_{N'}, z_{m'+1})$.

Step 2. *$(z_{N''}, p^{M'})$ is a minimum price Walrasian equilibrium of the economy E'.*[31]

[30] To see this, suppose that for some $x \in M'$, $q^x \geq p^x$. Then, there is $j \in N \setminus N'$ such that $(x,p^x)\;R_j\;z_j$. Since $x_j \in D(R_j,p)$, $x \in D(R_j,p)$. Thus, $j \in N'$. This contradicts $j \in N \setminus N'$.

[31] Let $W^{M',N''}(\bar{R}_{N''})$ and $W^{M',N''}_{\min}(\bar{R}_{N''})$ be the sets of Walrasian and minimum price Walrasian equilibria of the economy with objects M', agents N'', and their preference profile $\bar{R}_{N''}$, and let $P^{M',N''}(\bar{R}_{N''})$ and $p^{M',N''}_{\min}(\bar{R}_{N''})$ be the projections of $W^{M',N''}(\bar{R}_{N''})$ and $W^{M',N''}_{\min}(\bar{R}_{N''})$ onto $\mathbb{R}^{\#M'}_{+}$, respectively.

Proof of Step 2. Let $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W_{\min}^{M',N''}(R_{N'}, \hat{R}_{m'+1})$. Since $(z_{N''}, p^{M'}) \in W^{M',N''}(R_{N'}, \hat{R}_{m'+1})$, $\tilde{p}^{M'} \leq p^{M'}$. Let $M^- \equiv \{x \in M' : \tilde{p}^x < p^x\}$. We show that $M^- = \emptyset$. Suppose that $M^- \neq \emptyset$. Let $N^- \equiv \{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\}$.

Step 2.1. *For each $i \in N^-$, $\tilde{x}_i \in M^-$.*

Proof of Step 2.1. Let $i \in N^-$. Then, there is $x \in D(R_i, p^{M'}) \cap M^-$. Thus, $x \in M'$ and $\tilde{p}^x < p^x$. Since $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W_{\min}^{M',N''}(R_{N'}, \hat{R}_{m'+1})$, $\tilde{x}_i \in D(R_i, \tilde{p}^{M'})$. Then, Lemma A.2-(ii) implies that $\tilde{x}_i \in M'$ and $\tilde{p}^{\tilde{x}_i} < p^{\tilde{x}_i}$. Thus, $\tilde{x}_i \in M^-$. □

Step 2.2. $M^- = M'$, $N^- = N'$, and $\#M^- = \#N^-$.

Proof of Step 2.2. Since no two agents in N^- receive the same object, Step 2.1 implies $\#M^- \geq \#N^-$.

Suppose $M^- \neq M'$. Then, since $M^- \subsetneq M'$ and M' is minimal among the weakly underdemanded sets at p for R, M^- is not weakly underdemanded at $p^{M'}$ for $(R_{N'}, \hat{R}_{m'+1})$.[32] Thus, since for each $x \in M^-$, $p^x > 0$, we have $\#N^- \geq \#M^- + 1$. This contradicts $\#M^- \geq \#N^-$. Thus, $M^- = M'$.

By the definition of N^-, $M^- = M'$ implies $N^- = N'$.

Since M' is weakly underdemanded, $\#N' = \#M'$. By the above results, $\#M^- = \#M' = \#N' = \#N^-$. □

Step 2.3. *For each $x \in M'$, $\tilde{p}^x \geq q^x$.*

Proof of Step 2.3. Suppose that there is $x \in M'$ such that $\tilde{p}^x < q^x$. Hence, $q^x > 0$. Then, by $\tilde{x}_{m'+1} \in D(\hat{R}_{m'+1}, \tilde{p}^{M'})$ and $\tilde{p}^x < q^x = \widehat{CV}_{m'+1}(x; \mathbf{0})$, $\tilde{x}_{m'+1} \in M'$. By $M^- = M'$ and $N^- = N'$ (Step 2.2), Step 2.1 implies that for each $i \in N'$, $\tilde{x}_i \in M'$. Since $\#M' = m'$, this is a contradiction. □

Let $(\bar{z}, \bar{p}) \in Z \times \mathbb{R}_+^m$ be such that $\bar{z}_{N'} = \tilde{z}_{N'}$, $\bar{z}_{-N'} = z_{-N'}$, $\bar{p}^{M'} = \tilde{p}^{M'}$, and $\bar{p}^{-M'} = p^{-M'}$.

Step 2.4. *$(\bar{z}, \bar{p})$ is a Walrasian equilibrium of the original economy, i.e., $(\bar{z}, \bar{p}) \in W(R)$.*

Proof of Step 2.4. By Step 2.3, for each $y \in M'$, $\tilde{p}^y \geq q^y$. Let $h \in N \setminus N'$. Then, for each $y \in L$, if $y \notin M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) = (x_h, p^{x_h})\, R_h\, (y, p^y) = (y, \bar{p}^y),$$

where the preference relation follows from $x_h \in D(R_h, p)$, and if $y \in M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) = (x_h, p^{x_h})\, R_h\, (y, q^y)\, R_h\, (y, \bar{p}^y),$$

where the first preference relation follows from the definition of q^y, and the last from $\bar{p}^y = \tilde{p}^y \geq q^y$. Thus, for each $h \in N \setminus N'$, $\bar{x}_h \in D(R_h, \bar{p})$.

Let $h \in N'$. Then, for each $y \in L$, if $y \notin M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) = (\tilde{x}_h, \tilde{p}^{\tilde{x}_h})\, R_h\, (x_h, \tilde{p}^{x_h})\, R_h\, (x_h, p^{x_h})\, R_h\, (y, p^y) = (y, \bar{p}^y),$$

[32] By $M^- \subseteq M'$, $\{i \in N : D(R_i, p) \cap M^- \neq \emptyset\} \subseteq \{i \in N : D(R_i, p) \cap M' \neq \emptyset\} = N'$. Then, $\{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\} = \{i \in N : D(R_i, p) \cap M^- \neq \emptyset\}$. Hence, $\#N^- = \#\{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \emptyset\} = \#\{i \in N : D(R_i, p) \cap M^- \neq \emptyset\} > \#M^-$.

where the first preference relation follows from $\tilde{x}_h \in D(R_h, \tilde{p}^{M'})$, the second from $\tilde{p}^{M'} \leq p^{M'}$, and the third from $x_h \in D(R_h, p)$, and if $y \in M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) = (\tilde{x}_h, \tilde{p}^{\tilde{x}_h})\, R_h\, (y, \tilde{p}^y) = (y, \bar{p}^y),$$

where the preference relation follows from $\tilde{x}_h \in D(R_h, \tilde{p}^{M'})$. Thus, for each $h \in N'$, $\bar{x}_h \in D(R_h, \bar{p})$. Since (z, p) and $(\tilde{z}_{N''}, \tilde{p})$ satisfy (WE-ii), $(\bar{z}, \bar{p})$ also satisfies (WE-ii). Thus, $(\bar{z}, \bar{p}) \in W(R)$. □

Remember that $p = p_{\min}(R)$. However, since $M^- \neq \emptyset$, $\bar{p} \leq p$ and $\bar{p} \neq p$. This is a contradiction. Thus, $M^- = \emptyset$. This completes the proof of Step 2.

Without loss of generality, let $x_1 \equiv 1, \ldots, x_{m'} \equiv m'$. Denote by Π the set of the permutations of M' and by $\{x(k)\}_{k=1}^{m'}$ its generic element. Given $\{x(k)\}_{k=1}^{m'} \in \Pi$, let $\{i(k)\}_{k=1}^{m'}$ be such that

$$x_{i(1)} = x(1),\ x_{i(2)} = x(2),\ \ldots\ , x_{i(m')} = x(m'),$$

and $\{t(k)\}_{k=1}^{m'}$ be such that

$$t(1) \leq \widehat{CV}_{m'+1}(x(1); \mathbf{0}),\ t(2) \equiv CV_{i(1)}(x(2); z_0(1)),\ \ldots\ , t(m') \equiv CV_{i(m'-1)}(x(m'); z_0(m'-1)),$$

where for each $k \in \{1, \ldots, m'\}$, $z_0(k) \equiv (x(k), t(k))$. We call such a pair $\{z_0(k), i(k)\}_{k=1}^{m'}$ an **assignment sequence**. See Figure 2 for an illustration.

Step 3. *There is $b < p^1$ such that for any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$ constructed as above, and for k with $x(k) = 1$, $t(k) < b$.*

Proof of Step 3. For any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$, since $t(1) \leq q^{x(1)} < p^{x(1)}$, the following holds inductively: for each $k \geq 2$,

$$\begin{aligned}&(x(k), t(k))\ R_{i(k-1)}\ z_0(k-1)\ P_{i(k-1)}\ (x(k-1), p^{x(k-1)})\ R_{i(k-1)}\ (x(k), p^{x(k)}),\\ &\text{and } t(k) < p^{x(k)},\end{aligned}$$

where the first preference relation follows from $t(k) = CV_{i(k-1)}(x(k); z_0(k-1))$, the second from $t(k-1) < p^{x(k-1)}$, and the third from $x(k-1) \in D(R_{i(k-1)}, p)$. Since the cardinality of Π is finite ($m'!$), there is $b < p^1$ such that for any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$, and for k with $x(k) = 1$, $t(k) < b$. □

Let $\hat{R}_1$ be such that (i) $\hat{R}_1$ is the d_1-truncation of R_1, and (ii) $b < \widehat{CV}_1(x_1; \mathbf{0}) < p^1$.[33] Consider the economy $\widehat{E}$ with objects M', agents $N'' \equiv \{1, \ldots, m'+1\}$, and their preference profile $(\hat{R}_1, \hat{R}_{m'+1}, R_{N' \setminus \{1\}})$. Let $(\hat{z}, \hat{p})$ be a minimum price Walrasian equilibrium of $\widehat{E}$.

Step 4. $\hat{x}_1 \neq 0$.

Proof of Step 4. Suppose that $\hat{x}_1 = 0$. We use Claim A.1 below. It implies that m' agents (agents $2, \ldots, m'+1$) receive m' different objects in $M' \setminus \{x_1\}$. By $\#M' = m'$, this is a contradiction. Thus, proving Claim A.1 completes Proof of Step 4.

[33] Note that $d_1 > 0$.

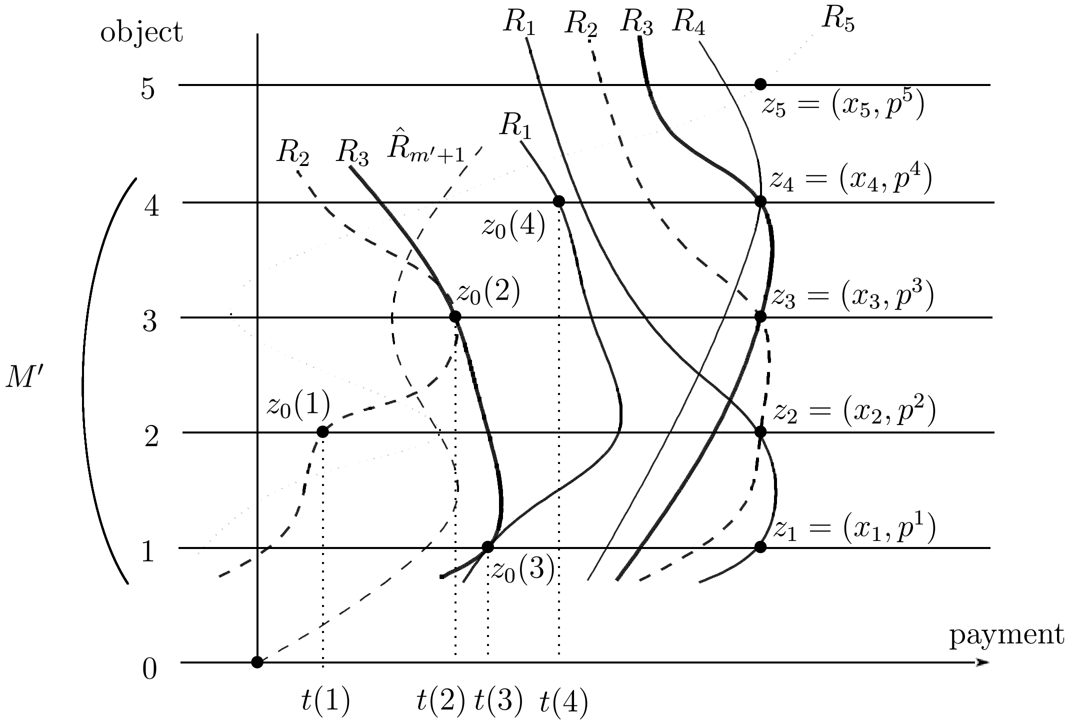

Figure 2: Illustration of assignment sequence for the case of $m' = 4$, $x(1) = x_2$, $x(2) = x_3$, $x(3) = x_1$, and $x(4) = x_4$.

Claim A.1. *The following sequences $\{i(k)\}$ and $\{z_0(k) \equiv (x(k), t(k))\}$, $k = 1, \ldots, m'$, can be constructed:*

$$x(1) \equiv \hat{x}_{m'+1},\ x_{i(1)} = x(1),\ \text{and } t(1) \equiv \hat{p}^{x(1)},\ \text{and}$$
$$\forall k \in \{2, \ldots, m'\}, \qquad x(k) \equiv \hat{x}_{i(k-1)},\ x_{i(k)} = x(k),\ \text{and } t(k) \equiv CV_{i(k-1)}(x(k); z_0(k-1)).$$

Furthermore, for each $k \in \{1, \ldots, m'\}$, $x(k) \neq 0$, $x(k) \neq x_1$, $\hat{p}^{x(k)} \leq t(k)$ and $\hat{p}^{x(k)} < p^{x(k)}$.

Proof of Claim A.1. The proof is by induction.

Part I. First, we show $x(1) \equiv \hat{x}_{m'+1} \neq 0$. Suppose $\hat{x}_{m'+1} = 0$. Then, since two agents (1 and $m'+1$) in N'' receive no object and $\#N'' = \#M'+1$, there is $x \in M$ such that for each $h \in N''$, $\hat{x}_h \neq x$. By (WE-ii), $\hat{p}^x = 0$. Since $\widehat{CV}_{m'+1}(x; \mathbf{0}) > 0$, $(x, \hat{p}^x)\, \hat{P}_{m'+1}\, \mathbf{0}$. This is a contradiction since $\hat{x}_{m'+1} = 0$ and $(\hat{z}, \hat{p}) \in W^{M',N''}_{\min}(\hat{R}_1, \hat{R}_{m'+1}, R_{N'\setminus\{1\}})$. Thus, $x(1) \neq 0$.

Note that by Step 1, $x(1) \neq 0$ implies that agent $i(1)$ with $x_{i(1)} = x(1)$ uniquely exists. Thus, $x(1)$, $i(1)$, and $t(1)$ are well-defined.

Second, we show that $x(1) \neq x_1$. Suppose that $x(1) = x_1$. Then, by Step 3 and (ii) of $\hat{R}_1$, $\hat{p}^{x(1)} \equiv t(1) < b < \widehat{CV}_1(x_1; \mathbf{0})$, that is, $(x(1), \hat{p}^{x(1)})\, \hat{P}_1\, \mathbf{0}$. Thus, by $\hat{x}_1 = 0$, $\hat{x}_1 \notin D(\hat{R}_1, \hat{p})$. However, since $(\hat{z}, \hat{p}) \in W^{M',N''}_{\min}(\hat{R}_1, \hat{R}_{m'+1}, R_{N'\setminus\{1\}})$, this is a contradiction. Thus, $x(1) \neq x_1$.

Third, by $x(1) \equiv \hat{x}_{m'+1} \in D(\hat{R}_{m'+1}, \hat{p})$, $\hat{p}^{x(1)} \leq \widehat{CV}_{m'+1}(x(1); \mathbf{0}) < p^{x(1)}$.

Part II (Induction argument). Let $k \in \{2, \ldots, m'\}$. Assume that Claim A.1 holds until $k-1$. Since $x(k-1) \in D(R_{i(k-1)}, p)$, $\hat{x}_{i(k-1)} \in D(R_{i(k-1)}, \hat{p})$, and $\hat{p}^{x(k-1)} < p^{x(k-1)}$, Lemma A.2-(ii) implies that $x(k) \equiv \hat{x}_{i(k-1)} \neq 0$ and $\hat{p}^{x(k)} < p^{x(k)}$.

Note that by Step 1, $x(k) \neq 0$ implies that agent $i(k)$ with $x_{i(k)} = x(k)$ uniquely exists. Thus, $x(k)$, $i(k)$, and $t(k)$ are well-defined.

If $\hat{p}^{x(k)} > t(k) = CV_{i(k-1)}(x(k); z_0(k-1))$, then

$$(x(k-1), \hat{p}^{x(k-1)}) \, R_{i(k-1)} \, z_0(k-1) \, P_{i(k-1)} \, (x(k), \hat{p}^{x(k)}),$$

contradicting $x(k) \equiv \hat{x}_{i(k-1)} \in D(R_{i(k-1)}, \hat{p})$. Thus, $\hat{p}^{x(k)} \leq t(k)$.

We show $x(k) \neq x_1$. Suppose that $x(k) = x_1$. Then, by Step 3 and (ii) of $\hat{R}_1$, $\hat{p}^{x(k)} \leq t(k) < b < \widehat{CV}_1(x_1; \mathbf{0})$. Thus, $(x(k), \hat{p}^{x(k)}) \, \hat{P}_1 \, \mathbf{0}$. Then, by $\hat{x}_1 = 0$, $\hat{x}_1 \notin D(\hat{R}_1, \hat{p})$. However, since $(\hat{z}, \hat{p}) \in W^{M', N''}_{\min}(\hat{R}_1, \hat{R}_{m'+1}, R_{N' \setminus \{1\}})$, this is a contradiction. Thus, $x(k) \neq x_1$. $\square$

Step 5. *We derive a contradiction to conclude that no set of objects is weakly underdemanded at p for R.*

Note that by (i) and (ii) of $\hat{R}_1$, $d_1 > 0$. Since $(\hat{z}, \hat{p}) \in W^{M', N''}_{\min}(\hat{R}_1, \hat{R}_{m'+1}, R_{N' \setminus \{1\}})$, Step 2 and Fact 3.5 imply that $\hat{p} \leq p^{M'}$. Note that

$$(\hat{x}_1, \hat{p}^{\hat{x}_1}) \, \hat{R}_1 \, \mathbf{0} \, \hat{I}_1 \, (x_1, \widehat{CV}_1(x_1; \mathbf{0})) \, \hat{P}_1 \, (x_1, p^{x_1}),$$

where the first preference relation follows from $\hat{x}_1 \in D(\hat{R}_1, \hat{p})$, the second from the definition of compensating valuation, and the third from (ii) of $\hat{R}_1$.

By Steps 1 and 4, $x_1 \neq 0$ and $\hat{x}_1 \neq 0$. Since (i) of $\hat{R}_1$, by Remark 3.1, $(\hat{x}_1, \hat{p}^{\hat{x}_1}) \, P_1 \, (x_1, p^{x_1})$. Then,

$$(\hat{x}_1, \hat{p}^{\hat{x}_1}) \, P_1 \, (x_1, p^{x_1}) \, R_1 \, (\hat{x}_1, p^{\hat{x}_1}),$$

where the second preference relation follows from $x_1 \in D(R_1, p)$. Thus, $\hat{p}^{\hat{x}_1} < p^{\hat{x}_1}$.

By (i) and (ii) of $\hat{R}_1$, R_1 is the $(-d_1)$-truncation of $\hat{R}_1$ and $-d_1 \leq 0 \leq -\widehat{CV}_1(0; \hat{z}_1)$. Then, Lemma A.1 implies that $\hat{p} \in P^{M', N''}(R_{N'}, \hat{R}_j)$. However, by Step 2, $p^{M'} = p^{M', N''}_{\min}(R_{N'}, \hat{R}_j)$. Since $\hat{p} \leq p^{M'}$ and $\hat{p}^{\hat{x}_1} < p^{\hat{x}_1}$, this is a contradiction. $\square$

Proof of Corollary 3.1. Suppose that for each $i \in N$, $p^{x_i}_{\min}(R) > 0$. Then, for each $i \in N$, $x_i \neq 0$. Let $\bar{M} \equiv \{x_1, \ldots, x_n\}$. Then, $\#\bar{M} \equiv \#N$. Since $\#\bar{M} = \#\{i \in N : D(R_i, p) \cap \bar{M} \neq \emptyset\}$, $\bar{M}$ is weakly underdemanded at p for R. This is a contradiction to Theorem 3.1. $\square$

Proof of Corollary 3.2. Let $x \in M$ be such that $p^x > 0$. Then, by (WE-ii) in Definition 3.1, there is $j_1 \in N$ such that $x_{j_1} = x$. By Theorem 3.1, the set $\{x\}$ is demanded at p by at lease two agents, and so, there is $j_2 \in N \setminus \{j_1\}$ such that $x \in D(R_{j_2}, p)$. If $x_{j_2} = 0$ or $p^{x_{j_2}} = 0$, then by letting $i_1 \equiv j_2$ and $i_2 \equiv j_1$, we obtain the desired conclusion. Thus, we assume that $x_{j_2} \neq 0$ and $p^{x_{j_2}} > 0$. Then, the set $\{x_{j_1}, x_{j_2}\}$ is demanded at p by at lease three agents, and so, there is $j_3 \in N \setminus \{j_1, j_2\}$ such that $x \in D(R_{j_3}, p)$. If $x_{j_3} = 0$ or $p^{x_{j_3}} = 0$, then by letting $i_1 \equiv j_3$, $i_2 \equiv j_2$, and $i_3 \equiv j_1$, we obtain the desired conclusion. Thus, we assume that $x_{j_3} \neq 0$ and $p^{x_{j_3}} > 0$. Repeating this argument inductively, there is a sequence $\{j_k\}_{k=1}^{K}$ of K distinct agents such that (a) $x_{j_K} = 0$ or $p^{x_{j_K}} = 0$, (b) $x_{j_1} = x$, and (c) for each $k \in \{2, \ldots, K\}$, $\{x_{j_k}, x_{j_{k-1}}\} \subseteq D(R_{j_k}, p)$. For each $k \in \{1, \ldots, K\}$, let $i_k \equiv j_{K-(k-1)}$. Then, the desired conclusion follows from (a), (b), and (c). $\square$

B Proofs for Section 4 (Main results: Fact 4.1 and Theorem 4.1)

Proof of Fact 4.1. Let $\mathcal{R} \subseteq \mathcal{R}^E$. Let g be a minimum price Walrasian rule on $\mathcal{R}^n$. By contradiction, suppose that there exist $R \in \mathcal{R}^n$, $\hat{N} \subseteq N$, and $\hat{R}_{\hat{N}} \in \mathcal{R}^{\#\hat{N}}$ such that for each $i \in \hat{N}$, $g_i(\hat{R}_{\hat{N}}, R_{-\hat{N}})\, P_i\, g_i(R)$. Let $z \equiv g(R)$ and $\hat{z} \equiv g(\hat{R}_{\hat{N}}, R_{-\hat{N}})$. Let p and $\hat{p}$ be the equilibrium prices associated with z and $\hat{z}$, respectively. Without loss of generality, let $\hat{N} = \{1, \ldots, \hat{n}\}$. Let $M^+ \equiv \{x \in M : 0 < p^x\}$ and $m^+ \equiv \#M^+$. Note that, if $n > m$, then $n > m^+$, and if $n \leq m$, then by Corollary 3.1, $m^+ \leq n - 1 < n$.

In this paragraph, we show that for each $i \in \hat{N}$, $\hat{x}_i \neq 0$, and $\hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$. Let $i \in \hat{N}$. Note that $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (x_i, p^{x_i})\, R_i\, \mathbf{0}$, where the first preference relation follows from $g_i(\hat{R}_{\hat{N}}, R_{-\hat{N}})\, P_i\, g_i(R)$, and the second from $x_i \in D(R_i, p)$. Thus, $\hat{x}_i \neq 0$. Note that $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (x_i, p^{x_i})\, R_i\, (\hat{x}_i, p^{\hat{x}_i})$, where the last preference relation also follows from $x_i \in D(R_i, p)$. Thus, $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (\hat{x}_i, p^{\hat{x}_i})$ implies that $\hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$.

Note that, for each $i \in \hat{N}$, since $0 \leq \hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$, $\hat{x}_i \in M^+$. Then, if $m^+ < \hat{n}$, more than m^+ agents receive the objects in M^+, which is a contradiction. Thus, assume that $m^+ \geq \hat{n}$. By Theorem 3.1, there is $i' \in N \setminus \hat{N}$ such that $D(R_{i'}, p) \cap \{\hat{x}_1, \ldots, \hat{x}_{\hat{n}}\} \neq \emptyset$. Without loss of generality, let $i' \equiv \hat{n} + 1$. Note that $R_{\hat{n}+1}$ itself is its $d_{\hat{n}+1}$-truncation. Thus, by Lemma A.2-(ii), $\hat{x}_{\hat{n}+1} \neq 0$, and $0 \leq \hat{p}^{\hat{x}_{\hat{n}+1}} < p^{\hat{x}_{\hat{n}+1}}$. Thus, $\hat{x}_{\hat{n}+1} \in M^+$. Then, by Theorem 3.1, there is $i'' \in N \setminus \{1, \ldots, \hat{n}+1\}$ such that $D(R_{i''}, p) \cap \{\hat{x}_1, \ldots, \hat{x}_{\hat{n}+1}\} \neq \emptyset$. Without loss of generality, let $i'' \equiv \hat{n} + 2$. Note that $R_{\hat{n}+2}$ itself is its $d_{\hat{n}+2}$-truncation. Thus, by Lemma A.2-(ii), $\hat{x}_{\hat{n}+2} \neq 0$, and $0 \leq \hat{p}^{\hat{x}_{\hat{n}+2}} < p^{\hat{x}_{\hat{n}+2}}$. Thus, $\hat{x}_{\hat{n}+2} \in M^+$. Repeat this argument $(m^+ - \hat{n} + 1)$ times. Then, more than m^+ agents receive objects in M^+. This is a contradiction. □

Next, we prove Theorem 4.1. Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. Since *if* part of Theorem 4.1 follows from the discussion in Subsection 4.1, we give the proof of the *only if* part of the theorem.

Part 1: Preliminary results (Proofs of Lemmas B.1–B.5, and Fact B.1)

Lemma B.1 (Zero-payment for losers). *Let f be a rule satisfying individual rationality and no subsidy for losers on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $f_i^x(R) = 0$. Then, $f_i^t(R) = 0$.*

Proof of Lemma B.1. By *no subsidy for losers*, $f_i^t(R) \geq 0$. By *individual rationality*, $f_i^t(R) \leq 0$. Thus, $f_i^t(R) = 0$. □

Lemma B.2. *Let f be a rule satisfying efficiency, individual rationality, and no subsidy for losers on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $x \in M$. Then, there is $i \in N$ such that $f_i^x(R) = x$.*

Proof of Lemma B.2. By contradiction, suppose that for each $i \in N$, $f_i^x(R) \neq x$. Then, by $n > m$, there is $j \in N$ such that $f_j^x(R) = 0$. By Lemma B.1, $f_j^t(R) = 0$. Let $\hat{z} \in Z$ be such that $\hat{z}_j \equiv (x, 0)$ and for each $i \in N \setminus \{j\}$, $\hat{z}_i \equiv f_i(R)$. Then, since $(x, 0)\, P_i\, (0, 0)$, $\hat{z}_j\, P_j\, f_j(R)$. Note that for each $i \in N \setminus \{j\}$, $\hat{z}_i\, I_i\, f_i(R)$, and $\sum_{i \in N} \hat{t}_i = \sum_{i \in N} f_i^t(R)$. Thus, $\hat{z}$ Pareto-dominates $f(R)$ at R, which contradicts *efficiency.* □

Lemma B.3. *Let $R \in \mathcal{R}^n$, $i, j \in N$, and $z \in Z$ with $x_i \neq 0$. Assume that (a) $0 \leq t_j - CV_i(x_j; z_i) < CV_j(x_i; z_j) - t_i$. Then, there is $\hat{z} \in Z$ that Pareto-dominates z at R.*

Proof of Lemma B.3. Let $d \equiv t_j - CV_i(x_j; z_i)$, and let $\hat{z} \in Z$ be such that $\hat{z}_i \equiv (x_j, t_j - d)$, $\hat{z}_j \equiv (x_i, t_i + d)$, and for each $k \in N \setminus \{i, j\}$, $\hat{z}_k \equiv z_k$. Since $\hat{z}_i = (x_j, CV_i(x_j; z_i))$, $\hat{z}_i \, I_i \, z_i$. By (a) and $\hat{z}_j = (x_i, t_i + t_j - CV_i(x_j; z_i))$, $\hat{z}_j \, P_j \, (x_i, CV_j(x_i; z_j)) \, I_j \, z_j$. For each $k \in N \setminus \{i, j\}$, $\hat{z}_k \, I_k \, z_k$, and $\sum_{k \in N} \hat{t}_k = t_j - d + t_i + d + \sum_{k \neq i,j} t_k = \sum_{k \in N} t_k$. Thus, $\hat{z}$ Pareto-dominates z at R. □

Given a bundle $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$ with $x_i \neq 0$, let $\mathcal{R}_{NCV}(z_i)$ be the set of preferences $\hat{R}_i \in \mathcal{R}$ such that for each $y \in L \setminus \{x_i\}$, $\widehat{CV}_i(y; z_i) < 0$, that is, for each object except for x_i, the compensating valuation of $\hat{R}_i$ from z_i is negative.

Lemma B.4. *Let f be a rule satisfying strategy-proofness and no subsidy on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $f_i^x(R) \neq 0$. Let $\hat{R}_i \in \mathcal{R}_{NCV}(f_i(R))$. Then, $f_i(\hat{R}_i, R_{-i}) = f_i(R)$.*

Proof of Lemma B.4. First, we show $f_i^x(\hat{R}_i, R_{-i}) = f_i^x(R)$. Suppose not. Let $x \equiv f_i^x(\hat{R}_i, R_{-i})$. By *strategy-proofness*, $f_i(\hat{R}_i, R_{-i}) \, \hat{R}_i \, f_i(R)$, and so, $f_i^t(\hat{R}_i, R_{-i}) \leq \widehat{CV}_i(x; f_i(R))$. Since $\hat{R}_i \in \mathcal{R}_{NCV}(f_i(R))$, $\widehat{CV}_i(x; f_i(R)) < 0$. Thus, $f_i^t(\hat{R}_i, R_{-i}) < 0$, contradicting *no subsidy*.

Next, we show $f_i^t(\hat{R}_i, R_{-i}) = f_i^t(R)$. Suppose that $f_i^t(\hat{R}_i, R_{-i}) < f_i^t(R)$. (The opposite case can be treated symmetrically.) Then, $f_i(\hat{R}_i, R_{-i}) \, P_i \, f_i(R)$, contradicting *strategy-proofness*. □

We introduce some additional notations. Given $R \in \mathcal{R}^n$, $x \in M$, and $z \in (L \times \mathbb{R})^n$, let $\pi^x(R) \equiv (\pi_1^x(R), \ldots, \pi_n^x(R))$ be the permutation on N such that $CV_{\pi_n^x(R)}(x; z_{\pi_n^x(R)}) \leq \cdots \leq CV_{\pi_1^x(R)}(x; z_{\pi_1^x(R)})$. That is, $\pi_n^x(R)$ is the agent with the lowest compensating valuation of object x from z, $\pi_{n-1}^x(R)$ is the agent with the second lowest compensating valuation of object x from z, and so on. For each $k \in N$, let $C^k(R, x; z) \equiv CV_{\pi_k^x(R)}(x; z_{\pi_k^x(R)})$. That is, $C^k(R, x; z)$ is *the k-th highest compensating valuation (CV) of object x from z*. We simply write $C^k(R, x; (\mathbf{0}, \ldots, \mathbf{0}))$ as $C^k(R, x)$.

Lemma B.5. *Let f be a rule satisfying strategy-proofness, efficiency, individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$. If $f_i^x(R) = x$, then, $f_i^t(R) \geq C^{m+1}(R, x)$.*

Proof of Lemma B.5. Note that for each $y \in M$ and each $i \in N$, $(y, 0) \, P_i \, (0, 0)$. Thus, for each $y \in M$, $C^{m+1}(R, y) > 0$. By contradiction, suppose that $f_i^x(R) = x$ and $f_i^t(R) < C^{m+1}(R, x)$. Let $\hat{R}_i \in \mathcal{R}^Q$ be a quasi-linear preference such that for each $y \in M$, $0 < \widehat{CV}_i(y; \mathbf{0}) < C^{m+1}(R, y)$, and $f_i^t(R) < \widehat{CV}_i(x; \mathbf{0})$. Let $\hat{y} \equiv f_i^t(\hat{R}_i, R_{-i})$. Then, by *strategy-proofness*, $f_i^t(\hat{R}_i, R_{-i}) \leq \widehat{CV}_i(\hat{y}; f_i(R))$. Since $\widehat{CV}_i(0; f_i(R)) < 0$, it follows from *no subsidy for losers* that $\hat{y} \neq 0$.

Since $\#\{j \in N \setminus \{i\} : CV_j(\hat{y}; \mathbf{0}) \geq C^{m+1}(R, \hat{y})\} \geq m$, there is $j \in N \setminus \{i\}$ such that $CV_j(\hat{y}; \mathbf{0}) \geq C^{m+1}(R, \hat{y})$ and $f_j^x(\hat{R}_i, R_{-i}) = 0$. By Lemma B.1, $f_j^t(\hat{R}_i, R_{-i}) = 0$.

Let $\hat{z}_i \equiv (0, \widehat{CV}_i(0; f_i(\hat{R}_i, R_{-i}))$, $\hat{z}_j \equiv (\hat{y}, \widehat{CV}_i(\hat{y}; \mathbf{0}))$, and for each $k \neq i, j$, $\hat{z}_k \equiv f_k(\hat{R}_i, R_{-i})$. Then, $\hat{z}_i \, \hat{I}_i \, f_i(\hat{R}_i, R_{-i})$, and for each $k \neq i, j$, $\hat{z}_k \, I_k \, f_k(\hat{R}_i, R_{-i})$. By $CV_j(\hat{y}; \mathbf{0}) > \widehat{CV}_i(\hat{y}; \mathbf{0})$, $\hat{z}_j \, P_j \, f_j(\hat{R}_i, R_{-i})$. Since $\hat{R}_i \in \mathcal{R}^Q$, $\widehat{CV}_i(0; f_i(\hat{R}_i, R_{-i})) = f_i^t(\hat{R}_i, R_{-i}) - \widehat{CV}_i(\hat{y}; \mathbf{0})$. Thus, $\hat{t}_i + \hat{t}_j = \widehat{CV}_i(0; f_i(\hat{R}_i, R_{-i})) + \widehat{CV}_i(\hat{y}; \mathbf{0}) = f_i^t(\hat{R}_i, R_{-i})$. Then, by $f_j^t(\hat{R}_i, R_{-i}) = 0$, $\sum_{k \in N} \hat{t}_k = \sum_{k \in N} f_k^t(\hat{R}_i, R_{-i})$. Thus, $\hat{z}$ Pareto-dominates $f(\hat{R}_i, R_{-i})$ at $(\hat{R}_i, R_{-i})$, which contradicts *efficiency*. □

Fact B.1. *Strategy-proofness, efficiency, individual rationality, and no subsidy for losers imply no subsidy.*

Proof of Fact B.1. Let f satisfy the four properties om $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$, $i \in N$, and $x \equiv f_i^x(R)$. If $x = 0$, Fact B.1 follows from *no subsidy for losers*. Thus, we assume that $x \neq 0$. Then, by Lemma B.1, $f_i^t(R) \geq C^{m+1}(R, x)$. Since for each $y \in M$ and each $i \in N$, $(y, 0)\, P_i\, (0, 0)$, for each $y \in M$, $C^{m+1}(R, y) > 0$. Thus, $f_i^t(R) > 0$. □

Hereafter, we maintain the assumption that f is a rule on $\mathcal{R}^n$, and that the rule f satisfies *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy for losers*. Then, by Fact B.1 above, *no subsidy* is implied by the four properties. Therefore, hereafter, we also assume that the rule f satisfies *no subsidy*.

Part 2: Proof of Proposition B.1 (Proofs of Lemmas B.6 and B.7, and Proposition B.1)

We establish Proposition B.1 below, which says that for each preference profile, the allocation chosen by the rule f satisfying *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy for losers* should (weakly) dominate the minimum price Walrasian equilibrium.

Proposition B.1.[34] *Let $R \in \mathcal{R}^n$ and $z \in Z_{\min}(R)$. For each $i \in N$, $f_i(R)\, R_i\, z_i$.*

We introduce two lemmas to prove Proposition B.1.

Lemma B.6. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$. If $f_i^x(R) = x$, then, $CV_i(x; \mathbf{0}) \geq C^m(R, x)$.*

Proof of Lemma B.6. By contradiction, suppose that $f_i^x(R) = x$ and $CV_i(x; \mathbf{0}) < C^m(R, x)$. Then, by Lemma B.5, $C^{m+1}(R, x) \leq f_i^t(R)$. By *individual rationality*, $f_i^t(R) \leq CV_i(x; \mathbf{0})$. Then, by $CV_i(x; \mathbf{0}) \leq C^{m+1}(R, x)$, $f_i^t(R) = CV_i(x; \mathbf{0})$. Since $\#\{j \in N : CV_j(x; \mathbf{0}) \geq C^m(R, x)\} = m$, there is $j \in N \setminus \{i\}$ such that $CV_j(x; \mathbf{0}) \geq C^m(R, x)$ and $f_j^x(R) = 0$. By Lemma B.1, $f_j^t(R) = 0$. Then, by $CV_i(x; f_i(R)) = 0$ and $f_i^t(R) = CV_i(x; \mathbf{0}) < C^m(R, x) \leq CV_j(x; \mathbf{0})$, $0 = f_j^t(R) - CV_i(x; f_i(R)) < CV_j(x; \mathbf{0}) - f_i^t(R)$. Note that $x \neq 0$. Thus, by Lemma B.3, there is $\hat{z} \in Z$ that Pareto-dominates $f(R)$ at R, which contradicts *efficiency*. □

Given $R \in \mathcal{R}^N$, let $Z^{IR}(R)$ be the set of *individually rational* allocations, that is, $Z^{IR}(R) \equiv \{z \in Z : \text{ for each } i \in N, z_i\ R_i\ \mathbf{0}\}$.

Lemma B.7. *Let $R \in \mathcal{R}^n$, $x \in M$, and $i \in N$ be such that for each $y \in M \setminus \{x\}$, $CV_i(y; \mathbf{0}) < C^m(R, y)$. Let $z \in Z^{IR}(R)$, $CV_i(x; \mathbf{0}) > C^1(R_{-i}, x; z)$, and $f_j(R)\, R_j\, z_j$ for each $j \in N \setminus \{i\}$. Then, $f_i^x(R) = x$.*

Proof of Lemma B.7. (Figure 3) By contradiction, suppose that $f_i^x(R) \neq x$. Then, by Lemma B.2, there is $j \in N \setminus \{i\}$ such that $f_j^x(R) = x$. Since $f_j(R)\, R_j\, z_j$, $f_j^t(R) \leq CV_j(x; z_j) < CV_i(x; \mathbf{0})$. Since $z \in Z^{IR}(R)$, for each $y \in M$, $CV_j(y; z_j) \leq CV_j(y; \mathbf{0})$. Let $\hat{R}_j \in \mathcal{R}_{NCV}(f_j(R))$ be such that (i) $-\widehat{CV}_j(0; f_j(R)) < CV_i(x; \mathbf{0}) - f_j^t(R)$, and (ii) for each $y \in M \setminus \{x\}$, $\widehat{CV}_j(y; \mathbf{0}) = CV_j(y; \mathbf{0})$. Then, by Lemma B.4, $f_j(\hat{R}_j, R_{-j}) = f_j(R)$. Since $f_j^x(\hat{R}_j, R_{-j}) = x$, $f_i^x(\hat{R}_j, R_{-j}) \neq x$. Next, we show that $f_i^x(\hat{R}_j, R_{-j}) \notin M \setminus \{x\}$. Suppose that there is $y \in M \setminus \{x\}$ such that $f_i^x(\hat{R}_j, R_{-j}) = y$. By (ii), $C^m(\hat{R}_j, R_{-j}, y) = C^m(R, y)$. Since $CV_i(y; \mathbf{0}) < C^m(R, y)$,

[34] This result also holds for any Walrasian equilibrium allocation z.

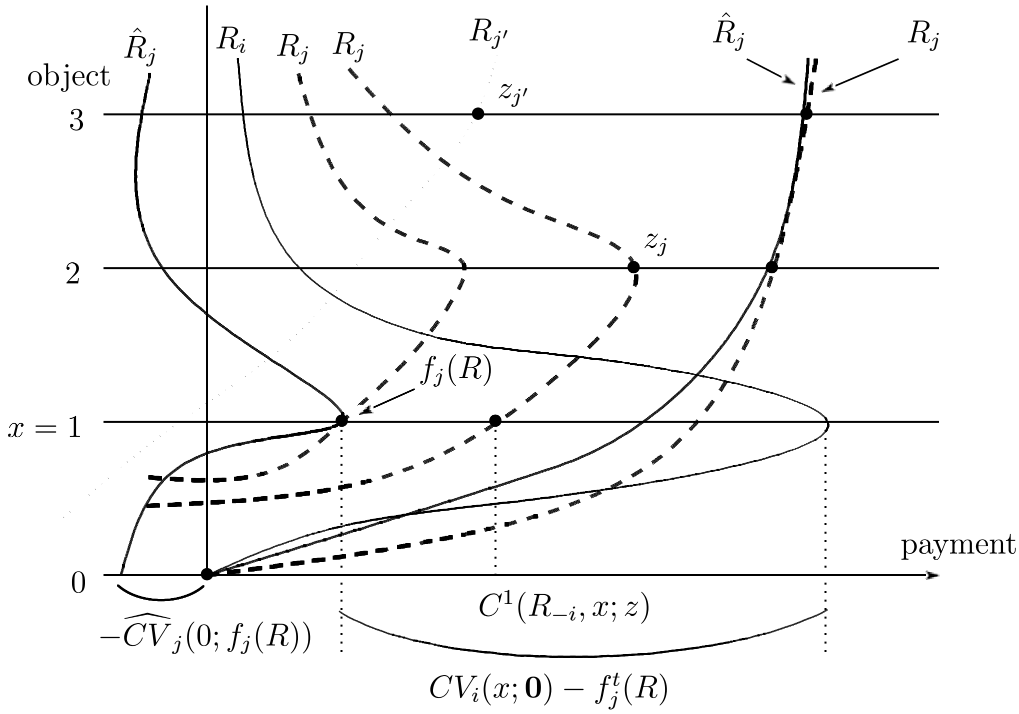

Figure 3: Illustration of proof of Lemma B.7.

$CV_i(y;\mathbf{0}) < C^m(\hat{R}_j, R_{-j}, y)$, which contradicts Lemma B.6. Thus, $f_i^x(\hat{R}_j, R_{-j}) = 0$. By Lemma B.1, $f_i^t(\hat{R}_j, R_{-j}) = 0$. Then, by (i) and Lemma B.3, there is $\hat{z} \in Z$ that Pareto-dominates $f(\hat{R}_j, R_{-j})$ at $(\hat{R}_j, R_{-j})$, which contradicts *efficiency*. $\square$

We now proceed to prove Proposition B.1.

Proof of Proposition B.1. We only show $f_1(R)\, R_1\, z_1$ since the case of any other agent can be treated in the same way. If $x_1 = 0$, then $z_1 = \mathbf{0}$, and so, by *individual rationality*, $f_1(R)\, R_1\, z_1$. Thus, we assume that $x_1 \neq 0$. Let $N^+ \equiv \{j \in N : x_j \neq 0\}$. Note that $\# N^+ = m$.

By contradiction, suppose that $z_1\, P_1\, f_1(R)$. We prove Claim B.1 below by induction. (iv-$(k+1)$) of Claim B.1 induces a contradiction by the finiteness of N^+.

Claim B.1. *For each $k \geq 0$, there exist a set $N(k+1)$ of $k+1$ distinct agents, say $N(k+1) \equiv \{1, \ldots, k+1\}$, and $\hat{R}_{N(k+1)} \in \mathcal{R}^{k+1}$ such that*
(i-$(k+1)$) $z_{k+1}\, P_{k+1}\, f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)})$,
(ii-$(k+1)$) for each $j \in N(k+1)$ and each $y \in M \backslash \{x_j\}$, $\widehat{CV}_j(y;\mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, y)$,
(iii-$(k+1)$) $t_{k+1} < \widehat{CV}_{k+1}(x_{k+1};\mathbf{0}) < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)}))$, and
(iv-$(k+1)$) $N(k+1) \subsetneq N^+$,
where $N(k) \equiv \{1, \ldots, k\}$.

Proof of Claim B.1.
Step 1. Let $k = 0$ and $N(1) \equiv \{1\}$. By $z_1\, P_1\, f_1(R)$, (i-1) holds, and so, $t_1 < CV_1(x_1; f_1(R))$. Note that for each $y \in M$, $C^n(R, y) > 0$. Thus, there is $\hat{R}_1 \in \mathcal{R}$ such that (ii-1): for each

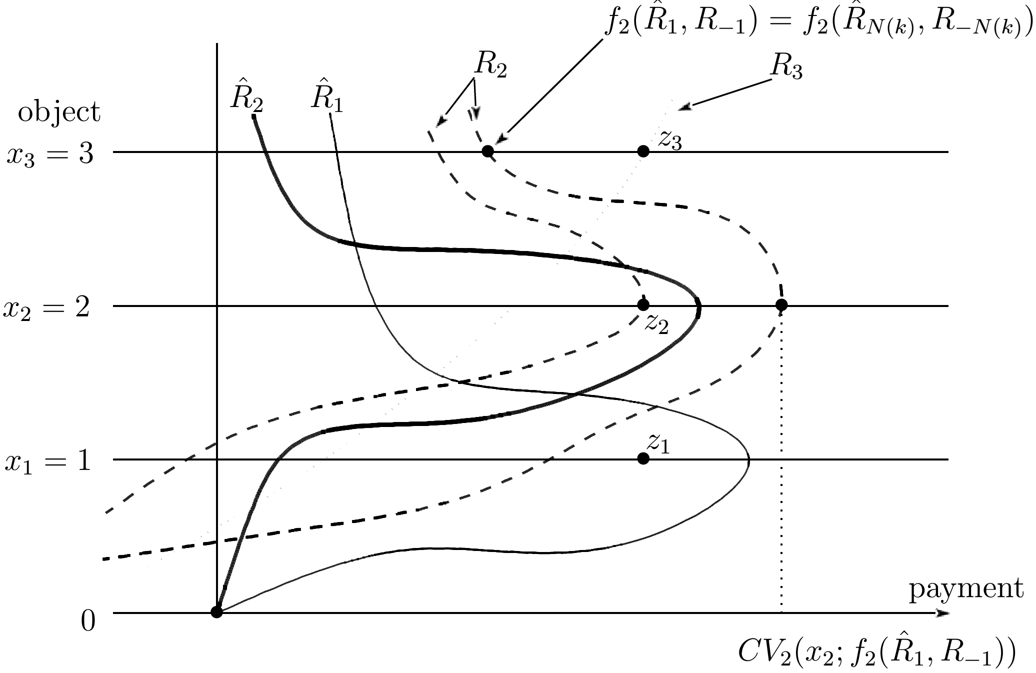

Figure 4: Illustration of (i-$(k+1)$), (ii-$(k+1)$) and (iii-$(k+1)$) in the proof of Proposition B.1 for $k = 1$.

$y \in M \setminus \{x_1\}$, $\widehat{CV}_1(y; \mathbf{0}) < C^n(R, y)$, and (iii-1): $t_1 < \widehat{CV}_1(x_1; \mathbf{0}) < CV_1(x_1; f_1(R))$.

Note that $\{1\} \subseteq N^+$. Suppose that $\{1\} = N^+$. Since $\#N^+ = m$, $m = 1$. Thus, by $x_1 \neq 0$, for each $j \in N \setminus \{1\}$, $z_j = \mathbf{0}$. Since $z \in W(R)$, for each $j \in N \setminus \{1\}$, $z_j \, R_j \, z_1$, and so, $CV_j(x_1; \mathbf{0}) \leq t_1$. Thus, by (iii-1), $C^1(R_{-1}, x_1; z) \leq t_1 < \widehat{CV}_1(x_1; \mathbf{0})$. By *individual rationality*, for each $j \in N \setminus \{1\}$, $f_j(\hat{R}_1, R_{-1}) \, R_j \, \mathbf{0} = z_j$. Since $z \in Z^{IR}(\hat{R}_1, R_{-1})$, Lemma B.7 implies $f_1^x(\hat{R}_1, R_{-1}) = x_1$. By *individual rationality*, $f_1^t(\hat{R}_1, R_{-1}) \leq \widehat{CV}_1(x_1; \mathbf{0})$. However, by (iii-1), $f_1^t(\hat{R}_1, R_{-1}) < CV_1(x_1; f_1(R))$. Thus, $f_1(\hat{R}_1, R_{-1}) \, P_1 \, f_1(R)$, contradicting *strategy-proofness*. Therefore, (iv-1): $\{1\} \subsetneq N^+$.

Step 2 (Induction argument). Let $k \geq 1$. As induction hypothesis, we assume that there exist a set $N(k) \supseteq N(1)$ of k distinct agents, say $N(k) \equiv \{1, \ldots, k\}$, and $\hat{R}_{N(k)} \in \mathcal{R}^k$ such that
(i-k) $z_k \, P_k \, f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}})$,
(ii-k) for each $j \in N(k)$ and each $y \in M \setminus \{x_j\}$, $\widehat{CV}_j(y; \mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, y)$,
(iii-k) $t_k < \widehat{CV}_k(x_k; \mathbf{0}) < CV_k(x_k; f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}))$, and
(iv-k) $N(k) \subsetneq N^+$.

See Figure 4 for an illustration of (i-$(k+1)$), (ii-$(k+1)$) and (iii-$(k+1)$) for $k = 1$.

By (iv-k), $N^+ \setminus N(k) \neq \emptyset$. The proof consists of the following two steps.

Step 2-1. *There is $k' \in N^+ \setminus N(k)$ such that $z_{k'} \, P_{k'} \, f_{k'}(\hat{R}_{N(k)}, R_{-N(k)})$.*

Proof of Step 2-1. By contradiction, suppose that for each $j \in N^+ \setminus N(k)$, $f_j(\hat{R}_{N(k)}, R_{-N(k)}) \, R_j \, z_j$.

First, we show that $f_k^x(\hat{R}_{N(k)}, R_{-N(k)}) = x_k$. By (ii-$k$), for each $y \in M \setminus \{x_k\}$,

$$\widehat{CV}_k(y;\mathbf{0}) < C^n(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}, y) = C^{n-1}(\hat{R}_{N(k)}, R_{-N(k)}, y) \leq C^m(\hat{R}_{N(k)}, R_{-N(k)}, y).$$

Let $\hat{z} \in Z$ be such that for each $j \in N \setminus N(k)$, $\hat{z}_j \equiv z_j$, and for each $j \in N(k)$, $\hat{z}_j \equiv \mathbf{0}$. Then, $\hat{z} \in Z^{IR}(\hat{R}_{N(k)}, R_{-N(k)})$. By the supposition of Step 2-1, for each $j \in N^+ \setminus N(k)$, $f_j(\hat{R}_{N(k)}, R_{-N(k)})\, R_j\, z_j \equiv \hat{z}_j$. By *individual rationality*, for each $j \in N(k) \cup (N \setminus N^+)$, $f_j(\hat{R}_{N(k)}, R_{-N(k)})\ R_j\ \mathbf{0} = \hat{z}_j$.

Since $z \in W(R)$, for each $j \in N \setminus N(k)$, $CV_j(x_k; \hat{z}_j) = CV_j(x_k; z_j) \leq t_k$. By (ii-$k$), for each $j \in N(k) \setminus \{k\}$,

$$\widehat{CV}_j(x_k; \hat{z}_j) = \widehat{CV}_j(x_k; \mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, x_k) \leq C^n(R, x_k) \leq t_k.$$

Thus, by (iii-k), $C^1(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)}, x_k; \hat{z}) \leq t_k < \widehat{CV}_k(x_k; \mathbf{0})$.

Since the assumptions of Lemma B.7 hold for the profile $(\hat{R}_{N(k)}, R_{-N(k)})$ as above, Lemma B.7 implies that $f_k^x(\hat{R}_{N(k)}, R_{-N(k)}) = x_k$.

By *individual rationality*, $f_k^t(\hat{R}_{N(k)}, R_{-N(k)}) \leq \widehat{CV}_k(x_k; \mathbf{0})$. However, (iii-$k$) implies that $f_k^t(\hat{R}_{N(k)}, R_{-N(k)}) < CV_k(x_k; f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}))$.

Thus, $f_k(\hat{R}_{N(k)}, R_{-N(k)})\, P_k\, f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}})$, contradicting *strategy-proofness*. □

Step 2-2. *We complete the proof of Claim B.1.*

Proof of Step 2-2. Without loss of generality, let $k+1 \equiv k'$ and $N(k+1) \equiv N(k) \cup \{k+1\}$. Then, $N(k+1) \supsetneq N(k)$, and (i-$(k+1)$) follow from $z_{k'}\ P_{k'}\ f_{k'}(\hat{R}_{N(k)}, R_{-N(k)})$. By (i-$(k+1)$), $t_{k+1} < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)}))$. Also, for each $y \in M$, $C^n(\hat{R}_{N(k)}, R_{-N(k)}, y) > 0$. Thus, there is $\hat{R}_{k+1} \in \mathcal{R}$ such that

$$t_{k+1} < \widehat{CV}_{k+1}(x_{k+1}; \mathbf{0}) < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)})),$$

and for each $y \in M \setminus \{x_{k+1}\}$, $\widehat{CV}_{k+1}(y; \mathbf{0}) < C^n(\hat{R}_{N(k)}, R_{-N(k)}, y)$. Let $\hat{R}_{N(k+1)} \equiv (\hat{R}_{N(k)}, \hat{R}_{k+1})$. Then, (ii-$(k+1)$) and (iii-$(k+1)$) follow from (ii-$k$).

By (iv-k) and $\{k+1\} \subseteq N^+$, $N(k+1) \subseteq N^+$.

Finally, we show (iv-$(k+1)$): $N(k+1) \subsetneq N^+$. Suppose that $N(k+1) = N^+$. Then, $\#N(k+1) = \#N^+ = m$. Thus, for each $j \in N \setminus N(k+1)$, $z_j = \mathbf{0}$.

By (ii-$(k+1)$), for each $y \in M \setminus \{x_{k+1}\}$,

$$\widehat{CV}_{k+1}(y; \mathbf{0}) < C^n(\hat{R}_{N(k)}, R_{-N(k)}, y) = C^{n-1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}, y) \leq C^m(\hat{R}_{N(k+1)}, R_{-N(k+1)}, y).$$

Let $\hat{z} \in Z$ be such that for each $j \in N$, $\hat{z}_j \equiv \mathbf{0}$. Then, $\hat{z} \in Z^{IR}(\hat{R}_{N(k+1)}, R_{-N(k+1)})$.

By *individual rationality*, for each $j \in N \setminus \{k+1\}$, $f_j(\hat{R}_{N(k+1)}, R_{-N(k+1)})\, R_j\, \mathbf{0} = \hat{z}_j$. Since $z \in W(R)$, for each $j \in N \setminus N(k+1)$, $CV_j(x_{k+1}; \hat{z}_j) = CV_j(x_{k+1}; z_j) \leq t_{k+1}$. By (ii-$(k+1)$), for each $j \in N(k+1) \setminus \{k+1\}$,

$$\widehat{CV}_j(x_{k+1}; \hat{z}_j) = \widehat{CV}_j(x_{k+1}; \mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, x_{k+1}) \leq C^n(R, x_{k+1}) \leq t_{k+1}.$$

Thus, by (iii-$(k+1)$), $\widehat{CV}_{k+1}(x_{k+1};\mathbf{0}) > t_{k+1} \geq C^1(\hat{R}_{N(k)}, R_{-N(k+1)}, x_{k+1}; \hat{z})$, and the assumptions of Lemma B.7 hold for the profile $(\hat{R}_{N(k+1)}, R_{-N(k+1)})$. Lemma B.7 implies that $f^x_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}) = x_{k+1}$.

By *individual rationality*, $f^t_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}) \leq \widehat{CV}_{k+1}(x_{k+1};\mathbf{0})$. However, by (iii-$(k+1)$), $f^t_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}) < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)}))$.

Thus, $f_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)})\ P_{k+1}\ f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)})$, contradicting *strategy-proofness.* □

Part 3: Proofs of Lemmas B.8–B.11.

Given $z \in Z(R)$, let $\mathcal{R}^I(z)$ be the set of preferences $R_i \in \mathcal{R}$ such that for each $i, j \in N$, $z_i\ I_i\ z_j$, that is, all the assignments under z are indifferent.

Lemma B.8. *Let $R \in \mathcal{R}^n$, $(z^*, p) \in W_{\min}(R)$, $N' \subseteq N$, $\hat{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$, and $\hat{R} \equiv (\hat{R}_{N'}, R_{-N'})$. Then, (a) $z^* \in Z_{\min}(\hat{R})$, (b) for each $i \in N$, $f_i(\hat{R})\,\hat{R}_i\, z^*_i$ and (c) for each $i \in N$, if $f^x_i(\hat{R}) = 0$, then $0 \in D(\hat{R}_i, p)$.*

Proof of Lemma B.8. First, we show (a). Let $M' \subseteq M$. Since $z^* \in Z_{\min}(R)$, it follows from Theorem A.1 that (i) $\#\{i \in N : D(R_i, p) \subseteq M'\} \leq \#M'$ and (ii) $\#\{i \in N : D(R_i, p) \cap M' \neq \emptyset\} > \#M'$. Note that for each $i \in N'$, $D(\hat{R}_i, p) = L$ and for each $j \in N \setminus N'$, $D(\hat{R}_j, p) = D(R_j, p)$. Thus, for each $i \in N'$, $D(\hat{R}_i, p) \not\subseteq M'$ and $D(\hat{R}_i, p) \cap M' \neq \emptyset$. Then,

$$\#\{i \in N : D(\hat{R}_i, p) \subseteq M'\} \leq \#\{i \in N : D(R_i, p) \subseteq M'\} \leq \#M', \text{ and}$$
$$\#\{i \in N : D(\hat{R}_i, p) \cap M' \neq \emptyset\} \geq \#\{i \in N : D(R_i, p) \cap M' \neq \emptyset\} > \#M'.$$

That is, no set of objects is overdemanded nor weakly underdemanded at p for $\hat{R}$. Thus, (a) follows from Theorem 3.1. Then, (b) also follows from Proposition B.1.

Finally, we show (c). Let $i \in N$. By contradiction, suppose that $f^x_i(\hat{R}) = 0$ and $0 \notin D(\hat{R}_i, p)$. Then, by Lemma B.1, $z^*_i\ \hat{P}_i\ \mathbf{0} = f_i(\hat{R})$. This contradicts (b). □

Given $p \in \mathbb{R}^m_{++}$ and $R \in \mathcal{R}^n$, let $N(R, p)$ denote the set of demanders of the non-null objects at the price p, that is, $N(R, p) \equiv \{i \in N : D(R_i, p) \cap M \neq \emptyset\}$.

Lemma B.9. *Let $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R)$. Let $N' \subseteq N$ with $1 \leq \#N' \leq m$, $\bar{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$, $\bar{R} \equiv (\bar{R}_{N'}, R_{-N'})$ and $N'' \equiv N(R, p) \setminus N'$. Assume that (9-i) for each $i \in N \setminus N'$, and each $x \in M$, if $f^x_i(\bar{R}) = x$, then $f^t_i(\bar{R}) \geq p^x$, and (9-ii) for each $j \in N'$, $f^x_j(\bar{R}) \neq 0$. Then,*
(9-a) for each $j \notin N(R, p) \cup N'$, $f^x_j(\bar{R}) = 0$, and
(9-b) there is a sequence $\{i_k\}^K_{k=1}$ of K distinct agents such that (i) $K \in \{2, \ldots, m+1\}$,
(ii) $f^x_{i_1}(\bar{R}) = 0$, (iii) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and
(iv) for each $k \in \{1, \ldots, K-1\}$, $\{f^x_{i_k}(\bar{R}), f^x_{i_{k+1}}(\bar{R})\} \subseteq D(R_{i_k}, p)$.

See Figure 5 for an illustration of (9-b).

Proof of (9-a) of Lemma B.9. Suppose that for some $j \notin N(R, p) \cup N'$, $x \equiv f^x_j(\bar{R}) \neq 0$. By $D(\bar{R}_j, p) = 0$, *individual rationality* implies $f^t_j(\bar{R}) \leq CV_j(x; \mathbf{0}) < p^x$. This contradicts (9-i).

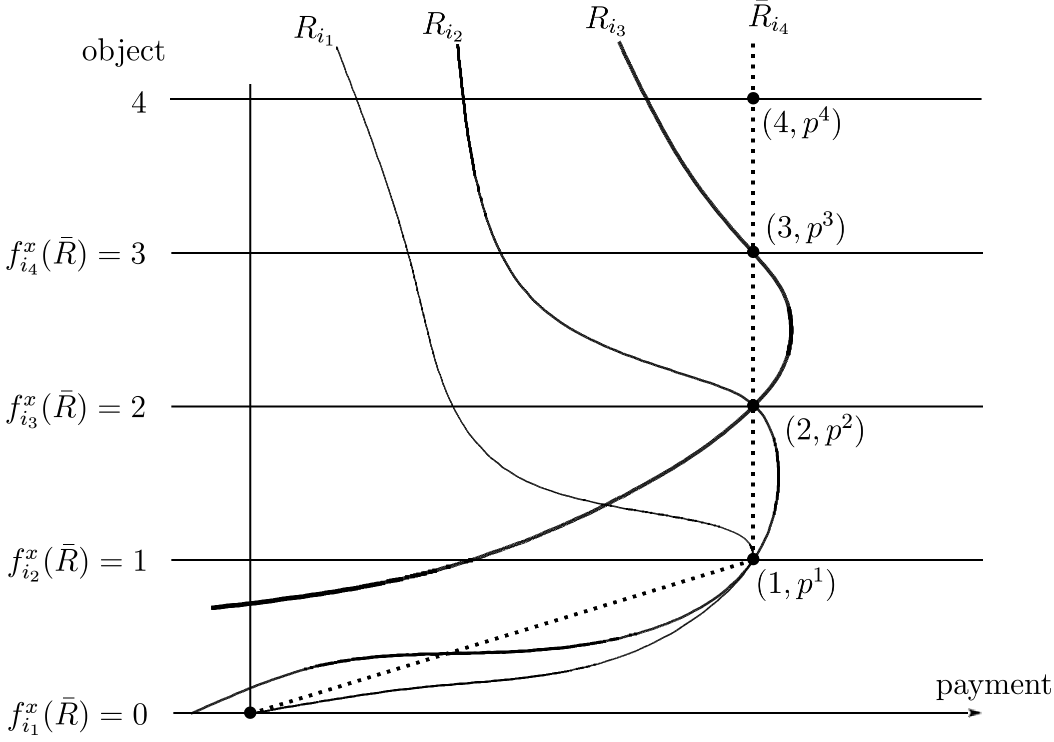

Figure 5: Illustration of (9-b) of Lemma B.9 for $K = 4$.

Proof of (9-b) of Lemma B.9. Let $N''_1 \equiv \{i \in N'' : f^x_i(\bar{R}) = 0\}$. We show that (9-1-b): $N''_1 \neq \emptyset$. Since $N'' \equiv N(R,p) \setminus N'$, $N'' \cup N' \supseteq N(R,p)$. Thus, $\#N'' + \#N' \geq \#N(R,p)$. By Lemma B.8-(a), $z^* \in Z_{\min}(\bar{R})$. Thus, by Theorem 3.1, there is no weakly underdemanded set at p for $\bar{R}$, and so, $\#N(R,p) \geq m+1$. Therefore, $\#N'' + \#N' \geq m+1$. By (9-ii), for each $j \in N'$, $f^x_j(\bar{R}) \neq 0$. Thus, at least one agent in N'' receives no object, that is, (9-1-b) holds.

Since $N''_1 \subseteq N(R,p)$, for each $i \in N''_1$, $D(R_i,p) \cap M \neq \emptyset$. Thus, by (9-1-b), we have (9-1-d): there is $i_1 \in N''_1$ such that $D(R_{i_1},p) \cap M \neq \emptyset$.

Let $N(1) \equiv N''_1$ and $D_1 \equiv [\bigcup_{i \in N(1)} D(R_i,p)] \setminus \{0\}$. Given $k \geq 2$, let $N''_k \equiv \{j \in N'' \setminus N(k-1) : f^x_j(\bar{R}) \in D_{k-1}\}$, $N(k) \equiv N(k-1) \cup N''_k$, and $D_k \equiv [\bigcup_{j \in N''_k} D(R_j,p)] \setminus [\bigcup_{j \in N(k-1)} D(R_j,p)]$.

We introduce Claim B.2 below to show (9-b) inductively. Note that Assumptions (9-($k-1$)-b) and (9-($k-1$)-d) of Claim B.2 follow from (9-1-b) and (9-1-d) when $k = 2$, that (9-k-b) implies $N(k) \supsetneq N(k-1)$, and that Assumptions except for (9-($k-1$)-a*) hold recursively. Thus, for any $k \geq 2$, as long as (9-($k-1$)-a*) holds, Claim B.2 is applied and $N(k)$ increases as k increases. Since $N(k) \subseteq N''$ and N'' is finite,[35] there is $k \leq m$ such that (9-k-a*) does not hold. Let k be the first number that violates (9-k-a*) in this iteration.

By (9-k-b), for each $k' \in \{1, \ldots, k\}$, $N''_{k'} \neq \emptyset$. Since (9-k-a*) does not hold, there are $j_k \in N''_k$ and $j_{k+1} \in N'$ such that $f^x_{j_{k+1}}(\bar{R}) \in D(R_{j_k},p)$. Then,

[35]By (9-ii) of Lemma B.9 and feasibility of object allocation, it should be $\#N'' \leq m$.

for each $k' \in \{1, \ldots, k-1\}$, there is $j_{k'} \in N''_{k'}$ such that $f^x_{j_{k'+1}}(\bar{R}) \in D(R_{j_{k'}}, p)$.

To show that the sequence $\{j_{k'}\}_{k'=1}^{k+1}$satisfies (iv) of (9-b), we prove

for each $k' \in \{1, \ldots, k\}$, $f^x_{j_{k'}}(\bar{R}) \in D(R_{j_{k'}}, p)$.

By $j_1 \in N''_1$, $f^x_{j_1}(\bar{R}) = 0$. Then, by Lemma B.8-(c), $f^x_{j_1}(\bar{R}) \in D(R_{j_1}, p)$. Let $k' \in \{2, \ldots, k\}$. By contradiction, suppose that $f^x_{j_{k'}}(\bar{R}) \notin D(R_{j_{k'}}, p)$. Let $y \equiv f^x_{j_{k'}}(\bar{R})$. Then, by $x^*_{j_{k'}} \in D(R_{i_{k'}}, p)$, $z^*_{j_{k'}}\ P_{i_{k'}}\ (y, p^y)$. By Lemma B.8-(b), $f_{j_{k'}}(\bar{R})\ R_{j_{k'}}\ z^*_{j_{k'}}$. Thus, $f_{j_{k'}}(\bar{R})\ R_{j_{k'}}\ z^*_{j_{k'}}\ P_{i_{k'}}\ (y, p^y)$, which implies $f^t_{j_{k'}}(\bar{R}) < p^y$. This contradicts (9-i) of Lemma B.9.

Then, the sequence $\{j_{k'}\}_{k'=1}^{k+1}$ satisfies (i), (ii), (iii), and (iv) of (9-b). Thus, for the rest of the proof of (9-b), we prove Claim B.2 below.

Claim B.2. *Let $k \geq 2$. Assume that*
(9-$(k-1)$-a) for each $i \in N(k-2)$ and each $j \in N'$, $f^x_j(\bar{R}) \notin D(R_i, p)$,[36]
(9-$(k-1)$-b) for each $k' \in \{1, \ldots, k-1\}$, $N''_{k'} \neq \emptyset$,
(9-$(k-1)$-c) for each $k' \in \{2, \ldots, k-1\}$, $\#N''_{k'} = \#D_{k'-1}$,[37]
(9-$(k-1)$-d) there is $i_{k-1} \in N''_{k-1}$ such that $D(R_{i_{k-1}}, p) \cap [M \setminus \bigcup_{k' \leq k-2} D_{k'}] \neq \emptyset$,[38] *and*
(9-$(k-1)$-a) for each $i \in N''_{k-1}$ and each $j \in N'$, $f^x_j(\bar{R}) \notin D(R_i, p)$.*
Then,
(9-k-a) for each $i \in N(k-1)$ and each $j \in N'$, $f^x_j(\bar{R}) \notin D(R_i, p)$,
(9-k-b) for each $k' \in \{1, \ldots, k\}$, $N''_{k'} \neq \emptyset$,
(9-k-c) for each $k' \in \{2, \ldots, k\}$, $\#N''_{k'} = \#D_{k'-1}$, and
(9-k-d) there is $i_k \in N''_k$ such that $D(R_{i_k}, p) \cap [M \setminus \bigcup_{k' \leq k-1} D_{k'}] \neq \emptyset$.

Proof of Claim B.2. First, (9-k-a) follows from (9-$(k-1)$-a) and (9-$(k-1)$-a*). By (9-$(k-1)$-d), there is $i_{k-1} \in N''_{k-1}$ such that $D(R_{i_{k-1}}, p) \cap [M \setminus \bigcup_{k' \leq k-2} D_{k'}] \neq \emptyset$. Thus, $D_k \neq \emptyset$. By Lemma B.2, for each $x \in D_k$, there is $i(x) \in N$ such that $f^x_{i(x)}(\bar{R}) = x$. Note that, by (9-a), $i(x) \in N(R, p) \cup N'$. By (9-k-a) and the definition of $N(k-1)$, for each $x \in D_k$, $i(x) \in N'' \setminus N(k-1)$. Thus, $N''_k \neq \emptyset$. Then, (9-k-b) follows from (9-$(k-1)$-b).

Since $f^x(\bar{R}) \in X$, no two agents receive the same object, *i.e.*, for each $x, y \in D_k$ with $x \neq y$, $i(x) \neq i(y)$. Thus, $\#N''_k = \#D_{k-1}$. Then, (9-k-c) also follows from (9-$(k-1)$-c).

Finally, we show (9-k-d). By contradiction, suppose that for each $i \in N''_k$, $D(R_i, p) \cap [M \setminus \bigcup_{k' \leq k-1} D_{k'}] = \emptyset$. See Figure 6 for an illustration of proof of (9-k-d). Then,

[36]Define $N(0) = \emptyset$. When $k = 2$, (9-$(k-1)$-a) holds vacantly.
[37]When $k = 2$, (9-$(k-1)$-c) holds vacantly.
[38]When $k = 2$, (9-$(k-1)$-d) requires that there is $i_1 \in N''_1$ such that $D(R_{i_1}, p) \cap M \neq \emptyset$.

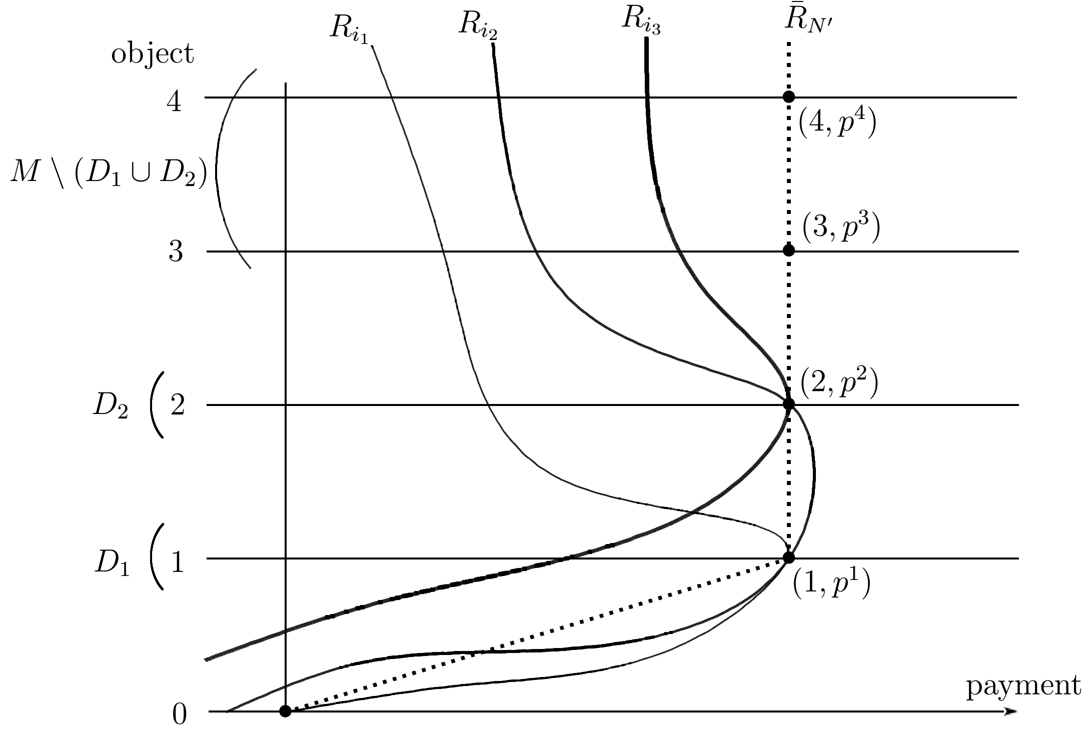

Figure 6: Illustration of (9-k-d) of Lemma B.9 for the case of $k = 3$, $m = 4$, $n = 5$, $N_1'' \equiv \{i_1\}$, $N_2'' \equiv \{i_2\}$, $N_3'' \equiv \{i_3\}$, and $N' \equiv N \setminus \{i_1, i_2, i_3\}$. In this case, $D_1 = \{1\}$, $D_2 = \{2\}$, and $\{j \in N : D(\bar{R}_j, p) \cap [M \setminus (D_1 \cup D_2)] \neq \emptyset\} = N'$.

$$
\begin{aligned}
\#\left\{ j \in N : D(\bar{R}_j, p) \cap \left[M \setminus \bigcup_{k' \leq k-1} D_{k'} \right] \neq \emptyset \right\} &= \#N' + \#N'' - \#N_1'' - \sum_{k'=2}^{k} \#N_{k'}'' \\
&= \#M - \sum_{k'=2}^{k} \#D_{k'-1} \\
&= \#\left\{ M \setminus \bigcup_{k' \leq k-1} D_{k'} \right\},
\end{aligned}
$$

where the first equality follows from $\#\left\{j \in N : D(\bar{R}_j, p) \cap M \neq \emptyset\right\} = \#N' + \#N''$, and for each $k' \in \{1, \ldots, k\}$ and each $i \in N_{k'}''$, $D(R_i, p) \cap \left[M \setminus \bigcup_{k'' \leq k-1} D_{k''}\right] = \emptyset$, and the second from $\#N' + \#N'' - \#N_1'' = m$ and (9-k-c): for each $k' \in \{2, \ldots, k\}$, $\#N_{k'}'' = \#D_{k'-1}$.

Therefore, the set $\left[M \setminus \bigcup_{k' \leq k-1} D_{k'}\right]$ is weakly underdemanded at p for $\bar{R}$. However, by Lemma B.8-(a), $z^* \in Z_{\min}(\bar{R})$, and so, by Theorem 3.1, there is no weakly underdemanded set at p for $\bar{R}$. This is a contradiction. $\square$

Lemma B.10. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$ be such that $f_i^x(R) = x$ and $CV_i(0; f_i(R)) < 0$. Let $j \in N \setminus \{i\}$. Assume that (10-i) $-CV_i(0; f_i(R)) < CV_j(x; \mathbf{0}) - f_i^t(R)$. Then, $f_j^x(R) \neq 0$.*

Proof of Lemma B.10. Suppose that $f_j^x(R) = 0$. By Lemma B.1, $f_j^t(R) = 0$. By assumption (10-i), $-CV_i(0; f_i(R)) < CV_j(x; \mathbf{0}) - f_i^t(R)$. Then, by Lemma B.3, there is $\hat{z} \in Z$ that Pareto-dominates $f(R)$ at R, which contradicts *efficiency*. $\square$

Lemma B.11. *Let $R \in \mathcal{R}^n$, $(z^*, p) \in W_{\min}(R)$, and $N' \subseteq N$. Assume that (11-i) for each $\bar{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$, each $i \in N \backslash N'$, and each $x \in M$, if $f_i^x(\bar{R}_{N'}, R_{-N'}) = x$, $f_i^t(\bar{R}_{N'}, R_{-N'}) \geq p^x$. Let $\hat{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$. Then, for each $i \in N'$ and each $x \in M$, if $f_i^x(\hat{R}_{N'}, R_{-N'}) = x$, then $f_i^t(\hat{R}_{N'}, R_{-N'}) \geq p^x$.*

Proof of Lemma B.11. Let $\hat{R} \equiv (\hat{R}_{N'}, R_{-N'})$. Without loss of generality, let $N' \equiv \{1, 2, \ldots, n'\}$. We only show that if $f_1^x(\hat{R}) = x \in M$, $f_1^t(\hat{R}) \geq p^x$ since we can treat similarly the other agents in N'. Let $f_1^x(\hat{R}) \equiv x \in M$. By contradiction, suppose that $f_1^t(\hat{R}) < p^x$. Let $N'' \equiv N(R, p) \setminus N'$.

Case 1. $\#N' \geq m + 1$.

Since $f_1^t(\hat{R}) < p^x$, there is $\bar{R}_1 \in \mathcal{R}_{NCV}(f_1(\hat{R}))$ such that (ii) $-\overline{CV}_1(0; f_1(\hat{R})) < p^x - f_1^t(\hat{R})$. Then, by Lemma B.4, $f_1(\bar{R}_1, \hat{R}_{-1}) = f_1(\hat{R})$. Note that for each $j \in N' \setminus \{1\}$,

$$-\overline{CV}_1(0; f_1(\hat{R})) < p^x - f_1^t(\hat{R}) = \widehat{CV}_j(x; \mathbf{0}) - f_1^t(\hat{R}),$$

where the inequality follows from (ii) and the equality from $\hat{R}_j \in \mathcal{R}^I(z^*)$. Thus, by Lemma B.10, for each $j \in N' \setminus \{1\}$, $f_j^x(\bar{R}_1, \hat{R}_{-1}) \neq 0$. Since $\#N' \geq m + 1$, this is a contradiction.

Case 2. $\#N' \leq m$.

First, we show the following step.

Step 1. *Let $S \subseteq N'$, $\bar{R}_S \in \mathcal{R}^I(z^*)^{\#S}$, and $\bar{R} \equiv (\bar{R}_S, \hat{R}_{-S})$. For each $i \in N'$, let $\overline{x}_i \equiv f_i^x(\bar{R})$. Assume that*
(11-1-i) for each $i \in N'$, $\overline{x}_i \neq 0$,
(11-1-ii) for each $i \in S$ and each $z_i \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$,
(11-1-iii) there is $j \in S$ such that $f_j^t(\bar{R}) < p^{\overline{x}_j}$, and
(11-1-iv) there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m + 1$, (ii*) $f_{i_1}^x(\bar{R}) = 0$, (iii*) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv*) for each $k \in \{1, \ldots, K-1\}$, $\{f_{i_k}^x(\bar{R}), f_{i_{k+1}}^x(\bar{R})\} \subseteq D(\bar{R}_{i_k}, p)$.*
Then, (11-a) $f_{i_K}^t(\bar{R}) < p^{\overline{x}_{i_K}}$, and (11-b) $i_K \notin S$.

Proof of Step 1.
Proof of (11-a). If $i_K = j$, $f_{i_K}^t(\bar{R}) < p^{\overline{x}_{i_K}}$ follows from (11-1-iii). Thus, let $i_K \neq j$. By Lemma B.8-(b), $f_{i_K}^t(\bar{R}) \leq p^{\overline{x}_{i_K}}$. By contradiction, suppose that $f_{i_K}^t(\bar{R}) = p^{\overline{x}_{i_K}}$. Let $z' \in Z$ be such that

$$\begin{aligned}
&z'_j \equiv (0, \overline{CV}_j(0; f_j(\bar{R}))), \\
&z'_{i_K} \equiv (\overline{x}_j, f_j^t(\bar{R}) - \overline{CV}_j(0; f_j(\bar{R}))), \\
&\text{for each } k \in \{1, \ldots, K-1\},\ z'_{i_k} \equiv f_{i_{k+1}}(\bar{R}), \text{ and} \\
&\text{for each } i \in N \setminus (\{i_k\}_{k=1}^K \cup \{j\}),\ z'_i \equiv f_i(\bar{R}).
\end{aligned}$$

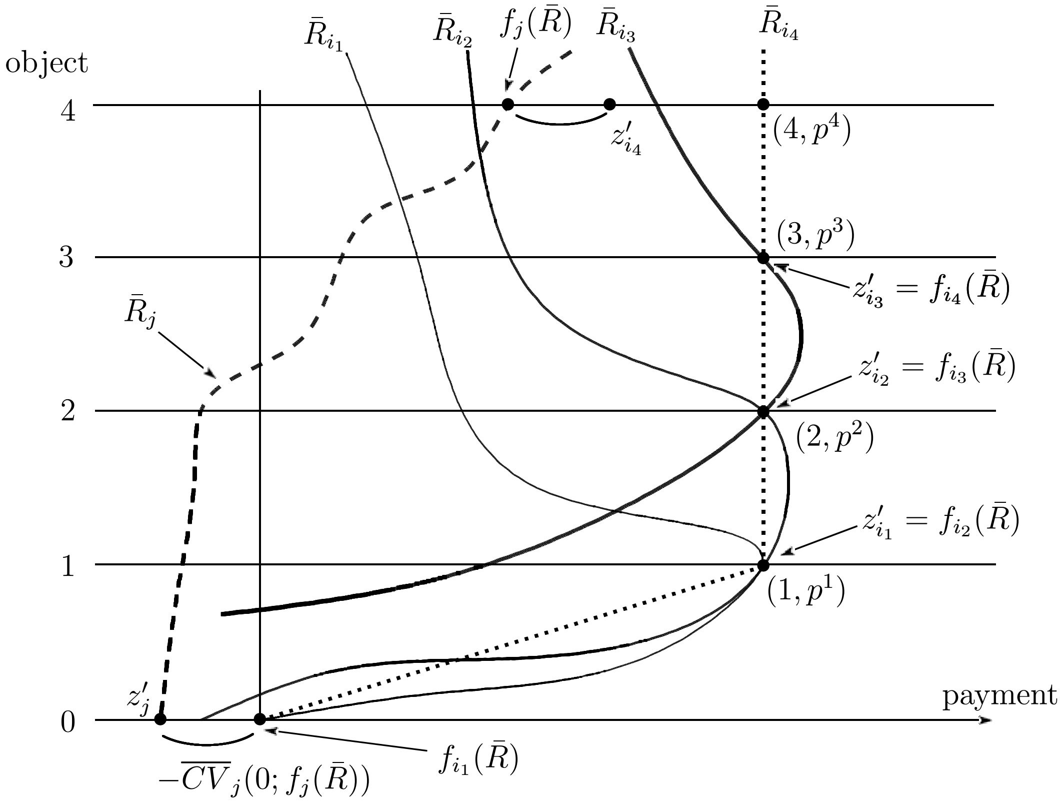

Figure 7: Illustration of z' in (11-a) of Lemma B.11 for $K = 4$.

See Figure 7 for the illustration of z'.

We show z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$. By the definition of $\overline{CV}_j(0; f_j(\bar{R}))$, $z'_j \, \bar{I}_j \, f_j(\bar{R})$. Note that

$$z'_{i_K} \, \bar{P}_{i_K} \, (\overline{x}_j, p^{\overline{x}_j}) \, \bar{I}_{i_K} \, f_{i_K}(\bar{R}),$$

where the first preference relation follows from $z'_{i_K} \equiv (\overline{x}_j, f^t_j(\bar{R}) - \overline{CV}_j(0; f_j(\bar{R})))$, (11-1-iii): $f^t_j(\bar{R}) < p^{\overline{x}_j}$, and (11-1-ii): $-\overline{CV}_j(0; f_j(\bar{R})) < p^{\overline{x}_j} - f^t_j(\bar{R})$, and the indifference relation from $f^t_{i_K}(\bar{R}) = p^{\overline{x}_{i_K}}$ and $i_K \in N'$, which implies $\bar{R}_{i_K} \in \mathcal{R}^I(z^*)$.

Lemma B.8-(b) and (11-i) imply that for each $k \in \{1, \ldots, K-1\}$, $f^t_{i_k}(\bar{R}) = p^{x_{i_k}}$. Thus, by (11-1-iv)-(iv*), for each $k \in \{1, \ldots, K-1\}$, $z'_{i_k} = f_{i_{k+1}}(\bar{R}) \, \bar{I}_{i_k} \, f_{i_k}(\bar{R})$.

For each $i \in N \setminus (\{i_k\}_{k=1}^K \cup \{j\})$, by $z'_i \equiv f_i(\bar{R})$, $z'_i \, \bar{I}_i \, f_i(\bar{R})$.

Note that

$$\sum_{i\in N} t'_i = \overline{CV}_j(0; f_j(\bar{R})) + f^t_j(\bar{R}) - \overline{CV}_j(0; f_j(\bar{R})) + \sum_{k=1}^{K-1} f^t_{i_{k+1}}(\bar{R}) + \sum_{i\in N\setminus(\{i_k\}_{k=1}^K \cup\{j\})} f^t_i(\bar{R})$$
$$= f^t_j(\bar{R}) + \sum_{k=2}^{K} f^t_{i_k}(\bar{R}) + \sum_{i\in N\setminus(\{i_k\}_{k=1}^K \cup\{j\})} f^t_i(\bar{R})$$
$$= \sum_{i\in N} f^t_i(\bar{R}),$$

where the last equality follows from (11-1-iv)-(ii*): $f^t_{i_1}(\bar{R}) = 0$. Thus, z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$, which contradicts *efficiency*. □

Proof of (11-b). By contradiction, suppose that $i_K \in S$. By (11-1-i) and (11-1-iv)-(iii*), $\overline{x}_{i_K} \neq 0$. By Step 1-(11-a), $f^t_{i_K}(\bar{R}) < p^{\overline{x}_{i_K}}$.

Let $z' \in Z$ be such that

$$\begin{aligned}
&z'_{i_K} \equiv (0, \overline{CV}_{i_K}(0; f_{i_K}(\bar{R}))),\\
&z'_{i_{K-1}} \equiv (\overline{x}_{i_K}, f^t_{i_K}(\bar{R}) - \overline{CV}_{i_K}(0; f_{i_K}(\bar{R})))\\
&\text{for each } k \in \{1,\ldots,K-2\},\ z'_{i_k} \equiv f_{i_{k+1}}(\bar{R}),\ \text{and}\\
&\text{for each } i \in N\setminus\{i_k\}_{k=1}^K,\ z'_i \equiv f_i(\bar{R}).
\end{aligned}$$

See Figure 8 for the illustration of z'.

We show z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$. By the definition of $\overline{CV}_j(0; f_j(\bar{R}))$, $z'_{i_K}\, \bar{I}_{I_K}\, f_{i_K}(\bar{R})$. Lemma B.8-(b) and (11-i) imply that for each $k \in \{1,\ldots,K-1\}$, $f^t_{i_k}(\bar{R}) = p^{x_{i_k}}$. By (11-1-iv)-(iv*), for each $k \in \{1,\ldots,K-2\}$, $z'_{i_k} = f_{i_{k+1}}(\bar{R})\, \bar{I}_{i_k}\, f_{i_k}(\bar{R})$.

Note that

$$z'_{i_{K-1}}\, \bar{P}_{i_{K-1}}\, (\overline{x}_{i_K}, p^{\overline{x}_{i_K}})\, \bar{I}_{i_{K-1}}\, (\overline{x}_{i_{K-1}}, p^{\overline{x}_{i_{K-1}}})\, \bar{I}_{i_{K-1}}\, f_{i_{K-1}}(\bar{R}),$$

where the strict preference relation follows from $i_K \in S$, $z'_{i_{K-1}} = (\overline{x}_{i_K}, f^t_{i_K}(\bar{R}) - \overline{CV}_{i_K}(0; f_{i_K}(\bar{R})))$, (11-a): $f^t_{i_K}(\bar{R}) < p^{\overline{x}_{i_K}}$, and (11-1-ii): $-\overline{CV}_{i_K}(0; f_{i_K}(\bar{R})) < p^{\overline{x}_{i_K}} - f^t_{i_K}(\bar{R})$, the first indifference relation from (11-1-iv)-(iv*): $\{\overline{x}_{i_{K-1}}, \overline{x}_{i_K}\} \subseteq D(\bar{R}_{i_{K-1}}, p)$, and the second from $f^t_{i_{K-1}}(\bar{R}) = p^{x_{i_{K-1}}}$.

For each $i \in N\setminus\{i_k\}_{k=1}^K$, by $z'_i = f_i(\bar{R})$, $z'_i\, \bar{I}_i\, f_i(\bar{R})$. Note that

$$\sum_{i\in N} t'_i = \overline{CV}_{i_K}(0; f_{i_k}(\bar{R})) + f^t_{i_K}(\bar{R}) - \overline{CV}_{i_K}(0; f_{i_K}(\bar{R})) + \sum_{k=1}^{K-2} f^t_{i_{k+1}}(\bar{R}) + \sum_{i\in N\setminus(\{i_k\}_{k=1}^K)} f^t_i(\bar{R})$$
$$= f^t_{i_K}(\bar{R}) + \sum_{k=2}^{K-1} f^t_{i_k}(\bar{R}) + \sum_{i\in N\setminus(\{i_k\}_{k=1}^K)} f^t_i(\bar{R})$$
$$= \sum_{i\in N} f^t_i(\bar{R}),$$

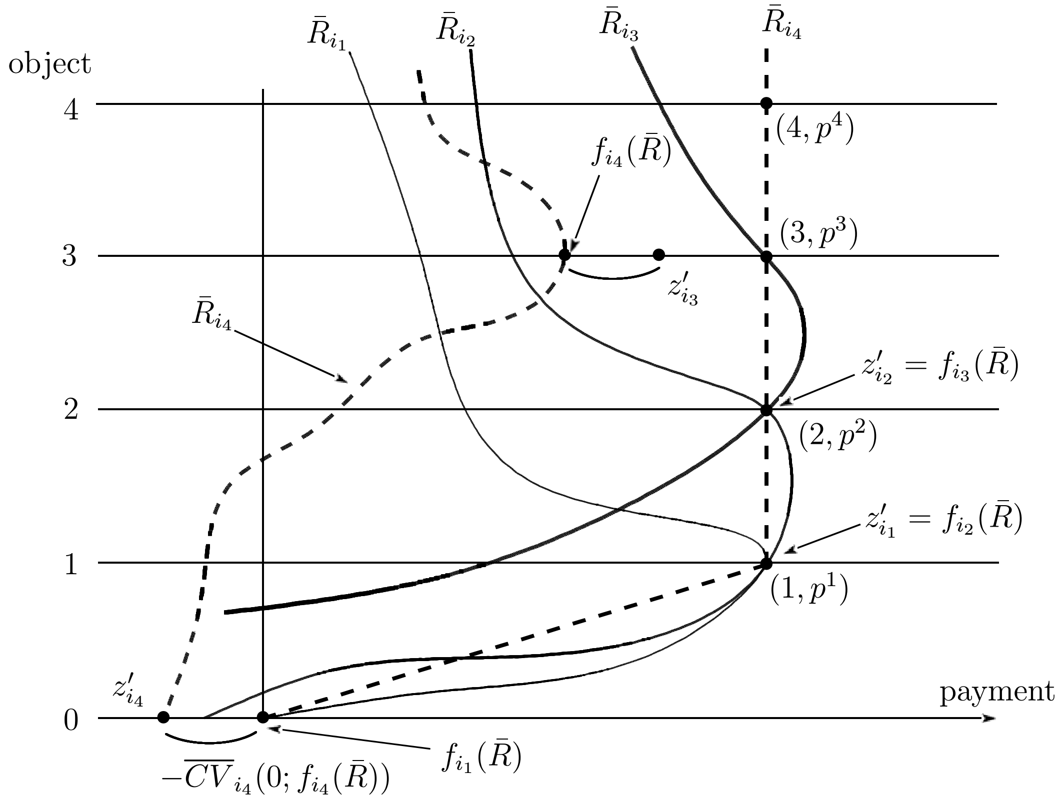

Figure 8: Illustration of z' in (11-b) of Lemma B.11 for $K = 4$.

where the last equality follows from (11-1-iv)-(ii*): $f^t_{i_1}(\bar{R}) = 0$. Thus, z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$, which contradicts *efficiency*. $\square$

Step 2. *We derive a contradiction to conclude that $f^t_1(\hat{R}) \geq p^x$.*

Since $f^t_1(\hat{R}) < p^x$, there is $\bar{R}_1 \in \mathcal{R}^I(z^*) \cap \mathcal{R}_{NCV}(f_1(\hat{R}))$ such that

(11-1-a) for each $z_1 \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $\;-\overline{CV}_1(0; z_1) < p^y - t$.

Then, by $\bar{R}_1 \in \mathcal{R}_{NCV}(f_1(\hat{R}))$ and Lemma B.4, $f_1(\bar{R}_1, \hat{R}_{-1}) = f_1(\hat{R})$. Thus,

(11-1-b) $f^x_1(\bar{R}_1, \hat{R}_{-1}) = x \in M$ and $f^t_1(\bar{R}_1, \hat{R}_{-1}) < p^x$.

Note that $\{1\} \subseteq N'$. Suppose that $\{1\} = N'$. Since $f_1(\bar{R}_1, \hat{R}_{-1}) = f_1(\hat{R})$ and $f^x_1(\hat{R}) = x \neq 0$, $f^x_1(\bar{R}_1, \hat{R}_{-1}) = x \neq 0$. Then, by (11-i) of Lemma B.11, it follows from (9-b) of Lemma B.9 that there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m + 1$, (ii) $f^x_{i_1}(\bar{R}_1, \hat{R}_{-1}) = 0$, (iii) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv) for each $k \in \{1, \ldots, K-1\}$, $\{f^x_{i_k}(\bar{R}_1, \hat{R}_{-1}), f^x_{i_{k+1}}(\bar{R}_1, \hat{R}_{-1})\} \subseteq D(\hat{R}_{i_k}, p)$. Then, by Step 1 (11-b), $i_K \notin \{1\}$. Since $\{1\} = N'$, $i_K \notin N'$, which contradicts (iii): $i_K \in N'$. Thus, if $\{1\} = N'$, we obtain a contradiction. Therefore, we assume that

(11-1-c) $\{1\} \subsetneq N'$.

Induction argument:

Let $s \geq 1$ and $N(1) \equiv \{1\}$. As induction hypothesis, we assume that there exist a set $N(s) \supseteq N(1)$ of s distinct agents and $\bar{R}_{N(s)} \in \mathcal{R}^I(z^*)^s$ such that

(11-s-a) for each $i \in N(s)$ and each $z_i \equiv (y,t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$,

(11-s-b) for some $j \in N(s)$, $f_j^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) \equiv x' \in M$ and $f_j^t(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) < p^{x'}$, and

(11-s-c) $N(s) \subsetneq N'$.

Note that (11-s-a), (11-s-b), and (11-s-c) follow from (11-1-a), (11-1-b), and (11-1-c) if $s = 1$.

We show that there exist a set $N(s+1) \supsetneq N(s)$ of $s+1$ distinct agents and $\bar{R}_{N(s+1)} \in \mathcal{R}^I(z^*)^{s+1}$ such that

(11-$(s+1)$-a) for each $i \in N(s+1)$ and each $z_i \equiv (y,t) \in M \times \mathbb{R}$ with $t < p^y$,

$$-\overline{CV}_i(0; z_i) < p^y - t, \text{ and}$$

(11-$(s+1)$-b) for some $j' \in N(s+1)$,

$$f_{j'}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) \equiv x'' \in M \text{ and } f_{j'}^t(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) < p^{x''}.$$

First, we show (11-$(s+1)$-a). Since $(\bar{R}_{N(s)}, \hat{R}_{-N' \setminus N(s)}) \in \mathcal{R}^I(z^*)^{\#N'}$, (11-$s$-b) and Lemma B.10 imply that

(B-1) for each $i \in N'$, $f_i^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) \neq 0$.

Then, by (11-i) of Lemma B.11, it follows from (9-b) of Lemma B.9 that there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m+1$, (ii) $f_{i_1}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) = 0$, (iii) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv) for each $k \in \{1, \ldots, K-1\}$, $\{f_{i_k}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}), f_{i_{k+1}}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)})\} \subseteq D(\hat{R}_{i_k}, p)$. Let $x_{i_K} \equiv f_{i_K}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)})$.
Then, by Step 1-(11-a),

(B-2) $f_{i_K}^t(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) < p^{x_{i_K}}$.

Also, by Step 1-(11-b),

(B-3) $i_K \in N' \setminus N(s)$.

Next, let $j' \equiv i_K$ and $N(s+1) \equiv N(s) \cup \{j'\}$. Then, by (B-3), $N(s+1) \supsetneq N(s)$. Also, (B-1) and (B-2) imply that $f_{i_K}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) \neq 0$ and $f_{j'}^t(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) < p^{x_{j'}}$. Thus, there is $\bar{R}_{j'} \in \mathcal{R}^I(z^*) \cap \mathcal{R}_{NCV}(f_{j'}(\bar{R}_{N(s)}, \hat{R}_{-N(s)}))$ such that

$$\text{for each } z_{j'} \equiv (y,t) \in M \times \mathbb{R} \text{ with } t < p^y, \ -\overline{CV}_{j'}(0; z_{j'}) < p^y - t,$$

Thus, (11-$(s+1)$-a) follows from (11-s-a).

Next, we show (11-$(s+1)$-b). By $\bar{R}_{j'} \in \mathcal{R}_{NCV}(f_{j'}(\bar{R}_{N(s)}, \hat{R}_{-N(s)}))$, Lemma B.4 implies that $f_{j'}(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) = f_{j'}(\bar{R}_{N(s)}, \hat{R}_{-N(s)})$. Then, by (B-1), $f_{j'}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) \neq 0$. By (B-2), $f_{j'}^t(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) < p^{x_{j'}}$. Thus, (11-$(s+1)$-b) holds.

Since $N(s) \subsetneq N'$ and $j' \in N'$, $N(s+1) \subseteq N'$. Suppose that $N(s+1) = N'$. Since $(\bar{R}_{N(s+1)}, \hat{R}_{-N' \setminus N(s+1)}) \in \mathcal{R}^I(z^*)^{\# N'}$, (11-$(s+1)$-b) and Lemma B.10 imply that

(B-4) for each $i \in N'$, $f_i^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) \neq 0$.

Then, by (11-i) of Lemma B.11, it follows from (9-b) of Lemma B.9 that there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m+1$, (ii) $f_{i_1}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) = 0$, (iii) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv) for each $k \in \{1, \ldots, K-1\}$, $\{f_{i_k}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}), f_{i_{k+1}}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)})\} \subseteq D(\hat{R}_{i_k}, p)$.

Then, by Step 1-(11-b), $i_K \notin N(s+1)$. Since $N(s+1) = N'$, $i_K \notin N'$, which contradicts (iii): $i_K \in N'$. Thus, if $N(s+1) = N'$, we obtain a contradiction.

If $N(s+1) \subsetneq N'$, we obtain a contradiction by repeating the induction argument $(\# N' - \# N(s+1))$ times. □

Part 4: Proof of Theorem 4.1.

Proof of Theorem 4.1. Let $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R)$. By Lemma B.5, for each $\bar{R} \in \mathcal{R}^I(z^*)^n$, each $i \in N$, and each $x \in M$, if $f_i^x(\bar{R}) = x$, then, $f_i^t(\bar{R}) \geq p^x$. Next, we prove the following claim.

Claim B.3. *Let $k \in \{1, \ldots, n\}$ and $N_k \subseteq N$ be such that $\# N_k = k$. Then, for each $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\# N \setminus N_k}$, each $i \in N$, and each $x \in M$, if $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$, then, $f_i^t(R_{N_k}, \bar{R}_{-N_k}) \geq p^x$.*

Proof of Claim B.3. We prove Claim B.3 by induction on k. Let $k = 1$. Let $N_1 \subseteq N$ with $\# N_1 = 1$. Let $\bar{R}_{-N_1} \in \mathcal{R}^I(z^*)^{\# N \setminus N_1}$, $i \in N_1$, and $x \in M$ be such that $f_i^x(R_{N_1}, \bar{R}_{-N_1}) = x$. Suppose that $f_i^t(R_{N_1}, \bar{R}_{-N_1}) < p^x$. Let $\bar{R}_i \in \mathcal{R}^I(z^*)$ and $\hat{x} \equiv f_i^x(\bar{R})$. Then, since $f_i^t(\bar{R}) \geq p^{\hat{x}}$, $f_i(R_{N_1}, \bar{R}_{-N_1}) \, \bar{P}_i \, f_i(\bar{R})$, contradicting *strategy-proofness*. Thus, for each $\bar{R}_{-N_1} \in \mathcal{R}^I(z^*)^{\# N \setminus N_1}$, each $i \in N_1$, and each $x \in M$, if $f_i^x(R_{N_1}, \bar{R}_{-N_1}) = x$, then, $f_i^t(R_{N_1}, \bar{R}_{-N_1}) \geq p^x$. Then, it follows from Lemma B.11 that for each $\bar{R}_{-N_1} \in \mathcal{R}^I(z^*)^{\# N \setminus N_1}$, each $i \in N \setminus N_1$, and each $x \in M$, if $f_i^x(R_{N_1}, \bar{R}_{-N_1}) = x$, then, $f_i^t(R_{N_1}, \bar{R}_{-N_1}) \geq p^x$.

Let $k \in \{2, \ldots, n\}$. As induction hypothesis, we assume that

B.3.1: *for each $N_{k-1} \subseteq N$ with $\# N_{k-1} = k-1$, each $\bar{R}_{-N_{k-1}} \in \mathcal{R}^I(z^*)^{\# N \setminus N_{k-1}}$, each $i \in N$, and each $x \in M$, if $f_i^x(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) = x$, then, $f_i^t(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) \geq p^x$.*

Let $N_k \subseteq N$ be such that $\# N_k = k$. Let $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\# N \setminus N_k}$, $i \in N_k$ and $x \in M$ be such that $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$. Suppose that $f_i^t(R_{N_k}, \bar{R}_{-N_k}) < p^x$. Let $N_{k-1} \equiv N_k \setminus \{i\}$. Let $\hat{x} \equiv f_i^x(R_{N_{k-1}}, \bar{R}_{-N_{k-1}})$. Then, by induction hypothesis (B.3.1), $f_i^t(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) \geq p^{\hat{x}}$. Thus, $f_i(R_{N_k}, \bar{R}_{-N_k}) \, \bar{P}_i \, f_i(R_{N_{k-1}}, \bar{R}_{-N_{k-1}})$, which contradicts *strategy-proofness.* Thus, for each $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\# N \setminus N_k}$, each $i \in N_k$, and each $x \in M$, if $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$, then, $f_i^t(R_{N_k}, \bar{R}_{-N_k}) \geq p^x$. Then, it follows from Lemma B.11 that for each $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\# N \setminus N_k}$, each $i \in N \setminus N_k$, and each $x \in M$, if $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$, then, $f_i^t(R_{N_k}, \bar{R}_{-N_k}) \geq p^x$. □

We show that $f(R)$ satisfies (WE-i) in Definition 3.1. Let $i \in N$ and $y \equiv f_i^x(R)$. By Proposition B.1, $f_i^t(R) \leq CV_i(y; z_i^*)$. By $z_i^* \in D(R_i, p)$, $CV_i(y; z_i^*) \leq p^y$, where $p^y = 0$ if $y = 0$. If $y = 0$, $p^y = 0 = f_i^t(R)$. If $y \neq 0$, by Claim B.3, $p^y \leq f_i^t(R)$. Then, by $f_i^t(R) \leq CV_i(y; z_i^*) \leq p^y \leq f_i^t(R)$. $CV_i(y; z_i^*) = p^y = f_i^t(R)$. Thus, $f_i(R) \, I_i \, z_i^*$. Since $z_i^* \in D(R_i, p)$, for each $z_i' \in B(p)$, $f_i(R) \, I_i \, z_i^* \, R_i \, z_i'$. Thus, for each $i \in N$, $f_i^x(R) \in D(R_i, p)$.

Next, we show that $f(R)$ satisfies (WE-ii) in Definition 3.1. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, for each $x \in M$, $p^x > 0$. By Lemma B.2, for each $x \in M$, there is $i \in N$ such that $f_i^x(R) = x$.

Since $p = p_{\min}(R)$, we conclude that $f(R) \in Z_{\min}(R)$. □

C Proofs for Section 5 (Proposition 5.1)

Proof of Proposition 5.1. Let $\mathcal{R} \subseteq \mathcal{R}^E$ and $R \in \mathcal{R}^n$. Consider a simultaneous ascending (SA) auction defined in Section 5. By the definition of the SA auction, the price path $p(t)$ generated by the SA auction is nondecreasing with respect to time t. Next, for each $x \in M$, let $\bar{p}^x > C^1(R, x)$. Then, each agent demands only the null object at the price vector $\bar{p}$, that is, no overdemanded set exists at $\bar{p}$. Thus, the price path $p(\cdot)$ is bounded above, that is, for each $t \in \mathbb{R}_+$, $p(t) \leq \bar{p}$. Note that the prices are raised at a speed at least $d > 0$. Thus, there is a price vector p^* such that the price path $p(\cdot)$ converges to p^* in a finite time.

Let T be the final time of the SA auction. We show that the final price $p(T) = p_{\min}(R)$. By the definition of SA auctions, no overdemanded set exists at the price $p(T)$. If no weakly underdemanded set exists at $p(T)$, then the desired conclusion follows from Theorem 3.1. Thus, we show that no weakly underdemanded set exists at $p(T)$. The proof consists of the following two steps.

Step 1. *Let $t' \in (0, T]$. Assume that there is a set M' of objects that is weakly underdemanded at $p(t')$. Let $N' \equiv \{i \in N : D(R_i, p(t')) \cap M' \neq \emptyset\}$. Then, (5-a) $\#N' \geq 2$, and (5-b) there exist $t'' \in (0, t')$ and $M'' \subsetneq M'$ such that $N'' \equiv \{i \in N : D(R_i, p(t'')) \cap M'' \neq \emptyset\} \subsetneq N'$ and M'' is underdemanded at $p(t'')$.*

Proof of Step 1. Since M' is weakly underdemanded at $p(t')$, for each $x \in M'$, $p^x(t') > 0$ and $\#N' \leq \#M'$. For each $i \in N$, let $z'_i \equiv (x'_i, t'_i) \in D(R_i, p(t'))$. Note that for each $i \in N \setminus N'$ and each $x \in M'$, $CV_i(x; z'_i) < p^x(t')$. For each $x \in M'$, let $q^x \equiv \max\{CV_j(x; z'_j) : j \in N \setminus N'\} \cup \{0\}$. Let $e > 0$ be such that for each $x \in M'$, $q^x < p^x(t') - e \equiv p^x$. Let $t'' \equiv \max\{t \in \mathbb{R}_+ : \text{for some } x \in M', p^x(t) \leq p^x\}$. Then, there is $x' \in M'$ such that $dp^{x'}(t'')/dt > 0$ and $p^{x'}(t'') = p^{x'}$. Since $dp^{x'}(t'')/dt > 0$, there is a minimal overdemanded set $\hat{M}$ at $p(t'')$ including x'. See Figure 9 for an illustration.

Let $\hat{M}' \equiv \hat{M} \cap M'$. Since $x' \in M'$, $\hat{M}' \neq \emptyset$. Let

$$\hat{N}' \equiv \{i \in N' : D(R_i, p(t'')) \cap \hat{M}' \neq \emptyset \text{ and } D(R_i, p(t'')) \subseteq \hat{M}\}.$$

We show that $\#\hat{N}' > \#\hat{M}'$. If $\hat{M} \subseteq M'$, then $\hat{M}' = \hat{M}$ and for each $i \in \hat{N}'$, $D(R_i, p(t'')) \subseteq \hat{M}'$. Since $\hat{M}$ is an overdemanded set at $p(t'')$, the desired conclusion holds. Thus, we assume that $\hat{M} \not\subseteq M'$. Let $\hat{M}'' \equiv \hat{M} \setminus M'$ and $\hat{N}'' \equiv \{i \in N : D(R_i, p(t'')) \subseteq \hat{M}''\}$. Then,

$$\begin{aligned}
&\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}\} \\
=&\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}''\} \cup \{i \in N : D(R_i, p(t'')) \cap \hat{M}' \neq \emptyset \text{ and } D(R_i, p(t'')) \subseteq \hat{M}\} \\
=&\hat{N}'' \cup \hat{N}',
\end{aligned}$$

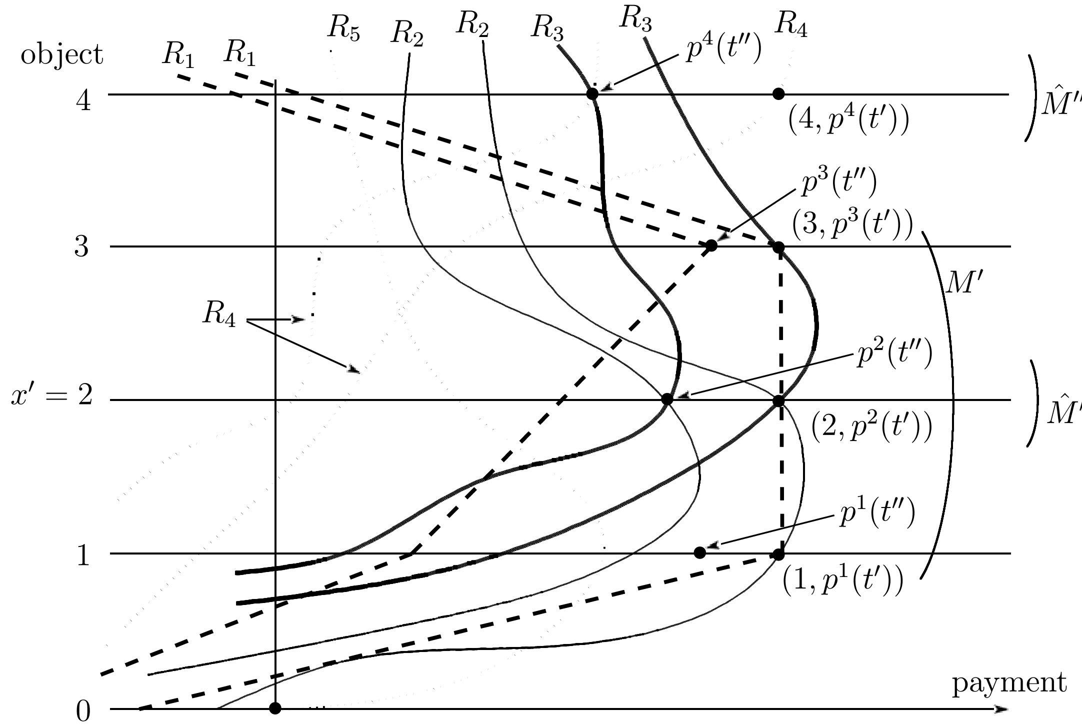

Figure 9: Illustration of proof of Step 1 of Proposition 5.1 for the case of $m = 4$, $M' \equiv \{1, 2, 3\}$, $N' \equiv \{1, 2, 3\}$, $x' \equiv 2$, and $\hat{M}' \equiv \{2, 4\}$. In this case, $\hat{M}' = \{2\}$, $\hat{N}' = \{2, 3\}$, $\hat{M}'' = \{4\}$, $\hat{N}'' = \{4\}$, $M'' = \{1, 3\}$, and $N'' = \{1\}$.

where the first equality follows from $\hat{M}'' \cup \hat{M}' = \hat{M}$ and $\hat{M}'' \cap \hat{M}' = \emptyset$, and the second from the fact that for each $i \in N \setminus N'$, $D(R_i, p(t'')) \cap \hat{M}' = \emptyset$. Note that for each $x \in M'$, $q^x < p^x \leq p^x(t'')$. Thus, for each $i \in N \setminus N'$ and each $x \in M'$,

$$(x'_i, p^{x'_i}(t''))\, R_i\, (x'_i, p^{x'_i}(t'))\, R_i\, (x, q^x)\, P_i\, (x, p^x(t'')).$$

Since $\hat{M}' \subseteq M'$, for each $i \in N \setminus N'$, $D(R_i, p(t'')) \cap \hat{M}' = \emptyset$. Thus, $\hat{N}'' \cap \hat{N}' = \emptyset$. Then,

$$\begin{aligned}\#\hat{N}'' + \#\hat{N}' &= \#\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}\} \\ &> \#\hat{M} \quad (\hat{M} \text{ is an overdemanded set at } p(t'')) \\ &= \#\hat{M}'' + \#\hat{M}'.\end{aligned}$$

Note that $\hat{M}'' \subsetneq \hat{M}$. Since $\hat{M}$ is a minimal overdemanded set at $p(t'')$, $\hat{M}''$ is not overdemanded at $p(t'')$, and so, $\#\hat{N}'' \leq \#\hat{M}''$. This implies that $\#\hat{N}' > \#\hat{M}'$.

We show (5-a). Since $\hat{M}' \neq \emptyset$, $1 \leq \#\hat{M}'$. By $\#\hat{N}' > \#\hat{M}'$ and $\hat{N}' \subseteq N'$, we have $1 \leq \#\hat{M}' < \#\hat{N}' \leq \#N'$, and thus, $\#N' \geq 2$.

Next, we show (5-b). Let $M'' \equiv M' \setminus \hat{M}'$. Since $\hat{M}' \subsetneq M'$,[39] $M'' \neq \emptyset$. By $\hat{M}' \neq \emptyset$, $M'' \subsetneq M'$.

[39]To see this, suppose that $\hat{M}' = M'$. Since M' is weakly underdemanded at $p(t')$, $\#N' \leq \#M'$. By $\hat{M}' = M'$ and $\#\hat{N}' > \#\hat{M}'$, $\#N' \leq \#M' = \#\hat{M}' < \#\hat{N}' \leq \#N'$, which is a contradiction.

First, we show that $N'' \subseteq N' \setminus \hat{N}'$, that is, for each $i \in N''$, $i \in N'$ and $i \notin \hat{N}'$. Let $i \in N''$. Then, $D(R_i, p(t'')) \cap M'' \neq \emptyset$. Since for each $x \in M'$, $q^x < p^x(t'')$ and $M'' \subseteq M'$, for each $j \in N \setminus N'$, $D(R_j, p(t'')) \cap M' = \emptyset$. This implies $i \in N'$. Since $\hat{M}' = M' \cap \hat{M}$ implies $M'' = M' \setminus \hat{M}$, $D(R_i, p(t'')) \cap M'' \neq \emptyset$ implies $D(R_i, p(t'')) \setminus \hat{M} \neq \emptyset$. Since $\hat{N}' \subseteq \{j \in N : D(R_j, p(t'')) \subseteq \hat{M}\}$, this implies $i \notin \hat{N}'$. Thus, $N'' \subseteq N' \setminus \hat{N}'$.

Since $\#\hat{N}' > \#\hat{M}' \geq 1$, $\#\hat{N}' \geq 2$, and so, $N'' \subsetneq N'$. Finally, it follows from the inequalities below that M'' is underdemanded at $p(t'')$.

$$\begin{aligned}
\#N'' &\leq \#N' - \#\hat{N}' \quad && \text{by } \hat{N}' \subseteq N' \\
&< \#N' - \#\hat{M}' && \text{by } \#\hat{N}' > \#\hat{M}' \\
&\leq \#M' - \#\hat{M}' && \text{by } \#N' \leq \#M' \\
&= \#M''.
\end{aligned}$$

□

Step 2. *There is no weakly underdemanded set at $p(T)$.*

Proof of Step 2. By contradiction, suppose that there is a set M_1 of objects that is weakly underdemanded at $p(T)$. Let $N_1 \equiv \{i \in N : D(R_i, p(T)) \cap M_1 \neq \emptyset\}$. Then, by Step 1, $\#N_1 \geq 2$, and there exist $t_1 < T$ and $M_2 \subsetneq M_1$ such that $N_2 \equiv \{i \in N : D(R_i, p(t_1)) \cap M_2 \neq \emptyset\} \subsetneq N_1$ and M_2 is underdemanded at $p(t_1)$. Since M_2 is underdemanded at $p(t_1)$, Step 1 also implies that $\#N_2 \geq 2$, and there exist $t_2 < t_1$ and $M_3 \subsetneq M_2$ such that $N_3 \equiv \{i \in N : D(R_i, p(t_2)) \cap M_3 \neq \emptyset\} \subsetneq N_2$ and M_3 is underdemanded at $p(t_2)$. Repeating this argument inductively, there is a sequence $\{N_k\} \subsetneq N_1$ such that for each $k \geq 2$, $\#N_k < \#N_{k-1}$ and $\#N_k \geq 2$. However, since N_1 is finite and for each $k \geq 2$, $N_k \subsetneq N_1$, this is a contradiction. □

D Proof of Fact 3.4.

The following theorem is useful to prove Fact 3.4.

Hall's Theorem (Hall, 1935). *Let $N \equiv \{1, \ldots, n\}$ and $M \equiv \{1, \ldots, m\}$. For each $i \in N$, let $D_i \subseteq M$. Then, (i) there is a one to one mapping $\hat{x}$ from N to M such that for each $i \in N$, $\hat{x}(i) \in D_i$ if and only if (ii) for each $N' \subseteq N$, $\# \bigcup_{i \in N'} D_i \geq \#N'$.*

Fact 3.4 (Mishra and Talman, 2010). *Let $\mathcal{R} \subseteq \mathcal{R}^E$ and $R \in \mathcal{R}^n$. A price vector p is a Walrasian equilibrium price for R if and only if no set of objects is overdemanded and no set of objects is underdemanded at p for R.*

Proof of Fact 3.4. First, we prove *only if* part of Fact 3.4. Then, we show *if* part.
Proof of "ONLY IF" part. Let $p \in P(R)$. Then, there is an allocation $z = (x_i, t_i)_{i \in N}$ satisfying conditions (WE-i) and (WE-ii) in Definition 3.1. Let $M' \subseteq M$.

We show that M' is not overdemanded at p for R. Let $N' \equiv \{i \in N : D(R_i, p) \subseteq M'\}$. Since for each $i \in N'$, $x_i \in D(R_i, p) \subseteq M'$, and each indivisible object is consumed at most one agent, $\#N' = \#\{x_i : i \in N'\}$. Since $\{x_i : i \in N'\} \subseteq M'$, $\#\{x_i : i \in N'\} \leq \#M'$. Thus, $\#N' \leq \#M'$.

We show that M' is not underdemanded at p for R. Let $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\}$. Suppose that for each $x \in M'$, $p^x > 0$ and $\#N' < \#M'$. Note that $\#N' < \#M'$ implies that there is $x \in M'$ such that for all $i \in N$, $x_i \neq x$. Then, condition (WE-ii) implies that $p^x = 0$. This is a contradiction. Thus, $\#N' \geq \#M'$.

Proof of "IF" part. Assume that no set of objects is overdemanded and no set of objects is underdemanded at p for R.

Let $Z^* \equiv \{z = (x_i, t_i)_{i \in N} \in Z : \forall\, i \in N, x_i \in D(R_i, p) \text{ and } t_i = p^{x_i}\}$. First, we show $Z^* \neq \emptyset$. Suppose that there is $N' \subseteq N$ such that for each $i \in N'$, $0 \notin D(R_i, p)$ and $\#\{\bigcup_{i \in N'} D(R_i, p)\} < \#N'$. Then $\{\bigcup_{i \in N'} D(R_i, p)\}$ is overdemanded at p for R. Thus, for each $N' \subseteq N$, if for each $i \in N'$, $0 \notin D(R_i, p)$, then $\#\{\cup_{i \in N'} D(R_i, p)\} \geq \#N'$. Then, by Hall's Theorem, there is $\bar{z} \in Z$ such that for each $i \in N$, if $0 \notin D(R_i, p)$, then $\bar{x}_i \in D(R_i, p)$ and $\bar{t}_i = p^{\bar{x}_i}$. Thus, $Z^* \neq \emptyset$.

By definition, for each $z \in Z^*$, (z, p) satisfies (WE-i). We show that there is $z \in Z^*$ such that (z, p) satisfies (WE-ii). Let $M^+(p) \equiv \{x \in M : p^x > 0\}$. Let

$$z \in \arg\max_{z' \in Z^*} \#\{y \in M^+(p) : \exists\, i \in N \text{ s.t. } x'_i = y\}, \tag{1}$$

that is, z maximizes over Z^* the number of objects in $M^+(p)$ that are assigned to some agents. Then, by the definition of Z^*, (z, p) satisfies (WE-i).

Let $M^0 \equiv \{y \in M^+(p) : \forall\, i \in N, x_i \neq y\}$. Note that, if $M^0 = \emptyset$, (z, p) also satisfies (WE-ii). Thus, we show that $M^0 = \emptyset$. By contradiction, suppose that $M^0 \neq \emptyset$.

Let $N^0 \equiv \{i \in N : D(R_i, p) \cap M^0 \neq \emptyset\}$. For each $k = 1, 2, \ldots$, let $M^k \equiv \{y \in M : \exists\, i \in N^{k-1} \text{ s.t. } x_i = y\}$ and $N^k \equiv \{i \in N : D(R_i, p) \cap M^k \neq \emptyset\} \setminus \{\bigcup_{k'=0}^{k-1} N^{k'}\}$. We claim by induction that for each $k \geq 0$, $M^k \subseteq M^+(p)$ and $N^k \neq \emptyset$.

Induction argument:

Step 1. By the definition of M^0, $M^0 \subseteq M^+(p)$. Since M^0 is not underdemanded at p for R, $\#N^0 \geq \#M^0$. Thus, $M^0 \neq \emptyset$ implies that $N^0 \neq \emptyset$.

Step 2. Let $K \geq 1$. As induction hypothesis, assume that for each $k \leq K - 1$, $M^k \subseteq M^+(p)$ and $N^k \neq \emptyset$.

First, we show that $M^K \subseteq M^+(p)$. Suppose that there is $x \in M^K \setminus M^+(p)$. Then, $p^x = 0$. By the induction hypothesis, there is a sequence $\{x(s), i(s)\}_{s=1}^K$ such that

$$\begin{array}{rr} x(1) = x, & x_{i(1)} = x(1), \\ x(2) \in D(R_{i(1)}, p) \cap M^{K-1}, & x_{i(2)} = x(2), \\ x(3) \in D(R_{i(2)}, p) \cap M^{K-2}, & x_{i(3)} = x(3), \\ \vdots & \vdots \\ x(K) \in D(R_{i_{(K-1)}}, p) \cap M^1, & x_{i(K)} = x(K). \end{array}$$

Let $x(K+1) \in D(R_{i(K)}, p) \cap M^0$. For each $s \in \{1, 2, \ldots, K\}$, let $\hat{z}_{i(s)} \equiv (x_{i(s+1)}, p^{x_{i(s+1)}})$, and for each $j \in N \setminus \{i(s)\}_{s=1}^K$, let $\hat{z}_j \equiv z_j$. Then, $\hat{z} \in Z^*$, and

$$\#\{y \in M^+(p) : \exists\, i \in N \text{ s.t. } \hat{x}_i = y\} = \#\{y \in M^+(p) : \exists\, i \in N \text{ s.t. } x_i = y\} + 1.$$

This is a contradiction to (1). Thus, $M^K \subseteq M^+(p)$.

Next, we show that $N^K \neq \emptyset$. By $M^K \subseteq M^+(p)$ and the induction hypothesis, $\bigcup_{k=1}^K M^k \subseteq M^+(p)$. Thus, since $\bigcup_{k=0}^K M^k$ is not underdemanded at p for R,

$$\# \bigcup_{k=0}^{K} N^k \geq \# \bigcup_{k=0}^{K} M^k. \tag{2}$$

By the definitions of M^k and N^k, for each $k, k' \in \{0, 1, \ldots, K\}$ with $k \neq k'$, $N^k \cap N^{k'} = \emptyset$, which also implies that $M^k \cap M^{k'} = \emptyset$. Thus,

$$\# \bigcup_{k=0}^{K} N^k = \sum_{k=0}^{K} \# N^k, \text{ and } \# \bigcup_{k=0}^{K} M^k = \sum_{k=0}^{K} \# M^k.$$

Then, by (2),

$$\sum_{k=0}^{K-1} \# N^k + \# N^K = \sum_{k=0}^{K} \# N^k \geq \sum_{k=0}^{K} \# M^k = \sum_{k=1}^{K} \# M^k + \# M^0. \tag{3}$$

For each $k \geq 1$, by $M^k \subseteq M^+(p)$, $\# M^k = \# N^{k-1}$. Thus, $\sum_{k=0}^{K-1} \# N^k = \sum_{k=1}^{K} \# M^k$. Then, by (3),

$$\# N^K \geq \# M^0.$$

Thus, by $M^0 \neq \emptyset$, $\# N^K \geq 1$, and so $N^K \neq \emptyset$.

Since $M^+(p)$ is finite, by the above induction argument, for large K, $\# \bigcup_{k=0}^K M^k = \sum_{k=0}^K \# M^k > \# M^+(p)$. Since $\bigcup_{k=0}^K M^k \subseteq M^+(p)$, this is a contradiction. □

E Proof of Fact 3.5.

Let $\mathcal{R} \subseteq \mathcal{R}^E$.

Lemma E.1. *Let $i \in N$ and $R_i \in \mathcal{R}$. Let $p, q \in \mathbb{R}^m_+$ and $x, y \in L$ be such that $x \in D(R_i, p)$ and $(y, q^y)\, P_i\, (x, p^x)$. Then, $y \in M$ and $q^y < p^y$.*

Proof of Lemma E.1. Since $(y, q^y)\, P_i\, (x, p^x)$ and $x \in D(R_i, p)$, we have $(y, q^y)\, P_i\, (x, p^x)\, R_i\, \mathbf{0}$. Thus, $y \in M$. Also, by $x \in D(R_i, p)$, $(y, q^y)\, P_i\, (x, p^x)\, R_i\, (y, p^y)$. Thus, $(y, q^y)\, P_i\, (y, p^y)$ implies that $q^y < p^y$. □

Given $R, \hat{R} \in \mathcal{R}^n$, $(z, p) \in W(R)$, and $(\hat{z}, \hat{p}) \in W(\hat{R})$, let

$N^1 \equiv \{i \in N : \hat{z}_i\, P_i\, z_i\}$, $M^2 \equiv \{x \in M : p^x > \hat{p}^x\}$,

$X^1 \equiv \{x \in L : \text{ for some } i \in N^1, x_i = x\}$, and $\widehat{X}^1 \equiv \{x \in L : \text{ for some } i \in N^1,\ \hat{x}_i = x\}$.

Lemma E.2: *Decomposition* (Demange and Gale, 1985). *Let $R \in \mathcal{R}^n$ and $(z, p) \in W(R)$. Let $\hat{R} \in \mathcal{R}^n$ be the d-truncation of R such that for each $i \in N$, $d_i \leq -CV_i(0; z_i)$, and let $(\hat{z}, \hat{p}) \in W(\hat{R})$. Then, $X^1 = \widehat{X}^1 = M^2$.*

Proof of Lemma E.2. First, we show that $\widehat{X}^1 \subseteq M^2$. Let $x \in \widehat{X}^1$. Then, there is $i \in N^1$ such that $\hat{x}_i = x$. By $i \in N^1$, $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (x_i, p^{x_i})$. Thus, by $x_i \in D(R_i, p)$, Lemma E.1 implies that $\hat{x}_i \in M$ and $\hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$, and so $x = \hat{x}_i \in M^2$. Thus, $\widehat{X}^1 \subseteq M^2$.

Next, we show that $M^2 \subseteq X^1$. Let $x \in M^2$. Then, $x \in M$ and $0 \leq \hat{p}^x < p^x$. Thus, by (WE-ii), there is $i \in N$ such that $x_i = x$. Since $d_i \leq -CV_i(0; z_i)$, Lemma A.2-(ii) implies that $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (x_i, p^{x_i})$. Thus, $i \in N^1$, and so $x = x_i \in X^1$. Thus, $M^2 \subseteq X^1$.

Note that by the definition of X^1 and $\widehat{X}^1$, $\#X^1 \leq \#N^1$ and $\#\widehat{X}^1 \leq \#N^1$. Since $\widehat{X}^1 \subseteq M^2 \subseteq M$, each agent in N^1 receives a different object, and so $\#\widehat{X}^1 = \#N^1 \geq \#X^1$. Since $\widehat{X}^1 \subseteq M^2 \subseteq X^1$, $\#\widehat{X}^1 \leq \#M^2 \leq \#X^1$. Thus, $\#\widehat{X}^1 = \#M^2 = \#X^1$. By $\#\widehat{X}^1 = \#M^2$ and $\widehat{X}^1 \subseteq M^2$, $\widehat{X}^1 = M^2$. By $\#M^2 = \#X^1$ and $M^2 \subseteq X^1$, $M^2 = X^1$. □

Lemma E.3: *Lattice Structure* (Demange and Gale, 1985). *Let $R \in \mathcal{R}^n$ and $(z, p) \in W(R)$. Let $\hat{R}$ be the d-truncation of R such that for each $i \in N$, $d_i \leq -CV_i(0; z_i)$, and let $(\hat{z}, \hat{p}) \in W(\hat{R})$. Then, (i) $p^{(-)} \equiv p \wedge \hat{p} \in P(R)$, and (ii) $p^{(+)} \equiv p \vee \hat{p} \in P(\hat{R})$.*[40]

Proof of Lemma E.3. Let $N^1 \equiv \{i \in N : \hat{z}_i\, P_i\, z_i\}$ and $M^2 \equiv \{x \in M : p^x > \hat{p}^x\}$.

Proof of (i). Let $z^{(-)}$ be an allocation such that for each $i \in N^1$, $z_i^{(-)} \equiv \hat{z}_i$, and for each $i \in N \setminus N^1$, $z_i^{(-)} \equiv z_i$. We show that $(z^{(-)}, p^{(-)}) \in W(R)$.

Step 1. $(z^{(-)}, p^{(-)})$ *satisfies (WE-i).*

Let $i \in N$ and $x \in L$. In the following two cases, we show that $(x_i^{(-)}, p^{(-)x_i^{(-)}})\, R_i\, (x, p^{(-)x})$, which implies $x_i^{(-)} \in D(R_i, p^{(-)})$.

Case 1. $i \in N^1$.

Since $x_i^{(-)} = \hat{x}_i$, by Lemma E.2, $x_i^{(-)} \in M^2$, and so $x_i^{(-)} \in M$ and $\hat{p}^{x_i^{(-)}} < p^{x_i^{(-)}}$. Thus, $p^{(-)x_i^{(-)}} = \hat{p}^{x_i^{(-)}}$.

First, we assume that $x \in M^2$. Then, by $p^{(-)x} = \hat{p}^x$,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = \hat{z}_i\, \hat{R}_i\, (x, \hat{p}^x) = (x, p^{(-)x}),$$

where the preference relation follows from $\hat{x}_i \in D(\hat{R}_i, \hat{p})$. Since $\hat{R}_i$ is the d_i-truncation of R_i, $x_i^{(-)} \neq 0$, and $x \neq 0$, Remark 3.1 implies that $(x_i^{(-)}, p^{(-)x_i^{(-)}})\, R_i\, (x, p^{(-)x})$.

Next, we assume that $x \notin M^2$. Then, by $p^{(-)x} = p^x$,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = \hat{z}_i\, P_i\, z_i\, R_i\, (x, p^x) = (x, p^{(-)x}).$$

where the strict preference relation follows from $i \in N^1$, and the second preference relation from $x_i \in D(R_i, p)$.

Case 2. $i \notin N^1$.

Since $x_i^{(-)} = x_i$, by Lemma E.2, $x_i^{(-)} \notin M^2$. Thus, $p^{x_i^{(-)}} \leq \hat{p}^{x_i^{(-)}}$ or $x_i^{(-)} = 0$. First, we assume that $x \in M^2$. Then, $p^{(-)x} = \hat{p}^x$. Note that $i \notin N^1$ implies $(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i\, R_i\, \hat{z}_i$.

Case 2-1. $\hat{x}_i \neq 0$.

[40] Denote $p \wedge \hat{p} \equiv (\min\{p^x, \hat{p}^x\})_{x \in M}$ and $p \vee \hat{p} \equiv (\max\{p^x, \hat{p}^x\})_{x \in M}$.

By $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, $\hat{z}_i \, \hat{R}_i \, (x, \hat{p}^x) = (x, p^{(-)x})$. Since $\hat{R}_i$ is the d_i-truncation of R_i, $\hat{x}_i \neq 0$, and $x \neq 0$, Remark 3.1 implies that $\hat{z}_i \, R_i \, (x, \hat{p}^x)$. Thus,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, \hat{z}_i \, R_i \, (x, \hat{p}^x) = (x, p^{(-)x}).$$

Case 2-2. $\hat{x}_i = 0$.

Then, $\hat{z}_i = \mathbf{0}$. Since $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, $\widehat{CV}_i(x; \mathbf{0}) \leq \hat{p}^x$. Thus, if $CV_i(x; \mathbf{0}) \leq \widehat{CV}_i(x; \mathbf{0})$, then, $\hat{z}_i \, R_i \, (x, \hat{p}^x)$, which implies that,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, \hat{z}_i \, R_i \, (x, \hat{p}^x) = (x, p^{(-)x}).$$

Next, assume that $CV_i(x; \mathbf{0}) > \widehat{CV}_i(x; \mathbf{0})$. Then, since $\hat{R}_i$ is the d_i-truncation of R_i, $d_i > 0$, which implies that $x_i \neq 0$.[41] Then, by $d_i \leq -CV_i(0; z_i)$, $CV_i(x; z_i) \leq \widehat{CV}_i(x; \mathbf{0}) \leq \hat{p}^x$, which implies that $z_i \, R_i \, (x, \hat{p}^x)$. Thus,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, (x, \hat{p}^x) = (x, p^{(-)x}).$$

Next, we assume that $x \notin M^2$. Then, $p^{(-)x} = p^x$. Since $x_i^{(-)} = x_i \in D(R_i, p)$,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, (x, p^x) = (x, p^{(-)x}).$$

Step 2. $(z^{(-)}, p^{(-)})$ *satisfies (WE-ii).*

Let $x \in M$ be such that $p^{(-)x} > 0$. We show that there is $i \in N$ such that $x_i^{(-)} = x$. Since $p^{(-)} = p \wedge \hat{p}$, $p^{(-)x} > 0$ implies that $p^x > 0$ and $\hat{p}^x > 0$.

Case 1. $x \in M^2$.

By Lemma E.2, there is $i \in N^1$ such that $\hat{x}_i = x$. Since $i \in N^1$, by construction of $z^{(-)}$, $x_i^{(-)} = \hat{x}_i$. Thus, $x_i^{(-)} = x$.

Case 2. $x \notin M^2$.

Since $p^x > 0$, there is $i \in N$ such that $x_i = x$. By Lemma E.2, $i \notin N^1$. This implies that $x_i^{(-)} = x_i$. Thus, $x_i^{(-)} = x$. □

Proof of (ii). Let $z^{(+)}$ be an allocation such that for each $i \in N^1$, $z_i^{(+)} \equiv z_i$, and for each $i \in N \setminus N^1$, $z_i^{(+)} \equiv \hat{z}_i$. We show that $(z^{(+)}, p^{(+)}) \in W(\hat{R})$.

Step 1. $(z^{(+)}, p^{(+)})$ *satisfies (WE-i).*

Let $i \in N$ and $x \in L$. In the following two cases, we show that $(x_i^{(+)}, p^{(+)x_i^{(+)}}) \, \hat{R}_i \, (x, p^{(+)x})$, which implies $x_i^{(+)} \in D(\hat{R}_i, p^{(+)})$.

Case 1. $i \in N^1$.

Since $x_i^{(+)} = x_i$, by Lemma E.2, $x_i^{(+)} \in M^2$, and so $x_i^{(+)} \in M$ and $\hat{p}^{x_i^{(+)}} < p^{x_i^{(+)}}$. Thus, $p^{(+)x_i^{(+)}} = p^{x_i^{(+)}}$. First, we assume that $x \in M^2$. Since $x_i^{(+)} = x_i \in D(R_i, p)$ and $p^{(+)x} = p^x$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = z_i \, R_i \, (x, p^x) = (x, p^{(+)x}).$$

[41] To see this, suppose that $x_i = 0$. Then, $d_i \leq -CV_i(0; z_i) = 0$, which contradicts $d_i > 0$.

Since $\hat{R}_i$ is the d_i-truncation of R_i, $x_i^{(+)} \neq 0$, and $x \neq 0$, Remark 3.1 implies that $(x_i^{(+)}, p^{(+)x_i^{(+)}}) \, \hat{R}_i \, (x, p^{(+)x})$.

Next, we assume that $x \notin M^2$. Then, $p^x \leq \hat{p}^x$ or $x = 0$.

Case 1-1. $x \neq 0$.

Since $x_i^{(+)} = x_i \in D(R_i, p)$ and $p^{(+)x} = \hat{p}^x \geq p^x$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = z_i \, R_i \, (x, p^x) \, R_i \, (x, p^{(+)x}).$$

Since $\hat{R}_i$ is the d_i-truncation of R_i and $x_i^{(+)} \neq 0$, $(x_i^{(+)}, p^{(+)x_i^{(+)}}) \, \hat{R}_i \, (x, p^{(+)x})$.

Case 1-2. $x = 0$.

Since $\hat{R}_i$ is the d_i-truncation of R_i and $d_i \leq -CV_i(0; z_i)$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = z_i \, \hat{R}_i \, \mathbf{0} = (x, p^{(+)x}).$$

Case 2. $i \notin N^1$.

Since $x_i^{(+)} = \hat{x}_i$, by Lemma E.2, $x_i^{(+)} \notin M^2$. Thus, $p^{x_i^{(+)}} \leq \hat{p}^{x_i^{(+)}}$ or $x_i^{(+)} = 0$. If $x_i^{(+)} = 0$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = \mathbf{0} = \hat{z}_i \, \hat{R}_i \, (x, \hat{p}^x) \, \hat{R}_i \, (x, p^{(+)x}),$$

where the first preference relation follows from $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, and the second from $p^{x(+)} = \max\{p^x, \hat{p}^x\}$.

Thus, we assume that $x_i^{(+)} \neq 0$. Then,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = \hat{z}_i \, \hat{R}_i \, (x, \hat{p}^x) \, \hat{R}_i \, (x, p^{(+)x}),$$

where the first equality follows from $p^{x_i^{(+)}} \leq \hat{p}^{x_i^{(+)}} = p^{(+)x_i^{(+)}}$, the first preference relation from $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, and the second preference relation from $p^{(+)x} = \max\{p^x, \hat{p}^x\}$.

Step 2. $(x_i^{(+)}, p^{(+)})$ *satisfies (WE-ii).*

Let $x \in M$ be such that $p^{(+)x} > 0$. We show that there is $i \in N$ such that $x_i^{(+)} = x$. Since $p^{(+)} = p \vee \hat{p}$, $p^{(+)x} > 0$ implies that $p^x > 0$ or $\hat{p}^x > 0$.

Case 1. $x \in M^2$.

By Lemma E.2, there is $i \in N^1$ such that $x_i = x$. Since $i \in N^1$, $x_i^{(+)} = x_i$. Thus, $x_i^{(+)} = x$.

Case 2. $x \notin M^2$.

If $\hat{p}^x = 0$, then $\hat{p}^x = 0 < p^x$. Thus, $x \in M^2$, which is a contradiction. Thus, $\hat{p}^x > 0$. Then, there is $i \in N$ such that $\hat{x}_i = x$. By Lemma E.2, $i \notin N^1$, which implies that $x_i^{(+)} = \hat{x}_i$. Thus, $x_i^{(+)} = x$. □

The following is obtained as a corollary of Lemma E.3.

Corollary E.1. *Let $R \in \mathcal{R}^n$ and $p, \hat{p} \in P(R)$. Then, (i) $p \wedge \hat{p} \in P(R)$ and (ii) $p \vee \hat{p} \in P(R)$.*

We now proceed to prove Fact 3.5.

Fact 3.5 (Roth and Sotomayor, 1990). *Let $R \in \mathcal{R}^n$ and $\hat{R}$ be the d-truncation of R such that for each $i \in N$, $d_i \geq 0$. Then, $p_{\min}(\hat{R}) \leq p_{\min}(R)$.*

Proof of Fact 3.5. Let $(\hat{z}, \hat{p}) \in W(\hat{R})$. Then, for each $i \in N$, since $\widehat{CV}_i(0; \hat{z}_i) \leq 0$ and $d_i \geq 0$, $-d_i \leq 0 \leq -\widehat{CV}_i(0; \hat{z}_i)$. Since R is the $(-d)$-truncation of $\hat{R}$, Lemma E.3 implies $p^{(-)} \equiv \hat{p} \wedge p_{\min}(R) \in P(\hat{R})$. Thus, since $p_{\min}(\hat{R}) \leq p^{(-)}$, $p_{\min}(\hat{R}) \leq p_{\min}(R)$. $\square$

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