Rewarding Trading Skills Without Inducing Gambling

Igor Makarov* and Guillaume Plantin†

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Abstract

This paper develops a model of active portfolio management in which fund managers may secretly gamble in order to manipulate their reputation and attract more funds. We show that such trading strategies may expose investors to severe losses and are more likely to occur when fund managers are impatient, their trading skills are scalable and generate higher profit per unit of risk. We characterize the optimal contracts that deter this behavior. Our model can explain a number of observed differences in performance between mutual and hedge fund. In particular, it explains why persistence in returns and net risk-adjusted returns can be higher for hedge funds, and offers a rationale for the prevalence of high-water mark contracts.

*London Business School. E-mail: imakarov@london.edu.

†Toulouse School of Economics and CEPR. E-mail: guillaume.plantin@tse-fr.eu.

> "Hedge funds are investment pools that are relatively unconstrained in what they do. They are relatively unregulated, charge very high fees, will not necessarily give you your money back when you want it, and will generally not tell you what they do. They are supposed to make money all the time, and when they fail at this, their investors redeem and go to someone else who has recently been making money. Every three or four years they deliver a one-in-a-hundred year flood."
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> -Cliff Asness, Journal of Portfolio Management, 2004.

1 Introduction

The last thirty years have witnessed two important evolutions in financial markets. First, the management of large amounts of capital has been delegated to agents who are subject to very few trading restrictions and disclosure rules about their trading strategies. As a result, investors seeking to allocate their funds across different investment vehicles have to rely predominantly on the history of realized returns to assess future performance.

Second, a rapid pace of financial innovation has made it possible to slice and combine a large variety of risks by trading a rich set of financial instruments. This created room for a particular type of agency problem. Managers running out of genuine arbitrage opportunities may find it tempting to secretly take gambles that can temporarily improve their reputation. Strategies that generate frequent small positive excess returns, and rarely experience very large losses are especially appealing as they help disguise luck as skill. The spectacular demises of LTCM, Amaranth, and several other large hedge funds have left many investors with nothing, and suggested that this type of risk-shifting was at play. Moreover, following the recent severe financial crisis, many observers argue that perverse incentives led to excessive and inefficient risk-taking by financial institutions.

The goal of this paper is to develop a new framework for studying this risk-shifting problem. We investigate whether it is possible to eliminate risk-shifting incentives given the array of instruments available to modern fund managers who enjoy limited liability. Implicit in the view of some policymakers calling for a ban on so-called toxic instruments is that risk-shifting incentives are impossible to curb via contracting when speculators have access to exotic derivative instruments.

Our model builds upon the frictionless benchmark of Berk and Green (2004), who study career concerns in delegated fund management. As in their model, a fund manager and investors discover the manager's skills by observing her realized returns. Expected excess rates of returns increase with the manager's skill, but decrease with the fund size. The fund size that optimally trades off scale and unit return increases with respect to skills. Competitive investors supply funds to the manager until they earn a zero net (after fees) expected return. At each period, the manager sets fees that enable her to reach the optimal fund size and extract the entire surplus that she generates. Thus, social learning and competition among investors imply that both fund flows and manager compensation strongly depend on the trading record.

We add a risk-shifting friction to this model. In contrast to many earlier papers on risk-taking, we consider a general setting in which the fund manager can secretly choose to trade any fairly-priced contract. This captures the large set of trading opportunities available in sophisticated markets. By tampering the distribution of realized returns, the manager may manipulate her reputation in order to increase future funds under management. A technical contribution of the paper is to show that when risk-shifting occurs, it takes a simple and natural form, a binary payoff analogous to the sale of earthquake insurance. This greatly simplifies our analysis of risk-taking incentives.

We show that there are three factors that are conducive to risk-shifting. The first one is the difference between the Sharpe ratios of a skilled and an unskilled manager. If this difference is large then the history of returns has a large impact on investors' beliefs about the manager's ability to generate future excess returns. The second factor is the scalability of trading skills - that is, the sensitivity of expected excess returns to fund size. If trading skills are scalable a good reputation translates into large future fund size, and thus into large future profits. Finally, because the manager can manipulate her reputation only temporarily, she finds it more valuable to do so when she is more impatient. These three factors determine the convexity of future expected gains as a function of realized returns, and thus drive inefficient risk-bearing incentives. In particular, the model predicts that "fallen star" managers (managers with a high initial potential who realize disappointing returns) are particularly prone to gambling. For a calibration similar to that by Berk and Green, we find that their frictionless equilibrium breaks down in the presence of the risk-shifting problem.

We then study investment management contracts that address this friction. We consider two empirically relevant alternatives. First, we analyze the case in which the fund manager can commit to a long-term contract while investors cannot. This represents, for example, the situation of a hedge fund facing investors with high demand for liquidity. In this case, risk-shifting incentives result from the fact that investors will withdraw funding from a manager with a poor track-record. In light of this fact, a poor-performer will generally have an incentive to gamble for resurrection. The only way to eliminate this risk-shifting incentive is for fund managers to have low-powered payoffs, with even good performers ceding virtually all rents to investors. It is noteworthy that this finding is the exact opposite of the commonplace contention that pay-for-performance is a cure-all. Our results show that risk-taking incentives can and will actually be created by high pay-for-performance sensitivity whenever investors cannot commit to locking up their funds. Consequently, the optimal contract here is low-powered with fund managers foregoing rents. In order to implement it, the manager must commit not only on future fees but also on her future fund size: She must turn down funds when successful. Thus, unlike the benchmark setup of Berk and Green, our model is able to explain why some funds may voluntarily restrict their size and predicts that there is persistence in performance (after fees) of the best funds, which is consistent with the findings of Jagannathan et al. (2006).

Second, we analyze the case in which investors can commit to a contract but the fund manager cannot. This represents, for example, the situation of a manager in a large bank who is free to move to seize better opportunities. In this case, to cope with the risk-shifting problem investors can commit not to punish harshly the manager if she does poorly. This reduces her incentive to gamble for resurrection. However, the scope of this insurance is severely limited by the manager's option to quit should she exhibit a good track-record. The lower the reputation, the higher is the value of the option to quit. Thus, only if there is sufficiently high reputation value, the risk-shifting incentive can be eliminated. Interestingly, the optimal contract resembles a contract with a high water mark provision. It consists in paying the manager a wage that increases when her reputation reaches a new high.

Our paper bridges two strands of literature: the literature on risk-shifting and the literature on career concerns. A large finance literature studies the impact of exogenous nonconcave objective functions on the risk-taking incentives of fund managers who have access to dynamically complete markets. Contributions include Basak, Pavlova, and Shapiro (2007), Carpenter (2000), and Ross (2004). Like this literature, we seek to identify the risk-taking strategies that optimally respond to nonconcave objectives. Within a simpler framework of risk-neutral agents, we extend this line of research in two directions. First, nonconcavities in the manager's objective arise endogenously in our model from the manager's career concerns. Second, we address the question of optimal contracting.

Cadenillas et al. (2007), Ou-Yang (2003), and Palomino and Prat (2003) also study how compensation contracts should be structured when a fund manager can secretly increase risk. Unlike in our paper, in which risk-shifting is problematic because career concerns introduce convexity in continuation utilities, they study the interplay of risk-shifting with the need to elicit effort from the manager. Thus, the papers deliver very different predictions. While Cadenillas et al., Ou-Yang and Palomino and Prat find that pay-for-performance is an optimal contract in their settings, we show that it can have devastating consequences for investors when career concerns enter the picture.

Our paper is also related to Goetzmann et al. (2007) who study manipulation-proof measures of managerial performance. They show that to be manipulation-proof a measure should take a form of a concave utility function averaged over the returns history. We also show that if the fund manager has nonconcave continuation utility she can engage in inefficient risk-shifting and that optimal contracts aim to concavify the manager's objective.

Finally, our study is related to Berk and Stanton (2007) who apply the Berk and Green setup to closed-end funds. In this case, learning affects the net asset value of the fund, not its size which is fixed by construction. Berk and Stanton (2007) show that the impact of learning explains several features of the closed-end fund discount, and that the behavior of this discount crucially depends on the nature of the compensation contract. While the focus of our paper is different we provide a closed form solution of the insurance contract they consider. Interestingly, Berk and Stanton (2007) conclude that any empirically observed persistence in investors' net returns must stem from some form of investors' market power. We show that in the presence of risk-shifting, persistence is actually the outcome of optimal mechanisms, even when investors are perfectly competitive and uninformed.

The paper is organized as follows. Section 2 defines and characterizes inefficient risk-bearing in a static framework. Section 3 introduces risk-shifting in the dynamic model of delegated asset management set forth by Berk and Green (2004). Section 4 analyzes contractual arrangements that mitigate the risk-shifting problem in this model. Most proofs are relegated to the appendix.

2 Risk-Shifting: Definition and Characterization

One of the central questions in delegated portfolio management is how much risk the manager takes. If investors and the manager have different risk preferences, then contracting imperfections may result in less than optimal risk-taking. Situations that are blatantly inefficient are those in which the manager has incentives to choose second-order dominated risk profiles. As is well-known, no risk-averse investor would agree with such a risk profile because it loads unnecessarily on idiosyncratic risk (see Rothschild and Stiglitz (1970)). This section provides a general characterization of the objective functions that lead a manager to engage in such inefficient risk-taking, and solves for the resulting risk profile in a simple static environment.

Suppose that a manager has one unit to invest at date 0. Her utility is an exogenously given function U of the date 1 gross return. At date 0, she can choose any distribution over the date 1 gross return subject to preserving its mean, which we normalize to one w.l.o.g. Formally, let M denote the set of Borelian probability measures over $[0, +\infty)$. Then the manager solves the following problem.

$$\begin{aligned} &\sup_{\mu \in M} \int_0^\infty U(R) d\mu(R) \\ &s.t. \quad \int_0^\infty R d\mu(R) = 1. \end{aligned} \tag{2.1}$$

Let $P(U)$ denote the solution of this problem. Clearly, if U is concave then $P(U) = U(1)$ and the solution is attained with the risk-free return. On the other hand, whenever $P(U) > U(1)$ the solution involves some exposure to idiosyncratic risk. Therefore, throughout the paper, we deem an objective function U to be conducive to risk-shifting if $P(U) > U(1)$.

Aumann and Perles (1965) have studied a very similar class of problems, and have established conditions under which the problem associated with a given objective function U has the same solution as the one associated with the concavification of U.[1] This section shows that the dual approach generates a simple and practical determination of $P(U)$.

Let $P^*(U)$ denote the solution of the dual problem. By definition,

$$\begin{aligned} P^*(U) &\equiv \inf_{(z_1, z_2) \in \mathbb{R}^2} z_1 + z_2 \\ s.t. \quad \forall y &\geq 0, z_1 + y z_2 \geq U(y). \end{aligned} \tag{2.2}$$

The dual problem minimizes the value at 1 of a straight line that is above the graph of U. The next proposition shows conditions under which these primal and dual problems have the same solution.

Proposition 1 *Let $U : [0, +\infty) \to \mathbb{R}$ be a continuous function such that*

$$\lim_{y \to +\infty} \frac{U(y)}{y} = 0, \tag{2.3}$$

then

$$P(U) = P^*(U).$$

Proof. See the Appendix.■

[1] The main difference is that they optimize over functions instead of measures.

Proposition 1 offers a simple graphical characterization of the solution. Consider the elementary example in which

$$U(R) = u(\max(R - D, 0)),$$

where u is increasing, concave, $u(0) = 0$, and $D > 0$. This corresponds to the situation in which an agent is protected by limited liability and faces a noncontingent liability equal to D at its consumption date. As is well-know, excessive leverage creates risk-shifting incentives. Figure 1 shows that if leverage is sufficiently low, then there is no risk-shifting. Conversely, as D increases, risk-shifting occurs. The optimal risk profile has a binary payoff: either 0 or a value which is strictly larger than 1. The solution to P^* is given by the value at 1 of the tangent to the objective that goes through the origin.

[Figure 1 About Here]

The following proposition shows that the generic solution to $P(U)$ also has this simple structure - either no gambling or a binary gamble.

Proposition 2 *Assume U satisfies (2.3), and $\lim_{+\infty} U = +\infty$. If there is risk-shifting, $P(U)$ can be attained with a binary payoff.*

Proof. See the Appendix.■

Thus, the solution to the general problem (2.1) boils down to finding an optimal binary gamble. Intuitively, as in the elementary example above, if the straight line corresponding to the solution of $P^*(U)$ is tangent to U in 1, then there is no risk-shifting. If this is not the case, then this straight line must have at least one intersection with the graph of U on the left of 1, and at least one on the right. In this case, a binary gamble that pays off the abscissae of two such intersections solves $P(U)$. If the intersection on the left of 1 is at 0 (as is the case in the elementary example above), then the optimal risk profile consists in earning a superior return most of the time or in losing everything with a small probability.

This result shows that if a manager's objective is conducive to risk-shifting then she may adopt a risk profile that exposes investors to severe potential losses. This suggests that endowing sophisticated managers with inappropriate incentives can have dramatic implications for investors' welfare and financial stability, in general. There are several reasons why a fund manager might have a nonconcave objective. For instance, in the presence of a moral-hazard problem - either an ex ante moral hazard problem as in Holmstrom (1979), or an ex post moral hazard as in Townsend (1979)) - providing the manager with incentives typically involves nonconcave payoffs. In the remainder of the paper, we focus on career concerns as the source of nonconcavities in the manager's preferences.

3 Risk Shifting and Career Concerns

As mentioned in the introduction, there is ample empirical evidence that investors chase past performance when allocating their funds. This means that a manager's reputation has an important impact on her future assets under management and, in turn, her future fees. Thus the manager has strong incentives to control her reputation even if it comes at some cost. The goal of this section is to study the interplay of managers' career concerns with the risk-shifting problem introduced in Section 1. In Subsection 2.1, we present a frictionless model of career concerns in delegated asset management that closely follows Berk and Green (2004). In Subsection 2.2, we introduce the risk-shifting friction and study its impact on the equilibrium.

3.1 The Berk and Green (2004) Model

Time is discrete and is indexed by $\{n\Delta t\}$, where $n \in \mathbb{N}$ and $\Delta t > 0$. There is a single consumption good which serves as the numéraire. Agents are of two types: a manager and several investors. Agents live forever, are risk-neutral, and discount future consumption at the instantaneous rate $r > 0$. The manager is protected by limited liability: She cannot have negative consumption. Investors receive a large endowment of the consumption good at each date $n\Delta t$, the manager does not. The manager has exclusive access to a storage technology. If the manager stores q consumption units at date t using her technology, she generates $\rho_{t+\Delta t}$ units at date $t + \Delta t$ such that

$$\rho_{t+\Delta t} = q_t e^{\left(r+a\theta-c(q_t)-\frac{\sigma^2}{2}\right)\Delta t+\sigma(B_{t+\Delta t}-B_t)}, \tag{3.1}$$

where $(B_t)_{t\geq 0}$ is a standard Wiener process, c is a continuous, nonnegative, and nondecreasing function with $c(0) = 0$, $\lim_{q\to+\infty} c(q) = +\infty$, and a and σ are strictly positive numbers. The parameter $\theta \in \{0; 1\}$ measures the manager's skills. This parameter is unobservable. All agents share the common date-0 *prior* that the manager is endowed with high skills with probability $\pi_0 \in (0, 1)$. The parameter a is the spread that a skilled manager can generate over the risk-free rate with her first dollar. As in Berk and Green (2004), the function c captures that many arbitrage opportunities or informational rents in financial markets are not perfectly scalable.

Except for the manager's skills, which nobody observes, each action and the manager's realized returns are publicly observable at each date $n\Delta t$. Let $(\pi_{n\Delta t})_{n\geq 0}$ denote the evolution of agents' beliefs about the probability that the manager is highly skilled - her perceived skills.

This baseline model is essentially identical to Berk and Green (2004). The main modelling difference is that the distribution of skills is binomial in our setup while it is Gaussian in theirs. The social optimum is achieved when a manager with perceived skills π at date t receives a quantity of funds $q(\pi)$ that maximizes the net expected surplus that she creates over $[t, t+\Delta t]$:

$$q(\pi) = \arg\max_q q\left(\pi e^{(a-c(q))\Delta t} + (1-\pi)e^{-c(q)\Delta t} - 1\right),$$

and thus

$$\lim_{\Delta t \to 0} q(\pi) = \arg\max_q q(\pi a - c(q)). \tag{3.2}$$

As in Berk and Green, we assume that the manager interacts with investors as follows:

Assumption 1 *At each date t, the manager quotes a fee in order to maximize her current surplus. The fee is the fraction of the date-$t + \Delta t$ assets under management (before new inflows/outflows of funds) that accrues to the manager. Investors supply funds competitively.*

Proposition 3 *Under Assumption 1, let $q(\pi)$ denote the fund size, and $w(\pi)\Delta t$ and $v(\pi)\Delta t$ denote respectively the fee and expected profit over $[t, t+\Delta t]$ when the manager's perceived skills at date t are π. We have*

$$\begin{aligned} \lim_{\Delta t \to 0} q(\pi) &= \arg\max_q q(\pi a - c(q)), \\ \lim_{\Delta t \to 0} w(\pi) &= \pi a - c(q(\pi)), \\ \lim_{\Delta t \to 0} v(\pi) &= \max_q q(\pi a - c(q)). \end{aligned} \tag{3.3}$$

The manager extracts the entire first-best surplus created by investments in her storage technology.

Proof.

If a manager with perceived skills π quotes a fee $w\Delta t$, competitive investors will supply q as long as their net expected rate of return is equal to r. Thus, the fund supply $q(w)$ solves

$$(1 - w\Delta t)\left(\pi e^{(r+a-c(q))\Delta t} + (1-\pi)e^{(r-c(q))\Delta t}\right) = e^{r\Delta t},$$

therefore for a given w, $\lim_{\Delta t \to 0} q(w)$ is the solution (if any) to

$$\pi a = c(q(w)) + w. \tag{3.4}$$

The manager chooses w such that:

$$w = \arg\max_{w} w\Delta t \times q(w) = \arg\max_{w} (\pi a - c(q(w))) \times q(w).$$

Further,

$$\max_{w} (\pi a - c(q(w))) \times q(w) = \max_{q} q(\pi a - c(q)),$$

meaning that the manager extracts the entire first-best surplus. ■

Corollary 1 *If there exists $\alpha \geq 1$, $\beta > 0$ such that $c(q) = \beta q^{\frac{1}{\alpha-1}}$, then*

$$\lim_{\Delta t \to 0} v(\pi) = \beta^{2-\alpha} \frac{(\alpha - 1)^{\alpha-1}}{\alpha^{\alpha}} (a\pi)^{\alpha}.$$

This power specification for c and thus v will play an important role in what follows because it generates a simple intuition on the impact of the scalability of trading skills on risk-shifting incentives. The larger α, the more scalable the manager's skills. In the hedge fund universe, global macro strategies would typically be quite scalable. On the other hand, strategies based on shareholder activism may be more difficult to spread over increasing amounts of capital.

Berk and Green specify a linear cost function c corresponding to $\alpha = 2$. They show that their model matches quantitatively well the empirically observed relationship between mutual funds realized returns and inflows/outflows. In the limiting case in which $\alpha \downarrow 1$, $v(\pi)$ tends to $\beta\pi$. This linear surplus is the one used in Harris and Holmstrom (1982) model of wage dynamics.

The Berk and Green benchmark assumes seamless delegation of asset management. This assumption seems more natural in the case of mutual funds that take only long positions in publicly traded stocks - Berk and Green's focus - than in situations in which managers face fewer portfolio restrictions and disclosure constraints. Managers employed by hedge funds or banks' proprietary desks typically have a free hand at taking a very large variety of bets that outsiders do not continuously monitor. The remainder of the paper studies the impact of the risk-shifting friction in this baseline model.

3.2 Risk Shifting

Assumption 2 *(Risk shifting). In addition to the efficient storage technology described in (3.1), the manager has access to an alternative technology whose returns are perfectly scalable and independent from the returns on the efficient technology. This technology enables her to generate a return with any arbitrary distribution over* $[0, \infty)$ *with mean* $e^{r'\Delta t}$*, where* $r' \leq r - \frac{\sigma^2}{2}$*. How the manager chooses this distribution and allocates her funds between these two technologies is private information to her.*

Assumption 2 adds informational asymmetry to the baseline model. An interpretation of this setup is the following. The manager's efficient storage technology can be viewed as a risky arbitrage opportunity in a new compartment of the market which is still inefficient, and whose risk is not yet spanned by existing, fairly priced instruments. The manager can also secretly invest in a rich set of orthogonal liquid instruments, but camouflaging these trades comes at a cost $r - r'$. The investors observe returns realized at the reporting and contracting dates $n\Delta t$, at which the manager's position is marked-to-market.

Remark. That secretly gambling comes at a cost is a natural assumption. The role of the lower bound on this cost is to rule out situations in which the manager secretly invests at the risk-free rate. With $r' > r - \frac{\sigma^2}{2}$, the manager would be tempted to do so as her reputation deteriorates (π close to 0). We will show that a sufficiently low r' eliminates this "secret risk-reducing" problem, which seems of limited practical relevance, and is not in the scope of our paper. Importantly, we will see that the value of r' is completely immaterial for the determination of the manager's incentives to increase risk as $\Delta t \to 0$. The intuition is that risk-shifting amounts to adding an instantaneous discrete zero-mean jump to the return process in the limit of a small Δt. In other words, gambles become instantaneous fair lotteries as $\Delta t \to 0$.

We first outline two mathematical results that yield a tractable analysis of incentive-compatibility constraints in the presence of the risk-shifting friction. Consider a situation in which the manager does not gamble and investors believe so. Assume that if investors believe that the manager is skilled with probability π_t at date t then the manager collects an expected surplus $v(\pi_t)\Delta t$ for the $[t, t + \Delta t]$-trading round, where v is a continuous function. Proposition 4 gives the expected surplus of the manager over her life-time when $\Delta t \to 0$ if she starts out with skills π.

Proposition 4 *Let* v *be a continuous function over* $[0, 1]$*. Let*

$$V(\pi, \Delta t) = E_0 \left(\sum_{n=0}^{\infty} e^{-rn\Delta t} v(\pi_{n\Delta t}) \Delta t \right), \quad \pi_0 = \pi, \tag{3.5}$$

and

$$V(\pi) = \lim_{\Delta t \to 0} V(\pi, \Delta t).$$

We have

$$V(\pi) = \int_0^1 G(\pi, x)\, v(x) dx, \tag{3.6}$$

where

$$G(\pi, x) = \frac{2\sigma^2}{\psi a^2 x^2 (1-x)^2} \begin{cases} g(1-\pi) g(x) & \text{if} \quad 0 \leq x \leq \pi \\ g(\pi) g(1-x) & \text{if} \quad \pi \leq x \leq 1 \end{cases}, \tag{3.7}$$

and

$$\begin{aligned} \psi &= \sqrt{1 + \frac{8r\sigma^2}{a^2}}, \\ g(u) &= u^{\frac{1}{2}+\frac{1}{2}\psi}(1-u)^{\frac{1}{2}-\frac{1}{2}\psi}. \end{aligned} \tag{3.8}$$

Convergence of $V(\pi, \Delta t)$ to $V(\pi)$ when $\Delta t \to 0$ is uniform over $\pi \in (0,1)$.

Proof. See the Appendix.■

The function $G(\pi, x)$ has an intuitive interpretation: It is a weighted probability that the manager will have a skill level x over her career if she starts with a skill level π. Note that as $r \to 0$, $\psi \to 1$ and all the contribution to the continuation utility comes from what the manager gets when her skill level is either 0 or 1. This is again intuitive because in the long-run, there is a complete revelation of the manager's skills.

Suppose now that the manager deviates and gambles during her first trading round but investors believe instead that she has invested in her storage technology at date 0. Let $R \geq 0$ denote the return that the manager realizes. Let $\pi^R_{\Delta t}$ be investors' perception of her skills after the return R is realized and $W(\pi, R, \Delta t)$ be the present value of her future earnings after the return R is realized and assuming she will no longer gamble from then on. We have

Proposition 5

$$\pi^R \equiv \lim_{\Delta t \to 0} \pi^R_{\Delta t} = \frac{\pi_0 R^{\frac{a}{\sigma^2}}}{1 - \pi_0 + \pi_0 R^{\frac{a}{\sigma^2}}}. \tag{3.9}$$

$$W(\pi, R) \equiv \lim_{\Delta t \to 0} W(\pi, R, \Delta t) = \int_0^1 G(\pi, x)\, v\left(\frac{x R^{\frac{a}{\sigma^2}}}{1 - x + x R^{\frac{a}{\sigma^2}}}\right) dx, \tag{3.10}$$

where G is defined in 3.7. Convergence is uniform over $(\pi, R) \in (0,1) \times [0, +\infty)$.

Proof. See the Appendix.■

In the remainder of the paper, all results will be established for Δt sufficiently small. All our results hinge on properties satisfied by the continuation value functions

$V(\pi, \Delta t)$ and $W(\pi, R, \Delta t)$ for Δt sufficiently small. To ease the exposition and focus on economic intuitions, we will only establish that these properties hold for their respective limits as $\Delta t \to 0$, $V(\pi)$ and $W(\pi, R)$. That the functions $V(\pi, \Delta t)$ and $W(\pi, R, \Delta t)$ also satisfy these properties if Δt is sufficiently small is a simple consequence of the uniform convergence properties established in Propositions 4 and 5.

From (3.9) we can see that

$$\frac{d^2\pi^R}{dR^2}\Big|_{R=1} = -\frac{a}{\sigma^2}\pi_0(1-\pi_0)\left(1-\frac{a}{\sigma^2}+2\frac{a}{\sigma^2}\pi_0\right). \tag{3.11}$$

This suggests that all else equal, the incentive to manipulate beliefs is strongest when π_0 is low, and that it decreases with π_0, because reputation becomes less concave in R as π_0 increases. Suppose that the manager tries to "pick up nickels in front of a steamroller", that is, she gambles and realizes an instantaneous return of $1+\varepsilon$ with probability $\frac{1}{1+\varepsilon}$ or loses everything. Then for π_0 small, in case of success, her new reputation is approximately

$$\pi^R \simeq \pi_0\left(1+\frac{a}{\sigma^2}\varepsilon\right).$$

If $v(\pi) = cst \times \pi^\alpha$ and the discount rate is not too small then the continuation utility of the manager also behaves as π^α.[2] Therefore, the manager's net expected gain from the gamble is approximately

$$\frac{1}{1+\varepsilon}\left(\pi^R\right)^\alpha - \pi_0^\alpha \simeq \pi_0^\alpha\left(\frac{\alpha a}{\sigma^2}-1\right)\varepsilon.$$

This suggests that whether there is risk-shifting or not depends on whether the ratio $\frac{\alpha a}{\sigma^2}$ is greater or less than 1. The next proposition formalizes this broad intuition.

Proposition 6 *Let $v(x) = kx^\alpha$, where $v(\cdot)$ is the expected surplus that the manager generates per period, $\alpha \geq 1$, $k > 0$. If $\sigma^2 > \alpha a$, then there exists an equilibrium in which the manager does not engage in risk-shifting. If $\sigma^2 < \alpha a$ and $r\sigma^2 > \frac{a^2}{2}\alpha(\alpha-1)$, then such an equilibrium does not exist.*

Proof. See the Appendix.■

Proposition 6 can be interpreted as follows. There are two ratios with simple economic interpretations that determine whether gambling occurs or not in the Berk and Green environment

$$R_1 = \frac{a}{\sigma^2}, \quad R_2 = \frac{r}{a}.$$

Risk-shifting does not occur when $R_1 \leq \frac{1}{\alpha}$ and occurs if $R_1 > \frac{1}{\alpha}$ and $2R_2 > R_1\alpha(\alpha-1)$. The condition on R_1 means that risk-shifting occurs if R_1 is sufficiently large, and the

[2] See Lemma 3 in the Appendix for a formal proof.

manager's skills are sufficiently scalable (α large). The ratio R_1 is large when skilled managers generate high excess returns with relatively low risks. The condition on R_2 is that the manager is sufficiently impatient. The intuition for this is the following. If a skilled manager generates high unit excess returns with low risk and her strategy is scalable, then small good news about her skill translate into large future fund size, and thus into large future profits. This creates strong incentives to try and boost returns with a gamble. If the manager is patient, however, she cares only for the long run in which she ends up with the reputation that she deserves regardless of earlier gambling attempts.

To assess whether the risk-shifting friction is likely to be important in practice, we consider a calibration similar to that of Berk and Green. We set $\alpha = 2$, $a = 5\%$, $\sigma = 25\%$, and $r = 5\%$. Then $a/\sigma^2 = 0.8$ and $a/r = 1$. Berk and Green show that such orders of magnitude lead to realistic sensitivities of fund flows to realized performance. Proposition 6 predicts that there is no equilibrium without risk-shifting in this case.

4 Contracting Risk Shifting Away

The previous section shows that when competitive investors and the manager interact according to the special way assumed by Berk and Green - that is, the manager simply quotes fees at every period - then it is not possible to sustain an equilibrium without risk-shifting for plausible parameter values. This section studies optimal arrangements between investors and the manager in these cases in which the risk-shifting friction matters. Note first that the contracting problem is trivial under full commitment. If both the manager and investors can fully commit to long-term contracts, then coping with risk-shifting is easy. For example, both parties can agree on a contract of infinite duration that features a unique upfront payment to the manager, and specifies future fund sizes as a function of future perceived skills. This does not induce risk-shifting while generating the same *ex ante* utilities as in the first-best described in Proposition 3. This "slavery" arrangement is of limited practical interest. Here, we study more interesting situations in which commitment is one-sided. Section 3.1 tackles the case in which the manager can commit but not investors. Section 3.2 studies the opposite situation in which the commitment problem is on the manager's side. Each situation seems empirically relevant. Limited commitment on the investors' side represents the situation of a hedge fund that has only access to investors with short horizons, e.g. because of important liquidity needs. The situation in which the manager cannot commit while investors can may be interpreted as one in which a proprietary manager in a large bank is free to move to seize better opportunities in the job market.

4.1 Investors Cannot Commit

In this section, we keep Assumption 2 and substitute Assumption 1 for Assumption 3.

Assumption 3 *At each date, a new cohort of investors enters the economy. Investors are competitive and live for one period. The manager can fully commit to any plan of future actions.*

We show that with the parameter values identified in Proposition 6, the only feasible contract is such that the manager earns zero surplus.

Proposition 7 *Suppose that $c(q) = cq^{\frac{1}{\alpha-1}}$, $c > 0$, $\alpha \geq 1$. If $\sigma^2 < \alpha a$ and $r\sigma^2 > \frac{a^2}{2}\alpha\left(\alpha - 1\right)$ then the manager can extract only zero surplus.*

Proof. See the Appendix.■

Thus, for a range of parameter values that are empirically plausible according to the Berk and Green calibration, the risk-shifting friction implies that the manager must commit to work for free if she wants to have any money to manage. When the manager works for free, she has of course no incentives to gamble, and can implement the first-best. Yet, all the surplus goes to the competitive investors. Proposition 7 shows that there exists no other feasible arrangement. This result is driven by the behavior of the continuation utility $V(\pi)$ for "bad" managers whose π is close to zero. Under the assumptions of Proposition 7, the continuation utility of the manager is very convex for π small. This implies that a manager who has any hope of a strictly positive stake in future surpluses has incentives to gamble if her reputation becomes close to zero. Because investors are not interested in earning r' in expectation, a manager with such a small π cannot receive any funds. But then, a manager whose reputation is slightly above this threshold where she cannot receive funds has incentives to gamble too. This contagion argument implies that any manager has no choice but committing to work for free in order to receive any money to manage.

To illustrate the key role that the curvature of v (and thus V) at 0 plays in this result, consider the situation - arguably realistic - in which the excess returns generated by the manager decrease with her fund size only when the assets under management exceed a threshold $\overline{q}$.

Proposition 8 *Assume $a < \sigma^2 < 2a$, $r\sigma^2 > a^2$ and $c(q) = (q - \overline{q})^+$, $\overline{q} > 0$. Then the manager can commit to not gamble and still extract an expected surplus at least equal to $\pi a\overline{q}\Delta t$ per period. If $\overline{q}$ is sufficiently small, the manager cannot extract the whole surplus that she generates.*

Proof. See the Appendix.■

Notice that the restrictions on the parameters a, r, and σ in Proposition 8 are the ones of Proposition 7 when $\alpha = 2$. Thus, the situation in Proposition 8 is not covered by Proposition 7 only to the extent that the cost c is equal to zero in an arbitrarily small right-neighborhood of zero. We show that this difference is important because it enables the manager to extract some of the surplus that she generates. This illustrates that with short-term investors, how much the manager can extract crucially depends on the behavior of the trading surplus when her skill level approaches zero.

Proposition 8 also shows that the maximal surplus that the manager can earn must still be strictly lower than the total surplus generated by the manager. This implies that the manager is still forced in this case to leave some surplus to investors. Thus, in the presence of a risk-shifting problem and short-term contracts, investors can earn persistent excess returns from investing in the best funds, which is consistent with the findings of Jagannathan et al. (2006). Glode and Green (2009) and Hochberg et al. (2008) explain persistence in private equity returns with informational rents and/or bargaining power accruing to first-round investors in subsequent rounds. Our explanation differs in that investors in our environment are uninformed and competitive throughout the entire timeline.

The contract with a linear surplus described in Proposition 8 cannot be implemented only by quoting fees because competitive investors always supply funds until they earn no expected surplus. Interestingly, the manager can implement this contract by quoting both a fee and a fund size at each period. The manager must commit to a fee per period that is smaller than the first-best fee described in Proposition 3 so as to leave some of the total surplus to investors. Since investors would supply more funds than the optimum fund size in Proposition 3 with such smaller fees, the manager must also impose a restriction on the fund's size and turn down inflows, all the more so because she has a good reputation. This implements the first-best. Some rents must be left to investors, however.

4.2 The Manager Cannot Commit

We now consider the symmetric situation in which investors can commit to a contract, while the manager cannot. We keep Assumption 2 and substitute Assumption 3 for Assumption 4.

Assumption 4 *Investors are long-lived, competitive, and can fully commit to a contract, while the manager cannot. At the end of each trading round, the manager can terminate a contract, and recontract with new investors with impunity.*

Such a "no-slavery" assumption is commonplace in the career-concern literature. Our problem in this case becomes related to the model of wage dynamics by Harris and Homstrom (1982). Harris and Holmstrom study a situation in which there is social learning about the productivity of a worker. The worker is risk-averse. Each firm can commit to a long-term labor contract. Firms are competitive, and the worker can leave her current job and take a new one at any date with impunity. The tension between consumption-smoothing and this commitment problem creates an interesting contracting problem. Here, the manager is risk-neutral, but there is also a similar tension between the same commitment problem and the need to cope with the risk-shifting friction.

In the case in which investors can commit studied in Section 3.1, the first-best is always attained because the manager can always commit to work for free. Risk-shifting affects only the maximal share of the surplus that the manager can extract. The picture is quite different when commitment power is on the investors' side. Since investors are competitive and managers are unable to commit to a contract, the manager who gets hired extracts all the expected surplus from the relationship. This creates too strong incentives to gamble for resurrection for the low skilled managers. The next proposition formalizes this intuition and shows that the first-best is out of reach if the manager has a sufficiently low initial reputation.

Proposition 9 *Suppose that* $c(q) = cq^{\frac{1}{\alpha-1}}$, $c > 0$, $\alpha > 1$. *If* $\sigma^2 < \alpha a$ *and* $r\sigma^2 > \frac{a^2}{2}\alpha(\alpha - 1)$ *then there exists* $\underline{\pi} > 0$ *such that no manager with initial skills smaller than* $\underline{\pi}$ *is being offered a contract that implements the first-best.*

Proof.

Suppose a manager with a given skill level π is hired. This means that this manager is expected to create some surplus. Then a manager with a lower reputation never has a zero expected future surplus if hired, because she will get at least the surplus associated with π if she reaches this level. The proof of Proposition 7 establishes that a manager with sufficiently low skills always has incentives to gamble whenever she has any hope of future strictly positive surplus. This completes the proof. ■

An interesting question then is whether there exists any ability level at all at which a manager can be hired and implement the first-best. If the manager could commit to a contract, paying her a constant wage would achieve this. However, with limited commitment on the manager's side, such full insurance is not feasible since the manager could renegotiate as soon as her reputation improves. Thus the contract for a given initial π_0 cannot be determined in isolation. Instead, all the contracts for all initial skill levels depend on each other through the channel of renegotiation. The next proposition shows that despite these contracting externalities, at least for π_0 sufficiently high, it is possible to hire the manager and have her implement the first-best. We establish this result in the case of a linear surplus ($\alpha = 1$).

Proposition 10 *Suppose that $\alpha = 1$, $\sigma^2 < a$, and $\frac{2r}{a} < \frac{\sigma^2}{a} + 1$. Then there exists $\overline{\pi} < 1$ such that all managers with initial skills above $\overline{\pi}$ are hired. They implement the first-best actions and extract the entire associated surplus. The contract consists in paying a wage per unit of time equal to $w(M_t)$, where*

$$\begin{aligned} M_t &= \sup_{0 \le s \le t} \pi_s, \\ w(x) &= \frac{x}{x + (1-x)\frac{\psi+1}{\psi-1}}. \end{aligned} \tag{4.1}$$

Proof. See the Appendix.■

The contract described in Proposition 10 is the optimal contract in Harris and Holmstrom (1982). It is worth noting that we characterize this contract in a very simple closed-form in continuous-time. This contract is somewhat reminiscent of financial contracts with high-water marks that grant performance fees only when cumulative returns reach a new maximum. It provides more insurance than typical 2/20 hedge-fund contracts with high-water marks because the payment per unit of time is downward rigid, but it has a similar history-dependence. To the best of our knowledge, this relationship between high-water marks and managers' incentives to gamble is novel.

From (4.1) we can see that the ratio of what the manager gets under this contract when she reaches new high to the wage that she would earn in the spot market, monotonically decreases from $(\psi + 1)/(\psi - 1)$ to one as a function of perceived ability. This reflects the lower value of the option to quit and renegotiate the contract as reputation gets better.

Appendix

Proof of Proposition 1

Lemma 1 *There exists* $(z_1, z_2) \in \mathbb{R} \times [0, +\infty)$ *such that* (z_1, z_2) *satisfies (2.2) and* $P^*(U) = z_1 + z_2$.

Proof. Condition (2.3) implies that the set $Z = \{(z_1, z_2) : \forall y \geq 0, z_1 + yz_2 \geq U(y)\}$ is nonempty. It is closed, and there exists K such that

$$(z_1, z_2) \in Z \to z_1 \geq K, z_2 \geq K.$$

The function $(z_1, z_2) \to z_1 + z_2$ is continuous. Thus, there exists $(z_1,\ z_2) \in Z$ such that $P^*(U) = z_1 + z_2$. Condition (2.3) readily implies that $z_2 \geq 0$.■

Let μ a probability measure that satisfies (2.1), and $(z_1,\ z_2) \in \mathbb{R}^2$ defined as in Lemma 1. We have

$$z_1 + z_2 = \int_0^\infty (z_1 + Rz_2)\, d\mu(R) \geq \int_0^\infty U(R) d\mu(R).$$

This implies that

$$P^*(U) \geq P(U).$$

Let us show that the reverse inequality also holds. Establishing the reverse inequality for U with compact support is without loss of generality: for all U satisfying (2.3), there clearly exists $V \in C_c([0, \infty))$ such that $V \leq U$ and $P^*(V) = P^*(U)$. We omit the straightforward proof of the following lemma.

Lemma 2

a. $P^*(U_1) \leq P^*(U_2)$ *for* $U_1,\ U_2 \in C_c([0, +\infty))$ *such that* $U_1 \leq U_2$,

b. $P^*(\alpha U) = \alpha P^*(U)$ *for* $U \in C_c([0, +\infty))$ *and* $\alpha \in [0, +\infty)$,

c. $P^*(U_1 + U_2) \leq P^*(U_1) + P^*(U_2)$ *for* $U_1,\ U_2 \in C_c([0, +\infty))$.

From Lemma 2, $P^*(\cdot)$ is a positively homogeneous and subadditive functional on $C_c([0, +\infty))$. The Hahn-Banach Theorem therefore implies that for any $U \in C_c([0, \infty))$, there exists a positive linear functional L_U on $C_c([0, \infty))$ such that $L_U \leq P^*$ and $L_U(U) = P^*(U)$. By the Riesz representation Theorem, there exists a Borelian measure μ_U on $[0, \infty)$ such that for all $V \in C_c([0, \infty))$

$$L_U(V) = \int_0^\infty V(R) d\mu_U(R).$$

For $M > 1$, let $u_M, v_M \in C_c([0,\infty)) \times C_c([0,\infty))$ such that

$$
\begin{aligned}
u_M(x) &= 1 \text{ on } [0,M], \, x \geq M \rightarrow u_M \leq 1, \\
v_M(x) &= x \text{ on } [0,M], \, x \geq M \rightarrow v_M \leq M.
\end{aligned}
$$

Clearly,

$$P^*(u_M) = P^*(v_M) = 1$$

Then

$$
\begin{aligned}
L_U(u_M) &= \int_0^\infty u_M(R)\, d\mu_U(R) \leq P^*(u_M) = 1, \\
L_U(v_M) &= \int_0^\infty v_M(R)\, d\mu_U(R) \leq P^*(v_M) = 1.
\end{aligned}
$$

Letting $M \rightarrow +\infty$ implies

$$\int_0^\infty d\mu_U(R) \leq 1, \int_0^\infty R d\mu_U(R) \leq 1,$$

and thus

$$P^*(U) = L_U(U) = \int_0^\infty U(R) d\mu_U(R) \leq P(U).$$

■

Proof of Proposition 2

Let U satisfying condition (2.3) and such that $\lim_{+\infty} U = +\infty$. Let $(z_1, z_2) \in \mathbb{R}^2$ defined as in Lemma 1. That $\lim_{+\infty} U = +\infty$ implies that for such a (z_1, z_2), $z_2 > 0$. (Note that the assumption $\lim_{+\infty} U = +\infty$ is only a sufficient condition for $z_2 > 0$, which is all that is required to establish the proposition). Let

$$S = \{y \geq 0 : z_1 + z_2 y = U(y)\}.$$

That U is continuous together with condition 2.3 implies that S is a nonempty compact set. Let

$$y_1 = \min S, \; y_2 = \max S.$$

We now proceed in two steps.

Step 1. We show that $y_2 \geq 1 \geq y_1$.

Proof. We show that $y_2 \geq 1$. The proof that $y_1 \leq 1$ is symmetric. Assume that

$y_2 < 1$. For some $\varepsilon \in (0, 1 - y_2)$, let

$$\eta(\varepsilon) = \min_{y \geq y_2 + \varepsilon} \left\{ \frac{z_1 - U(y)}{y} + z_2 \right\}.$$

By condition 2.3,

$$\lim_{y \to +\infty} \frac{z_1 - U(y)}{y} + z_2 = z_2 > 0,$$

and by definition of y_2

$$y \geq y_2 + \varepsilon \to \frac{z_1 - U(y)}{y} + z_2 > 0,$$

continuity of U therefore implies that $\eta(\varepsilon) > 0$. Let (z_1', z_2') defined as

$$z_1' = z_1 + (y_2 + \varepsilon)\,\eta(\varepsilon), \;\; z_2' = z_2 - \eta(\varepsilon).$$

First, the pair (z_1', z_2') satisfies (2.2). To see this, notice that

$$z_1' + y z_2' = z_1 + y z_2 + \eta(\varepsilon)(y_2 + \varepsilon - y).$$

Thus $z_1' + y z_2' > z_1 + y z_2 \geq U(y)$ for $y < y_2 + \varepsilon$.

Further $z_1' + y z_2' \geq z_1 + y z_2 - \eta(\varepsilon)\, y \geq U(y)$ for $y \geq y_2 + \varepsilon$ by definition of $\eta(\varepsilon)$.

Second, we have

$$z_1' + z_2' = z_1 + z_2 + (y_2 + \varepsilon - 1)\,\eta(\varepsilon) < z_1 + z_2,$$

which contradicts the definition of (z_1, z_2). Thus it must be that $y_2 \geq 1$.■

Step 2. Now, if $y_1 = y_2$, then step 1 implies that $S = \{1\}$, and there is no gambling. If $y_1 < y_2$,

from

$$\begin{aligned} z_1 + y_1 z_2 &= U(y_1), \\ z_1 + y_2 z_2 &= U(y_2), \end{aligned}$$

we have

$$z_1 + z_2 = \frac{1 - y_1}{y_2 - y_1} U(y_2) + \frac{y_2 - 1}{y_2 - y_1} U(y_1). \tag{A1}$$

From (A1), $P(U) = P^*(U)$ is attained with a payoff equal to y_1 with probability $\frac{y_2 - 1}{y_2 - y_1}$ and y_2 with probability $\frac{1 - y_1}{y_2 - y_1}$. These probabilities are well-defined from step 1. Notice that there is gambling if and only if $y_1 < 1$ and $y_2 > 1$.■

Proof of Proposition 4

We first show point-wise convergence. That is, we establish (3.6) for a fixed $\pi_0 = \pi$. By Bayes' theorem, $\pi_{n\Delta t}$, the perceived skills at date $n\Delta t$, satisfy

$$\pi_{n\Delta t} = \frac{\pi_0 \varphi_{n\Delta t}}{1 - \pi_0 + \pi_0 \varphi_{n\Delta t}}, \tag{A2}$$

where

$$\varphi_{n\Delta t} = \exp\left\{ \frac{a}{\sigma^2} \left(a \left(\theta - \frac{1}{2} \right) n\Delta t + \sigma B_{n\Delta t} \right) \right\} \tag{A3}$$

is the likelihood ratio process. Let us introduce the continuous-time process $(\pi_t)_{t \geq 0}$ that obeys

$$d\pi_t = \frac{a}{\sigma} \pi_t (1 - \pi_t) d\overline{B}_t, \quad \pi_0 = \pi,$$

where $\overline{B}_t = \frac{1}{\sigma} \left(\theta a t + \sigma B_t - a \int_0^t \pi_s ds \right)$. Then $(\overline{B}_t)_{t \geq 0}$ is a standard Wiener process under the agents' filtration (see Liptser and Shiryaev (1978)). Further, as $\Delta t \to 0$ and $n\Delta t \to t$, $\pi_{n\Delta t} \to \pi_t$ a.s. (see Liptser and Shiryaev (1978)). Hence, $V(\pi)$ can be re-written as

$$\begin{aligned} V(\pi) &= E_0 \int_0^\infty e^{-rt} v(\pi_t) dt, \\ s.to \quad d\pi_t &= \frac{a}{\sigma} \pi_t (1 - \pi_t) d\overline{B}_t, \quad \pi_0 = \pi. \end{aligned} \tag{A4}$$

By the Feynman-Kac formula, the function V solves the following linear second-order differential equation:

$$\frac{a^2}{2\sigma^2} \pi^2 (1 - \pi)^2 V''(\pi) - rV(\pi) + v(\pi) = 0. \tag{A5}$$

From (A4) it follows that

$$V(0) = v(0)/r, \quad V(1) = v(1)/r. \tag{A6}$$

The corresponding homogeneous equation

$$\frac{a^2}{2\sigma^2} \pi^2 (1 - \pi)^2 V''(\pi) - rV(\pi) = 0 \tag{A7}$$

has two regular singular points at 0 and 1. The solutions of the homogeneous equation are linear combinations of the two independent solutions

$$\begin{aligned} g(\pi) &= (1-\pi)^{\frac{1}{2}+\frac{1}{2}\psi}\pi^{\frac{1}{2}-\frac{1}{2}\psi}, \\ h(\pi) &= g(1-\pi). \end{aligned}$$

From here, formulas (3.6) and (3.7) are standard results in the theory of inhomogeneous differential equations. The function G is the Dirichlet-Green function for the differential operator associated with the homogeneous differential equation (see, e.g., Driver (2003)).

We now show that $V(\pi_0, \Delta t)$ converges to $V(\pi_0)$ uniformly in π_0 as $\Delta t \to 0$. We have

$$V(\pi_0) - V(\pi_0, \Delta t) = E_0 \sum_{n=0}^{\infty} \int_{n\Delta t}^{(n+1)\Delta t} \left(e^{-rt}v(\pi_t) - e^{-rn\Delta t}v(\pi_{n\Delta t})\right) dt.$$

Thus it enough to show that $\forall \varepsilon > 0$, $\exists \overline{\Delta} t$ such that $\forall \Delta t < \overline{\Delta} t$ and $\forall \pi \in [0,1]$

$$\sup_{s \leq \Delta t} \sup_{\pi_0 \in [0,1]} \left| E\left(v(\pi_s) - v(\pi_0)\right) \right| < \varepsilon. \tag{A8}$$

By change of variables (A8) can be re-written as

$$\sup_{s \leq \Delta t} \sup_{\pi_0 \in [0,1]} \left| E\left(\widehat{v}(\pi_0, B_s, s) - \widehat{v}(\pi_0, 0, 0)\right) \right| < \varepsilon,$$

where

$$\widehat{v}(\pi_0, x, t) = v\left(\frac{\pi_0 \exp\left\{\frac{a}{\sigma^2}\left(a\left(\theta - \frac{1}{2}\right)t + \sigma x\right)\right\}}{1 - \pi_0 + \pi_0 \exp\left\{\frac{a}{\sigma^2}\left(a\left(\theta - \frac{1}{2}\right)t + \sigma x\right)\right\}}\right). \tag{A9}$$

Since v is continuous over [0,1] and thus uniformly continuous, it is enough to show that $\forall \pi_0 \in [0,1]$ and $\forall \varepsilon > 0$, $\exists \overline{\Delta} t$ such that $\forall \Delta t < \overline{\Delta} t$

$$\sup_{s \leq \Delta t} \left| E\left(\widehat{v}(\pi_0, B_s, s) - \widehat{v}(\pi_0, 0, 0)\right) \right| < \varepsilon, \tag{A10}$$

which follows from the weak converge of the measures induced by B_s to the measure concentrated at 0 as $s \to 0$.■

Proof of Proposition 5

Suppose that the manager gambles and realizes a return R over $[0, \Delta t]$, and from then on invests in her own storage technology. Let $\left(\pi^R_{n\Delta t}\right)_{n\in\mathbb{N}}$ denote the process - *under the* *manager's filtration* - of her skills as perceived by investors who believe instead that she has invested in her storage technology at date 0. These investors believe that

$$R = e^{\left(r+\theta a - c(q_0) - \frac{\sigma^2}{2}\right)\Delta t + \sigma B_{\Delta t}},$$

and thus

$$\begin{aligned} \pi^R_{\Delta t} &= \frac{\pi_0 R^{\frac{a}{\sigma^2}} e^{\frac{a}{\sigma^2}\left(\frac{\sigma^2}{2}+c(q_0)-r-\frac{a}{2}\right)\Delta t}}{1-\pi_0+\pi_0 R^{\frac{a}{\sigma^2}} e^{\frac{a}{\sigma^2}\left(\frac{\sigma^2}{2}+c(q_0)-r-\frac{a}{2}\right)\Delta t}}, \\ \forall n &\geq 0, \quad \pi^R_{(n+1)\Delta t} = \frac{\pi^R_{\Delta t}\frac{\varphi_{(n+1)\Delta t}}{\varphi_{\Delta t}}}{1-\pi^R_{\Delta t}+\pi^R_{\Delta t}\frac{\varphi_{(n+1)\Delta t}}{\varphi_{\Delta t}}}, \end{aligned} \tag{A11}$$

As $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \pi^R_{\Delta t} = \frac{\pi_0 R^{\frac{a}{\sigma^2}}}{1-\pi_0+\pi_0 R^{\frac{a}{\sigma^2}}},$$

and

$$\lim_{\Delta t \to 0} \frac{\varphi_{(n+1)\Delta t}}{\varphi_{\Delta t}} = \frac{\pi_t}{1-\pi_t}\frac{1-\pi_0}{\pi_0}.$$

Therefore,

$$\lim_{\Delta t \to 0,\, n\Delta t \to t} \pi^R_{n\Delta t} = \frac{\pi_t R^{\frac{a}{\sigma^2}}}{1-\pi_t+\pi_t R^{\frac{a}{\sigma^2}}}. \tag{A12}$$

The result now follows from Proposition 4.■

Proof of Proposition 6

Part 1. We first show that if $\sigma^2 > \alpha a$ then the manager does not engage in risk-shifting in the Berk and Green equilibrium. There are two steps. In step 1, we show that any risky gamble is not desirable. Then in step 2, we demonstrate that the manager will not invest in the alternative technology at the risk-free rate.

Step 1. Suppose the manager believes that she is skilled with probability π_0. At the same time, investors believe that the manager is skilled with probability π_0' and that the manager never engages in risk-shifting. We show that the manager has no incentives to deviate by taking a one-shot risky gamble in this case. Let $W(\pi_0, \pi_0', R, \Delta t)$ be the expected utility of the manager conditional on realizing a first-period return R. Note that

$$W(\pi_0, \pi_0, R, \Delta t) = W(\pi_0, R, \Delta t).$$

Similar to the proof of Proposition 5 one can show that

$$\lim_{\Delta t \to 0} Z(\pi_0, \pi_0', R, \Delta t) = \int_0^1 G(\pi_0, x)\, v\left(\frac{xR^{\frac{a}{\sigma^2}}}{1 - x + \frac{(1-\pi_0)\pi_0'}{(1-\pi_0')\pi_0} xR^{\frac{a}{\sigma^2}}}\right) dx,$$

and that convergence is uniform over $(\pi_0, \pi_0', R) \in (0,1) \times (0,1) \times [R_0, +\infty)$ for any $R_0 > 0$. Differentiating twice w.r.t. R shows that this function is concave in R when $\sigma^2 > \alpha a$. Hence the manager has no incentives to take a one-shot risky gamble in this case. Because this holds for arbitrary heterogeneous priors π_0, π_0', this implies that multi-period deviations cannot be desirable by backward induction.

Step 2. We now show that investing in the alternative technology at the risk-free rate is also not desirable. If the manager invests in her efficient storage technology then π_t evolves according to (A4). If, on the other hand, she invests in the risk-free asset π_t evolves as

$$d\pi = \frac{a}{\sigma}\pi(1-\pi)\left(r' - r + \frac{\sigma^2}{2} - (\pi a - c(q_t))\right) dt.$$

Suppose that at time t the manager allocates x_t percentage of her funds to the efficient storage technology and invest the rest in the risk-free asset. Then her continuation utility is

$$\begin{aligned} V(\pi, x) &= E_0 \int_0^\infty e^{-rt} v(\pi_t) dt, && \text{(A13)} \\ s.to \quad d\pi_t &= \frac{a}{\sigma}\pi_t(1-\pi_t)\left((1-x_t)\left(r' - r + \frac{\sigma^2}{2} - (\pi a - c(q_t))\right) + x_t d\overline{B}_t\right), \ \pi_0 = \pi. \end{aligned}$$

The optimal investment policy x_t that maximizes $(A13)$ satisfies the HJB equation:

$$\sup_{x \in [0,1]} x^2 \psi_2^2 V'' + (1-x)\psi_1 V' - rV + v = 0, \tag{A14}$$

where

$$\begin{aligned} \psi_1 &= \frac{a}{\sigma}\pi_t(1-\pi_t)\left(r' - r + \frac{\sigma^2}{2} - (\pi a - c(q_t))\right) < 0, \\ \psi_2 &= \frac{a}{\sigma}\pi_t(1-\pi_t) > 0. \end{aligned}$$

If $x_t \equiv 1$ then π_t is a martingale and by Jensen's inequality

$$Ev(\pi_t) \geq v(\pi_0).$$

Therefore, at the optimal investment policy x_t, $rV(\pi,x) \geq v(\pi)$. Thus (A14) implies that the optimal policy is indeed $x_t \equiv 1$.

Part 2. Here we show that if $\sigma^2 < \alpha a$, $r\sigma^2 > \frac{a^2}{2}\alpha(\alpha-1)$ and there exists a finite limit $\lim_{x\to 0} v(x)x^{-\alpha} = \phi_0$ then for π_0 small enough, there exists a one-period gamble which makes the manager better off. Let $R > 1$, consider the following gamble:

$$\begin{cases} R & \text{Prob.} \quad 1/R \\ 0 & \text{Prob.} \quad 1-1/R. \end{cases}$$

Let $R = (1-\rho)^{-\sigma^2/a}$ and $\phi(x) = v(x)x^{-\alpha}$. From (3.10) the expected net gain from the above one-shot gamble over perpetual investment in the efficient storage technology is

$$\int_0^1 G(\pi,x)\, x^\alpha u(x,\rho)dx, \tag{A15}$$

where

$$u(x,\rho) = \phi\left(\frac{x}{1-\rho(1-x)}\right)\frac{(1-\rho)^{\sigma^2/a}}{(1-\rho(1-x))^\alpha} - \phi(x). \tag{A16}$$

Consider first a case $\phi_0 > 0$. Since $\sigma^2 < \alpha a$, there exists $\bar{x}$ and some $\widehat{\rho} \in (0,1)$ such that for all $x \in [0,\bar{x}]$, $\frac{(1-\widehat{\rho})^{\sigma^2/a}}{(1-\widehat{\rho}(1-x))^\alpha} > 1+\varepsilon$ for some $\varepsilon > 0$ and therefore, $u(x,\widehat{\rho}) > \phi_0\varepsilon > 0$. Thus for π small enough

$$\int_0^1 G(\pi,x)\, x^\alpha u(x,\widehat{\rho})dx > \int_\pi^1 G(\pi,x)\, x^\alpha u(x,\widehat{\rho})dx.$$

Using (3.7) we have

$$\int_\pi^1 G(\pi,x)\, x^\alpha u(x,\widehat{\rho})dx = \frac{2\sigma^2}{\psi a^2}g(\pi)\int_\pi^1 x^{\alpha-\frac{3}{2}-\frac{1}{2}\psi}(1-x)^{-\frac{3}{2}+\frac{1}{2}\psi}u(x,\widehat{\rho})dx > 0 \tag{A17}$$

The integral in (A17) is greater than zero because $r\sigma^2 > \frac{a^2}{2}\alpha(\alpha-1)$ implies that $\psi > 2\alpha+1$. Therefore, the integral diverges as $\pi \to 0$. In this case its sign is determined by the sign of $u(\cdot,\widehat{\rho})$ in the neighborhood of 0, which we just found is positive.

Suppose now that $\phi_0 = 0$. Then unless $\phi(x) \equiv 0$ there exists $\bar{x}$ such that $\phi(\bar{x}) = \widehat{\phi} > 0$ and for all $x \in [0,\bar{x}]$, $\phi(x) < \widehat{\phi}$ (here we assume that all the properties hold on sets of nonzero measure). Then by choosing $\widehat{\rho}$ large enough so that $\frac{\pi}{1-\widehat{\rho}(1-\pi)} = \bar{x}$ we can ensure that A17 still holds.■

Proof of Proposition 7

The proof follows closely Part 2 of the proof of Proposition 6. We first establish the following two lemmas.

Lemma 3 *Under the assumptions of Proposition 7 there exists a finite limit*

$$\lim_{\pi\to 0} V(\pi)\pi^{-\alpha} < \infty. \tag{A18}$$

Proof. We have

$$\int_0^1 G(\pi,x)\,x^{\alpha}dx = \frac{2\sigma^2}{a^2\psi}\left[\begin{array}{l}(1-\pi)^{\frac{1}{2}(1+\psi)}\pi^{\frac{1}{2}(1-\psi)}\int_0^{\pi}(1-x)^{-\frac{1}{2}(\psi+3)}x^{\alpha+\frac{1}{2}(\psi-3)}dx\\ +(1-\pi)^{\frac{1}{2}(1-\psi)}\pi^{\frac{1}{2}(1+\psi)}\int_{\pi}^{1}(1-x)^{\frac{1}{2}(\psi-3)}x^{\alpha-\frac{1}{2}(3+\psi)}dx\end{array}\right],$$

where ψ is defined in (3.8). Further $r\sigma^2 > \frac{a^2}{2}\alpha(\alpha-1)$ implies that $\psi > 2\alpha+1$. Therefore,

$$\exists\lim_{\pi\to 0}(1-\pi)^{\frac{1}{2}(1-\psi)}\pi^{\frac{1}{2}(1+\psi)}\int_{\pi}^{1}(1-x)^{\frac{1}{2}(\psi-3)}x^{\alpha-\frac{1}{2}(3+\psi)}dx\times\pi^{-\alpha} < \infty,$$

and

$$\exists\lim_{\pi\to 0}(1-\pi)^{\frac{1}{2}(1+\psi)}\pi^{\frac{1}{2}(1-\psi)}\int_{0}^{\pi}(1-x)^{-\frac{1}{2}(\psi+3)}x^{\alpha+\frac{1}{2}(\psi-3)}dx\times\pi^{-\alpha} < \infty.\blacksquare$$

Let $\chi(\pi)$ denote the expected surplus of the manager if she starts out with perceived skills $\pi\in(0,1)$. The next lemma shows that to preclude risk-shifting $\chi(\pi)$ must be a continuous function of π.

Lemma 4 *It must be that for all $\pi\in(0,1)$ $\exists\lim_{x\to\pi}\chi(x)=\chi(\pi)$. Otherwise, there is risk-shifting.*

Proof. If not, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n\to\pi,\quad \chi(x_n)-\chi(\pi)\to l\neq 0.$$

It means that the manager can take an arbitrarily small gamble to switch from x_n to π (or reciprocally depending on the sign of l).$\blacksquare$

Step 1. We first show that a manager with sufficiently small skills must work for free. Consider a gamble that generates R with prob. $1/R$ and 0 otherwise. Define ρ as $R=(1-\rho)^{-\sigma^2/a}$. Let $U(\pi,\rho)$ denote the net expected gain from such a gamble for a manager with $\pi_0 = \pi$ and let $\omega(\pi) = \chi(\pi)/V(\pi)$. Note that $0 \le \omega \le 1$. Then

$$U(\pi,\rho) = (1-\rho)^{\frac{\sigma^2}{a}}\,\omega\left(\frac{\pi}{1-\rho\,(1-\pi)}\right) V\left(\frac{\pi}{1-\rho\,(1-\pi)}\right) - \omega\,(\pi)\,V\,(\pi)\,. \qquad \text{(A19)}$$

Therefore, as $\pi \to 0$ by Lemma 3

$$U(\pi,\rho) \sim \pi^{\alpha}\left(\omega\left(\frac{\pi}{1-\rho\,(1-\pi)}\right)\frac{(1-\rho)^{\sigma^2/a}}{(1-\rho(1-x))^{\alpha}} - \omega\,(\pi)\right). \qquad \text{(A20)}$$

The rest of the proof is the same as that of Proposition 6 with function $u(\pi,\rho)$. The only difference that there is no need to take the integral.

Step 2. Now let

$$\underline{\pi} = \max\{\pi : \chi = 0 \text{ a.s. over } [0,\pi]\}$$

Step 1 shows that $\underline{\pi} > 0$. Assume $\underline{\pi} < 1$. If a manager's initial skills are arbitrarily close to but larger than $\underline{\pi}$, the probability of spending time below the zero-surplus boundary $\overline{\pi}$ becomes non-zero. It is always strictly better then for the manager to gamble so as to loose all the expected surplus with the same probability in exchange for the upside return. Thus $\underline{\pi} = 1$.■

Proof of Proposition 8

With this cost function, the first-best surplus has the following limit as $\Delta t \to 0$:

$$\begin{array}{ll} \pi a\overline{q} & \text{if } \ \pi \le \frac{\overline{q}}{a}, \\ \left(\frac{\pi a+\overline{q}}{2}\right)^2 & \text{if } \ \pi \ge \frac{\overline{q}}{a}. \end{array}$$

Propositions 6 implies that a linear surplus does not induce risk-shifting for these values of a, r, and σ. The largest linear surplus that can be implemented is $\pi a\overline{q}$. Proposition 6 implies that a quadratic surplus induces gambling for π sufficiently small. It implies that for $\overline{q}$ sufficiently small any contract that yields the total surplus will induce risk-shifting as long as the manager's skills become sufficiently small.■

Proof of Proposition 10

Step 1. First we solve for the Harris and Holmstrom contract. Let us remind that

$$\pi_t = \frac{\pi_0\phi_t}{1-\pi_0+\pi_0\phi_t}, \quad \phi_t = e^{\frac{a}{\sigma}\overline{B_t^{\theta}}}, \quad \overline{B_t^{\theta}} = \frac{a}{\sigma}\left(\theta - \frac{1}{2}\right)t + B_t.$$

Let

$$M_t^\theta = \sup_{0\le s\le t} \overline{B_t^\theta}. \tag{A21}$$

Let $p_{\theta,t}$ denote the density of M_t^θ and let

$$\widehat{p}_{\theta,t} = \int_0^\infty e^{-rt} p_{\theta,t}. \tag{A22}$$

From formula 1.1.2 of Borodin and Salminen (page 250) we can deduce that

$$\widehat{p}_{1,t}(y) = \frac{a}{2\sigma r}(\psi-1)\, e^{-\frac{a}{2\sigma}(\psi-1)y}, \tag{A23}$$

$$\widehat{p}_{0,t}(y) = \frac{a}{2\sigma r}(\psi+1)\, e^{-\frac{a}{2\sigma}(\psi+1)y}, \tag{A24}$$

where ψ is defined in (3.8).

Let the wage w is a function of $\sup_{0\le s\le t}\pi_t$. Since the manager gets all the surplus we must have

$$\frac{\pi_0}{r} \equiv \int_0^\infty w\left(\frac{\pi_0 e^{\frac{a}{\sigma}y}}{1-\pi_0+\pi_0 e^{\frac{a}{\sigma}y}}\right)\left(\pi_0\widehat{p}_{1,t}(y) + (1-\pi_0)\widehat{p}_{0,t}(y)\right) dy. \tag{A25}$$

Let $z = e^{\frac{a}{\sigma}y} \Rightarrow y = \frac{\sigma\ln(z)}{a}$ and $dy = \frac{\sigma}{az}$. Hence, (A25) becomes

$$2\pi_0 \equiv \int_1^\infty w\left(\frac{\pi_0 z}{1-\pi_0+\pi_0 z}\right)\left(\pi_0(\psi-1)z + (1-\pi_0)(\psi+1)\right) z^{-\frac{3+\psi}{2}} dz. \tag{A26}$$

If

$$w(x) = \frac{x}{x+(1-x)\frac{\psi+1}{\psi-1}}. \tag{A27}$$

Then

$$w\left(\frac{\pi_0 z}{1-\pi_0+\pi_0 z}\right)\left(\pi_0(\psi-1)z + (1-\pi_0)(\psi+1)\right) = \pi_0 z(\psi-1). \tag{A28}$$

Therefore, the RHS of (A29) is always equal to the LHS of (A29).

Step 2. Next, we proof that for sufficiently high π_0 the Harris and Holmstrom contract does not induce risk-shifting. It is clear that the manager has the strongest incentives to engage in risk-shifting when her perceived ability achieves an all time maximum. Let it be π_0. Suppose that the manager deviates and gambles during her first trading round but investors believe instead that she has invested in her storage technology at date 0. Let $R \ge 0$ denote the return that she realizes. The present value of her future earnings after the return R is realized and assuming she will no longer gamble from then on is

$$W(\pi_0, R) = \frac{1}{2}\int_1^\infty \max\left\{\frac{\pi_0 R^{\frac{a}{\sigma^2}} z}{\pi_0 R^{\frac{a}{\sigma^2}} z + (1-\pi_0)\frac{\psi+1}{\psi-1}}, \frac{\pi_0}{\pi_0 + (1-\pi_0)\frac{\psi+1}{\psi-1}}\right\} \times$$

$$\times \left(\pi_0(\psi-1)z + (1-\pi_0)(\psi+1)\right) z^{-\frac{3+\psi}{2}} dz. \tag{A29}$$

Direct computations show that $W(\pi_0, R)$ as a function of R is first convex and then concave. Therefore, applying Proposition 1 we can see that the necessary and sufficient conditions for the absence of risk-shifting in this case is that

$$W'_R(\pi_0, 1) \leq W(\pi_0, 1) - W(\pi_0, 0). \tag{A30}$$

Indeed, (A30) implies that $W(\pi_0, R)$ as a function of R is concave over $[1; \infty)$ and therefore, the tangent to $W(\pi_0, R)$ at $R = 1$ is the straight line that solves $P^*(W(\pi_0, R))$. Direct computations show that (A30) takes the following form:

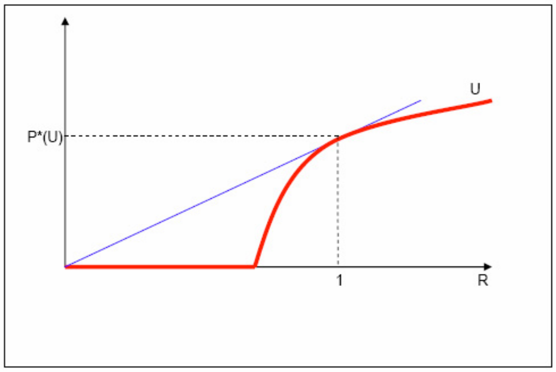
$$\frac{a}{2\sigma^2}(1-\pi_0)\pi_0 \int_1^\infty \frac{(\psi^2-1)\, z^{-\frac{1+\psi}{2}} dz}{\pi_0(\psi-1)z + (1-\pi_0)(\psi+1)} \leq \pi_0 - \frac{\pi_0}{\pi_0 + (1-\pi_0)\frac{\psi+1}{\psi-1}}, \tag{A31}$$

or

$$\frac{1}{2}\int_1^\infty \frac{(\psi^2-1)\left(\pi_0(\psi-1) + (1-\pi_0)(\psi+1)\right) z^{-\frac{1+\psi}{2}} dz}{\pi_0(\psi-1)z + (1-\pi_0)(\psi+1)} \leq \frac{2\sigma^2}{a}. \tag{A32}$$

We are interested in π_0 sufficiently close to 1. Therefore, for no risk-shifting we need

$$(\psi - 1) < \frac{2\sigma^2}{a} \Leftrightarrow \frac{2r}{a} < \frac{\sigma^2}{a} + 1. \blacksquare \tag{A33}$$



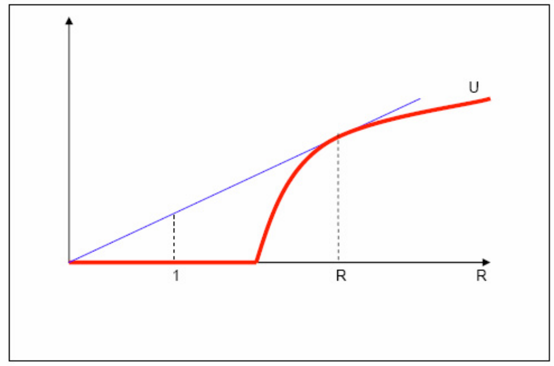


Figure 1

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