Efficient Contracts, Inefficient Equilibria, and Renegotiation*

Edward J. GREEN
Pennsylvania State University

Jia PAN
Fudan University

INCOMPLETE DRAFT: 2009.06.15

Abstract

Consider truthful reporting in a continuum-of-agents model of risk sharing with private 'shocks'. A 'Law of Large Numbers' constrains shock profiles. Any unilateral deviation from truthfulness might also have been truthful under that assumption, so outcomes need only be defined on such profiles in order to verify Bayesian incentive compatibility. Might there be another, Pareto inferior, equilibrium? This phenomenon, akin to a bank run in the Diamond-Dybvig (*JPE,* 1983) model, can occur in an Atkeson-Lucas (*REStud,* 1993) model, reformulated with production and costly capital adjustment, if the full contract is required to be renegotiation proof.

1 Introduction

A theoretical model of a continuum of infinite-lived agents is widely used to study macroeconomic aspects of optimal risk sharing. Agents in this model economy each maximize the expectation of a time-separable utility function, and they experience idiosyncratic, private shocks (independent across both agents and also, for each agent, across time periods) to their respective endowments or preferences. A feasible allocation (subject to both technical and incentive constraints) in this environment is represented as an *incentive-compatible contract.* That is, each agent's consumption is a function of his *reported* shocks, and truth telling is a Bayesian Nash equilibrium. An assumption that the realized distribution of shocks in the population almost surely mirrors the probability law of the shock distribution tightly constrains the shock profiles that can occur. Any unilateral deviation from truthfulness results in a reporting profile that also might have been truthful under that assumption, so the contract need only be defined on such profiles in order to

*This article is a revised and extended version of chapter 2 of Pan (2008) verify its Bayesian incentive compatibility. However, this partial-definition shortcut begs the question of whether there may be another equilibrium, in which *many agents* report untruthfully, that yields an allocation that is Pareto inferior to the one that results from truthful reporting. We will show here that this phenomenon, akin to a bank run in the Diamond and Dybvig (1983) model, does occur in a suitably parametrized version of the Atkeson and Lucas (1993) model with production and costly capital adjustment, if the contract is extended to cover simultaneous deviations in one, natural, way that is partially renegotiation proof.

In the language of implementation theory, a *direct mechanism* is a function that maps every possible profile of reports (or, in an intertemporal model, every possible profile of sequences of reports) to the allocation (of dated commodities, in the intertemporal setting) that will result from that profile.[1] A *contract* is the a pair of functions: a *reporting map* that maps each possible state of the world to a profile of reports, and an *outcome map* that maps profiles of reports to resulting allocations. The outcome map resembles a direct mechanism, but it has a smaller domain. Specifically, the outcome map is defined only on profiles in the image of the state space under the reporting map, and on profiles that are unilateral deviations from such image profiles. A direct mechanism *implements* an allocation as a Bayesian Nash equilibrium (BNE) if it has a unique BNE strategy profile (which is a reporting map), and the mechanism assigns the allocation in question to that strategy profile. A direct mechanism *partially implements* an allocation if it has a BNE strategy profile, to which the mechanism assigns the profile. Similarly, we will say that a contract partially implements a state-contingent allocation if its reporting map is a BNE (which is well defined, as we observed in the previous paragraph) and the state-contingent allocation is the image of the reporting map under the outcome map.[2]

A threshold issue is, does the issue of implementation versus partial implementation matter? In some circumstances, it seems reasonable to argue that truthful reporting is a 'focal point' equilibrium. That is, if it is common knowledge among the agents that truthful reporting is an equilibrium, then they will report truthfully, regardless of whether there are other BNE. The case for this view is particularly strong, perhaps, in a case where truthful reporting results in a Pareto efficient allocation. However, there are other features of an equilibrium that various game theorists have proposed as also constituting a plausible focal point. What both a truthful and a non-truthful equilibrium exist, and the non-truthful equilibrium possesses some such focal-point feature? Diamond and Dybvig (1983) have argued that bank runs (and perhaps, by implication, some other forms of financial-instability episode) should be understood in precisely this way.[3]

[1]The concepts of direct mechanism, Bayesian implementation, and partial implementation are defined formally by Jackson (1991). Definitions specific to the model studied here will be formulated later in this paper.

[2]Strictly speaking, the condition is that the allocation in each state of the world is the image under the outcome map of the image of that state under the reporting map.

[3]Diamond and Dybvig's analysis is not a fully satisfactory application of implementation theory. Ennis

If one is not satisfied with partial implementation, then there is a way of extending the outcome function of a contract to a direct mechanism in a way that guarantees full implementation, in the context of a continuum-of-agents model. This domain-extension procedure, due to Mas-Colell and Vives (1993), involves reversion to a 'punishment allocation' after a detectable, simultaneous deviation from truth telling by a positive measure of agents.[4]

However, there is an issue that, from an ex post perspective, the punishment allocation is Pareto inefficient. It might be supposed, then, that agents would unanimously agree to re-write the contract after a deviation that triggered the punishment allocation had occurred. If the punishment allocation would not actually be imposed in the event that it had been triggered, then the proposed extension of the domain of the outcome function to such an event should not be taken seriously

Immunity to renegotiation is a subtle issue. In principle, the contract could be renegotiated after profiles of reports that are made in equilibrium, as well as after those that are made out of equilibrium. In the type of model to be studied here, if what agents receive in the current period (that is, immediately following the receipt of their reports) can be renegotiated, then the incentive for truthful reporting is undermined. We take a pragmatic approach to this issue. We believe that we observe economic arrangements that are modeled well as incentive-compatible contracts, and that those arrangements are largely immune to renegotiation on a day-to-day basis. However, when those arrangements perform in ways that have not been anticipated to occur in equilibrium, then significant renegotiation does occur. Typically, the renegotiation process takes time, so what agents receive immediately when the unanticipated event occurs is not renegotiated. We assume that the *current-period* allocation specified as a consequence of a report profile cannot be renegotiated, but that the *future* allocation could be renegotiated, and that it must thus be ex post Pareto efficient in order that there should be no scope for renegotiation. In particular, the form of punishment allocation imposed by Mas-Collel and Vives cannot be imposed.

In the specific model environment that we will consider, a deviation from equilibrium by a positive measure of agents implies a loss of capital when contractual promises of current consumption are honored. In such a situation, it is natural to consider proportionately scaling down promises regarding future consumption so that the aggregate of promised consumption is feasible to produce using the diminished capital stock. In the model environment, this form of renegotiation is ex post Pareto efficient, so it satisfies the constraint that has just been proposed. If agents' coefficient of relative risk aversion is greater than

and Keister (2008) provide a fully satisfactory theoretical example of a bank run in an amended version of their model.

[4]recall that, by the 'Law of Large Numbers' assumption made at the beginning of this discussion, some such deviations are detectable, although the identities of the deviators may not be known. Mas Collel and Vives specifically considered autarky as the punishment allocation, but both their idea and the issue regarding renegotiation that will be discussed next are more general.

1, then the direct mechanism derived from an efficient contract by such extension-by-rescaling has at least two Bayesian Nash equilibria. One BNE is truthful reporting, which leads to a Pareto efficient allocation. The other BNE qualitatively resembles a bank run in the Diamond-Dybvig model, and is Pareto inefficient.

2 Economic Environment

Time is discrete, $t = 1, 2, \ldots$.

- Production Technology:

  Assume there is only one type of good in the economy, which can be used either in production or consumption. There is an illiquid production technology with a linear gross rate of return $R_I > 1$ in each period. It is illiquid in the sense that any interruption from the technology will encounter an adjustment cost. Assume the adjustment cost is linear in the value that is adjusted, i.e. given the investment level of the current period I_t, and the investment decision of the next period I_{t+1}, the adjustment cost is given by $\beta(R_I I_t - I_{t+1})$, where β is called the adjustment cost factor, and given exogenously.

- Agents:

  There are a continuum of agents in the economy $i \in A$. Without loss of generality, assume the measure of agents is 1. Agents are born at the beginning of period 1 with an investment portfolio I_1. To consume in later periods, agents have 2 choices. They can either keep their endowments and manage their investment portfolio themselves, or they may transfer their endowments to the bank and get consumption from the bank in later periods. In each period t, agents face an idiosyncratic consumption time preference shock, which can be specified as the way follows: let CRRA function $\rho(c) = \dfrac{c^\gamma}{\gamma}$, where $\gamma \leq 1$, be the utility function for one period and $\theta_t \in (\theta^l, \theta^h)$ be the multiplication factor to $\rho(c_t)$. Without loss of generality, assume $\delta^h > \delta^l$. Thus the utility of an agent for one period is given by $\theta_t \rho(\cdot)$. Agents with θ^h in the period t are relatively impatient compared with agents with θ^l, since they value current consumption higher[5]. Assume agents' consumption time preference types are private information.

[5]Although the utility function of this model is the same as that in Atkeson and Lucas (1983), we can not directly use their solution. This is because in Atkeson and Lucas (1983), they assume the feasible constraint is a constant in every period. While in this model, the feasible is endogenously determined by the investment decision of the bank

If agents decide to deposit into the bank in the initial period, then the initial investment level of the bank is given by $I_1 = \int_A I_1 d\, i$.[6] In each period, agents can withdraw (get consumption goods) from the bank by reporting their current preference types.

In period $t+1$, the agent i's history of preference types can be denoted by $\theta^{i,t} = \{\theta_1^i, \ldots, \theta_t^i\}$. Correspondingly, the history of the consumption level of an agent i is $c^{i,t} = \{c_1^i, \ldots, c_t^i\}$, and the history of the bank's investment portfolio is $I^{t+1} = \{I_1, \ldots, I_{t+1}\}$[7].

Definition 2.1 *Let Θ^1 be the space of consumption preference shocks for one period, $\{\theta^l, \theta^h\}$, and Θ^{t+1} be the $(t+1)$-fold product space. Let μ be the measure on Θ, and μ^{t+1}, the product measure, be the distribution of the history of consumption time preference shocks. Let Θ^∞ and μ^∞ be the corresponding infinite product space and probability measure.*

- the Bank:
  The bank acts as a social planner in the economy. It maximizes the ex ante expected discounted utility of all depositors. The bank accepts the deposits from agents at the beginning of the initial period and then manages the investment portfolio for all depositors. The bank satisfies depositors' consumption needs by adjusting its investment level. The withdrawal amounts of agents' depend on their current and historical reporting profiles.

- Reporting Strategy:
  The reported current consumption time preference type of a single agent depends on his/her realized current preference type as well as his/her own history of preference types, consumption levels and the bank's investment decisions.

Definition 2.2 *Let $j_{t,z^i} = z_t^i(\theta^t, c^{t-1}, I^t)$ be the report type of any agent i in period t given the historical and current preference types θ^{t-1}, θ_t, the history of consumption values c^t and the history of the bank's investment decisions I^t. Denote $z^i = \{z_t^i\}_{t=1}^\infty$ as the reporting strategy of any agent i. Let Z be the space of reporting strategies that an agent can adopt. Denote $j_{z^i}^t$ as the history of the reported types of agent i.*

[6] Since all agents are identical ex ante, if there exists an agent who deposits his/her endowment into the bank, all agents deposit into the bank. Thus in this case we consider that all agents deposit in the bank.

[7] Note that the bank's investment level of current period is determined in the previous period.

Definition 2.3 *The truth telling reporting strategy is the reporting strategy* $z^* \in Z$*, which satisfies, for all* $t \geq 1$, $\theta^t \in \Theta^t$, $c^{t-1} \in \mathcal{R}_+^{t-1}$, *and* $I^t \in \mathcal{R}_+^t$, $j_{t,z^*} = z_t(\theta^t, c^{t-1}, I^t) = \theta_t$.

3 An Incentive Compatible Deposit Contract

Due to the assumption that the time preference types of agents are private information, the first best allocation (full insurance allocation) violates the incentive compatible condition[8]. In this section, we want to design an optimal deposit such that it maximizes the agents' utility. At the same time it is subject to the incentive compatible constraints that prevent full insurance being offered[9].

The incentive compatible problem here is solved by conditioning the current consumption value (current withdrawal) of agents not only on their current consumption preference reports but also on the history of their reported preference types.
Denote the agents' expected discounted utility in each periodas their **bank balance**of that period.

$$w_{t+1} = \sum_{\tau=1}^{\infty} \delta^{t+\tau-1} \int_{\Theta^\tau} \theta_{t+\tau}\rho(C_{t+\tau}(w_1, \{\theta^t, \theta_{t+1}, \ldots, \theta_\tau\}))d\mu^\tau.$$

Assume when agents get the consumption goods c_t from the bank, they will be notified of their updated balance w_{t+1}.
Thus by allowing the current impatient agents to have a higher current consumption and a lower bank balance in the next period, it is incentive compatible for them to report their true types. For patient agents, although they will have a lower current consumption, the bank balance in the next period is higher. Therefore, it is also incentive compatible for patient agents to report their true consumption types.

Let C_t be the allocation function in each period t, $C_t : R_+ \times \Theta^t \mapsto R_+$, that is the consumption value of an agent in each period depends on his/her own current reported preference type $j_{t,z}$ as well as the historical reports j_z^t. As all agents are identical ex ante, moreover assume they all use the truthful reporting strategy, they will have an identical initial bank balance, which is uniquely determined by the endowment level I_1. Denote it as w_1.

[8]Under the first best allocation, in each period, all agents have the same marginal utility value. That is, patient agents can get a higher consumption value by claiming to be impatient. Thus it is not incentive compatible for current patient agents to tell the truth.

[9]Recall that the allocation of a deposit contract only specifies the consumption of agents on the truth telling equilibrium path. In this section, we do not consider the joint deviation case.

Let D be the space of utility values[10]. $\rho^{-1}(v)$ denotes its inverse function of $\rho(c)$, $\rho^{-1} : D \mapsto \mathcal{R}_+$ characterizes the consumption goods which is needed to have utility value v in one period. Thus ρ defines a one-to-one mapping between an allocation $\{C_t(w_1, \theta^t)\}$ and a sequence of utility functions $\{u_t(w_1, \theta^t)\}_{t=1}^{\infty} = \{\rho(c_t(w_1, \theta^t))\}_{t=1}^{\infty}$. Let u_C be the utility sequence corresponding to a given allocation C. Afterwards, we focus on studying the utility sequence as well.

Definition 3.1 *Define U as the ex ante expected discounted utility function given the allocation C and reporting strategy z,*

$$U(w_1, u_C, z) = \sum_{t=1}^{\infty} \delta^{t-1} \int_{\Theta^t} \theta_t u(w_1, j_z^t) d\mu^t.$$

Definition 3.2 *Define U_t as the present expected discounted utility function in period t for $t > 1$, given the ex ante expected discounted utility value w_1, the allocation C, the reporting history $j_{\hat{z}}^t$, and the future reporting strategy z,*

$$U_t(w_1, u_C, j_{\hat{z}}^t, z) = \sum_{\tau=t}^{\infty} \delta^{t-1} \int_{\Theta^{\tau-t}} \theta_\tau u(w_1, (j_{\hat{z}}^t, j_z^{\tau-t})) d\mu^{\tau-t}.$$

Next we focus on the allocations of the deposit contract whose corresponding utility value function sequence satisfies the conditions described below:

1. Transversality condition: $\{u_t(w_1, \theta^t)\}$ satisfies:

$$\lim_{t\to\infty} \sum_{\tau=1}^{\infty} \delta^{t+\tau-1} \int_{\Theta^\tau} \theta_{t+\tau} u_{t+\tau}(w_1, \theta^{t+\tau})\, d\mu^\tau = 0. \tag{1}$$

2. Promise keeping condition: $\{u_t(w_1, \theta^t)\}_{t=1}^{\infty}$ delivers the ex ante expected discounted utility value to each agent,

$$w_1 = U(w_1, u, z^*) \tag{2}$$

3. Incentive compatible condition: $\{u_t(w_1, \theta^t)\}_{t=1}^{\infty}$ satisfies

$$U(w_1, u, z^*) \geq U(w_1, u, z) \tag{3}$$

   for all $z \in Z$

[10] Assume that both the utility value for one period $\rho(c_t)$ and the ex ante expected discounted utility value $\sum_t \delta^{t-1}\theta_t \rho(c_t)$ belong to D, i.e. D is an interval on R.

4. Feasibility condition: it is feasible for u_C to deliver the ex ante expected discounted utility w_1 to all agents with initial investment level I_1[11]:

$$\sum_{t=1}^{\infty} \left(\frac{1}{R_I(1-\beta)} \right)^t \{ \int_A \rho^{-1}(u_t)\, di \} \leq I_1 \tag{4}$$

Let S be the space of all utility sequences u which satisfy the above conditions (1), (2), (3) and (4). Thus, for any $u = \{u_t\} \in S$, $\{\rho^{-1}(u_t)\}$ defines a feasible allocation of the deposit contract.

Following the same approach as in Atkeson and Lucas (1993), instead of solving the maximization problem of the bank we consider the following problem: let $\mathcal{D}$ be the Borel measurable subsets on D and let M be the space of all probability measures on $\mathcal{D}$. Denote $\varphi^*(\psi)$ as the greatest lower bound of initial investment level needed to attain the expected discounted utility distribution ψ, which maps distributions of utility to the real line.

$$\varphi^*(\psi_1) = inf_{\{u_t\} \in S} \sum_{t=1}^{\infty} \int_{D \times \Theta^t} (\frac{1}{R_I})^t (1+\beta) \rho^{-1}[u_t(w_1, \delta^t)]\, d\psi d\mu^t \}.$$

For the problem stated here, the expected discounted utility value distribution in the initial period is a degenerate distribution with $P(w = w_1) = 1$,

In order to solve the functional form of φ^*, we formulate the problem in a recursive way. The consumption value in period 1 $C_1(w_1, j_{1,z})$ and the updated balance value $w_2 = g_1(w_1, j_{1,z})$ depend on the initial bank balance value w_1 and the reported consumption preference type $j_{1,z}$ in period 1. In period 2, the consumption value $C_2(w_2, j_{2,z})$ and the updated bank balance $w_3 = g_2(w_2, j_{2,z})$ are given according to the updated balance value w_2 and consumption preference report $j_{2,z}$. In period 3, the bank chooses another pair of functions of $w_3, j_{3,z}$ and so on. Identify each agent with their bank balance value w_t, all agents with the bank balance value w_t will have the same treatment.

Fix $\rho(\cdot)$ as the utility function for one period, the above problem can be reformulated in the following way: given the bank balance w_t and current reporting preference type $j_{t,z}$, the bank chooses a pair of Borel measurable functions (f_t, g_t) in each period, where $f_t(w_t, j_{t,z}) = \rho(C_t(w_t, j_{t,z}))$ is the utility value from current period consumption, and $g_t(w_t, j_{t,z}) = w_{t+1}$ is the updated bank balance. Thus the agents' bank balance values will be updated every time they make a preference type report. Moreover, the current

[11] The allocation function fully determines the investment plan of the bank by $R_I I_t - I_{t+1} = \int_A c(w_1, j_{z^i}^t) di + \beta(R_I I_t - I_{t+1})$. Thus if it is feasible for u_C to deliver ex ante expected discounted utility value with the initial investment level I_1, then there must exist a feasible investment plan $\{I_t\}_{t=2}^{\infty}$, for all $t \geq 2$, $I_t > 0$.

consumption value and the updated balance value depend on their current balance values and current preference reports.

Given the distribution of initial balance values ψ_1 in the initial period, g_1 defines an operator $S_{g_1} : M \mapsto M$ for any $D_0 \in \mathcal{D}$

$$(S_{g_1}\psi_1)(D_0) = \int_{B_{g_1}(D_0)} d\mu d\psi_1$$

where $B_{g_1}(D_0) = \{(w_1, \theta_1) \in D \times \Theta : g_1(w_1, \theta) \in D_0\}$. Thus $S_{g_1}\psi_1$ determines the utility distribution in period 2. Repeating the steps above, $\{f_t, g_t\}$ generates a sequence of utility distribution $\{\psi_t\}_{t=1}^{\infty}$, where $\psi_{t+1} = S_{g_t}\psi_t$.

Let σ be defined as $\sigma = \{f_t, g_t\} :\ f_t, g_t :\ D \times \Theta :\mapsto D \times D$, σ is called an allocation rule if it satisfies the following conditions:

1. Transversality condition:
   a. if $u_t(w_1, j_z^t) = f_t[W_t(w_1, j_z^{t-1}), j_{t,z}]$ is the utility sequence generated by σ, then it satisfies the equation (1);
   b.For $t > 1$, let W_t be a function induced by w_t, $W_t(w_1, j_z^t) = w_t : D \times \Theta^{t-1} \mapsto D$. $\{W_t\}$ satisfies the transversality condition as follows: given $w_1 \in D$ and all $\{\theta^t\} \in \Theta^{\infty}$

$$\lim_{t\mapsto\infty} \delta^{t-1} W_t(w_1, \theta^t) = 0; \tag{5}$$

2. Temporary promise keeping condition: in each period t, f_t, g_t delivers the expected discounted utility value w_t to all agents with w_t as their bank balance value,

$$w_t \ = \ \int_{\Theta} [\theta_t f_t(w_t, \theta) + \delta g_t(w_t, \theta)\, d\mu]. \tag{6}$$

3. Temporary incentive compatible condition [12]: for all $t \geq 0$ all $w_t \in D$, and $\theta_t = \theta^i, \theta^j \in \Theta,\ i \neq j$,

$$\theta^i f_t(w_t, \theta^i) + \delta g_t(w_t, \theta^i) \ \geq \ \theta^i f_t(w_t, \theta^j) + \delta g_t(w_t, \theta^j). \tag{7}$$

4. Feasibility constraint: it is feasible for the allocation rule σ to attain the utility distribution ψ_1 with initial investment level I_1, if

$$\sum_{t=1}^{\infty} (\frac{1}{(1-\beta)R_I})^t \{\int_{D\times\Theta} \rho^{-1}(f_t(w_t, \theta_t))\, d\psi_t d\mu\} \leq I_1. \tag{8}$$

[12]The terminology follows the name in Green (1987)

The following proposition shows the equivalence between an allocation C (u_C) and an allocation rule σ. Thus the optimal allocation of the deposit contract can be characterized by the optimal allocation rule.

Proposition 3.3 *Let ψ_1 be a degenerate distribution of the ex ante expected discounted utility value $P(w = w_1) = 1$ for any $w_1 \in D$,. If there exists an allocation C, such that it is feasible for C to deliver the ex ante expected discounted utility w_1 with the initial investment level I_1, then there exists a feasible allocation rule σ that attains ψ_1 with initial investment I_1. If there exists an allocation rule σ attains ψ_1 with initial investment I_1, let $\{u_t\}$ be the expected discounted utility sequence generated by σ, and $\{u_t\} \in S$.*

In order to show the equivalence between these 2 problems, we need the following lemma, which says that if an allocation is incentive compatible, then given an arbitrary reporting history $\hat{j}_{\hat{z}}^{r-1}$, it is always optimal to adopt the truthful reporting strategy from period r and on.

Lemma 3.4 *A utility sequence satisfies (3) if and only if it satisfies :*

$$\begin{aligned} \theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, \theta_r)) \quad &+ \quad \delta U_{r+1}(w_1, u, j_{\hat{z}}^{r-1}, z^*) \\ &\geq \quad \theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, j_{z,r})) + \delta U_{r+1}(w_1, u, j_{\hat{z}}^{r-1}, z) \end{aligned} \tag{9}$$

for any $w_1 \in D$, $r \geq 0$, $\hat{j}_{\hat{z}}^{r-1} \in J^{r-1}$, reporting strategies $z \in Z$ and $j_{z,r} \neq \theta_r \in \Theta$.

With lemma (3.4) we can verify proposition (3.3). The logic of this proof is very similar to the proof of Lemma (3.2) and Lemma (3.3) in Atkeson and Lucas (1993). Their feasibility constraint models an exogenous, constant endowment in each period, while in our problem, the feasibility constraint is endogenously determined by the bank's investment decision. The detail of the proof is given in the appendix.

Recall that the definition of $\varphi^* : M \mapsto R_+ \cup \{+\infty\}$: for any utility distribution $\psi \in M$, $\varphi^*(\psi)$ is the infinimum of the investment levels needed such that there exists an allocation rule σ that attains ψ. Let $\varphi^*(\psi) = +\infty$ if the distribution of utility value ψ can not be attained by any finite investment level. Define that an allocation rule σ is efficient if it attains ψ with investment level $\varphi^*(\psi)$.

To characterize $\varphi^*(\cdot)$, we consider a Bellman equation as follows: denote B as the set of Borel measurable functions $D \times \Theta \mapsto D \times D$,

$$\varphi(\psi) = \inf_{f,g \in B} \{ \frac{1}{R_I} \int_{D \times \Theta} (1+\beta)\rho^{-1}[f(w,\theta)] + \varphi(S_g \psi) d\mu d\psi \} \tag{10}$$

such that $f <$ and g[13] satisfy the temporary promise keeping condition: for all $w \in D$,

$$w = \int_\Theta [\theta f(w,\theta) + \delta g(w,\theta)] d\mu; \tag{11}$$

and the temporary incentive compatible condition,

$$\theta^i f(w,\theta^i) + \delta g(w,\theta^i) \geq \theta^i f(w,\theta^j) + \delta g(w,\theta^j) \tag{12}$$

for all $w \in D$ and $\theta^i, \theta^j \in \Theta$ and $i \neq j$.
In the remaining part of this section, we will show that φ^* is the solution to the Bellman equation (10) first. And then we will illustrate the steps to solve this Bellman equation.

Let X be the set of functions $\varphi : M \mapsto R_+ \cup \{+\infty\}$. Define the operator $T : X \mapsto X$ as follows:

$$(T\varphi)(\psi) = \inf_{f,g \in B} \{ \frac{1}{R_I} \int_{D\times\Theta} (1+\beta)\rho^{-1}[f(w,\theta)] + \varphi(S_g\psi) d\mu d\psi \} \tag{13}$$

subject to condition (11) and (12).

Proposition 3.5 *φ^* is the fixed point of T.*

See the detail of the proof in the appendix.

Since we assume the return from the liquid investment is homogeneous of degree 1 and we use a CRRA function as the utility function for one period $\rho(c) = \dfrac{c^\gamma}{\gamma}$ with $\gamma < 1$, the distribution of utility value (the bank balance value) is multiplicate to the cost minimization problem. Thus we can solve the Bellman equation by solving a static problem as follows:

$$\phi(\alpha) = \min_{r,y} \{ \frac{1}{R_I} \int_\Theta \frac{\rho^{-1}(r(\theta))}{(1-\beta)} + \alpha(\frac{\gamma}{|\gamma|} y(\theta))^{\frac{1}{\gamma}} \, d\mu \} \tag{14}$$

subject to

$$\frac{\gamma}{|\gamma|} = \int_\Theta \theta r(\theta) + \delta y(\theta) \, d\mu \tag{15}$$

and

$$\theta^i r(\theta^i) + \delta y(\theta^i) \geq \theta^i r(\theta^j) + \delta y(\theta^j) \tag{16}$$

for $i \neq j \in \{l, h\}$.
The static problem above is in fact a special case of the Bellman equation (10). The fixed point α^* to the static problem (14) is the greatest lower bound of the initial investment level to attain the degenerate distribution of the ex ante expected discounted utility value, $P(w_1 = \dfrac{\gamma}{|\gamma|}) = 1$.

[13]Since $(f_t \; g_t)$ is the solution to the time invariant Bellman equation (10), $(f_t \; g_t)$ is in fact time invariant. Thus afterwards, it is equivalent to write $(f \; g)$ as well.

Lemma 3.6 *For any $\alpha > 0$, the minimum in problem (14) is attained by a unique* $(r(\theta,\alpha), y(\theta,\alpha)$.

Lemma 3.7 *The function ϕ defined in problem (14) has a unique fixed point $\alpha^* \in [\alpha_c, \alpha_a]$ and* $\lim_{n\to\infty} \phi^n(\alpha) = \alpha^*$.

The details of the proof for lemma (3.6) and lemma (3.7) can be found in the appendix.

Lemma (3.6) and lemma (3.7) guarantee that there exists a unique fixed point to the static problem (14).

Let α^* be the fix point of the static problem (3.5) and $(r(\theta,\alpha^*), y(\theta,\alpha^*))$ be the solution to the problem $\phi(\alpha^*)$. Denote $r^*(\theta) = r(\theta,\alpha^*)$, and $y^*(\theta) = y(\theta,\alpha^*)$. $(r^*(\theta), y^*(\theta))$ can generate the optimal solution to the Bellman equation (10) as follows:

$$\begin{aligned} f(w,\theta) &= \frac{\gamma}{|\gamma|} w r^*(\theta) \\ g(w,\theta) &= \frac{\gamma}{|\gamma|} w y^*(\theta). \end{aligned} \tag{17}$$

$\varphi^*(\psi)$ can be derived in the following way:

$$\varphi^*(\psi) = \alpha \int_D (\frac{\gamma}{|\gamma|} w)^{\frac{1}{\gamma}}\, d\psi. \tag{18}$$

To characterize the solution to the static problem (14), we rewrite the discrete form of the static problem (14) as follows: given $\mu(\theta = \theta^l) = p$, $\mu(\theta = \theta^h) = 1 - p$,

$$\begin{aligned} \phi(\alpha) &= \min_{(r(\theta^l),y(\theta^l)),(r(\theta^h),y(\theta^h))} \{ \frac{1}{R_I} \{ p \left[\frac{\rho^{-1}(r(\theta^h))}{(1-\beta)} + \alpha(\frac{\gamma}{|\gamma|} y(\theta^h))^{\frac{1}{\gamma}} \right] \\ &\quad +(1-p) \left[\frac{\rho^{-1}(r(\theta^l))}{(1-\beta)} + \alpha(\frac{\gamma}{|\gamma|} y(\theta^l))^{\frac{1}{\gamma}} \right] \} \end{aligned} \tag{19}$$

subject to:

$$p(\theta^h r(\theta^h) + \delta y(\theta^h)) + (1-p)(\theta^l r(\theta^l) + \delta y(\theta^l)) = \frac{\gamma}{|\gamma|}; \tag{20}$$

$$\theta^h r(\theta^h) + \delta y^h \geq \theta^h r(\theta^l) + \delta y(\theta^l); \tag{21}$$

$$\theta^l r(\theta^l) + \delta y(\theta^l) \geq \theta^l r(\theta^h) + \delta y(\theta^h) \tag{22}$$

Let λ, ζ, η be the Lagrange coefficient to the above 3 constraints correspondingly. The first order condition with respect to r^l, r^h, y^l, y^h can be written as follows:

$$
\begin{aligned}
\frac{1}{R_I(1-\beta)}p(\rho^{-1})'(r^l) &= \lambda\theta^h p + \zeta\theta^h - \eta\theta^l; &(23)\\
\frac{1}{R_I(1-\beta)}(1-p)(\rho^{-1})'(r^h) &= \lambda\theta^l(1-p) - \zeta\theta^h + \eta\theta^l; &(24)\\
\frac{1}{R_I}\alpha\frac{1}{|\gamma|}p(\frac{\gamma}{|\gamma|}y^l)^{\frac{1}{\gamma}-1} &= \lambda p\delta + \zeta\delta - \eta\delta; &(25)\\
\frac{1}{R_I}\alpha\frac{1}{|\gamma|}(1-p)(\frac{\gamma}{|\gamma|}y^h)^{\frac{1}{\gamma}-1} &= \lambda(1-p)\delta - \zeta\delta + \eta\delta &(26)
\end{aligned}
$$

Lemma 3.8 *The optimal solution to problem (19) satisfies:*

$$
\begin{aligned}
\theta^h r(\theta^h,\alpha^*) + \delta y(\theta^h,\alpha^*) &> \theta^h r(\theta^l,\alpha^*) + \delta y(\theta^l,\alpha^*)\\
\theta^l r(\theta^l,\alpha^*) + \delta y(\theta^l,\alpha^*) &= \theta^l r(\theta^h,\alpha^*) + \delta y(\theta^h,\alpha^*)
\end{aligned}
$$

See the appendix for the proof.

Corollary 3.9 *Under the optimal allocation rule, patient agents in the current period are indifferent with claiming to be impatient or patient, and impatient agents are strictly better off by telling the truth.*

Proof The proof of this corollary directly follows from lemma (3.8) and equation (17). □

Given the set of parameter values of R_I, β, θ^h, θ^l, p, γ and δ, the static problem (14) can be solved numerically.[14]

Example 3.10 *Let $R_I = 1.2$, $\beta = 0.2$, $\theta^l = 0.5$, $\theta^h = 1.5$, $p = 0.5$ and $\gamma = -1$. The numerical solution to (14) is given by $\alpha = 54.3344$, $r(\theta^h,\alpha) = -0.1139$, $y(\theta^h,\alpha) = -0.9845$ and $r(\theta^l,\alpha) = -0.1916$, $y(\theta^l,\alpha) = -0.9414$.*

According to the computation result, if the initial investment level of the bank's portfolio is given by $I_1 = 1$, then the ex ante discounted utility value of each agent by depositing in the bank is given by $w_1 = -(\frac{1}{54.3344})^{-1} = -54.3344$. The expected discounted utility value in the autarkic case is $\hat{w} = -54.6664$.[15] As expected, depositing in the bank can give agents a higher ex ante discounted utility value than in the autarkic case $w_1 > \hat{w}_1$.

[14] See the appendix for the numerical algorithm

[15] The procedure can be found in the appendix.

In the initial period, an agent with current preference type θ^l *will have the current utility value* $f(w_1, \theta^l) = -(-54.3344)(-0.1916) = -10.4105$*, that is, the current consumption value is given by* $c_1 = \dfrac{1}{10.4105} = 0.0961$*, and the updated bank balance is* $g(w_1, \theta^l) = -50.2090$*. Similarly an agent with current preference type* θ^h *will have the current utility value* $f(w_1, \theta^h) = -(-54.3344)(-0.1139) = -6.1887$*, that is, the current consumption value is given by* $c_1 = \dfrac{1}{6.249} = 0.1616$*, and the updated bank balance is* $g(w_1, \theta^h) = -52.5077$*.*

4 A Direct Mechanism

This section will describe an efficient direct payment mechanism such that the truth telling reporting strategy can be supported as an equilibrium result. This allocation of the mechanism is constructed according to the optimal allocation rule of the incentive compatible deposit contract in the previous section. Moreover, we show that except for the truth telling equilibrium, there may also exist an inefficient equilibrium where all agents claim to be impatient regardless of their true consumption preference types.[16]

The allocation rule of the incentive compatible deposit contract only characterizes what happens when all agents report truthfully. In this section, we would like to specify the allocation rule given any arbitrary reporting strategy profile, i.e. the case when there exists a joint deviation.[17] Then we will show that the truth telling reporting strategy is a Perfect Bayesian Nash equilibrium of this mechanism, according to our specification. Further analysis shows that truth telling may not be the unique equilibrium. In fact, if $\gamma < 0$, then we verify that all agents claim to be impatient regardless of their true preference types is also an equilibrium of the mechanism. While for $\gamma > 0$, we use a numerical examples, when $\gamma = 0.5$, to show that truth telling may be the unique equilibrium reporting strategy, and we believe this is correct for all $0 < \gamma < 1$.

4.1 Equilibrium

4.1.1 Reporting Strategy Profile

Since all agents are identical ex ante, it is adequate to consider the set of symmetric reporting strategy profiles.

[16]This inefficient equilibrium is an analogue of "run equilibrium" in Diamond and Dybvig (1983).

[17]In this section, "joint deviation" means that a group of agents with positive measure will deviate. Since we only consider the symmetric reporting strategy profile, there are only 3 possible scenarios of the reporting profiles: all agents are telling the truth; all agents claim tobe patient regardless of their true preference types or all agents claim to be impatient regardless of their true preference types.

Definition 4.1 *Let* $\widetilde{Z}_t = \{\widetilde{z}_t : I \mapsto Z_t | \widetilde{z}_t$ *is a constant map*$\}$*. Let* $\widetilde{j}_{t,\widetilde{z}} = \prod_{i \in I} j_{t,z^i}$*, where* $\widetilde{z}_t = \prod_{i \in I} z_t^i \in \widetilde{Z}_t$*. Thus* $\widetilde{Z} = \{\widetilde{z} | \widetilde{z} = \lim_{t \to \infty} \widetilde{z}_t\}$ *is the space of symmetric reporting strategy profiles and* $\widetilde{J}^t = \Theta^{I \times t}$*.*

4.1.2 Allocation Function

Given any history of reporting profiles, the allocation function of an agent $\{\widetilde{C}_t\}$ depends on the investment portfolio of the bank in the current period [18] as well as the ex ante expected discounted utility value [19], his/her current, and historical reporting preference type.

Definition 4.2 *Define the allocation function of an agent in period* t*,* $\widetilde{C}_t(w_1, j^t_{z^i}, I_t) : D \times \Theta^t \times \mathcal{R}_+ \mapsto \mathcal{R}_+$*.*

4.1.3 Feasibility Constraint

Definition 4.3 *The bank's investment decision function in each period* K_t *is a function of the reporting profile of the current period* $\widetilde{j}_{\widetilde{z}_t} \in \Theta^I$ *and the investment level of the current period* I_t*:* $K_t : \Theta^I \times \mathcal{R}_+ \mapsto \mathcal{R}_+$*.*

Definition 4.4 *Given the reporting strategy profile* $\widetilde{z} = \prod_{i \in A} z^i \in \widetilde{Z}$*, define* $\hat{p}_{t,\widetilde{z}}$ *as the measure of agents claiming to be impatient in period* t*,*

$$\hat{p}_{t,\widetilde{z}} = \mu(i \in A | j_{t,z^i} = \theta^l).$$

Thus the feasibility constraint of the bank is given by the following:
for $t \geq 0$ and all $\widetilde{z} \in \widetilde{Z}$:

$$\begin{aligned} 0 &\leq I_{t+1} = K(\widetilde{j}_{t,\widetilde{z}}, I_t), \\ \int_\Theta \widetilde{C}_t(w_1, j^t_z, I_t)\, d\mu + \beta(R_I I_t - I_{t+1}) &= R_I I_t - I_{t+1}, \end{aligned} \tag{27}$$

where $I_{t+1} = K(\widetilde{j}_{t,\widetilde{z}}, I_t)$.

[18] The investment portfolio of the bank in the current period contains the information of the reporting profile history

[19] Given the initial endowment level, and if all agents choose to use the truthful reporting strategy, then each agent will have an identical ex ante expected discounted utility value w_1. In turn, the initial endowment level is determined, given the allocation function and the ex ante expected discounted utility value when all agents adopt the truthful reporting strategy.

4.1.4 Definition of Equilibrium

The equilibrium concept we will use in this paper is the Perfect Bayesian Nash equilibrium, which means in each period, agents choose their reporting types to maximize their utility conditional on his/her information of that period.

In this economy, agents can use one reporting strategy first and switch to another one at any time with no cost.[20]
As in the previous section, the allocation sequence determines a unique utility sequence $\widetilde{u}_{\widetilde{C}}$ corresponding to the given allocation $\widetilde{C}$.

For any $\widetilde{z} \in \widetilde{Z}$ and $z^i \in Z$, denote $(\widetilde{z} \backslash_i z^i)(j)$ as

$$(\widetilde{z} \backslash_i z^i)(j) = \begin{cases} z^j & \text{if } j \neq i \\ z^i & \text{if } j = i \end{cases} \tag{28}$$

With the above notation, we define the total expected discounted utility value function as follows:

Definition 4.5 *Denote $\widetilde{U}(w_1, u_C, z^i \times \widetilde{z} \backslash z^i)$ as the total expected discounted utility function of agent i.*

$$\widetilde{U}(w_1, u_{\widetilde{C}}, z^i \times \widetilde{z} \backslash z^i) = \sum_{t=1}^{\infty} \delta^{t-1} \int_{\Theta^t} \theta_t \widetilde{u}_t(w_1, j_{t,z^i}, I_t) d\mu^t \tag{29}$$

Similarly we can define $\widetilde{U}_t$ as the present expected discounted utility function in period t:

Definition 4.6 *In period $t > 1$, the function of the present expected discounted utility in period t, $\widetilde{U}_t(w_1, u_{\widetilde{C}}, j_{\hat{z}^i}^{t-1}, z^i \times \widetilde{z} \backslash z^i, I_t)$ is a function of the ex ante expected discounted utility value w_1 the allocation $\widetilde{C}$, his/her reporting history $j_{\hat{z}^i}^{t-1}$, his/her reporting strategy planned to use in the future z^i, and the investment portfolio of the bank in the current period I_t,*

$$\widetilde{U}_t(w_1, u_{\widetilde{C}}, j_{\hat{z}^i}^{t-1}, z^i \times \widetilde{z} \backslash z^i, I_t) = \sum_{\tau=1}^{\infty} \int_{\Theta^\tau} \delta^\tau \theta_{t+\tau} \widetilde{u}_{t+\tau}(w_1, (j_{\hat{z}^i}^t, j_{z^i}^\tau), I_{t+\tau}))\, d\mu^\tau.$$

Thus for each agent, $i \in A$, will choose $z^i \in Z$ to maximize their total expected discounted utility value.

Definition 4.7 *Given the ex ante expected discounted utility w_1, the allocation $\widetilde{C}$, $\widetilde{z}$ is then a Bayesian Nash Equilibrium reporting strategy profile if it satisfies for all $i \in A$, and $z^i \in Z$,*

$$\widetilde{U}(w_1, \widetilde{u}_{\widetilde{C}}, \widetilde{z}) \geq \widetilde{U}(w_1, \widetilde{u}_{\widetilde{C}}, z^i \times \widetilde{z} \backslash_i z^i). \tag{30}$$

[20]In general, this switch can be done repeatedly in any period.

Definition 4.8 *An allocation is incentive compatible if the truthful reporting strategy satisfies the condition (30).*

Definition 4.9 *Given the ex ante expected discounted utility* w_1*, and the allocation* $\widetilde{C}$*, the truthful reporting strategy* z^* *is a Perfect Bayesian Nash Equilibrium if it is incentive compatible and satisfies the following condition: for any* I_r*,* $\widetilde{j}^r_{\hat{z}}, z^i \in Z$*,* $r > 1$*,*

$$\widetilde{U}_r(w_1, u_{\widetilde{C}}, \widetilde{j}^r_{\hat{z}}, \widetilde{z}^*) \geq \widetilde{U}_r(w_1, u_{\widetilde{C}}, \widetilde{j}^r_{\hat{z}}, z^i \times \widetilde{z}^* \backslash_i z^i).$$

Given the underlying probability of agents being impatient in each period is p, denote $\widetilde{\sigma}^p = \{\widetilde{f}^p, \widetilde{g}^p\}$ as the optimal allocation rule of the direct mechanism, where $\widetilde{f}^p(w_t, \theta_t)$ is the current utility value, and it is only a function of the current bank balance w_t and the current preference report $j_{t,z}$; $\widetilde{g}^p(w_t, \theta_t)$ is the updated bank balance and it is a function of the investment level of the bank in the next period I_t as well as the current bank balance w_t, and the current preference report θ_t.

Let $\{\widetilde{u}_t\}$ be the utility sequence generated by the allocation rule of the direct mechanism,

$$\widetilde{u}_t(w_1, j^t_z, I_{t-1}) = \widetilde{f}^p(w_t, j_{t,z}),$$

where w_t is given by

$$w_t = \widetilde{g}^p_{t-1}(w_{t-1}, j_{t-1,z}, I_t).$$

The allocation rule of the direct mechanism can be specified as follows:

1. The allocation rule of the direct mechanism coincides with the optimal allocation rule of the deposit contract if there is no joint deviation.

2. If there does exist a joint deviation:
   the current consumption $\widetilde{f}^p(w_t, j_{t,z})$ is not affected by the deviation,

   $$\widetilde{f}^p(w_t, j_{t,z}) = f^p(w_t, j_{t,z}),$$

   while the bank balance is updated according to the actual investment level of the bank.

   Given the reporting strategy profile $\widetilde{z}$, the law of motion of the bank's investment portfolio $I_{t+1} = K(\widetilde{j}_{t,\widetilde{z}}, I_t)$ is:

$$\begin{aligned}
K(\widetilde{j}_{t,\widetilde{z}}, I_t) &= R_I I_t - \frac{1}{1-\beta}\int_{D\times\Theta}(\rho^{-1})(f^p(w_t, j_{t,z}, I^{t-1}))\,d\psi_t\,d\mu \\
&= R_I I_t - \frac{1}{1-\beta}\int_{D\times\Theta}(\rho^{-1})(w_t r^p(j_{t,z}))\,d\psi_t\,d\mu \\
&= R_I I_t - \frac{1}{1-\beta}\int_{D\times\Theta}(\frac{\gamma}{|\gamma|}w_t)^{\frac{1}{\gamma}}(\rho^{-1})(r^p(j_{t,\widetilde{z}})\,d\psi_t\,d\mu \\
&= R_I I_t - \frac{1}{1-\beta}(\hat{p}_{t,\widetilde{z}}(\rho^{-1})(r^p(\theta^l)) \\
&\quad +(1-\hat{p}_{t,\widetilde{z}})(\rho^{-1})(r^p(\theta^h))\cdot\int_D(\frac{\gamma}{|\gamma|}w_t)^{\frac{1}{\gamma}}\,d\psi_t
\end{aligned}$$

The updated bank balance of agents should be feasible for the actual investment portfolio of the bank:

$$\alpha^p\cdot(\frac{\gamma}{|\gamma|}\int_D w_{t+1}\,d\psi_{t+1})^{\frac{1}{\gamma}} \leq I_{t+1} = K(\widetilde{j}_{t,\widetilde{z}}, I_t)$$

and this constraint is binding in the optimal case.

In this paper we consider the most straight forward way of adjustment in which $N(\cdot,\cdot)$ is linear in its first argument[21]:

$$\begin{aligned}
w_{t+1} = \widetilde{g}^p_t(w_t, j_{t,z}, K(\widetilde{j}_{t,\widetilde{z}}, I_t)) &= N(g^p_t(w_t, j_{t,z}), K(\widetilde{j}_{\widetilde{z}^*,t}, I_t)) \\
&= \frac{\gamma}{|\gamma|}(\frac{K(\widetilde{j}_{t,\widetilde{z}}, I_t)}{\alpha(p)})^{\gamma}\frac{g^p_t(w_t, j_{t,z})}{\int_{D\times\Theta} g^p_t(w_t, j_{t,z})\,d\mu\,d\psi_t}
\end{aligned}$$

Thus in each period t, given the expected discounted utility value w_t, and current preference type shock θ_t, an agent chooses the reporting type $j_{t,z} \in \Theta$ to maximize his/her expected discounted utility value:

$$\max_{j_{t,z}\in\Theta}\{\theta_t\widetilde{f}^p_t(w_t, j_{t,z}) + \delta\widetilde{g}^p_t(w_t, j_{t,z}, K(\widetilde{j}_{t,\widetilde{z}}, I_t))\} \tag{31}$$

Theorem 4.10 *Given the allocation rule specified as above, the truth telling reporting strategy is an equilibrium reporting strategy of the direct mechanism.*

Proof According to the definition of an equilibrium reporting strategy, given the ex ante expected discounted utility value w_1, and utility sequence $\{\widetilde{u}_t\}$, where $\widetilde{u}_t(w_1, j^t_z, I_t) =$

[21]Due to the homogeneity 1 of both investment and preference, this functional form works.

$f_t^p(w_t, j_{t,z})$, and $w_t = N(g_{t-1}^P(w_{t-1}, j_{t,z}), K(\widetilde{j}_{t-1,\widetilde{z}}, I_{t-1}))$ are generated by the allocation rule we defined above, we need to show that for any $r > 1$, $j_{\hat{z}^i}^{r-1}$, I_t, and $\hat{z}^i, z^i \in Z$,

$$\widetilde{U}_t(w_1, u_{\widetilde{C}}, j_{\hat{z}^i}^{t-1}, \widetilde{z}^*, I_t) \geq \widetilde{U}_t(w_1, u_{\widetilde{C}}, j_{\hat{z}^i}^{t-1}, z^i \times \widetilde{z}^* \backslash z^i, I_t).$$

Since that all the other agents are using the truth telling reporting strategy in period t, then $\widetilde{g}^p(w_t, j_{t,z}, K(\widetilde{j}_{t,\widetilde{z}^*}, I_t)) = g_t^p(w_t, j_{t,z})$. Thus for any $w_t \in D$ and $\theta_t = \theta^i$,

$$\begin{aligned} & \theta^i \widetilde{f}_t^p(w_t, \theta^i) + \delta \widetilde{g}_t^p(w_t, \theta^i, K(\widetilde{j}_{t,\widetilde{z}^*}, I_t)) \\ = \; & \theta^i f_t^p(w_t, \theta^i) + \delta g_t^p(w_t, \theta^i) \\ \geq \; & \theta^i f_t^p(w_t, \theta^j) + \delta g_t^p(w_t, \theta^j) \\ = \; & \theta^i \widetilde{f}_t^p(w_t, \theta^j) + \delta \widetilde{g}_t^p(w_t, \theta^j, K(\widetilde{j}_{t,\widetilde{z}^*}, I_t)) \end{aligned}$$

for $j \neq i$ and $\theta^i, \theta^j \in \Theta$.

Therefore, given any arbitrary reporting history, if all the other agents are telling the truth, it is temporary incentive compatible for an agent to tell the truth in the current period. As shown in the proof of proposition (3.3), the truth telling reporting strategy is a Nash equilibrium reporting strategy. Apply lemma (3.4), the truth telling reporting strategy is a Perfect Bayesian Nash equilibrium. □

Proposition 4.11 *If $\gamma < 0$, the reporting strategy that all agents claiming to be impatient regardless of their true preference types is an equilibrium of the direct mechanism.*

To prove the proposition, we need the following lemma:

Lemma 4.12 *If $\gamma < 0$, $N(g^p(w_t, \theta^h), K(\widetilde{j}_{t,\widetilde{z}}, I_t)) - N(g^p(w_t, \theta^l), K(\widetilde{j}_{t,\widetilde{z}}, I_t))$ is a decreasing function of $\hat{p}_{t,\widetilde{z}}$.*

Proof Since the expected discounted utility level is multiplicate to $\widetilde{f}_t^p$ and $\widetilde{g}_t^p$, and according to the proof of proposition (3.3), the temporary incentive compatibility implies the total incentive compatibility, it is enough to consider the following case: assume in the initial period, all agents have the identical expected utility value $w_1 = \dfrac{\gamma}{|\gamma|}$. Thus given an arbitrary reporting strategy profile $\widetilde{z}$, the updated bank balance of an impatient agent is given by:

$$\begin{aligned} N(g_1^p(w_1, \theta^l), K(\widetilde{j}_{1,\widetilde{z}}, I_1)) &= \frac{\gamma}{|\gamma|}\left(\frac{K(\widetilde{j}_{1,\widetilde{z}}, I_1)}{\alpha(p)}\right)^\gamma \frac{g_1^p(w_t, \theta^l)}{\int_\Theta g_1^p(w_1, j_{z_t})\, d\mu} \\ &= \frac{\gamma}{|\gamma|}\left(\frac{R_I\alpha(p) - \frac{1}{(1-\beta)}(\hat{p}_{1,\widetilde{z}}\rho^{-1}(r^p(\theta^l)) + (1-\hat{p}_{1,\widetilde{z}})\rho^{-1}(r^p(\theta^h)))}{\alpha(p)}\right)^\gamma \\ &\cdot \quad \frac{y^p(\theta^l)}{\hat{p}_{1,\widetilde{z}} y^p(\theta^l) + (1-\hat{p}_{1,\widetilde{z}}) y^p(\theta^h)} \end{aligned}$$

and the adjusted bank balance value of a patient agent is

$$
\begin{aligned}
N(g_1^p(w_1,\theta^h), K(\widetilde{j}_{1,\widetilde{z}}, I_1)) &= \frac{\gamma}{|\gamma|}(\frac{K(\widetilde{j}_{1,\widetilde{z}}, I_1)}{\alpha(p)})^{\gamma} \frac{g_1^p(w_t,\theta^h)}{\int_\Theta g_1^p(w_1, j_{z_t})\, d\mu} \\
&= \frac{\gamma}{|\gamma|}(\frac{R_I\alpha(p) - \frac{1}{1-\beta}(\hat{p}_{1,\widetilde{z}}\rho^{-1}(r^p(\theta^l)) + (1-\hat{p}_{1,\widetilde{z}})\rho^{-1}(r^p(\theta^h)))}{\alpha(p)})^{\gamma} \\
&\cdot \frac{y^p(\theta^h)}{\hat{p}_{1,\widetilde{z}} y^p(\theta^l) + (1-\hat{p}_{1,\widetilde{z}} y^p(\theta^h)},
\end{aligned}
$$

where $K(\widetilde{j}_{1,\widetilde{z}}, I_1)$ is the actual investment level of the bank at the beginning of period 2 with reporting strategy profile $\widetilde{z}$.
Denote $d(\hat{p}_{1,\widetilde{z}}) = N(g^p(w_1,\theta^h), K(\widetilde{j}_{1,\widetilde{z}}, I_1)) - N(g^p(w_1,\theta^l), K(\widetilde{j}_{1,\widetilde{z}}, I_1))$,

$$
d(\hat{p}_{1,\widetilde{z}}) = \frac{\gamma}{|\gamma|}(\frac{R_I\alpha(p) - \frac{1}{1-\beta}(\hat{p}_{1,\widetilde{z}}\rho^{-1}(r^p(\theta^l)) + (1-\hat{p}_{1,\widetilde{z}})\rho^{-1}(r^p(\theta^h)))}{\alpha(p)})^{\gamma} \frac{y^p(\theta^h) - y^p(\theta^l)}{\hat{p}_{1,\widetilde{z}} y^p(\theta^l) + (1-\hat{p}_{1,\widetilde{z}}) y^p(\theta^h)}.
$$

Taking the derivative of $d(\hat{p}_{1,\widetilde{z}})$ with respect to $\hat{p}_{1,\widetilde{z}}$,

$$
\begin{aligned}
\frac{d\, d(\hat{p}_{1,\widetilde{z}})}{d\, \hat{p}_{1,\widetilde{z}}} &= \frac{\gamma^2}{|\gamma|}(\frac{K(\widetilde{j}_{\widetilde{z}_1}, I_1)}{\alpha(p)})^{(\gamma-1)} \frac{-\frac{1}{1-\beta}(\rho^{-1}(r^p(\theta^l)) - \rho^{-1}(r^p(\theta^h)))}{\alpha(p)} \frac{y^p(\theta^h) - y^p(\theta^l)}{\hat{p}_{1,\widetilde{z}} y^p(\theta^l) + (1-\hat{p}_{j_{1,\widetilde{z}}}) y^p(\theta^h)} + \\
&\frac{\gamma}{|\gamma|}(\frac{K(\widetilde{j}_{1,\widetilde{z}}, I_1)}{\alpha(p)})^{(\gamma)} \frac{(y^p(\theta^h) - y^p(\theta^l))^2}{(\hat{p}_{j_{1,\widetilde{z}}} y^p(\theta^l) + (1-\hat{p}_{1,\widetilde{z}}) y^p(\theta^h))^2}.
\end{aligned}
$$

Thus $\frac{d\, d(\hat{p}_{1,\widetilde{z}})}{d\, \hat{p}_{1,\widetilde{z}}} < 0$ if $\gamma < 0$,, which concludes the proof. □

Next we will use the conclusion of the lemma to prove the proposition (4.11).

Proof With the same reason stated in the proof of lemma (4.12), it is enough to consider the following case: assume in the initial period, all agents have the identical expected utility value $w_1 = \frac{\gamma}{|\gamma|}$.
Then we show that a patient agent chooses to lie about being impatient, given that almost all patient agents claim to be impatient.

In fact, an impatient agent chooses a reporting type by comparing the difference in the expected discounted utility between telling the truth or lying about being patient:

$$
\begin{aligned}
& (\theta^h f^p(w_1,\theta^h) + \delta N(g^p(w_1,\theta^h), K(\widetilde{j}_{1,\widetilde{z}}, I_1))) \\
- & (\theta^h f^p(w_1,\theta^l) + \delta N(g^p(w_1,\theta^l), K(\widetilde{j}_{1,\widetilde{z}}, I_1))) \\
= & (\theta^h r^p(\theta^h) + \delta N(y^p(\theta^h), K(\widetilde{j}_{1,\widetilde{z}}, I_1))) - (\theta^h r^p(\theta^l) + \delta N(y^p(\theta^l), K(\widetilde{j}_{1,\widetilde{z}}, I_1)))
\end{aligned} \tag{32}
$$

If the above value is greater than 0, then an impatient agent will choose to tell the truth. While if it is less than 0, he/she chooses to lie.

Similarly a patient agent will make a report by evaluating the following:

$$\begin{array}{rl} & (\theta^l f^p(w_1,\theta^l) + \delta N(g^p(w_1,\theta^l), K(\widetilde{j}_{1,\widetilde{z}}, I_1))) \\ - & (\theta^l f^p(w_1,\theta^h) + \delta N(g^p(w_1,\theta^h), K(\widetilde{j}_{1,\widetilde{z}}, I_1))) \\ = & (\theta^l r^p(\theta^l) + \delta N(y^p(\theta^l), K(\widetilde{j}_{1,\widetilde{z}}, I_1))) - (\theta^l r^p(\theta^h) + \delta N(y^p(\theta^h), K(\widetilde{j}_{1,\widetilde{z}}, I_1))) \end{array} \quad (33)$$

The reported type of a patient agent also relies on the sign of the above equation.

If almost all patient agents choose in the first period, that is, $1 = \hat{p}_{1,\widetilde{z}} > p$, then according to lemma (4.12), the value of (32) is greater than 0, which means an impatient agent will tell the truth given that almost all patient agents lie about being impatient.

Next we consider the value of (33). When $\hat{p}_{1,\widetilde{z}} = p$, (33) is equal to 0. Since the value of (33) is decreasing with the increase of $\hat{p}_{1,\widetilde{z}}$, if $\hat{p}_{\widetilde{z}_1} = 1$, then (33) is less than 0. Thus a patient agents will choose to lie if almost all other patient agents claim to be impatient.

Thus all agents claiming to be impatient can also be supported as an equilibrium result of the direct mechanism if $\gamma < 0$. $\square$

The following numerical example shows that if $\gamma \in (0, 1)$, then the truth telling reporting strategy might be supported as the unique equilibrium.

Example 4.13 *Let* $R_I = 1.2$, $\theta^l = 0.5$, $\theta^h = 1.5$, $\beta = 0.2$, $\delta = 0.9$ $p = 0.5$ *and* $\gamma = 0.5$. *We find the optimal allocation rule for the incentive deposit contract first by solving the static problem (14),* $r(\theta^l) = 0.0.0076$, $y(\theta^l) = 1.0944$, $r(\theta^h) = 0.0225$, $y(\theta^h) = 1.0861$ *and* $\alpha = 0.0078$.

Now we can verify numerically truth telling is the unique equilibrium in this case. Since the utility level is multiplicate to function f *and* g*,and according to proposition (3.3), it is enough to verify that agents are willing to tell the truth in current period for an given investment portfolio. Assume each agent is born with an investment portfolio* $I_1 = 0.0078$. *Consequently, the expected promised utility level of each agent is* 1 *at the beginning of the initial period.*

- *First we verify that all agents claim to be impatient is not an equilibrium. In this case* $\hat{p}_{1,z} = 1$, $f^{0.5}(1, \theta^l) = 0.0076$, $f^{0.5}(1, \theta^h) = 0.0225$, $I_2 = K(\prod_{i \in I} \theta^l, 0.0078) = 0.0092$, $N(g^{0.5}(1, \theta^l), 0.0092) = 1.09434$, *and* $N(g^{0.5}(1, \theta^h), 0.0092) = 1.08604$.

For patient agents

$$\begin{aligned} f^{0.5}(1,\theta^l)\theta^l + \delta N(g^{0.5}(1,\theta^l), 0.0092) &= 0.0076 * 0.5 + 0.9 * (1.09434) \\ &= 0.98871 \\ f^{0.5}(1,\theta^h)\theta^l + \delta N(g^{0.5}(1,\theta^h), 0.0092) &= 0.0225 * 0.5 + 0.9 * (1.08604) \\ &= 0.98869 \end{aligned}$$

$$f^{0.5}(1,\theta^l)\theta^l + \delta N(g^{0.5}(1,\theta^l), 0.0092) > f^{0.5}(1,\theta^h)\theta^l + \delta N(g^{0.5}(1,\theta^h), 0.0092). \quad (34)$$

This means that a patient agent will tell the truth even if almost all other patient agents will lie about being impatient in the first period.

Then consider impatient agents:

$$\begin{aligned} f^{0.5}(1,\theta^h)\theta^h + \delta N(g^{0.5}(1,\theta^h), 0.0088) &= 0.0225 * 1.5 + 0.9 * (1.0832) \\ &= 1.01119 \\ f^{0.5}(1,\theta^l)\theta^h + \delta N(g^{0.5}(1,\theta^l), 0.0088) &= 0.0076 * 1.5 + 0.9 * (0.915) \\ &= 0.99631 \end{aligned}$$

$$f^{0.5}(-1,\theta^h)\theta^h + \delta N(g^{0.5}(-1,\theta^h), 0.0088) > f^{0.5}(-1,\theta^l)\theta^h + \delta N(g^{0.5}(-1,\theta^l), 0.0088). \quad (35)$$

This means an impatient agent will tell the truth even almost all patient agents lie. We can conclude under this set of parameters, that always claiming to be impatient is not an equilibrium reporting strategy.

- *Next we show that always claiming to be patient is also not an equilibrium. In this case* $\hat{p}_{1,z'} = 0$, $f^{0.5}(1,\theta^l) = 0.0076$, $f^{0.5}(1,\theta^h) = 0.0225$, $I_2 = K(\prod_{i\in I}\theta^h, 0.0075) = 0.009342$, $N(g^{0.5}(1,\theta^l), 0.009342) = 1.09349$, *and* $N(g^{0.5}(-1,\theta^h), 0.009342) = 1.08691$. *For patient agents*

$$\begin{aligned} f^{0.5}(1,\theta^l)\theta^l + \delta N(g^{0.5}(1,\theta^l), 0.0092) &= 0.0076 * 0.5 + 0.9 * (1.09439) \\ &= 0.988751 \\ f^{0.5}(-1,\theta^h)\theta^l + \delta N(g^{0.5}(-1,\theta^h), 0.0092) &= 0.0225 * 0.5 + 0.9 * (1.08609) \\ &= 0.988731 \end{aligned}$$

$$f^{0.5}(-1,\theta^l)\theta^l + \delta N(g^{0.5}(-1,\theta^l), 0.009) > f^{0.5}(-1,\theta^h)\theta^l + \delta N(g^{0.5}(-1,\theta^h), 0.009). \quad (36)$$

This means it is optimal for a patient agent to tell the truth even if almost all impatient agents lie.

Consider impatient agents:

$$\begin{aligned} f^{0.5}(1,\theta^h)\theta^h + \delta N(g^{0.5}(1,\theta^h), 0.009) &= 0.0225 * 1.5 + 0.9 * (1.0871) \\ &= 1.011231 \\ f^{0.5}(1,\theta^l) + \theta^h N(g^{0.5}(1,\theta^l), 0.0739) &= 0.5563 + 0.9 * (0.102) \\ &= 0.996351 \end{aligned}$$

$$f^{0.5}(-1,\theta^h)\theta^h + \delta N(g^{0.5}(-1,\theta^h), 0.009) > f^{0.5}(-1,\theta^l)\theta^h + \delta N(g^{0.5}(-1,\theta^l), 0.009). \tag{37}$$

This means an impatient agent will tell the truth even if almost all other impatient agents lie.

Thus we can conclude, always claiming to be patient is also not an equilibrium reporting strategy.

All in all, truth-telling reporting strategy can be supported as the unique equilibrium.

5 Conclusion

This paper is aimed at formulating a modern idea of policy making. When policy makers design a mechanism, they tend to neglect the existence of multiple equilibria of the mechanism and focus on the best equilibrium. While in practice, there is no way to prevent all agents in the economy playing a strategy such that a "bad equilibrium" happens. Ennis and Keister (2008) concretely show the existence of "bank run" equilibrium in addition to an optimal risk sharing one. Based on the work by Atkeson and Lucas (1993), we show that the truth telling reporting strategy can be implemented as a Perfect Bayesian Nash equilibrium of a direct mechanism by an efficient, incentive compatible allocation. Moreover, we demonstrate the existence of a "bad" equilibrium which is an analogue to the "bank run" equilibrium in Diamond and Dybvig (1983). We consider this as the first step of our research. One direction for further investigation might be looking for a mechanism which shares the optimal equilibrium we have now but without the "bad" one.

6 Appendix

6.1 Proof

Proof of lemma(3.4):

Proof Sufficiency is given by the fact that (3) is the special case of (9), given $r = 0$.

Necessity is proved by contradiction.
Suppose that (3) holds, but (9) fails to hold. Then given $w_1 \in D$, for some $r \geq 1$, a preference type history (θ^{r-1}, θ_r), a reporting history $j_{\hat{z}}^{r-1}$, there exists a reporting strategy $z \in Z$, $z \neq z^*$[22], such that the following inequality holds:

$$\begin{aligned}\theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, \theta_r)) \quad + \quad & \delta U_{r+1}(w_1, u, j_{\hat{z}}^{r-1}, z^*) \\ < \quad & \theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, j_{r,z})) + \delta U_{r+1}(w_1, u, j_{\hat{z}}^{r-1}, z).\end{aligned}$$

Thus we can construct a reporting strategy z' such that if (1) $t < r$, let $j_{t,z'} = j_{t,z^*}$, for all $\theta^t \in \Theta^t$; (2) for $t \geq r$, continue truth telling unless $(j_{\hat{z}}^{r-1}, \theta_r)$[23] is realized; (3) if $(j_{\hat{z}}^{r-1}, \theta_r)$ is realized, switch to reporting strategy z from period r.

Thus this newly defined reporting strategy z' yields the same utility as the truthful reporting one z^* does in the first $r-1$ periods. If (θ^{r-1}, θ_r) has not been realized, it also yields the same utility as z^* in t periods, for $t \geq r$. While if (θ^{r-1}, θ_r) is realized, z' yields a strictly higher expected discounted utility than that of the truth telling one. Since the probability of having the preference type history as (θ^{r-1}, θ_r) is positive, $U(w_1, u, z') > U(w_1, u, z^*)$, which is a contradiction to (3). □

The proposition (3.3) can be proved in the following way:
First we show that an allocation of the deposit contract can induce a sequence of pairs of Boral measurable functions $\{(f_t, g_t)\}_{t=1}^{\infty}$, which in fact satisfies all the condition of being a feasible allocation rule.
For the inverse direction, we construct a feasible allocation through a feasible allocation rule. The key step of this direction is to show that the temporary incentive compatibility implies total incentive compatibility. It can be done in the following manner: first, according to the transversality condition, if it is optimal to deviate from the truth telling reporting strategy in infinitely many periods, then it is also optimal to deviate in N periods, with a sufficiently large N. Next, using the induction method, we show that it is not optimal to deviate from truth telling in finite periods. Thus follows the conclusion that if the allocation satisfies the temporary incentive compatible condition, then the utility sequence it generates satisfies the inequality (3). According to lemma (3.4), the corresponding allocation sequence is incentive compatible.

Here is the proof for proposition (3.3):
Proof

1. First we prove that if there exists a feasible allocation C to deliver the ex ante expected discounted utility value w_1 to all agents with the initial investment level

[22] Generally speaking, z is not necessarily the same as $\hat{z}$.
[23] According to the reporting strategy $\hat{z}$, in the first r periods, the agent is not necessarily truthfully reporting his/her preference types.

I_1, then there is an allocation rule $\sigma \in \Sigma$ such that it attaints ψ_1 with the initial investment level I_1, where ψ_1 is a degenerate utility value distribution given by $P(w = w_1) = 1$.

Since the utility function for one period, ρ, defines a one-to-one mapping from an allocation $\{C_t\}$ to a sequence of utility value for one period $\{u_t\}$, the existence of a feasible allocation C implies the existence of $u_C \in S$, which satisfies the bounded condition (1) , promise keeping condition (2), incentive compatible condition (3), and feasible condition (4).

Let $w_t = U_t(w_1, u, \theta^{t-1}, z^*)$, and define (f_t, g_t) as follows:

$$f_t(w_t, \theta_t) = u_t(w_1, (\theta^{t-1}, \theta_t)),$$
$$g_t(w_t, \theta_t) = U_{t+1}(w_1, u, \theta^t, z^*).$$

Thus the bounded condition (1) and transversality condition (5) hold by the way that (f_t, g_t) is defined. According to its definition of U_{t+1} and lemma (3.4), (f_t, g_t) satisfies the temporary promise keeping condition (6), and the temporary incentive compatible condition (7).

For the feasibility condition: according to the definition of ψ_t, we have

$$\int_{D\times\Theta} \rho^{-1}(f_t(w_t, \theta_t))d\psi_t d\mu \;=\; \int_{\Theta} \left(\int_{D\times\Theta} \rho^{-1}(f_t(g_{t-1}(w_{t-1}, \theta_{t-1}), \theta_t))d\psi_{t-1}d\mu \right) d\mu$$

Repeating the above step:

$$\begin{aligned}
\int_{D\times\Theta} \rho^{-1}(f_t(w_t, \theta_t))d\psi_t d\mu &= \int_{D\times\Theta^{t-1}} \{\int_{\Theta} \rho^{-1}(f_t(g_{t-1}(U_{t-1}(w_1, u, \theta^{t-1}, z^*), \theta_t))d\mu\} d\mu^{t-1} d\psi_1 \\
&= \int_{D\times\Theta^t} \rho^{-1}(f_t(U_t(w_1, u, \theta^t, z^*), \theta_t))d\psi_1 d\mu^t \\
&= \int_{D\times\Theta^t} \rho^{-1}(u_t(w_1, (\theta^{t-1}, \theta_t)))d\psi_1 d\mu^t
\end{aligned}$$

Thus if the allocation C is feasible, the allocation rule σ defined above is feasible too.

2. Next we show the inverse direction: if there exists a feasible allocation rule which satisfies the bounded condition (1) , transversality condition (5) , the temporary promise keeping condition (6), the temporary incentive compatible condition (7), and the feasible condition (8), then there is a feasible allocation, whose corresponding utility sequence satisfies the bounded condition (1) , the promise keeping condition (2), the incentive compatible condition (3), and the feasible condition (4).

Given an ex ante expected discounted utility value $w_1 \in D$, $u_t,\ W_t : D \times \Theta^t \mapsto \mathcal{R}_+$ can be constructed as follows:

$$u_t(w_1, j_z^t) = f_t(w_t, j_{t,z}),$$

and

$$W_{t+1}(w_1, j_z^t) = g_t(w_t, j_{t,z}) = w_{t+1}.$$

Instead of showing equation (2), which is the total promise keeping condition, we prove the following eqaution: for any $t \geq 1$,

$$W_t(w_1, \theta^{t-1}) = U_t(w_1, u, \theta^{t-1}, z^*). \tag{38}$$

Equation (2) is the special case of (38), given $t = 1$.
In fact, according to the definition of W_t, and the fact that (f_t, g_t) satisfies equation (6), for all t, w_1, and θ^{t-1},

$$W_t(w_1, \theta^{t-1}) = \int_\Theta \{\theta_t u_t(w_1, (\theta^{t-1}, \theta_t)) + \delta W_{t+1}(w_1, \theta^t) d\mu\},$$

while

$$U_t(w_1, u, \theta^{t-1}, z^*) = \int_\Theta \{\theta_t u_t(w_1, (\theta^{t-1}, \theta_t)) + \delta U_{t+1}(w_1, u, \theta^t, z^*) d\mu\}.$$

Subtracting the above two equations, we have

$$|W_t(w_1, \theta^{t-1}) - U_t(w_1, u, \theta^{t-1}, z^*)| \leq (\delta)^s \sup_{w_1, \theta^{t+s-1}} |W_{t+s}(w_1, \theta^{t+s-1}) - U_{t+s}(w_1, u, \theta^{t+s-1}, z^*)|$$

for all s and t. As $s \to \infty$, the right hand side of the above inequality goes to 0. Thus we prove equation (38), as well as equation (2).

For the total incentive compatible condition (3), according to the lemma (3.4), it is enough to verify that the utility sequence generated by the allocation rule satisfies inequality (9).

We show this by contradiction.

Consider infinite deviations first. Given an arbitrary reporting history, suppose there exists a reporting strategy which differs from the truth telling reporting strategy in infinitely many periods, and yields a higher expected utility than that of the truth telling one. Since the allocation rule satisfies the transversality condition (1), given a sufficiently large N, there exists a reporting strategy, which only differs from the truth telling reporting strategy in N periods and yields higher expected discounted utility.

By induction method, we show that, given the ex ante expected discounted utility value w_1, and arbitrary reporting preference type history $j_{\hat{z}}^{r-1}$, there is no reporting strategy involving a finite-period deviation that yields a higher expected discounted utility than that of the truth telling one.

Given $j_{\hat{z}}^{r-1}$, an arbitrary reporting history, suppose z is the reporting strategy different from the truth telling one z^* only in the first period. According to the definition of u_r, the fact that $U_r = W_r$, and since (f_t, g_t) satisfies the temporary incentive compatible condition (7),

$$\begin{aligned}\theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, \theta_r)) &+ \delta U_{r+1}(w_1, u, j_{\hat{z}}^{r-1}, z^*) \geq \theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, j_{r,z})) \\ &+ \delta U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, j_{r,z}), z^*).\end{aligned}$$

Denote z^N as a reporting strategy which differs from the truth telling one in N periods and continues with truth telling after that.

Assume that there is no reporting strategy that differs from the truth telling one in N periods and yields a higher expected discounted utility value,

$$\begin{aligned}\theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, \theta_r)) &+ \delta U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, \theta_r), z^*) \geq \theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, j_{r,z})) \\ &+ \delta U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, j_{r,z}), z^{N-1}),\end{aligned} \tag{39}$$

we verify that an analogue of inequality (39) holds for $N+1$ periods.

According to the definition of U_{r+1}:

$$\begin{aligned}U_{r+1}(w_1, u, j_{\hat{z}}^{r-1}, j_{r,z}, z^N) &= \int_\Theta \{\theta_{r+1} u_{r+1}(w_1, (j_{\hat{z}}^{r}, j_{r+1,z})) + \\ &\quad \delta U_{r+2}(w_1, u, (j_{\hat{z}}^{r}, j_{r+1,z}), z^{N-1})\} d\mu.\end{aligned}$$

By assumption, inequality (39) holds for N,

$$\begin{aligned}U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, j_{r,z}), z^N) &\leq \int_\Theta \{\theta_{r+1} u_{r+1}(w_1, (j_{\hat{z}}^{r-1}, j_{r,z}, \theta_{r+1})) \\ &+ \delta U_{r+2}(w_1, u, (j_{\hat{z}}^{r-1}, j_{r,z}, \theta_{r+1}), z^*)\} d\mu \\ &= U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, j_{r,z}), z^*)\end{aligned}$$

for all $j_{r,z} \in \Theta$.

Thus,

$$\begin{aligned}\theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, j_{z,r})) &+ \delta U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, j_{r,z}), z^N) \\ &\leq \theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, j_{r,z})) + \delta U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, j_{r,z}), z^*) \\ &\leq \theta_r u_r(w_1, (j_{\hat{z}}^{r-1}, \theta_r)) + \delta U_{r+1}(w_1, u, (j_{\hat{z}}^{r-1}, \theta_r), z^*)\end{aligned}$$

Therefore, the analogue of inequality (39) holds for $N+1$ periods. This concludes the induction.

So an analogue of inequality (39) holds for any reporting strategy which differs from the truth telling one.

For the feasibility condition, it holds directly.

□

Proof of proposition (3.5).

Proof By contradiction, we first show $\varphi^* \leq T\varphi^*$, then $\varphi^* \geq T\varphi^*$.
Suppose $\varphi^*(\psi) > (T\varphi^*)(\psi)$, then there exists some promised utility level distribution ψ, and $\varepsilon_0 > 0$, for some f, $g \in B$,

$$\varphi^*(\psi) - \frac{1}{(1-\beta)R_I} \int_{D\times\Theta} [\rho^{-1}[f(w,\theta)] + \varphi^*(S_g\psi)] d\mu d\psi > \varepsilon_0.$$

Let $\psi' = S_g\psi$. According to the definition of φ^* and proposition 3.3, there exists an allocation rule σ' such that it attains ψ' with $\varphi^*(\psi') + \frac{R_I(1-\beta)}{2}\varepsilon_0$. Define $\sigma_0 = \{(f,g), \sigma'\}$. Thus σ_0 is an allocation rule that uses (f,g) for the first period, then switches to the allocation rule σ'. Moreover, it attains ψ with resource $\varphi^*(\psi) - \frac{1}{2}\varepsilon_0$, which is a contradiction.

Next we prove that $\varphi^* \geq T\varphi^*$.
Suppose $\varphi^* < T\varphi^*$, then there exists some utility value distribution ψ, $\varepsilon_0 > 0$, and an allocation rule $\sigma = \{(f_t, g_t)\}_{t=1}^{\infty}$ such that it attains ψ with $(T\varphi^*)(\psi) - \varepsilon_0$. Consider the pair of Borel measurable function (f_1, g_1), it belongs to the feasible set of $(T\varphi^*)(\psi)$, and

$$\frac{1}{(1-\beta)R_I} \int_{D\times\Theta} [\rho^{-1}[f_1(w,\theta)] + \varphi^*(S_{g_1}\psi)] d\mu d\psi < (T\varphi^*)(\psi) - \varepsilon_0,$$

which is a contradiction. □

Proof of lemma(3.6):

Proof Given γ, arbitrarily choose a pair of Borel measurable functions (r^0, y^0) that satisfy equation (15) and equation (16). Let Y be the set of pairs of Borel measurable functions $r, y \in B$, which satisfy constraints equation (15), equation (16), and the following inequalities:

$$\begin{aligned} \rho^{-1}(r) \le \rho^{-1}(r^0), (\frac{\gamma}{|\gamma|}y)^{\frac{1}{\gamma}} \le (\frac{\gamma}{|\gamma|}y^0)^{\frac{1}{\gamma}}, \quad & \text{if } \gamma \neq 0; \\ e^r \le e^{r^0}, e^y \le e^{y^0}, \quad & \text{if } \gamma = 0. \end{aligned} \tag{40}$$

Evidently, set Y is a nonempty set, and the coordinates of set Y are bounded above. Since $r(\theta)$ and $y(\theta)$ satisfy equation (15) and $\theta^h > \theta^l > 0$ guarantee that set Y is bounded below.

Since both θ^l and θ^l have a positive measure in Θ, the set Y is a compact set. Thus there exists a solution to the problem (14) in set X. The uniqueness is guaranteed by the concavity of ρ. $\square$

Proof of lemma (3.7)

Proof Denote α_a as the lowest initial investment level needed to have a normalized ex ante expected discounted utility value $\frac{\gamma}{|\gamma|}$ in the autarkic case. Since the allocation in the autarkic case (r_a^l, y_a^l), (r_a^h, y_a^h) must be incentive compatible, $\phi(\alpha_a) \le \alpha_a$.

Denote α_a as the lowest initial investment level needed in the full insurance case, to have a normalized ex ante expected discounted utility value $\frac{\gamma}{|\gamma|}$. Let α_c be the solution to problem (14) with the incentive compatible constraints discarded. Thus $\phi(\alpha_c) \ge \alpha_c$.

The uniqueness of the fixed point is proved by contradiction. Suppose there exist 2 fixed points α', α. Without loss of generality, assume $0 < \alpha < \alpha'$. Thus the solution $((r^l(\alpha), y^l(\alpha)), (r^h(\alpha), y^h(\alpha)))$ is feasible to $\phi(\alpha')$.

$$\begin{aligned} \alpha' - \alpha &= \phi(\alpha') - \phi(\alpha) \\ &\le \int_\Theta \left(\frac{\gamma}{|\gamma|}y(\alpha)\right)^{\frac{1}{\gamma}} (\alpha' - \alpha) \\ &< \frac{\phi(\alpha)}{\alpha}(\alpha' - \alpha), \end{aligned} \tag{41}$$

which is a contradiction. $\square$

Proof of lemma (3.8)

Proof By subtracting inequality (22) from inequality (21), we can conclude $(\theta^h -$

$\theta^l)(r(\theta^h, \alpha^*) - r(\theta^l, \alpha^*)) \geq 0$. $\theta^h > \theta^l$ implies $r(\theta^h, \alpha^*) - r(\theta^l, \alpha^*) \geq 0$.

First, suppose $r(\theta^h, \alpha^*) = r(\theta^l, \alpha^*)$, then inequality (21) and inequality (22) imply $y(\theta^h, \alpha^*) = y(\theta^l, \alpha^*)$. While according to inequality (25) and inequality (26), $\zeta = \eta = 0$, which is a contradiction.

Next we assume $r(\theta^h, \alpha^*) > r(\theta^l, \alpha^*)$. Since the consumption function ρ^{-1} is increasing and convex in the utility level, $(\rho^{-1})'(r(\theta^h)) > (\rho^{-1})'(r(\theta^l))$. Subtracting equation (23) from equation (24), and since $\zeta, \eta \geq 0$, we can conclude $\zeta = 0$, $\eta > 0$. The conclusion that the constraint (21) is not binding and the constraint (22) is binding directly follows. □

6.2 The Autarkic Case

Consider the autarkic problem, the agent maximizes his/her ex ante expected discounted utility value with initial investment level I_0. According to the principle of optimality, we can write down the Bellman equation of the agent in autarky:

$$v(I_t) = \max_{I_{t+1} \geq 0} \int_{\Theta} (1 - \theta)\rho(c_t) + \theta v(I_{t+1})\, d\mu, \tag{42}$$

subject to

$$c_t + \beta(R_I I_t - I_{t+1}) = R_I I_t - I_{t+1}. \tag{43}$$

Since the gross rate of return of the investment technology is homogeneous of degree 1, and we use CRRA function as the utility function for one period, the initial investment level is multiplicate to the ex ante expected discounted utility value,

$$v(I_t) = (I_t)^{\gamma} v(1). \tag{44}$$

Thus it is enough to find out the value $v(1)$.

Since the probability that $P(\theta = \theta^l) = p$, we can rewrite equation (42) as follows:

$$v(1) = \max_{(c^l, I^l), (c^h, I^h)} [p(\rho(c^l)(1 - \theta^l) + \theta^l (I^l)^{\gamma} v(1)) + (1 - p)\rho(c^h)(1 - \theta^h) + \theta^h (I^h)^{\gamma} v(1)], \tag{45}$$

where $c^i + \beta(R_I - I^i) = R_I - I^i$, for $i \in \{l, h\}$.

Thus equation (45) is a discrete static problem of $v(1)$. Given the set of parameter values, this problem can be computed numerically.

References

Andrew G. Atkeson and Jr. Lucas, Robert E. On efficient distribution with private information. *Review of Economic Studies*, 59(3):427–453, July 1993.

Douglas W. Diamond and Philip H. Dybvig. Bank runs, deposit insurance, and liquidity. *Journal of Political Economy*, 91(3):401–419, 1983.

Huberto M. Ennis and Todd Keister. Run equilibria in a model of financial intermediation. Staff Report 312, Federal Reserve Bank of New York, 2008. URL `http://www.ny.frb.org/research/staff_reports/sr312.html`.

Matthew O. Jackson. Bayesian implementation. *Econometrica*, 59(2):461–477, 1991.

Andreu Mas-Colell and Xavier Vives. Implementation in economies with a continuum of agents. *Review of Economic Studies*, 60(3):613–629, 1993.

Jia Pan. *Optimal intermediated investment in a liquidity-driven business cycle*. PhD thesis, Pennsylvania State University, 2008.