Auctions with Resale and Bargaining Power

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Abstract

We establish the bid-equivalence between an independent private-value (IPV) first-price auction model with resale and a model of first-price common-value auction, when the resale market satisfies a minimal efficiency property and the common value is defined by the transaction price. This implies that the speculator-buyer model of auction with resale is observationally equivalent to the Wilson drainage tract common-value model. With an application of the Coase Theorem, we show two polar cases in which auctions with resale have opposite properties. We examine the effects of bargaining power on the revenue and efficiency of first-price auctions with resale. This is done for three types of bargaining models: (a) bargaining with commitment, (b) bargaining with delay costs, and (c) k-double auctions resale market. We also provide conditions under which the first-price auction generates higher revenue than the second-price auction when resale is allowed.

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1 Introduction

In this paper we study the way resale opportunities after the auction may affect bidders's behavior in the auction. When resale is allowed after the auction, we refer to it as an auction with resale, and represent it by a two-stage game. It is intuitively understood in the profession that resale is an important source of common- value among the bidders. In the survey for their book, Kagel and Levin (2002, page 2) said that "There is a common-value element to most auctions. Bidders for an oil painting may purchase for their own pleasure, a private-value element, but they may also bid for investment and eventual resale, reflecting the common-value element". Haile (2001)[1] studied the empirical evidence of the effects of resale in the U.S. forest timber auctions. In spectrum auctions held by many governments, there are often restrictions on resale. For example, in the British 3-G spectrum auctions[2] of 2000, resale restrictions were imposed despite economists' recommendation to the contrary. It is not clear why the restrictions were imposed. It is possible that the government may look bad when the bidders can turn around and resell for quick profits after the auction[3]. Bidders, however, find ways to circumvent such restrictions in the form of a change of ownership control. For example, a month after the British 3-G auction, Orange, the winner of the license E, was acquired by France Telecom, yielding a profit of 2 billion pounds to Vodafone[4]. The winner of the most valuable licence A is TIW (Telesystem International Wireless). In July 2000, Hutchison then sold 35% of its share in TIW to KPN and NTT DoCoMo with an estimated profit of 1.6 billion pounds[5].

Although resale is sometimes conducted so that parties who did not or could not participate in the auction has a chance to acquire the object sold during the

[1] His model of resale is different from our specifications here. In his model, there is no asymmetry among bidders before auctions, and trade occurs after the auction because of information differences after the auction. In our model, bidders are asymmetric before auctions.Haile, Hong, and Shum (2003) studied the U.S. forest lumber auctions and found the bidding data to conform to private-value auctions in some and common-value auctions in others. An explanation may be due to the presence or lack of resale.

[2] The third-generation technology allows high speed data access to the internet. It was held on Mar 6, 2000, and concluded on April 27, 2000, raising 22.5 billion pounds (2.5% of the GNP of UK). This revenue is seven times the original estimate. Five licenses A,B,C,D,E were offered. Licence A was available for bidding by non-incumbent operators only. A more detailed account is given in Klemperer (2004).

[3] Beyond the political and legal reasons, resale may facilitate collusions in the English auction as is shown in Garrat, Troger and Zheng (2007).

[4] Orange paid 4 billion pounds for the licence. In May of 2000, France telecom paid 6 billion pounds more than the price Mannesmann had paid for it in October 1999 before the auction. Orange was the number three UK mobile group at the time. The reason for the resale is due to a divestment agreement by Vodafone with the British government after acquiring Mannesmann.

[5] TIW was a Canadian company based in Montreal and largely owned by Hutchison Whampoa. The Hong Kong conglomerate) gained the upper hand when NTL Mobile, a joint venture of the UK cable operator and France Telecom, withdrew from the bidding. The price of the licence is 4.4 billion pounds. The profit is based on the implicit valuation of the license at 6 billion in the transaction.

auction. We will focus on resale among bidders of the original auction. There are situations in which resale opportunities to a third party only affect a bidder's valuation of the object, and thus can be indirectly represented by a change in valuations. In this case, our model may allow third party participation. However third party participation may give rise to issues that are not explored here. The main idea of the paper is that it is the resale price, not the private valuation, which determines the bidder behavior, and we expect this idea to apply to a more general model with broader participation.

By focusing on the resale among bidders, we study an interesting interaction between resale in the second stage and bidding behavior in the first stage. Hortascu and Kastl (2008) showed that bidding data for 3-month treasury bills are more like private-value auctions, but not so for 12-month treasure bills. The difference may be due to the relevance of resale in long term treasury bills. When an asset is held in a longer period, there may be a need for resale, and this may affect bidding behavior. We will in fact show that in private-value auctions with resale, the bidding data will behave as if it is a common-value auction. This has been observed In Gupta and Lebrun (1999), and Lebrun (2007) in cases when the resale market is a simple monopoly or monopsony market. We will provide a theoretical examination for this intuition in more general resale environments.

We describe the phenomenon by the term "bid-equivalence". It means that the bidding behavior of the an independent private-value auction with resale is the same as a pure common-value auction in which the common-value is defined by the transaction price. The two auctions have the same equilibrium bid distributions. The auctioneer has no way of knowing the difference between the two from the bidding behavior in the auctions, nor can an econometrician from the bidding data. The concept of bid-equivalence is similar to the observational equivalence used in Green and Laffont (1987). We prefer to use a different term because the observational equivalence concept is often associated with the identification problem in econometrics. Here we want to focus on the theoretical implications. This concept is different from equilibrium equivalence as the auction with resale is a two-stage game, while the common-value auction is a one-stage game. Furthermore, the equilibrium payoffs of the two auctions for the bidders are in general different. Laffont and Vuong (1996) showed that for any fixed number of bidders in a first-price auction, any symmetric affiliated values model is observationally equivalent to some symmetric affiliated private-values model. In a symmetric model, there is no incentive for resale. In our paper, we look at the asymmetric IPV auctions with resale. We show that when bidders anticipate trading activities after the auction, the bidding data is observationally equivalent to a common-value auction.

We assume that there are only two bidders in the auction. This is partly due to the substantial complexity of auctions with resale when there are more than two bidders, a model which is still poorly understood in the literature. This assumption is however justified here as we are looking at the issue of bargaining power effects in auctions with resale, and the single seller and buyer framework in the resale game gives us a clear setting to address this issue. We adopt an axiomatic approach to the description of the resale stage game. This includes the case of a general bilateral trading game with incomplete information in the noncooperative approach, or equivalently a mechanism design formulation. It does not preclude a mediated or cooperative bargaining model, as long as the model yields an outcome satisfying the properties of the model. The framework is flexible enough to include a resale process in which bidders make sequential offers, or simultaneous offers in the bargaining. The main assumption is a minimal efficiency property which says that trade should occur with probability one when the trade surplus is the highest possible. It is easily satisfied in most Bayesian bargaining equilibrium with sequential or simultaneous offers. It rules out the no-trade equilibrium in which there cannot be bid-equivalence. Our formulation of the minimal efficiency property is a variation of the sure-trade property in Hafalir and Krishna (2008)[6].

The bid-equivalence result may be somewhat surprising. One would expect that resale only contributes a common-value "component" to the bidding behavior, and there is still a private-value component. Our result however says that the bidding behavior is the same as if it is a pure common-value model. What happens to private-value component? The answer is that bid-equivalence is true only in equilibrium, hence the private-value is still relevant out-of-equilibrium. Furthermore, the private-value is incorporated in the definition of the common-value, and is therefore indirectly affecting the equilibrium bidding behavior. In the proof of the bid-equivalence result, we show that once the belief system is given, the payoffs of the bidders in the IPV auction with resale and the common-value auction only differ by a constant (independent of the bid). This means that there is a strong tautological element in the equivalence result. For this reason, we think that the result should be true in a more general environment than is adopted here.

Auction with resale in general can be a very complicated game. The resale game may involve potentially complicated sequences of offers, rejections, and counter-offers. In-between the auction stage and the resale stage, there may be many possible bid revelation rules that affect the beliefs of the bargainers in the resale stage. We cannot deal with so many issues at the same time. For the bid revelation rule, we adopt the simplest framework of minimal information, i.e. that is there is no bid revelation in-between the two stages. Despite the lack of bid information, the bidders update their beliefs after winning or losing the auction. Since bidders with different valuations bid differently in the first stage, the updated beliefs depend on the bid in the first stage. For this reason, there are heterogeneous beliefs in the resale stage. Furthermore, a bidder may become a seller or a buyer in the resale stage depending on the bidding behavior

[6] Our assumption is a bit stronger. Hafalir and Krishna (2008) use it to show the symmetry property in equilibrium. We need a version that is needed for the bid-equivalence result. In more general models (such as affiliated signals), our condition may yield bid equivalence even though the symmetry property typically fails.

in the first stage. This distinguishes the bargaining in the resale stage from the standard bargaining game in the literature in which there is a fixed seller or buyer, and the beliefs are homogeneous. The outcome of the bargaining in general is different. For instance, the updating may improve the efficiency of the bargaining compared to the standard homogenous model, as bargainers have better information.

One important implication of the bid equivalence result is that the equilibrium analysis of the auction with resale is reduced to the simpler equilibrium analysis of the one-stage common-value model. The revenue of the auction with resale is completely determined by the common-value function, and the efficiency of the auction with resale is determined by the trading set as well as the common-value function. We will apply the bid-equivalence result to address some of these questions. More applications of this approach can be found in Cheng and Tan (2007). Gupta and Lebrun (1999) showed that there is a reversal of revenue ranking between the first-price and second-price auctions with resale for the maximum and minimum (common-value) case. We show that there is also a reversal of revenue ranking between the first-price auction with and without resale for the two cases. Gupta and Lebrun (1999) assume that there is complete information during the resale stage, and the maximum case represents the monopoly market, while the minimum case represents the monopsony market. We combine the bid-equivalence result and the Coase Theorem to argue that in repeated bargaining, when there is commitment problems in the offers, and the buyer is sufficiently patient, then the seller loses all the bargaining power, and in the limit, the common-value function converges to the maximum function in the limit. Thus the maximum case provides the upper bound of the revenue of all possible revenue of the auction with resale. Similarly, the minimum case provides the lower bound of the revenue of all possible revenue of the auction with resale. In this sense, the two cases provide polar cases of the revenue of the auction with resale.

Three kinds of questions are of interest regarding auctions with resale. Is it a good idea to allow resale[7]? Is it more efficient to allow resale? For first-price and second-price auctions with resale, which one gives a higher revenue? Some of these questions have been investigated in Hafalir and Krishna (2007,2008). Examples of the application of the bid-equivalence to these questions are given in section 6. We look at the three types of resale markets: bargaining with commitment, sequential bargaining with delay costs, and k-double auctions in sections 6.1,6.2,6.3 respectively. These relatively simple applications also raise new interesting questions regarding bargaining with heterogeneous beliefs. For instance, we would expect the seller and the buyer to choose an efficient mechanism in the resale stage. In this case, we would like to know how to characterize

[7] According to Myerson (1981), the optimal auction can be achieved by selecting the optimal reserve price, one for each bidder. No resale is needed for the highest revenue. However, this is not easy to do in practice. Here we assume no reservation price or a single reservation price low enough to make little difference.

efficient bargaining mechanisms with heterogeneous beliefs such as extensions of Williams (1987) result to this case.

In section 6.1, we examine how the valuation distributions may affect bargaining power and the ranking results in auctions with resale. It is in the commitment case of Hafalir and Krishna (2008) in which the offer-maker makes a take-it-or-leave-it offer to the offer-receiver. We show a general ranking result of the first-price and second-price auctions with resale using a simple property of the common-value function. This property is satisfied when the resale market is either a monopoly or a monopsony market, and the offer-receiver has a convex valuation distribution. It can also be applied to the k-double auction resale markets of section 6.3. Part of the purpose of this section is to provide a partial explanation of the unambiguous ranking result of Hafalir and Krishna (2008) and polar ranking results of section 5. A more complete answer is provided in Cheng and Tan (2007). In section 6.2, we look at the issue of delay costs and bargaining power in a two-period sequential bargaining model of Sobel and Takahashi (1983). The weak bidder has very high delay costs, while the strong bidder has little delay costs. As a result, the revenue of the first-price auction with resale is substantially depressed so that it is lower than the auction without resale. In this example, it is also true that the first-price auction with resale has lower revenue than the second-price auction. In section 6.3, we examine the bargaining power in the linear k-double auction resale market of Chatterjee and Samuelson (1983). We show that the auctioneer's revenue is an increasing function of the weak bidder's bargaining power. Furthermore, we show that there is a trade-off between efficiency and revenue. As the revenue gets higher, the efficiency of the auction with resale is lower. We also touch on the question of how bid revelation affects the outcome. It is interesting to note that in the k-double auction model of section 6.3, if the auctioneer announces the winning bid as often is the case in the real world auctions, the revenue of the auctioneer is higher, and the revelation is profitable for the monopolist. The revenue is lower if the losing bid is announced. Thus the auctioneer has the incentive to reveal the winning bid, but not the losing bid. The bargaining position of the monopolist is enhanced by information disclosure of the monopolist's valuation.

There are interesting implications of the idea offered in this paper on some policy issues regarding auction design. Higher prices in resale will raise the bid in the auction and improve the revenue of the auctioneer. The fact that resale prices were high after the British 3-G auction meant that it would have been better for the government to allow resale. Since then the British government has become more receptive to the idea of allowing resale (Klemperer (2004)).

Section 2 illustrates the ideas of the paper with a simple discrete model. Section 3 presents the common value model, while section 4 formulates the auction with resale model and gives the bid equivalence result. Section 5 applies the bid-equivalence result and the Coase Theorem to provide two polar cases of the auctions with resale model. Section 6 gives applications to three types of bargaining problems.

2 An Illustrative Example

In this section, we shall use a simple discrete model to illustrate the important issue of bargaining power to the study of auctions with resale. We also illustrate the ideas of ranking reversal and bid-equivalence.

Assume that there are two bidders in an independent private-value asymmetric auction. Bidder one valuation is either 0 or 1 with probability 0.7 for 0. Bidder two valuation is either 0 or 2 with probability 0.4 for 0. We call bidder one the weak bidder, and bidder two the strong bidder.

Let H_i denote the equilibrium cumulative bid distribution of bidder i. Consider the first-price auction without resale. Bidder one with valuation 1 chooses b to maximize

$$H_2(b)(1-b).$$

The first-order condition is

$$\frac{H_2'(b)}{H_2(b)} = \frac{1}{1-b}.$$

Similarly, the first-order condition for bidder two is

$$\frac{H_1'(b)}{H_1(b)} = \frac{1}{2-b}.$$

The boundary conditions are $H_1(0) = 0.7, H_2(0) = 0.4$. These probabilities have been chosen so that the other boundary condition is satisfied: $H_1(b^*) = H_2(b^*) = 1$ for some b^*.

We have the following simple equilibrium without resale:

$$H_1(b) = \frac{1.4}{2-b}, H_2(b) = \frac{0.4}{1-b} \text{ for } b \in [0.0.6].$$

Now we consider the first-price auction with resale. The winner of the auction acts as a monopolist in the resale game, and chooses an optimal monopoly price. Because of the two-point distribution assumption, the optimal monopoly price is 2 when bidder one wins the auction. There is no resale if bidder two has valuation 0, or if bidder two wins the object and has valuation 2. The payoff of bidder one with valuation 1 bidding $b > 0$ is

$$(H_2(b) - 0.4)(2-b) + 0.4(1-b) = H_2(b)(2-b) - 0.4$$

The first-order condition for this maximization problem is

$$H_2'(b)(2-b) - H_2(b) = 0$$

or

$$\frac{H_2'(b)}{H_2(b)} = \frac{1}{2-b}. \tag{1}$$

For a bidder one with valuation 0, the payoff from bidding $b > 0$ is

$$(H_2(b) - 0.4)(2-b) + 0.4(0-b) = H_2(b)(2-b) - 0.8,$$

and the first-order condition is also (1). We have the same first-order condition for a bidder one with different valuations, because the resale opportunity has changed his valuation to 2 rather than 0 or 1.

For bidder two with valuation 2, there is no profit in the resale whether or not there is resale. The payoff from bidding $b > 0$ is

$$H_1(b)(2-b)$$

and the first-order condition is also given by (1). From the boundary conditions, $H_1(0) = 0.4 = H_2(0)$, we get the auction with resale equilibrium[8]

$$H_1^r(b) = H_2^r(b) = \frac{0.8}{2-b} \text{ for } b \in [0, 1.2].$$

This is a symmetric equilibrium in bid distributions. The symmetry of the equilibrium bid distribution was first discovered in Engelbrecht-Wiggans, Milgrom, and Weber (1983) for the Wilson track model and proved more generally in Parreiras (2006) and Quint (2006) with independent signal (see equations (6)). This property also holds in first-price auctions with resale in Hafalir and Krishna (2008). The identical first-order conditions strongly suggest that there is some equivalence relationship between the auction with resale equilibrium and the common-value auction properly defined. We will in fact show that the equilibrium bid distributions of the auction with resale is the same as a common-value auction in which the common-value of the two bidders is the transaction price 2 when bidder two receives a high signal regardless of the signal of bidder one. This idea has great generality and is formulated in the paper.

If we let the loser of the auction make offers, so that the resale is a monopsony game. The optimal monopsony price to offer is 1 when bidder two loses the auction. There is resale if the bidder two has valuation 2 and loses the auction. When bidder two with valuation 2 bids $b > 0$ and the payoff is

$$H_1(b)(2-b) + (1 - H_1(b))(2-1) = H_1(b)(1-b) + 1$$

The first-order condition is

$$\frac{H_1'(b)}{H_1(b)} = \frac{1}{1-b}. \tag{2}$$

In other words, bidder two bids as if his valuation is 1. When bidder one with valuation 1 bids $b > 0$, the payoff is

$$0.4(1-b) + (H_2(b) - 0.4)(1-b) = H_2(b)(1-b)$$

[8]This equilibrium is unique for any tie-breaking rule adopted. For the auction without resale model, the equilibrium is also unique for this example. In auctions with resale, bidder one with valuation 0 may bid positive amount because of future resale.

and we have the same first-order condition (2). The model is bid equivalent to a common-value model in which both bidders value the object at either 0 or 1. The only equilibrium with full extraction of surplus, determined by the boundary conditions $H_1(0) = 0.7 = H_2(0)$, by the monopsonist is given by[9]

$$H_1^r(b) = H_2^r(b) = \frac{0.7}{1-b} \text{ for } b \in [0, 0.3]. \tag{3}$$

The equilibrium bid distributions are the same as that of a common-value auction in which the common-value is 1 when both bidders have high signals, and 0 otherwise.

Now, we want to compare the revenues with and without resale. In the auction without resale, bidder one has a positive profit only if the valuation is 1, and in this case the profit is 0.4. Bidder two has a profit profit only if the valuation is 2, and in this case the profit is 1.4. The revenue is just the realized surplus minus the expected profit of the bidders.

For the equilibrium (2) in the aution with monopoly resale, bidder one has a positive profit only if the valuation is 1, and in this case the profit is 0.4. Note that this is the same for the auction without resale. The reason is that even though resale promises more profit to bidder one, it also makes the bidder two bid more aggressively, and the two effects happen to cancel each other in this example. Bidder two has a positive profit only if the valuation is 2, and in this case, the profit is 0.8. This lower profit is due to the more aggressive bidding behavior of bidder one, while bidder two gets no benefit from resale. Note that for each bidder the expected profit is the same or smaller in the auction with resale, and the realized surplus is higher in the auctions with resale because it is efficient. Therefore it is clear intuitively that the resale opportunity increases the revenue and is beneficial to the auctioneer when the resale is a monopoly market.

For the equilibrium (3) in the auction with monopsony resale, bidder one has a positive profit only if the valuation is 1, and in this case the profit is 0.7. Bidder two has a positive profit only if the valuation is 2, and in this case the profit is 1.7. Although the realized trade surplus is higher than that of the auction with resale, the revenue is lower with resale. In fact, the equilibrium bid distribution of the auction without resale first-order stochastically dominates that of the auction with resale as

$$\frac{1.4}{2-b}\frac{0.4}{1-b} \leq \frac{0.49}{(1-b)^2} .$$

This example illustrates the idea that if the resale market is the seller's market, allowing resale benefits the auctioneer, but the opposite is true when the resale market is the buyer's market. Again this reversal is quite general, and will be proved later.

[9] This is an equilibrium when ties are broken in favor of bidder two. Other equilibria exist. This particular equilibrium has the lowest revenue among all possible equilibria and the only equilibrium with full extraction of surplus by the monopsonist.

It can also be easily computed that when the resale market is the seller's market, the first-price auction revenue is higher than the second-price auction revenue, but the opposite is true when the resale market is the buyer's market. This again is a general property with full rent extraction, and will be shown later.

More generally, different trading rules in the resale game change the transaction price of the object in resale, which is used to define the common-value function of the common-value auction. Auction with resale can then be shown to be bid equivalent to the common-value auction with a specifically defined common-value function.

3 The Common-Value Model

There are two risk neutral bidders in an auction for a single object. Our model begins with an independent private-value (IPV) model. A pure common-value model will be constructed from the IPV model and the resale process. To make the notations more compatible with the common-value model, we shall adopt a signal representation of the IPV model. This representation (called a distributional approach) is first proposed in Milgrom and Weber (1985) and discussed extensively in Milgrom (2004).

In this representation, a bidder is described by an increasing valuation function $v_i(t_i) : [0,1] \to [0, a_i]$, with the interpretation that $v_i(t_i)$ is the private valuation of bidder i when he or she receives the private signal t_i. We can normalize the signals so that both signals are uniformly distributed over $[0,1]$. The two signals are assumed to be independent. The word "private" refers to the important property that bidder i's valuation is not affected by the signal t_j of the other bidder, while in the common-value model, this is not the case. The function $v_i(t_i)$ induces a distribution on $[d_i, a_i]$, whose cumulative probability distribution is given by $F_i(x_i) = v_i^{-1}(x_i)$ when v_i is one-to-one. Since we want to include the case of discrete distributions, v_i may have a finite number of values, hence need not be strictly increasing. In this case v_i also induces a discrete distribution over $[0, a_i]$. The standard IPV model is often described by $F_i's$ rather than $v_i's$. If $v_1(t) \leq v_2(t)$ for all t, we say that bidder one is the weak bidder, and bidder two is the strong bidder, and the pair is a weak-strong pair of bidders[10]. This is equivalent to saying that F_1 is first-order stochastically dominated by F_2.

To define a pure common-value auction model, we need to define a common value function $V = w(t_1, t_2)$. This common value function represents the (expected) common value V of both bidders when both signals t_1, t_2 are known.

[10] Here we only require that F_1 is dominated by F_2 in the sense of the first order stochastic dominance. Note that this concept is weaker than that of Maskin and Riley (2000a), in which conditional stochastic dominance is imposed.

The common value can be defined through $v_1(t_1), v_2(t_2)$, and in this case we write $V = p(v_1(t_1), v_2(t_2))$. For instance, in a double auction between the seller and the buyer, $V = p(v_1(t_1), v_2(t_2))$ may be defined as the transaction price when the seller offer is lower than the buyer offer. We will assume that the function w is continuous and increasing in each t_i.

One condition of p will be useful for our revenue ranking and can be stated as follows:

Condition (C): for all x_1, x_2 in $[0, \min(a_1, a_2)]$, *we have*

$$p(x_1, x_2) \geq \frac{p(x_1, x_1) + p(x_2, x_2)}{2} \tag{4}$$

Since v_1, v_2 may have different ranges, $p(x, x)$ is not be defined if $x > \min(a_1, a_2)$. In this case, we assume that $p(x, x) = 0$. By this convention, the above inequality is trivial when $x_i > \min(a_1, a_2)$ for some i. Therefore, it is understood that the above inequality holds for all $x_i \in [0, a_i], i = 1, 2$. Note that in (C), we do not necessarily impose symmetry. When p is symmetric, the submodular property p implies (C). However, when p is not symmetric, condition (C) does not follow from submodularity. For example, $p(x_1, x_2) = \frac{2}{3}x_1 + \frac{1}{3}x_2$ is submodular but does not satisfy condition (C).

In fact condition (4) will only be required for pairs (x_1, x_2) such that $F_1(x_1) = F(x_2), (x_1, x_2) \in [0, \min(a_1, a_2)]$. We shall refer to this as:

Condition (C'): for all pairs (x_1, x_2) such that $F_1(x_1) = F_2(x_2), (x_1, x_2) \in [0, \min(a_1, a_2)$, (4) holds.

When $p(x, x) = x$ holds, condition (4) can be written as

$$p(x_1, x_2) \geq \frac{x_1 + x_2}{2}. \tag{5}$$

Condition (C) cannot hold for all (x_1, x_2) when p is of the form $p(x_1, x_2) = rx_1 + (1-r)x_2$. Condition (C) holds for all pairs when p is of the form $p(x_1, x_2) = \max\{rx_1 + (1-r)x_2, (1-r)x_1 + rx_2\}$, and in this case, we have a kink on the diagonal.

The distribution function F_i is called regular if the following virtual value function is strictly increasing in x :

$$x - \frac{1 - F_i(x)}{f_i(x)}.$$

Let $b_i(t_i)$ be the strictly increasing equilibrium bidding strategy of bidder i in the first-price common-value auction, and $\phi_i(b)$ be its inverse. The following first order condition is satisfied by the equilibrium bidding strategy

$$\frac{d \ln \phi_i(b)}{db} = \frac{1}{p(v_1(\phi_1(b), v_2(\phi_2(b))) - b} \text{ for } i = 1, 2. \tag{6}$$

with the boundary conditions $\phi_i(0) = 0, \phi_1^{-1}(1) = \phi_2^{-1}(1)$. The ordinary differential equation system with the boundary conditions determine the equilibrium inverse functions. Hafalir and Krishna (2008) have shown the same first-order condition for auctions with resale. We will also prove it in our formulation.

4 Auctions with Resale

The first-price auction with resale is a two-stage game. The bidders participate in a standard sealed-bid first-price auction in the first stage. In the second stage, there is a resale game. At the end of the auction and before the resale stage, some information about the submitted bids may be available. The disclosed bid information in general changes the beliefs of the valuation of the other bidder. This may further change the outcome of the resale market. We shall adopt the simplest formulation in which no bid information is disclosed[11]. We call this the minimal information case. It should be noted that there is valuation updating even if there is no disclosure of bid information, as information about the identity of the winner alone leads to updating of the beliefs.

We shall adopt a rather general formulation of the resale process. We assume that trade takes place either with probability 0 or 1 almost surely[12]. Let Q be the set of (t_1, t_2) when trade occurs with probability 1. Let $w(t_1, t_2)$ be the transaction price when trade occurs. Each bidder may be a winner in the first stage, and become the seller in the second stage. Therefore whether a bidder becomes a seller or a buyer is endogenously determined. Let $b_i(t), \phi_i = b_i^{-1}, i = 1, 2$, be the increasing bidding strategies and their inverse functions in the first stage. Let $[0, b_i^*]$ be the range of b_i. Without loss of generality, assume that $b_1^* \leq b_2^*$. Let $h(t_1)$ be defined by $b_1(t_1) = b_2(h(t_1))$. We make the following assumptions:

(A1) We have $(t_1, t_2) \notin Q$, if either $v_1(t_1) < v_2(t_2), h(t_1) < t_2$, or $v_1(t_1) > v_2(t_2), h(t_1) > t_2$.

(A2) If $(t_1, t_2) \in Q$, $v_1(t_1) < v_2(t_2)$, then $(t, t_2) \in Q$ for all $t \leq t_1, h(t) \geq t_2$,and $(t_1, t) \in Q$ for all $t \geq t_2, h(t) \geq t_2$.

(A3) If $(t_1, t_2) \in Q$, $v_1(t_1) > v_2(t_2)$, then $(t_1, t) \in Q$ for all $t \leq t_2, h(t_1) \leq t$, and $(t, t_2) \in Q$ for all $t \geq t_1, h(t_1) \leq t_2$.

(A4) The pricing function $w(t_1, t_2)$ is continuous in Q and monotonic in $t_i, i = 1, 2$.

(A5) For all $(t_1, t_2) \in Q$, we have $\min(v_1(t_1), v_2(t_2)) \leq w(t_1, t_2) \leq \max(v_1(t_1), v_2(t_2))$.

[11]Although the equivalence result may be established in a broader context with disclosure of different bid information, it is sufficient to restrict ourselves to the resale market with no disclosure of bid information in this paper. We shall deal with a more genereal formulation of the observational equivalence result in a later paper.

[12]When bids are the same, each bidder has equal chance of winning, hence we can only have the almost sure property. In Hafalir and Krishna (2007)'s formulation, a more general description is adopted in which trade may take place with a probability lower than one. However, trade occurs with probability one when the trade surplus is the maximum possible amount.

Property (A1) is a natural requirement. It says that if a bidder has lower valuation, but loses the auction, then there is no resale trade. Note that in (A2), bidder one must be the seller (the winner of the auction). By the monotonicity of $v_i, i = 1, 2$, the condition means that trade occurs with probability 1 if the seller valuation (cost) becomes lower, or the buyer valuation becomes higher. Note that we do not specify how Q is determined or by what process the transaction price $w(t_1, t_2)$ is determined. We leave this unspecified.

Note that the determination of the resale outcome depends on the bidding strategies in the first stage as the bidding strategies determine the belief system in the second stage and the belief system affects the outcome of the resale game. For a seller i, let $F_i|_x$ be the conditional distribution of F_i over the support $[x, a_i]$, and for a buyer j, let $F_j|_y$ is the conditional distribution of F_j over the support $[0, y]$. When bidder one with signal t_1 wins the auction, he becomes a seller, and the updated belief about the buyer is described by $F_2|_{v_2(h(t_1))}$. Therefore different types of bidder one have different updated beliefs. Similarly, when bidder two loses the auction, she becomes the buyer, and her updated belief about the seller (bidder one) is described by $F_1|_{v_1(h^{-1}(t_2))}$, where it is understood that if $b_2(t_2) > b_1^*$, we mean $h^{-1}(t_2) = a_1$. Because of the difference in updated beliefs among different types of bidders, the resale game after the auction here differs from the standard bilateral bargaining model. In the standard bilateral bargaining, the beliefs of different types of players are the same. This will make the equilibrium behavior in the second stage resale game different from the standard bargaining models.

We shall assume that the resale game satisfies the following property:

(ME): For all t, we have $(t, h(t)) \in Q$. If $v_1(t) \neq v_2(h(t))$, then $(t, h(t))$ is an interior point of Q.

This is a minimal efficiency property of the resale mechanism, as it says that trade occurs with probability one at the price $w(t_1, t_2)$ when the trade surplus is close to the maximum possible amount.For the bid-equivalence result, it is sufficient that Q and the price system $w(t_1, t_2)$ is common knowledge. By our assumptions, for a bidder one with signal t_1, winning the auction and $v_1(t_1) < v_2(h(t_1))$, there is an infimum $k(t_1) < h(t_1)$ such that trade always occurs whenever bidder two receives the signal $t_2 \in (k(t_1), h(t_1)]$. By our definition, there is no trade when $t_2 < k(t_1)$.

Given the strictly increasing bidding strategies $b_i(t_i), i = 1, 2$ in the first stage, and the trading set Q, and the resale price function $w(t_1, t_2)$ on Q (which depend on the bidding strategies $b_i(t_i), i = 1, 2$) in the second stage, we want to consider the optimal bidding behavior. A bidder one with signal $t_1, v_2(h(t_1)) \geq v_1(t_1)$ may consider a bid $b \geq b_1(t_1)$. If he loses the auction, the winning bidder two has valuation above $v_2(h(t_1))$, hence there is no need for resale and no payoff in this case. Thus payoff is possible only when he wins the auction. By our notation, trade occurs when $t_2 > k(t_1)$ and the payoff is given by

$$U(t_1, b) = \frac{1}{k(t_1)} \left[\int_{k(t_1)}^{\phi_2(b)} (w(t_1, t_2) - b) dt_2 + k(t_1)(v_1(t_1) - b) \right]. \qquad (7)$$

The payoff formula (7) is valid for any bid b with $\phi_2(b) \geq k(t_1)$ as well. When $\phi_2(b) < k(t_1)$, the payoff is given by $U(t_1, b) = \frac{\phi_2(b)}{k(t_1)}(v_1(t_1) - b)$. We say that $b_1(t_1)$ is an optimal bid if $b_1(t_1)$ maximizes $U(t_1, b)$.

Now consider the other case when bidder one receives a signal t_1 with $v_2(h(t_1)) \leq v_1(t_1)$. If he bids $b \leq b_1(t_1)$ and wins the auction, there is no need for resale and the payoff is $\frac{\phi_2(b)}{k(t_1)}(v_1(t_1) - b)$. If he loses the auction, there is resale (and he becomes a buyer) if $t_2 < k(t_1)$. The payoff is therefore given by

$$U(t_1, b) = \frac{1}{k(t_1)}\left[\phi_2(b)(v_1(t_1) - b) + \int_{\phi_2(b)}^{k(t_1)} (v_1(t_1) - w(t_1, t_2))dt_2\right] \quad (8)$$

The payoff formula (8) is valid for b with $\phi_2(b) \leq k(t_1)$. For b with $\phi_2(b) > k(t_1)$, there is no resale after losing the auction. The payoff is given by $U(t_1, b) = \frac{\phi_2(b)}{k(t_1)}(v_1(t_1) - b)$ in this case. Again we say that $b_1(t_1)$ is an optimal bid if $b_1(t_1)$ maximizes $U(t_1, b)$. If this is true for all t_1, we say that the strategy b_1 is optimal with respect to b_2. The optimality of b_2 with respect to b_1 is defined similarly. When b_i is optimal with respect to $b_j, j \neq i$, we say that the pair of bidding strategies $\{b_1, b_2\}$ is an equilibrium in the auction with resale. A simple standard argument shows that in equilibrium we must have $b_1^* = b_2^*$. We let b^* denote this common maximum bid.

In this definition, a particular trading equilibrium in the resale game has been implicitly assumed (given the belief system induced by the bidding strategies in the first stage), and the equilibrium of the first-stage auction is defined with respect to this resale game outcome. Since it is often possible that there are different equilibria in the resale game, we can have different equilibrium bidding behavior in the first stage game. Therefore in general there are many such equilibria in the auction with resale.

The following result is similar to the symmetry property proved in Hafalir and Krishna (2008). We need to prove it for our formulation.

Theorem 1 *If the inverse equilibrium bidding functions $\phi_i, i = 1, 2$ are differentiable in $(0, b^*]$, then the following first-order conditions are satisfied*

$$\frac{\phi_i'(b)}{\phi_i(b)} = \frac{1}{w(\phi_1(b), \phi_2(b)) - b}, i = 1, 2, b \in (0, b^*]. \quad (9)$$

and we have $\phi_1(b) = \phi_2(b)$ for all $b \in [0, b^*]$.

Proof. Let $t_1 \in (0, 1]$ be the signal of bidder one, and assume that $v_2(h(t_1)) > v_1(t_1)$. By (ME) and properties (A.2),(A3), the payoff from bidding b close to $b_1(t_1)$ is given by (7). Taking derivative of the payoff function with respect to b, we must have the following equilibrium property

$$(w(t_1, \phi_2(b)) - b)\phi_2'(b) - \phi_2(b) = 0.$$

Since $t_1 = \phi_1(b)$, we have

$$(w(\phi_1(b), \phi_2(b)) - b)\phi_2'(b) = \phi_2(b),$$

which is the same as (9). If $v_2(h(t_1)) < v_1(t_1)$ instead, then the payoff is given by (8), and the derivative of the payoff with respect to b yields

$$(v_1(t_1) - b)\phi_2'(b) - \phi_2(b) - (v_1(t_1) - w(t_1, \phi_2(b)))\phi_2'(b) = 0,$$

or

$$(w(t_1, \phi_2(b)) - b)\phi_2'(b) - \phi_2(b) = 0.$$

By substituting $t_1 = \phi_1(b)$, we get the same property (9). If $v_2(h(t_1)) = v_1(t_1)$, then the payoff is given by (7),(8) for $b > b_1(t_1), b < b_1(t_1)$ respectively. Since the derivatives of the two functions are the same as shown above, the payoff function is differentiable at $b_1(t_1)$ and must be equal to 0 by the equilibrium property. This gives us (9) is all cases. The proof for bidder two is entirely the same. The symmetry property $\phi_1(b) = \phi_2(b)$ follows from (**??** in standard arguments. ■

Theorem 2 *The pair of bidding strategies*

$$b_i(t_i) = \frac{1}{t_i}\int_0^{t_i} w(s,s)ds, t_i > 0, b_i(0) = 0$$

Proof. Since $w(s,s)$ is a continuous function, b_i is continuously differentiable

in $(0,1]$. By the L'Hopital rule, $b_i(t_i) \to 0$ as $t_i \to 0$. Hence b_i is continuous on $[0,1]$. Let $\phi_i = b_i^{-1}$ be the inverse bidding function. Then ϕ_i is also continuously differentiable in $(0,1]$. We have

$$t_i b_i(t_i) = \int_0^{t_i} w(s,s)ds \tag{10}$$

Taking derivative of (9), we have

$$t_i b_i'(t_i) + b_i(t_i) = w(t_i, t_i).$$

Since $\phi_i'(b(t_i)) = \frac{1}{b_i'(t_i)}$, we have

$$\frac{t_i}{\phi_i(b_i(t_i))} = w(t_i, t_i) - b_i(t_i)$$

or

$$\frac{\phi_i(b)}{\phi_i'(b)} = w(\phi_1(b), \phi_2(b)) - b \text{ for all } b > 0,$$

and we know that the first-order conditions (9) are satisfied by $\phi_i, i = 1, 2$. We want to apply the Sufficiency Theorem (Theorem 4.2 of Milgrom (2004)) ■

From Theorem 1, we have $h(t_1) = t_1$ in an equilibrium of auction with resale. The description of the resale process includes most of the well-known equilibrium models of bilateral bargaining between the seller and the buyer. For example, the monopoly resale of Hafalir and Krishna (2008) is a special case in which the winner of the auction is the monopolist seller in the resale game. The monopolist makes a take-it-or-leave-it offer, and the transaction price is the optimal monopoly price. Assume that bidder one is the weak bidder. Bidder one with signal t_1 has the valuation $v_1(t_1)$ and the belief that bidder two's valuation is $F_2|_{h(t)}$. Bidder one is the seller when $h(t_1) \geq t_2$. Assume that there is a uniquely determined optimal offer (equilibrium) price $P(t_1)$ of the seller. In this case, $Q = \{(t_1, t_2) : v_2(t_2) \geq P(t_1), h(t_1) \geq t_2\}$, and the pricing function $w(t_1, t_2) = P(t_1)$ is defined for $(t_1, t_2) \in Q$. Hence trade occurs if and only if $(t_1, t_2) \in Q$, and the trading price is the optimal offer price. The (ME) property must be satisfied in this case, as we know $P(t_1) \leq v_2(h(t_1))$ when $h(t_1) \geq t_2$. It is also clear that Q satisfies the assumptions we make.

Similarly, in a monopsony resale mechanism with a take-it-or-leave-it offer by the buyer, the buyer chooses an optimal monopsony price higher than the lowest possible valuation of the seller. The offer is accepted when the seller has the lowest valuation, hence the (ME) property also holds, and the transaction price is the optimal monopsony price.

Another possibility is to designate one of the bidder, say bidder one, as the offer-maker. When it is not a weak-strong pair, bidder one may become a seller or a buyer depending on the realized signals. Thus it is a mixture of the monopoly and the monopsony market. The choice of the offer-maker or the market type affects the bargaining power of the bidders and the outcome of the resale. In the case of the monopoly market mechanism, the choice of the offer-maker is not fixed in the beginning, and is contingent on the outcome of the auction. More generally, there can be simultaneous offers by both, or repeated offers with delay costs in a sequential bargaining model of resale.

For any general bilateral trade mechanism R between the seller and the buyer satisfying the property that trade takes place with probability 1 or 0, and a Bayesian equilibrium e of the mechanism, we can apply the revelation principle to define a direct trade mechanism M such that truthful-reporting is incentive compatible and individually rational and yields the same payoffs as the equilibrium payoffs in e for each seller or buyer with valuations $v_i(t_i), v_j(t_j)$ respectively. In the direct trade mechanism M, given the reported valuations $v_i(t_i), v_j(t_j)$, there is a payment $p(v_i(t_i), v_j(t_j))$ from the buyer to the seller when trade occurs. The set Q is the set of pairs (t_1, t_2) of signals in which trade occurs with probability 1, and $w(t_1, t_2) = p(v_1(t_1), v_2(t_2))$ is the transaction price for (t_1, t_2).

Now we show how the multiple-offer bargaining with a discount factor δ can be represented by $w(t_1, t_2)$ on Q satisfying our assumptions. Consider a bargaining model with two rounds of offers by the seller. Assume that signals are independent, and we have a weak-strong pair. The seller with the signal t_1 and the valuation $v_1(t_1)$ has the belief $F_2|_{h(t_1)}$ and makes an offer P_1 in the first period. This offer is either accepted or rejected, with the threshold of acceptance represented by Z, *i.e.* a buyer accepts the first offer if and only if his or her valuation is above Z. If the first offer is accepted, the game ends. If it is not accepted, the seller makes a second offer P_2 which is a take-it-or-leave-it offer. An equilibrium analysis of this model is provided in section 6.2. Let $P_1(t_1), P_2(t_1), Z(t_1)$ denote the equilibrium first-period, second-period prices and threshold level in this bargaining problem. The equilibrium prices in the bargaining model can be used to define the pricing function $w(t_1, t_2)$. Given the reported $(v_1(t_1), v_2(t_2))$, bidder one is the seller if $h(t_1) > t_2$. There is no trade if $v_2(t_2) < P_2(t_1)$. Trade occurs (with probability one) with the transaction price $p(v_1(t_1), v_2(t_2)) = P_1(t_1)$ if $v_2(t_2) \geq Z(t_1)$, and the transaction price $p(v_1(t_1), v_2(t_2)) = \delta P_2(t_2)$ if $P_2(t_1) \leq v_2(t_2) < Z(t_1)$. The set Q is

$$Q = \{(t_1, t_2) : h(t_1) \geq t_2, v_2(t_2) \geq Z(t_1) \text{ or } P_2(t_1) \leq v_2(t_2) < Z(t_1)\}$$

The (ME) property is satisfied because we must have $Z(t_1) < v_2(t_2)$, and we have $p(v_1(t_1), v_2(t_2)) = P_1(t_1)$. The (ME) property holds in a monopoly resale mechanism with many rounds of offers from the seller, if the equilibrium first offer is lower than the highest valuation of the buyer. This is true if the monopolist has a strictly positive payoff in the equilibrium.

The resale market may allow simultaneous offers made by both the buyer and the seller similar to a double auction game. We now give a resale game with simultaneous offers to illustrate the formulation of the model. Assume that the signals are independent and $v_1(t) = t, v_2(t) = 2t$ so that $F_1(x) = x, F_2(x) = \frac{x}{2}$. The first stage is a first-price auction. In the resale game, let p_s, p_b be the offer price by the seller and buyer respectively. The transaction takes place if and only if $p_s \leq p_b$, and the transaction price is given by

$$p = \frac{p_s + p_b}{2}.$$

Let the inverse bidding strategy in the first-price auction with resale be ϕ_1, ϕ_2 and in equilibrium we have $\phi_2(b) = 2\phi_1(b)$ by the symmetry property. To find an equilibrium with linear strategies in the resale game, let $p_s(v_1) = c_1 v_1 + d_1, p_b(v_2) = c_2 v_2 + d_2$ be the equilibrium strategies as functions of valuations. Bidder one with valuation v_1 chooses $p \leq 2c_2 v_1 + d_2$ to maximize

$$\int_{\frac{p-d_2}{c_2}}^{2v_1} \left[\frac{p + c_2 v_2 + d_2}{2} - v_1 \right] dv_2.$$

The derivative of the payoff with respect to p is given by

$$\begin{aligned} & -\frac{p - v_1}{c_2} + \frac{1}{2} \int_{\frac{p-d_2}{c_2}}^{2v_1} dv_2 \\ = \quad & \frac{1}{c_2}(-\frac{3}{2}p + (1 + c_2)v_1 + \frac{1}{2}d_2) \end{aligned}$$

which is decreasing in p. Therefore the payoff function is concave. The first-order condition of optimality gives us

$$p_s(v_1) = \frac{2}{3}(1 + c_2)v_1 + \frac{1}{3}d_2. \quad (11)$$

For the bidder two with valuation v_2, the price offer $p \geq \frac{v_2}{2}c_1 + d_1$ maximizes

$$\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} \left[v_2 - \frac{c_1 v_1 + d_1 + p}{2} \right] dv_1.$$

The first-order condition for the optimal offer is

$$\frac{v_2 - p}{c_1} - \frac{1}{2} \int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} dv_1 = 0$$

or

$$v_2 - p - \frac{c_1}{2}\left(\frac{p - d_1}{c_1} - \frac{v_2}{2}\right) = 0,$$

and we have the optimal offer of the buyer

$$p_b(v_2) = \frac{4 + c_1}{6} v_2 + \frac{1}{3} d_1.$$

To be an equilibrium, we must have

$$d_1 = \frac{1}{3} d_2, d_2 = \frac{1}{3} d_1$$

$$c_1 = \frac{2}{3}(1 + c_2), c_2 = \frac{4 + c_1}{6}$$

Solving the equations, we have

$$d_1 = d_2 = 0, c_1 = \frac{5}{4}, c_2 = \frac{7}{8}.$$

The (piecewise) linear equilibrium in the resale game is then given by

$$\begin{aligned} p_s(v_1) &= \frac{5}{4} v_1, v_1 \in [0, 1], \\ p_b(v_2) &= \frac{7}{8} v_2 \text{ for } v_2 \leq \frac{10}{7}, \\ &= \frac{5}{4} \text{ for } v_2 > \frac{10}{7}. \end{aligned}$$

The transaction price in the direct mechanism corresponding to this resale game equilibrium is given by

$$\begin{aligned} p(v_1(t_1), v_2(t_2)) &= \frac{1}{2}\left(\frac{5}{4} v_1(t_1) + \frac{7}{8} v_2(t_2)\right) = \frac{5}{8} t_1 + \frac{7}{8} t_2 \text{ if } v_2(t_2) \leq \frac{10}{7}, \\ &= \frac{5}{8} t_1 + \frac{5}{8} \text{ if } v_2(t_2) > \frac{10}{7}. \end{aligned}$$

Here $Q = \{(t_1, t_2) : t_1 \geq t_2, \min(\frac{7}{8} v_2(t_2), \frac{5}{4}) \geq \frac{5}{4} v_1(t_1)\} = \{(t_1, t_2) : t_1 \geq t_2, , \min(t_2, \frac{5}{7}) \geq \frac{5}{7} t_1\}$, or $Q = \{(t_1, t_2) : t_1 \geq t_2 \geq \frac{5}{7} t_1\}$. Trade occurs with probability one if and only if $(t_1, t_2) \in Q$, and there is no trade outside Q. Trade

occurs if and only if $2v_1 \geq v_2 \geq \frac{10}{7} v_1$.

Remark 3 *With homogeneous beliefs of the traders, the optimal offer functions are* $p_s(v_1) = \frac{2}{3}v_1 + \frac{1}{2}$, $v_1 \in [0,1]$ *for and seller, and* $p_b(v_2) = \frac{2}{3}v_2 + \frac{1}{6}, v_2 \in [\frac{1}{2}, \frac{3}{2}]; = \frac{1}{2}$ *when* $v_2 \leq \frac{1}{2}; = \frac{7}{6}$ *when* $v_2 \geq \frac{3}{2}$ *for the buyer. Trade offers if and only if* $v_2 \geq v_1 + \frac{1}{2}$. *Since* $v_2 \geq v_1 + \frac{1}{2}$ *implies* $v_2 \geq \frac{10}{7}v_1$, *trade is less efficient in the homogeneous case. This is because the updating of beliefs improves efficiency of trade in our model.*

We now use the resale game example with simultaneous offers above to illustrate the intuition of the bid-equivalence result. Given the equilibrium of the IPV auction with resale, let ϕ_1, ϕ_2 be the inverse bidding functions of the equilibrium bidding strategies. We have the equilibrium bidding strategies:

$$\begin{aligned} b_1(t_1) &= \frac{1}{t_1}\int_0^{t_1} w(t,t)dt = \frac{1}{t_1}\int_0^{t_1}(\frac{5}{8}t + \frac{7}{8}t)dt = \frac{3}{4}t_1, \text{ for } t_1 \leq \frac{5}{7} \\ &= \frac{1}{t_1}\int_0^{\frac{5}{7}} \frac{3}{2}tdt + \frac{1}{t_1}\int_{\frac{5}{7}}^{t_1}(\frac{5}{8}t + \frac{5}{8})dt = \frac{5}{8}(1 + \frac{1}{2}t_1 - \frac{5}{14t_1}), \text{ for } t_1 \geq \frac{5}{7} \end{aligned}$$

and the same formula applies to $b_2(t_2)$. Hence

$$\phi_1(b) = \phi_2(b) = \frac{4}{3}b \text{ for } b \leq \frac{15}{28}.$$

When bidder one with signal t_1 chooses the bid b, and wins the auction, there is trade in the resale game if and only if $t_2 \geq \frac{5}{7}t_1$. Hence the payoff is

$$\int_{\frac{5}{7}t_1}^{\phi_2(b)} w(t_1, t_2)dt_2 + \frac{5}{7}t_1 v_1(t_1) - \phi_2(b)b \tag{12}$$

if $b \geq b_2(\frac{5}{7}t_1)$. If $b < b_2(\frac{5}{7}t_1)$, there is no resale, and the payoff is $(v_1(t_1) - b)\phi_2(b)$. The optimal bid is $b_1(t_1)$. We can now define the common-value function corresponding to the resale game as follows. For $(t_1, t_2) \in Q$, let

$$w(t_1, t_2) = p(v_1(t_1), v_2(t_2)).$$

In order to extend the definition of p to all pairs $(v_1(t_1), v_2(t_2))$, let $v_2 < \frac{10}{7}v_1$. and define $p(v_1, v_2) = p(v_1, \frac{10}{7}v_1)$. For $v_2 > 2v_1$, let $p(v_1, v_2) = p(v_1, 2v_1)$. We have the property

$$p(x, x) = p(x, \frac{10}{7}x) = \frac{5}{4}x > x.$$

We now define w outside Q also as

$$w(t_1, t_2) = p(v_1(t_1), v_2(t_2)).$$

In the common-value model, when bidder one with signal t_1 bids $b \geq b_2(\frac{5}{7}t_1)$, the payoff is

$$\int_{\frac{5}{7}t_1}^{\phi_2(b)} w(t_1, t_2)dt_2 + \int_0^{\frac{5}{7}t_1} w(t_1, t_2)dt_2 - \phi_2(b)b$$

$$= \int_{\frac{5}{7}t_1}^{\phi_2(b)} w(t_1,t_2)dt_2 + \int_0^{\frac{5}{7}t_1} w(t_1,t_2)dt_2 - \phi_2(b)b,$$

which differs from (12) by a constant term not involving the variable b. Hence the bid $b_1(t_1)$ is optimal when $b \geq b_2(\frac{5}{7}t_1)$. If $b < b_2(\frac{5}{7}t_1) = \frac{15}{28}t_1$, we have $\phi_2(b) = \frac{4}{3}b$, and the payoff is

$$\int_0^{\phi_2(b)} w(t_1,t_2)dt_2 - \phi_2(b)b = (\frac{5}{4}t_1 - b)\phi_2(b) = \frac{4}{3}(\frac{5}{4}t_1 - b)b.$$

This is an increasing function of b when $b \leq \frac{5}{8}t_1$. Since $b < \frac{15}{28}t_1 < \frac{5}{8}t_1$, the payoff for $b \leq b_2(\frac{5}{7}t_1)$ attains the highest value at $b = b_2(\frac{5}{7}t_1)$. We conclude that $b_1(t_1)$ is the optimal bid for the common-value auction as well.

We now look at bidder two in the auction with resale. When bidder two with signal t_2 chooses the bid b, and loses the auction, there is resale if and only if $t_1 \leq t^* = \min\{1, \frac{7}{5}t_2\}$. Hence the payoff is

$$(v_2(t_2) - b)\phi_1(b) + \int_{\phi_1(b)}^{t^*} (v_2(t_2) - w(t_1,t_2))dt_1$$

$$= t^*v_2(t_2) - \int_{\phi_1(b)}^{t^*} w(t_1,t_2)dt_1 - b\phi_1(b) \quad (13)$$

if $b \leq b_1(t^*)$. If $b > b_1(t^*)$, there is no resale and the payoff is $(v_2(t_2) - b)\phi_1(b)$. In the common-value auction, when bidder two with signal t_2 bid b, the payoff is

$$\int_0^{\phi_1(b)} (w(t_1,t_2) - b)dt_1 = \int_0^{\phi_1(b)} w(t_1,t_2)dt_1 - b\phi_1(b)$$

$$= \int_0^{t^*} w(t_1,t_2)dt_1 - \int_{\phi_1(b)}^{t^*} w(t_1,t_2)dt_1 - b\phi_1(b), \quad (14)$$

if $b \leq b_1(t^*)$. If $b > b_1(t^*)$, there is no resale, and the payoff is

$$\int_0^{\phi_1(b)} (w(t_1,t_2) - b)dt_1 = \int_0^{t^*} p(t_1,t_2)dt_1 + \int_{t^*}^{\phi_1(b)} w(t_1,t_2)dt_1 - b\phi_1(b). \quad (15)$$

The optimal bid in the auction with resale is $b_2(t_2)$. The difference between the payoffs in (13) and (14) is a constant term which does not involve the variable b. Therefore, $b_2(t_2)$ is optimal for the common-value auction for $b \leq b_1(t^*)$. If $t^* = 1$, this means that $b_2(t_2)$ is the optimal bid in the common-value auction. If $t^* < 1$, then $t^* = \frac{7}{5}t_2$. For $t_1 \geq t^*$, we have $w(t_1,t_2) = \frac{5}{4}t_1$, hence

$$\int_{t^*}^{\phi_1(b)} w(t_1,t_2)dt_1 - b\phi_1(b) = \frac{5}{4}\int_{t^*}^{\phi_1(b)} t_1 dt_1 - b\phi_1(b)$$

$$= \frac{5}{8}\phi_1(b)^2 - b\phi_1(b) - \frac{5}{8}t^{*2} = (\frac{5}{8}\phi_1(b) - b)\phi_1(b) - \frac{5}{8}t^{*2}. \quad (16)$$

We want to show that this payoff function is decreasing in b. When $b \leq \frac{15}{28}$, we have

$$(\phi_1(b) - \frac{8}{5}b)\phi_1(b) = (\frac{4}{3}b - \frac{8}{5}b)\frac{4}{3}b = -\frac{16}{45}b^2,$$

which is a decreasing function of b .

When $t_1 \geq \frac{5}{7}$, we have

$$(t_1 - \frac{5}{8}b_1(t_1))t_1 = t_1^2 - (1 + \frac{1}{2}t_1 - \frac{5}{14t_1})t_1 = \frac{1}{2}t_1^2 - t_1 + \frac{5}{14}$$

The derivative of this function is $t_1 - 1 < 0$. This implies that for $b > \frac{15}{28}$, the function

$$(\phi_1(b) - \frac{5}{8}b)\phi_1(b)$$

is a decreasing function of b. Therefore for $b \geq b_1(t^*)$, the payoff (16) is highest at $b = b_1(t^*)$. Hence $b_2(t_2)$ is the optimal bid in the common-value auction as well. We have shown that $b_i(t_i), i = 1, 2$ is an equilibrium in the common-value auction.

There are alternative definitions of the common value outside Q yielding the same bid-equivalence result. For $v_2 < \frac{10}{7}v_1$,and define $p(v_1, v_2) = p(\frac{7}{10}v_2, v_2)$. Then outside Q we let

$$w(t_1, t_2) = p(\frac{7}{10}v_2(t_2), v_2(t_2)) = \frac{7}{4}t_2 \text{ for } t_2 \leq \frac{5}{7}.$$

To check the optimal property of $b_i(t_i)$, consider bidder one bidding $b < b_2(\frac{5}{7}t_1) \leq b_2(\frac{5}{7})$, we have $\phi_2(b) = \frac{4}{3}b$, and the payoff is

$$\int_0^{\phi_2(b)} w(t_1, t_2)dt_2 - \phi_2(b)b = \frac{7}{4}\int_0^{\frac{4}{3}b} t_2 dt_2 - \frac{4}{3}b^2 = \frac{2}{9}b^2.$$

This is an increasing function of b. Since the payoff for $b \leq b_2(\frac{5}{7}t_1)$ attains the highest value at $b = b_2(\frac{5}{7}t_1)$. We conclude that $b_1(t_1)$ is the optimal bid for the common-value auction as well. For bidder two, if $b > b_1(t^*), t^* < 1$, then $t_2 < \frac{5}{7}$, $t^* = \frac{7}{5}t_2$. For $t_1 \geq t^*$, we have $w(t_1, t_2) = \frac{7}{4}t_2$, hence

$$\int_{t^*}^{\phi_1(b)} w(t_1, t_2)dt_1 - b\phi_1(b) = \frac{7}{4}t_2(\phi_1(b) - \frac{7}{5}t_2) - b\phi_1(b) = (\frac{7}{4}t_2 - b)\phi_1(b) - \frac{49}{20}t_2^2.$$

We want to show that this payoff function is decreasing in b. When $b \leq \frac{15}{28}$, we have

$$(\frac{7}{4}t_2 - b)\phi_1(b) = \frac{4}{3}(\frac{7}{4}t_2 - b)b.$$

This is a decreasing function if $b \geq \frac{7}{8}t_2$ which is true as $b > b_1(t^*) = \frac{3}{4}\frac{7}{5}t_2 = \frac{21}{20}t_2 > \frac{7}{8}t_2$. For $t_1 \geq \frac{5}{7}$, we have

$$(\frac{7}{4}t_2 - b_1(t_1))t_1 = (\frac{7}{4}t_2 - \frac{5}{8})t_1 - \frac{5}{16}t_1^2 + \frac{25}{112}$$

The derivative with respect to t_1 is

$$\frac{7}{4}t_2 - \frac{5}{8} - \frac{5}{8}t_1 \leq \frac{7}{4}t_2 - \frac{5}{8} - \frac{7}{8}t_2 = \frac{7}{8}t_2 - \frac{5}{8} < \frac{7}{8}\frac{5}{7} - \frac{5}{8} = 0.$$

Therefore, the payoff is decreasing for $b > b_1(t^*)$, and $b_2(t_2)$ is the optimal bid for bidder two. We have shown that $b_i(t_i), i = 1, 2$ is an equilibrium in the common-value auction with an alternative definition of the common value outside Q. For this choice of p, we have $p(x, x) = \frac{7}{8}x < x$.

To state the equivalence result, we need to define a common-value model with a common-value function $w(t_1, t_2)$ defined by the resale game after the auction. The common-value function we define is also determined by the equilibrium bidding strategy of the auctions with resale model. Let the strictly monotone equilibrium bidding functions of the bidders be $b_i(t_i), i = 1, 2$.

Given the equilibrium bidding strategies $b_i(t_i), i = 1, 2$ in the auction with resale and the pricing function $w(t_1, t_2)$ defined over Q, we define a common-value model with the common-value function

$$w(t_1, t_2) = w(t_1, t_2) \text{ for } (t_1, t_2) \in Q.$$

For (t_1, t_2) outside Q, let $k(t_1)$ be the maximum t_2 such that $(t_1, t_2) \in Q$. we define $w(t_1, t_2) = w(t_1, k(t_1))$ for $(t_1, t_2) \notin Q$.

We now state the bid-equivalence result.

Theorem 4 *Assume that there is no disclosure of bid information in between the auction stage and the resale stage, and the resale process satisfies (A1)˜(A4). Assume that there is a strictly monotone equilibrium bidding strategy profile* $b_i(t), i = 1, 2$ *in the auction with resale satisfying the minimal efficiency property (ME). Then there is common-value first-price auction with a common-value function defined by the pricing function of the resale game R whenever trade occurs, such that* $b_i(t), i = 1, 2$ *is also an equilibrium of the common-value auction, and we have bid-equivalence between the auction with resale and the common-value auction.*

Proof of Theorem 4:

Consider the determination of the equilibrium bidding strategy in the first stage of the auction with resale. Let $\phi_i = b_i^{-1}, i = 1, 2$ be the inverse bidding functions. Let bidder one signal be t_1, and $t_2 = h(t_1)$. Assume that $v_1(t_1) < v_2(t_2)$. Since trade occurs when the valuation of bidder two is $v_2(t_2)$, bidder one must be the seller when bidder two receives the signal t_2. By assumption $(t_1, t_2) \in Q$, hence there exists a minimum $t_2^0 \geq t_2$ such that $(t_1, t) \in Q$ for all $t \in (t_2^0, t_2]$. We also have $(t_1, t) \notin Q$ for all $t < t_2^0$. Let $b_2^0 = b_2(t_2^0)$.

When bidder one with signal t_1 chooses the bid $b \geq b_2^0$, there is resale after the auction at the price $p(t_1, t)$ if $t \in (t_2^0, \phi_2(b))$, and the payoff is

$$\int_{t_2^0}^{\phi_2(b)} w(t_1, t)dt + v_1(t_1)t_2^0 - \phi_2(b)b. \tag{17}$$

If $b < b_2^0$, then there is no resale and the payoff is $(v_1(t_1) - b)\phi_2(b)$. The same bidder bidding $b \geq b_2^0$ in the common-value auction has payoff

$$\int_{t_2^0}^{\phi_2(b)} w(t_1, t)dt + \int_0^{t_2^0} w(t_1, t)dt - \phi_2(b)b$$

$$= \int_{t_2^0}^{\phi_2(b)} w(t_1, t)dt + w(t_1, t_2^0)t_2^0 - \phi_2(b)b \tag{18}$$

Since (17) and (18) differ only by a constant term (with respect to b), and $b_1(t_1)$ is optimal for (17) , we know that $b_1(t_1)$ is also optimal for the payoff (18) within the range $b \geq b_2^0$. For $b < b_2^0$, we have the payoff

$$(p(t_1, t_2^0) - b)\phi_2(b).$$

The derivative with respect to b (with the possible exception of a finite number of points) is

$$(w(t_1, t_2^0) - b)\phi_2'(b) - \phi_2(b) = \phi_2'(b)(w(t_1, t_2^0) - b - (p(\phi_1(b), \phi_2(b)) - b))$$

$$= \phi_2'(b)[w(t_1, t_2^0) - w(\phi_1(b), \phi_2(b))].$$

We have $\phi_2(b) < t_2^0$, and $b < b_2^0 \leq b_1(t_1)$, hence $\phi_1(b) < t_1$, and the derivative is ≥ 0. Therefore the payoff is increasing within the range $b < b_2^0$, and this implies that $b_1(t_1)$ is optimal for the payoff (18) in the common-value auction.

In the auction with resale model, we have $(h^{-1}(t_2), t_2) \in Q$, hence there exists a maximum $t_1^0 \geq h^{-1}(t_2)$ such that $(t, t_2) \in Q$ for all $t \in [t_1, t_1^0)$. We also have $(t, t_2) \notin Q$ for all $t > t_1^0$. Let $b_1^0 = b_1(t_1^0)$. When bidder two with signal t_2

offers the bid $b \leq b_1^0$, the payoff is

$$(v_2(t_2) - b)\phi_1(b) + \int_{\phi_1(b)}^{t_1^0} (v_2(t_2) - w(t, t_2))dt$$

$$= \int_0^{t_1^0} v_2(t_2)dt - \int_{\phi_1(b)}^{t_1^0} w(t, t_2)dt - b\phi_1(b)$$

In the common-value model, when bidder two with signal t_2 bid $b \leq b_1^0$, the payoff is

$$\int_0^{\phi_1(b)} (w(t, t_2) - b)dt = \int_0^{t_1^0} w(t, t_2)dt - \int_{\phi_1(b)}^{t_1^0} w(t, t_2)dt - b\phi_1(b).$$

The difference between the payoff functions in the two different auctions is a constant term not involving b. Therefore, $b_2(t_2)$ is optimal in the range $b \leq b_1^0$. For $b > b_1^0$, the payoff is

$$\int_0^{\phi_1(b)} (w(t,t_2) - b)dt = \int_0^{t_1^0} w(t,t_2)dt + \int_{t_1^0}^{\phi_1(b)} w(t,t_2)dt - b\phi_1(b)$$

We want to show that this payoff function is decreasing in $b > b_1^0$. The derivative with respect to b is

$$(w(\phi_1(b),t_2) - b)\phi_1'(b) - \phi_1(b) = \phi_1'(b)[w(\phi_1(b),t_2) - b - (w(\phi_1(b),\phi_2(b)) - b)]$$

$$= \phi_1'(b)[w(\phi_1(b),t_2) - w(\phi_1(b),\phi_2(b))].$$

Since $b > b_1^0 \geq b_2(t_2)$, we also have $\phi_2(b) \geq t_2$. Hence the derivative is ≤ 0. Therefore, $b_2(t_2)$ is the optimal bid in the common-value auction as well. The other case $v_1(t_1) > v_2(t_2)$ is similar. Finally, if $v_1(t_1) = v_2(t_2)$, either one of the bidder may be a winner and hence a seller. The other one is a buyer. The arguments then proceed in a similarly way as well.

In a bilateral trade equilibrium, trade need not occur with probability one even when the trade surplus is the highest possible amount. In this case, the minimal efficiency property fails, and the bid equivalence result need not hold. To show necessity of the minimal efficiency condition for the bid equivalence result, let us consider the monopoly resale example in section 2. Assume that in the resale stage, trade is artificially restricted so that it only occurs with probability 0.5 when the buyer is willing to accept the offer by the seller. There is no trade when the buyer does not accept the offer. With this trade mechanism, the optimal offer by the seller is the same as the monopoly price without restriction. Hence the optimal offer price is 2 whenever there is a positive probability that the buyer has valuation 2. The weak bidder with valuation 1 chooses b to maximize the following payoff

$$0.5(H_2(b)-0.4)(2-b)+0.5(H_2(b)-0.4)(1-b)+0.4(1-b) = H_2(b)(1.5-b)-0.2.$$

The first-order condition is

$$\frac{H_2'(b)}{H_2(b)} = \frac{1}{1.5-b}. \tag{19}$$

Bidder one with valuation 0 chooses b to maximize

$$0.5(H_2(b)-0.4)(2-b)+0.5(H_2(b)-0.4)(0-b)+0.4(0-b) = H_2(b)(1-b)-0.4$$

and the first-order condition is

$$\frac{H_2'(b)}{H_2(b)} = \frac{1}{1-b}. \tag{20}$$

Bidder two with valuation 2 chooses b to maximize the payoff

$$H_1(b)(2-b)$$

with the first-order condition

$$\frac{H_1'(b)}{H_1(b)} = \frac{1}{2-b}. \tag{21}$$

Let b^* be the common maximum bid of both bidders. We have the following boundary conditions

$$H_2(0) = 0.4, H_2(b^*) = H_1(b^*) = 1.$$

Note that the first-order conditions of the two bidders are not symmetric, and we don't expect the equilibrium bid distributions of the two bidders to satisfy the symmetry property. If the symmetry property fails, then the equilibrium of the auction with this restricted monopoly resale market cannot be bid equivalent to any equilibrium of a common-value auction. To find the equilibrium, let $H_1(c) = 0.7$. We have

$$H_1(b) = \frac{A}{2-b} \text{ for } b \in [0, b^*],$$

where $A = 2 - b^*, 0.7(2-c) = 2 - b^*$. For bidder two, we have

$$H_2(b) = \frac{B}{1.5-b} \text{ for } b \in [c, b^*],$$

hence $B = 1.5 - b^*$. For $b \in [0, c]$, we must have $H_2(0) = 0.4$, hence

$$H_2(b) = \frac{0.4}{1-b} \text{ for } b \in [0, c].$$

We must have

$$H_2(c) = \frac{0.4}{1-c} = \frac{B}{1.5-c}$$

or

$$\frac{0.4(1.5-c)}{1-c} = B = 1.5 - b^* = 1.5 - (2 - 0.7(2-c)) = 0.9 - 0.7c$$

We have the following quadratic equation in c :

$$(0.9 - 0.7c)(1-c) - 0.4(1.5-c) = 0.7c^2 - 1.2c + 0.3 = 0$$

and the solution is $c = 0.303\,86$. From this number, we have $b^* = 0.687\,30, A = 1.31270, B = 0.81270$. Hence we have the following equilibrium cumulative bid distributions when ties are broken in favor of bidder one:

$$H_1(b) = \frac{1.3127}{2-b}, b \in [0, 0.6873],$$

$$H_2(b) = \frac{0.4}{1-b}, b \in [0, 0.30386]$$
$$= \frac{0.8127}{1.5-b}, b \in [0.30386, 0.6873].$$

Clearly, this equilibrium does not satisfy the symmetry property, and the bid equivalence result fails.

Later on in section 6.1, we will compare revenues of the first-price and second-price auctions with resale. In the second-price auction with resale, the game differs only in the first stage, in which the first-price auction is replaced by the second-price auction. In a second-price auction with resale, the winner of the auction knows the losing bid if the payment is made, as the losing bid is the price he pays in the auction. To conceal this information, the payment can be deferred after the resale game. There is in fact a continuum of equilibria (see Blume and Heidhues (2004)) in the second-price auction with resale. It is an equilibrium for both bidders to bid their valuation (see Proposition 2 in Hafalir and Krishna (2008)), and this is an efficient equilibrium. The efficiency means that there is no need for resale after the auction, so that the revenue is the same with or without resale. When there is no resale, the "bid-your-value" strategies constitute a weakly dominant equilibrium strategy. With resale, it is no longer weakly dominant. However it is robust in the sense of Borgers and McQuade (2007), and is the only robust equilibrium (see the supplement to Hafalir and Krishna (2008)). This is the equilibrium used in the revenue ranking of the auctions with resale. Since there is no resale transaction in the bilateral trade mechanisms, the second-price auction revenue does not depend on the different trade mechanisms in the second stage.

4.1 Speculator-Buyer Model of Auction with Resale

A special interesting case is worth mentioning here. If bidder one is a speculator having no value for the object, but participates in the auction for resale, and bidder two is a regular buyer, we call this the speculator-buyer model of auction with resale. From our bid equivalence result, the equilibrium bidding behavior of this model is the same as the Wilson Drainage Tract Common Value Model.

Assume that bidder two has a cumulative valuation distribution $F(x)$ on $[0, a]$. and satisfies the condition that the virtual value

$$x - \frac{1 - F(x)}{f(x)} \tag{22}$$

is increasing. Assume that the speculator is a monopolist in the resale stage. Let $v = F^{-1}$, and $p(t)$ be the optimal monopoly price when the speculator faces a buyer with the valuation distribution F conditional on the support $[0, v(t)]$. The first order condition for $p(t)$ is the equation in p :

$$p - \frac{F(v(t)) - F(p)}{f(p)} = 0.$$

The equation has a unique solution when (1) is increasing.

According to the bid equivalence result, the equilibrium bid distributions of this auction with resale model are the same as the Wilson drainage tract common value model in which the common value function is $w(t_1, t_2) = p(t_1)$ where t_1, t_2 are the signals received by the drainage tract owner and the non-neighbor respectively. The set Q can be described as $Q = \{(t_1, t_2) : t_1 \geq t_2, v_2(t_2) \geq p(t_1)\}$. Interestingly, the speculator in the auction with resale model corrsponds to the drainage tract owner who receives the private informative signal, but in the auction with resale model, the regular buyer is the one having the private information. It is an intersting question to ask whether bidders behave the same way in both models in practice or in experiments, given the evidence that winner's curse is observed in practice, but we don't expect similar irrationality in the auction with resale.

5 Two Polar Cases

The existence and uniqueness of the equilibrium in the first-price common-value auctions have been studied in the literature[13]. There is an explicit formula for the equilibrium in Parreiras (2006). In the pure common-value model with independent signals, it is well-known that in equilibrium, the winning probabilities of the two bidders are the same when they bid the same amount[14]. The symmetric property of the winning probabilities is the property that both bidders have identical bidding strategies (as functions of t). In other words, we have $b_1(t) = b_2(t)$. Note that there is asymmetry in the signals as v_1, v_2 are different, and bidding strategies as functions of v_i are not symmetric. However, bidding strategies in terms of t are symmetric.

The symmetry property of the equilibrium bidding strategy gives us very simple formulas for the bidding strategy and the revenue. The following proposition summarizes the formulas in Gupta and Lebrun (1999) in the context of auction with resale and in Parreiras (2006) in the context of common-value auctions. We adopt a different notation, as we use the signal representation

[13] The existence of a non-decreasing equilibrium in the common value model is established in Athey (2001). The existence of a strictly increasing equilibrium has been shown in Rodriguez (2000). The uniqueness of equilibrium of the first price auction of the common value model can be found in Lizzeri and Persico (1998) and Rodriguez (2000).

[14] This can be found in Engelbrecht-Wiggans, Milgrom, and Weber (1983) for the Wilson track model and more generally in Parreiras (2006) and Quint (2006). This property also holds in first-price auctions with resale in Hafalir and Krishna (2007).

(or distributional) approach[15]. In this representation, the formulas take a very simple form.

Proposition 5 *With two bidders, the equilibrium bidding strategy in the first-price common-value auction is symmetric and is given by*

$$b(t) = \frac{1}{t}\int_0^t p(v_1(r), v_2(r))dr$$

with the revenue given by

$$R^F = 2\int_0^1 (1-t)p(v_1(t), v_2(t))dt.$$

We shall first apply the formulas to the two polar cases: $w(t_1, t_2) = \max\{v_1(t_1), v_2(t_2)\}$, or $w(t_1, t_2) = \min\{v_1(t_1), v_2(t_2)\}$. These common-value functions correspond to two types of resale markets. In Gupta and Lebrun (1999), they assume that after the first-stage auction, the valuations of both bidders become common knowledge (with no particular explanation of how this is achieved). In the monopoly resale market, the complete information leads to the transaction price $\max\{v_1(t_1), v_2(t_2)\}$. In the monopsony resale market, the complete information leads to the transaction price $\min\{v_1(t_1), v_2(t_2)\}$. We have the following equilibrium strategies and revenues

$$b_{\max}(t) = \frac{1}{t}\int_0^t \max(v_1(r), v_2(r))dr, R^F_{\max} = 2\int_0^1 (1-t)\max(v_1(t), v_2(t))dt,$$

for the maximum case, and

$$b_{\min}(t) = \frac{1}{t}\int_0^t \min(v_1(r), v_2(r))dr, R^F_{\min} = 2\int_0^1 (1-t)\min(v_1(t), v_2(t))dt,$$

for the minimum case. Note that the equilibrium bidding strategies are strictly increasing. The two values give us the upper bound and the lower bound of the revenue of the auction with resale.

Another simple example without perfect information in the resale stage in which the two polar cases represent the seller's and buyer's market is the one in section two. In that example, the common value functions are $w(t_1, t_2) = \max\{v_1(t_1), v_2(t_2)\}$, or $w(t_1, t_2) = \min\{v_1(t_1), v_2(t_2)\}$ respectively, even though there is still incomplete information about the valuations. Furthermore, even if

[15] We want to thank Jeremy Bulow for pointing out that the bidding formula can also be obtained from the theorem in Milgrom and Weber (1982) by using symmetric signals but asymmetric common value functions.

all the bids are announced in that example, the equilibrium still holds. In the maximum case, if bidder two lowers the bid to 0, the profit is lower or the same (when ties are broken in favor of bidder two). Therefore there is no incentive for bidder two to lower the bid to signal low valuation in this case. In the minimum case, If bidder one with valuation 0 bids $b > 0$, and signals a high valuation, the object is sold at the price 1 to bidder two. This yields the profit $-0.7b + (H_2^r(b) - 0.7)(1-b) = 0$. Hence there is no profit from such signaling strategy. Bidder one has no incentive to signal high valuation either. Thus the announcement of all the bids by the auctioneer does not destroy the respective equilibrium.

In general, however, when all the bids are announced, the signalling strategy can destroy the increasing equilibrium. Take the following example $v_1(t) = t, v_2(t) = 2t$. We have the symmetric bidding strategy for the maximum common value without bid disclosure

$$b_{\max}(t) = \frac{1}{t}\int_0^t 2r dr = t.$$

If bids are announced, bidder two with valuation $2t$ can lower the bid to $b < t$, and buys the object from bidder one at the lower price $2b$ when the auction is lost. The profit becomes

$$\phi_1(b)(2t-b) + (1-\phi_1(b))(2t-2b) = \phi_1(b)b - 2b + 2t = b^2 - 2b + 2t.$$

The derivative of the profit function is $2b - 2 < 0$. This means that when the bid b is lowered, the profit will increase, thus destroying the above equilibrium strategy when bids are announced, and signalling is effective. Similarly, the symmetric equilibrium for the minimum common value case with no bid disclosure is

$$b_{\min}(t) = \frac{1}{t}\int_0^t r dr = \frac{t}{2}.$$

When bids are announced, bidder one with valuation t can raise the bid $b > \frac{t}{2}$, and sells the object to bidder two at the price $2b$. The profit from this deviation is

$$\phi_2(b)(2b - b) = 2b^2,$$

which is an increasing function of b. The signalling strategy destroys the no disclosure equilibrium.

As we shall see in a minute, if we assume that only the winning bid is disclosed (which is normally the case in the real world auctions), the Coase Theorems (Gul, Sonnenschein, and Wilson (1986)) will allow us to obtain similar equilibrium outcome in auctions with resale in the limit. For convenience, we shall refer to the maximum case as the case of seller's market. This means that the resale market is a monopoly market in which the seller captures all the surplus from trade. Similarly, the minimum case is referred to as the buyer's market in which the buyer captures all the trade surplus.

Gupta and Lebrun (1999) have shown that there is a reversal of ranking between the first-price auction with resale and the second-price auction with resale in this case. We now show a similar result for the comparison of revenues in auctions with and without resale. Let R^r denote the revenue of the first-price auction with resale, and R^o denote the revenue of the first-price auction without resale.

Theorem 6 *Assume that $v_1(t) \leq v_2(t)$, for all t and $v_1(t)) \neq v_2(t)$ for a subset of $[0,1]$ of non-zero measure. If $w(t_1,t_2) = \max\{v_1(t_1), v_2(t_2)\}$, then $R^r > R^o$.*

If $w(t_1,t_2) = \min\{v_1(t_1), v_2(t_2)\}$, then we have $R^o > R^r$.

Proof. Let
$$F(x) = \min\{F_1(x), F_2(x)\}.$$
and $v(t) = F^{-1}(t)$. Then we have $v(t) = \max\{v_1(t), v_2(t)\}$. Then R^r is the revenue of a symmetric auction with the valuation distribution $F(x)$ for each bidder. Let H be equilibrium bid distribution of each bidder in this symmetric auction. Let H_1, H_2 be the equilibrium bid distribution of bidder one and two respectively in the asymmetric auction without resale. According to Proposition of Maskin and Riley (2000), H first-order stochastically dominate H_2 which also first-order stochastically dominates H_1. Hence we have H^2 first-order stochastically dominates H_1H_2, and we have $R^r > R^o$. The result for the case of $w(t_1,t_2) = \min\{v_1(t_1), v_2(t_2)\}$ is completely similar. ∎

Remark 7 *There is a difference between full extraction of surplus and monopoly or monopsony. For the example in section 2, when ties are broken in favor of bidder two, the following is an equilibrium in the auction with the monopsony resale market,*
$$H_1(b) = \frac{0.55}{1-b} = H_2(b), b \in [0, 0.45].$$
This is not a full extraction equilibrium, as bidder two fails to extract all the surplus of a bidder one with valuation 0. *The revenue of this equilibrium is*
$$\int_0^{0.45} (1 - \frac{0.55^2}{(1-b)^2})db = 0.202\,5.$$
The bid-your-valuation equilibrium in the second-price auction with resale yields the revenue 0.18. *Hence the equilibrium in the first-price auction with monopsony resale has higher revenue than the second-price auction with resale. The full-extraction monopsony equilibrium for the same example has the revenue*
$$\int_0^{0.3} (1 - \frac{0.7^2}{(1-b)^2})db = 0.09,$$

which is lower than the second-price auction with resale. The revenue of the first-price auction without resale is

$$\int_0^{0.6} (1 - \frac{1.4 * 0.4}{(1-b)(2-b)})db = 0.286\,62.$$

Hence this is also an example in which the first-price auction without resale may yield a higher revenue than the auction with a monopsony resale without full extraction of surplus.

We now look at the implications of the Coase Theorem. Assume that the object for sale is a durable good, and both the seller and the buyer are patient (with a discount factor close to 1) or when the offers are made in increasingly short intervals. The Coase (1972) conjecture in fact says that the monopolist may lose all bargaining power if the buyer anticipates lower prices in future offers. This has been formalized in Gul, Sonnenschein and Wilson (1986)[16]. In their model, the monopolist makes a sequence of offers until the offer is accepted by the buyer. Assume that the seller's valuation is common knowledge (as in the case when the winning bid is announced). Consider the set of equilibria with the stationary property that the state of the market, after any price that is lower than all preceding prices. is independent of the earlier price history in the market. Such equilibria are said to be stationary, since the acceptance and rejection decisions and future offers depend only on the current price. They show that for such equilibria, all prices including the first offer goes to the marginal cost of the monopolist (Theorem 3).

Applying this result to our model, the winner of the auction is the seller, and the winning bidder's valuation is the marginal cost of the seller. When the winning bid is announced, the seller's valuation becomes common knowledge. Since the first offer price is accepted by the buyer with the highest valuation, this price is the common value. When the first offer converges to the marginal cost, the common value function then converges to $\min\{v_1(t), v_2(t)\}$[17]. If only the losing bid is announced, while the loser makes offers, similar arguments lead us to the common value function $\max\{v_1(t), v_2(t)\}$ in the limit. Thus we can see these two cases as extreme cases of auctions with resale.

In the literature on Coase conjecture, the seller's cost is usually fixed, and equal to 0. In our resale model, the seller's cost can be any number within the

[16] Fudenberg, Levine, and Tirole (1985) have a Coase Theorem in the "gap" case in an infinite horizon model of bargaining when the discount rate is close to 1. Our model does not allow the "gap" case.

[17] When alternating offers are allowed, Ausubel and Deneckere (1992, Theorem 3.2) show that a version of the Coase conjecture also holds. Assume that the equilibrium satisfies stationarity, monotonicity, pure strategies, and no free screening. As the time interval between successive periods is made sufficiently short, the initial serious offer by the seller or buyer in an alternating-offer bargaining game must be close to the marginal cost of the monopolist for an entire family of distribution functions. Thus the buyer's market could also arise with alternating offers. In private communication, Ausubel has indicated that this result does not imply the authors's belief in the generality of the Coase conjecture for bargaining with alternating offers.

range $[0, a_1]$. To show how the Coase Theorem can be adapted for any cost of the seller with heterogeneous beliefs due to updating, we illustrate with the finite horizon model of Sobel and Takahashi (1983). We show that for any given discount factor $\delta_1 < 1$ of the seller, the Coase conjecture holds as $\delta_2 \to 1$, and the number of periods goes to infinity. We focus on the linear case of Sobel and Takahashi (1983).

Assume that bidder one and two have uniform IPV distributions over the intervals $[0, a_1], [0, a_2]$ respectively and $a_1 < a_2$. After the first-price auction in stage one, the winning bid is announced. In stage two, the winner of the auction makes no commitment offers (except the last one which is a take-it-or-leave-it offer) to the loser for n periods. In this case, only bidder one will make offers after winning the auction. First we derive the unique perfect Bayesian equilibrium of this finite-offer game and show that the revenue ranking is reversed. Let the seller has the valuation x and in equilibrium she believes that the buyer's valuation is uniformly distributed over $[0, y], y = \frac{a_2}{a_1}x$. We denote this bargaining game by $L_n(x, y)$. The proof of the following result is given in the appendix.

Proposition 8 *The first period offer of the bargaining game $L_n(x, y)$ in the resale stage with n periods of offers is given by*

$$p = c_n y + (1 - c_n)x$$

where c_n is defined recursively by

$$c_1 = \frac{1}{2}, c_k = \frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}}.$$

Fix $\delta_1 < 1$, and let $\delta_2 \to 1$, we have

$$c_k \to \frac{c_{k-1}}{2 - \delta_1} \text{ for all } k.$$

Since $c_1 = \frac{1}{2}$, we have $c_n = \frac{1}{2(2-\delta_1)^{n-1}} \to 0$, as $n \to \infty$. Therefore the first period offer p converges to $x = \min\{x, y\}$ as $n \to \infty$. By Theorem **??**, the revenue ranking is reversed if $\delta_1 < 1$ is fixed, δ_2 is close to 1, and the number of offer periods n is sufficiently large. In this example, Coase Theorem holds as long as the buyer is sufficiently patient, and the number of bargaining period is sufficiently large.

6 Effects of Bargaining Power

The main purpose of this section is to illustrate the applications of the bid-equivalence result and explore the effects of bargaining power on many important questions regarding auctions with resale. The bid equivalence result makes it possible to analyze the revenue and efficiency questions in the auction with resale when the resale market is more complicated than the monopoly or monopsony market in Hafalir and Krishna (2008).

We examine the impact of the valuation distribution on bargaining power and the ranking results of first-price and second-price auctions with resale. In section 6.2, we look at the way delay costs affect bargaining power and consequently the ranking results as well as the revenue comparisons between auctions with or without resale.

In subsection 6.3, we examine the resale market with simultaneous offers such as the k-double auction. Lebrun (2007) shows that the revenue for the monopoly market is higher than that for the monopsony market. This accords with our intuition with bargaining power effect.

We shall first offer a simple way of formulating the idea that the revenue of the auction with resale is an increasing function of the bargaining power of the seller. Let bidder one and two be a pair of weak-strong bidders. Consider the common value function

$$w(t_1, t_2) = (1 - k)v_1(t_1) + kv_2(t_2) \tag{23}$$

where k represents the bargaining power of bidder one. The higher k is, the higher the transaction price in the resale stage. Gupta and Lebrun (1999) have noted that as k changes from 0 to 1, there is a reversal between the ranking of first-price auction and second price auction with resale. There is $k_0 \in (0, 1)$ such that the two auctions yield the same revenue. This implies that $R^F > (<)R^S$ if and only if $k > (<)k_0$.

There is a similar result for the revenue comparisons between the auction with resale and auction without resale.

Theorem 9 *For a weak-strong pair, the auctioneer's revenue is an increasing function of the bargaining power k of the weak bidder. There exists $k_1 \in (0, 1)$ such that $R^r > (<)R^o$ if and only if $k > (<)k_1$.*

What makes this statement not completely satisfactory is that we don't know how the common value function (23) arises from the bargaining process. In the following subsections, we examine the resale markets in more details and endogenize the determination of the bargaining power by many factors in the bargaining process.

6.1 Bargaining power and valuation distributions

We will present two results. One is a general ranking result based on a property of the common value function. When the property is satisfied, the first-price auction with resale has higher revenue than the second-price auction. The second result says that when the resale market is either a monopoly or a monopsony, the property is always satisfied if the offer-receiver has a convex valuation distribution. The two results combined explain an intuition in the unambiguous ranking result of Hafalir and Krishna (2008) and reconcile it with the polar ranking results of the last section.

It is quite intuitive that one's valuation distribution affects the bargaining power in the resale stage. For instance, assume that we have a monopoly resale market, and the buyer's valuation tends to be on the high side. We would expect the seller to offer a relatively high monopoly price to the buyer. The higher price benefits the auctioneer, and raises the revenue in the first-stage auction. One way to formalize this idea is to assume that the buyer's valuation distribution is convex. In the monopoly resale market, condition (C) becomes

$$p(x_1, x_2) \geq \frac{x_1 + x_2}{2},$$

which says that the monopoly price is above the average of the seller's and buyer's valuations. We show in Theorem 10 below that in this case, the revenue of the first-price auction with resale dominates that of the second price auction. Furthermore, Proposition 11 shows that condition (C) holds when the buyer's valuation distribution is convex and the resale market is a monopoly. We should emphasize that Theorem 10 can be applied to more general resale markets. For example, we will see in the subsection that it applies to the k-double auction case when the distributions are uniform.

The above observation allows us to reconcile the polar results of the above section with that of Hafalir and Krishna (2008). They show that when the valuation distributions are regular, the first-price auction with resale is always superior to the second-price auction. Convexity is a strong form of regularity, and the implication of regularity is similar to convexity which insures that the seller has sufficient bargaining power to raise the first-price auction revenue above that of the second-price auction revenue. The implication of regularity is beyond the scope of this paper, and is treated in Cheng and Tan (2008) in which it is shown that regularity is necessary for the unambiguous ranking result.

The first result we offer is based on condition (C') of the common-value function. When condition (C) holds, the ranking holds without detailed knowledge of the valuation distributions $F_i, i = 1, 2$. The common-value function $p(x_1, x_2) = \max\{x_1, x_2\}$ satisfies condition (C), and the ranking result applies to this case.

Theorem 10 *Suppose p satisfies condition (C'), $p(x,x) \geq x, p(0,0) = 0$, and $v_1(t)) \neq v_2(t)$ for a subset of $[0,1]$ of non-zero measure. Then $R^F > R^S$.*

Proof of Theorem 10:

Let $a = \min(a_1, a_2)$. Note that $p(x,x) = 0$ for $x > a$ by our convention. We have

$$R^S = \int_0^a x d[1-(1-F_1(x))(1-F_2(x)] \leq \int_0^a p(x,x) d[1-(1-F_1(x))(1-F_2(x)]$$

$$= -\int_0^a p(x,x) d[(1-F_1(x))(1-F_2(x)] = \int_0^a (1-F_1(x))(1-F_2(x) dp(x,x)$$

$$< \frac{1}{2}\int_0^a [(1-F_1(x))^2 + (1-F_2(x))^2] dp(x,x) \leq \frac{1}{2}\int_0^{a_1} (1-F_1(x))^2 dp(x,x) + \frac{1}{2}\int_0^{a_2} (1-F_2(x))^2 dp(x,x)$$

Using integration by parts, we have

$$R^S < \int_0^{a_1} (1-F_1(x)) p(x,x) dF_1(x) + \int_0^{a_2} (1-F_2(x)) p(x,x) dF_2(x)$$

$$= \int_0^1 (1-t) p(v_1(t), v_1(t)) dt + \int_0^1 (1-t) p(v_2(t), v_2(t)) dt$$

$$= \int_0^1 (1-t) \left[p(v_1(t), v_1(t)) + p(v_2(t), v_2(t)) \right] dt.$$

Condition (C') now implies that

$$R^S < 2\int_0^1 (1-t) p(v_1(t), v_2(t)) dt = R^F,$$

and the theorem is proved.

The following result, which is of independent interest on its own, says that if an offer-maker (a monopolist or monopsonist) makes an offer to the offer receiver (the buyer or the seller, respectively), and the offer receiver has a convex valuation distribution, then the optimal offer price satisfies condition (C).

Proposition 11 *Let the seller valuation be x, and the buyer has a convex valuation distribution F_j over $[b_j, a_j], a_j \geq x$. Let $y \in [b_j, a_j]$ be the maximum valuation of the buyer. Then the optimal monopoly price function $p(x,y)$ satisfies condition (C). Similarly, let the buyer valuation be x, and the seller has a convex valuation distribution F_j over $[b_j, a_j]$ with $b_j \leq x$. Let $y \in [b_j, a_j]$ be the minimum valuation of the seller. The optimal monopsony price $r(x,y)$ also satisfies condition (C).*

Proof of Proposition 11:

Since $p(x,x) = x$, the optimal monopoly price $p(x,y)$ satisfies condition (C) if

$$p(x,y) \geq \frac{x+y}{2}.$$

As $z = p(x,y)$ maximizes the following objective function in variable z

$$K(z) = [F_j(y) - F_j(z)](z - x),$$

it is sufficient to show that

$$K'(\frac{x+y}{2}) > 0,$$

or

$$F_j(y) - F_j(\frac{x+y}{2}) - F_j'(\frac{x+y}{2})(\frac{x+y}{2} - x) > 0.$$

Equivalently, we need to show that

$$\frac{F_j(y) - F_j(\frac{x+y}{2})}{\frac{y-x}{2}} > F_j'(\frac{x+y}{2}). \tag{24}$$

Note that the left-hand side (24) is the slope of the line through the two points $(\frac{x+y}{2}, F_j(\frac{x+y}{2})), (y, F_j(y))$, while the right-hand side is the slope of F_j at $\frac{x+y}{2}$. The convexity of F_j is sufficient for (24) to hold.

For the monopsony case, the arguments are very similar. Since $z = r(x,y)$ maximizes the following objective function in variable z

$$K(z) = (F_j(z) - F_j(x))(y - z),$$

it is sufficient to show that

$$K'(\frac{x+y}{2}) > 0,$$

or

$$F_j'(\frac{x+y}{2})(y - \frac{x+y}{2}) - F_j(\frac{x+y}{2}) + F_j(x) > 0.$$

Equivalently, we need to show that

$$F_j'(\frac{x+y}{2}) > \frac{F_j(\frac{x+y}{2}) - F_j(x)}{\frac{y-x}{2}}. \tag{25}$$

Note that the right-hand side (25) is the slope of the line through the two points $(x, F_j(x)), (\frac{x+y}{2}, F_j(\frac{x+y}{2}))$, while the left-hand side is the slope of F_j at $\frac{x+y}{2}$. The convexity of F_j is sufficient for (25) to hold. The proof is complete.

6.2 Bargaining power and delay costs

When there is a single offer (which is equivalent to a commitment equilibrium in the bargaining literature) in the resale mechanism, the convexity or regularity assumption insures that the bidders derive sufficient benefits from resale so that the general ranking is possible. If we allow repeated offers with no commitment, it is well-known (Sobel and Takahashi (1983), Fudenberg and Tirole (1983)) that high delay costs weaken the bargaining power of the monopolist. The weakened bargaining power may lead to low trade prices when the auction winner makes offers to the loser. We show by an example that the opposite ranking can occur when the bargaining power is substantially reduced in bargaining with repeated offers.

The resale market is one in which the winner of the auction makes a first offer to the buyer. If the offer is accepted, the game ends. If the first order is rejected, a second final offer is made to the buyer. The bargaining problem with repeated offers from one-side to the other with delay costs is similar to that of Sobel and Takahashi (1983). However, there is a main difference: the seller may have different non-zero costs (or valuations) and different valuations imply different beliefs about the buyer's valuations due to the outcome of the first-stage auction. The delay costs are expressed by discount factors δ_1, δ_2 for bidder one, two respectively. Our example assumes that bidder one has low δ_1 (close to 0), and bidder two has high δ_2 (close to 1). Note that when there are delay costs in repeated offers, the common-value is the first offer price and later offers are not involved in the equilibrium revenue.

Consider the weak-strong pair of bidders $v_1(t) = t, v_2(t) = 1.5t$ over $[0, 1]$. There are only two rounds of offers. For the example, we adopt the notations x, y for x_i, x_j respectively. We have $F_1(x) = x, F_2(y) = \frac{2}{3}y$. In equilibrium, bidder one with valuation x believes that bidder two valuation distribution is $F_2|_{1.5x}$, after she wins the auction. We let $y = 1.5x$. Given the first price offer p_1, bidder two has a threshold of acceptance z. The offer will be accepted if and only if bidder two's valuation is higher than z. When bidder two rejects the offer, the equilibrium period two offer is given by $p_2(x, z) = \frac{x+z}{2}$. The following equation determines the equilibrium z

$$z - p_1 = \delta_2(z - \frac{z+x}{2}),$$

and we have

$$z = \frac{p_1 - 0.5\delta_2 x}{1 - 0.5\delta_2}.$$

The optimal first offer p_1 maximizes the profit function

$$\frac{2}{3}(y-z)(p_1-x) + \frac{2}{3}\delta_1(z-p_2)(p_2-x) = \frac{2}{3}(y-z)(p_1-x) + \frac{2}{3}\frac{\delta_1}{4}(z-x)^2$$

$$= \frac{2}{3}(y - \frac{p_1 - 0.5\delta_2 x}{1-0.5\delta_2})(p_1 - x) + \frac{2}{3}\frac{\delta_1}{4}(\frac{p_1 - x}{1-0.5\delta_2})^2.$$

The first order condition for p_1 is

$$y - \frac{2p_1 - (1+0.5\delta_2)x}{1-0.5\delta_2} + \frac{\delta_1}{2(1-0.5\delta_2)^2}(p_1 - x) = 0,$$

and we get the optimal first period offer

$$p_1(x, y, \delta_1, \delta_2) = \frac{(1-0.5\delta_2)^2}{2-\delta_2-0.5\delta_1} y + \frac{1-0.5\delta_1-0.25\delta_2^2}{2-\delta_2-0.5\delta_1} x.$$

where $y = 1.5x$.

Since the first price auction revenue R^F with resale is increasing in p_1, and p_1 is increasing in δ_1, and decreasing in δ_2, we know that R^F is increasing in δ_1 and decreasing in δ_2. Therefore we know that a higher delay cost (or lower bargaining power) for bidder one hurts the revenue in the first price auction, while the opposite is true for bidder two. When $\delta_1 = 0, \delta_2 = 1$, we have the lowest revenue in the first price auction. In this case, we have $p(x, y) = \frac{1}{4}y + \frac{3}{4}x = 1.125x$, hence

$$R^F = \int_0^1 2(1-t)1.125t dt = 0.375,$$

which is lower than the revenue from the second price auction

$$\int_0^1 (1-x)(1-\frac{2}{3}x)dx = 0.388\,89.$$

We now compute the revenue without resale. By Plum (1992), the equilibrium bidding strategies of the two bidders are given by

$$b_w(t) = \frac{9}{5}\frac{1-\sqrt{1-\frac{5}{9}t^2}}{t}, b_s(t) = \frac{6}{5}\frac{\sqrt{1+\frac{5}{4}t^2}-1}{t}.$$

with the maximum bid $\frac{3}{5}$. The inverse bidding functions can be easily computed as

$$\varphi_w(b) = \frac{18b}{9+5b^2}, \varphi_s(b) = \frac{12b}{9-5b^2}, b \in [0, \frac{3}{5}]$$

From this, we can compute the revenue of the auction without resale:

$$R = \int_0^{\frac{3}{5}} bd(\frac{18b}{9+5b^2}\frac{12b}{9-5b^2}) = \frac{3}{5} - \int_0^{\frac{3}{5}} (\frac{18b}{9+5b^2}\frac{12b}{9-5b^2})db = 0.404\,62$$

which is higher than the revenue in the auction with resale.

6.3 Bargaining power in k-double auctions

The k-double auction with linear bidding strategies has been studied in Chatterjee and Samuelson (1983). In the bargaining after the auction, we have noted a difference from their model: the heterogeneous beliefs of the seller and the buyer. Because of this difference, the equilibrium strategy is different, but is still linear. In the standard bargaining with homogeneous beliefs, Williams (1987) has shown that such linear equilibrium satisfies efficient properties, and in experimental studies (Radner and Schotter (1989)), it seems to be the relevant one in practice. Williams (1987) also shows that the parameter k can be a proxy of the utility payoffs of the bargainer. Thus in this case, k can be a proxy of the bargaining power as well. As k takes the extreme values $0, 1$, we get the monopoly or monopsony market. Although other non-linear equilibria exist, the linear equilibrium case deserves special attention. With heterogeneous beliefs, it is desirable that the efficiency properties of Williams (1987) continue to hold, so that we can argue that the linear equilibrium is the one to study, as the bargainers will tend to adopt an efficient bargaining mechanism. We can not explore this question here, but will take it with a grain of faith that such equilibrium is important to study. It is also the only tractable one so far to analyze.

As another application of the bid equivalence, we consider the k-double resale market in which both the seller and the buyer make simultaneous offers. Let p_s,p_b be the seller and buyer offers respectively. The transaction takes place if and only of $p_s \leq p_b$ at the price $(1-k)p_s+kp_b$. When k increases, the bargaining power shifts from the seller to the buyer. This may appear to be counter-intuitive as a higher k means a higher price for the seller, and the seller benefits if there is no change in the offer strategies. Chatterjee and Samuelson ((1983), example one and two) have shown that the seller profit is actually decreasing in k as the offer strategies do change when k changes. This result is for the case of homogeneous beliefs. We will show that with heterogeneous beliefs, the conclusion is the same. Notice however that the beliefs of the players are not homogeneous, and hence the solution we get is different from theirs. With the uniform distribution of the valuations, there exist a unique (piecewise) linear equilibrium in our model as in Chatterjee and Samuelson (1983).

In this framework, we show that the auctioneer's revenue is an increasing function of the bargaining power $1-k$ of the weak bidder. When $k = 0, 1$ respectively, the equilibrium in the resale market is the same as the monopoly and monopsony market respectively. However, there is a trade-off between the revenue and efficiency. The efficiency, measured by the realized trade surplus of the auction with resale, is a decreasing function of the bargaining power $1-k$ of the weak bidder.

We first note the result of Lebrun (2007) that the auctioneer's revenue is higher in an auction with a monopoly resale market than with a monopsony resale market.

Assume that the signals are independent and the valuations are given by[18] $v_1(t) = t, v_2(t) = 2t$ so that $F_1(x) = x, F_2(x) = \frac{x}{2}$. The first stage is a first-price auction. We have the following result. Let $R^F(k)$ be the revenue of the first-price auction with the k-double auction resale market. Let the $E(k)$ be the efficiency of the auction measured by the realized surplus of the auction with resale. We also have exploit solution of the equilibrium bidding strategy of the auction with resale.

Proposition 12 *The revenue $R^F(k)$ is a decreasing function of k, while the efficiency $E(k)$ is increasing in k. The equilibrium bidding strategy is given by*

$$b_1(t) = b_2(t) = \frac{3}{4}t \text{ for } t \leq \frac{3-k}{4-k}, \tag{26}$$

$$= \frac{3-k}{2}\left[k + \frac{1-k}{2}t - \frac{k(3-k)}{2(4-k)t}\right] \text{ for } t \geq \frac{3-k}{4-k}.$$

Proof. In the resale game, let p_s, p_b be the offer price by the seller and buyer respectively. The transaction takes place if and only if $p_s \leq p_b$, and the transaction price is given by

$$p = (1-k)p_s + kp_b.$$

■

Let the inverse bidding strategy in the first-price auction with resale be ϕ_1, ϕ_2 and in equilibrium we have $\phi_2(b) = 2\phi_1(b)$ by the symmetry property. To find an equilibrium with linear strategies in the resale game, let $p_s(v_1) = c_1v_1 + d_1, p_b(v_2) = c_2v_2 + d_2$ be the equilibrium strategies as functions of valuations. Bidder one with valuation v_1 chooses $p \leq 2c_2v_1 + d_2$ to maximize

$$\int_{\frac{p-d_2}{c_2}}^{2v_1} \left[(1-k)p + k(c_2v_2 + d_2) - v_1\right] dv_2.$$

The derivative of the payoff with respect to p is given by

$$-\frac{p - v_1}{c_2} + (1-k)\int_{\frac{p-d_2}{c_2}}^{2v_1} dv_2$$

$$= \frac{1}{c_2}\left[-(2-k)p + (1 + 2(1-k)c_2)v_1 + (1-k)d_2\right],$$

which is decreasing in p. Therefore the payoff function is concave. The first-order condition of optimality gives us

$$p_s(v_1) = \frac{1 + 2(1-k)c_2}{2-k}v_1 + \frac{1-k}{2-k}d_2$$

[18] More general parameters can be allowed without affecting the results as long as the distributions are uniform.

For the bidder two with valuation v_2, the price offer $p \geq \frac{v_2}{2}c_1 + d_1$ maximizes

$$\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} [v_2 - (1-k)(c_1 v_1 + d_1) - kp]\, dv_1.$$

The first-order condition for the optimal offer is

$$\frac{v_2 - p}{c_1} - k \int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} dv_1 = 0$$

or

$$\frac{1}{c_1}((1 + \frac{kc_1}{2})v_2 - (1+k)p + kd_1) = 0,$$

and we have the optimal offer of the buyer

$$p_b(v_2) = \frac{1 + \frac{kc_1}{2}}{1+k} v_2 + \frac{k}{1+k} d_1.$$

To be an equilibrium, we must have

$$d_1 = \frac{1-k}{2-k} d_2, d_2 = \frac{k}{1+k} d_1,$$

$$c_1 = \frac{1 + 2(1-k)c_2}{2-k}, c_2 = \frac{1 + \frac{kc_1}{2}}{1+k}.$$

Solving the equations, we have

$$d_1 = d_2 = 0, c_1 = \frac{3-k}{2}, c_2 = \frac{4-k}{4}.$$

The (piecewise) linear equilibrium in the resale game is then given by

$$\begin{aligned} p_s(v_1) &= \frac{3-k}{2} v_1, v_1 \in [0,1], \\ p_b(v_2) &= \frac{4-k}{4} v_2 \text{ for } v_2 \leq \frac{6-2k}{4-k}, \\ &= \frac{3-k}{2} \text{ for } v_2 > \frac{6-2k}{4-k}. \end{aligned}$$

Note that when $k = 0$, the strong bidder bids the true valuation when $v_2 \leq \frac{6-2k}{4-k}$, and the transaction takes place if and only if $2v_1 \geq v_2 \geq p_s(v_1) = \frac{3}{2}v_1$. This is equivalent to the monopoly case in which the seller offers the optimal monopoly price and the buyer accepts if and only if the buyer valuation is above the price. Similarly, when $k = 1$, the weak bidder bids the true valuation, and the transaction takes place if and only if $\frac{1}{2}v_2 \leq v_1 \leq p_b(v_2) = \frac{3}{4}v_2$ when $v_2 \leq \frac{6-2k}{4-k}$. This is equivalent to the monopsony case in which the buyer offers the optimal monopsony price and the seller accepts if and only if the seller valuation is below the price.

The transaction price is given by

$$\begin{aligned} w(t_1,t_2) &= \frac{(1-k)(3-k)}{2}v_1(t_1)+\frac{k(4-k)}{4}v_2(t_2)=\frac{(1-k)(3-k)}{2}t_1+\frac{k(4-k)}{2}t_2 \text{ if } v_2(t_2)\le \frac{6-2k}{4-k}, \\ &= \frac{3-k}{2}((1-k)t_1+k) \text{ if } v_2(t_2)>\frac{6-2k}{4-k}. \end{aligned}$$

Here $Q=\{(t_1,t_2):t_1\ge t_2, \min(\frac{4-k}{4}v_2(t_2),\frac{3-k}{2})\ge \frac{3-k}{2}v_1(t_1)\}$, or $Q=\{(t_1,t_2): t_1\ge t_2\ge \frac{3-k}{4-k}t_1\}$. Trade occurs with probability one if and only if $(t_1,t_2)\in Q$, and there is no trade outside Q. This means that the object is in the hand of the strong bidder if and only if $v_2\ge \frac{6-2k}{4-k}v_1$. The slope $\frac{6-2k}{4-k}=2(1-\frac{1}{4-k})$ is decreasing in k. This implies that the efficiency of the auction increases as k becomes larger with the highest efficiency for the monopsony market.

We have

$$\begin{aligned} w(t,t) &= [(1-k)(3-k)+k(4-k)]\frac{t}{2}=\frac{3}{2}t \text{ for } t\le\frac{3-k}{4-k} \\ &= \frac{3-k}{2}((1-k)t+k) \text{ for } t\ge \frac{3-k}{4-k}. \end{aligned}$$

We want to show that $w(t,t)$ is decreasing in k when $t\ge\frac{3-k}{4-k}$. The derivative with respect to k is

$$\frac{3-k}{2}(1-t)-\frac{(1-k)t+k}{2}=\frac{3-2k}{2}-(2-k)t\le\frac{3-2k}{2}-\frac{(2-k)(3-k)}{4-k}$$

$$=\frac{-k}{2(4-k)}<0,$$

hence $p(t,t)$ is decreasing on the range $t\ge\frac{3-k}{4-k}$. Since the revenue of the first-price auction with resale is

$$2\int_0^1(1-t)p(t,t)dt,$$

the revenue is a decreasing function of k.

We have the equilibrium bidding strategies:

$$b_1(t_1)=\frac{1}{t_1}\int_0^{t_1}w(t,t)dt=\frac{1}{t_1}\int_0^{t_1}\frac{3}{2}tdt=\frac{3}{4}t_1, \text{ for } t_1\le\frac{3-k}{4-k}.$$

For $t_1\ge\frac{3-k}{4-k}$, we have

$$\begin{aligned} b_1(t_1) &= \frac{3(3-k)^2}{4(4-k)^2t_1}+\frac{1}{t_1}\int_{\frac{3-k}{4-k}}^{t_1}\frac{3-k}{2}((1-k)t+k)dt \\ &= \frac{3(3-k)^2}{4(4-k)^2t_1}+\frac{3-k}{2t_1}\left[\frac{1-k}{2}t_1^2+kt_1-\frac{1-k}{2}(\frac{3-k}{4-k})^2-\frac{k(3-k)}{4-k}\right] \end{aligned}$$

$$= \frac{3-k}{2}\left[k + \frac{1-k}{2}t_1 - \frac{k(3-k)}{2(4-k)t_1}\right].$$

Next we show that Theorem 10 can be applied to this k-double auction resale market.

Theorem 10 can be applied to the k-double auction model. For instance, in the $\frac{1}{2}$-double auction example, for $F_1(x_1) = F_2(x_2), x_2 \leq 1$, we have

$$p(x_1, x_2) = \frac{5}{8}x_1 + \frac{5}{16}x_2 = \frac{5}{4}x_1 = \frac{p(x_1, x_1) + p(x_2, x_2)}{2},$$

and condition (C') holds for this example. In fact, it holds for all k-double auctions for this example.

In a k-double auction, if the winning bid is announced, the valuation of the winner is known in the resale stage. As a result, the equilibrium in the resale game is the same as the equilibrium in the monopoly market. This implies that the auctioneer revenue is higher when the winning bid is announced. Similarly, if the losing bid is announced, then the revenue is lower compared to the k-double auction.

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Appendix:

Proof of Proposition 8:

Let the number of periods remaining be k, and denote the optimal offer by p_k. The updated belief of the highest valuation z_k of the buyer is the threshold of acceptance in the period before. By backward induction, p_k depends only on x, z_k, and we use the notation $p_k(x, z_k)$. Let $\pi_k(x, z_k)$ be the expected profit function when k periods are remaining. Again by backward induction, z_k depends only on x and z_{k+1}. Given p_k, p_{k-1}, bidder two has a threshold level of acceptance z_{k-1}. Bidder two will accept the offer p_k whenever his or her valuation is above z_{k-1}. Given p_k, p_{k-1}, we can determine z_{k-1} by the condition

$$z_{k-1} - p_k = \delta_2(z_{k-1} - p_{k-1})$$

Thus we have the equation

$$(1 - \delta_2)z_{k-1} + \delta_2 p_{k-1} = p_k \tag{27}$$

If the offer p_k is rejected, the bidder i updates his belief of the valuation of bidder j, and the new highest (lowest) valuation of the buyer (seller) is now z_{k-1}. Let $p_{k-1}(x_i, z_{k-1})$ be the optimal offer with $k-1$ periods remaining with the updated z_{k-1}. We can rewrite (27) as

$$(1-\delta_2)z_{k-1} + \delta_2 p_{k-1}(x, z_{k-1}) = p_k \tag{28}$$

If the optimal offer p_{k-1} with $k-1$ periods remaining has been determined by backward induction and is increasing in z_{k-1}. The left-hand side of (28) is increasing in z_{k-1}, and there is a unique solution denoted by $z_{k-1}(x_i, p_{k-1})$. Thus we know how z_{k-1} is determined once p_k is chosen.

The choice of p_k is determined by the maximization of the profit function of the seller given by

$$[F_2(z_k) - F_2(z_{k-1}(x, p_k))]\,(p_k - x) + \delta_1 \pi_{k-1}(x, z_{k-1}) \tag{29}$$

The first order condition for p_k is

$$F_2(z_k) - F_2(z_{k-1}) - f_2(z_{k-1})(p_k - x)\frac{\partial z_{k-1}}{\partial p_k} + \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}} \frac{\partial z_{k-1}}{\partial p_k} = 0.$$

Take the implicit derivative of (27) with respect to p_k, we have

$$(1-\delta_2)\frac{\partial z_{k-1}}{\partial p_k} + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}} \frac{\partial z_{k-1}}{\partial p_k} = 1,$$

or

$$\frac{\partial z_{k-1}}{\partial p_k} = \frac{1}{(1-\delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}}. \tag{30}$$

Substitute (30) into the first order condition, we have

$$F_2(z_k) - F_2(z_{k-1}) - \frac{f_2(z_{k-1})(p_k - x) - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1-\delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0.$$

For uniform distributions, we have $f_2 = 1$. Hence we have the first order condition

$$z_k - z_{k-1} - \frac{p_k - x - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1-\delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0 \tag{31}$$

When $k = 1$, we have

$$p_1(x, y) = \frac{x+y}{2}, \pi_1(x, y) = (\frac{y-x}{4})^2$$

and $p_1(x, z_1) = \frac{x+z_1}{2}, \pi_1(x, z_1) = (\frac{z_1 - x}{2})^2$. Hence

$$\frac{\partial p_1}{\partial z_1} = \frac{1}{2}, \frac{\partial \pi_1}{\partial z_1} = \frac{z_1 - x}{2}.$$

The theorem holds for $k = 1$ with $c_1 = \frac{1}{2}$. More generally, by mathematical induction, assume that the theorem holds for $k - 1$, and we have

$$p_{k-1} = c_{k-1}z_{k-1} + (1 - c_{k-1})x, \pi_{k-1} = 0.5c_{k-1}(z_{k-1} - x)^2$$

$$\frac{\partial p_{k-1}}{\partial z_{k-1}} = c_{k-1}, \frac{\partial \pi_{k-1}}{\partial z_{k-1}} = c_{k-1}(z_{k-1} - x).$$

The first order condition (31) for z_{k-1}, p_k is

$$y - z_{k-1} = \frac{(1 - \delta_2)z_{k-1} + \delta_2(c_{k-1}z_{k-1} + (1 - c_{k-1})x) - x - \delta_1 c_{k-1}(z_{k-1} - x)}{1 - \delta_2 + \delta_2 c_{k-1}}$$

or

$$y - z_{k-1} = \frac{(1 - \delta_2 + \delta_2 c_{k-1} - \delta_1 c_{k-1})(z_{k-1} - x)}{1 - \delta_2 + \delta_2 c_{k-1}}$$

or

$$\frac{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}}{1 - \delta_2 + \delta_2 c_{k-1}} z_{k-1} = y + \frac{1 - \delta_2 + \delta_2 c_{k-1} - \delta_1 c_{k-1}}{1 - \delta_2 + \delta_2 c_{k-1}} x$$

and we have

$$z_{k-1} = \frac{1 - \delta_2 + \delta_2 c_{k-1}}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}} y + \frac{1 - \delta_2 + \delta_2 c_{k-1} - \delta_1 c_{k-1}}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}} x.$$

Let

$$d_{k-1} = \frac{1 - \delta_2 + \delta_2 c_{k-1}}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}},$$

then

$$z_{k-1} = d_{k-1}y + (1 - d_{k-1})x.$$

We have

$$p_k = (1 - \delta_2)z_{k-1} + \delta_2 p_{k-1} = (1 - \delta_2)z_{k-1} + \delta_2(c_{k-1}z_{k-1} + (1 - c_{k-1})x)$$
$$= (1 - \delta_2 + \delta_2 c_{k-1})z_{k-1} + \delta_2(1 - c_{k-1})x = c_k y + (1 - c_k)x$$

where

$$c_k = \frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}} = (1 - \delta_2 + \delta_2 c_{k-1})d_{k-1}.$$

The expected profit can be written as

$$\pi_k = (y - z_{k-1})(p_k - x) + \delta_1 \pi_{k-1}$$
$$= c_k(1 - d_{k-1})(y - x)^2 + 0.5\delta_1 c_{k-1}(z_{k-1} - x)^2$$
$$= (y - x)^2(c_k - c_k d_{k-1} + 0.5\delta_1 c_{k-1} d_{k-1}^2)$$
$$= (y - x)^2(c_k - (1 - \delta_2 + \delta_2 c_{k-1})d_{k-1}^2 + 0.5\delta_1 c_{k-1} d_{k-1}^2)$$
$$= (y - x)^2(c_k - 0.5d_{k-1}^2(2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}))$$
$$= (y - x)^2(c_k - 0.5\frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}})$$
$$= (y - x)^2(c_k - 0.5c_k) = 0.5c_k(y - x)^2.$$

By mathematical induction, the proof is complete.