Common Belief Foundations of Global Games

Stephen Morris Hyun Song Shin

October 2007

Global Games

- Coordination games have multiple equilibria, but...

- Carlsson and van Damme (1993) relaxed common knowledge through small "noise" to get unique, dominance solvable outcome in 2×2 games.

  – Coined "global games"

- Subsequent applications to currency crises, bank runs, etc.

Recent Criticisms/Questions

- Global game uniqueness arguments turn on relative precisions of noisy private and public signals. Endogenous public information (e.g. prices) serve as coordination device, restores multiplicity (Atkeson 2001, Angeletos and Werning 2006, Hellwig, Mukherji and Tsyvinski 2006)

- If we don't know what these "noisy" signals are in real life, debates about relative precisions have no conceptual basis (e.g., Sims 2005)

- What about other ways of relaxing common knowledge assumptions (Weinstein and Yildiz 2007))? What are the higher order beliefs that correspond to global game deviations from common knowledge?

Outline of Talk

- Characterize higher-order beliefs that underpin play in global games.
  - Belief operator on type space resembling p-belief operators
  - Rationalizability equivalent to common belief
- Re-examine argument for uniqueness
  - Separate features of noisy signal information structure that are important for uniqueness from those that are merely incidental
- Two sufficient conditions for uniqueness (without talk of noisy signals)
  - Common certainty of rank beliefs for undominated types
  - Common certainty of beliefs about differences for undominated types

Example

Combine features of Rubinstein's (1989) e-mail game and Carlsson and van Damme's (1993) global game.

Finite number I of players

Binary choice from $\{$invest, not invest$\}$

Cost of investing, $p \in (0, 1)$, gross payoff to success in investing is 1

Fundamental state $\theta \in \Theta$ (countable), with prior μ

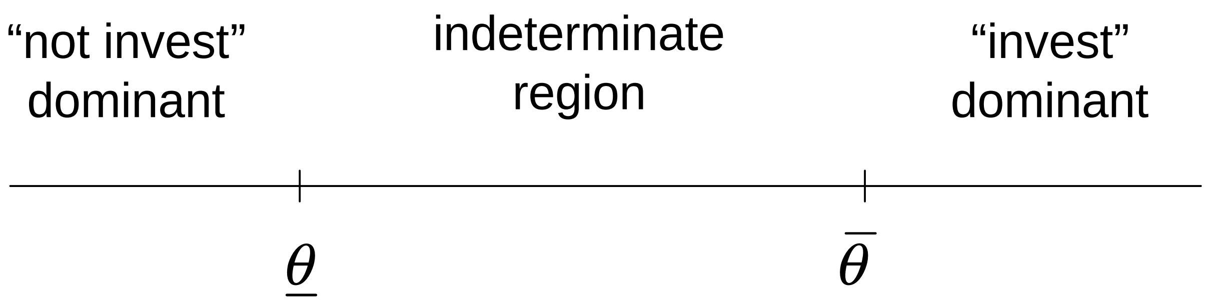


Figure 1: Tripartite Partition of Θ

Critical mass q for successful investment in middle region

$\theta < \underline{\theta}$	at least q invest	less than q invest
invest	$-p$	$-p$
not invest	0	0

$\underline{\theta} \leq \theta \leq \bar{\theta}$	at least q invest	less than q invest
invest	$1-p$	$-p$
not invest	0	0

$\theta > \bar{\theta}$	at least q invest	less than q invest
invest	$1-p$	$1-p$
not invest	0	0

Revealed Beliefs

Stance of an outside observer.

Player i is seen to invest.

What does this action reveal about his beliefs?

Revealed Beliefs

Stance of an outside observer.

Player i is seen to invest.

What does this action reveal about his beliefs?

Either player i has a dominant action to invest

Revealed Beliefs

Or player i p-believes

1. $\theta \geq \underline{\theta}$

Revealed Beliefs

Or player i p-believes

1. $\theta \geq \underline{\theta}$

2. proportion q or more either have a dominant action to invest or p-believe that $\theta \geq \underline{\theta}$

Revealed Beliefs

Or player i p-believes

1. $\theta \geq \underline{\theta}$

2. proportion q or more either have a dominant action to invest or p-believe that $\theta \geq \underline{\theta}$

3. proportion q or more either have a dominant action to invest or p-believe that [proportion q or more either have a dominant action to invest or p-believe that $\theta \geq \underline{\theta}$]

Revealed Beliefs

Or player i p-believes

1. $\theta \geq \underline{\theta}$

2. proportion q or more either have a dominant action to invest or p-believe that $\theta \geq \underline{\theta}$

3. proportion q or more either have a dominant action to invest or p-believe that [proportion q or more either have a dominant action to invest or p-believe that $\theta \geq \underline{\theta}$]

4. and so on...

Information Structure

$I = 2n + 1$ players.

Conditional on θ, signal realization of i in $[\theta - n, \theta + n]$ is uniform

Every realization in $[\theta - n, \theta + n]$ received by precisely one player

E.g. Conditional on θ, Nature selects highest signal $\theta + n$ with uniform density, then choose next highest with uniform density among remaining players, etc.

- Players ranked ex post

- Equal chance of being ranked anywhere between first to last conditional on θ

Posterior beliefs

$$\frac{\mu\left(\theta \mid s_i\right)}{\mu\left(\theta' \mid s_i\right)} = \frac{\mu\left(\theta\right)}{\mu\left(\theta'\right)}$$

Beliefs over rank

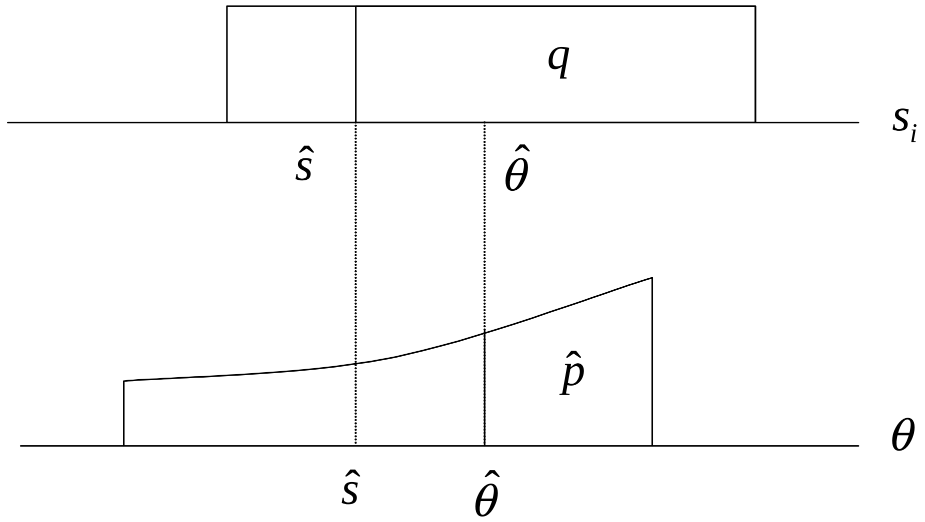
$$\rho_k\left(s_i\right) = \mathsf{Prob}\left(\#\left\{j | s_j < s_i\right\} = k - 1 | s_i\right)$$

Rank beliefs

$$\rho\left(s_i\right) \equiv \left(\rho_1\left(s_i\right), \rho_2\left(s_i\right), \cdots, \rho_I\left(s_i\right)\right)$$

Evident Events

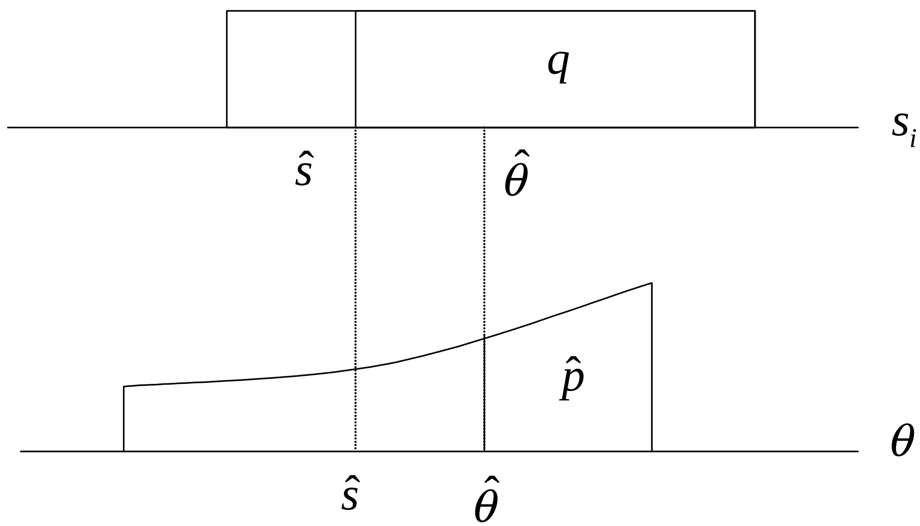
Fix $\hat{\theta}$. Define $\hat{s}$ and $\hat{p}$.



1. When $\theta \geq \hat{\theta}$, proportion q or more players receive signal $\hat{s}$ or higher.

2. When $s_i \geq \hat{s}$, player i $\hat{p}$-believes that $\theta \geq \hat{\theta}$.

Evident Events



When $\theta \geq \hat{\theta}$, proportion q or more players $\hat{p}$-believes that $\theta \geq \hat{\theta}$.

$\Rightarrow \left\{\theta | \theta \geq \hat{\theta}\right\}$ is $(q, \hat{p})$-*evident* (Monderer and Samet (1989))

Claim. "Invest" is rationalizable for i if and only if i p-believes some (q,p)-evident subset of $\{\theta|\theta \geq \underline{\theta}\}$. "Not invest" is rationalizable for i if and only if i $(1-p)$-believes some $(1-q,1-p)$-evident subset of $\left\{\theta|\theta \leq \bar{\theta}\right\}$.

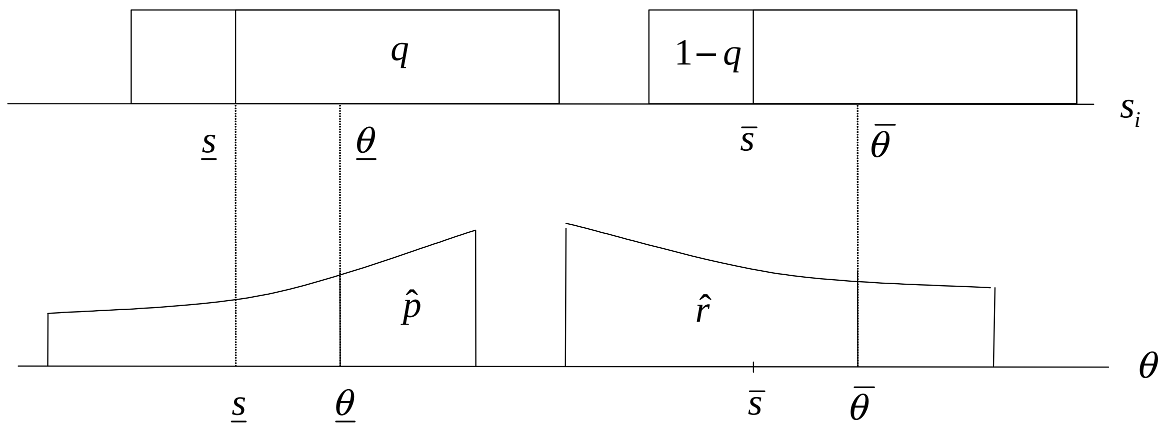
Claim. "Invest" is rationalizable for i if and only if i p-believes some (q,p)-evident subset of $\{\theta|\theta \geq \underline{\theta}\}$. "Not invest" is rationalizable for i if and only if i $(1-p)$-believes some $(1-q,1-p)$-evident subset of $\left\{\theta|\theta \leq \bar{\theta}\right\}$.

Invest rationalizable (for non-dominant type) if and only if i p-believes

1. $\theta \geq \underline{\theta}$
2. proportion q or more p-believe that $\theta \geq \underline{\theta}$
3. proportion q or more p-believe that [proportion q or more p-believe that $\theta \geq \underline{\theta}$]
4. $\cdots$

"Either-Or" clause is redundant, and Θ is countable $\Rightarrow$ existence of evident event.

Case of Multiple Rationalizable Actions



Case when $\hat{p} \geq p$ and $\hat{r} \geq 1 - p$

- $\{\theta | \theta \geq \underline{\theta}\}$ is (q, p)-evident: "Invest" rationalizable
- $\left\{\theta | \theta \leq \bar{\theta}\right\}$ is $(1 - q, 1 - p)$-evident: "Not Invest" rationalizable

Monotone Rank Beliefs

$\rho\left(s_i'\right) \trianglerighteq \rho\left(s_i\right)$ when $\rho\left(s_i'\right)$ weakly dominates $\rho\left(s_i\right)$ in the sense of first degree stochastic dominance.

Rank beliefs are *weakly increasing* when $s_i' \geq s_i$ implies $\rho\left(s_i'\right) \trianglerighteq \rho\left(s_i\right)$.

Rank beliefs are *weakly decreasing* when $s_i' \geq s_i$ implies $\rho\left(s_i'\right) \trianglelefteq \rho\left(s_i\right)$.

Monotone Rank Beliefs

$\rho(s_i') \trianglerighteq \rho(s_i)$ when $\rho(s_i')$ weakly dominates $\rho(s_i)$ in the sense of first degree stochastic dominance.

Rank beliefs are *weakly increasing* when $s_i' \geq s_i$ implies $\rho(s_i') \trianglerighteq \rho(s_i)$.

Rank beliefs are *weakly decreasing* when $s_i' \geq s_i$ implies $\rho(s_i') \trianglelefteq \rho(s_i)$.

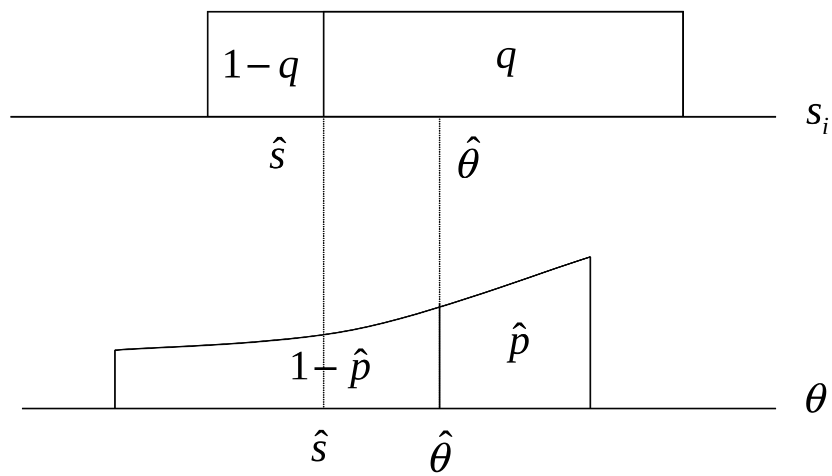
Exam. I only know my own score in an exam. Am I ranked high or low? How typical am I?

Voting. My political views change after major national event. How much is this just me, and how much a change in the "national mood" as a whole?

Speculative Attack. Central bank of target country has raised interest rates. How typical are my losses?

Uniqueness

Claim. If rank beliefs are weakly decreasing in signals in $\{s_i \mid \underline{s} \leq s_i \leq \bar{s}\}$, then there is a unique rationalizable outcome in the investment game, except possibly at one value of θ.



- $\left\{\theta|\theta \geq \hat{\theta}\right\}$ is (q, p)-evident iff $\left\{\theta|\theta < \hat{\theta}\right\}$ is not $(1-q, 1-p)$-evident (neglecting atoms)

- Define $\hat{p}(\theta) =$ largest h such that $\left\{\theta'|\theta' \geq \theta\right\}$ is (q, h)-evident.

- $\hat{p}(\theta)$ is increasing when rank beliefs are decreasing.

Uniqueness

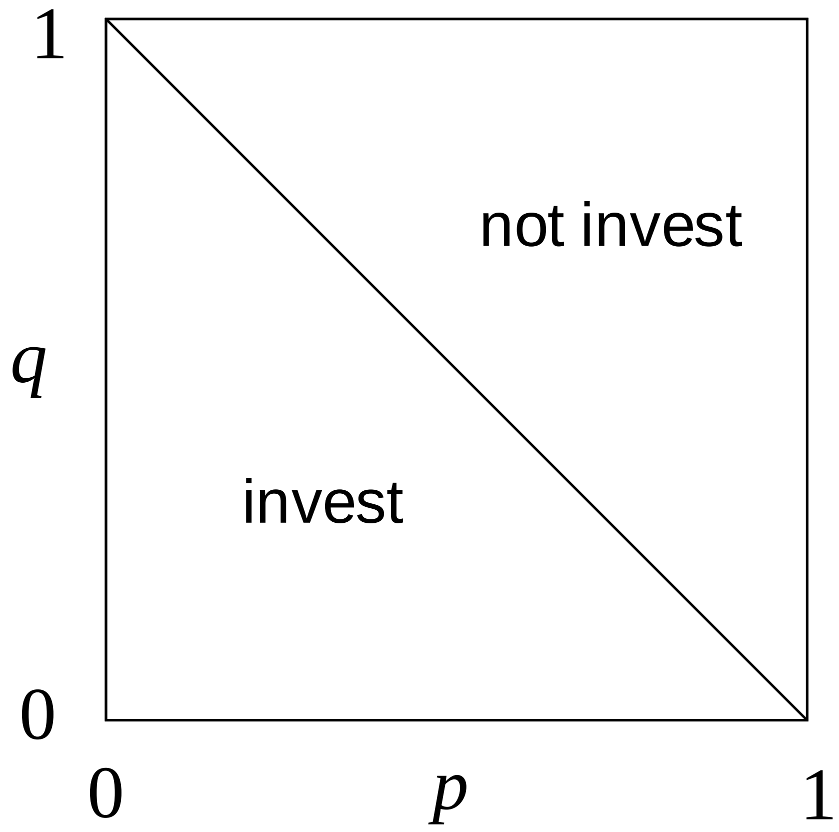
Corollary. If $\rho(\cdot)$ is a constant function over $\{s_i \mid \underline{s} \leq s_i \leq \bar{s}\}$, then there is a unique rationalizable outcome in the investment game.

(Cf. Izmalkov and Yildiz (2006))

Corollary. If $\rho(s_i) = \left(\frac{1}{I}, \frac{1}{I}, \cdots, \frac{1}{I}\right)$ over $\{s_i \mid \underline{s} \leq s_i \leq \bar{s}\}$, then, "invest" is the unique rationalizable action in the first-order undominated region when $p + q < 1$. "Not invest" is the unique rationalizable action in the first-order undominated region when $p + q > 1$.

If ρ uniform, $\hat{p} = 1 - q$. "Invest" is rationalizable when $\hat{p} > p$. That is, when $p + q < 1$. "Not Invest" is rationalizable when $1 - \hat{p} > 1 - p$. That is, when $p + q > 1$.

Uniqueness with Constant, Uniform Rank Beliefs



Gaussian Information Structure

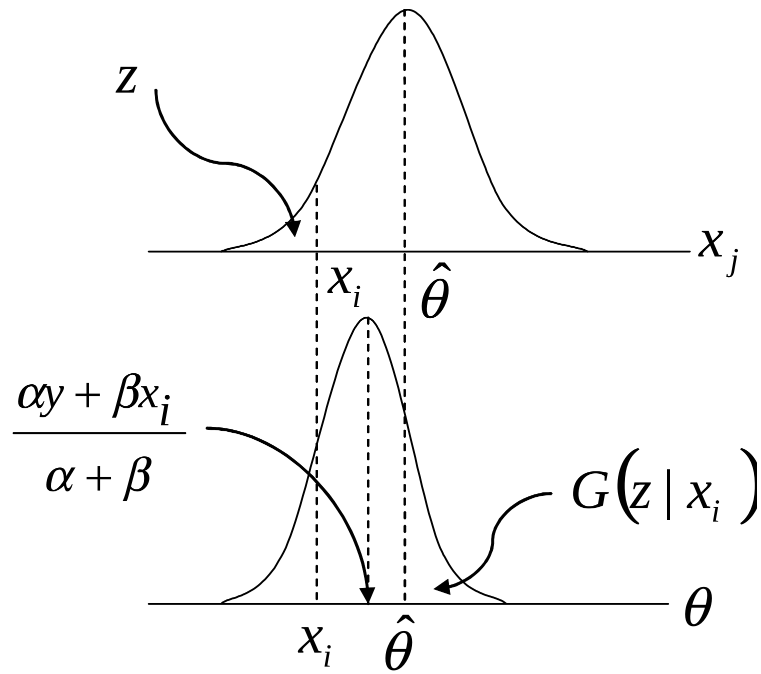
Noisy signal:

$$x_i = \theta + \varepsilon_i$$

$\theta \sim N(y, 1/\alpha)$, $\varepsilon_i \sim N(0, 1/\beta)$, mutual independence, and with θ

$\lambda(x)$ is proportion of players whose signal is x or less.

$G(z|x_i) \equiv \Pr(\lambda(x_i) \leq z|x_i)$, c.d.f. of $\lambda(x_i)$ conditional on x_i



$\hat{\theta}$ solves $\Phi\left(\sqrt{\beta}\,(x_i - \theta)\right) = z$. Then

$$\lambda(x_i) \le z \text{ whenever } \theta \ge \hat{\theta}.$$

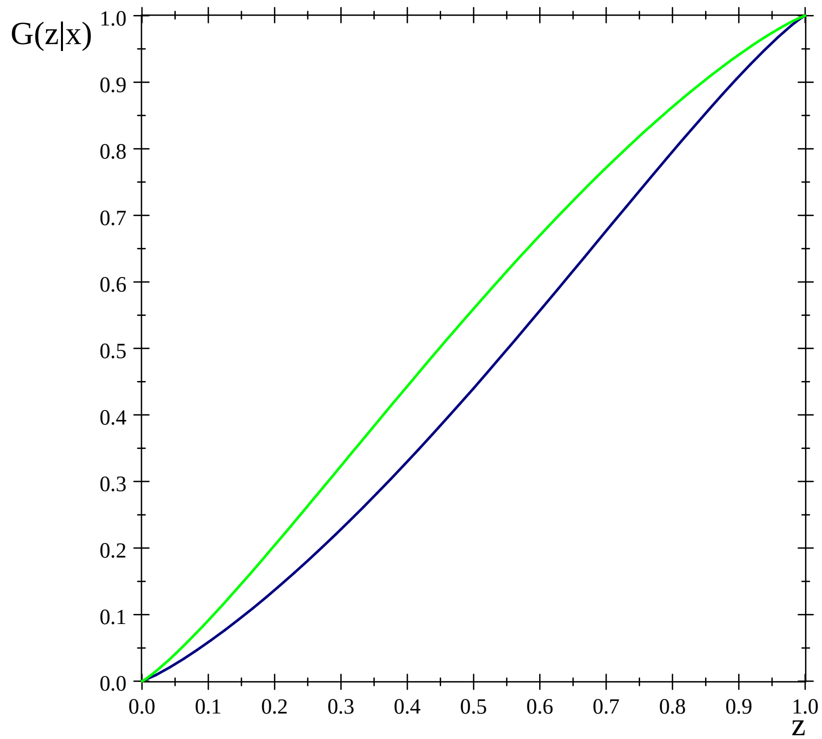
$$
\begin{aligned}
G\left(z|x_i\right) &= \Pr\left(\left\{\theta|\theta \geq \hat{\theta}\right\} \mid x_i\right) \\
&= 1 - \Phi\left(\sqrt{\alpha+\beta}\left(\hat{\theta} - \tfrac{\alpha y + \beta x_i}{\alpha+\beta}\right)\right) \\
&= \Phi\left(\tfrac{\alpha}{\sqrt{\alpha+\beta}}\left(y - x_i\right) + \sqrt{\tfrac{\alpha+\beta}{\beta}}\Phi^{-1}\left(z\right)\right)
\end{aligned}
$$

$x_i \leq x_i'$ implies $G\left(z|x_i\right) \trianglelefteq G\left(z|x_i'\right)$

Gaussian information structure builds in increasing rank beliefs

Increasing Rank Beliefs

Example: $\alpha = 1, \beta = 3$



$y = 0.5, x = 0.2$ (green), $x = 0.8$ (blue)

Limiting Case

Limiting case $\beta \to \infty$

$$G\left(z|x\right) \to \Phi\left(\Phi^{-1}\left(z\right)\right) = z$$

Density over $\lambda\left(x_i\right)$ is uniform. Player i believes he is "typical" in strong sense (puts equal weight on every realization of $\lambda\left(x_i\right)$).

Framework

1. Background

   - Players $\mathcal{I} = \{1, ..., I\}$
   - Finite "payoff states" θ

Framework

1. Background
   - Players $\mathcal{I} = \{1, ..., I\}$
   - Finite "payoff states" θ
2. Type Space $\mathcal{T} = (T_i, \pi_i)_{i=1}^{I}$
   - i's types: T_i
   - i's belief: $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$

Framework

1. Background

2. Type Space $\mathcal{T} = (T_i, \pi_i)_{i=1}^{I}$

3. Binary Action Game with Strategic Complementarities $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^{I}$

   - i chooses $a_i \in \{0, 1\}$
   - $\lambda_i(Z, \theta)$ is payoff gain to action 1 over 0 in state θ if Z is the set of opponents choosing 1, i.e.

$$u_i(1, a_{-i}, \theta) - u_i(0, a_{-i}, \theta) = \lambda_i(\{j \neq i | a_j = 1\}, \theta)$$

   - $\lambda_i : 2^{\mathcal{I}/\{i\}} \times \Theta \rightarrow \mathbb{R}$, increasing in Z

Key Question

What joint restriction on higher order beliefs ($\mathcal{T}$) and payoffs ($\boldsymbol{\lambda}$) gives unique rationalizable outcomes?

Generalized Belief Operators

- Product event
  - $F = \underset{i=1,I}{\times} F_i$
  - each $F_i \subseteq T_i$
  - F is an event in $T = \underset{i=1,I}{\times} T_i$

- Two interpretations of product events
  - Event in type space T
  - Strategy profile

- Product events closed under $\cap, \vee, \neg$ where

$$\begin{aligned} E \vee F &\equiv \times_{i=1}^{I} (E_i \cup F_i) && \textit{(join)} \\ \neg \times_{i=1}^{I} F_i &\equiv \times_{i=1}^{I} \backsim F_i && \textit{(negation)} \end{aligned}$$

Then

$$\neg\neg F = F, \quad \neg\emptyset = T, \quad \neg T = \emptyset$$
$$\neg (E \vee F) = \neg E \cap \neg F$$

- Generalized Belief Operator

$$Z_{F,i}(t_1, \cdots, t_I) = \{j \in \mathcal{I} \mid j \neq i \text{ and } t_j \in F_j\}$$

Define the operator $B_i^{\lambda_i}(\cdot)$ on product events as

$$B_i^{\lambda_i}(F) = \{t_i \in F_i \,| E_{t_i}(\lambda_i(Z_{F,i}, \theta)) \geq 0\}$$
$$B^{\boldsymbol{\lambda}}(F) = \times_{i=1}^{I} B_i^{\lambda_i}(F)$$

$t_i \in B_i^{\lambda_i}(F)$ reveals that type t_i puts high weight on $t_j \in F_j$ for many $j \neq i$

monotonicity: $F \subseteq F' \Rightarrow B_i^{\lambda_i}(F) \subseteq B_i^{\lambda_i}(F')$

DEFINITION: There is $common$ $\boldsymbol{\lambda}$-$belief$ of F at t if

$$t \in C^{\boldsymbol{\lambda}}(F) \equiv \bigcap_{k \geq 1} \left[B^{\boldsymbol{\lambda}}\right]^k (F).$$

DEFINITION: Event F is $\boldsymbol{\lambda}$-$evident$ if

$$F \subseteq B^{\boldsymbol{\lambda}}(F).$$

PROPOSITION (cf, Aumann 1976, Monderer and Samet 1989): Event F is common $\boldsymbol{\lambda}$-belief at t ($t \in C^{\boldsymbol{\lambda}}(F)$) if and only if there exists a $\boldsymbol{\lambda}$-evident event F' such that $t \in F' \subseteq F$.

PROPOSITION: Action 1 is rationalizable for type t_i if and only if $t_i \in B_i^{\lambda_i}\left(C^{\boldsymbol{\lambda}}(T)\right)$.

Proof. Define dual operator

$$\begin{aligned} S^{\lambda}(F) &\equiv \neg B^{\lambda}(\neg F) \\ &= \times_{i=1}^{I} \backsim B_i^{\lambda_i}\left(\times_{i=1}^{I} \backsim F_i\right) \end{aligned}$$

$S^{\lambda}(F)$ is set of type profiles who strictly prefer to play action 0 when action zero is played on F.

$\left(S^{\lambda}\right)^{k+1}(\emptyset)$ is the set of type profiles who strictly prefer action 0 when faced with types who do not use kth order dominated actions.

Action 1 is rationalizable for t_i if only if

$$\begin{aligned} t_i &\in B_i^{\lambda_i}\left(\neg \bigvee_{k\geq 1}\left(S^{\lambda}\right)^k(\emptyset)\right) \\ &= B_i^{\lambda_i}\left(\bigcap_{k\geq 1}\neg\left(S^{\lambda}\right)^k(\emptyset)\right) \\ &= B_i^{\lambda_i}\left(\bigcap_{k\geq 1}\left(B^{\lambda}\right)^k(T)\right) \\ &= B_i^{\lambda_i}\left(C^{\lambda}(T)\right) \end{aligned}$$

Inverse operator:

$$\widetilde{\lambda}_i(Z,\theta) = -\lambda_i(\mathcal{I}/Z,\theta)$$

PROPOSITION: Action 0 is rationalizable for type t_i if and only if $t_i \in B_i^{\widetilde{\lambda}_i}\left(C^{\widetilde{\lambda}}(T)\right)$.

Example. Linear Regime Change Game

There is a cost of investing: $c \in (0,1)$. The return to investing is 1 if proportion investing is at least $1-\theta$, 0 otherwise

$$\lambda_i\left(Z,\theta\right)=\begin{cases} 1-c, \text{ if } \frac{\#Z}{I-1}\geq 1-\theta \\ -c, \text{ otherwise}\end{cases}$$

Morris and Shin (1998, 2004), Metz (2001, 2003), Dasgupta (2006), Hellwig (2002), Heinemann, Nagel and Ockfels (2005), Rochet and Vives (2004), Angeletos, Hellwig and Pavan (2006, 2007), Angeletos and Werning (2006), Hellwig, Mukherji and Tsyvinski (2006) ...

Action 1 if rationalizable for player i if only if

1. Player i c-belives $\theta \geq 0$ i.e., $\Pr_i(\theta \geq 0) \geq c$

2. Player i c-believes that [the proportion who c-believe $\theta \geq 0$ is at least $1 - \theta$]
$$\Pr_i\left(\frac{\#\{j\,|\Pr_j(\theta \geq 0) \geq c\}}{I-1} \geq 1 - \theta\right) \geq c$$

3. Player i c-believes that [the proportion who c-believe that [the proportion who c-believe that $\theta \geq 0$ is at least $1 - \theta$] is at least $1 - \theta$]

4. and so on....

Uniqueness: Common Certainty of Rank Beliefs

- Separable symmetric payoffs

$$\lambda_i (Z, \theta) = g (\# Z) + h (\theta)$$

- Define complete order on the union of all types across players:

$$t_i \succeq t_j \Leftrightarrow \sum_{t_{-i}, \theta} \pi_i (t_i) [t_{-i}, \theta] \, h (\theta) \geq \sum_{t_{-j}, \theta} \pi_j (t_j) [t_{-j}, \theta] \, h (\theta)$$

- Limit dominance

$$
\begin{aligned}
g(0) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) &> 0 \text{ for some } t_i \\
g(I-1) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) &< 0 \text{ for some } t_i
\end{aligned}
$$

- Let $\rho_i : T_i \to \Delta(\{1, ..., I\})$ be an agent's belief about his rank.

$$
\begin{aligned}
\rho_i(t_i)[k] &= \sum_{t_{-i},\theta} \pi_i(t_i)[\{(t_{-i},\theta)\,|\#\{j \neq i | t_j \succ t_i\}\} = k-1] \\
\rho_i(t_i) &\equiv (\rho_i(t_i)[1], \rho_i(t_i)[2], \cdots, \rho_i(t_i)[I])
\end{aligned}
$$

- Common certainty of rank beliefs for undominated types: $\rho_i(.)$ is a constant function for all undominated types and all players.

“Technical” Assumptions

- Type profile t has no rank ties if $t_i \nsim t_j$ for all $i \neq j$.

- Type profile t has no payoff ties if for all i

$$\sum_{n=0}^{I-1} \rho(n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)\, [t_{-i},\theta]\, h(\theta) \neq 0$$

- Uniform separation: there exists $\varepsilon > 0$ such that

$$t_i \nsim t_j \Rightarrow \left| \sum_{t_{-i},\theta} \pi_i(t_i)\, [t_{-i},\theta]\, h(\theta) - \sum_{t_{-j},\theta} \pi_j(t_j)\, [t_{-j},\theta]\, h(\theta) \right| \geq \varepsilon$$

Sufficient Condition for Uniquness

Proposition. Under the auxiliary assumptions, constant rank beliefs implies dominance solvability.

Paraphrase: *Common certainty of common rank beliefs (for strategic types) implies dominance solvability*

Sufficient Condition for Uniquness

Proposition. Under auxiliary assumptions, constant rank beliefs implies dominance solvability.

If r^* is the common rank belief then action 1 is the unique rationalizable action for type t_i of player i if

$$\sum_{n=0}^{I-1} r^*(n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) > 0;$$

and action 0 is the unique rationalizable action of type t_i of player i if

$$\sum_{n=0}^{I-1} r^*(n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) < 0.$$

Idea behind proof.

Limit dominance implies there is $\bar{t}_j$ such that

$$c = g(0) + \sum_{t_{-j},\theta} \pi_j\left(\bar{t}_j\right)[t_{-j},\theta]\, h(\theta) > 0.$$

First step in induction

$$\begin{aligned}\left\{t_i \in T_i \,\middle|\, t_i \succeq \bar{t}_j\right\} &= \left\{t_i \,\middle|\, g(0) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\, h(\theta) \geq c\right\} \\ &\subseteq S_i^{\lambda_i}(\varnothing)\end{aligned}$$

For each i and $k = 0, 1, ...$

$$\left\{ t_i \;\middle|\; \begin{array}{c} \sum_{n=0}^{I-1} r^* (n+1)\, g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i}, \theta]\, h(\theta) \\ \geq c + \sum_{n=0}^{I-1} r^* (n+1)\, g(n) - g(0) - \varepsilon^* k > 0 \end{array} \right\} \tag{1}$$

$$\subseteq \quad S_i^{\lambda_i} \left[S^{\boldsymbol{\lambda}} \right]^k (\varnothing)$$

$$\left\{ t_i \in T_i \left| \sum_{n=0}^{I-1} r^* (n+1) g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta] h(\theta) > 0 \right. \right\}$$

$$\subseteq \quad \cup_{k\geq 1} S_i^{\lambda_i} \left[S^{\boldsymbol{\lambda}} \right]^k (\varnothing)$$

Similarly, for "Not Invest"

$$\left\{ t_i \in T_i \left| \sum_{n=0}^{I-1} r^* (n+1) g(n) + \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta] h(\theta) < 0 \right. \right\}$$

$$\subseteq \quad \cup_{k\geq 1} S_i^{\widetilde{\lambda}_i} \left[S^{\widetilde{\boldsymbol{\lambda}}} \right]^k (\varnothing)$$

Generalizations

- Decreasing rank beliefs
- Near-constant rank beliefs

Uniqueness: Common Certainty of Beliefs about Differences

- Separability: there exist increasing $\lambda_i^1 : \mathcal{I}/\{i\} \to \mathbb{R}$ and $\lambda_i^2 : \Theta \to \mathbb{R}$ such that
$$\lambda_i (Z, \theta) = \lambda_i^1 (Z) + \lambda_i^2 (\theta)$$

- Each player's type can be decomposed into two components. The first component is completely ordered and we identify it with the set of integers $\mathcal{Z}$. The second component any finite set Ψ_i. Thus, for each i, we have a bijection $g_i : T_i \to \mathcal{Z} \times \Psi_i$.

- Uniform Monotonicity: Expectation of $\lambda_i^2(\theta)$ is uniformly increasing in the first component: there exists $\varepsilon > 0$ such that

$$\begin{aligned} g_{i1}(t_i) &> g_{i1}(t_i') \\ &\Rightarrow \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\,\lambda_i^2(\theta) > \sum_{t_{-i},\theta} \pi_i(t_i)[t_{-i},\theta]\,\lambda_i^2(\theta) + \varepsilon \end{aligned}$$

for all i, t_i, t_i'.

- Limit Dominance: For each i, there exist $\underline{t}_i$ and $\bar{t}_i$ such that

$$\begin{aligned} \lambda_i^1(\mathcal{I}/\{i\}) + \sum_{t_{-i},\theta} \pi_i(\underline{t}_i)[t_{-i},\theta]\,\lambda_i^2(\theta) &< 0 \\ \text{and } \lambda_i^1(\varnothing) + \sum_{t_{-i},\theta} \pi_i(\bar{t}_i)[t_{-i},\theta]\,\lambda_i^2(\theta) &> 0. \end{aligned}$$

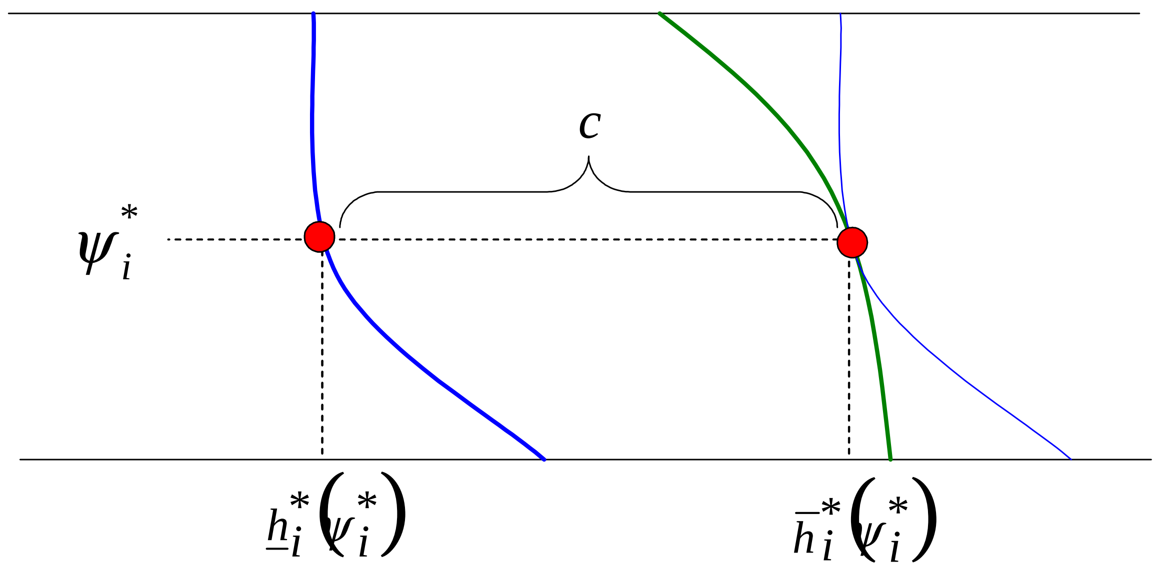
- η-Diffuse Beliefs: There exists $\eta > 0$ such that, for each i and, for each $j \neq i$, $h_j : \Psi_j \to \mathcal{Z}$,

$$\sum_{\left\{t_{-i}: g_{j1}\left(t_j\right) = h_j\left(g_{j2}\left(t_j\right)\right) \text{ for some } j\right\}, \theta} \pi_i\left(t_i\right)\left[t_{-i}, \theta\right] < \eta$$

- Beliefs about Differences: Define player i's beliefs about differences $\xi_i : T_i \to \Delta\left(\left(\mathcal{Z} \times \ominus_j\right)_{j \neq i}\right)$ as follows:

$$\xi_i\left(t_i\right)\left[\left(\left(\delta_j, \psi_j\right)_{j \neq i}, \theta\right)\right] = \pi_i\left(t_i\right)\left[\left\{\left(g_j^{-1}\left(g_{i1}\left(t_i\right) + \delta_j, \psi_j\right)\right)_{j \neq i}\right\} \times \Theta\right].$$

PROPOSITION 2. Assume uniform monotonicity and limit dominance. Then there exists $\overline{\eta} > 0$ such that, if $\eta \leq \overline{\eta}$ and there are η-diffuse beliefs, then common knowledge of beliefs in differences implies a unique rationalizable outcome.



Conclusions

- Characterized higher-order beliefs that underpin play in global games.
  - There is departure from common knowledge, but the departure has to be a special kind that preserves high degree of "common belief"
- Re-examined argument for uniqueness
  - Separated features of noisy signal information structure that are important for uniqueness from those that are merely incidental
- Two sufficient conditions for uniqueness (without talk of noisy signals)
  - Common certainty of rank beliefs for undominated types (distills examples from the applied literature)
  - Common certainty of beliefs about differences for undominated types (potentially new - applications to multi-dimensional global games)