# Lexicographic Preference and Arrow's General Impossibility Theorem*

Susumu Cato†

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**Abstract**

The purpose of this paper is to investigate Arrow's theorem under the domain with lexicographic preference, and obtain the possibility result.

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**Keywords** Lexicographic Ordering; Arrow's impossibility theorem; Domain Restriction

## 1 Introduction

Resolution of Arrow's general impossibility theorem is one of the most important topic in the social choice theory. One direction of this is restricting the preference domain, and many papers appeared. Traditional studies of domain restriction is single-peaked preference. On this issue, see Sen (1970). Kalai, Muller and Satterswaite (1979) discuss sufficient condition for a preference domain to give impossibility results. Gaertner (2002) gives the summary of approachs and results of domain restriction.

In this paper, we show that the exsistance of person who have lexicographic preference implies the exsistace of social welfare function that satisfies Pareto condition, Independence of Irrelevant Alternatives and Non-Dictaorship. Recentry, Suzumura and Xu (2004) consider the extended framework and define "non-consequentialist", and show the domain with non-consequentialst is Arrow consistent. In thier paper, preference of non-consequentialst is lexicographic order, and play the essential role to escape impossibility. Our claim is that with lexicographic preference, we can obtain the possibility result without the extended framework.

Arrovian framework with economic environment have lately attracted considerable attention. In this approach, it is often assumed that preference satisfies continuity, monotonicity and convexity. Unfortunatry, interenting economic domain with such regular indivudual preference is Arrow inconsistent.[1] Our results implies that in economic envrironment, if individual's preference is either such regular preference or lexicographic preference, a preference domain is Arrow consistent if and only if there exists a person who have lexicographic preference. Note that lexicographic preference is not continuous and can not be represented by utility function. Therefore, our results is meanful for social choice with economic environment.

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†mailto:ksusumu@hotmail.co.jp, Graduate school of Economics, University of Tokyo
[1]See Le Breton and Weymark (1996).

## 2 Notation and Definition

For the lexicographic order, the set of social states $X$ is a Catesian set of alternatives; $X = \prod_{k \in K} X_k$ with $|X_k| \geq 2$. Thus, a social alternative $x \in X$ is a vecter $(x_k)_{k \in K}$ where $x_k \in X_k$. $N = \{1, 2, \ldots, n\}$ with $n \geq 2$ be the finite set of individuals on the society. Let $\wp$ be the set of all logically possible ordering over $X$. $R \in \wp$ stand for a social preference relation on $X$. Symmmetric and Assymmetric part of $R$ are $I$ and $P$, respectively. Each individual $i \in N$ have the preference ordering $R_i$ over the set of social alternatives $X$. Then a profile $\mathbf{R} = (R_1, R_2, \cdots, R_n)$ is a $n-$tuple of individual preference, and is an element of $\wp^n$. $\mathcal{D} \subset \wp^n$ is the admssible preference domain. A social welfare function(SWF) with $\mathcal{D}$ is a function $f$ which maps each and every profile in some subset $\mathcal{D}$ of $\wp^n$ into $\wp$.

Now, we define the $k$-lexicographic ordering. For $i$th person, $\succeq_i^k$ is complete order on $X_k$ and $\succeq_i^{h \neq k}$ is complete order on $\prod_{h \neq k} X_h$.

**Definition 1.** $i$th person have the $k$-lexicographic order $R_i^{Lk}$ is the binary relation on $X$ defined by $xR_i^{Lk}y$ iff $x_k \succ_i^k y_k \Rightarrow xR_i^{Lk}y$ and $x_k \sim_i^k y_k \Rightarrow [x_{h \neq k} \sim_i^{h \neq k} y_{h \neq k} \Leftrightarrow xR_i^{Lk}y]$.

**Definition 2.** $\mathcal{D}_L \subset \wp^n$ is the domain with $k$-lexicographic preference if there exists at least one person who have the $k$-lexicographic ordering on $X$.

We next introduce three axioms on the social welfare function $f$.

**Axiom 1.** Strong Pareto Priciple(SP)
For all $x, y \in X$, and for all $(R_i)_{i \in N} \in D_f$, if $xP_iy$ holds for all $i \in N$, then we have $xPy$, and if $xI_iy$ holds for all $i \in N$, then we have $xIy$

**Axiom 2.** Independence of Irrelevant Alternatives(IIA)
For all $\mathbf{R} = (R_1, R_2, \cdots, R_n)$, $\mathbf{R}' = (R_1', R_2', \cdots, R_n') \in D_f$, and for all $x, y \in X$, if $[xR_iy \Leftrightarrow xR_i'y]$ for all $i \in N$, then $[xRy \Leftrightarrow xR'y]$ holds, where $R = f(\mathbf{R})$ and $R' = f(\mathbf{R}')$.

**Axiom 3.** Non-Dictaorship(ND)
There exists no $i \in N$ such that $xP_iy \Rightarrow xPy$ for all $x, y \in X$.

## 3 Results

**Theorem 1.** *There exists an social welfare fanction $f$ with the domain $D_L$ which satisfies (SP), (IIA) and (ND).*

*Proof.* In this proof, to show the existence of SWF, we construct $f$ first, and show $f$ satisfies three conditions.

**Step 1.** By assumption, there exists $l \in N$ whose preference is $k$-lexicographic order. Consider the following SWF: For all $x, y \in X$,

$$x_1 \succ_l^k y_1 \Rightarrow xPy, \tag{1}$$

$$x_1 \sim_l^k y_1 \Rightarrow [xRy \Leftrightarrow xR_iy, i \in N \setminus l]. \tag{2}$$

**Step 2.** By constriction, this SWF satisfies (SP), (IIA), and (ND). $R$ by this SWF is clealy reflexive and complete. We have only to show that $R$ is transitive, $\forall x, y, z \in X, (x \succeq y \wedge y \succeq z) \Rightarrow x \succeq z$. This SWF impose that $x \succeq y \Leftrightarrow$ (a)$x_k \sim_l^k y_k \wedge xR_iy$ or (b)$x \succ_l^k y$. Then, to check transitivity, we must investigate four cases.
(i)case of $(x_k \sim_l^k y_k \wedge xR_iy)$ and $(y_k \sim_l^k z_k \wedge yR_iz)$.
$(x_k \sim_l^k y_k \wedge y_k \sim_l^k z_k)$ and $(xR_iy \wedge yR_iz) \Rightarrow x_k \sim_l^k z_k$ and $xR_iz$. This imply $xRz$.

(ii)case of $(x_k \sim_l^k y_k \wedge xR_iy)$ and $y_k \succ_l^k z_k$.
$(x_k \sim_l^k y_k \wedge y \succ_l^k z) \Rightarrow x_k \succ_l^k z_k$. This imply $xRz$.
(iii)case of $x \succ_l^k y$ and $(y_k \sim_l^k z_k \wedge yR_iz)$.
$(x_k \sim_l^k y_k \wedge y \succ_l^k z) \Rightarrow x_k \succ_l^k z_k$. This imply $xRz$.
(iv)case of $x \succ_l^k y$ and $y \succ_l^k z$.
$(x_k \succ_l^k y_k \wedge y \succ_l^k z) \Rightarrow x_k \succ_l^k z_k$. This imply $xRz$.

Therefore, $R$ is transitive. Q.E.D.

**Remark 1.** This domain $D_L$ is not common and not satuating. Concepts of "common" and "satiating" are defined by Kalai, Muller and Satterswaite (1979).

**Remark 2.** Regular preference condition for public goods;
We assume that $X$ is any connecting subset of $\mathbb{R}^m$. The alternative is interepleted as vecters of public goods. Suppose that every individual's preference is either continuous, convex and strictry monotonic, or lexicographic. There exists an social welfare fanction $f$ satisfing (SP), (IIA) and (ND) if and only if there exsit at least one person who have $k$-lexicographic preference.

Proof is straightforward. By Theorem 2 in Kalai Muller and Satterswaite (1979), if all individual's preference are continuous, convex and strictry monotonic, then the preference domain is common and satuating and there exist no SWF satisfing (SP), (IIA) and (ND). Therefore, the existence of the person who have lexicographic order is necessary and sufficient condition of the existence of SWF satisfing (SP), (IIA) and (ND).

**Remark 3.** Example of economic environment with private goods;
Cosider an environment with 2 induvuduals and 2 private goods; $N = \{1, 2\}$ and $X = \prod_{k\in\{1,\dots 4\}} X_k = \mathbb{R}^4$ with $x_1 + x_3 = 1$ and $x_2 + x_4 = 1$. Then, in this example, social alternative is allocation of private goods.

In this case, we assume $(x_1, x_2) \in X_1 \times X_2$ is 1's consumption bundle, and $(x_3, x_4) \in X_3 \times X_4$ is 2's one. $x_1 \in X_1$ and $x_2 \in X_2$ is 1's consumption level of goods 1 and 2, respctively. Similary, $x_3 \in X_3$ and $x_4 \in X_4$ is 2's consumption level of goods 1 and 2, respctively. Suppose both individual's preference is monotonic and selfish for thier consimption bundle. By theorem, if 1st person have 1-lexicographic preference, then there exist SWF satisfing (SP), (IIA) and (ND). By using the social welfare function in the proof of theorem, $(1, 0, 0, 1)$ is best element, all of goods 1 is consumed by 1st person and all of goods 2 is consumed by 2nd person.

# References


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