A Continuous-Time Analysis of Optimal Defaultable Debt Contracts: Theory and Applications

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This paper presents a new approach for modeling continuous-time defaultable debt contracts. It studies an optimal competitive debt contract in continuous time by exploring a dynamic costly monitoring model under asymmetric information in a common-agency setting. Consequently, it shows that, under an optimal debt contract, a fully informed debtor defaults strategically and recurrently. On the other hand, a less informed creditor expects default to occur stochastically based on an exponential probability distribution under which the arrival rate of default is increasing in monitoring ability. This paper provides a mathematically tractable framework to analyze firms' financial structure and dynamic auditing problems in labor and insurance contracts.

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1. INTRODUCTION

Debt contracts are a key element in any modern economy. When entrepreneurs hold informational and control advantages over their creditors, debt claims are a device to mitigate the entrepreneurs' misbehavior under a loan contract. One crucial characteristic of debt contracts is that a debtor provides a creditor with the right to foreclose on the debtor's assets when he misses principal or interest payments under the borrowing agreements (i.e., in a default state). In many cases, a defaulting debtor repudiates the debt when being foreclosed. However, occasionally, a debtor can reconstruct the contract in a default state and continue to maintain the relationship with the same creditor. This default is not a terminal event, but rather is a restructuring. When the debt structuring is a possible default event under a loan contract, default is strategically undertaken in equilibrium, in some cases recurrently.[1]

In practice, debt restructuring is important in actual economies. Sovereign defaults are a typical example. Assets held in a defaulting country are usually difficult to seize across its border. In addition, a sovereign debt is not covered directly by a bankruptcy code.[2] Accordingly, Western governments and commercial investors often hold negotiations with defaulting sovereigns to partially exempt their liability and restructure the debts without a termination (e.g., the Paris and London Club). Also, in corporate finance, most defaulting firms first try to restructure their debt either in or out of court.[3] In particular, most large companies reorganize under Chapter 11 (court-supervised debt restructuring), rather than liquidate under Chapter 7 (termination of contract), when filing for a bankruptcy in the United States. Hence, the impact of the debt restructuring on actual economies is huge.

Despite such practical importance of debt restructuring, economists know surprisingly little about it. In asset pricing literature, continuous-time defaultable bond models have studied the dynamics of this type of default in a reduced form, in which a default time arrives based on an exogenously given intensity probability distribution.[4] These reduced-form models succeed in capturing empirically several features of debt restructuring from actual financial data. However, due to lack of a game-theoretic treatment of the default, they fail to detect strategic default incentives of debtors. In fact, in the empirical work, there appears to be some instability of default intensity parameters as borrowers' credit qualities change. Moreover, financial contract literature often treats default as a terminal event, not as a restructuring

[1] In academic literature, default is sometimes used, in a narrow sense, only for a termination of a loan contract. This paper regards debt restructuring as a default event.

[2] See Bulow and Rogoff [6, 7], Eaton and Gersovitz [11].

[3] See Gilson [16], Gilson et al. [17]

[4] Theoretically, see, for example, Duffie and Singleton [10], Jarrow and Turnbull [23]. As for empirical applications, see Duffee [8], Duffie, Pedersen, and Singleton [9].

one.[5] Due to such an oversimplified default structure, this literature is unsuccessful in empirical application. In short, there is a gap between contract theory on the one hand, and asset pricing models and actual financial data on the other.

The purpose of this paper is to present a new approach for modeling an optimal continuous-time defaultable debt contract by bridging this gap. More precisely, it formalizes an optimal competitive design of a continuous-time communication game in environments with Markov income shocks and asymmetric information. Consequently, it shows that, under an optimal debt contract, a fully informed debtor defaults strategically and recurrently. On the other hand, a less informed creditor expects default to occur stochastically based on an exponential probability distribution under which the arrival rate of default is increasing in monitoring ability. This model provides strategic insights into reduced-form defaultable bond models. Due to mathematical tractability of the continuous-time, competitive structure, this paper provides a better framework to study defaultable debts in actual firms' complex financial structure.

This paper presents a model that is a dynamic extension of a classical finite-period costly monitoring model. In particular, the model uses a component game that is similar to the finite-period costly state verification (CSV, hereafter) model presented by Townsend [34]. Specifically, a borrower's project produces single non-storable goods by using one unit of capital that a lender invests. The income process from the project is uncertain and its realization is privately observable to the borrower. A deterministic state verification (or disclosure) technology is available and costly.[6] Income is allocated at each grid according to the terms in the contract.

Previous finite-period CSV models have been successful in capturing a role of default as a device to achieve a direct revelation principle. This paper uses a similar approach. The lender has an incentive to exempt the borrower from his payment liability and continues the contract after restructuring only when the borrower verifies his bad shape via a costly disclosure. This equilibrium costly exemption that results from restructuring is a specific definition of default in this paper.

This model extends Townsend's model mainly in three points. First, the income process is Markov in infinite horizon $\{0_-\} \cup [0, \infty)$.[7] The Markovian technological

[5] For example, Anderson and Sundaresan [2], Hart and Moore [21, 22]. A model presented by Bulow and Rogoff [6] is an exception. They study a repeated recontracting of sovereign debts in an infinite horizon model in which a credit country can impose direct, finite sanctions on a defaulting debtor by costing him his ability to transact freely in the financial and goods markets. A main difference of their model from mine is that their paper focuses on ex post recontracting behavior in a given complete, competitive world debt market, not on any optimal long-term defaultable debt contract. Their model does not study why such restructuring is permitted in a long-term contract from an ex ante point of view.

[6] Deterministic disclosure means that if a player demands a disclosure, then the disclosure will be undertaken with probability one. Note that, contrary to Townsend [34]'s model, the borrower has the right to demand the disclosure by incurring the costs. This twist simplifies the outcome function form in the contract in that a less informed lender designs a contract ex ante whereas the fully informed borrower undertakes all the ex post actions.

[7] Algebraically, $t_- := \lim_{u \uparrow t} u$ for $u < t$ and $t \geq 0$, i.e., t's left-limit time.

shocks are more realistic than either a one-shot shock or i.i.d. shocks.[8] Second, this model has a common agency structure like the one presented by Peters [29]. There are three infinitely-lived risk-averse players: one borrower and two identical lenders. I elaborate equilibria in which each of the lenders maximizes his ex ante lifetime utility by designing a contract non-cooperatively subject to the borrower's optimization with respect to ex post state verification actions under a contract. This paper restricts attention to equilibria in stationary Markovian strategies. In addition, it focuses only on symmetry of the contracts that the two lenders design in equilibrium. This construction has competitive implications regarding the optimal payout policy. Finally, this model has a continuous-time structure, which makes the complex, dynamic Bayesian game tractable to achieve complete characterizations of the equilibrium. In particular, this framework makes it possible to use a Markov operator method, which is useful to provide observable implications of potentially rich families of Markov processes.[9] However, as Fudenberg and Tirole [14] show, a continuous-time game has no natural notion of extensive-form stages in general. My model is not an exception.[10] To remedy this problem, this paper specifies an appropriate topological, not algebraic, notion of left-limit time and formalizes the dynamic game as a continuous limit of discrete-time games with fine grids $\{t_-\} \cup [t, t+\triangle)$ for time t.[11]

In addition to those departures from Townsend's model, a solution method that this paper uses is also new. Precisely, I solve for the optimal contract via an impulse control method of Bensoussan and Lions [3]. Contrary to continuous control problems, the state of the system is subject to jumps (i.e., "impulses") in an impulse control problem. The timing, number, jump size, and intensity of impulses are decision variables in the control problem. The borrower's default decision in this dynamic CSV model is a typical impulse control. Based on this method, I show that, under several assumptions, there is a stationary equilibrium in which defaults occur recurrently. In particular, as a part of the solution method, I characterize the borrower's optimization program via a stochastic maximization principle of Bismut [5] given some boundary conditions on a finite horizon $[0, \tau]$ where τ denotes the first

[8]In the previous infinite-horizon CSV models, Wang [35] presumes i.i.d. income shocks. Because of the lack of intertemporal links across stages, the equilibrium disclosure strategy is static in that only a current shock triggers a disclosure that is history independent. In Nakamura [26], Wang's model is extended to have Markov income shocks. However, for mathematical convenience, Nakamura [26] restricts the shock process to follow a two-state Markov chain. In contrast, this paper generalizes the process to have a continuum of states in continuous time.

[9]For example, Aït-Sahalia, Hansen, and Scheinkman [1], Hansen and Scheinkman [19].

[10]Most recently, several versions of continuous-time contracting models have been intensively studied. (e.g., Biais et al. [4], Sannikov [32], Williams [36]) Most models study an instantaneously repeated game in which information does not flow strategically across infinitesimal stages. Accordingly, they do not have the difficulty mentioned by Fudenberg and Tirole. In this paper, by contrast, the strategic disclosure action influences the lender's information set. Hence, this model needs some appropriate notion of stages in an infinitesimal extensive-form component game.

[11]The preceding draft of this paper, namely Nakamura [27], includes a mathematical appendix showing that $\{t_-\} \cup [t, t+\triangle)$ is a relevant grid of an infinitesimal component game at time t. This draft is available on request from the author.

default time after instant 0. Using the results of Williams [36], two Hamiltonian adjoint processes (i.e., differentials of the Hamiltonian) associated with the income and payment variables stand for the borrower's endogenous reservation utility and the shadow price of the hidden state. These adjoint processes encode history dependence for the borrower's interim individual rationality and truth revealing actions.

Such mathematical tractability expands the practical applicability of this optimal contract model to analyze actual firms' complex financial structure and dynamic auditing problems in labor and insurance contracts beyond standard costly monitoring models. This paper provides two applications: (1) hidden entrepreneurial efforts and (2) human capital investment in disclosure technology. The first application results in a moral-hazard premium for the entrepreneur's potential laziness in an optimal payout policy. The second one shows that if the human capital of the disclosure ability is endogenously accumulated, then a countercyclical payout policy is optimal. These results seem consistent with empirical observations.

This paper is organized as follows. The next two sections define the physical and institutional environments. Section 4 defines the strategies and the equilibrium notion. Section 5 characterizes the optimal contract. Section 6 provides two practical applications. The final section concludes.

2. ENVIRONMENT IN CONTINUOUS TIME

Consider an economy with single non-storable consumption goods under uncertainty in infinite-horizon continuous time $\mathbb{T} = \{t | t \in \{0_-\} \cup [0, \infty)\}$.[12] Note that contrary to the control-theoretic convention, $\{0_-\} \neq \{0\}$.

2.1. Uncertainty and dated commodity space

The stochastic basis is a filtered space $R = (R, \mathcal{R}, \pi)$: a compact real-valued function space $R = \{\varphi : \mathbb{T} \mapsto \mathbb{R}\}$ with $L_2(\pi)$-norm, a non-trivial right-continuous Borel set $\mathcal{R}$ of subsets of R, and a uniquely common Lebesgue measure π. The space R consists of a Cartesian product of two disjoint probability spaces: a finite initial state space Θ with N elements (a finite integer $N \geq 2$) and a driving state space $\Omega = (\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{\mathbb{T}}$ with $\lim_{t\to\infty} \mathcal{F}_{t_-} = \mathcal{F}_{\infty_-} = \mathcal{F}_\infty = \mathcal{F}$. The initial space Θ governs the set of possible initial states at 0_-; the driving state space Ω governs the evolution of the economy over time. Then, a Cartesian product filtration $\mathcal{R} = \{\mathcal{R}_t\}_{\mathbb{T}}$ with $\lim_{t\to\infty} \mathcal{R}_{t_-} = \mathcal{R}_{\infty_-} = \mathcal{R}_\infty = \mathcal{R}$ is well-defined.

Further, there is a two-dimensional standard Brownian motion $\omega = \{\omega(t)\}_{\mathbb{T}}$ where $\omega(t) = \begin{bmatrix} W_s(t) & W_0(t) \end{bmatrix}^\top$,[13] the elements of which are independent of each other and

[12] Nakamura [27] includes a mathematical appendix showing longer, technical descriptions.

[13] The superscript $\top$ represents a transpose of the matrix.

form independent driving state space $(\Omega^s, \mathcal{F}^s, P^s)$, $(\Omega^0, \mathcal{F}^0, P^0)$ for $(\Omega, \mathcal{F}, P)$. The sample paths of ω specify all the distinguishable events after 0_-. Therefore, a complete filtration generated by ω is equivalent to $\mathcal{F}$. Note that, as defined below, W_0 drives a sequence of income shocks while W_s drives a sequence of payment randomization shocks. Also, the initial payment level is deterministic according to the contract. Thus, there is no payment uncertainty at 0_-.

Next, the commodity space, denoted by Φ, is a convex set of semimartingales ϕ such that $\phi \in \Phi$ is L^2-reducible and that its cumulative return process $\mathcal{L}n_t(\phi)$ is special semimartingale for all $\phi \in \Phi$.[14] Let $\mathbb{L}$ denote the set of the cumulative return process in Φ. Let $\mathbb{M}$ denote the set of the local martingale parts n in the unique canonical decomposition in $\mathbb{L}$. For a filtration $\mathcal{R}$, there exists a density process $M(t) := E\left[\frac{d\tilde{\pi}}{d\pi} \mid \mathcal{R}_{t_-}\right]$ such that for each semimartingale $\phi \in \Phi$ and for each $t \geq 0$, $M(t)\phi(t)$ is a martingale under an equivalent measure $\tilde{\pi}$. Let $\mathbb{M}_\pi$ denote the space of m such that $M(t) = \mathcal{E}_t[m]$ and that $E\left[\sup_{t\in\mathbb{T}} |m(t)|\right] < \infty$. Define a bilinear form $(m, n) : \mathbb{M}_\pi \times \mathbb{M} \mapsto \mathbb{R} := E\left[\langle m, n\rangle_\infty\right]$.[15] $\mathbb{L}$ is a Hilbert lattice under the inner product $\langle l_1 \mid l_2\rangle := E\left[\int_0^\infty l_1 l_2 dt\right]$ for any $l_1, l_2 \in \mathbb{L}$.

2.2. Players

The economy is populated with three infinitely-lived players: one borrower and two identical lenders, indexed by $i = 1, 2, 2'$. Each of them consumes the consumption goods at each instant: $\{\gamma_{si}(t)\}_{t\in\mathbb{T}} \in \Phi$ for $i = 1, 2, 2'$. Player i has a time-separable utility of consumption characterized by an instantaneous utility function $f_i : \Phi^{\mathbb{T}} \to \mathbb{R}$ and a common instantaneous discount rate δ. In particular, f_i is of a CRRA type with the coefficient of relative risk aversion $0 < \psi_i < 1$ with $\psi_2 = \psi_{2'}$. Given $\{\gamma_{si}(t)\}_{t\in\mathbb{T}}$, player i's ex ante lifetime utility level is $E\left[\int_{\mathbb{T}} e^{-\delta t} \frac{\gamma_{si}(t)^{1-\psi_i}}{1-\psi_i} dt\right]$. Also, player 1's ex ante autarky utility level $U_0 > 0$; player $2, 2'$'s are zero $V_0 = 0$.

2.3. Technology

Each player $2, 2'$ has one unit of indivisible physical input (or capital). Player 1 has a project that could produce a positive, predictable income process of the goods, denoted by $X \in \Phi$, when either player 2 or $2'$ (not both) transfers one unit of his capital to player 1. The capital transfer (or investment) may take place only at the outset of the whole game, i.e., before 0_-. If player 1 has no input, then the economy has no income forever.

[14] For each t, $\mathcal{L}n_t(\phi)$ has a unique canonical decomposition into a local martingale part and a predictable part with finite variation where $\mathcal{L}n$ denotes a stochastic logarithm operator $\mathcal{L}n = \mathcal{E}^{-1}$ and $\mathcal{E}$ denotes a (modified) Doléans-Dade exponential. A semimartingale with bounded jumps is a special semimartingale (Protter ([30], Theorem 18-20, p. 107).

[15] E denotes an expectation operator.

Suppose that player 2's capital is delivered to player 1. Then, player 1's project starts its production $\{X(t)\}_{\mathbb{T}}$. The initial state space Θ completely characterizes the initial income $X(0_{-})$: $\Theta = \left(\Theta, \tilde{\Theta}, \eta\right)$ where $\Theta = \{\theta_1, \theta_2, ..., \theta_N\}$ is a finite set with $0 < \theta_1 < \theta_2 < ... < \theta_N < +\infty$. $\hat{\theta} := \sum_{\Theta} \theta\eta(\theta)$. θ is randomly drawn from Θ by nature before 0_{-} and is revealed accurately to player 1 ex ante. Call $\theta \in \Theta$ as the type of the player. After 0_{-}, the income process is driven by one-dimensional standard P^0-Wiener process W_0:

$$dX(t) = \mu_0 X(t)dt + \sigma_0 X(t)dW_0(t)$$

where $\mu_0, \sigma_0 \in \mathbb{R}_+$ are constant. The pair (μ_0, σ_0) is public information. The realization of income is privately observable to player 1.

A state verification (or disclosure) technology reveals player 1's current income level to player 2 (or 2′) with perfect accuracy. The technology is available to player 1. A disclosure process $d \in \mathbb{D} = \{0,1\}_{t \in \mathbb{T}}$ is predictable. A point of time t is said to be a disclosure time if the disclosure occurs at t_{-} (i.e., $d(t_{-}) = 1$). When the disclosure is undertaken, it requires resources from player 1 at the disclosure time and causes the process of player 1's state variable to decrease permanently relative to what it would otherwise be. The resource loss is deadweight loss. Call the loss as disclosure costs. Let $\hat{X}(t_{-})$ denote the disclosed income level at the disclosure time t.[16] Let $S(t)$ denote the amount of the ex post observable payment to player 2 (or 2′) at t.[17] The disclosure costs are represented by $\lambda \begin{bmatrix} S(\tau_{-}) & \hat{X}(\tau_{-}) \end{bmatrix}^{\top}$ with a non-negative two-dimensional row vector $\lambda = \begin{bmatrix} \lambda_s & \lambda_x \end{bmatrix}$. The elements are constant and public information. By using the costly disclosure technology, player 1 could control his own income process downward impulsively. Therefore, X is an impulsively controlled process.

3. INSTITUTIONAL STRUCTURE: CONTRACT AND DEFAULT

This section describes the institutional structure as a continuous-time game in an extensive form. As Simon and Stinchcombe [33] discuss, a continuous-time game often faces difficulty with defining extensive-form stages in an infinitesimal component game. In particular, there is no natural notion of the previous stage before a point of t. In fact, for a continuous-time game, there may not exist a sequence of the discrete-time games in general that would converge to the the continuous-time game (with some relevant topology) as the discrete-time grid goes to zero. By construction, player 1's report and strategic actions at a stage might cause some information

[16] I will write the revealed income level simply as $X(t_{-})$ (i.e., without a hat) below unless it causes any confusion.
[17] Obviously, $\gamma_{s2}(t) = S(t)$ and $\gamma_{s1}(t) = X(t) - S(t)$.

flows across the following stages in an infinitesimal component game in this model. Accordingly, this model is not an exception like several continuous-time contracting models (e.g., Biais et al. [4], Sannikov [32], Williams [36]). By defining an appropriate topology, this continuous-time game is a continuous limit. For each t, I can define the very fine time grid $\{t_-\} \cup [t, t+dt)$ during which a component game is played.[18]

To begin with, I describe the institutional structure informally. Before 0_-, each player $2, 2'$ announces a contract, which prescribes (1) a goods transfer process to his own, characterized by a function of player 1's messages, the observed actions and outcomes and the calendar time conditional on the participation, and (2) a recommended participation probability that he sends to player 1. During the announcements, neither player 2 nor $2'$ can observe the contract announced by the other, while player 1 can observe the two announced contracts. Next, player 1 communicates with player $2, 2'$. Specifically, he sends a message to player $2, 2'$ regarding the announced contracts. For simplicity, player 1 reports player 2 (or $2'$)'s contract itself to player $2'$ (or 2).[19] Player 1 does not necessarily tell the truth. In turn, each player $2, 2'$ reports his recommended participation probability to player 1 as prescribed in the contract. Then, player 1 chooses which contract he participates in or neither. He can use a mixed strategy regarding the participation decision, although he will enter into only one contract. Once the choice is made, then it is revealed to player $2, 2'$. If player 1 chooses neither of them, then all must live in autarky forever. If player 1 does not choose player 2, then player 2 must live in autarky forever. If player 1 chooses player 2, then they make a bilateral contract by signing an agreement. The contracting players can commit to the agreement except for player 1's costly disclosure. The process until the agreement is called an ex ante negotiation stage (Figure 1). For notational convenience, let $\{-1\}$ denote this stage. By construction, the contracts are exclusive after the agreement. Still, there exists contractual externality through the participation probability. After the agreement, player 2 (or $2'$) transfers his capital to the contracting player.

From the initial point 0_-, a game starts according to the contract. Player 1's production process starts with the invested capital, and the output is allocated between the bilaterally contracting players over time according to the contract. For notational convenience, suppose that player 1 and 2 enter into a contract in an equilibrium. For each t, a component game evolves for a very fine time duration $\{t_-\} \cup [t, t+dt)$ (or grid t). The component game consists of three stages. First, at a disclosure stage t_-, player 1 decides whether or not to disclose the true state. If disclosure occurs, then disclosure cost is imposed on player 1, and player 2 observes the instant state $X(t_-)$

[18] Nakamura [27] includes a mathematical appendix proving these results.

[19] The message could be general in the degree and nature of the communication. But the simplification does not lose any generality.

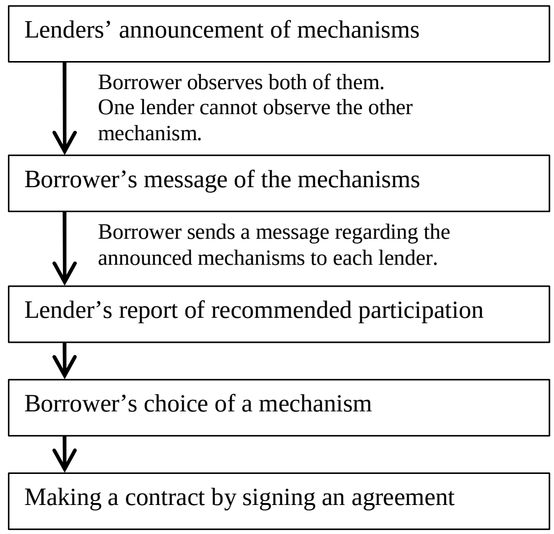

FIG. 1 Timing of events in the ex ante negotiation stage

at t_-. Then, the instant-t restructured payment level is resolved deterministically in a way that meets the contractual requirement based on the information set generated by the revealed true state. If player 1 does not disclose, then no restructuring occurs. The output and payment processes evolve continuously from the left-limit time t_- to t. The second stage is a production stage. At this step, the grid-t output is produced based on the driving force of the one-dimensional geometric Brownian motion and reveals the true output amount (i.e., grid-t true state) only to player 1. Third, the component game proceeds to a payment stage. At this stage, player 1 sends a message of his current state to player 2.[20] The message is not necessarily true. Then, player 1 sends a payment to player 2 according to the contract. At the end of the grid, they consume the allocated goods. The dynamic game moves on continuously (Figure 2).

Formally, at the ex ante negotiation stage $\{-1\}$, first, each player $2, 2'$ proposes a contract to player 1, which is characterized by a message space $\mathbb{C}$ and an outcome function γ. The message space $\mathbb{C} := \mathbb{C}_1 \cup \mathbb{C}_2$ in which $\mathbb{C}_1$ denotes player 1's message space regarding the announced contracts at the ex ante negotiation stage; $\mathbb{C}_2$ denotes player 1's dated message space regarding his own states over $\mathbb{T}$. The message spaces are measurable. Let $\{C(t)\}_{\{-1\}\cup\mathbb{T}} \in \mathbb{C}$ denote the message process over $\{-1\} \cup \mathbb{T}$.

[20]The message could be general in the degree and nature of the communication. But the simplification does not lose any generality.

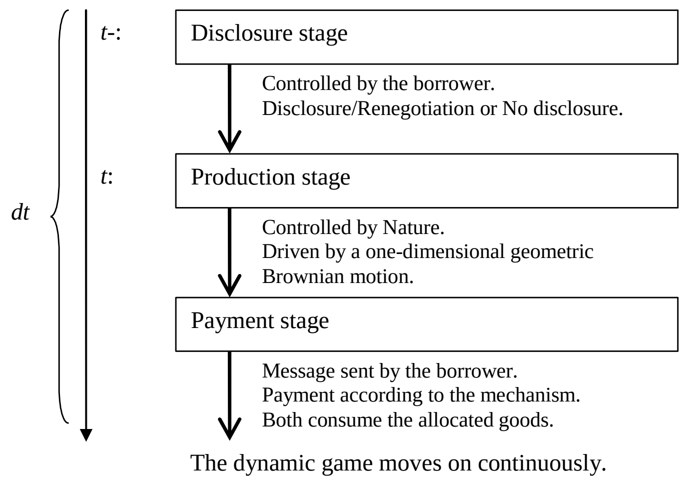

FIG. 2 Timing of events in a component game

Based on the message space, define the outcome function as $\gamma : \mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{D} \to \Phi^P \times \triangle(P)$ where Φ denotes the set of payment processes S from player 1 to player 2, $\mathbb{D} = \{0, 1\}_{\mathbb{T}}$ denotes player 1's dated disclosure space with $d(t)$ (for $t \in \mathbb{T}$, $d(t) = 1$ if a disclosure takes place; $d(t) = 0$ otherwise), $P = \{0, 1\}$ (1 denotes participation; 0 no participation), and $\triangle(P) = [0, 1]$ (its element p_γ given a contract γ) denotes the space of the recommended participation probability that player 2 announcing γ gives to player 1 before 0_- after receiving player 1's message $C(-1) \in \mathbb{C}_1$. Note that Φ^P denotes the set of the payment processes conditional on player 1's participation. Participation and disclosure actions are player 1's incontractible efforts. I will call the outcome function γ as well as a contract. Write $\gamma = (\gamma_s, \gamma_p)$ where γ_s denotes a payment rule and γ_p denotes a recommended participation probability.

A contract is said to be *feasible* if the sum of the allocated consumption is not larger than the whole income for any state at each time, i.e., $0 \leq \gamma_s(C(t), d(t)) \leq X(t)$ for any $C(t) \in \mathbb{C}$, $d(t) \in \mathbb{D}$, and $t \geq 0$ almost everywhere (a.e., hereafter), almost surely (a.s., hereafter) conditional on the participation. Notice that at initial point $t = 0_-$, $S(0_-) > X(0_-)$ is possible instantaneously. Then, player 1 strategically takes a disclosure action. So, the payment $S(0_-)$ is not realized in this case. Let Γ_0 denote the set of the feasible contracts, endowed with some topology.

At the ex ante negotiation stage, the type of player 1 is his own private information, although the type space is public information. The type determines the initial level of his income process. Player 2 chooses γ to make player 1's optimal choices as favorable as possible to player 2 unless he loses player 1's participation. Player 1 sends a message $C(-1) \in \mathbb{C}_1$ after observing the two announced contracts before 0_-, receives player 2's recommended participation probability according to terms of the contract, and chooses a contract γ or γ' or neither. Importantly, since player 2's contract terms include his recommendation of player 1's entering into his contract, player 2's contract may depend whether player $2'$'s contract depends on whether player 2's contract depends... and so on. That is, the ex ante negotiation process has a "nesting" structure. In equilibrium, player 1 is supposed to decide a state verification episode $d(t) \in \mathbb{D}$, makes a payment $S(t)$ to player 2, and participation $p \in [0,1]$ such that for some path $(\bar{S}, \bar{p})$, (S, p) coincides with $(\bar{S}, \bar{p})$ on $[0, t]$.

From 0_- onwards, a dynamic game begins under the chosen contract. There exist two kinds of decision nodes: (1) a distinguished node $\{0_-\}$, which is the root of the game, and (2) regular nodes indexed by a point of left-limit time t_-. The information set is denoted by $\mathcal{R}_{i,t_-}$ for each t_- ($t \geq 0$) and i – call it player i's private filtration – which is generated by the processes distinguishable to player i prior to t (i.e., at or prior to the left-limit time t_-). Because of the private information, $\mathcal{R}_{2,t_-}$ is coarser than $\mathcal{R}_{1,t_-}$ for all t. In particular, for the distinguished point $\{0_-\}$, $\mathcal{R}_{1,0_-} = \sigma\left\{S\left(0_-\right)\right\} \cup \sigma\left\{\theta_i\right\}$ and $\mathcal{R}_{2,0_-} = \sigma\left\{S\left(0_-\right)\right\}$. For a regular node at t_-, $\mathcal{R}_{1,t_-} = \sigma\left\{S\left(u_-\right), 0 \leq u \leq t\right\} \cup \sigma\left\{X(u_-), 0 \leq u \leq t\right\}$ and $\mathcal{R}_{2,t_-} = \sigma\left\{S\left(u_-\right), 0 \leq u \leq t\right\} \cup \sigma\left\{\hat{X}(u_-), 0 \leq u \leq t\right\}$. Let $\mathcal{R}_1 := \left\{\mathcal{R}_{1,t}\right\}_{\mathbb{T}}$ and $\mathcal{R}_2 := \left\{\mathcal{R}_{2,t}\right\}_{\mathbb{T}}$. In summary, a decision node is characterized by a point of time and the two players' private filtrations: $(0_-, \mathcal{R}_{1,0_-}, \mathcal{R}_{2,0_-})$; for $t > 0$, $(t_-, \mathcal{R}_{1,t_-}, \mathcal{R}_{2,t_-})$. Let $E\left[\cdot|\mathcal{R}_{i,t_-}\right] = E_t^i\left[\cdot\right]$ denote player i's conditional expectation operator given $\mathcal{R}_{i,t_-}$. At each node, there is an arbitrarily small time duration dt such that for a time interval $\{t_-\} \cup [t, t+dt)$, an instant-t component game is played – call this interval grid t as well.[21]

Under the contract, for each $t \geq 0$, the timing of events during the grid $\{t_-\} \cup [t, t+dt)$ given $\left(\mathcal{R}_{1,t_-}, \mathcal{R}_{2,t_-}\right)$ is as follows. First, at the disclosure stage t_-, player 1 decides whether or not to disclose his current income level via the costly disclosure technology given the information set $\mathcal{R}_{1,t_-}$. If player 1 does not disclose, then no restructuring occurs. The instant-t output and payment processes evolve continuously from the left-limit time to t: $X(t) = X(t_-)$ and $S(t) = S(t_-)$. If player 1 discloses, then the disclosure costs are imposed on player 1's left-limit income $X(t_-)$. The income process is lowered discontinuously by the amount of the disclosure costs: the instant-t income level is $X(t) = X(t_-) - \lambda \begin{bmatrix} S(t_-) & X(t_-) \end{bmatrix}^{\top}$. Then, the instant-$t$ payment is restructured in a "deterministic" way according to the terms in the ex ante contract. That is, at the ex ante negotiation stage $\{-1\}$, player 2 provides player 1 with a take-it-or-leave-it offer of the contract that prescribes not only the payment level in the "normal" situation (i.e., in which no disclosure is undertaken)

[21] For details, see Nakamura [27].

but also a restructuring plan. Player 2 intends to obtain as high payoffs as possible in the restructuring plan unless the payoffs prevent player 1 from participating. At the same time, because of the non-cooperative competition of such contract designing, the restructured payment level is no lower than player 1's willing-to-pay level. Note that by construction, player 2 knows the current state in this situation. This model assumes that the restructuring mechanism is deterministic in the sense that it has no randomization. Therefore, I focus on an instantaneous restructured payment level that is equivalent to player 1's willing-to-pay level – denoted by $\hat{S}$ – which the contract prescribes as a function of observed actions and disclosed true states. Second, at the production stage, given the instantaneous state variable levels at t, the grid-t output is produced based on the one-dimensional geometric Brownian motion and reveals player 1's true state only to himself accurately: $dX(t) = \mu_0 X(t)dt + \sigma_0 X(t)dW_0(t)$. Third, the component game proceeds to the payment stage. At this stage, player 1 sends a message $C(t) \in \mathbb{C}_2$ to player 2. Then, player 2 receives a payment according to the contract. At the end of the infinitesimal component game, they consume the goods allocated for the whole grid $\{t_-\} \cup [t, t + dt)$. Then, the dynamic game moves on continuously.

In the game, both the contracting players commit to the contract except for the borrower's "defaults." The specific meaning of the default in this paper is as follows. Player $2, 2'$ as well as player 1 have an incentive to minimize the disclosure costs in order to save as much income as possible. Accordingly, each player $2, 2'$ draws up a contract that seeks to balance two goals conditional on player 1's participation: (1) to make player 1 reveal his true state as frequently as possible in order to prevent player 1 from excessive exploitation of the informational rents and (2) to make player 1 reveal his true state as infrequently as possible in order to reduce the disclosure costs. Following the standard CSV discussions, we can conjecture that in case player 1 discloses his bad state, each player $2, 2'$ strategically provides him with a partial payment exemption. That is, player 1 has an option to break his payment promise by resorting to a costly verification of his current low-income state. Player 1's disclosure decision is not specifically anticipated as a response to the terms of the contract, although it is averse to player $2, 2'$. Since player 1's underlying state evolves continuously in the CSV environment, a downward discontinuity in the payment path, which immediately follows after the disclosure action, is a good candidate to be treated as a default in this sense. So, henceforth, I define a default to be player 1's instantaneous choice to lower discontinuously his payment level.

In summary, player 2 designs a contract competitively to maximize his lifetime utility; after choosing a contract, player 1 makes payment and, if necessary, demands state verifications and restructures the payment profiles according to the terms of the contract that he has chosen to maximize his lifetime utility. Hence, the contract is modeled to be contingent and partially enforceable, i.e., player 1 cannot make a sequence of decisions incompatible with the payment rule except for defaults.

4. CONTINUOUS CONTRACTS, STRATEGIES, AND EQUILIBRIUM NOTION

4.1. Continuous contracts and strategies

This paper focuses on a particular form of contracts in the following five points. First, the payment $S(t)$ is predictable $\mathcal{F}^s_{t\_}$-measurable. Second, the payment is stationary Markovian in the sense that it is dependent only on the current actions and outcomes that player 2 can distinguish. Third,

Assumption 4.1. *A payment process $S \in \Phi$ is continuous on a.e. sample path, except for a countable, discrete set of discontinuities caused by costly disclosures, adapted only to $\mathcal{F}^s$.*

Since the initial payment level is deterministically decided according to the contract, there is no payment uncertainty at 0_-. For a payment process $S \in \Phi$ and a local martingale $M(t) = \mathcal{E}_t(m)$,

$$M(t)\frac{S(t)}{S(0)} = \mathcal{E}_t(m)\mathcal{E}_t(s_0) = \mathcal{E}_t(m + s_0 + [m, s_0])$$

where $s_0(t) = \mathcal{L}n_t(\frac{S}{S(0)})$ denote the stochastic logarithm of the payment process normalized by its own date-0 value. By those assumptions, we obtain two lemmas as follows.

Lemma 4.1. *$\forall n \in \mathbb{M}$, $\ln S(0) + d(t) + \langle m, n\rangle_t = 0$.*

The proof is similar to Proposition 10.1.8 of Musiela and Rutkowski ([25], p. 246). There exists a price-of-risk process ς_u such that $\int_{\mathbb{T}} \varsigma_u d\langle m\rangle_u = \ln S(0) + d(t)$ and $P(\int_{\mathbb{T}} |\varsigma_u|\, d\langle m\rangle_u < +\infty) = 1$. Hence, the Girsanov transformation parameters are deterministic in the sense that they are independent of the state W_s. Next, the continuity of the process means that jumps may occur endogenously only at disclosure times. By Lemma 4.1 and the Martingale Representation Theorem for a local martingale,

Lemma 4.2. *There exist $\mathcal{F}^s$-predictable processes μ, σ such that for $\tau_m \le t < \tau_{m+1}$ $(m = 1, 2, ...)$, $dS(t) = \mu S(t)dt + \sigma S(t)dW_s(t)$.*

where τ_m denotes the mth state verification time.

Fourth, for some technical reasons, σ is independent of either player 1's controls or time. In addition, μ is independent of t except through player 1's controls. Let $\Sigma := \mathbb{R}_+ \cup \{0\}$. Let $\mathbb{A}$ denote an equicontinuous family of real-valued functions on

$\mathbb{C}_2 \times \mathbb{D}$ that is uniformly bounded on any closed interval on $\mathbb{C}_2 \times \mathbb{D}$. By the Ascoli-Arzelá Theorem, $\mathbb{A}$ is relatively compact in a set of all the continuous mappings (Royden [31], theorems 40,41, p. 169).

ASSUMPTION 4.2. *$\sigma \in \Sigma$ and $\mu(C(t), d(t)) \in M$.*

The diffusion coefficient is a non-negative scalar. By Lemma 4.2, the continuous payment profile is characterized by $S(\tau_m) \in \mathbb{R}_+$ and its subsequent evolution by a geometric Brownian motion between immediate state verifications: for $\tau_m \leq t < \tau_{m+1}$ $(m = 1, 2, ...)$,

$$dS(t) = \mu(C(t), d(t))S(t)dt + \sigma S(t)dW_s(t).$$

In words, $dW_s(t)$ is a randomization to conceal a pure choice of the payment level at each t. $\sigma S(t)$ is the amplitude of the randomization. In particular, $\sigma = 0$ would mean no randomization in the contract.

Finally, I characterize the restructured payment function by $\hat{S}(S(t_-), X(t_-))$ at each $t \geq 0$. Let $\mathbb{B}$ denote an equicontinuous family of real-valued functions on $\Phi \times \Phi$ that are uniformly bounded on any closed interval on $\Phi \times \Phi$. Again, by the Ascoli-Arzelá Theorem, $\mathbb{B}$ is relatively compact in a set of all the continuous mappings. In summary, γ_s is characterized by $\Big(S(0_-), \hat{S}, \mu, \sigma\Big)$ in $\mathbb{R}_+ \times \mathbb{B} \times \mathbb{A} \times \Sigma$, rather than designs S complexly. Write $\gamma_s = \Big(S(0_-), \hat{S}, \mu, \sigma\Big)$ as well. A contract is said to be *continuous* if it satisfies those five specifications. Let Γ denote the set of continuous contracts in Γ_0. Assume that the set of continuous contracts Γ is non-empty. Define Γ_s as the set of γ_s corresponding to each $\gamma \in \Gamma$. Assume that there is no randomization across the elements of Γ.

Next, player 1's strategies are stationary Markovian in the sense that player 1 chooses a fixed-dimensional function that maps his current information to the communication, disclosure and participation actions. They are pure except for the participation strategies. Precisely, the ex ante negotiation communication strategy is a mapping $\tilde{C}_1 : \Theta \times \Gamma^2 \to \mathbb{C}_1$, that is, a message is sent to player 2 under player 2's contract $\gamma \in \Gamma$ when player $2'$ is using $\gamma' \in \Gamma$ given his true type θ at -1. Since player 1 chooses a contract either γ or γ' or neither after observing the two announced contracts at -1, the participation strategy γ is a map $\tilde{p} : \Theta \times \Gamma^2 \times \mathbb{C}_1^2 \times \triangle(P)^2 \to [0, 1]$ after player 1 sends a message to player $2, 2'$ and receives the recommended participation probabilities from player $2, 2'$ when the other contract is γ' given his true type θ. For consistency, for any $\theta \in \Theta$, $\gamma, \gamma' \in \Gamma$, $\tilde{p}(\theta, \gamma, \gamma', p_\gamma, p_{\gamma'}) + \tilde{p}(\theta, \gamma', \gamma, p_{\gamma'}, p_\gamma) \leq 1$.

Player 1's predictable disclosure strategy conditional on the participation is $\tilde{d}$: $\Phi^2 \times \Gamma^2 \to \mathbb{D}$, that is, $d(t) = \tilde{d}(S(t_-), X(t_-), \gamma, \gamma')$. In particular, define player 1's pure-strategy control policy conditional on the participation as two increasing predictable processes of finite variation: $l_s : \Phi^2 \times \Gamma^2 \to \Phi$ and $l_x : \Phi^2 \times \Gamma^2 \to \Phi$.

$l_s(t)$ (resp. $l_x(t)$) represents the cumulative amount of decrease in the logarithm value of income process (payment process) controlled by player 1 up to t: $l_s(t) = l_s(0) + \int_0^t dl_s(s)$ and $l_x(t) = l_x(0) + \int_0^t dl_x(s)$. The two time-paths jump at the same time. $l(t) := \begin{bmatrix} l_s(t) & l_x(t) \end{bmatrix}^\top$. For the control process, a disclosure action $\tilde{d}$ is characterized by $\mathcal{R}_1$-predictable stopping times. Let $\{\tau\} = \{\tau_1, \tau_2, \ldots, \tau_m, ...\}$ denote a sequence of the stopping times. Suppose that player 1 requests a state verification and pays the resource costs for the mth time at t_- in state $r \in \mathcal{R}$. Then, $\tau_m(r) = t$ denotes a default time. Therefore, as a policy function representation, we can replace d with τ. Mathematically, as Harrison [20] formulates, $\{\tau\}$ is generated implicitly in the process l. Still, this paper represents a default time as an additional control. For notational convenience, set $\tau_0 = 0$, $\tau(-1)$ denotes one default time before each τ, and $\tau_m(t)$ denotes the latest state verification time up to t (including t).

The communication strategy over $\mathbb{T}$ conditional on the participation is a mapping $\tilde{C}_2 : \Phi^2 \times \mathbb{D} \times \Gamma^2 \to \mathbb{C}_2$, that is, for $t \in \mathbb{T}$, a message $C(t) = \tilde{C}_2(S(t), X(t), d(t), \gamma, \gamma')$ is sent to player 2 under the contract $\gamma \in \Gamma$ (given player $2'$'s contract $\gamma' \in \Gamma$) after taking a disclosure action $d(t)$ and observing $S(t), X(t)$ given his own true type θ. Write player 1's control policy as $\tilde{c} = \left\{\tilde{C}, \tilde{d} \text{ (or } \tau), dl, \tilde{p}\right\}$ in a well-defined policy set.

4.2. Equilibrium notion

For an arbitrary control policy of his own $\tilde{c}$, given the type θ, the contracts $\gamma, \gamma' \in \Gamma$, player 1's ex ante utility is:

$$U(\tilde{c}; \gamma, \gamma', \theta) \tag{4.1}$$

$$= \left\{ \begin{array}{c} \tilde{p}(\theta, \gamma, \gamma', p_\gamma, p_{\gamma'}) \lim\limits_{m\to\infty} \sum\limits_{\tau=\tau_1}^{\tau_m} \int\limits_{\tau(-1)}^{\tau} e^{-\delta t} E\left[f_1(X(t) - S(t)) \mid \mathcal{R}_{1,t_-}\right] dt \\ +\tilde{p}(\theta, \gamma', \gamma, p_{\gamma'}, p_\gamma) \lim\limits_{m\to\infty} \sum\limits_{\tau=\tau_1'}^{\tau_m'} \int\limits_{\tau'(-1)}^{\tau'} e^{-\delta t} E\left[f_1(X'(t) - S'(t)) \mid \mathcal{R}'_{1,t_-}\right] dt \end{array} \right\}$$

subject to for any t,

$dX(t) = \mu_0 X(t)dt + \sigma_0 X(t)dW_0(t) - dl_x(t)$ given $X(0) = \theta_i$

$dS(t) = \mu(\tilde{C}(t), \tilde{d}(t))S(t)dt + \sigma S(t)dW_s(t) - dl_s(t)$ given $S(0_-)$

$dl_x(t) = X(t_-) - X(t) = \lambda_x X(t_-) + \lambda_s S(t_-)$

if $t = \tau_m$ for $m = 1, 2, ...$ and $dl_x(t) = 0$ otherwise.

$dl_s(t) = S(t_-) - S(t)$

if $t = \tau_m$ for $m = 1, 2, ...$ and $dl_s(t) = 0$ otherwise.

$dX'(t) = \mu_0 X'(t)dt + \sigma_0 X'(t)dW_0(t) - dl'_x(t)$ given $X(0) = \theta'_i$

$$
\begin{aligned}
&dS'(t) = \mu'(\tilde{C}'(t), \tilde{d}'(t))S'(t)du + \sigma' S'(t)dW_s(t) - dl'_s(t) \text{ given } S'(0)\\
&dl'_x(t) = X'(t_-) - X'(t) = \lambda_x X'(t_-) + \lambda_s S'(t_-)\\
&\text{if } t = \tau'_m \text{ for } m = 1, 2, ... \text{ and } dl'_x(t) = 0 \text{ otherwise.}\\
&dl'_s(t) = S'(t_-) - S'(t)\\
&\text{if } t = \tau'_m \text{ for } m = 1, 2, ... \text{ and } dl'_s(t) = 0 \text{ otherwise.}
\end{aligned}
$$

where X' (resp. S', $\mathcal{R}'_1$) denotes player 1's income process (or payment process, information set) when he participates in the contract γ', and $\chi_{\{\tau_m \leq \tau_m(t)\}}$ denotes an indicator that, if $\tau_m \leq \tau_m(t)$, means 1, or else 0. Now, a control policy $\tilde{c}$ is said to be a *continuation equilibrium* relative to Γ if player 1 has no incentive to deviate from $\tilde{c}$ for any realization $X \in \Phi$, any pair of announced contracts $\gamma, \gamma' \in \Gamma$. Call the optimization program Problem (4.1). Assume the existence of a continuation equilibrium relative to Γ. Let c^* denote the continuation equilibrium strategy.

Next, define the equilibrium notion in the contract design. Player 2's preference is: for an arbitrary $\gamma \in \Gamma$ and given $\gamma' \in \Gamma, c^*$,

$$
V(\gamma; \gamma', c^*) = E\left[p_\gamma \int_{\mathbb{T}} e^{-\delta t} E\left[f_2(S(t)) \mid \mathcal{R}_{2,t_-}\right] dt\right]. \tag{4.2}
$$

A pair (γ^*, c^*) is said to be a (*symmetric*) *equilibrium* relative to Γ if

$$
\gamma^* \in \arg\max_{\gamma \in \Gamma} V(\gamma; \gamma^*, c^*),\ S(\tau^*_-) + l^*_s = \hat{S}^*, \text{ and } p_{\gamma^*} = \tilde{p}^*
$$

where $S(\tau^*_-)$ denotes the payment level just before each continuation equilibrium disclosure time. Assume the unique existence of a symmetric equilibrium. I confine attention to symmetry on the strategies that the players use in equilibrium.[22] Call the optimization program Problem (4.2).

To summarize, player 2 maximizes his expected discounted utility by designing ($S(0_-)$,$\hat{S}$, μ,σ,p,p') ex ante; player 1 pays and, if necessary, executes state verification strategies by controlling $\tilde{c}$. In particular, this paper focuses here only on the symmetric equilibrium. The homogeneity of the instantaneous utility (degree of $1 - \psi_1$) and technological constraints (degree of one) leads to the homogeneity of value functions. By substituting $\tilde{p}^*$ and $\tilde{c}'^*$, define $\bar{U}(S, X) = \max_{\tilde{c}} U(\tilde{c}; \gamma, \gamma')$ given γ, γ'. Write $\bar{U}(t) = \bar{U}(S(t), X(t))$ as well.

[22] This paper does not investigate the multipilicity of equilibria including asymmetric ones and their Pareto ranks. Rather, it focuses on the competitive implications of the symmetric equilibrium.

This section describes an optimal contract. In this common agency problem, the principals' contracts can depend on one another in complex ways: player 2's contract may depend whether player $2'$'s contract depends on player 2 contract depends... and so on. Therefore, the set of the agent' true states that matter in the contract designs must come from an infinite dimensional space even when the set of states are finite in the conventional sense. As to player 1's reports of the true states, the contract choice process reveals private information beyond the reports of his own type. Therefore, a standard direct-revelation principle does not hold for the communication games Γ. Hence, instead of the communication games, this paper initially confines attention to decentralized menus of payoff-relevant actions like Peters [29] does.

Especially in CSV environments, player 1's messages are not necessarily able to deliver credible information because of the disclosure costs. Player 1 has an incentive to make a lie-telling report as low as a critical level when the true income level is higher than the critical level. Player 2 also has an incentive to minimize the disclosure costs. Importantly, as Townsend [34] shows, player 1's welfare must be indifferent with respect to whether or not to disclose at the critical point. This result suggests that the original equilibrium outcome in the communication mechanisms could remain an equilibrium in some decentralized menu game.

Therefore, this section, initially, focuses on the set of menu contracts of payoff-relevant choices conditional on the participation and solves for the optimal contract in a constrained strategy space, which is coarsened into the set of ex post verifiable payoff-relevant variables. Then, I will show that the original equilibrium outcome in the communication mechanisms remains an equilibrium in the optimal contract in the constrained space.

This optimization program consists of two steps: first, player 1 optimizes his continuation utility level with respect to $\tilde{c}$ given $\{S(0_-), \hat{S}, \mu, \sigma\}$ and, second, player 2 optimizes his lifetime utility with respect to $\{S(0_-), \hat{S}, \mu, \sigma\}$ given player 1's optimal control mappings $\tilde{c}^*(S(0_-),\ \hat{S}, \mu, \sigma)$. The process $\hat{S}$ is consistent with the payment control process l_s in equilibrium.

5.1. Player 1's problem and implementability of contracts

This subsection studies player 1's optimization with respect to $\{\tau, dl(\tau)\}$ given $\left\{S(0_-), \hat{S}, \mu, \sigma\right\}$. Specifically, it (1) elaborates player 1's problem as a stochastic impulse control formulation on the infinite horizon conditional on his participation in player 2's contract γ given $\left\{S(0_-), \hat{S}, \mu, \sigma\right\}$, given γ' for the appropriate $\tilde{p}, \tilde{p}'$, and then (2) characterizes the implementability of the contracts.[23]

For some technical convenience, assume that the first default occurs at 0_-. In the model, since the initial shock is unobservable to the contract designer, this assumption does not lose any generality. So, $X(0) \neq X(0_-)$. The construction is contrary to the conventional stochastic-process discussions, but is convenient to achieve a recursive structure from 0. In addition, assume that for an arbitrarily large finite time T, there exists the second stopping time τ in $[0, T]$. Then, I focus attention on $[0, \tau]$. I will revisit the economic implications of the restrictions in Subsection 5.3.

First, with some regularities, there exists a semigroup formulation associated with the above problem (4.1):[24] on the finite horizon, given $\{X(0), S(0), \mu, \sigma\}$ and appropriate $\bar{U}(\tau)$,

$$\bar{U}(0) = \max_{\{\tau, dl(\tau)\}} E\left[\int_0^\tau e^{-\delta u} f_1\left(X(u) - S(u)\right) du + e^{-\delta\tau}\bar{U}(\tau) \mid \mathcal{R}_{1,0}\right] \tag{5.1}$$

subject to

$$\begin{aligned}
&dX(t) = \mu_0 X(t)dt + \sigma_0 X(t)dW_0(t) \text{ for } 0 \leq t < \tau \text{ given } X(0) \\
&dS(t) = \mu S(t)dt + \sigma S(t)dW_s(t) \text{ for } 0 \leq t < \tau \text{ given } S(0) \\
&X(\tau) = (1-\lambda_x)X(\tau_-) - \lambda_s S(\tau_-) \qquad (5.2) \\
&\bar{U}(0) \geq U_0.
\end{aligned}$$

Call this program as Problem (5.1). This paper does not necessarily mean by "being associated" an exact equivalence between Problems (4.1) and (5.1). In fact, under the regularities, Problem (5.1) is a weak formulation of Problem (4.1).[25] In the remaining section, I focus on the semigroup formulation (5.1). That is, for an arbitrary $\gamma_s = \left\{S(0_-), \hat{S}, \mu, \sigma\right\} \in \Gamma_s$, there exists a maximal element of the set of the stationary value functions $\bar{U}$. A key of this solution method is to treat a state verification decision as an impulse control by player 1.

I use a stochastic maximization principle on a finite horizon $[0, \tau]$. Since W_0 is not observable to player 1, the contract cannot depend on W_0. By construction of the income process X, define a predictable process:

$$\Pi(t) = \exp\left(\int_0^t \sigma_0^{-1}\mu_0 d\bar{W}_0(u) - \frac{1}{2}\int_0^t \left|\sigma_0^{-1}\mu_0\right|^2 dt\right)$$

where $W_0(t) = \bar{W}_0(t) - \int_0^t \sigma_0^{-1}\mu_0 du$. By the above hypotheses, Novikov's condition is satisfied. Therefore, $\Pi(t)$ is a martingale with $E\left[\Pi(\tau_-)\right] = \Pi(0) = 1$. By the

[23] As Fleming and Soner ([13], p. 153) point out, the solution method in the bilinear forms uses quasi-variational inequalities, rather than the viscosity method, because of branch points on the paths.

[24] For details, see Nakamura [27].

[25] For more discussions, see Nakamura [27].

Girsanov theorem, I have a new measure $\bar{P}^0$:

$$\frac{dP^0}{d\bar{P}^0} = \Pi(\tau_-).$$

Call the distribution process Π the relative density process. Also, define $z = \frac{S}{X}$ and $\bar{z} = z\Pi$. By the homogeneity structure of the objective function (degree $1 - \psi_1$) and of the cost structure and state evolution equations (degree 1), the value function is homogeneous of degree $1 - \psi_1$. Define $\bar{U}(\frac{S}{X}, 1) = u(z)$. Then, I replace the original state variable pair (S, X) with a new state variable pair $(\bar{z}, \Pi)$. The evolution equations of the new state variables are rewritten as: for $0 \leq t < \tau$, given $\Pi(0), \bar{z}(0)$,

$$d\Pi(t) = \Pi(t)\sigma_0^{-1}\mu_0 d\bar{W}_0(t)$$

$$d\bar{z}(t) = \bar{z}(t)\left\{\mu(t)dt + \sigma(t)dW_s(t) + \left(\sigma_0^{-1}\mu_0 - \sigma_0\right)d\bar{W}_0(t)\right\}.$$

Hence, by the measure change, Problem (5.1) is rewritten into:

$$\begin{aligned}
& X(0)^{1-\psi_1}u(z(0)) && (5.3)\\
&= \max_{\{\tau, dl(\tau)\}} \bar{E}_0^1 \left[\begin{array}{l} \int_0^\tau e^{-\delta t}\Pi(t)X(t)^{1-\psi_1}f_1\left(1 - z(t)\right)dt \\ +e^{-\delta\tau}\Pi(\tau)\left\{X(\tau_-)\kappa\right\}^{1-\psi_1}u(z(\tau)) \end{array}\right] \\
& \text{subject to for } 0 \leq t < \tau, \text{ given } \tau, \Pi(0), \bar{z}(0), \\
& d\Pi(t) = \Pi(t)\sigma_0^{-1}\mu_0 d\bar{W}_0(t) \\
& d\bar{z}(t) = \bar{z}(t)\left\{\left(\mu(t) - \mu_0\right)dt + \sigma(t)dW_s(t) + \left(\sigma_0^{-1}\mu_0 - \sigma_0\right)d\bar{W}_0(t)\right\} \\
& \kappa(\tau) = (1 - \lambda_x) - \lambda_s z(\tau_-) \\
& \bar{U}(0) \geq U_0
\end{aligned}$$

where $\kappa(\tau) = \frac{X(\tau)}{X(\tau_-)}$ and $\bar{E}_0^1$ denotes the expectation operator conditional on the information set $\mathcal{R}_{1,0}$ under the changed measure. First, fix the first disclosure time τ and deal with the maximization with respect to $dl(\tau)$. Hence, given $\tau, \bar{z}(\tau_-)$,

$$\max_{dl(\tau)} \bar{U}(\tau) = \max_{z(\tau)} X(\tau_-)^{1-\psi_1}\left\{(1 - \lambda_x) - \lambda_s z(\tau_-)\right\}^{1-\psi_1}u(z(\tau)). \quad (5.4)$$

Since the value function u is convex in the state variables,

Assumption 5.1. *There exits a $z(\tau)$ satisfying $u_z(z(\tau)) = 0$ and $u_{zz}(z(\tau)) < 0$.*

Denote the maximand by $z^*(\tau)$. By using the optimal control $z^*(\tau)$ (or $dl^*(\tau)$), I replace $\max_{dl(\tau)}\bar{U}(\tau)$ with:

$$\bar{U}^*(\tau_-) = X(\tau_-)^{1-\psi_1}\left\{(1 - \lambda_x) - \lambda_s z(\tau_-)\right\}^{1-\psi_1}u(z^*(\tau)). \quad (5.5)$$

Move further to the maximization with respect to τ (or equivalently τ_-). Following

Williams [36], I use a stochastic maximization principle in continuous time for $0 \leq t \leq \tau_-$. Rewrite the variables into stacked forms:

$$y(t) = \begin{bmatrix} \bar{z}(t) \\ \Pi(t) \end{bmatrix}, \bar{\omega}(t) = \begin{bmatrix} W_s(t) \\ \bar{W}_0(t) \end{bmatrix}, A(t) = \Pi(t) \cdot \begin{bmatrix} z(t)\left(\mu(t) - \mu_0\right) \\ 0 \end{bmatrix},$$

$$B(t) = \Pi(t) \cdot \begin{bmatrix} z(t)\sigma(t) & z(t)\left(\sigma_0^{-1}\mu_0 - \sigma_0\right) \\ 0 & \sigma_0^{-1}\mu_0 \end{bmatrix}.$$

Let $\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$ denote the adjoint processes associated with y. Also, let $\Lambda(t) = \begin{bmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) \\ \Lambda_{21}(t) & \Lambda_{22}(t) \end{bmatrix}$ denote the 2×2 target volatility matrix of the adjoint processes. Given τ, for $0 \leq t \leq \tau_-$, the Hamiltonian for this problem with the adjoint equations is:

$$\mathcal{H}_\Pi(\tau) = \Psi^\top A + \operatorname{tr}\left(\Lambda^\top B\right) + \Pi f_1 \tag{5.6}$$

subject to for $0 \leq t < \tau$,

$$dy(t) = A(t)dt + B(t)d\bar{\omega}(t)$$

$$d\Psi(t) = -\left[\frac{\partial \mathcal{H}_\Pi}{\partial y}(t)\right] dt + \Lambda(t)d\bar{\omega}(t) \tag{5.7}$$

given $y(0)$ and $\Psi(\tau_-) = \dfrac{\partial\left(\Pi(\tau_-)\bar{U}^*(\tau_-)\right)}{\partial y(\tau_-)}$.

For any variable, let a superscript $*$ denote the optimal value of the variable. By a standard stochastic maximum principle,

Lemma 5.1. *There exist $\mathcal{R}_{1,t}$-predictable adjoint processes $\{\Psi(t), \Lambda(t)\}$, which satisfy the evolution equation (5.7). In addition, given $dl^*(\tau)$ induced by Equation (5.4), τ^* satisfies for almost every $t \in [0, \tau_-]$ a.s., $\mathcal{H}_\Pi(\tau^*) = \max_{\{\tau\}} \mathcal{H}_\Pi(\tau)$ in the Hamiltonian (5.6).*

From the boundary conditions with respect to the backward variables,

$$\Psi_2^*(\tau_-) = u(\tau_-) + z^*(\tau_-)\Psi_1^*(\tau_-),$$

$$\Psi_1^*(\tau_-) = \frac{\partial u(\tau_-)}{\partial z^*(\tau_-)}.$$

By the assumption that a default occurs at 0_-,

$$u(0) = \Psi_2^*(0) - z^*(0)\Psi_1^*(0).$$

For $0 \leq t \leq \tau_-$, $\Psi_2^*(t) - z^*(t)\Psi_1^*(t)$ represents player 1's reservation continuation utility level, which player 1 would accept without requesting a default at t. The other adjoint process Ψ_1^* represents the shadow price process of the hidden payment/income ratio. Hence, they constitute additional state variables in this optimization program. Specifically, in terms of the standard dynamic optimal contract literature (e.g., Fernandes and Phelan [12]), the evolution of $\Psi_2^* - z^*\Psi_1^*$ characterizes player 1's interim individual rationality condition (that is, a promise-keeping condition). On the other hand, Ψ_1^* characterizes player 1's truth revelation condition via the disclosure (i.e., a threat-keeping condition). Now, rewrite the control policy $\{\tau^*, dl^*\}$ as a function of t, y, Ψ given γ_s. By substituting the control policy function $\{\tau^*, dl^*(\tau^*)\}$ into a function, rewrite the function with a tilde.

Finally, I characterize a class of the implementable contracts. Define player 1's target controls as $\{\hat{\tau}, \hat{dl}\}$ from player 2's viewpoint. Then, let the association between a contract γ_s and player 1's target controls $\{\hat{\tau}, \hat{dl}\}$ be denoted by a contract correspondence $\Gamma_{\{\hat{\tau}, \hat{dl}\}}$, which is induced as a result of player 2's optimal contract designs for each target $\{\hat{\tau}, \hat{dl}\}$. Then, a contract is said to be *implementable* if $\{\tau^*, dl^*\}$ is an optimal control when player 1 faces the contract correspondence $\Gamma_{\{\hat{\tau}, \hat{dl}\}}$. By Lemma 5.1 above and Theorem 4.1 of Williams [36]:

PROPOSITION 5.1. *A contract is implementable if and only if (1) the contract satisfies $\bar{U}(0) \geq U_0$, (2) the contract and its optimal control $\{\tau^*, dl^*\}$ satisfies the solutions of the Hamiltonian (5.6) for τ, and (3) for almost every $t \in [0, \tau_-]$, a.s. $\mathcal{H}_\Pi(\tau^*) = \max_{\{\tau\}} \mathcal{H}_\Pi(\tau)$.*

Let Γ^* denote the set of implementable contract γ. Correspondingly, define Γ_s^* with elements γ_s^*.

5.2. Player 2's problem: Optimal contract design

Finally, this subsection addresses player 2's optimization problem. Define $Y^\gamma := \begin{bmatrix} z \\ X \end{bmatrix}$. If there exists a unique, globally stable solution for player 2's optimization problem, then an equilibrium contract uniquely exists. By adding some relevant parametric restrictions, this subsection solves for the optimal contract. In particular, I focus on a case of $\bar{U}(0) \geq U_0$ under a symmetric, competitive contract.

Formally, Problem (4.2) is solved in a form of forward-backward stochastic differential equations subject to the implementability of the contracts. For state variables $\tilde{y}^\gamma = \begin{bmatrix} \tilde{Y}^\gamma \\ \tilde{\Psi} \end{bmatrix}$, let the corresponding adjoint variables be denoted by $\begin{bmatrix} \Psi_1^\gamma \\ \Psi_2^\gamma \end{bmatrix}$. The

Hamiltonian $\mathcal{H}^\gamma$ for player 2 is as follows:

$$\mathcal{H}^\gamma = \begin{bmatrix} \Psi_1^\gamma \\ \Psi_2^\gamma \end{bmatrix}^\top A^\gamma + \mathrm{tr}\left((\Lambda^\gamma)^\top B^\gamma\right) + f_2(\tilde{S}) \tag{5.8}$$

$$\text{subject to given } \tilde{\Psi}(\tau_-^*) = \frac{\partial \Pi(\tau_-^*)\bar{U}^*(\tau_-^*)}{\partial y(\tau_-^*)}, \tilde{Y}^\gamma(0),$$

$$d\tilde{y^\gamma}(t) = A^\gamma dt + B^\gamma d\omega(t),$$

$$\text{given } \Psi_1^\gamma(\tau_-^*) = \frac{\partial V(\tau_-^*)}{\partial \tilde{Y}^\gamma(\tau_-^*)}, \Psi_2^\gamma(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$d \begin{bmatrix} \Psi_1^\gamma(t) \\ \Psi_2^\gamma(t) \end{bmatrix} dt = -\left[\frac{\partial \mathcal{H}^\gamma}{\partial \tilde{y^\gamma}}(t)\right] dt + \Lambda^\gamma d\omega(t) \tag{5.9}$$

We can define A^γ, B^γ directly from the above representation. From the analogue of Lemma 5.1, there exist $\mathcal{R}_{2,t}$-predictable adjoint processes $\left\{\begin{bmatrix} \Psi_1^\gamma(t) \\ \Psi_2^\gamma(t) \end{bmatrix}, \Lambda^\gamma(t)\right\}$, which satisfy the evolution equation (5.9). In particular, Ψ_2^γ denotes the shadow price of player 1's interim individual rationality and truth revelation (i.e., Ψ). In other words, the first (second) element in Ψ_2^γ represents whether or not the individual rationality (truth revelation) condition is binding. More precisely, the individual rationality (truth revelation) condition is binding at instant t if $\Psi_2^\gamma(t)\{1\} > 0$ ($\Psi_2^\gamma(t)\{2\} > 0$) (otherwise 0) where $\Psi_2^\gamma(t)\{i\}$ ($i = 1, 2$) is the ith element of $\Psi_2^\gamma(t)$. Note that by the above assumption, $\Psi_2^\gamma(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Based on our equilibrium notion, in the optimal contract, for some t, $\Psi_2^\gamma(t)\{1\} > 0$ if and only if $\Psi_2^\gamma(t)\{2\} > 0$ a.s.. In other words, a default occurs when, and only when, a state verification is undertaken.

PROPOSITION 5.2. *Suppose that player 2's control space Γ_s^* is convex. If $\gamma^* \in \Gamma^*$ is an optimal contract design, $\frac{\partial \mathcal{H}^\gamma}{\partial \gamma_s} \cdot (\gamma_s - \gamma_s^*) \geq 0\ \forall \gamma_s \in \Gamma_s^*$ a.e., a.s.. In particular, the optimal contract requires the first-order, value-matching, smooth-pasting conditions when and only when default occurs in Problem (5.1).*

For the optimal utility process Ψ_2, the binding promise-keeping condition implies a value matching at $t = \tau_-$ in Problem (5.1) For the shadow price process Ψ_1 with respect to the true states, the binding threat-keeping condition implies a smooth pasting at $t = \tau_-$ in Problem (5.1). Most stochastic control problems treat value-matching and smooth-pasting conditions as necessary conditions in their optimization procedures from a technical perspective. In contrast, this model provides the above game-theoretic interpretation to both the value-matching and smooth-pasting conditions.

5.3. Solutions

First, this section studies the diffusion coefficient $\sigma S(t)$ for the payment process. Denote $M_s^*(\sigma)$ is a subset of Γ_s^* given $\sigma \in \Sigma$.

LEMMA 5.2. *Suppose that for each given σ, player 2's control subspace $M_s^*(\sigma)$ is convex. If γ is an optimal contract, then $\sigma = 0$. In addition, there must be a constant $\mu_m > 0$ for each default interval $m > 0$ such that for every payment path S that is a component of γ, there are $\tau_0 < \tau_1 < ... < \tau_m < ...$ and $k_0 > k_1 > ... > k_m > ...$ such that $S(t) = k_m \exp \mu_m (t - \tau_m)$ for $\tau_m \leq t < \tau_{m+1}$.*

The intuition for the first part is as follows. γ is constrained to be sample paths for stochastic processes with either 0 or positive diffusion coefficient. Positive diffusion coefficient would mean the borrower's ex ante (i.e., instant-t_-) ignorance of player 2's pure choice of payment levels. This game, however, is not of a pure conflict. Ex ante uncertainty of the pure payment choice would make the borrower more anxious of a bad case and choose defaults more often. It would lead to a smaller size of the whole pie and less profit of the lender. Removing the uncertainty would be favorable to the lender as well as to player 1. Therefore, any positive diffusion coefficient process would be inefficient. Under the above assumptions, an optimal payment process would be non-stochastic.

A rough sketch of the proof for the second part is as follows. Suppose that $\mu_m \leq 0$ for some m. These have payment paths on which the lender receives virtually nothing a.e. since the system would be divergent if $\frac{-(\mu-\mu_0)}{\sigma_0} \geq 0$ for each t. Obviously, the lender would not permit the borrower to choose such path. Therefore, for $s \in \mathbb{S}$, $s(t) = s(\tau_m) + \mu_m(t - \tau_m)$ for each discontinuity interval $\tau_m \leq t < \tau_{m+1}$ $\forall m$.

Next, I solve for player 1's optimal default behavior given $\mu, S(0)$. This framework is a typical impulse control problem. Due to the asymmetric information, the impulse control problem has a one-sided boundary. Given Equation (5.4), because of the players' CRRA-type utility form, there exists an interval $(0, b) \subset \mathbb{R}$ and a point $z^* \in (0, b)$ such that the process jumps to z^* at instant 0 if $z(0_-)$ is in $[b, \infty)$, while it also jumps to z^* if for some t, $z(t)$ is inside $(0, b)$ and subsequently the process hits b from below (Figure 3). For $z \in (0, b)$, the evolution of player 1's value function $u(z)$ follows the Hamiltonian-Jacobi-Bellman (or HJB) equation:

$$\rho^1 u(z) = \frac{(1-z)^{1-\psi_1}}{1-\psi_1} + (\mu - \mu_0)\, z u_z(z) + \frac{1}{2}\sigma_0^2 z^2 u_{zz}(z). \tag{5.10}$$

where $\rho^1 = \delta - (1-\psi_1)\left(\mu_0 + \frac{\sigma_0^2}{2}\right)$. Note that ρ^1 is an adjusted instantaneous discount rate. Given $z \in (0, b)$, I can choose an appropriate particular solution $u^p(z)$. Also, due to the one-sided boundary, the homogeneous solution is represented by a one-

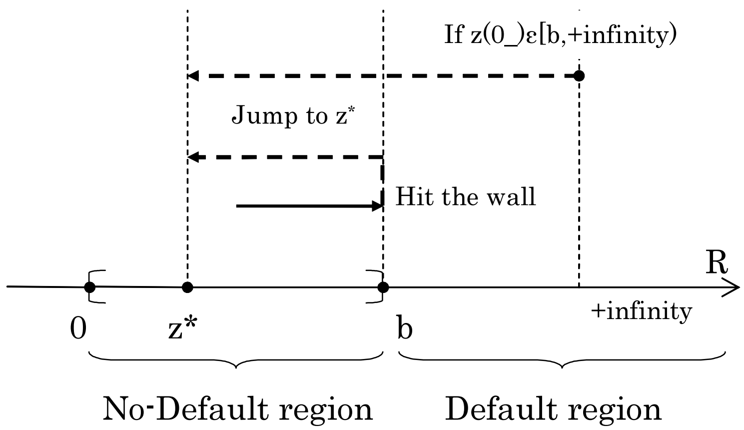

FIG. 3 Impulsive control

sided convergent form:

$$u^h(z) = az^{\nu^1}$$

where $a < 0$ and $\nu^1 = (\frac{1}{2} + \frac{\mu-\mu_0}{\sigma_0^2}) - \sqrt{\frac{2\rho^1}{\sigma_0^2} + (\frac{1}{2} + \frac{\mu-\mu_0}{\sigma_0^2})^2} < 0$. Then, the general solution $u(z)$ is:

$$u(z) = u^p(z) + u^h(z).$$

By Proposition 5.2, the following first-order, value-matching, and smooth-pasting conditions are satisfied at default instant τ under the optimal contract:

$$u_z(z^*) = 0,$$

$$u^p(b) + ab^{\nu^1} = \{(1-\lambda_x) - \lambda_s b\}^{1-\psi_1} u(z^*),$$

$$u_z^p(b) + a\nu^1 b^{\nu^1 - 1} = -\lambda_s(1-\psi_1)\{(1-\lambda_x) - \lambda_s b\}^{-\psi_1} u(z^*).$$

Since we have three equations for unknown parameters (z^*, b, a), they are solvable. Assume that there exists a solution triple (z^*, b, a). Since $u_{zz}(z^*) < 0$, by the implicit function theorem, the solution (z^*, b, a) is locally stable. Note that it is independent of the income level except through defaulting actions.

Finally, I solve for an optimal contract. This model does not have an appropriate closed-form representation because the particular solution $u^p(z)$ has no explicit closed-form solution. Hence, assume that there is an optimal contract μ^*, although the assumption is at a high level. A contract γ is said to take the form of a *defaultable debt* (*in an exponential sense*) if the logarithm of payment level is constant between immediate defaults and if γ allows player 1 to default for a payment allowance. The following two main theorems are obtained: the first one describes the form of the optimal contract, and the second one characterizes equilibrium performance under this contract. First,

THEOREM 5.1. *Suppose that there is an optimal contract μ^*. Then, an optimal contract takes the form of a defaultable debt, which is characterized by three factors: (1) a constant drift rate μ^*, (2) a constant non-default region $(0, b)$ of payment/income ratio, and (3) a constant renewal payment/income ratio $z^* \in (0, b)$. In particular, the original equilibrium outcome in the communication games remains an equilibrium in the optimal contract.*

A rough sketch of the proof is as follows. By definition, default is not specifically predicted to occur. Since the drift rate control problem given $S(0)$ has a convex structure with respect to the state variable, the elements of Γ are non-redundant with respect to the utilities of the players. By the relative compactness of the state space, there exists a unique, locally stable optimal contract. In particular, because of the homogeneity of instantaneous utility function and technological constraints, the parameters μ^*, z^*, b are constant over time. Non-default region $(0, b)$ of the payment/income ratio is completely specified as a function of μ^* ex ante. By the recursive formulation, the initial payment level is set as $S^*(0_-) = z^* \exp \hat{\theta}$. The transition probability converges weakly to an invariant distribution.

Now, we move back to the original communication mechanisms. Player 1 makes a lie-telling report as low as the critical level b when the true payment/income level is higher than the critical level. Player 2 also has an incentive to minimize the disclosure costs. Importantly, player 1's welfare must be indifferent with respect to whether or not to disclose at the critical point. Since the critical point and the optimal restructured payment level depend only on ex post observable, payoff-relevant variables, the original equilibrium outcome in the communication games remains an equilibrium in the optimal contract. Hence, the optimality of the above-described defaultable debt contract is established.

Until now, I have assumed that the first default occurs at 0_-. The assumption means that $z(0_-)$ is outside $(0, b)$. But I can now generalize the results. The payment profile imposed by player 1 is deterministic except at stochastic default times. When the payment/income ratio z is inside $[b, \infty)$ at 0_-, then the liability of the payment is too high relative to the initial income level, and a default occurs immediately. Then, the payment level is rescheduled, and the payment/income ratio z jumps to z^* instantaneously and then restarts from the level. If the payment/income ratio z is inside $(0, b)$ at 0_-, then player 1 make payments as the contract prescribes. If z subsequently hits a floor level b, then a default occurs. Then, the payment/income ratio z is rescheduled into z^* instantaneously (Figure 4).

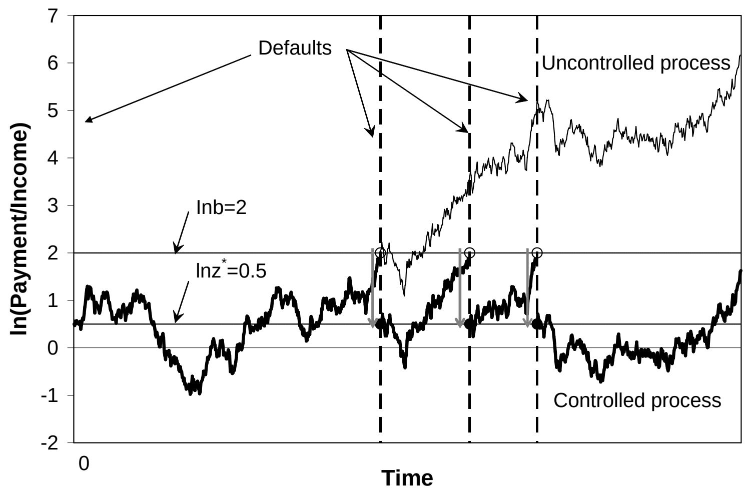

FIG. 4 Simulation: Controlled Payment/Income Process

With respect to the equilibrium default behavior, by the Markovian perfection property, we derive the second theorem:

THEOREM 5.2. *Under the optimal contract in Theorem 5.1, with respect to the equilibrium default performance, the following four results are obtained: (1) the contract requires state verification when, and only when, a default occurs, (2) the equilibrium income involves paths with arbitrarily large finite numbers of defaults within any time interval, (3) player* 1 *defaults infinitely many times a.e., and (4) from player* 2*'s viewpoint, a default time is expected to arrive based on an exponential distribution while from player* 1*'s viewpoint, a default occurs as a contingent claim.*

Regardless of player 2's nonlinear utility ($0 < \psi_2 < 1$), the optimality of the defaultable debt structure is kept because of the homogeneity of the instantaneous utility functions and the technological structure. The optimal payout process is piece-wise deterministic and its stochastic logarithm has jump of a constant size.

As a consequence, as a continuous-time analogue of the discrete-time model in Nakamura [26], an optimal contract takes the form of a payoff-relevant, contingent contact, although a costly default itself is incontractible. The debt restructuring plays a positive role as a necessary evil: a costly default is a trigger into a payment allowance while it gives the lender a tool to make the borrower reveal his true states at intervals and to prevent the borrower from exploiting too much informational rents. The borrower's verification of his bad state works as a credible exemption clause for a default and its resulting payment allowance. In contrast to Nakamura [26], a state-revealing episode degenerates to a singleton. Except for countable disclosure times, the truth-concealing episode is true a.e.. In addition, from the lender's viewpoint, the equilibrium default probability follows an exponential distribution while from the borrower's viewpoint, a default jump occurs as a strategic impulse. By comparative statics, under the optimal debt contract, the arrival rate (jump size, resp.) of default is increasing (decreasing) in monitoring ability. Note that too high disclosure costs would negate the contracting opportunity. As a result, this model provides some strategic insights into Duffie and Singleton [10]'s continuous-time reduced-form defaultable bond model.

6. EXTENSIONS BEYOND STANDARD CSV MODELS

The continuous-time structure is characterized by its high mathematical tractability as compared with most of the previous discrete-time game-theoretic models. The continuous-time construction is a powerful tool to analyze dynamic contracts under informational asymmetry. In fact, this structure has many possibilities for practical extensions to incorporate actual firms' complex capital structure and other dynamic auditing problems in labor and insurance markets beyond standard CSV models. This section provides two examples: (1) hidden entrepreneurial efforts of borrower and (2) human capital accumulation of disclosure ability.

6.1. Hidden entrepreneurial efforts of the borrower

So far, this paper has assumed that the income process is exogenous. So long as actual default problems are concerned, however, one of the lenders' considerations is that a borrower's lazy performance may cause a default when the lenders cannot verify the borrower's effort. In such circumstances, from an ex ante viewpoint, the lender would try to make a contract that could prevent the borrower from being too lazy. This is a typical moral hazard problem.

This subsection adds hidden entrepreneurial efforts of the borrower to the above benchmark model. Player 1's continuous effort can have influence on the average logarithmic income path: the drift rate of stochastic logarithm income process μ_0 depends continuously on player 1's hidden entrepreneurial effort except for a countable, discrete set of discontinuities. Specifically, we formulate the effort effect on his income process as a concavely increasing drift functional $\mu_0(e_1(t))$ with $\frac{d\mu_0}{de_1(t)} > 0, \frac{d^2\mu_0}{de_1(t)^2} < 0, \lim_{e_1(t)\uparrow 1} \frac{d^2\mu_0}{de_1(t)^2} = 0$ for $e_1(t) \in [0, 1)$. In addition, assume that $\mu_0(e_1(t))$ is invertible for every $e_1(t) \in \mathbb{R}_+$. For each t, the effort requires borrower's non-negative allocated goods $e_1(t)X(t)dt \geq 0$ with $e_1(t) \in [0, 1) \subset \mathbb{R}_+$ at grid t. The effort is made at the end of the disclosure stage at each fine grid. Informationally, the borrower himself can privately observe his efforts.

By construction, the moral hazard problem is independent of the CSV problem. Because of the concavity of μ_0, the equilibrium correspondence with respect to both effort and disclosure is convex. By the stochastic maximization principle,

$$(\Lambda_{12}S + \Lambda_{22})\,\sigma_0 \frac{d\mu_0}{de_1} \leq \{(1-e_1)X - S\}^{-\psi_1}$$

with inequality only if $e_1 = 0$. Suppose that there exists an optimal level $e_1^* \in (0,1)$ a.e., a.s.. Then, the moral hazard problem causes a premium in the payout paths a.e..

6.2. Human capital accumulation of disclosure ability

The above benchmark model has assumed that the disclosure (or monitoring) technology is unchanged throughout the dynamic games. In practice, the disclosure ability is time-varying, especially dependent on human capital accumulation with respect to disclosure ability. As a matter of fact, several empirical results show that the disclosure costs are negatively correlated with the business cycles in the US: disclosure procedures tend to be more costly in economic recessions than in booming periods. Such countercyclical characteristic of disclosures implies that deteriorated monitoring ability might delay an economic recovery in a recession phase.

To investigate the economic aspect, this subsection introduces human capital accumulation of disclosure ability. There exists disclosure capital $D_t \in [0,1] \subset \mathbb{R}$ owned by player 1. Any overstock of the disclosure capital over one unit would be discarded instantaneously. Instead of the above cost structure (5.2), a new structure of the disclosure costs is characterized by:

$$X(\tau) = D(\tau_-)\left\{(1-\lambda_x)X(\tau_-) - \lambda_s S(\tau_-)\right\}.$$

In other words, when a disclosure occurs, the renewed income level is discounted in proportion to the accumulated disclosure capital level at its left-limit time. The disclosure capital is a communication infrastructure that makes it easier to disclose player 1's current state given the costly disclosure technology λ. Player 1 invests a non-negative part of his allocated income (say, $e_2(t)X(t)$ with $e_2(t) \geq 0$) into his own disclosure capital at the payment stage at each very fine grid $\{t_-\} \cup [t, t+dt)$.[26] The transformation rate from goods to capital is $\frac{D(t)}{X(t)}$. The capital depreciates at a constant rate ϑ continuously. Because of the continuous control structure, there is no serious problem of branch points with respect to the disclosure capital process.

[26] Instead, I can model monitoring ability owned and invested by financial sectors. For analytic convenience, however, this paper assumes that player 1 possesses and invests the capital.

Hence,

$$dD(t) = (-\vartheta + e_2(t))\, D(t)dt$$

for continuous sample paths a.e. subject to $D(t) \in [0, 1]$. So, the disclosure costs depend on history of player 1's continuous investment efforts. The investment efforts and disclosure capital levels are public information.

In player 1's maximization program, D is added to the state variable set, and e_2 is also added to the control set γ_s. By modifying Equation (5.5),

$$\bar{U}^*(\tau_-) = \{D(\tau_-)X(\tau_-)\}^{1-\psi_1}\{(1-\lambda_x) - \lambda_s z(\tau_-)\}^{1-\psi_1} u(z^*(\tau), \frac{D}{X}).$$

Assume that the equilibrium correspondence is convex-valued in state variables. Then, by the analogue of Proposition 5.2, there exists an equilibrium satisfying:

$$\frac{\partial \mathcal{H}^\gamma}{\partial \gamma_s} \cdot (\gamma_s - \gamma_s^*) \geq 0 \forall \gamma_s \in \Gamma_s^*$$

a.e., a.s.. For simplicity, assume that $D < 1$ in equilibrium. The HJB evolution (5.10) is modified into:

$$0 \geq \max_{e_2} \left[\begin{array}{c} -\left\{\delta - (1-\psi_1)\left(\mu_0 + \frac{\sigma_0^2}{2} - \vartheta + e_2\right)\right\} u(z, \frac{D}{X}) \\ +\frac{(1-z-e_2)^{1-\psi_1}}{1-\psi_1} + (\mu - \mu_0)\, z u_z(z, \frac{D}{X}) - (\vartheta - e_2 + \mu_0)\, \frac{D}{X} u_{\frac{D}{X}}(z, \frac{D}{X}) \\ +\frac{\sigma_0^2}{2} z^2 u_{zz}(z, \frac{D}{X}) + \frac{\sigma_0^2}{2}\left(\frac{D}{X}\right)^2 u_{\frac{D}{X}\frac{D}{X}}(z, \frac{D}{X}) \end{array} \right].$$

with inequality only if $e = 0$. The adjusted discount rate $\delta - (1-\psi_1)(\mu_0 + \frac{\sigma_0^2}{2} - \vartheta + e_2)$ is increased by the depreciation rate, net of the effort, of the disclosure capital. So, the local optimal effort is characterized necessarily as:

$$(1 - z - e_2)^{-\psi_1} \geq (1-\psi_1)u(z, \frac{D}{X}) + \frac{D}{X} u_{\frac{D}{X}}(z, \frac{D}{X}).$$

with inequality only if $e = 0$. The second-order condition is satisfied. By the implicit function theorem, with the mild assumption that the cross derivative $u_{z\frac{D}{X}} < 0$, the optimal investment is decreasing in the payment/income ratio. Intuitively, when player 1 is in good shape, he makes large investments (vice versa). As a result, a default would boost the disclosure investment at its successive very fine grid, because a default gives player 1 the payment allowance to lower the payment/income ratio and makes some room for the investment (Figure 5).

From a business-cycle perspective, the implications are as follows. When the borrower's payment liability relative to his income level is increasing, the monitoring-ability investment effort level is getting lower even in good economic conditions. So,

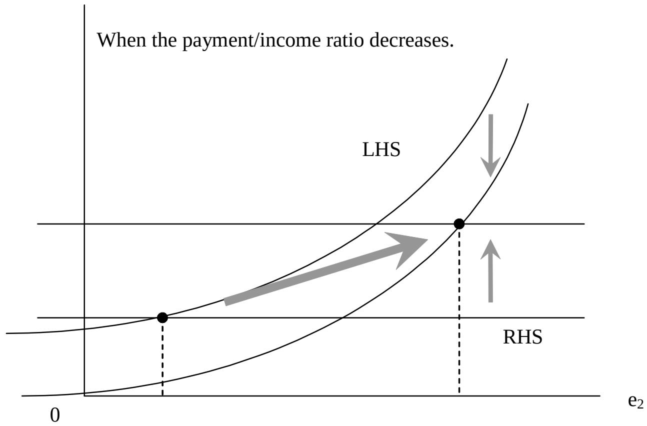

FIG. 5 Monitoring ability investment after a default

the disclosure capital tends to depreciate when the payment liability increases sufficiently in a while after a default. When a default occurs again in an economic recession, the lowered disclosure capital is used for the disclosure. Right after the default, the borrower increases investment efforts in his disclosure capital. In summary, the disclosure capital process tends to be countercyclical. In this situation, the depreciated monitoring ability would increase the deadweight loss in a recession and delay an economic recovery compared with the above benchmark case. From an empirical perspective, these results show that the optimal price-of-risk is nonlinear in liquidity factors. In particular, the liquidity premium process tends to be countercyclical. Also, it implies that the fixed-effect formulation of a liquidity effect, as in Duffie and Singleton [10]'s reduced form model, could encounter a model misspecification problem. This intuition looks consistent with the empirical results of Duffee [8].

7. CONCLUDING REMARKS

This paper studied dynamic CSV in continuous time in competitive environments and established a continuous-time, competitive model of the Markov communication game in costly information environments. This paper shows, first, that an optimal contract takes the form of a defaultable debt in the sense that the payment profile is deterministic almost everywhere except for a countable, discrete set of the downward discontinuities at the default times and that the contract permits the borrower to default at any instant recurrently. The optimal contract is ex ante describable, although the costly default itself is incontractible. Second, with respect to the equilibrium default behavior, the contract requires a state verification when, and only when, a default occurs. Moreover, the equilibrium income involves paths with arbitrarily large finite numbers of defaults within any time interval. The less informed lender expects the equilibrium default time to arrive based on an exogenous exponential probability distribution, while the fully informed borrower defaults strategically. As a consequence, this model provided a game-theoretic interpretation to Duffie and Singleton [10]'s reduced-form defaultable bond model. Furthermore, beyond standard CSV models, this paper studied two applications: (1) hidden entrepreneurial efforts of the borrower and (2) human capital accumulation of the disclosure ability. This model provided a better framework than before to analyze actual financial data regarding defaultable debts.

However, this model still faces limited applicability to actual defaultable debt contracts. First, this model presumes deterministic monitoring. In practice, however, stochastic monitoring is often undertaken in financial contracts in some industries (See Krasa and Villamil [24]). The assumption of stochastic monitoring could result in suboptimality of a debt-type contract. Second, and more importantly, this model does not study any kind of debt markets. Most notably, the contract is ex post exclusive in this common agency model. In practice, most debt defaults are observable to non-contracting lenders without a cost in the markets. In such circumstances, if the debts are tradable, then any potential lender might not have an incentive to enter into a contract that permits costly disclosures. A contract presented in this above model might not be sustainable in security markets.[27] Future work should involve such ex post contract non-exclusivity.[28]

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[27] This logic looks close to Grossman and Stiglitz [18].

[28] For example, see Nakamura [28].

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