Kidney Exchange with Good Samaritan Donors: A Characterization

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1 Introduction

- Transplantation is the preferred treatment for the most serious forms of kidney disease.

- More than 60,000 patients on the waitlist for deceased donor kidneys in the U.S., about 15,000 waiting more than 3 years. In 2004 about 3,800 patients died while on the waitlist while only 14,500 patients received a transplant form deceased (about 8,500) or live donors (about 6,000).

- Buying and selling a body part is illegal in many countries in the world including the U.S. Donation is the only source of kidneys in many countries.

Sources of Donation:

1. *Deceased Donors:* In the U.S. and Europe a centralized priority mechanism is used for the allocation of deceased donor kidneys, which are considered national treasure.

2. *Living Donors:* Live donations have been the increasing source of donations in the last decade. Two types:

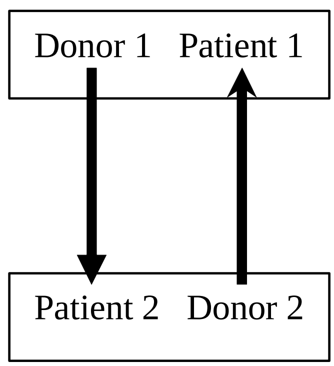
   (a) *Directed donation*: Generally friends or relatives of a patient specifically want to donate their kidney to their loved ones.

   (b) *Undirected donation*: "*Good Samaritans*" (GS) who anonymously donate one of their kidneys. Usually GS kidney is treated as a deceased donor kidney and is transplanted to the highest priority patient in the deceased donor waiting list.

2 Donations and Live Donor Exchanges

- There are two tests that a donor should pass before she is deemed compatible with the patient:
  - Blood compatibility test: O type kidneys compatible with all patients; A type kidneys compatible with A and AB type patients; B type kidneys compatible with B and AB type patients; AB type kidneys compatible with AB type patients.
  - Tissue compatibility test (crossmatch test): HLA proteins play two roles (1) determine tissue rejection or compatibility and (2) how close the tissue match is.
- If either test fails, the patient remains on the deceased donor waiting list. If the donor is a directed donor, she goes home unutilized.
- Medical community has already come up with a way of utilizing these "unused" directed donors.

- A paired exchange involves two incompatible patient-donor couples such that the patient in each couple feasibly receives a transplant from the donor in the other couple. This pair of patients exchange donated kidneys.



- Larger exchanges can also be utilized (Two 3-way exchanges have been utilized in Johns Hopkins University Transplant Center)

3 Kidney Exchange Developments

- Kidney exchange mechanisms were proposed by

  Roth, Sönmez and Ünver *QJE* (2004), *JET* (2005) (also see *AER-P&P* (2005), NBER wp (2005))

- New England Kidney Exchange (NEPKE) was established by the proposals of by Alvin Roth, Drs. Francis Delmonico Susan Saidman, and us in 2004

- A national exchange program is being proposed.

4 Integrating GS Donations with Paired Exchanges

In May 2005, surgeons at Johns Hopkins performed an exchange between a *Good Samaritan donor*, two incompatible patient-donor pairs, and a patient on the deceased-donor priority list.

- In the recent exchange at Johns Hopkins,
  - the kidney from the GS-donor is transplanted to the patient of the first incompatible pair,
  - the kidney from the first incompatible pair is transplanted to the patient of the second incompatible pair, and
  - the kidney from the second incompatible pair is transplanted to the highest priority patient on the deceased-donor priority list.
- What are plausible mechanisms to integrate GS donations with paired exchanges?

5 Other Related Literature

- Shapley and Scarf *JME* (1974) - housing market
- Roth *EL* (1982) - strategy-proofness of core as a mechanism in housing markets
- Ma *IJGT* (1994) - characterization of core in housing markets
- Svensson *SCW* (1999) - characterization of serial dictatorships in house allocation
- Abdulkadiroğlu and Sönmez *JET* (1999) - house allocation problem with existing tenants
- Ergin *JME* (2000) - another characterization of serial dictatorships in house allocation

6 The Model

- $\mathcal{I}$: a finite set of patients

- $\mathcal{D}$: a finite set of donors such that $|\mathcal{D}| \geq |\mathcal{I}|$.

- Each patient $i \in \mathcal{I}$ has a *paired-donor* $d_i \in \mathcal{D}$ and has strict preferences P_i on all donors in $\mathcal{D}$.
  - Let R_i denote the weak preference relation induced by R_i and
  - For any $D \subset \mathcal{D}$, let $\mathcal{R}(D)$ denote the set of all strict preferences over D.

A *kidney exchange problem with good samaritan donors*, or simply a *problem*, is a triple $\langle I, D, R\rangle$ where:

- $I \subseteq \mathcal{I}$ is any set of patients,
- $D \subseteq \mathcal{D}$ is any set of donors such that $d_i \in D$ for any $i \in I$, and,
- $R = (R_i)_{i\in I} \in [\mathcal{R}(D)]^{|I|}$ is a preference profile.

Given a problem $\langle I, D, R\rangle$, the set of "unattached" donors $D \setminus \{d_i\}_{i\in I}$ is referred as *Good Samaritan donors* (or in short *GS-donors*).

- Paired-donor d_j of a patient j is formally a GS-donor in a problem $\langle I, D, R\rangle$ if $d_j \in D$ although $j \notin I$.

- Given $I \subseteq \mathcal{I}$ and $D \subseteq \mathcal{D}$, a *matching* is a mapping $\mu : I \rightarrow D$ such that

$$\forall i, j \in I,\ i \neq j \ \Rightarrow \ \mu(i) \neq \mu(j).$$

- We denote a problem $\langle I, D, R \rangle$ simply by its preference profile R

- A *mechanism* is a systematic procedure that selects a matching for each problem.

7 Axioms

7.1 Individual Rationality, Pareto Efficiency and Strategy Proofness

Fixed population axioms:

- A matching is *individually rational* if no patient is assigned a donor worse than her paired-donor.
  - A mechanism is *individually rational* if it always selects an individually rational matching.

- A matching is *Pareto efficient* if there is no other matching that makes every patient weakly better off and some patient strictly better off.
  - A mechanism is *Pareto efficient* if it always selects a Pareto efficient matching.

- A mechanism is *strategy-proof* if no patient can ever benefit by misrepresenting her preferences.

7.2 Weak Neutrality and Consistency

Variable population axioms:

- A mechanism is *weakly neutral* if labeling of GS-donors has no affect on the outcome of the mechanism.

Let for any $i \in I$, $R_i \in \mathcal{R}(D)$ for $D \subset \mathcal{D}$ and $I \subset D$. For any $J \subset I$ and $C \subset D$, let $R_J^C = (R_i^C)_{i \in J}$ be the *restriction of profile R to patients in J and donors in C.*

We refer $\left\langle J, C, R_J^C \right\rangle$ as the *restriction of problem $\langle I, D, R \rangle$ to patients in J and donors in C*. The triple $\left\langle J, C, R_J^C \right\rangle$ itself is a *well-defined reduced problem* if whenever a patient is in J then her paired-donor is in C.

Given a problem $\langle I, D, R \rangle$, the removal of a set of patients $J \subset I$ together with their assignments $\phi[R](J)$ under ϕ and a set of unassigned donors $C \subset D$ under ϕ results in a well-defined reduced problem

$$\left\langle I \setminus J,\ D \setminus (\phi[R](J) \cup C),\ R_{-J}^{-\phi[R](J) \cup C} \right\rangle$$

if

$$(\phi[R](J) \cup C) \cap \{d_i\}_{i \in I \setminus J} = \emptyset.$$

- A mechanism is *consistent* if the removal of
  - a set of patients,
  - their assignments, and
  - some unassigned donors

  does not affect the assignments of remaining patients provided that the removal results in a well-defined reduced problem.

- Once a mechanism finds a matching, actual operations can be done months apart in different exchanges. Moreover, some unassigned donors (who are either GS-donors or donors of patients who already received a transplant) may be assigned to the deceased donor waiting list in the mean time. Therefore, *consistency* of the mechanism ensures that once the operations in an exchange are done and some unassigned donors become unavailable, there is no need to *renege* the determined matching, since the mechanism will determine the same matching in the reduced problem.

8 You Request My Donor-I Get Your Turn Mechanism

- Abdulkadiroğlu and Sönmez *JET* (1999) introduced in the context of *house allocation with existing tenants*(see also Chen and Sönmez *JET* (2006) and Sönmez and Ünver *GEB* (2005)

- A (priority) ordering f : $f(1)$ indicates the patient with the highest priority in $\mathcal{I}$, $f(2)$ indicates the patient with the second highest priority in $\mathcal{I}$, and so on.

- Given a set of patients $J \in \mathcal{I}$, the restriction of f to J is an ordering f_J of the patients in J which orders them as they are ordered in f.

- Each ordering $f \in \mathcal{F}$ defines a YRMD-IGYT mechanism.

– For any problem $\langle I, D, R\rangle$, let $\psi^f[R]$ denote the outcome of YRMD-IGYT mechanism induced by ordering f.

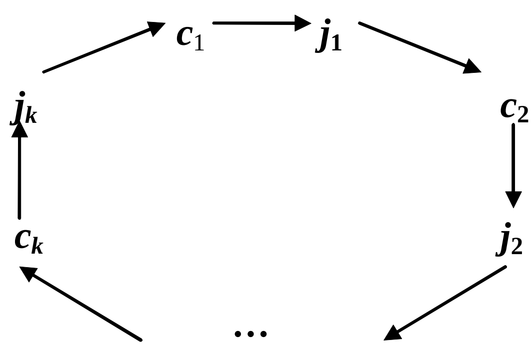
– Let $\psi^f[R_J^C]$ denote the outcome of the YRMD-IGYT mechanism induced by ordering f_J for problem $\left\langle J, C, R_J^C \right\rangle$.

For any problem $\langle I, D, R \rangle$, matching $\psi^f[R]$ is obtained with the following YRMD-IGYT algorithm in several rounds.

Round 1(a): Construct a graph in which each patient and each donor is a node. In this graph:

- each patient "points to" her top choice donor (i.e. there is a directed link from each patient to her top choice donor),
- each paired-donor $d_i \in D$ points to her paired-patient i in case $i \in I$, and to the highest priority patient in I otherwise,
- and each GS-donor points to the patient with the highest priority in I.

Define: a cycle is an ordered list $(c_1, j_1, \ldots, c_k, j_k)$ of donors and patients where donor c_1 points to patient j_1, patient j_1 points to donor c_2, donor c_2 points to patient j_2, ..., donor c_k points to patient j_k, and patient j_k points to donor c_1.



Since there is a finite number of patients and donors, there is at least one cycle. If there is no cycle *without a GS-donor* then skip to Round 1(b). Otherwise consider each cycle without a GS-donor. (Observe that if there is more than one such cycle, they do not intersect.) Assign each patient in such a cycle the donor she points to and remove each such cycle from the graph. Construct a new graph with the remaining patients and donors such that

- each remaining patient points to her first choice among the remaining donors,

- each remaining paired-donor $d_i \in D$ points to her paired-patient i in case her paired patient i remains in the problem, and to the highest priority remaining patient otherwise,

- and each GS-donor points to the highest priority remaining patient.

There is a cycle. If there is no cycle without a GS-donor then skip to Round 1(b); otherwise carry out the implied exchange in each such cycle and proceed similarly until either no patient is left or there exists no cycle without a GS-donor.

Round 1(b): There is a unique cycle in the graph, and it includes both the highest priority patient among remaining patients and a GS-donor. Assign each patient in such a cycle the donor she points to and remove each such cycle from the graph. Proceed with Round 2.

In general, at

Round t(a): Construct a new graph with the remaining patients and donors such that

- each remaining patient points to her first choice among the remaining donors,

- each remaining paired-donor $d_i \in D$ points to her paired-patient i in case her paired patient i remains in the problem, and to the highest priority remaining patient otherwise,

- and each remaining GS-donor points to the highest priority remaining patient.

There is a cycle. If the only remaining cycle includes either a GS-donor or a paired-donor whose paired-patient has left, then skip to Round t(b); otherwise carry out the implied exchange in each such cycle and proceed similarly until either no patient is left or the only remaining cycle includes either a GS-donor or a paired-donor whose paired-patient has left.

Round t(b): There is a unique cycle in the graph, and it includes the highest priority patient among remaining patients and either a GS-donor or a paired-donor whose paired-patient has left. Assign each patient in such a cycle the donor she points to and remove each such cycle from the graph. Proceed with Round t+1.

The algorithm terminates when there is no patient left in the graph.

9 Characterization of the YRMD-IGYT Mechanisms

Our main result is a characterization of the YRMD-IGYT mechanism:

Theorem 1: A mechanism is *Pareto efficient*, *individually rational*, *strategy-proof*, *weakly neutral*, and *consistent* if and only if it is a YRMD-IGYT mechanism.

We present our main result through two propositions:

Proposition 1: For any ordering $f \in \mathcal{F}$, the induced YRMD-IGYT mechanism ψ^f is *Pareto efficient*, *individually rational*, *strategy-proof*, *weakly neutral* and *consistent*.

Proposition 2: Let ϕ be a *Pareto efficient*, *individually rational*, *strategy-proof*, *weakly neutral,* and *consistent* mechanism. Then $\phi = \psi^f$ for some $f \in \mathcal{F}$.

Sketch of Proof of Proposition 2:

- Construct f as follows: Let $d_{gs} \in \mathcal{D}$ be a GS-donor.

  – Construct R^1 as follows

R_1^1	R_2^1	$\cdots$	$\cdots$	R_n^1
d_{gs}	d_{gs}			d_{gs}
d_1	d_2			d_n
$\vdots$	$\vdots$			$\vdots$

  Pareto efficiency of $\phi \Rightarrow$ for some i, $\phi\left[R^1\right](i) = d_{gs}$. Let $f(1) = i$.

  – Construct R^2 as follows:

$R_{f(1)}^2$	R_1^2	R_2^2	$\cdots$	R_n^2
$d_{f(1)}$	d_{gs}	d_{gs}		d_{gs}
$\vdots$	d_1	d_2		d_n
	$\vdots$	$\vdots$		$\vdots$

  Individual rationality of $\phi \Rightarrow \phi\left[R^2\right](f(1)) = d_{gs}$.

  Pareto efficiency of $\phi \Rightarrow$ for some $i \neq f(1)$, $\phi\left[R^2\right](i) = d_{gs}$. Let $f(2) = i$.

– similarly construct R^3 by changing $f(2)$'s preferences so that only $d_{f(2)}$ is acceptable. We continue similarly... This gives a unique ordering f.

- Let $R \in \mathcal{R}(D)^{|I|}$ for $I \subseteq \mathcal{I}$ and $D \subseteq \mathcal{D}$. We will prove that $\psi^f[R] = \phi[R]$.

- To prove this result we construct an interim preference profile R' using R. Use YRMD-IGYT algorithm to construct $\psi^f[R]$.

  – Let A^t be the patients removed in round t(a) for any t.

  – Let B^t be the patients removed in round t(b) for any t.

- R'_i is constructed in two different ways for a patient $i \in I$ depending on how she leaves the algorithm. Suppose she leaves the algorithm in round t Two cases are possible: She leaves

  1. (i) in round t(a) or (ii) in round t(b) and she is not the highest priority patient in this cycle.

  2. in round t(b) and she is the highest priority patient in this cycle

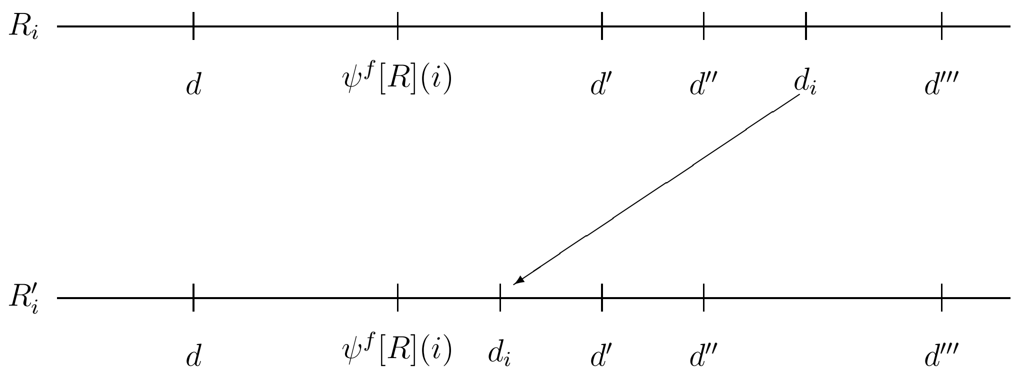


Figure 1: Construction of Preference R'_i for Case 1

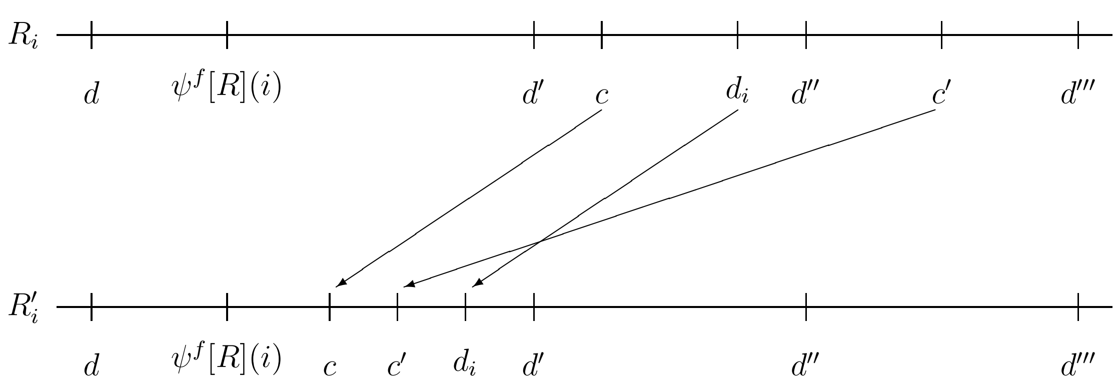


Figure 2: Construction of Preference R'_i for Case 2 when $\psi^f[R](B^t) = \left\{\psi^f[R](i), c, c'\right\}$

By construction, $\psi^f[R'] = \psi^f[R]$. We will prove four claims that will facilitate the proof of Proposition 2.

We consider the patients in A^1 in the first two claims.

Claim 1: For any $\hat{R}_{-A^1} \in \mathcal{R}^{|I \setminus A^1|}$ and $i \in A^1$, we have $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](i) = \psi^f[R](i)$.

The proof uses *individual rationality* and *Pareto efficiency* of ϕ.

Claim 2: For any $\hat{R}_{-A^1} \in \mathcal{R}^{|I \setminus A^1|}$, and any $i \in A^1$, we have $\phi\left[R_{A^1}, \hat{R}_{-A^1}\right](i) = \psi^f[R](i)$.

The proof uses Claim 1, *strategy-proofness* in addition to *individual rationality* and *Pareto efficiency* of ϕ.

We consider the patients in B^1 in the next two claims.

Claim 3: $\phi\left[R'_{B^1}, R_{-B^1}\right](i) = \psi^f[R](i)$ for all $i \in B^1$.

The proof uses Claim 2, *consistency* and *weak neutrality* in addition to *strategy-proofness*, *individual rationality* and *Pareto efficiency* of ϕ.

Claim 4: $\phi[R](i) = \psi^f[R](i)$ for all $i \in B^1$.

The proof uses Claims 2 and 3, *strategy-proofness, consistency*, and *individual rationality* of ϕ.

For the rest of the patients, we use *consistency* of ϕ and the above 4 claims.

By Claim 2 and Claim 4,

$$\phi[R](i) = \psi^f[R](i) \qquad \text{for all } i \in A^1 \cup B^1.$$

By invoking *consistency*, we can remove patients in $A^1 \cup B^1$ and their assigned donors and we can similarly prove

$$\phi[R](i) = \psi^f[R](i) \qquad \text{for all } i \in A^2 \cup B^2.$$

Iteratively we continue to prove that

$$\phi[R] = \psi^f[R].$$

10 Independence of the Axioms

The following examples establish the independence of the axioms.

Example 1: Individually rational, *strategy-proof*, *weakly neutral* and *consistent* but not *Pareto efficient mechanism:* Let mechanism ϕ assign each patient $i \in I$ her paired-donor d_i for each problem $\langle I, D, R \rangle$.

Example 2: Pareto efficient, *strategy-proof*, *weakly neutral* and *consistent* but not *individually rational mechanism:* Fix an ordering $f \in \mathcal{F}$ and let mechanism ϕ be the *serial dictatorship induced by* f.

Example 3: Pareto efficient, *individually rational*, *weakly neutral* and *consistent* but not *strategy-proof mechanism:* Fix an ordering $f \in \mathcal{F}$. Let $g \in \mathcal{F}$ be constructed from f by demoting patient $f(1)$ to the very end of the ordering. For any problem $\langle I, D, R \rangle$, let

$$\phi[R] = \begin{cases} \psi^g[R] & \text{if } dR_i d_{f(1)} \text{ for all } i \in I \text{ and } d \in D, \\ \psi^f[R] & \text{if otherwise.} \end{cases}$$

Example 4: Pareto efficient, *individually rational*, *strategy-proof*, and *consistent* but not *weakly neutral mechanism*:

Let $\mathcal{I}, \mathcal{D}$ be such that $|\mathcal{I}| \geq 2$ and $|\mathcal{D}| \geq |\mathcal{I}| + 2$. Let $i_1, i_2 \in \mathcal{I}$ and $d^* \in \mathcal{D} \setminus \{d_i\}_{i \in \mathcal{I}}$. Let $f, g \in \mathcal{F}$ be such that $f(1) = g(2) = i_1$, $f(2) = g(1) = i_2$ and $f(i) = g(i)$ for all $i \in \mathcal{I} \setminus \{i_1, i_2\}$. For any problem $\langle I, D, R \rangle$, let

$$\phi[R] = \begin{cases} \psi^f[R] & \text{if } i_1 \in I,\ d^* \in D \text{ and} \\ & d^* R_{i_1} d \text{ for all } d \in D \setminus \{d_i\}_{i \in I} \\ \psi^g[R] & \text{if otherwise.} \end{cases}$$

Example 5: Pareto efficient, *individually rational*, *strategy-proof*, and *weakly neutral* but not *consistent mechanism:*

Let $f, g \in \mathcal{F}$ be such that $f \neq g$. For any problem $\langle I, D, R \rangle$, let

$$\phi[R] = \begin{cases} \psi^f[R] & \text{if there are odd number of GS-donors,} \\ \psi^g[R] & \text{if there are even number of GS-donors.} \end{cases}$$

11 Conclusions

- The result can be generalized to a setting in which the deceased donor waiting patients (without any paired donors) are also explicitly modeled. (A similar domain with house allocation existing tenants problem).

- New England Program for Kidney Exchange (NEPKE) has started to integrate GS donations with paired exchanges.