Stochastic Private Values in Auctions: Identification and Estimation*

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Abstract

This paper addresses the identification and estimation of the stochastic private value model for first-price sealed-bid auctions. Under a constant absolute risk aversion specification for the bidders' utility function, the model is semiparametrically identified from the winner's ex post private value and the winning bid. A semiparametric method is proposed for estimating the risk aversion parameter, the risk premium, and the distributions of the ex ante private signal and the ex post shock. The semiparametric estimator for the risk aversion parameter and the risk premium converges at the parametric rate. A Monte Carlo study confirms its small sample good behavior.

Keyword: Stochastic private value; Risk aversion; First-price auction; Semiparametric identification, Semiparametric estimation

JEL classifications: D44; C14

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1 Introduction

Many auction situations suggest that bidders are uncertain about the ex post value of the auctioned object when forming their bids. Examples include construction procurements and natural resource auctions among others.[1] In construction procurement, unpredictable weather conditions may affect the cost for performing the contract. Given the length of the contract, firms may also face important variations in raw material input prices. Both may contribute to a significant change in the firm's costs. In online auctions, bidders do not perfectly know the quality of the auctioned object due to the lack of information or assessment.

Risk aversion is a widely adopted assumption in economic models. While considering auction models, Maskin and Riley (1984) consider the optimal design when the bidders are risk averse. Matthews (1987) compares different auction designs with risk averse bidders from a buyer's point of view. McAfee and McMillan (1987) consider an auction with a stochastic number of risk averse bidders. Experimental studies have provided support for bidders' risk aversion.[2] Athey and Levin (2001), Campo, Guerre, Perrigne and Vuong (2003) and Perrigne (2003) provide empirical evidence for risk-averse bidders using timber auction data.

Eso and White (2003) consider auction models with uncertain ex post values and risk averse bidders and compare different auction formats from the point of view of both

[1] Athey and Levin (2001) consider the US forest service auctions in which the uncertain final payment in these auctions is related to the quantity of timber harvested, which is imperfectly known at the time of bidding.

[2] Cox, Smith and Walker (1988) use a constant relative risk aversion (CRRA) utility function to explain the observed overbidding relative to the risk neutral Nash equilibrium. Cox, Roberson and Smith (1982) reject the risk neutral Nash equilibrium bidding behavior in favor of a CRRA model. While considering a quantal response equilibrium, Goeree, Holt and Palfrey (2002) provide some evidence for risk averse bidders. See Kagel (1995) for a survey of this literature.

auctioneer and bidders. In a stochastic private value (SPV) framework they put forward, each bidder receives an (ex ante) signal about his ex post private value for the auctioned object. The (ex post) private value is modeled as the sum of the (ex ante) private signal and a zero mean random shock. At the time of bidding, each bidder knows his private signal, his utility function as well as the private signal distribution and the random shock distribution. The bidders are symmetric in the sense that they share the same risk attitude and the same distributions of signals and random shocks. Moreover, it is assumed that private signals and random shocks are all independent.

This paper extends the structural auction literature by addressing the identification and estimation of the stochastic private value model as described above in a first-price sealed-bid auction setting.[3] Campo, Guerre, Perrigne and Vuong (2003) consider the identification and estimation of a standard private value auction model with independent private values and risk averse bidders. They show that the model is in general not identified from the observed bids only and that identification can be achieved through additional parametric restrictions. Following their semiparametric identification result, they propose a semiparametric estimator, which has a non standard consistency rate. My paper considers a more general model with stochastic private values. In particular, the model that I consider encompasses the deterministic case considered by Campo, Guerre, Perrigne and Vuong (2003) in which there is no ex post risk. Nonetheless, identification of the SPV model is achieved through exploiting the observability of the winner's ex post value. Moreover, the semiparametric estimator converges at the standard $\sqrt{N}$ rate.

The structure of the SPV model is defined as the bidders' utility function, the private signal distribution and the ex post shock distribution. It is assumed that the utility function is concave, i.e. bidders are risk averse. Identification consists in recovering uniquely the structure of the model from observations. In a first-price sealed-bid auction, one

[3]It will be shown that the same identification result also holds for the descending auction setting.

usually observes the bids and the number of bidders. It is clear that the SPV model is not identified from the bids only since these observations do not provide enough information to recover the utility function and two distributions nonparametrically. A first natural restriction is to parameterize the utility function. A constant absolute risk aversion (CARA) specification is chosen for its mathematical simplicity. As a matter of fact, it leads to a constant risk premium. This parametric restriction is insufficient to achieve identification of the model. In particular, additional restrictions such as parametric conditional quantiles as in Campo, Guerre, Perrigne and Vuong (2003) will at most allow us to identify the private signal distribution up to a shift equal to the risk premium. The ex post shock distribution remains unidentified even if the risk premium can be identified, because the same risk premium can result from different shock distributions for a given CARA function. That is why more information is needed to identify the distribution of the random shock. The ex post private value of the winning bidder conveys information on both the ex post shock and the private signal. The observability of the winner's ex post private value together with the winning bid allows us to identify the CARA risk aversion parameter, the private signal distribution and the shock distribution without making additional parametric restrictions. Since only information about the winning bid and winner's ex post private value is required, this identification result also holds for the descending auction setting. This identification result relies on convolution theorem. First, the distribution of the winner's ex ante private signal is identified, as the winner's ex ante signal is the expectation of the winner's ex post private value conditional on the winning bid. Then the distribution of a typical bidder's private signal is identified, since the private signals of the bidders are i.i.d. distributed and the winner's ex ante private signal is the first order statistics of all the private signals of the bidders. Second, since the winner's ex post private value is the sum of his ex ante private signal and the ex post shock, and the private signal is independent of the ex post shock, the ex post shock distribution is then identified as a result of the convolution theorem. Moreover, the identification of the ex ante private signal and ex post shock distributions does not rely on the specification of the bidders' utility function. The risk aversion parameter and the risk premium can then be identified through considering the inverse bidding function.

The semiparametric identification result naturally leads to a semiparametric estimation procedure, which will be conducted in three steps. Using the winners' ex post values and bids, the first step consists in estimating the risk aversion parameter and the risk premium using a nonstandard nonlinear least square estimator. The dependent variable is the winner's ex post private value, the explanatory variables are the winning bid and a nonparametric estimator of the ratio of winning bid distribution function over its density. The ratio is evaluated at the observed winning bids. The asymptotic distribution of the estimator in this step is derived. The uncertainties in the private value can thus be examined by testing whether the risk premium equals to zero. In particular, the standard parametric rate $\sqrt{N}$ is achieved in spite of the nonparametric component involved in the estimation cannot be estimated at the rate of $\sqrt{N}$. This result is easily understood as the method suggested fits into the framework of Newey and Mcfadden (1994). This is the first time in the structural empirical auction literature, the parametric rate is achieved in estimating the risk aversion parameter. Using the estimated values for the risk aversion parameter and the risk premium, the second and third step recover respectively the pseudo values for winners' private ex ante signals and ex post shocks through the inverse bidding function. These pseudo values are used to estimate nonparametrically the distributions of the private signal and ex post shock.

Since the standard parametric rate $\sqrt{N}$ is achieved to estimate risk aversion parameter and the risk premium. It is expected that the estimation procedure can provide satisfying results with a relatively small sample size. I then conduct a Monte Carlo study with simulated auction data. The results show the good behavior of the semiparametric estimator.

My paper contributes to the structural analysis of auction data by extending the structural approach to a stochastic private value framework. Specifically, it provides useful tools to analyze auctions in which uncertainty on the ex post value of the auctioned object is important such as in procurement and natural resource auctions among others. As such, some auction data could be reanalyzed in this perspective. Though the observability of the winner's ex post value may be restrictive, it is required for the winner only. As a matter of fact, some data such as the US Forest Service timber auction data provides some information on the amount of harvested timber, which could be used to assess the ex post private value.

The paper is organized as follows. A second section is devoted to the identification of the SPV model after a brief introduction of the model. A third section presents the semiparametric estimation procedure as well as its asymptotic properties. A fourth section contains the simulation exercise, while a fifth section concludes. The appendix contains all the proofs.

2 Identification

This section presents the identification of the SPV model under a CARA utility specification. I first briefly introduce the symmetric SPV model with risk averse bidders for first-price sealed-bid auctions.

2.1 The SPV Model

A single and indivisible object is sold through a first-price sealed-bid auction. All sealed bids are collected simultaneously. The object is sold to the highest bidder who pays his bid. Within the SPV paradigm, each bidder is assumed to have an (ex ante) private signal about his random ex post private value. The signal is equal to the expectation of the random ex post private value. The signal is denoted by v_i, which is private information across bidders. The v_is are i.i.d. distributed with a density $f(\cdot)$ and a cumulative distribution function $F(\cdot)$ both defined on the support $[\underline{v}, \overline{v}]$. Bidder i's ex post private value is denoted by $\tilde{v}_i$. For any $i \in \{1, 2, ..., n\}$, $\tilde{v}_i = v_i + \varepsilon_i$, where n is the number of bidders in the auction and the ε_is are i.i.d. zero-mean ex post shocks with a density $h(\cdot)$ and a cumulative distribution function $H(\cdot)$. The shocks ε_i are assumed to be independent of the signals v_j, $\forall i, j \in \{1, 2, ..., n\}$. Both distributions $F(\cdot)$ and $H(\cdot)$ are common knowledge across bidders. Bidders are assumed to be risk averse and to evaluate their monetary gain from the auction by a strictly concave utility function.[4] Bidders is assumed to have initial wealth denoted by w. Their von Neumann Morgenstern utility function is then $U_{vNM}(\cdot + w)$. If bidder i wins the auction, his monetary gain is $\tilde{v}_i - b_i$, while $\tilde{v}_i$ is unknown at the time of the auction. If he loses the auction, his gain is zero.

A bidder i with a signal v_i and a bid b_i has an expected utility equal to $\mathrm{E}_{\varepsilon_i} U_{vNM}(w + v_i + \varepsilon_i - b_i)\Pr(b_i \geq b_j, j \neq i) + U_{vNM}(w)(1 - \Pr(b_i \geq b_j, j \neq i))$. As usual, I consider strictly increasing and symmetric equilibrium bidding strategies denoted by $b(\cdot)$. With independent private signals, the probability of winning the auction reduces to $F^{n-1}(v_i)$. I define $U(\cdot) = U_{vNM}(w + \cdot) - U_{vNM}(w)$. Note that $U(0) = 0$. Bidder i chooses his report $\hat{v}_i$ to maximize his expected utility in the following problem

$$\max_{\hat{v}_i \in [\underline{v}, \overline{v}]} F^{n-1}(\hat{v}_i)\mathrm{E}_{\varepsilon_i} U(v_i + \varepsilon_i - b(\hat{v}_i)).$$

Differentiating with respect to $\hat{v}_i$ and requiring $\hat{v}_i = v_i$ at the Bayesian Nash equilibrium, I obtain the Bayesian Nash equilibrium strategy $b(\cdot, U, F, H, n)$, which satisfies the following differential equation

$$1 = (n-1)\frac{f(v_i)}{F(v_i)}\lambda(v_i - b(v_i))\frac{1}{b'(v_i)}, \tag{1}$$

[4]The model can be easily extended to accommodate the risk neutrality case.

where $\lambda(x) = \mathrm{E}_{\varepsilon_i} U(x + \varepsilon_i)/(d\mathrm{E}_{\varepsilon_i} U(x + \varepsilon_i)/dx)$.

Differential equation (1) with the boundary condition $b(\underline{v}) = \underline{v} - \pi$ determines the equilibrium bidding strategy $b(\cdot)$, where π satisfies $\mathrm{E}_{\varepsilon_i} U(\pi + \varepsilon_i) = U(0) = 0$.[5] The risk premium π is the amount to be given to any bidder with a utility function $U(\cdot)$ in order for him to take the risk of ε_i if his initial wealth were equal to 0. Thus π depends on $U(\cdot)$ and $H(\cdot)$. When $\varepsilon_i \equiv 0$, the SPV model degenerates to the deterministic private value model considered by Campo, Guerre, Perrigne and Vuong (2003). Hereafter, I define $[U(\cdot), F(\cdot), H(\cdot)]$ as the structure of the SPV model.

2.2 Identification Under a CARA Specification

This section addresses the identification problem of the structure $[U(\cdot), F(\cdot), H(\cdot)]$ from observations. Generally in first-price sealed-bid auctions, the number of bidders n and their bids $b_i, i = 1, \ldots, n$ are observed. Because signals are random, bids are also random and distributed as $G(\cdot)$. First, let us consider whether the structure $[U(\cdot), F(\cdot), H(\cdot)]$ can be recovered uniquely from the knowledge of $(n, G(\cdot))$. Following Campo, Guerre, Perrigne and Vuong (2003) who consider a simpler structure $[U(\cdot), F(\cdot)]$, it is clear that the SPV model is not identified from $(n, G(\cdot))$ even when the bidders' utility function is parameterized as CARA or CRRA. Hereafter, I consider the identification of the model under the CARA specification for the bidders' utility function. The CARA utility function has been frequently adopted when modeling risk aversion in economic models. I consider the CARA utility function for its mathematical simplicity.[6] Under a CARA utility specification, the effect of the ex post risk in the SPV model is reduced to the introduction of a risk premium.

[5] At the lower boundary, the competition drives the $\mathrm{E}_{\varepsilon_i} U(\underline{v} - b(\underline{v}) + \varepsilon_i) = U(0)$.

[6] I will show that the identification of the distributions of the ex ante private signal and ex post shock does not depend on the specification of the bidders' utility function.

Denote by $\mathcal{U}_{vNM}^{CARA}$ the set of all CARA utility functions. The Arrow-Pratt coefficient of absolute risk aversion at x is defined as $-U''_{vNM}(x)/U'_{vNM}(x)$. Therefore, the CARA utility functions takes the form $U_{vNM}(x) = \alpha\frac{1-\exp(-rx)}{r} + \beta$, $\forall r > 0$, $\forall \alpha > 0$, $\forall \beta \in I\!R$, where r is the measure of absolute risk aversion. This leads to $U(x) = \tilde{\alpha}(1 - \exp(-rx))$ with $\tilde{\alpha} = \alpha\frac{\exp(-rw)}{r}$. The following lemma provides the $\lambda(\cdot)$ function for a CARA utility function. Appendix A provides the proof of this result.

Lemma 1: *If* $U(\cdot)$ *is* $CARA(r)$ *and* ε *is a zero-mean random shock, then* $\lambda(\cdot) = (\exp(r(\cdot - \pi)) - 1)/r$, *where* π *is the constant risk premium defined by* $\mathrm{E}_\varepsilon U(\varepsilon + \pi) = U(0) = 0$ *in the boundary condition.*

Following Guerre, Perrigne and Vuong (2000), one can express the differential equation (1) using the equilibrium bid distribution G$(\cdot)$.[7] For every $b \in [b(\underline{v}), b(\overline{v})]$, I have $G(b) = F(b^{-1}(b)) = F(v)$ with a density $g(b) = f(v)/b'(v)$. Thus the differential equation (1) can be written equivalently as

$$1 = (n-1)\frac{g(b_i)}{G(b_i)}\lambda(v_i - b_i). \tag{2}$$

Because $\lambda'(\cdot) \geq 1$, $\lambda(\cdot)$ is strictly increasing. Thus solving (2) for v_i gives

$$\begin{aligned} v_i &= b_i + \lambda^{-1}\left(\frac{1}{n-1}\frac{G(b_i)}{g(b_i)}\right) \\ &= b_i + \pi(r, H) + \frac{1}{r}\log\left(1 + \frac{r}{n-1}\frac{G(b_i)}{g(b_i)}\right) \\ &= \xi(b_i, r, G, H, n), \end{aligned} \tag{3}$$

where $\lambda^{-1}(\cdot)$ denotes the inverse of $\lambda(\cdot)$. This equation gives each bidder's private value signal as a function of its corresponding bid, the bid distribution, the number of bidders, the shock distribution and the CARA risk aversion parameter.[8] Equation (3) tells us that

[7]Note that if only the winning bid is observed, one have $G(\cdot) = G^w(\cdot)^{1/n}$, where $G^w(\cdot)$ is the distribution of the winning bid.

[8]The boundary condition can be derived from (3) by considering the lower boundary.

the private signal will be larger by the amount of the risk premium $\pi(r, H)$ relative to the deterministic private value model with no ex post shock.

Note that the parameters α, β and w do not appear in (3) or in the boundary condition as they do not affect the bidding strategy. It follows naturally that these parameters cannot be identified.[9] In order to achieve identification of $\tilde{\alpha}$, I can normalize $U(\cdot)$ such that $U(1) = 1$, which gives $\tilde{\alpha} = 1/(1 - \exp(-r))$. Hereafter, I impose the following assumptions on the utility function $U(\cdot)$ and the distributions $F(\cdot)$ and $H(\cdot)$.

Definition 1: *The set $\mathcal{U}^{CARA}$ is defined as the set of CARA utility functions $U(\cdot)$ satisfying U(0)=0 and U(1)=1.*

Definition 2: *For $R \geq 1$, let $\mathcal{F}_R$ be the set of distribution functions $F(\cdot)$ satisfying*

(i) $F(\cdot)$ is a c.d.f. with support $[\underline{v}, \overline{v}]$, where $0 \leq \underline{v} < \overline{v} < \infty$,

(ii) $F(\cdot)$ admits $R + 1$ continuous derivatives on $[\underline{v}, \overline{v}]$,

(iii) $f(\cdot) = F'(\cdot) > 0$ on $[\underline{v}, \overline{v}]$,

(iv) The monotone hazard rate property holds i.e., $\frac{d\frac{F(v)}{f(v)}}{dv} > 0$.

Following Theorem 1 in Campo, Guerre, Perrigne and Vuong (2003), when $U(\cdot) \in \mathcal{U}^{CARA}$ and $F(\cdot) \in \mathcal{F}_R$, the equilibrium strategy $b(\cdot)$ admits $R + 1$ continuous derivatives on $[\underline{v}, \overline{v}]$. In addition, I have $b'(v) > 0$. The log concavity of $F(v)$ in item (iv) of Definition 2 ensures that $b'(v) < 1$, thus $v - b(v)$ is increasing wrt. v.[10]

[9] When the initial wealth varies across bidders, the bidding strategies remain symmetric across bidders for the same reason.

[10] First, I show that I must have $b'(v) \leq 1$. Note $b'(v) = (n-1)\frac{f(v)}{F(v)}\lambda(v - b(v))$. Taking limit on both sides when v goes to $\underline{v}$ leads to the result that $b'(\underline{v}) = \frac{n-1}{n} < 1$. Note also $b''(v) = \frac{1}{n-1}\frac{f(v)}{F(v)}\lambda'(v - b(v))(1 - b'(v)) + \frac{1}{n-1}\lambda(v - b(v))\frac{d^2 \log F(v)}{dv^2}$. It is easy to see that $b'(v)$ starts from $\frac{n-1}{n}$ at $\underline{v}$ and has no chance to go strictly above 1 as v increases because whenever it has a chance to reach 1 at a point of v^*, it has to drop below 1 since $b''(v^*) = \frac{1}{n-1}\lambda(v^* - b(v^*))\frac{d^2 \log F(v^*)}{dv^2} < 0$. Second, I show $b'(v) < 1$. Suppose there exists $v' \in (\underline{v}, \overline{v}]$, $b'(v') = 1$, then $b''(v') < 0$. Thus I must have $v'' < v'$ which gives $b'(v'') > 1$. But this is contradictory to $b'(v) \leq 1$ on $[\underline{v}, \overline{v}]$.

Definition 3: *For $R \geq 1$, let $\mathcal{H}_R$ be the set of c.d.f. functions $H(\cdot)$ satisfying*

(i) $H(\cdot)$ is a c.d.f. with support $[\underline{\epsilon}, \overline{\epsilon}]$, where $-\infty < \underline{\epsilon} \leq \overline{\epsilon} < \infty$,

(ii) $H(\cdot)$ has zero mean,

(iii) $H(\cdot)$ admits $R+1$ continuous derivatives on $[\underline{\epsilon}, \overline{\epsilon}]$,

(iii) $h(\cdot) = H'(\cdot) > 0$ on $[\underline{\epsilon}, \overline{\epsilon}]$.

Note that an additional restriction should be imposed on the structure $[U(\cdot), F(\cdot), H(\cdot)]$ since observation provides positive values for the bids. Define $(\mathcal{U}^{CARA} \times \mathcal{F}_R \times \mathcal{H}_R)^* = \left\{[U(\cdot), F(\cdot), H(\cdot)] \;\middle|\; \underline{v} - \pi(r, H) \geq 0, [U(\cdot), F(\cdot), H(\cdot)] \in \mathcal{U}^{CARA} \times \mathcal{F}_R \times \mathcal{H}_R\right\}$ as the set of structures leading to nonnegative bids.[11]

The equilibrium bid distribution $G(\cdot)$ then satisfies some regularity properties implied by the regularity assumptions on $[U, F]$ and the smoothness of the equilibrium bid strategy $b(\cdot)$. These regularity properties are summarized in the following definition.[12]

Definition 4: *For $R \geq 1$, let $\mathcal{G}_R$ be the set of distribution functions $G(\cdot)$ satisfying*

(i) $G(\cdot)$ is a c.d.f. with support $[\underline{b}, \overline{b}]$, where $0 \leq \underline{b} < \overline{b} < +\infty$,

(ii) $G(\cdot)$ admits R+1 continuous derivatives on $[\underline{b}, \overline{b}]$,

(iii) $g(\cdot) = G'(\cdot) > 0$ on $[\underline{b}, \overline{b}]$,

(iv) $G(\cdot)/g(\cdot)$ admits $R+1$ continuous derivatives on $[\underline{b}, \overline{b}]$.

(v) The monotone hazard rate property holds i.e., $\frac{d\frac{G(b)}{g(b)}}{db} > 0$.

Items (ii) and (iv) in Definition 4 imply that $g(\cdot)$ admits R+1 continuous derivatives on $(\underline{b}, \overline{b}]$, i.e. $g(\cdot)$ is smoother than $f(\cdot)$. Item (v) means that the bid distribution $G(b)$ must be log concave.

Additional restrictions such as those on parametric conditional quantiles as in Campo,

[11] Considering (3) at the lower bound $\underline{b}$ gives $\underline{v} = \underline{b} + \pi$. Assuming $\underline{b} \geq 0$ implies $\underline{v} - \pi \geq 0$. Note that $b'(v) > 0$.

[12] I do not provide a proof of the properties $(i) - (iv)$ as it is similar to Campo, Guerre, Perrigne and Vuong (2003). Property (v) holds because $\frac{G(b)}{g(b)} = \frac{F(v)}{f(v)} b'(v) = (n-1)\lambda(v - b(v))$.

Guerre, Perrigne and Vuong (2003) will at most allow us to identify the private signal distribution up to a shift equal to the risk premium. The ex post shock distribution remains unidentified nonparametrically from $(n, G(\cdot))$ only, even if the risk premium can be identified, because the same risk premium can result from different shock distributions for a given CARA function. That is why more information is needed to identify the distribution of the random shock. I then assume the observability of the winner's ex post private value. The ex post private value of the winning bidder conveys information on both the ex post shock and the private signal. Though this seems to be restrictive in practice, some auction data such as that on the timber and oil track auctions provide information that could be used to assess the winner's private value. Note that having the winner's (ex post) private value does not solve trivially the identification problem because the ex ante signal and ex post shock are not observed.

Hereafter, I assume that the joint distribution $J(\cdot,\cdot)$ of the equilibrium winning bid and the winning (ex post) private value is known, when considering the identification of he model.[13] The identification problem is whether the structure $[U, F, H] \in (\mathcal{U}^{CARA} \times \mathcal{F}_R \times \mathcal{H}_R)^*$ can be uniquely recovered from the knowledge of $J(\cdot,\cdot)$.

Let v^w denote the winner's private signal, ε^w his ex post shock, b^w his bid and $\tilde{v}^w$ his (ex post) private value. Since the bidding strategy $b(\cdot)$ is strictly increasing, I have

$$v^w = \max_{i=1,\dots,n} v_i, \qquad b^w = \max_{i=1,\dots,n} b_i = b(v^w), \qquad \tilde{v}^w = v^w + \varepsilon^w, \tag{4}$$

and

$$\tilde{v}^w = \sum_{i=1}^{n} \tilde{v}_i \mathbb{I}(v_j < v_i, \forall j \neq i), \text{if } v^w > v^{(n-1)}, \ \ \varepsilon^w = \sum_{i=1}^{n} \varepsilon_i \mathbb{I}(v_j < v_i, \forall j \neq i), \text{if } v^w > v^{(n-1)}, \tag{5}$$

where $v^{(n-1)}$ is the second highest private signal. Note that in contrast to that b^w and v^w are the maximum of b_i and v_i respectively, $\tilde{v}^w$ and ε^w are not the maximum of $\tilde{v}_i$ and ε_i

[13] The knowledge of $J(\cdot,\cdot)$ from observed winning bid and observed winning (ex post) private value is an estimation issue.

respectively. For the winner, (3) can then be written as

$$\begin{aligned} v^w &= b^w + \pi(r,H) + \frac{1}{r}\log\left(1 + \frac{r}{n-1}\frac{G(b^w)}{g(b^w)}\right) \\ &= \xi(b^w, r, G, H, n). \end{aligned} \tag{6}$$

Assuming that only the winning bid is observed, (6) can be written equivalently as

$$\begin{aligned} v^w &= b^w + \pi(r,H) + \frac{1}{r}\log\left(1 + \frac{n\ r}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) \\ &= \tilde{\xi}(b^w, r, G^w, H, n), \end{aligned} \tag{7}$$

since $\frac{G(\cdot)}{g(\cdot)} = n\frac{G^w(\cdot)}{g^w(\cdot)}$, where $G^w(\cdot)$ is the distribution of the winning bid and $g^w(\cdot)$ its corresponding density. As $\frac{G^w(b)}{g^w(b)} = \frac{G(b)}{g(b)}/n$, items (iv) and (v) in Definition 4 imply that $G^w(\cdot)$ is also log concave, and $g^w(\cdot)$ admits $R+1$ continuous derivatives on $(\underline{b}, \overline{b}]$.

Equation (7) with $\tilde{v}^w = v^w + \varepsilon^w$ gives

$$\begin{aligned} \varepsilon^w &= \tilde{v}^w - b^w - \pi(r,H) - \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) \\ &= \tilde{v}^w - \tilde{\xi}(b^w, r, G^w, H, n). \end{aligned} \tag{8}$$

Lemma 2 provides a result, which will be used further in the identification and estimation of the SPV model. The proof of Lemma 2 is given in Appendix A.

Lemma 2: *Suppose* $[U,F,H] \in \mathcal{U}^{CARA} \times \mathcal{F}_R \times \mathcal{H}_R$, *then* v^w *and* ε^w *are independent, and* ε^w *is distributed as* ε_i. *Furthermore,* b^w *and* ε^w *are independent.*

The observability of the winner's ex post private value together with the winning bid allows us to identify the CARA risk aversion parameter, the private signal distribution and the shock distribution without making additional parametric restrictions. Identification means that for a given equilibrium joint distribution of winning bid and winner's (ex post) private value $J(\cdot,\cdot)$, there exists a unique structure $[U(\cdot), F(\cdot), H(\cdot)] \in (\mathcal{U}^{CARA} \times \mathcal{F}_R \times \mathcal{H}_R)^*$ that leads to this joint distribution. This is the object of Proposition 1, which relies on the convolution theorem. Moreover, the identification of the ex ante private signal and ex post shock distributions does not rely on the specification of the bidders' utility function.

Proposition 1: *Let $n \geq 2$. Any structure $[U(\cdot), F(\cdot), H(\cdot)] \in (\mathcal{U}^{CARA} \times \mathcal{F}_R \times \mathcal{H}_R)^*$ is identified from the joint distribution of $J(b^w, \tilde{v}^w)$, where b^w is the equilibrium winning bid and $\tilde{v}^w$ is the winner's ex private value in a first-price sealed-bid auction with structure $[U(\cdot), F(\cdot), H(\cdot)]$.*

Proof of Proposition 1: Equation (8) implies

$$\tilde{v}^w = b^w + \pi(r, H) + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) + \varepsilon^w. \tag{9}$$

Because b^w and ε^w are independent as shown in Lemma 2, then

$$\begin{aligned}
\mathrm{E}(\tilde{v}^w|b^w) &= b^w + \pi(r, H) + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) + \mathrm{E}(\varepsilon^w|b^w) \\
&= b^w + \pi(r, H) + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) + \mathrm{E}(\varepsilon^w) \\
&= b^w + \pi(r, H) + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) \\
&= v^w.
\end{aligned} \tag{10}$$

Equation (10) implies that the distribution of v^w is identified as the distribution of $\mathrm{E}(\tilde{v}^w|b^w)$. Thus the distribution of private signals $F(\cdot)$ is identified because v^w is the first order statistics of all v_i and v_is are i.i.d. distributed. Moreover, since $\tilde{v}^w = v^w + \varepsilon^w$ and v^w, ε^w are independent with each other, the distribution of ε^w, which is also the distribution of ε_i, is identified as a result of convolution theorem. Note that the above identification results do not rely on the specification of the bidders' utility function.

Furthermore, $\pi(r, H)$ can be identified from the boundary condition

$$\pi(r, H) = \mathrm{E}(\tilde{v}^w|b^w = \underline{b}) - \underline{b}.$$

The risk aversion parameter can be identified from the following equation, as $(1/r)\log(1+$

$nr/((n-1)g^w(\overline{b}))$ is a strictly decreasing function with respect to r,

$$\mathrm{E}(\tilde{v}^w | b^w = \overline{b}) = \overline{b} + \pi(r,H) + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{1}{g^w(\overline{b})}\right).$$

Alternatively, the shock distribution $H(\cdot)$ can be identified as the distribution of ε^w in the following equation

$$\begin{aligned}\varepsilon^w &= \tilde{v}^w - v^w \\ &= \tilde{v}^w - b^w - \pi(r,H) - \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right).\end{aligned} \tag{11}$$

The following minimization problem which will be used for the estimation of r and π, provides an alternative way to identify the parameters r and $\pi(r,H)$ since it has a unique solution at the true value of the parameters. Namely,

$$(r,\pi(r,H)) = \mathrm{Argmin}_{(\tilde{r},\tilde{\pi})\in\Theta}\mathrm{E}_0\left(\tilde{v}^w - b^w - \tilde{\pi} - \frac{1}{\tilde{r}}\log\left(1 + \frac{n\tilde{r}}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right)^2, \tag{12}$$

where Θ is a compact set with the true value $(r,\pi(r,H))$ as an inner point. This is shown in the consistency proof of my estimator for $(r,\pi(r,H))$ in Appendix B. □

3 Estimation

3.1 General Procedure

I consider now L auctions selling similar objects to same number of bidders.[14] Following the identification result, I observe the winning bid and winner's ex post private value $(b_\ell^w, \tilde{v}_\ell^w), \ell = 1, ..., L$. The semiparametric identification naturally leads to a semiparametric estimation procedure, which will be conducted in three steps.

[14]Heterogeneity across auctioned objects could be easily introduced through a discrete variable. The method can be also extended to continuous variables.

Step 1: Equation (11), where the ratio $G^w(\cdot)/g^w(\cdot)$ is replaced by its nonparametric estimate from observed winning bids, is used to define a nonlinear least square estimator as in (12) to estimate the risk aversion parameter and the risk premium from the winners' ex post values and bids.

Step 2: Using the estimated values for the risk aversion parameter and the risk premium obtained in Step 1, winners' pseudo private signals can be computed using (10). The private signal density $f(\cdot)$ is estimated nonparametrically from these pseudo private signals.

Step 3: Using the estimated values for the risk aversion parameter and the risk premium obtained in Step 1, winners' pseudo ex post shocks can be computed using (11). The shock density $h(\cdot)$ is estimated nonparametrically using these pseudo shocks.

Specifically, from (10), I have

$$\mathrm{E}(\tilde{v}^w|b^w) = b^w + \pi(r,h) + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right),\ \forall\ b^w \in [\underline{b}, \overline{b}], \tag{13}$$

and from (9)

$$\tilde{v}^w_\ell = b^w_\ell + \pi(r,h) + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{G^w(b^w_\ell)}{g^w(b^w_\ell)}\right) + \varepsilon^w_\ell,\ \ \ell = 1, ..., L. \tag{14}$$

The ratio $G^w(\cdot)/g^w(\cdot)$ can be estimated nonparametrically from the observations b^w_ℓ, $\ell = 1, ..., L$.[15] With $G^w(\cdot)/g^w(\cdot)$ replaced by its estimate, (14) can be used to estimate r and $\pi(r,h)$ by a nonlinear least square (NLLS) estimator, which will be detailed below.

Using the observed $\tilde{v}^w_\ell$ and $b^w_\ell, \ell = 1, ..., L$, the pseudo signals v^w_ℓ can be estimated nonparametrically as

$$\hat{v}^w_\ell = \hat{\mathrm{E}}(\tilde{v}^w|b^w = b^w_\ell), \forall\ \ell = 1, ..., L, \tag{15}$$

[15] If all the bids $b_{i\ell}, i = 1, \ldots, n, \ell = 1, \ldots, L$ are observed, it is more efficient to use all these observations rather than the winning bid only to estimate $nG^w(\cdot)/g^w(\cdot) = G(\cdot)/g(\cdot)$.

where $\hat{\mathrm{E}}(\tilde{v}^w|b^w)$ is a nonparametric estimate for $\mathrm{E}(\tilde{v}^w|b^w)$. Alternatively, from (13), the winner's private signal $\hat{v}_\ell^w$ can be computed using the first step estimator $(\hat{r}, \hat{\pi}(r, H))$ as

$$\hat{v}_\ell^w = b_\ell^w + \hat{\pi}(r, h) + \frac{1}{\hat{r}} \log\left(1 + \frac{n\hat{r}}{n-1} \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}\right), \ \forall\ \ell = 1, ..., L. \tag{16}$$

Thus the private signal distribution can be nonparametrically estimated from the pseudo values $\hat{v}_\ell^w,\ \ell = 1, ..., L$. Since $f(\cdot) = f^w(\cdot)/(n[F^w(\cdot)]^{(n-1)/n})$, where $f(\cdot)$ is the marginal density for v_i, $F^w(\cdot)$ and $f^w(\cdot)$ are the distribution and density of v^w, respectively. A natural estimator for $f(v)$ is then $\hat{f}(v) = \hat{f}^w(\cdot)/(n[\hat{F}^w(\cdot)]^{(n-1)/n})$, where $\hat{F}^w(\cdot)$ and $\hat{f}^w(\cdot)$ are nonparametric estimators for $F^w(\cdot)$ and $f^w(\cdot)$ respectively, constructed using the pseudo signals $\hat{v}_\ell^w,\ \ell = 1, ..., L$.

The pseudo shocks $\hat{\varepsilon}_\ell^w$ can be estimated as

$$\hat{\varepsilon}_\ell^w \ = \ \tilde{v}_\ell^w - \hat{v}_\ell^w, \forall\ \ell = 1, ..., L. \tag{17}$$

The distribution of ε^w, which is also the distribution of ε_i, can then be nonparametrically estimated from $\hat{\varepsilon}_\ell^w,\ \ell = 1, ..., L$.

Note that there are at least two restrictions that can be used to test the model. First, $\mathrm{E}(\tilde{v}^w|b^w = b_\ell^w)$ can be compared with $b_\ell^w + \hat{\pi}(r, h) + (1/\hat{r}) \log(1 + n\hat{r}\hat{G}^w(b_\ell^w)/((n-1)\hat{g}^w(b_\ell^w)))$ as they are two different estimates for v_ℓ^w. Second, $\hat{\pi}(r, h)$ can be compared with $\pi(\hat{r}, \hat{h})$ as they are two different estimates for the risk premium.

3.2 Asymptotic Properties for the Estimators for r and $\pi(r, H)$

Equation (11) suggests a NLLS estimator for (r_0, π_0) which maximizes

$$\frac{1}{L} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - b_\ell^w - \pi - \frac{1}{r} \log\left(1 + \frac{nr}{n-1} \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}\right)\right)^2,$$

where (r_0, π_0) is the true value for $(r, \pi(r, H))$. As $G^w(\cdot)/g^w(\cdot)$ is unknown, this estimator is infeasible. Thus I need to replace $G^w(\cdot)/g^w(\cdot)$ by its estimate obtained from a nonparametric estimator.

For estimating the ratio $G^w(\cdot)/g^w(\cdot)$, I propose to use the following kernel estimators following Guerre, Perrigne and Vuong (2000)

$$\hat{G}^w(b^w) = \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}(b_\ell^w \leq b^w), \tag{18}$$

$$\hat{g}^w(b^w) = \frac{1}{L}\sum_{\ell=1}^{L} K_h(b_\ell^w - b^w), \tag{19}$$

where $K_h(\cdot) = (1/h)K(\cdot/h)$ with $K(\cdot)$ a kernel function and h a bandwidth. Note that the estimator for $G^w(\cdot)$ is a simple counting process.

The nonparametric kernel estimator of a density is known to suffer from the boundary effect, which consists in a bias close to the boundaries. A convenient method to correct this problem is to introduce a trimming. As a matter of fact, adopting a weight function is equivalent to doing a trimming, which takes care of the boundary effect of the nonparametric estimator $\hat{g}^w(\cdot)$.

The previous discussion eventually leads to using the following method relying on a NLLS estimator to estimate the risk aversion parameter r and the risk premium π. In particular,

$$(\hat{r}_L, \hat{\pi}_L) = \text{Argmin}_{(r,\pi)\in\Theta} \hat{Q}_L(r,\pi), \tag{20}$$

where $\Theta \subset (0,+\infty) \times [0,+\infty)$ is a compact set containing the true value $r_0 > 0, \pi_0 \geq 0$. The function $\hat{Q}_L(r,\pi)$ is defined as

$$\hat{Q}_L(r,\pi) = \frac{1}{L}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w - b_\ell^w - \pi - \frac{1}{r}\log\left(1 + \frac{nr}{n-1}\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}\right)\right)^2 w(b_\ell^w), \tag{21}$$

where $w(\cdot)$ is a weight on $(-\infty,+\infty)$ taking strictly positive values on $(\underline{b}^*, \overline{b}^*)$ and a zero value elsewhere. I restrict $w(\cdot)$ to have bounded $(R+1)$th derivative. Here $[\underline{b}^*, \overline{b}^*]$ can be any subset of $(\underline{b}, \overline{b})$. Any interval $[\underline{b}^*, \overline{b}^*] \subset (\underline{b}, \overline{b})$ guarantees the identification of r and π as shown in Appendix B.

3.2.1 Consistency

This section addresses the consistency of the estimator $(\hat{r}_L, \hat{\pi}_L)$ suggested above. I first need to make some assumptions.

Assumption 1: $\mathrm{E}_0(\tilde{v}^w)^2 < \infty$.

Assumption 2: $\sup_{b^w \in [\underline{b}^*, \overline{b}^*]} \left| \frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)} - \frac{G^w(b^w)}{g^w(b^w)} \right| = O(\frac{1}{r_L}),$ *a.s., where* $r_L \to \infty$.

Note that Assumption 1 is a standard moment condition, and Assumption 2 is always satisfied for the estimators defined in (18) and (19) as indicated in the following Lemma 3 under appropriate choice of kernel function and bandwidth. Under Assumptions 1 and 2, I have the following consistency result, whose proof is given in Appendix B.

Proposition 2: *Under Assumptions 1 and 2, the NLLS estimator as defined in (20) and (21) for* r *and* $\pi(r, H)$ *is consistent.*

3.2.2 Asymptotic Normality

Before presenting the asymptotic normality result, I need to address the consistency rate for $(\hat{r}_L, \hat{\pi}_L)$ and to make the following assumptions on the kernel function $K(\cdot)$ and the bandwidth h.

Assumption 3: *Let* $R \geq 1$. *Suppose*

(i) $K(\cdot)$ *is a* $(R{+}1)$*th-order kernel on a compact subset of* $I\!R$, *i.e.,* $\int K(u)du = 1$, $\int K(u)u^s du = 0$, $\forall\ 1 \leq s \leq R$, *and* $\int K(u)u^{R+1}du$ *is finite. Moreover,* $\sup_{u \in I\!R} |K(u)|$ *and* $\int K^2(u)du$ *are finite.*

(ii) $Lh^2 \to \infty$, $Lh^{2(R+1)} \to 0$ *and* $L\Big(h^{R+1} + \frac{1}{\sqrt{\tilde{L}h}}\Big)^4 \to 0$ *as* $L \to \infty$, *where* $\tilde{L} = \frac{L}{\log L}$.

Note that $h = (\log L/L)^k$ with $k \in (1/(2R+2), 1/2)$ satisfies Assumption 3(ii). Since $k > 1/(2R+2) > 1/(2R+3)$, h is less than $h^* = (\log L/L)^{1/(2R+3)}$, where h^* is the standard optimal bandwidth. Thus Assumption 3(ii) requires undersmoothing as usually required for a semiparametric estimator to achieve the $\sqrt{N}$ consistency.

From Lemma B2 in Guerre, Perrigne and Vuong (2000), I have the following lemma as $g^w(b^w)$ is bounded away from zero on any support of the form $[\underline{b}+\delta, \overline{b}]$, $\forall\, \delta \in (0, \overline{b}-\underline{b})$.

Lemma 3: *Under Assumption 3,*

$$\sup_{b^w \in [\underline{b}, \overline{b}]} \left(|\hat{G}^w(b^w) - G^w(b^w)|\ w(b^w) \right) = O(r_{G^w}^{-1}),\ a.s. \tag{22}$$

where $r_{G^w} = \left(\frac{L}{\log L}\right)^{\frac{1}{2}}$, *and*

$$\sup_{b^w \in [\underline{b}, \overline{b}]} \left(|\hat{g}^w(b^w) - g^w(b^w)|\ w(b^w) \right) = O(r_{g^w}^{-1}) = o(L^{-\frac{1}{4}}),\ a.s., \tag{23}$$

where $r_{g^w} = \left(h^{R+1} + \frac{1}{\sqrt{\tilde{L}h}}\right)^{-1}$ *where* $\tilde{L} = \frac{L}{\log L}$.

From Lemma 3, Assumption 3 implies Assumption 2 for the nonparametric estimators (18) and (19). Thus under Assumptions 1 and 3, the NLLS estimator as defined in (20) and (21) for r and $\pi(r, H)$ is consistent following Proposition 2.

Relying on the projection theorem of the U-statistics in Serfling (1980) and Lemma 3.1 in Powell, Stock and Stoker (1989), I can show the asymptotic normality of the semi-parametric estimators for r and $\pi(r, H)$. I need first to introduce some new notations. Namely, $\theta = (r, \pi)$ and $\varphi(\cdot,\cdot,\cdot)$ is a function on $[\underline{b}, \overline{b}] \times [0, +\infty] \times \Theta$ taking values in $[\underline{b}, +\infty]$ defined as $\varphi(b, x, \theta) = b + \pi + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}x\right)$ as (B.1) in Appendix B.

Proposition 3: *Under Assumptions 1 and 3,* $\sqrt{L}(\hat{\theta}_L - \theta_0) \xrightarrow{d} \mathcal{N}\left(0, \Omega\right)$ *for the NLLS estimator defined in (20), where the nonparametric estimators* $\hat{G}^w(b^w)$ *and* $\hat{g}^w(b^w)$ *are defined in (18) and (19). Here,* $\Omega = A^{-1}BA^{-1}$ *and*

$$\begin{aligned}
A &= \mathrm{E}_0\left[\frac{\partial\varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial\theta}\frac{\partial\varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial\theta'} w(b^w)\right], \\
B &= \mathrm{Var}_0\left\{ w(b^w)\left[\tilde{v}^w - \varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right) - \frac{G^w(b^w)}{g^w(b^w)}\frac{\partial\varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial x}\right]\right. \\
&\quad \left. \frac{\partial\varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial\theta} + \int_{b^w}^{\overline{b}} \frac{\partial\varphi\left(t, \frac{G^w(t)}{g^w(t)}, \theta_0\right)}{\partial x}\frac{\partial\varphi\left(t, \frac{G^w(t)}{g^w(t)}, \theta_0\right)}{\partial\theta} w(t)dt \right\}.
\end{aligned}$$

The proof of Proposition 3 is given in Appendix B. Proposition 3 says that the semiparametric estimators for r and $\pi(r, H)$ achieve the parametric consistency rate, namely $\sqrt{L}$. This result follows the semiparametric literature as the model that I consider fits into the general framework considered by Newey and McFadden (1994, Section 8). The reason why the parametric rate is achieved, while it is not in Campo, Guerre, Perrigne and Vuong (2003) lies in the observability of the winner's ex post private value assumed in this paper. Since Proposition 3 shows the asymptotic normality of the estimator for (r, π) and provides an expression for the covariance matrix for the estimate $(\hat{r}, \hat{\pi})$, a test can be performed to test the significancy of π. As a matter of fact, the null hypothesis $H_0: \ \pi = 0$ corresponds to the case of deterministic private values.

3.3 Nonparametric Estimation of $f(\cdot)$

For any inner interval $[v_l, v_u] \subset (\underline{v}, \overline{v})$, on which the density $f(\cdot)$ is to be estimated, I consider a particular fixed trimming defined as follows. Let $b_l = \xi^{-1}(v_l, r, G, H, n), b_u = \xi^{-1}(v_u, r, G, H, n)$. Take $\underline{b}_0 \in (\underline{b}, b_l)$ and $\overline{b}_0 \in (b_u, \overline{b})$. Thus, $[b_l, b_u] \subset (\underline{b}_0, \overline{b}_0) \in (\underline{b}, \overline{b})$. [16]

Instead of using (16) directly to recover the pseudo signals, I trim some $\hat{v}_\ell^w$s from the estimation of $f(\cdot)$ when the corresponding b_ℓ^ws are close to the boundaries. In particular, this gives

$$\hat{v}_\ell^w = \begin{cases} b_\ell^w + \hat{\pi}(r, h) + \frac{1}{\hat{r}} \log\left(1 + \frac{n\hat{r}}{n-1} \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}\right) & \text{if} \quad b_\ell^w \in [\underline{b}_0, \overline{b}_0], \\ -\infty & \text{if} \quad b_\ell^w \in [\underline{b}, \underline{b}_0), \\ +\infty & \text{if} \quad b_\ell^w \in (\overline{b}_0, \overline{b}], \end{cases} \quad \forall \ell = 1, ..., L, \tag{24}$$

where $\hat{r}$ and $\hat{\pi}(r, H)$ are estimates from the first step and $\hat{G}^w(\cdot)$ and $\hat{g}^w(\cdot)$ are defined

[16] Note that $[b_l, b_u]$ and $[\underline{b}, \underline{b}_0]$ are not restricted to be smaller than the trimming interval $[\underline{b}^*, \overline{b}^*]$ used when (r, π) are estimated.

by equations (18) and (19), respectively. The bandwidth used in (24) to estimate $g^w(\cdot)$ vanishes at the optimal rate $(L/\log(L))^{1/(2R+3)}$. Note that this bandwidth does not vanish at the same rate as the bandwidth used to estimate $g^w(\cdot)$ in the first step of the estimator. Assumption 3 and Lemma 3 require a particular vanishing rate, which must undersmooth the corresponding density. This rate allows us to obtain a $\sqrt{L}$ rate for the estimator of r and π. In Step 2, the objective is to estimate $f(\cdot)$ at the fastest possible rate from the recovered private signals. Thus, another smoothing parameter, which corresponds to the standard vanishing (optimal) rate, is needed in step 2 in order to recover the winners' pseudo signals in the fastest possible rate.

I then define the following nonparametric estimator for winner's signal density $f^w(\cdot)$ as

$$\hat{f}^w(v^w) = \frac{1}{Lh_f}\sum_{\ell=1}^{L} K\left(\frac{\hat{v}^w_\ell - v^w}{h_f}\right), \tag{25}$$

where $K(\cdot)$ is a kernel of order R defined on a compact support, h_f is a bandwidth vanishing at the rate $(L/\log(L))^{1/(2R+3)}$. Note that this bandwidth corresponds to a oversmoothing as the density $f^w(\cdot)$ is R-order differentiable. This oversmoothing provides the fastest rate of estimating $f^w(\cdot)$ because the pseudo values instead of the true values of v^w_ℓ are used to estimate $f^w(\cdot)$. Following a similar reasoning as in Theorem 3 in Guerre, Perrigne and Vuong (2000), I have $\hat{f}^w(v^w)$ converging uniformly and almost surely at the rate $(L/\log L)^{R/(2R+3)}$ to $f^w(v^w)$ on $[v_l, v_u]$.

I then define the following nonparametric estimators for $F^w(\cdot)$

$$\hat{F}^w(v^w) = \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{1}(\hat{v}^w_\ell \leq v^w). \tag{26}$$

Thus, $f(\cdot)$ can be estimated as follows

$$\hat{f}(v) = \frac{\hat{f}^w(v)}{n[\hat{F}^w(v)]^{(n-1)/n}}, \forall\ v \in [v_l, v_u]. \tag{27}$$

It is well known that $\hat{F}^w(\cdot)$ converges uniformly and almost surely to $F^w(\cdot)$ on $[v_l, v_u]$ at a faster rate than $\hat{f}^w(\cdot)$ does. Thus $\hat{f}(\cdot)$ converges uniformly and almost surely to $f(\cdot)$ at the rate $(L/\log L)^{R/(2R+3)}$ on $[v_l, v_u]$.

3.4 Nonparametric Estimation of $h(\cdot)$

From Lemma 2, ε_i and ε^w follow the same distribution $H(\cdot)$. Thus $h(\cdot)$ can be estimated using an estimator for the density of ε^w. The pseudo $\hat{\varepsilon}_\ell^w$ can be computed as

$$\hat{\varepsilon}_\ell^w \quad = \quad \tilde{v}_\ell^w - \hat{v}_\ell^w, \forall \ \ell = 1, ..., L. \tag{28}$$

Because of the boundary effect in the nonparametric estimator $\hat{g}^w(\cdot)$, $\hat{v}_\ell^w$ can be a biased estimate when the corresponding b_ℓ^w is close to the boundaries. In this case, $\hat{\varepsilon}_\ell^w$ may be a biased estimate of ε_ℓ^w as well. In order to eliminate this boundary effect, I need to do some trimming, which leads to eliminating the $\hat{\varepsilon}_\ell^w$s from the estimation of $h(\cdot)$ when the corresponding b_ℓ^ws are close to the boundaries. This gives

$$\hat{h}(\varepsilon) = \frac{1}{L_0 h_h} \sum_{\ell=1}^{L} 1\!\!1(b_\ell^w \in [\underline{b}_0, \overline{b}_0]) K\left(\frac{\hat{\varepsilon}_\ell^w - \varepsilon}{h_h}\right), \tag{29}$$

where $L_0 = \sum_{\ell=1}^{L} 1\!\!1(b_\ell^w \in [\underline{b}_0, \overline{b}_0])$, $K(\cdot)$ is a kernel function of order R defined on a compact support and h_h is a bandwidth vanishing at the rate $(L/\log L)^{1/(2R+3)}$. This bandwidth corresponds to an oversmoothing as the density $h(\cdot)$ is R-order differentiable. This oversmoothing gives the fastest rate of estimating $h(\cdot)$ because the pseudo values instead of the true values of ε_ℓ^w are used to estimate $h(\cdot)$. This trimming is rather simple: Only the observations corresponding to $b_\ell^w \in [\underline{b}_0, \overline{b}_0]$ are used in the estimation of $h(\cdot)$. Note that this trimming is different from the one adopted for the estimation of $f^w(\cdot)$. In particular, $\hat{f}(\cdot)$ is obtained through a smoothing based on L observations, while $\hat{h}(\cdot)$ is obtained through a smoothing based on only $L_0 \leq L$ observations. Since ε^w and b^w are independent, ignoring a subsample based on b^w will still provide a consistent estimate for the density of ε^w. This is not the case for $f(\cdot)$ because v^w is an increasing function of b^w. Based on a similar reasoning as in Theorem 3 in Guerre, Perrigne and Vuong (2000), $\hat{h}(\cdot)$ converges uniformly and almost surely to $h(\cdot)$ at the rate $(L/\log L)^{R/(2R+3)}$ on any inner support $[\underline{\varepsilon}^*, \overline{\varepsilon}^*] \in (\underline{\varepsilon}, \overline{\varepsilon})$.

4 Monte Carlo Study

This section provides a step by step guide of the estimation procedure presented in Section 3 relying on simulated auction data. Moreover, the results show the good behavior of the estimator on small samples.

4.1 Monte Carlo Design

All the bidders participating in different auctions are assumed to have the same degree of absolute risk aversion. Specifically, I consider the CARA parameter $r = 1$. For simplicity, I consider auctions of similar auctioned objects and $n = 2$ bidders. If heterogeneity across auctioned objects can be characterized by a discrete variable and/or the number of bidders varies across auctions, the same procedure can apply for each pair of values for the auction characteristics variable and number of bidders.

To run the Monte Carlo experiments, I need to find an auction structure, which leads to a simple equilibrium bidding strategy. I consider the following truncated exponential distribution for the private signals

$$F(v) = \frac{1 - \exp(-\frac{v - \underline{v}}{2})}{1 - \exp(-\frac{\overline{v} - \underline{v}}{2})} \tag{30}$$

on the interval $[\underline{v}, \overline{v}] \in (0, +\infty)$. Note that this $F(\cdot)$ satisfies the properties in Definition 2. The ex post shocks ε_i are assumed to be uniformly distributed on the interval $[-\underline{v}, \underline{v}]$.[17]

[17] The ex post gain of the winning bidder is nonnegative in this setting.

Note that this $H(\cdot)$ satisfies the properties in Definition 3. With a CARA(1) utility function and a uniform $H(\cdot)$ distribution, the risk premium is

$$\pi(r=1,H)=\log\left(\frac{\exp(\underline{v})-\frac{1}{\exp(\underline{v})}}{2\underline{v}}\right). \tag{31}$$

Note that π is a strictly increasing function of $\underline{v}$. Moreover, $\pi(r=1,H)<\underline{v}$.

This setting of the auction structure leads to that the equilibrium bidding strategy takes the following linear form

$$b(v)=\underline{v}-\pi(r=1,H)+(v-\underline{v})/2. \tag{32}$$

This linear form is especially convenient when simulating the equilibrium bids. The complex numerical computation involved in calculating the equilibrium bid is then avoided. I assume $\underline{v}=2$, and $\overline{v}=10$. The Monte Carlo study consists in 500 replications indexed by $j=1,2,...,500$ in the following procedure:

1. Let $L=100$. Private signals $v_{i\ell}, i=1,2, \ell=1,\ldots,L$ are random draws from the distribution $F(v)$ given in (30), while winners' ex post shocks $\varepsilon_\ell^w, \ell=1,\ldots,L$ are random draws from the uniform distribution on $[-2,2]$.

2. The winners' private values are computed as $\tilde{v}_\ell^w=v_\ell^w+\varepsilon_\ell^w$, where $v_\ell^w=\max_{i=1,2}\{v_{i\ell}\}$, $\ell=1,\ldots,L$. The risk premium $\pi(r=1,H)$ is computed from (31) and is equal to 0.5952.

3. Equilibrium winning bids $b_\ell^w, \ell=1,\ldots,L$ are computed from (32) using the $v_\ell^w, \ell=1,\ldots,L$.

4.2 A Step by Step Guide

The observations are the winning bids $b_\ell^w, \ell=1,\ldots,100$ and the winner's private value $\tilde{v}_\ell^w, \ell=1,\ldots,100$. I implement the estimation procedure given in Section 3 from these observations to recover the CARA risk aversion parameter r, the risk premium π, the private signal density $f(\cdot)$ and the shock density $h(\cdot)$. Hereafter, I consider $R = 1$.

The first step consists in developing a NLLS estimator from equation (8). The ratio $G^w(\cdot)/g^w(\cdot)$ needs first to be estimated using a standard counting process for the numerator and a kernel density estimator for the denominator as defined in (18) and (19). I choose a triweight kernel of the form $K(u) = (35/32)(1 - u^2)^3$ when $|u| \leq 1$ and $K(u) = 0$ when $|u| \geq 1$. The bandwidth h requires special attention. In particular, the $\sqrt{N}$ consistency rate for the estimator of (r, π) as given in Proposition 3 requires some assumptions on the bandwidth used in the estimator of $g^w(\cdot)$ as described in Assumption 3(ii). As a matter of fact, some undersmoothing is necessary in the estimation of $g^w(\cdot)$. To satisfy such a requirement, I choose a bandwidth of the form $h = c_g(L/\log L)^{-5/16}$, since $5/16$ belongs to the interval $(1/4, 1/2)$ for $R = 1$. See the discussion after Assumption 3. Regarding the constant c_g, I simply set $c_g = \sigma_{b^w}$, where σ_{b^w} is the empirical standard deviation of the observed winning bids b_ℓ^w. See Härdle (1991). Using the estimated ratio $\hat{G}^w(\cdot)/\hat{g}^w(\cdot)$, the NLLS estimator as defined in (20) can be implemented. For simplicity the weight function $w(\cdot)$ is chosen to be equal to one on $[\underline{b}, \overline{b}]$. This step provides an estimate for r and π denoted as $\hat{r}$ and $\hat{\pi}$.

The second step consists in recovering the winners' signals as defined in (24) using $\hat{r}$ and $\hat{\pi}$. This equation requires a different estimate for the ratio $G^w(\cdot)/g^w(\cdot)$. Note that such an estimate has been performed with undersmoothing in the first step. The second step requires another convergence rate for the bandwidth used in estimating $g^w(\cdot)$ as the objective is to estimate $f(\cdot)$. The distribution $G^w(\cdot)$ will be estimated using a counting process as in (18), while the density $g^w(\cdot)$ will be estimated using a kernel estimator as in (19) with a bandwidth of the form $h = c_g(L/\log L)^{-1/5}$, where $c_g = \sigma_{b^w}$. This bandwidth corresponds to the optimal rate as defined by Stone (1982). I do not perform any trimming as suggested in (24) as I prefer the Monte Carlo results to display the impact of any potential boundary effect when estimating $g^w(\cdot)$. Using the pseudo winners' signals $\hat{v}_\ell^w, \ell = 1, \ldots, 100$, the winner's signal density can be estimated using a kernel estimator as defined in (25). I choose a triweight kernel function and a bandwidth of the form $h_f = c_f(L/\log L)^{-1/5}$ following Guerre, Perrigne and Vuong (2000), which is an oversmoothing bandwidth for $R = 1$. The constant c_f is simply set as the empirical standard deviation of the pseudo signals $\hat{v}_\ell^w$. Thus, I estimate the distribution $F^w(\cdot)$ using a counting process as in (26). This allows me to estimate the signal density $f(\cdot)$ using the estimated winners' signal density and distribution as in (27).

The third step consists, for each auction, in recovering the ex post shocks ε_ℓ^w from the difference between the observed winner's private value $\tilde{v}_\ell^w$ and the recovered winner's signal $\hat{v}_\ell^w$ recovered in step 2. These pseudo shocks are then used to estimate nonparametrically their density $h(\cdot)$. I use a similar kernel function with a bandwidth of the form $h_h = c_h(L/\log L)^{-1/5}$ with c_h obtained from the empirical standard deviation of the pseudo shocks. As in step 2, no trimming is conducted. This allows the estimation for $h(\cdot)$ to display the impact of the boundary effect in the kernel estimator for $g^w(\cdot)$.

4.3 Estimation Results

The above procedure is performed 500 times, which gives $(\hat{r}_j, \hat{\pi}_j, \hat{f}_j(\cdot), \hat{h}_j(\cdot), j = 1, \ldots, 500)$. Using these 500 estimates, I construct 95-percent confidence intervals for r and π. I estimate the densities $f(\cdot)$ and $h(\cdot)$ at 100 equally spaced values on the intervals $[2, 10]$ and $[-2, 2]$, respectively. For each of these values, I have 500 different estimates for both densities, from which I eliminate 12 of the lowest values as well as 12 of the highest values to obtain the 95% confidence intervals.

The 95-percent confidence interval for the risk aversion parameter is $[0.5994, 1.4985]$, which covers the true value $r = 1$. The median of these 500 estimates is equal to 0.9990, which is very close to 1. The 95-percent confidence interval for the risk premium is

$[0.2136, 0.9976]$, which covers the true value $\pi = 0.5952$. The median of the 500 estimates is equal to 0.5985, which is very close to the true value. The results show that on average one can expect to recover the true values though the precision is a little bit low. Note that the precision could be improved by doing some trimming at every step of the method. Nonetheless, given the boundary effects, one can consider that the estimation method provides very good results for a sample size of 100 auctions (one pair of winning bid and winning ex post private value in each auction).

Figure 1 displays the 95-percent confidence interval for the private signal density estimated at 100 equally spaced points on $[2, 10]$, the median of these 500 estimates at every estimation point, as well as the true density. The median perfectly superimposes the true density, when excluding the values between 2 and 3, which corresponds to some boundary effects. The 95-percent confidence interval captures the shape of the density almost everywhere. Note that the width of the confidence interval is large for values between 2 and 3, while it becomes small on the rest of the interval for the signal density. This is due to the fact that the variance of the kernel density estimator is proportional to the true value of the density.

[Figure 1 here]

Figure 2 displays the 95-percent confidence interval for the ex post shock density estimated at 100 equally spaced points on $[-2, 2]$, the median of these 500 estimates at every estimation point, as well as the true density. The median superimposes the true density on the interval [-1.6,1.6]. On the boundaries, the estimator tends to underestimate the true density, which correspond to some boundary effects. For this reason, the 95-percent confidence interval does not capture all the shape of the true density. The width of the confidence interval is relatively small for such a number of observations and a nonparametric estimator.

[Figure 2 here]

Overall, the Monte Carlo results show the good behavior of the semiparametric multistep procedure presented in Section 3 given the relatively small number of auctions. These results clearly display the impact of some boundary effects, which could be further corrected by using some trimming. Some Monte Carlo experiments have been also conducted for a smaller number of auctions ($L = 50$). While the median values show a good match between the estimates and the true values of the parameters and density functions, the confidence intervals become wider. Results are available upon request to the author.

5 Conclusion

This paper extends the empirical structural auction literature to the stochastic private value model, while addressing its identification and estimation under a CARA specification for the bidders' utility function in a first-price sealed-bid auction setting. The model is not identified from bids only and more information/observation is needed to pin down the risk premium and the shock distribution. Thus additional observation, which conveys information on the ex post shock is necessary to identify the SPV model. When the winner's ex post private value is observed, the SPV model is shown to be identified. In particular, the identification of the distributions of the ex ante private signal and ex post shock does not depend on the specification of the bidders' utility function. Since only information about the winning bid and winner's ex post private value is required, the identification result also holds for the descending auction setting.

Following the semiparametric identification result, a semiparametric estimation procedure is suggested to estimate the CARA parameter, the risk premium as well as the distributions of the private signals and the ex post shocks. Asymptotic properties for the estimator of the risk premium and risk aversion parameter are derived and the standard

$\sqrt{N}$ consistency rate is achieved, in contrast to Campo, Guerre, Perrigne and Vuong (2003). A Monte Carlo study is conducted to illustrate the estimation procedure using simulated data. The results show the good behavior of the estimation method on small samples.

Several extensions could be entertained. First, the estimation procedure could be generalized to the case of continuous characteristics. Though the implementation is a straightforward extension of the current method, the derivation of its asymptotic properties becomes more involved. Second, other families of utility functions could be considered such as the constant relative risk aversion utility functions. In this case, the risk premium is no longer a constant and becomes a function of the private signal. This greatly complicates the estimation problem, while the model is still identified under similar conditions as in this paper. Third, the independence of private signals may seem to be a restrictive assumption, which could be relaxed to affiliated private signals. Identification of the model is likely to be obtained if the private signals are assumed to be independent of the ex post shocks. Fourth, the requirement of additional information such as the winner's ex post value may seem restrictive as some auction data do not contain such an information or contain some imperfect information, which could be used to assess the winner's private value. Thus, other possibilities could be explored to identify the SPV model. As a matter of fact, some asymmetry among bidders through their attitude toward risk aversion can help in identifying the model. Nonetheless, considering stochastic private values represents an important step in the analysis of auction data as many auction situations suggest that the value of the object is not known with certainty by the bidders at the time of bidding. In this respect, many auction data for which the ex post value of the auctioned object may be subject to some uncertainty or fluctuations could be studied within the perspective of stochastic private values.

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Appendix A

Proof of Lemma 1: Since $U(\cdot)$ is a CARA utility function, $\mathrm{E}_\varepsilon U(x+\varepsilon) = U(x-\pi)$, thus $d\mathrm{E}_\varepsilon U(x+\varepsilon)/dx = U'(x-\pi)$. It is then easy to verify that $\lambda(x) = \mathrm{E}_\varepsilon U(x+\varepsilon)/(d\mathrm{E}_\varepsilon U(x+\varepsilon)/dx) = U(x-\pi)/U'(x-\pi) = (\exp(r(x-\pi))-1)/r$. $\square$

Proof of Lemma 2: Since $[U, F, H] \in \mathcal{U}^{CARA} \times \mathcal{F}_R \times \mathcal{H}_R$, $b'(\cdot, r, F, H, n) > 0$ following Theorem 1 in Campo, Guerre, Perrigne and Vuong (2003). Thus (4) and (5) hold. Moreover, $\forall v_0 \in [\underline{v}, \overline{v}]$, $\forall \varepsilon_0 \in [\underline{\varepsilon}, \overline{\varepsilon}]$,

$$\begin{aligned} P(v^w \le v_0) &= \sum_{i=1}^n P(v^w \le v_0 | i \text{ wins}) P(i \text{ wins}) = \sum_{i=1}^n P(v^w \le v_0 | i \text{ wins}) \frac{1}{n} \\ &= \sum_{i=1}^n P(v^i \le v_0 |\ i \text{ wins}) \frac{1}{n} = \sum_{i=1}^n \frac{P(v^i \le v_0, v_j < v_i, \forall j \ne i)}{P(i \text{ wins})} \frac{1}{n} = F^n(v_0), \quad \text{(A.1)} \end{aligned}$$

$$\begin{aligned} P(\varepsilon^w \le \varepsilon_0) &= \sum_{i=1}^n P(\varepsilon^w \le \varepsilon_0 |\ i \text{ wins}) P(i \text{ wins}) = \sum_{i=1}^n P(\varepsilon^w \le \varepsilon_0 | i \text{ wins}) \frac{1}{n} \\ &= \sum_{i=1}^n P(\varepsilon^i \le \varepsilon_0 |\ i \text{ wins}) \frac{1}{n} = \sum_{i=1}^n P(\varepsilon^i \le \varepsilon_0) \frac{1}{n} = H(\varepsilon_0), \quad \text{(A.2)} \end{aligned}$$

$$\begin{aligned} P(v^w \le v_0, \varepsilon^w \le \varepsilon_0) &= \sum_{i=1}^n P(v^w \le v_0, \varepsilon^w \le \varepsilon_0 | i \text{ wins}) P(i \text{ wins}) \\ &= \sum_{i=1}^n P(v^w \le v_0, \varepsilon^w \le \varepsilon_0 | i \text{ wins}) \frac{1}{n} = \sum_{i=1}^n P(v^i \le v_0, \varepsilon^i \le \varepsilon_0 | i \text{ wins}) \frac{1}{n} \\ &= \sum_{i=1}^n \frac{P(\varepsilon^i \le \varepsilon_0, v^i \le v_0, v_j < v_i, \forall j \ne i)}{P(i \text{ wins})} \frac{1}{n} = F^n(v_0) H(\varepsilon_0). \quad \text{(A.3)} \end{aligned}$$

Equations (A.1), (A.2) and (A.3) imply that v^w and ε^w are independent with each other and that ε^w follows the distribution of ε_i. Note that v^w and ε^w are independent with each other implies $b^w = b(v^w)$ and ε^w are independent with each other. $\square$

Appendix B

Proof of Proposition 2: Let define

$$Q_L(r,\pi) = \frac{1}{L}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w - b_\ell^w - \pi - \frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}\right)\right)^2 w(b_\ell^w),$$

$$Q(r,\pi) = \mathrm{E}_0\left(\tilde{v}^w - b^w - \pi - \frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right)^2 w(b^w).$$

In order to establish the consistency of the NLLS estimator, I need to show that (i) $\hat{Q}_L(r,\pi)$ uniformly converges to $Q(r,\pi)$ in probability with $(r,\pi)\in\Theta$ when $\hat{G}^w(b^w)/\hat{g}^w(b^w)$ uniformly converges on $[\underline{b}^*,\overline{b}^*]$ and, (ii) the minimization problem $\min_{(r,\pi)\in\Theta}Q(r,\pi)$ has a unique solution.

First, I show that $\hat{Q}_L(r,\pi)$ converges to $Q(r,\pi)$ uniformly and almost surely when $(r,\pi)\in\Theta$. The proof consists in two steps. Based on Jennrich (1969) Strong Law of Large Number (SLLN), the first step consists in showing that $Q_L(r,\pi)$ converges to $Q(r,\pi)$ almost surely and uniformly with $(r,\pi)\in\Theta$, while the second step shows that $\hat{Q}_L(r,\pi)$ converges to $Q_L(r,\pi)$ almost surely and uniformly with $(r,\pi)\in\Theta$. Let first define

$$\psi(v^w,b^w,r,\pi) = \left(\tilde{v}^w - b^w - \pi - \frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right)^2 w(b^w).$$

Since $\log(1+x)\leq x$, when $x\in[0,+\infty)$, I have

$$\begin{aligned}
&\sup_{(r,\pi)\in\Theta}|\psi(v^w,b^w,r,\pi)|\\
&= \sup_{(r,\pi)\in\Theta}\left(\tilde{v}^w - b^w - \pi - \frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right)^2 w(b^w)\\
&\leq \sup_{(r,\pi)\in\Theta}\left(|\tilde{v}^w| + \left|b^w + \pi + \frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right|\right)^2 w(b^w)\\
&= \sup_{(r,\pi)\in\Theta}\left(|\tilde{v}^w| + b^w + \pi + \frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right)^2 w(b^w)\\
&\leq \left(|\tilde{v}^w| + b^w + \max_{(r,\pi)\in\Theta}(\pi) + \frac{n}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)^2 w(b^w).
\end{aligned}$$

Let

$$\tilde{\psi}(v^w,b^w) = \left(|\tilde{v}^w| + b^w + \max_{(r,\pi)\in\Theta}(\pi) + \frac{n}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)^2 w(b^w).$$

I have $\mathrm{E}_0\tilde{\psi}(v^w, b^w) = 2\mathrm{E}_0\Big[\Big(|\tilde{v}^w|+b^w+\sup(\pi)+\frac{n}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\Big)^2 w(b^w)\Big]$. Thus as long as Assumption 1 holds, I have $\mathrm{E}_0\tilde{\psi}(v^w, b^w) < \infty$, which implies that $Q_L(r,\pi)$ converges to $Q(r,\pi)$ almost surely and uniformly with $(r,\pi) \in \Theta$ from Jennrich (1969) SLLN as $\psi(v^w, b^w, r, \pi)$ is continuous on Θ.

I then have to show that under Assumptions 1 and 2, $\hat{Q}_L(r,\pi)$ converges to $Q_L(r,\pi)$ almost surely and uniformly with $(r,\pi) \in \Theta$. Let $\theta = (r,\pi)$ and define function $\varphi(\cdot,\cdot,\cdot)$ from $[\underline{b}, \overline{b}] \times [0, +\infty] \times \Theta$ to $[\underline{b}, +\infty]$ as follows

$$\varphi(b, x, \theta) = b + \pi + \frac{1}{r}\log\left(1 + \frac{nr}{n-1}x\right). \tag{B.1}$$

Before proceeding further, I need to establish the following lemma.

Lemma B.1: *If $x, \tilde{x} \geq 0$, then*

$$\Big|\varphi(b, \tilde{x}, \theta) - \varphi(b, x, \theta)\Big| \leq \frac{n}{n-1}\Big|\tilde{x} - x\Big|, \ \forall \theta \in \Theta.$$

The proof of Lemma B.1 is in Appendix C.

From Lemma B.1, if $\hat{G}^w(b^w)/\hat{g}^w(b^w) \geq 0$,

$$\left|\varphi(b^w, \frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)}, \theta) - \varphi(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta)\right| \leq \frac{n}{n-1}\left|\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)} - \frac{G^w(b^w)}{g^w(b^w)}\right|, \ \forall \theta \in \Theta.$$

I can now consider $|\hat{Q}_L(r,\pi) - Q_L(r,\pi)|$. Let $\hat{B}_\ell = \tilde{v}^w_\ell - \varphi\left(b^w_\ell, \frac{\hat{G}^w(b^w_\ell)}{\hat{g}^w(b^w_\ell)}, \theta\right)$ and $B_\ell = \tilde{v}^w_\ell - \varphi\left(b^w_\ell, \frac{G^w(b^w_\ell)}{g^w(b^w_\ell)}, \theta\right)$. I apply the inequality $|\hat{B}_\ell^2 - {B_\ell}^2| \leq (\hat{B}_\ell - B_\ell)^2 + 2|B_\ell||\hat{B}_\ell - B_\ell|$ to show the desired result. In particular,

$$\begin{aligned}
&|\hat{Q}_L(r,\pi) - Q_L(r,\pi)| \\
&= \left|\frac{1}{L}\sum_{\ell=1}^{L}\left\{\left[\left(\tilde{v}^w_\ell - \varphi\left(b^w_\ell, \frac{\hat{G}^w(b^w_\ell)}{\hat{g}^w(b^w_\ell)}, \theta\right)\right)^2 - \left(\tilde{v}^w_\ell - \varphi\left(b^w_\ell, \frac{G^w(b^w_\ell)}{g^w(b^w_\ell)}, \theta\right)\right)^2\right] w(b^w_\ell)\right\}\right| \\
&\leq \frac{1}{L}\sum_{\ell=1}^{L}\left\{\left|\left(\tilde{v}^w_\ell - \varphi\left(b^w_\ell, \frac{\hat{G}^w(b^w_\ell)}{\hat{g}^w(b^w_\ell)}, \theta\right)\right)^2 - \left(\tilde{v}^w_\ell - \varphi\left(b^w_\ell, \frac{G^w(b^w_\ell)}{g^w(b^w_\ell)}, \theta\right)\right)^2\right| w(b^w_\ell)\right\} \\
&\leq \frac{1}{L}\sum_{\ell=1}^{L}\left|2\left(\tilde{v}^w_\ell - \varphi\left(b^w_\ell, \frac{G^w(b^w_\ell)}{g^w(b^w_\ell)}, \theta\right)\right)\left(\left(\varphi(b^w_\ell, \frac{\hat{G}^w(b^w_\ell)}{\hat{g}^w(b^w_\ell)}, \theta\right) - \varphi\left(b^w_\ell, \frac{G^w(b^w_\ell)}{g^w(b^w_\ell)}, \theta\right)\right) w(b^w_\ell)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{L}\sum_{\ell=1}^{L}\left[\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\theta\right)-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta\right)\right]^2 w(b_\ell^w)\\
&\leq \sup_{b^w\in[\underline{b}^*,\overline{b}^*]}\left|\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\theta\right)-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta\right)\right|\\
&\qquad\times\frac{1}{L}\sum_{\ell=1}^{L}\left|2\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta\right)\right)w(b_\ell^w)\right|\\
&+\sup_{b^w\in[\underline{b}^*,\overline{b}^*]}\left|\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\theta\right)-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta\right)\right|^2\times\frac{1}{L}\sum_{\ell=1}^{L}w(b_\ell^w).
\end{aligned}
$$

By SLLN, $\frac{1}{L}\sum_{\ell=1}^{L}\left|2\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta\right)\right)w(b_\ell^w)\right|$ converges almost surely to $\mathrm{E}_0\left|2\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta\right)\right)w(b_\ell^w)\right|<\infty$, and $\frac{1}{L}\sum_{\ell=1}^{L}w(b_\ell^w)$ converges almost surely to $\mathrm{E}_0 w(b^w)$. Thus under Assumptions 1 and 2, Lemma B.1 implies that, as $L\to\infty$,

$$
\sup_{(r,\pi)\in\Theta}|\hat{Q}_L(r,\pi)-Q_L(r,\pi)|=O(\frac{1}{r_L}),\ a.s. \tag{B.2}
$$

Equation (B.2) implies that $\hat{Q}_L(r,\pi)$ converges to $Q_L(r,\pi)$ almost surely and uniformly with $(r,\pi)\in\Theta$ under Assumptions 1 and 2. Combining the above results, I have under Assumptions 1 and 2,

$$
\sup_{(r,\pi)\in\Theta}|\hat{Q}_L(r,\pi)-Q(r,\pi)|=o(1),\ a.s.
$$

It remains to show that there exists $[\underline{b}^*,\overline{b}^*]\in(\underline{b},\overline{b})$ such that the minimization problem $\mathrm{Argmin}_{(r,\pi)\in\Theta}Q(r,\pi)$ has a unique solution $r^*=r_0,\pi^*=\pi_0$, where

$$
\begin{aligned}
Q(r,\pi)&=\mathrm{E}_0\left(\tilde{v}^w-b^w-\pi-\frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right)^2 w(b^w)\\
&=\mathrm{E}_{b^w}\mathrm{E}_{\varepsilon^w|b^w}\left(\tilde{v}^w-b^w-\pi-\frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right)^2 w(b^w)\\
&=\mathrm{E}_{b^w}w(b^w)\Bigg\{\mathrm{Var}_0\varepsilon^w\\
&\quad+\left[\pi_0-\pi+\frac{1}{r_0}\log\left(1+\frac{nr_0}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)-\frac{1}{r}\log\left(1+\frac{nr}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right)\right]^2\Bigg\}.
\end{aligned} \tag{B.3}
$$

From (B.3), the solution $(r=r_0,\pi=\pi_0)$ does solve the above minimization problem. Item (v) in Definition 4 implies that for any $[\underline{b}^*,\overline{b}^*]\in(\underline{b},\overline{b})$ I have $\forall b^w\in[\underline{b}^*,\overline{b}^*]$, $\frac{d\left(\frac{G^w(b^w)}{g^w(b^w)}\right)}{db^w}>0$. Consider any interval $[\underline{b}^*, \overline{b}^*] \in (\underline{b}, \overline{b})$, I claim that the minimization problem $\text{Argmin}_{(r,\pi)\in\Theta} Q(r,\pi)$ has a unique solution $(r = r_0, \pi = \pi_0)$. Suppose that $(\tilde{r}, \tilde{\pi})$ also solves the problem. Thus I must have $\pi_0 - \tilde{\pi} + \frac{1}{r_0}\log\left(1 + \frac{nr_0}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) - \frac{1}{\tilde{r}}\log\left(1 + \frac{n\tilde{r}}{n-1}\frac{G^w(b^w)}{g^w(b^w)}\right) = 0$ on $[\underline{b}^*, \overline{b}^*]$. Taking the first derivative for both sides gives

$$\frac{d\left(\frac{G^w(b^w)}{g^w(b^w)}\right)}{db^w}\left(\frac{1}{\frac{n-1}{n} + r_0\frac{G^w(b^w)}{g^w(b^w)}} - \frac{1}{\frac{n-1}{n} + \tilde{r}\frac{G^w(b^w)}{g^w(b^w)}}\right) = 0, \ \forall\, b^w \in [\underline{b}^*, \overline{b}^*].$$

As $\frac{G^w(b^w)}{g^w(b^w)} > 0, \ \forall\, b^w \in [\underline{b}^*, \overline{b}^*]$, thus I must have $\tilde{r} = r_0$, which leads to $\tilde{\pi} = \pi_0$.

Based on the above results, for any interval $[\underline{b}^*, \overline{b}^*] \in (\underline{b}, \overline{b})$, $(\hat{r}_L, \hat{\pi}_L)$ almost surely converges to the true value (r_0, π_0) if Assumptions 1 and 2 hold. □

Proof of Proposition 3: From Lemma 3, Assumption 3 implies Assumption 2 for the nonparametric estimators (18) and (19). Thus from Proposition 2, under Assumptions 1 and 3 I have

$$\hat{\theta}_L - \theta_0 = o(1), \ a.s.. \tag{B.4}$$

The first-order condition of (20) is given by

$$\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \hat{\theta}_L\right)\right)\left(\frac{\partial\varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \hat{\theta}_L\right)}{\partial\theta}\right)_{2\times 1} w(b_\ell^w) = 0_{2\times 1}. \tag{B.5}$$

Taking the Taylor expansion of the left-hand side at $\theta = \theta_0$ gives

$$\begin{aligned}
&\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0\right)\right)\left(\frac{\partial\varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0\right)}{\partial\theta}\right)_{2\times 1} w(b_\ell^w) \\
&+\Bigg\{\sum_{\ell=1}^{L}\Bigg[\left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L\right)\right)\left(\frac{\partial^2\varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L\right)}{\partial\theta\partial\theta'}\right)_{2\times 2} \\
&-\left(\frac{\partial\varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L\right)}{\partial\theta}\frac{\partial\varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L\right)}{\partial\theta'}\right)_{2\times 2}\Bigg] w(b_\ell^w)\Bigg\}(\hat{\theta}_L - \theta_0)_{2\times 1} = 0_{2\times 1},
\end{aligned}$$

where $\tilde{\theta}_L$ is a middle point between $\hat{\theta}_L$ and θ_0. Thus

$$\begin{aligned}\sqrt{L}(\hat{\theta}_L - \theta_0) &= \left\{ \frac{-1}{L} \sum_{\ell=1}^{L} \left[\left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right) \right) \frac{\partial^2 \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right)}{\partial\theta\partial\theta'} \right.\right. \\ &\quad \left.\left. - \frac{\partial \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right)}{\partial\theta} \frac{\partial \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right)}{\partial\theta'} \right] w(b_\ell^w) \right\}^{-1} \\ &\quad \times \left\{ \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0 \right)}{\partial\theta} w(b_\ell^w) \right\}. \end{aligned} \tag{B.6}$$

Therefore the conclusion of Proposition 3 holds from Lemmas B.2 and B.3 below.

Lemma B.2: *Under Assumptions 1 and 3,*

$$\begin{aligned}&\frac{1}{L} \sum_{\ell=1}^{L} \left[\frac{\partial \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right)}{\partial\theta} \frac{\partial \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right)}{\partial\theta'} \right. \\ &\quad \left. - \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right) \right) \frac{\partial^2 \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \tilde{\theta}_L \right)}{\partial\theta\partial\theta'} \right] w(b_\ell^w) \\ &\quad \to \mathrm{E}_0 \left[\frac{\partial \varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0 \right)}{\partial\theta} \frac{\partial \varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0 \right)}{\partial\theta'} w(b^w) \right] \ a.s., \end{aligned}$$

where the nonparametric estimators $\hat{G}^w(b^w)$ and $\hat{g}^w(b^w)$ are defined in (18) and (19).

Proof of Lemma B.2: See Appendix C.

Lemma B.3: *Under Assumptions 1 and 3, $S_L \xrightarrow{d} \mathcal{N}\Big(0, B\Big)$, where*

$$S_L = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0 \right)}{\partial\theta} w(b_\ell^w),$$

where the nonparametric estimators $\hat{G}^w(b^w)$ and $\hat{g}^w(b^w)$ are defined in (18) and (19).

Proof of Lemma B.3: See Appendix C. □

Appendix C

Proof of Lemma B.1:

$$\left|\varphi(b,\tilde{x},\theta)-\varphi(b,x,\theta)\right| = \left|\frac{\partial\varphi(b,x',\theta)}{\partial x}(\tilde{x}-x)\right| \leq \frac{n}{n-1}\left|\tilde{x}-x\right|,\ \forall\theta\in\Theta, \tag{C.1}$$

where x' is a value between x and $\tilde{x}$. □

Proof of Lemma B.2: The proof consists in two steps. The first step will show

$$\frac{1}{L}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right)\left(\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}\right)_{2\times2} w(b_\ell^w)=o(1)\ a.s., \tag{C.2}$$

while the second step will show

$$\begin{aligned}&\frac{1}{L}\sum_{\ell=1}^{L}\left[\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta}\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta'}w(b_\ell^w)\right]\\ &-\mathrm{E}_0\left[\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta}\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta'}w(b^w)\right]=o(1)\ a.s.\end{aligned} \tag{C.3}$$

The left-hand side of (C.2) can be written as

$$\begin{aligned}&\frac{1}{L}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right)\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}w(b_\ell^w)\\
&=\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)+\left(\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right)\right]\\
&\quad\times\left[\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}+\left(\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right)\right]w(b_\ell^w)\\
&=\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right]w(b_\ell^w)\\
&+\frac{1}{L}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\left(\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right)w(b_\ell^w)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right)\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right]w(b_\ell^w)\\
&+\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right)\right.\\
&\quad\left.\times\left(\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right)\right]w(b_\ell^w)\\
&=A1+B1+C1+D1.
\end{aligned}
\tag{C.4}
$$

I consider the four components $A1,\ B1,\ C1$ and $D1$ one by one. By SLLN,

$$
A1=\frac{1}{L}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}w(b_\ell^w)=O\left(\frac{1}{\sqrt{L}}\right),\ a.s. \tag{C.5}
$$

$$
\begin{aligned}
|B1|&=\left|\frac{1}{L}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\left(\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right)w(b_\ell^w)\right|\\
&\le\frac{1}{L}\sum_{\ell=1}^{L}\left|\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right|\left|\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right|w(b_\ell^w)\\
&\le\sup_{b^w\in[\underline{b},\overline{b}]}\left\{\left|\frac{\partial^2\varphi\left(b^w,\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right|w(b_\ell^w)\right\}\\
&\qquad\times\frac{1}{L}\sum_{\ell=1}^{L}\left|\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right|w(b_\ell^w)=o(1),\ a.s..
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
|C1|&=\left|\frac{1}{L}\sum_{\ell=1}^{L}\left(\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right)\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}w(b_\ell^w)\right|\\
&\le\frac{1}{L}\sum_{\ell=1}^{L}\left|\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right|\left|\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right|w(b_\ell^w)
\end{aligned}
$$

$$\leq \sup_{b^w\in[\underline{b},\overline{b}]}\left\{\left|\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right|w(b_\ell^w)\right\}$$

$$\times\frac{1}{L}\sum_{\ell=1}^{L}\left|\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right|w(b_\ell^w)=o(1),\ \ a.s. \tag{C.7}$$

$$|D1|=\left|\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right)\right.\right.$$

$$\left.\left.\times\left(\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right)\right]w(b_\ell^w)\right|$$

$$\leq\frac{1}{L}\sum_{\ell=1}^{L}\left|\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right|$$

$$\times\left|\frac{\partial^2\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right|w(b_\ell^w).$$

$$\leq \sup_{b^w\in[\underline{b},\overline{b}]}\left\{\left|\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)-\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)\right|w(b_\ell^w)\right\}$$

$$\times\sup_{b^w\in[\underline{b},\overline{b}]}\left\{\left|\frac{\partial^2\varphi\left(b^w,\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)},\tilde{\theta}_L\right)}{\partial\theta\partial\theta'}-\frac{\partial^2\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta\partial\theta'}\right|w(b_\ell^w)\right\}=o(1),\ \ a.s. \tag{C.8}$$

Thus (C.2) holds from (C.4)-(C.8).

Similarly, the left-hand side of (C.3) can be written as

$$\frac{1}{L}\sum_{\ell=1}^{L}\left[\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta}\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta'}w(b_\ell^w)\right]$$

$$=\frac{1}{L}\sum_{\ell=1}^{L}\left[\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta}\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta'}w(b_\ell^w)\right]$$

$$+\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta}-\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta}\right)\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta'}w(b_\ell^w)\right]$$

$$
\begin{aligned}
&+\frac{1}{L}\sum_{\ell=1}^{L}\left[\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta}\left(\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta'}-\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta'}\right)w(b_\ell^w)\right]\\
&+\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta}-\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta}\right)\right.\\
&\qquad\left.\times\left(\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta'}-\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta'}\right)w(b_\ell^w)\right]\\
&=A2+B2+C2+D2. \qquad \text{(C.9)}
\end{aligned}
$$

I consider the four components one by one. By SLLN,

$$
\begin{aligned}
A2&=\frac{1}{L}\sum_{\ell=1}^{L}\left[\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta}\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta'}w(b_\ell^w)\right]\\
&=\mathrm{E}_0\left[\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta}\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta'}w(b^w)\right]+O(\frac{1}{\sqrt{L}}),\ \ a.s. \qquad \text{(C.10)}
\end{aligned}
$$

$$
\begin{aligned}
|B2|&=\left|\frac{1}{L}\sum_{\ell=1}^{L}\left[\left(\frac{\partial\varphi\left(b_\ell^w,\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)},\tilde{\theta}_L\right)}{\partial\theta}-\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta}\right)\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta'}w(b_\ell^w)\right]\right|\\
&\le\sup_{b^w\in[\underline{b},\overline{b}]}\left\{\left|\frac{\partial\varphi\left(b^w,\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)},\tilde{\theta}_L\right)}{\partial\theta}-\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta}\right|w(b^w)\right\}\\
&\quad\times\frac{1}{L}\sum_{\ell=1}^{L}\left|\frac{\partial\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta'}w(b_\ell^w)\right|=o(1),\ \ a.s. \qquad \text{(C.11)}
\end{aligned}
$$

Similarly, I have

$$
|C2|\le o(1),\ \ |D2|\le o(1),\ \ a.s.. \qquad \text{(C.12)}
$$

Thus (C.3) holds from (C.9)-(C.12). This completes the proof of Lemma B.2. □

Proof of Lemma B.3: I consider the following decomposition of the S_L.

$$S_L = S_L^1 + S_L^2 + S_L^3 + S_L^4, \tag{C.13}$$

where

$$S_L^1 = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} w(b_\ell^w), \tag{C.14}$$

$$S_L^2 = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \times \left(\frac{\partial \varphi(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0)}{\partial \theta} - \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} \right) w(b_\ell^w), \tag{C.15}$$

$$S_L^3 = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) - \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} w(b_\ell^w), \tag{C.16}$$

$$S_L^4 = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) - \varphi(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0) \right) \times \left(\frac{\partial \varphi\left(b_\ell^w, \frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} - \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} \right) w(b_\ell^w). \tag{C.17}$$

Note that S_L^1 is regular and that SLLN directly applies. I then consider the other three components. Namely,

$$\begin{aligned} S_L^4 \quad & \leq \sqrt{L} \sup_{b^w \in [\underline{b}, \bar{b}]} \left\{ \left| \varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0 \right) - \varphi\left(b^w, \frac{\hat{G}^w(b^w)}{\hat{g}^w(b_\ell^w)}, \theta_0 \right) \right| w(b^w) \right\} \\ & \quad \times \sup_{b^w \in [\underline{b}, \bar{b}]} \left\{ \left| \frac{\partial \varphi\left(b^w, \frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)}, \theta_0 \right)}{\partial \theta} - \frac{\partial \varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0 \right)}{\partial \theta} \right| w(b^w) \right\} \\ & = \sqrt{L}\ O(r_{g^w}^{-2}) = o(1),\ a.s. \end{aligned} \tag{C.18}$$

$$S_L^2 \quad = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi^2(b_\ell^w, x_\ell, \theta_0)}{\partial \theta \partial x} \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) w(b_\ell^w)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \left(\frac{\partial \varphi^2 \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial\theta\partial x} \right. \\
&\qquad \left. + \frac{\partial \varphi^3(b_\ell^w, \tilde{x}_\ell, \theta_0)}{\partial\theta\partial x^2} \left(x_\ell - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \right) \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) w(b_\ell^w) \\
&= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi^2 \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial\theta\partial x} \\
&\qquad \times \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) w(b_\ell^w) \\
&+ \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \\
&\qquad \times \frac{\partial \varphi^3(b_\ell^w, \tilde{x}_\ell, \theta_0)}{\partial\theta\partial x^2} \left(x_\ell - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) w(b_\ell^w) \\
&= S_L^{21} + S_L^{22}.
\end{aligned} \tag{C.19}
$$

where x_ℓ takes a value between $G^w(b_\ell^w)/g^w(b_\ell^w)$ and $\hat{G}^w(b_\ell^w)/\hat{g}^w(b_\ell^w)$, and $\tilde{x}_\ell$ takes a value between x_ℓ and $G^w(b_\ell^w)/g^w(b_\ell^w)$, $\ell = 1, 2, ..., L$.

I then consider the two components S_L^{21} and S_L^{22}. In particular,

$$
\begin{aligned}
S_L^{21} &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi^2 (b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0)}{\partial\theta\partial x} \\
&\quad \times \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) w(b_\ell^w),
\end{aligned} \tag{C.20}
$$

$$
\begin{aligned}
S_L^{22} &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \\
&\quad \times \frac{\partial \varphi^3(b_\ell^w, \tilde{x}_\ell, \theta_0)}{\partial\theta\partial x^2} \left(x_\ell - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) w(b_\ell^w).
\end{aligned} \tag{C.21}
$$

Note that

$$
\begin{aligned}
\left| S_L^{22} \right| \quad &\leq \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left| \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \right. \\
&\qquad \left. \times \frac{\partial \varphi^3(b_\ell^w, \tilde{x}_\ell, \theta_0)}{\partial\theta\partial x^2} \left(x_\ell - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) w(b_\ell^w) \right| \\
&\leq \sup_{b^w \in [\underline{b}, \overline{b}]} \left\{ \left| \frac{\partial \varphi^3(b_\ell^w, \tilde{x}(b^w), \theta_0)}{\partial\theta\partial x^2} \left(x(b^w) - \frac{G^w(b^w)}{g^w(b^w)} \right) \left(\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)} - \frac{G^w(b^w)}{g^w(b^w)} \right) \right. \right.
\end{aligned}
$$

$$\times w(b^w)\Big|\Big\} \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left| \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) w(b_\ell^w) \right|$$
$$= \sqrt{L}\, O(r_{g^w}^{-2}) = o(1), \; a.s. \tag{C.22}$$

Thus I can focus on S_L^{21} which is the remaining part of S_L^2. I first need to consider the term $w(b^w)\left(\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)} - \frac{G^w(b^w)}{g^w(b^w)}\right)$ before proceeding further. Linearizing $\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)} - \frac{G^w(b^w)}{g^w(b^w)}$ gives

$$\begin{aligned}
& w(b^w)\left(\frac{\hat{G}^w(b^w)}{\hat{g}^w(b^w)} - \frac{G^w(b^w)}{g^w(b^w)}\right) \\
&= w(b^w)\left(\frac{1}{\tilde{g}^w(b^w)}\left(\hat{G}^w(b^w) - G^w(b^w)\right) - \frac{\tilde{G}^w(b^w)}{(\tilde{g}^w(b^w))^2}\left(\hat{g}^w(b^w) - g^w(b^w)\right) \right) \\
&= w(b^w)\left(\left(\frac{1}{g^w(b^w)} - \frac{1}{(\tilde{\tilde{g}}^w(b^w))^2}(\tilde{g}(b^w) - g^w(b^w)) \right)\left(\hat{G}^w(b^w) - G^w(b^w)\right) \right. \\
&\quad - (\hat{g}^w(b^w) - g^w(b^w))\left(\frac{G^w(b^w)}{(g^w(b^w))^2} + \frac{1}{(\hat{\tilde{g}}^w(b^w))^2}(\tilde{G}^w(b^w) - G^w(b^w)) \right. \\
&\quad \left.\left. - \frac{2\hat{\tilde{G}}^w(b^w)}{(\hat{\tilde{g}}^w(b^w))^3}(\tilde{g}^w(b^w) - g^w(b^w)) \right)\right) \\
&= w(b^w)\left(\frac{\hat{G}^w(b_\ell^w) - G^w(b_\ell^w)}{g^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{(g^w(b_\ell^w))^2}(\hat{g}^w(b_\ell^w) - g^w(b_\ell^w)) \right) + O(r_{g^w}^{-2}),
\end{aligned} \tag{C.23}$$

where $\tilde{g}^w(b^w)$ takes a value between $\hat{g}^w(b^w)$ and $g^w(b^w)$, $\tilde{G}^w(b^w)$ takes a value between $\hat{G}^w(b^w)$ and $G^w(b^w)$, $\tilde{\tilde{g}}^w(b^w)$ and $\hat{\tilde{g}}^w(b^w)$ take values between $\tilde{g}^w(b^w)$ and $g^w(b^w)$, and $\tilde{\tilde{G}}^w(b^w)$ and $\hat{\tilde{G}}^w(b^w)$ take values between $\tilde{G}^w(b^w)$ and $G^w(b^w)$, respectively.

Equations (C.20) and (C.23) imply that

$$\begin{aligned}
S_L^{21} &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi^2\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial\theta\partial x} \times \\
&\left(\frac{\hat{G}^w(b_\ell^w) - G^w(b_\ell^w)}{g^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{(g^w(b_\ell^w))^2}(\hat{g}^w(b_\ell^w) - g^w(b_\ell^w)) \right) w(b_\ell^w) + \sqrt{L} O(r_{g^w}^{-2}).
\end{aligned} \tag{C.24}$$

Thus I only need to consider the remaining part of S_L^{21}, which is denoted by

$$\tilde{S}_L^{21} = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right) \right) \frac{\partial \varphi^2\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial\theta\partial x}$$

$$\times\Big\{\frac{\hat{G}^w(b_\ell^w)-G^w(b_\ell^w)}{g^w(b_\ell^w)}-\frac{G^w(b_\ell^w)}{(g^w(b_\ell^w))^2}\left(\hat{g}^w(b_\ell^w)-g^w(b_\ell^w)\right)\Big\}w(b_\ell^w)$$

$$=\tilde{S}_L^{211}-\tilde{S}_L^{212}-\tilde{S}_L^{213}+\tilde{S}_L^{214}, \tag{C.25}$$

where

$$\tilde{S}_L^{211}=\frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\frac{\partial\varphi^2\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial x}\frac{w(b_\ell^w)}{g^w(b_\ell^w)}\hat{G}^w(b_\ell^w),$$

$$\tilde{S}_L^{212}=\frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\frac{\partial\varphi^2\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial x}\frac{G^w(b_\ell^w)w(b_\ell^w)}{(g^w(b_\ell^w))^2}\hat{g}^w(b_\ell^w),$$

$$\tilde{S}_L^{213}=\frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\frac{\partial\varphi^2\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial x}\frac{w(b_\ell^w)}{g^w(b_\ell^w)}G^w(b_\ell^w),$$

$$\tilde{S}_L^{214}=\frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w-\varphi\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)\right)\frac{\partial\varphi^2\left(b_\ell^w,\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)},\theta_0\right)}{\partial\theta\partial x}\frac{G^w(b_\ell^w)w(b_\ell^w)}{(g^w)(b_\ell^w)}.$$

I will apply the projection theorem of the U-statistics (Serfling, 1980) to show $\tilde{S}_L^{211}=\tilde{S}_L^{213}+o_p(1)$ and $\tilde{S}_L^{212}=\tilde{S}_L^{214}+o_p(1)$, which together eventually lead to $\tilde{S}_L^{21}=o_p(1)$.

I first consider $\tilde{S}_L^{211}$. Let $a^{211}(\tilde{v}^w,b^w))=\left(\tilde{v}^w-\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)\right)\frac{\partial\varphi^2\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta\partial x}\frac{w(b^w)}{g^w(b^w)}$, then I have

$$\begin{aligned}\tilde{S}_L^{211} &= \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}a^{211}(\tilde{v}_\ell^w,b_\ell^w)\ \hat{G}^w(b_\ell^w)=\sqrt{L}\left\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1}^{L}a^{211}(\tilde{v}_\ell^w,b_\ell^w)1\!\!1(b_{\ell'}^w-b_\ell^w\le 0)\right\}\\ &= \sqrt{L}\left\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1,\ell'\neq\ell}^{L}a^{211}(\tilde{v}_\ell^w,b_\ell^w)1\!\!1(b_{\ell'}^w-b_\ell^w\le 0)\right\}+\sqrt{L}\left\{\frac{1}{L^2}\sum_{\ell=1}^{L}a^{211}(\tilde{v}_\ell^w,b_\ell^w)\right\}.\end{aligned}$$

Since I have $\tilde{S}_L^{2112}=\sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}a^{211}(\tilde{v}_\ell^w,b_\ell^w)\Big\}=o_p(\frac{1}{\sqrt{L}})$, the main part of $\tilde{S}_L^{211}$ is

$$\begin{aligned}\tilde{S}_L^{2111} &= \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1,\ell'\neq\ell}^{L}a^{211}(\tilde{v}_\ell^w,b_\ell^w)\ 1\!\!1(b_{\ell'}^w-b_\ell^w\le 0)\Big\}\\ &= \sqrt{L}\left\{\frac{1}{L^2}\sum_{\ell=1}^{L-1}\sum_{\ell'=\ell+1}^{L}[a^{211}(\tilde{v}_\ell^w,b_\ell^w)1\!\!1(b_{\ell'}^w-b_\ell^w\le 0)+a^{211}(\tilde{v}_{\ell'}^w,b_{\ell'}^w)1\!\!1(b_\ell^w-b_{\ell'}^w\le 0)]\right\}\\ &= \sqrt{L}\Big\{\frac{L(L-1)}{L^2}U_L^{2111}\Big\},\end{aligned}$$

where

$$U_L^{2111} = \frac{2}{L(L-1)} \sum_{\ell=1}^{L-1} \sum_{\ell'=\ell+1}^{L} P_L^{2111}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w),$$

$$P_L^{2111}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w) = \frac{a^{211}(\tilde{v}_\ell^w, b_\ell^w)\, \mathbb{1}(b_{\ell'}^w - b_\ell^w \le 0) + a^{211}(\tilde{v}_{\ell'}^w, b_{\ell'}^w)\, \mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0)}{2}.$$

Note that $P_L^{2111}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)$ does not depend on L.

Let $\hat{U}_L^{2111}$ be the projection of U_L^{2111} on the observations, namely

$$\hat{U}_L^{2111} = \gamma_L^{2111} + \frac{2}{L} \sum_{\ell=1}^{L} (R_L^{2111}(\tilde{v}_\ell^w, b_\ell^w) - \gamma_L^{2111}),$$

where $R_L^{2111}(\tilde{v}_\ell^w, b_\ell^w) = \mathrm{E}_0[P_L^{2111}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w) | \tilde{v}_\ell^w, b_\ell^w]$, $\gamma_L^{2111} = \mathrm{E}_0[R_L^{2111}(\tilde{v}_\ell^w, b_\ell^w)] = \mathrm{E}_0[P_L^{2111}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)]$. Note that $R_L^{2111}(\tilde{v}_\ell^w, b_\ell^w)$ and γ_L^{2111} do not depend on L.

Direct computation gives

$$\begin{aligned} R_L^{2111}(\tilde{v}_\ell^w, b_\ell^w) &= \frac{a^{211}(\tilde{v}_\ell^w, b_\ell^w)}{2} \mathrm{E}_{b_{\ell'}^w}[\, \mathbb{1}(b_{\ell'}^w - b_\ell^w \le 0) | \tilde{v}_\ell^w, b_\ell^w] \\ &= \frac{a^{211}(\tilde{v}_\ell^w, b_\ell^w)}{2} \int \mathbb{1}(b_{\ell'}^w - b_\ell^w \le 0) g^w(b_{\ell'}^w) db_{\ell'}^w \\ &= \frac{a^{211}(\tilde{v}_\ell^w, b_\ell^w)}{2} G^w(b_\ell^w), \\ \gamma_L^{2111} &= \mathrm{E}_0[R_L^{2111}(\tilde{v}_\ell^w, b_\ell^w)] = 0. \end{aligned}$$

I first show $\sqrt{L}(U_L^{2111} - \hat{U}_L^{2111}) = o_p(1)$. According to Lemma 3.1 in Powell, Stock and Stoker (1989), it suffices to show $\mathrm{E}_0[P_L^{2111}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w) P'^{2111}_L(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)] = o(L)$. This is true as $\mathrm{E}_0[P_L^{2111}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w) P'^{2111}_L(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)]$ does not depend L.

Now I can consider the main component of $\sqrt{L} U_L^{2111}$, which is

$$\begin{aligned} \sqrt{L} \hat{U}_L^{2111} &= \sqrt{L} \gamma_L^{2111} + \sqrt{L} \frac{2}{L} \sum_{\ell=1}^{L} (R_L^{2111}(\tilde{v}_\ell^w, b_\ell^w) - \gamma_L^{2111}) \\ &= \sqrt{L} \Big(\frac{1}{L} \sum_{\ell=1}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\ g^w(b_\ell^w) \Big) = \tilde{S}_L^{213}. \end{aligned}$$

Thus when $L \to \infty$,

$$\tilde{S}_L^{211} = \tilde{S}_L^{2111} + \tilde{S}_L^{2112} = \sqrt{L} \Big\{ \frac{L(L-1)}{L^2} U_L^{2111} \Big\} + o_p(\frac{1}{\sqrt{L}})$$

$$= \frac{L(L-1)}{L^2}\Big\{\sqrt{L}\hat{U}_L^{2111} + \sqrt{L}(U_L^{2111} - \hat{U}_L^{2111})\Big\} + o_p(\frac{1}{\sqrt{L}})$$
$$= \frac{L(L-1)}{L^2}\Big\{\tilde{S}_L^{213} + o_p(1) + o_p(1)\Big\} + o_p(\frac{1}{\sqrt{L}}) = \tilde{S}_L^{213} + o_p(1), \qquad \text{(C.26)}$$

as desired.

Similarly, I will next show $\tilde{S}_L^{212} = \tilde{S}_L^{214} + o_p(1)$.

Let $a^{212}(\tilde{v}^w, b^w)) = \left(\tilde{v}^w - \varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)\right)\frac{\partial\varphi^2(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0)}{\partial\theta\partial x}\frac{G^w(b^w)w(b^w)}{(g^w(b^w))^2}$, then I have

$$\tilde{S}_L^{212} = \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\ \hat{g}^w(b_\ell^w) = \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\ K_h(b_{\ell'}^w - b_\ell^w)\Big\}$$
$$= \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1,\ell'\neq\ell}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\ K_h(b_{\ell'}^w - b_\ell^w)\Big\} + \sqrt{L}\Big\{\frac{K_h(0)}{L^2}\sum_{\ell=1}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\Big\}.$$

Since $\tilde{S}_L^{2122} = \sqrt{L}\Big\{\frac{K_h(0)}{L^2}\sum_{\ell=1}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\Big\} = o_p(\frac{1}{\sqrt{L}h})$, the remaining part of $\tilde{S}_L^{212}$ becomes

$$\tilde{S}_L^{2121} = \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1,\ell'\neq\ell}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\ K_h(b_{\ell'}^w - b_\ell^w)\Big\}$$
$$= \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L-1}\sum_{\ell'=\ell+1}^{L}(a^{212}(\tilde{v}_\ell^w, b_\ell^w) + a^{212}(\tilde{v}_{\ell'}^w, b_{\ell'}^w))K_h(b_\ell^w - b_{\ell'}^w)\Big\} = \sqrt{L}\Big\{\frac{L(L-1)}{L^2}U_L^{2121}\Big\},$$

where

$$U_L^{2121} = \frac{2}{L(L-1)}\sum_{\ell=1}^{L-1}\sum_{\ell'=\ell+1}^{L} P_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w),$$
$$P_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w) = \frac{a^{212}(\tilde{v}_\ell^w, b_\ell^w)\ + a^{212}(\tilde{v}_{\ell'}^w, b_{\ell'}^w)}{2}\ K_h(b_\ell^w - b_{\ell'}^w).$$

Let $\hat{U}_L^{2121}$ be the projection of U_L^{2121} on the observations, namely

$$\hat{U}_L^{2121} = \gamma_L^{2121} + \frac{2}{L}\sum_{\ell=1}^{L}(R_L^{2121}(\tilde{v}_\ell^w, b_\ell^w) - \gamma_L^{2121}),$$

where $R_L^{2121}(\tilde{v}_\ell^w, b_\ell^w) = \mathrm{E}_0[P_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)|\tilde{v}_\ell^w, b_\ell^w]$, $\gamma_L^{2121} = \mathrm{E}_0[R_L^{2121}(\tilde{v}_\ell^w, b_\ell^w)] = \mathrm{E}_0[P_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)]$.

I first show $\sqrt{L}(U_L^{2121} - \hat{U}_L^{2121}) = o_p(1)$. According to Lemma 3.1 in Powell, Stock and Stoker (1989), it suffices to show $\mathrm{E}_0[P_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w){P'}_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)] = o(L)$. Namely,

$$\mathrm{E}_0\left[P_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w){P'}_L^{2121}(\tilde{v}_\ell^w, b_\ell^w; \tilde{v}_{\ell'}^w, b_{\ell'}^w)\right]$$

$$= \frac{1}{4}\mathrm{E}_0\left[K_h^2(b_\ell^w - b_{\ell'}^w)(a^{212}(\tilde v_\ell^w, b_\ell^w) \ + a^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w))(a'^{212}(\tilde v_\ell^w, b_\ell^w) \ + a'^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w))\right]$$

$$= \frac{1}{4}\mathrm{E}_0\Big\{K_h^2(b_\ell^w - b_{\ell'}^w)\Big[a^{212}(\tilde v_\ell^w, b_\ell^w)a'^{212}(\tilde v_\ell^w, b_\ell^w)$$

$$+2\ a^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w)a'^{212}(\tilde v_\ell^w, b_\ell^w) + a^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w)a'^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w)\Big]\Big\}.$$

$$= \frac{1}{4}\mathrm{E}_{(b_\ell^w, b_{\ell'}^w)}\Big\{K_h^2(b_\ell^w - b_{\ell'}^w)\mathrm{E}_{\tilde v_\ell^w, \tilde v_{\ell'}^w|b_\ell^w, b_{\ell'}^w}\Big[a^{212}(\tilde v_\ell^w, b_\ell^w)a'^{212}(\tilde v_\ell^w, b_\ell^w)$$

$$+2\ a^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w)a'^{212}(\tilde v_\ell^w, b_\ell^w) + a^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w)a'^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w)\Big]\Big\}.$$

$$= \frac{1}{4}\mathrm{E}_{(b_\ell^w, b_{\ell'}^w)}\Big\{K_h^2(b_\ell^w - b_{\ell'}^w)(M(b_\ell^w) + M(b_{\ell'}^w))\Big\},$$

where

$$M(b^w) = \mathrm{E}_{\tilde v_\ell^w|b_\ell^w}[a^{212}(\tilde v_\ell^w, b_\ell^w)a'^{212}(\tilde v_\ell^w, b_\ell^w)]$$

$$= \mathrm{Var}_0(\varepsilon^w)\left(\frac{\partial\varphi^2\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial\theta\partial x}\right)\left(\frac{\partial\varphi^2\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial\theta\partial x}\right)'\left(\frac{G^w(b^w)w(b^w)}{(g^w(b^w))^2}\right)^2.$$

As $\sup_{b^w\in[\underline{b},\overline{b}]}|M(b^w)| < \infty$, $\mathrm{E}_0[P_L^{2121}(\tilde v_\ell^w, b_\ell^w; \tilde v_{\ell'}^w, b_{\ell'}^w){P'}_L^{2121}(\tilde v_\ell^w, b_\ell^w; \tilde v_{\ell'}^w, b_{\ell'}^w)]$ $\le \frac{1}{2}\sup_{b^w\in[\underline{b},\overline{b}]}|M(b^w)|\mathrm{E}_{(b_\ell^w, b_{\ell'}^w)}\{K_h^2(b_\ell^w - b_{\ell'}^w)\}$. Using the change of variable $u = \frac{b_\ell^w - b_{\ell'}^w}{h}$,

$$\mathrm{E}_{(b_\ell^w, b_{\ell'}^w)}\{K_h^2(b_\ell^w - b_{\ell'}^w)\} = \frac{1}{h}\int\int K^2(u)g^w(b^w)g^w(b^w + uh)db^w du$$

$$\le \frac{1}{h}\sup_{b^w\in[\underline{b},\overline{b}], u\in(-\infty,+\infty)}|g^w(b^w)g^w(b^w + uh)|\int\int K^2(u)db^w du$$

$$= \frac{1}{h}\sup_{b^w\in[\underline{b},\overline{b}], u\in(-\infty,+\infty)}|g^w(b^w)g^w(b^w + uh)|(\overline{b} - \underline{b})\int K^2(u)du.$$

Since $\int K^2(u)du < \infty$ and $\frac{1}{\sqrt{L}h} = o(1)$, I have

$$\mathrm{E}_0[P_L^{2121}(\tilde v_\ell^w, b_\ell^w; \tilde v_{\ell'}^w, b_{\ell'}^w){P'}_L^{2121}(\tilde v_\ell^w, b_\ell^w; \tilde v_{\ell'}^w, b_{\ell'}^w)] = o(\sqrt{L}) = o(L).$$

I can now consider the main component of $\sqrt{L}U_L^{2121}$, which is $\sqrt{L}\hat U_L^{2121} = \sqrt{L}\gamma_L^{2121} + \sqrt{L}\frac{2}{L}\sum_{\ell=1}^L(R_L^{2121}(\tilde v_\ell^w, b_\ell^w) - \gamma_L^{2121})$, with

$$R_L^{2121}(\tilde v_\ell^w, b_\ell^w) = \mathrm{E}_0[\frac{a^{212}(\tilde v_\ell^w, b_\ell^w) \ + a^{212}(\tilde v_{\ell'}^w, b_{\ell'}^w)}{2}\ K_h(b_\ell^w - b_{\ell'}^w)|\tilde v_\ell^w, b_\ell^w]$$

$$= \frac{a^{212}(\tilde v_\ell^w, b_\ell^w)}{2}\mathrm{E}_{b_{\ell'}^w}[K_h(b_\ell^w - b_{\ell'}^w)|\tilde v_\ell^w, b_\ell^w] = \frac{a^{212}(\tilde v_\ell^w, b_\ell^w)}{2}\int K_h(b_\ell^w - b_{\ell'}^w)g^w(b_{\ell'}^w)db_{\ell'}^w$$

$$= \frac{a^{212}(\tilde v_\ell^w, b_\ell^w)}{2}g^w(b_\ell^w) + \frac{a^{212}(\tilde v_\ell^w, b_\ell^w)}{2}\int K(u)(g^w(b_\ell^w + hu) - g^w(b_\ell^w))du$$

$$= R^{2121}(\tilde v_\ell^w, b_\ell^w) + t_L^{2121}(\tilde v_\ell^w, b_\ell^w),$$

where

$$R^{2121}(\tilde{v}_\ell^w, b_\ell^w) = \frac{a^{212}(\tilde{v}_\ell^w, b_\ell^w)}{2} g^w(b_\ell^w),$$
$$t_L^{2121}(\tilde{v}_\ell^w, b_\ell^w) = \frac{a^{212}(\tilde{v}_\ell^w, b_\ell^w)}{2} \int K(u)(g^w(b_\ell^w + hu) - g^w(b_\ell^w))du.$$

Since $\gamma_L^{2121} = \mathrm{E}_0[R_L^{2121}(\tilde{v}_\ell^w, b_\ell^w)] = \mathrm{E}_0[R^{2121}(\tilde{v}_\ell^w, b_\ell^w) + t_L^{2121}(\tilde{v}_\ell^w, b_\ell^w)] = 0$, thus

$$\begin{aligned}
\sqrt{L}\hat{U}_L^{2121} &= \sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L} R_L^{2121}(\tilde{v}_\ell^w, b_\ell^w)\Big) \\
&= \sqrt{L}\Big(\frac{1}{L}\sum_{\ell=1}^{L} a^{212}(\tilde{v}_\ell^w, b_\ell^w)\ g^w(b_\ell^w)\Big) + \sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L} t_L^{2121}(\tilde{v}_\ell^w, b_\ell^w)\Big) \\
&= \tilde{S}_L^{214} + \sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L} t_L^{2121}(\tilde{v}_\ell^w, b_\ell^w)\Big).
\end{aligned}$$

Let $T_L^{2121} = \sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L} t_L^{2121}(\tilde{v}_\ell^w, b_\ell^w)\Big)$, then $\mathrm{Var}_0(T_L^{2121}) = \mathrm{E}_0\Big[\Big(\int K(u)(g^w(b_\ell^w + hu) - g^w(b_\ell^w))du\Big)^2 a^{212}(\tilde{v}_\ell^w, b_\ell^w) a'^{212}(\tilde{v}_\ell^w, b_\ell^w)\Big] = O(h^{2(R+1)}) = o(1)$. Thus by Chebyshev's inequality, $T_L^{2121} = o_p(1)$.

Thus when Assumption 3(ii) is satisfied,

$$\begin{aligned}
\tilde{S}_L^{212} &= \tilde{S}_L^{2121} + \tilde{S}_L^{2122} \\
&= \sqrt{L}\Big\{\frac{L(L-1)}{L^2} U_L^{2121}\Big\} + o_p(\frac{1}{\sqrt{L}h}) \\
&= \frac{L(L-1)}{L^2}\Big\{\sqrt{L}\hat{U}_L^{2121} + \sqrt{L}(U_L^{2121} - \hat{U}_L^{2121})\Big\} + o_p(\frac{1}{\sqrt{L}h}) \\
&= \frac{L(L-1)}{L^2}\Big\{\tilde{S}_L^{214} + o_p(1) + o_p(1)\Big\} + o_p(\frac{1}{\sqrt{L}h}) = \tilde{S}_L^{214} + o_p(1), \qquad \text{(C.27)}
\end{aligned}$$

as desired.

When Assumption 3(ii) is satisfied, from (C.22), (C.25), (C.26), (C.27),

$$\begin{aligned}
S_L^2 &= S_L^{21} + S_L^{22} = S_L^{21} + o_p(1) \\
&= \tilde{S}_L^{21} + o_p(1) = \tilde{S}_L^{211} - \tilde{S}_L^{212} - \tilde{S}_L^{213} + \tilde{S}_L^{214} + o_p(1) \\
&= \tilde{S}_L^{213} - \tilde{S}_L^{214} - \tilde{S}_L^{213} + \tilde{S}_L^{214} + o_p(1) = o_p(1), \qquad \text{(C.28)}
\end{aligned}$$

as desired.

I now turn to S_L^3. Namely

$$
\begin{aligned}
S_L^3 &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \frac{\partial \varphi(b_\ell^w, x_\ell, \theta_0)}{\partial x} \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial \theta} w(b_\ell^w) \\
&= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial x} + \frac{\partial^2 \varphi(b_\ell^w, \tilde{x}_\ell, \theta_0)}{\partial x^2} \left(x_\ell - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \right) \\
&\qquad \times \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial \theta} w(b_\ell^w) \\
&= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial x} \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial \theta} w(b_\ell^w) \\
&+ \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \frac{\partial^2 \varphi(b_\ell^w, \tilde{x}_\ell, \theta_0)}{\partial x^2} \left(x_\ell - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \\
&\quad \times \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial \theta} w(b_\ell^w) = S_L^{31} + S_L^{32}, \qquad \text{(C.29)}
\end{aligned}
$$

where x_ℓ takes a value between $\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}$ and $\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)}$, and $\tilde{x}_\ell$ takes a value between x_ℓ and $\frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}$, $\ell = 1, 2, ..., L$, and

$$
\begin{aligned}
S_L^{31} &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial x} \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial \theta} w(b_\ell^w), \\
S_L^{32} &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \frac{\partial^2 \varphi(b_\ell^w, \tilde{x}_\ell, \theta_0)}{\partial x^2} \left(x_\ell - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \left(\frac{\hat{G}^w(b_\ell^w)}{\hat{g}^w(b_\ell^w)} - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)} \right) \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial \theta} w(b_\ell^w).
\end{aligned}
$$

Since $|S_L^{32}| \leq \sqrt{L}\ O(r_{g^w}^{-2}) = o(1)$, I can then focus on S_L^{31} which is the main part of S_L^3. Applying (C.23) I have

$$
\begin{aligned}
S_L^{31} &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \Big\{ \frac{1}{g^w(b_\ell^w)} \Big(\hat{G}^w(b_\ell^w) - G^w(b_\ell^w) \Big) - \frac{G^w(b_\ell^w)}{(g^w(b_\ell^w))^2} \Big(\hat{g}^w(b_\ell^w) - g^w(b_\ell^w) \Big) \Big\} \\
&\quad \times \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial x} \frac{\partial \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial \theta} w(b_\ell^w) + \sqrt{L}\ O(r_{g^w}^{-2}). \qquad \text{(C.30)}
\end{aligned}
$$

When Assumption 3(ii) is satisfied, $\sqrt{L}\ O(r_{g^w}^{-2}) = o(1)$. Consider the main component of

S_L^{31} which is

$$\tilde{S}_L^{31} = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \Big\{ \frac{1}{g^w(b_\ell^w)} \Big(\hat{G}^w(b_\ell^w) - G^w(b_\ell^w) \Big) - \frac{G^w(b_\ell^w)}{(g^w(b_\ell^w))^2} \Big(\hat{g}^w(b_\ell^w) - g^w(b_\ell^w) \Big) \Big\}$$
$$\times \frac{\partial \varphi \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial x} \frac{\partial \varphi \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} w(b_\ell^w) = \tilde{S}_L^{311} - \tilde{S}_L^{312},$$

where

$$\tilde{S}_L^{311} = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \frac{w(b_\ell^w)}{g^w(b_\ell^w)} \frac{\partial \varphi \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial x} \frac{\partial \varphi \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} \hat{G}^w(b_\ell^w),$$

$$\tilde{S}_L^{312} = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \Big\{ \frac{G^w(b_\ell^w)}{(g^w(b_\ell^w))^2} \frac{\partial \varphi \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial x} \frac{\partial \varphi \left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0 \right)}{\partial \theta} w(b_\ell^w) \hat{g}^w(b_\ell^w).$$

I will consider the above two components one by one.

Let $a^{311}(b^w) = \frac{w(b^w)}{g^w(b^w)} \frac{\partial \varphi \left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0 \right)}{\partial x} \frac{\partial \varphi \left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0 \right)}{\partial \theta}$, then

$$\begin{aligned}
\tilde{S}_L^{311} &= \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} a^{311}(b_\ell^w) \hat{G}^w(b_\ell^w) = \sqrt{L} \Big\{ \frac{1}{L^2} \sum_{\ell=1}^{L} \sum_{\ell'=1}^{L} a^{311}(b_\ell^w) \ \mathbb{1}(b_{\ell'}^w - b_\ell^w \leq 0) \Big\} \\
&= \sqrt{L} \Big\{ \frac{1}{L^2} \sum_{\ell=1}^{L} \sum_{\ell'=1, \ell' \neq \ell}^{L} a^{311}(b_\ell^w) \ \mathbb{1}(b_{\ell'}^w - b_\ell^w \leq 0) \Big\} + \sqrt{L} \Big\{ \frac{1}{L^2} \sum_{\ell=1}^{L} a^{311}(b_\ell^w) \Big\} \\
&= \tilde{S}_L^{3111} + \tilde{S}_L^{3112},
\end{aligned}$$

where $\tilde{S}_L^{3111} = \sqrt{L} \Big\{ \frac{1}{L^2} \sum_{\ell=1}^{L} \sum_{\ell'=1, \ell' \neq \ell}^{L} a^{311}(b_\ell^w) \ \mathbb{1}(b_{\ell'}^w - b_\ell^w \leq 0) \Big\}$, $\tilde{S}_L^{3112} = \sqrt{L} \Big\{ \frac{1}{L^2} \sum_{\ell=1}^{L} a^{311}(b_\ell^w) \Big\}$.

As $\tilde{S}_L^{3112} = O_p(\frac{1}{\sqrt{L}})$, the main part of $\tilde{S}_L^{311}$ is

$$\begin{aligned}
\tilde{S}_L^{3111} &= \sqrt{L} \Big\{ \frac{1}{L^2} \sum_{\ell=1}^{L} \sum_{\ell'=1, \ell' \neq \ell}^{L} a^{311}(b_\ell^w) \ \mathbb{1}(b_{\ell'}^w - b_\ell^w \leq 0) \Big\} \\
&= \sqrt{L} \Big\{ \frac{1}{L^2} \sum_{\ell=1}^{L-1} \sum_{\ell'=\ell+1}^{L} [a^{311}(b_\ell^w) \mathbb{1}(b_{\ell'}^w - b_\ell^w \leq 0) \ + a^{311}(b_{\ell'}^w) \ \mathbb{1}(b_\ell^w - b_{\ell'}^w \leq 0)] \Big\} \\
&= \sqrt{L} \Big\{ \frac{L(L-1)}{L^2} U_L^{3111} \Big\},
\end{aligned}$$

where

$$\begin{aligned}
U_L^{3111} &= \frac{2}{L(L-1)} \sum_{\ell=1}^{L-1} \sum_{\ell'=\ell+1}^{L} P_L^{3111}(b_\ell^w; b_{\ell'}^w), \\
P_L^{3111}(b_\ell^w; b_{\ell'}^w) &= \frac{a^{311}(b_\ell^w) \mathbb{1}(b_{\ell'}^w - b_\ell^w \leq 0) \ + a^{311}(b_{\ell'}^w) \ \mathbb{1}(b_\ell^w - b_{\ell'}^w \leq 0)}{2}.
\end{aligned}$$

Let $\hat{U}_L^{3111}$ be the projection of U_L^{3111} on the observations, i.e., $\hat{U}_L^{3111}=\gamma_L^{3111}+\frac{2}{L}\sum_{\ell=1}^{L}(R_L^{3111}(b_\ell^w)-\gamma_L^{3111})$, where $R_L^{3111}(b_\ell^w) = \mathrm{E}_0[P_L^{3111}(b_\ell^w; b_{\ell'}^w)|b_\ell^w]$, $\gamma_L^{3111} = \mathrm{E}_0[R_L^{3111}(b_\ell^w)] = \mathrm{E}_0[P_L^{3111}(b_\ell^w; b_{\ell'}^w)]$. I will first show $\sqrt{L}(U_L^{3111} - \hat{U}_L^{3111}) = o_p(1)$. According to Lemma 3.1 in Powell, Stock and Stoker (1989), it suffices to show $\mathrm{E}_0[P_L^{3111}(b_\ell^w; b_{\ell'}^w){P'}_L^{3111}(b_\ell^w; b_{\ell'}^w)] = o(L)$. Note this is true since $P_L^{3111}(b_\ell^w; b_{\ell'}^w)$ does not depend on L.

Now I are ready to consider the main component of $\sqrt{L}U_L^{3111}$, which is

$$\sqrt{L}\hat{U}_L^{3111} = \sqrt{L}\gamma_L^{3111} + \sqrt{L}\frac{2}{L}\sum_{\ell=1}^{L}(R_L^{3111}(b_\ell^w) - \gamma_L^{3111}). \tag{C.31}$$

Let $\gamma^{3111} = \mathrm{E}_0[\frac{G^w(b^w)}{g^w(b^w)}\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial x}\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)},\theta_0\right)}{\partial\theta}w(b^w)]$ and $R^{3111}(b_\ell^w) = \frac{a^{311}(b_\ell^w)}{2}G^w(b_\ell^w)$ $+\int\frac{a^{311}(b_{\ell'}^w)}{2}\mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0)g^w(b_{\ell'}^w)db_{\ell'}^w$. Direct computation gives

$$\begin{aligned}
R_L^{3111}(b_\ell^w) &= \frac{a^{311}(b_\ell^w)}{2}\mathrm{E}_{b_{\ell'}^w}[\mathbb{1}(b_{\ell'}^w - b_\ell^w \le 0)|b_\ell^w] + \mathrm{E}_0[\frac{a^{311}(b_{\ell'}^w)}{2}\ \mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0)|b_\ell^w]\\
&= \frac{a^{311}(b_\ell^w)}{2}\int\mathbb{1}(b_{\ell'}^w - b_\ell^w \le 0)g^w(b_{\ell'}^w)db_{\ell'}^w + \int\frac{a^{311}(b_{\ell'}^w)}{2}\mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0)g^w(b_{\ell'}^w)db_{\ell'}^w\\
&= \frac{a^{311}(b_\ell^w)}{2}G^w(b_\ell^w) + \int\frac{a^{311}(b_{\ell'}^w)}{2}\mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0)g^w(b_{\ell'}^w)db_{\ell'}^w = R^{3111}(b_\ell^w),
\end{aligned}$$

$$\begin{aligned}
\gamma_L^{3111} &= \mathrm{E}_0[R_L^{3111}(b_\ell^w)] = \mathrm{E}_0[R^{3111}(b_\ell^w)]\\
&= \frac{1}{2}\gamma^{3111} + \mathrm{E}_0[\int\frac{a^{311}(b_{\ell'}^w)}{2}\mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0)g^w(b_{\ell'}^w)db_{\ell'}^w],\\
&= \frac{1}{2}\gamma^{3111} + \int\int\frac{a^{311}(b_{\ell'}^w)}{2}\mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0)g^w(b_\ell^w)g^w(b_{\ell'}^w)db_\ell^w db_{\ell'}^w,\\
&= \frac{1}{2}\gamma^{3111} + \int\frac{a^{311}(b_{\ell'}^w)}{2}G^w(b_{\ell'}^w)g^w(b_{\ell'}^w)db_{\ell'}^w = \gamma^{3111}.
\end{aligned}$$

Thus (C.31) becomes

$$\sqrt{L}(\hat{U}_L^{3111} - \gamma^{3111}) = \sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(R^{3111}(b_\ell^w) - \gamma^{3111})\Big).$$

When Assumption 3(ii) is satisfied,

$$\begin{aligned}
\tilde{S}_L^{311} &= \tilde{S}_L^{3111} + \tilde{S}_L^{3112}\\
&= \sqrt{L}\Big\{\frac{L(L-1)}{L^2}U_L^{3111}\Big\} + O_p(\frac{1}{\sqrt{L}})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{L(L-1)}{L^2}\Big\{\sqrt{L}\hat{U}_L^{3111} + \sqrt{L}(U_L^{3111} - \hat{U}_L^{3111})\Big\} + O_p(\frac{1}{\sqrt{L}}) \\
&= \frac{L(L-1)}{L^2}\Big\{\sqrt{L}(\hat{U}_L^{3111} - \gamma^{3111}) + \sqrt{L}\gamma^{3111} + o_p(1)\Big\} + O_p(\frac{1}{\sqrt{L}}) \\
&= \sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(R^{3111}(b_\ell^w) - \gamma^{3111})\Big) + \sqrt{L}\gamma^{3111} + o_p(1) \qquad \text{(C.32)}
\end{aligned}
$$

Let consider $\tilde{S}_L^{312}$ now. Using the following notation

$$
a^{312}(b^w) = \frac{G^w(b^w)}{(g^w(b^w))^2}\frac{\partial\varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial x}\frac{\partial\varphi\left(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\right)}{\partial\theta}w(b^w),
$$

I obtain

$$
\begin{aligned}
\tilde{S}_L^{312} &= \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} a^{312}(b_\ell^w)\hat{g}^w(b_\ell^w) = \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1}^{L} a^{312}(b_\ell^w)\ K_h(b_{\ell'}^w - b_\ell^w)\Big\} \\
&= \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1,\ell'\neq\ell}^{L} a^{312}(b_\ell^w)\ K_h(b_{\ell'}^w - b_\ell^w)\Big\} + \sqrt{L}\Big\{\frac{K_h(0)}{L^2}\sum_{\ell=1}^{L} a^{312}(b_\ell^w)\Big\} \\
&= \tilde{S}_L^{3121} + \tilde{S}_L^{3122},
\end{aligned}
$$

where $\tilde{S}_L^{3121} = \frac{\sqrt{L}}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1,\ell'\neq\ell}^{L} a^{312}(b_\ell^w)K_h(b_{\ell'}^w - b_\ell^w)$ and $\tilde{S}_L^{3122} = \sqrt{L}\Big\{\frac{K_h(0)}{L^2}\sum_{\ell=1}^{L} a^{312}(b_\ell^w)\Big\}$. Under Assumption 3(ii), $\tilde{S}_L^{3122} = O_p(\frac{1}{\sqrt{L}h}) = o(1)$, thus the main part of $\tilde{S}_L^{312}$ is

$$
\begin{aligned}
\tilde{S}_L^{3121} &= \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L}\sum_{\ell'=1,\ell'\neq\ell}^{L} a^{312}(b_\ell^w)\ K_h(b_{\ell'}^w - b_\ell^w)\Big\} \\
&= \sqrt{L}\Big\{\frac{1}{L^2}\sum_{\ell=1}^{L-1}\sum_{\ell'=\ell+1}^{L} (a^{312}(b_\ell^w)\ + a^{312}(b_{\ell'}^w))\ K_h(b_\ell^w - b_{\ell'}^w)\Big\} \\
&= \sqrt{L}\Big\{\frac{L(L-1)}{L^2}U_L^{3121}\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
U_L^{3121} &= \frac{2}{L(L-1)}\sum_{\ell=1}^{L-1}\sum_{\ell'=\ell+1}^{L} P_L^{3121}(b_\ell^w; b_{\ell'}^w), \\
P_L^{3121}(b_\ell^w; b_{\ell'}^w) &= \frac{a^{312}(b_\ell^w)\ + a^{312}(b_{\ell'}^w)}{2}\ K_h(b_\ell^w - b_{\ell'}^w).
\end{aligned}
$$

Let $\hat{U}_L^{3121}$ be the projection of U_L^{3121} on the observations, i.e., $\hat{U}_L^{3121} = \gamma_L^{3121} + \frac{2}{L}\sum_{\ell=1}^{L}(R_L^{3121}(b_\ell^w) - \gamma_L^{3121})$, where $R_L^{3121}(b_\ell^w) = \mathrm{E}_0[P_L^{3121}(b_\ell^w; b_{\ell'}^w)|b_\ell^w]$, $\gamma_L^{3121} = \mathrm{E}_0[R_L^{3121}(b_\ell^w)] = \mathrm{E}_0[P_L^{3121}(b_\ell^w; b_{\ell'}^w)]$. I now want to show $\sqrt{L}(U_L^{3121} - \hat{U}_L^{3121}) = o_p(1)$. According to Lemma 3.1 in Powell, Stock and Stoker (1989), it suffices to show that $\mathrm{E}_0[P_L^{3121}(b_\ell^w; b_{\ell'}^w) P'^{3121}_L(b_\ell^w; b_{\ell'}^w)] = o(L)$. Namely,

$$\begin{aligned}
&\mathrm{E}_0[P_L^{3121}(b_\ell^w; b_{\ell'}^w) P'^{3121}_L(b_\ell^w; b_{\ell'}^w)] \\
&= \frac{1}{4}\mathrm{E}_0[K_h^2(b_\ell^w - b_{\ell'}^w)(a^{312}(b_\ell^w) \ + a^{312}(b_{\ell'}^w))(a'^{312}(b_\ell^w) \ + a'^{312}(b_{\ell'}^w))]. \\
&= \frac{1}{4}\mathrm{E}_0\{K_h^2(b_\ell^w - b_{\ell'}^w)[a^{312}(b_\ell^w)a'^{312}(b_\ell^w) + 2\ a^{312}(b_{\ell'}^w)a'^{312}(b_\ell^w) + a^{312}(b_{\ell'}^w)a'^{312}(b_{\ell'}^w)]\}. \\
&= \frac{1}{4}\mathrm{E}_{(b_\ell^w, b_{\ell'}^w)}\{K_h^2(b_\ell^w - b_{\ell'}^w)M(b_\ell^w, b_{\ell'}^w)\},
\end{aligned}$$

where $M(b_\ell^w, b_{\ell'}^w) = a^{312}(b_\ell^w)a'^{312}(b_\ell^w) + 2\ a^{312}(b_{\ell'}^w)a'^{312}(b_\ell^w) + a^{312}(b_{\ell'}^w)a'^{312}(b_{\ell'}^w)$.

Since $\sup_{b_\ell^w \in [\underline{b},\bar{b}], b_{\ell'}^w \in [\underline{b},\bar{b}]} |M(b_\ell^w, b_{\ell'}^w)| < \infty$, $\mathrm{E}_0[P_L^{3121}(b_\ell^w; b_{\ell'}^w) P'^{3121}_L(b_\ell^w; b_{\ell'}^w)] \leq \frac{1}{2} \sup_{b_\ell^w \in [\underline{b},\bar{b}], b_{\ell'}^w \in [\underline{b},\bar{b}]}$ $|M(b_\ell^w, b_{\ell'}^w)| \mathrm{E}_{(b_\ell^w, b_{\ell'}^w)}\{K_h^2(b_\ell^w - b_{\ell'}^w)\}$. Using the change of variable $u = \frac{b_\ell^w - b_{\ell'}^w}{h}$ gives

$$\begin{aligned}
\mathrm{E}_{(b_\ell^w, b_{\ell'}^w)}\{K_h^2(b_\ell^w - b_{\ell'}^w)\} &= \frac{1}{h}\int\int K^2(u) g^w(b^w) g^w(b^w + uh) db^w du \\
&\leq \frac{1}{h} \sup_{b^w \in [\underline{b},\bar{b}], u \in (-\infty,+\infty)} |g^w(b^w) g^w(b^w + uh)| \int\int K^2(u) db^w du \\
&= \frac{1}{h} \sup_{b^w \in [\underline{b},\bar{b}], u \in (-\infty,+\infty)} |g^w(b^w) g^w(b^w + uh)| (\bar{b} - \underline{b}) \int K^2(u) du.
\end{aligned}$$

Since $\int K^2(u)du < \infty$ and $\frac{1}{\sqrt{L}h} = o(1)$, I have $\mathrm{E}_0[P_L^{3121}(b_\ell^w; b_{\ell'}^w) P'^{3121}_L(b_\ell^w; b_{\ell'}^w)] = o(\sqrt{L}) = o(L)$.

Now I consider the main component of $\sqrt{L}U_L^{3121}$, which is $\sqrt{L}\hat{U}_L^{3121} = \sqrt{L}\gamma_L^{3121} + \sqrt{L}\frac{2}{L}\sum_{\ell=1}^{L}$ $(R_L^{3121}(b_\ell^w) - \gamma_L^{3121})$. Direct computation gives

$$\begin{aligned}
&R_L^{3121}(b_\ell^w) = \mathrm{E}_0\left[\frac{a^{312}(b_\ell^w) \ + a^{312}(b_{\ell'}^w)}{2}\ K_h(b_\ell^w - b_{\ell'}^w)|b_\ell^w\right] \\
&= \frac{a^{312}(b_\ell^w)}{2}\mathrm{E}_{b_{\ell'}^w}[\ K_h(b_\ell^w - b_{\ell'}^w)|b_\ell^w] + \mathrm{E}_{b_{\ell'}^w}\left[\frac{a^{312}(b_{\ell'}^w)}{2}\ K_h(b_\ell^w - b_{\ell'}^w)|b_\ell^w\right] \\
&= \frac{a^{312}(b_\ell^w)}{2}\int K_h(b_\ell^w - b_{\ell'}^w) g^w(b_{\ell'}^w) db_{\ell'}^w + \int \frac{a^{312}(b_{\ell'}^w)}{2} K_h(b_\ell^w - b_{\ell'}^w) g^w(b_{\ell'}^w) db_{\ell'}^w \\
&= \frac{a^{312}(b_\ell^w)}{2}\int K(u) g^w(b_\ell^w + hu) du + \int \frac{a^{312}(b_\ell^w + hu)}{2} K(u) g^w(b_\ell^w + hu) du \\
&= \frac{a^{312}(b_\ell^w)}{2} g^w(b_\ell^w) + \frac{a^{312}(b_\ell^w)}{2} g^w(b_\ell^w) + \frac{a^{312}(b_\ell^w)}{2}\int K(u)(g^w(b_\ell^w + hu) - g^w(b_\ell^w)) du \\
&\quad + \int K(u)\Big(\frac{a^{312}(b_\ell^w + hu)}{2} g^w(b_\ell^w + hu) - \frac{a^{312}(b_\ell^w)}{2} g^w(b_\ell^w)\Big) du \\
&= R^{3121}(b_\ell^w) + t_L^{3121}(b_\ell^w),
\end{aligned}$$

where

$$\begin{aligned}
R^{3121}(b_\ell^w) &= a^{312}(b_\ell^w)g^w(b_\ell^w),\\
t_L^{3121}(b_\ell^w) &= \frac{a^{312}(b_\ell^w)}{2}\int K(u)(g^w(b_\ell^w+hu)-g^w(b_\ell^w))du\\
&\quad+\int K(u)\Big(\frac{a^{312}(b_\ell^w+hu)}{2}g^w(b_\ell^w+hu)-\frac{a^{312}(b_\ell^w)}{2}g^w(b_\ell^w)\Big)du.
\end{aligned}$$

Since $w(\cdot)$ has a bounded $(R+1)$-th order direvative, I can show that $t_L^{3121}(b_\ell^w)=O(h^{R+1})$ and $E(t_L^{3121}(b_\ell^w))=O(h^{R+1})$ using standard argument.

Let $\gamma^{3121}=\mathrm{E}_0[R^{3121}(b_\ell^w)]=\mathrm{E}_0[\frac{G^w(b^w)}{g^w(b^w)}\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)};\theta_0\right)}{\partial x}\frac{\partial\varphi\left(b^w,\frac{G^w(b^w)}{g^w(b^w)};\theta_0\right)}{\partial\theta}w(b^w)]$, then I have $\gamma^{3121}=\gamma^{3111}$ by definition, and $\gamma_L^{3121}=\mathrm{E}_0[R_L^{3121}(b_\ell^w)]=\mathrm{E}_0[R^{3121}(b_\ell^w)+t_L^{3121}(b_\ell^w)]=\gamma^{3121}+\mathrm{E}_0[t_L^{3121}(b_\ell^w)]$. Thus

$$\begin{aligned}
&\sqrt{L}(\hat{U}_L^{3121}-\gamma^{3121})=\sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(R_L^{3121}(b_\ell^w)-\gamma_L^{3121})\Big)-\sqrt{L}(\gamma^{3121}-\gamma_L^{3121})\\
&=\sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(a^{312}(b_\ell^w)g^w(b_\ell^w)-\gamma^{3121})\Big)+\sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(t_L^{3121}(b_\ell^w)-\mathrm{E}_0t_L^{3121}(b_\ell^w))\Big)-\sqrt{L}(\gamma^{3121}-\gamma_L^{3121}).
\end{aligned}$$

Let $T_L^{3121}=\sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(t_L^{3121}(b_\ell^w)-\mathrm{E}_0t_L^{3121}(b_\ell^w)\Big)$, then

$$\begin{aligned}
&\mathrm{Var}_0(T_L^{2121})=4\mathrm{E}_0\Big[(t_L^{3121}(b^w)-\mathrm{E}_0t_L^{3121}(b^w))(t_L^{3121}(b^w)-\mathrm{E}_0t_L^{3121}(b^w))'\Big]\\
&=4\mathrm{E}_0\Big[t_L^{3121}(b^w){t'}_L^{3121}(b^w)\Big]-4\mathrm{E}_0t_L^{3121}(b^w)\mathrm{E}_0't_L^{3121}(b^w)=O(h^{2(R+1)})=o(1).
\end{aligned}$$

Thus by Chebyshev's inequality, $T_L^{3121}=o_p(1)$. I also have that Assumption 3(ii) implies

$$\begin{aligned}
\sqrt{L}(\gamma^{3121}-\gamma_L^{3121}) &= \sqrt{L}\mathrm{E}_0[t_L^{3121}(b^w)]\\
&= \sqrt{L}\mathrm{E}_0[\frac{a^{312}(b^w)}{2}\int K(u)(g^w(b^w+hu)-g^w(b^w))du\\
&\qquad+\int K(u)\Big(\frac{a^{312}(b^w+hu)}{2}g^w(b^w+hu)-\frac{a^{312}(b^w)}{2}g^w(b^w)\Big)du]\\
&= \sqrt{L}\;O(h^{R+1})=o_p(1).
\end{aligned}$$

Aggregating the above results gives

$$\tilde{S}_L^{312} = \tilde{S}_L^{3121}+\tilde{S}_L^{3122}=\sqrt{L}\Big\{\frac{L(L-1)}{L^2}U_L^{3121}\Big\}+O_p(\frac{1}{\sqrt{L}h})$$

$$
\begin{aligned}
&= \frac{L(L-1)}{L^2}\Big\{\sqrt{L}\hat{U}_L^{3121} + \sqrt{L}(U_L^{3121} - \hat{U}_L^{3121})\Big\} + O_p(\frac{1}{\sqrt{L}h}) \\
&= \frac{L(L-1)}{L^2}\Big\{\sqrt{L}(\hat{U}_L^{3121} - \gamma^{3121}) + \sqrt{L}\gamma^{3121} + o_p(1)\Big\} + O_p(\frac{1}{\sqrt{L}h}) \\
&= \frac{L(L-1)}{L^2}\Big\{\sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(a^{312}(b_\ell^w)\ g^w(b_\ell^w) - \gamma^{3121})\Big) + \sqrt{L}\gamma^{3121} + o_p(1)\Big\} + O_p(\frac{1}{\sqrt{L}h}) \\
&= \sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(a^{312}(b_\ell^w)\ g^w(b_\ell^w) - \gamma^{3121})\Big) + \sqrt{L}\gamma^{3121} + o_p(1). \qquad \text{(C.33)}
\end{aligned}
$$

Since $\gamma^{3121} = \gamma^{3111}$, I have

$$
\begin{aligned}
S_L^3 &= S_L^{31} + S_L^{32} = S_L^{31} + o_p(1) = \tilde{S}_L^{31} + o_p(1) = \tilde{S}_L^{311} - \tilde{S}_L^{312} + o_p(1) \\
&= \Big(\sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(R^{3111}(b_\ell^w) - \gamma^{3111})\Big) + \sqrt{L}\gamma^{3111}\Big) \\
&\qquad -\Big(\sqrt{L}\Big(\frac{2}{L}\sum_{\ell=1}^{L}(a^{312}(b_\ell^w)\ g^w(b_\ell^w) - \gamma^{3121})\Big) + \sqrt{L}\gamma^{3121}\Big) + o_p(1) \\
&= \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} 2\Big(R^{3111}(b_\ell^w) - a^{312}(b_\ell^w)\ g^w(b_\ell^w)\Big) + o_p(1) \\
&= \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\Big(\int a^{311}(b_{\ell'}^w)\mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0) g^w(b_{\ell'}^w) db_{\ell'}^w - a^{311}(b_\ell^w) G^w(b_\ell^w)\Big) + o_p(1). \qquad \text{(C.34)}
\end{aligned}
$$

Note that $\mathrm{E}_{b^w}\Big(\int a^{311}(t)\mathbb{1}(b^w - t \le 0) g^w(t) dt - a^{311}(b^w) G^w(b^w)\Big) = 0$ holds as $\gamma^{3121} = \gamma^{3111}$.

Aggregating S_L^1, S_L^2, S_L^3 and S_L^4 from (C.14), (C.18), (C.28) and (C.34) gives

$$
\begin{aligned}
S_L &= S_L^1 + S_L^2 + S_L^3 + S_L^4 \\
&= \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\left(\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)\right)\frac{\partial\varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial\theta} w(b_\ell^w) \\
&+ \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\Big(\int a^{311}(b_{\ell'}^w)\mathbb{1}(b_\ell^w - b_{\ell'}^w \le 0) g^w(b_{\ell'}^w) db_{\ell'}^w - a^{311}(b_\ell^w) G^w(b_\ell^w)\Big) + o_p(1) \\
&= \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L}\left\{\int_{b_\ell^w}^{\bar{b}} \frac{\partial\varphi\left(t, \frac{G^w(t)}{g^w(t)}, \theta_0\right)}{\partial x}\frac{\partial\varphi\left(t, \frac{G^w(t)}{g^w(t)}, \theta_0\right)}{\partial\theta} w(t) d\, t\right. \\
&\left. + \left[\tilde{v}_\ell^w - \varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right) - \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}\frac{\partial\varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial x}\right]\frac{\partial\varphi\left(b_\ell^w, \frac{G^w(b_\ell^w)}{g^w(b_\ell^w)}, \theta_0\right)}{\partial\theta} w(b_\ell^w)\right\} \\
&\quad + o_p(1).
\end{aligned}
$$

Since $\mathrm{E}_{b^w}\Big(\int a^{311}(t)\mathbb{I}(b^w - t \leq 0)g^w(t)dt - a^{311}(b^w)G^w(b^w)\Big) = 0$ and $\mathrm{E}_0\Big\{\Big(\tilde{v}^w - \varphi\Big(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\Big)\Big) \frac{\partial\varphi\Big(b^w, \frac{G^w(b^w)}{g^w(b^w)}, \theta_0\Big)}{\partial\theta} w(b^w)\Big\} = 0$, CLT gives

$$S_L \xrightarrow{d} \mathcal{N}\Big(0, B\Big). \ \square$$

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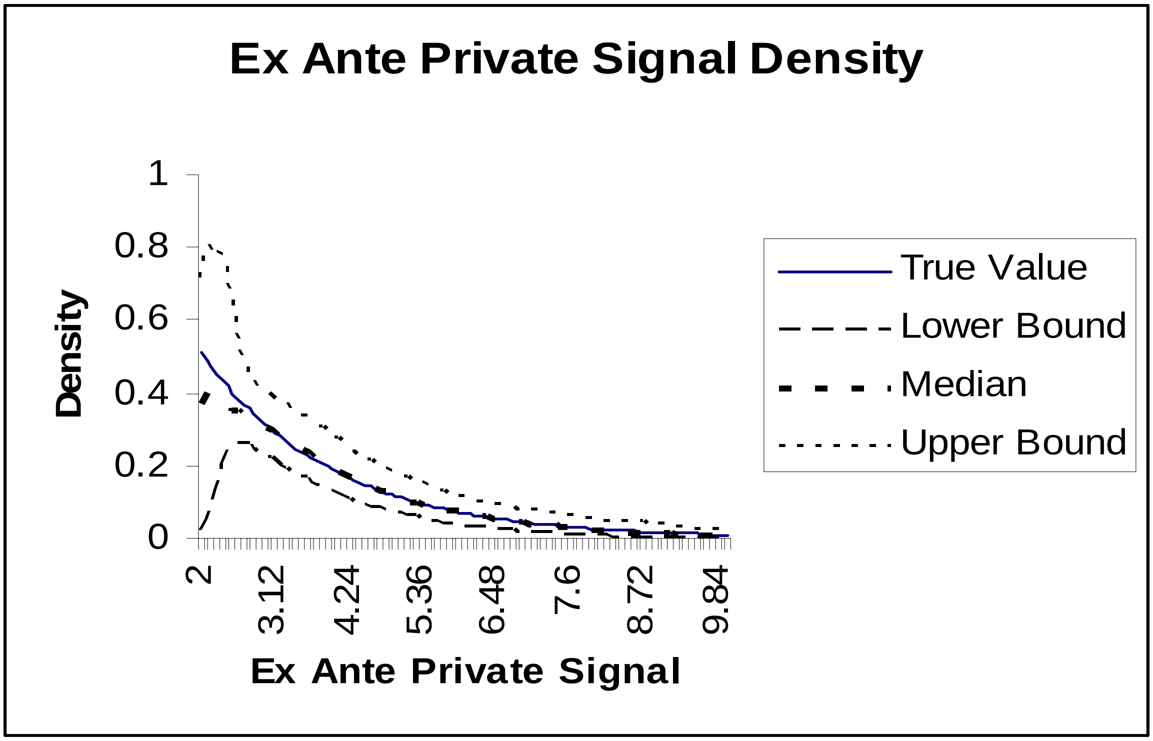

Figure 1: Ex ante Private Signal Density

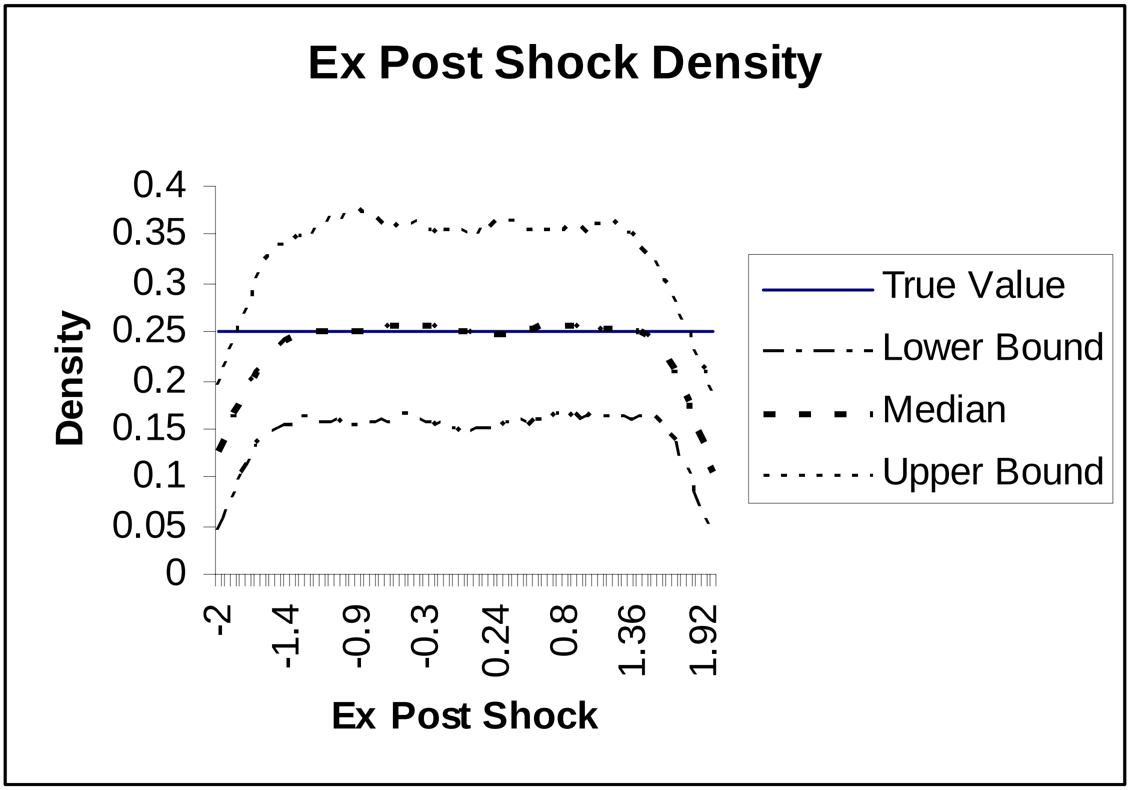

Figure 2: Ex post Shock Density