The Private Provision of Public Goods: Neutrality, Efficiency, Equity and Population

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Abstract

In this paper, we develop a new neutrality theorem in the theory of private provision of public good. Our new theorem is *qualitative* or *global* in the sense that it does not depend on the original equilibrium allocation. Applying the theorem repeatedly, we will establish that the equilibrium obtained under an appropriate income redistribution can be made more efficient in the strict Pareto's sense than the original allocation. It is noteworthy that the redistribution leading to the improvement also makes the inequity of income distribution extremely high. Finally, we establish a proposition that the inefficiency of Nash equilibria becomes big as the population expands.

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1 Introduction

The models of private provision of public good establish a remarkable theorem. It is the neutrality theorem stating that the total supply of public good as well as the allocation of private good in the equilibrium remains intact even when income redistributions are made (see Warr (1982), (1983), Kemp (1984) and Bergstrom-Blume-Varian (1986)). We call this property as the *neutrality property*. 'Nash equilibrium' is the common equilibrium concept in this filed. A key tool exploited in the theorem is the lump-sum income redistribution[1]. Suppose that there exists an equilibrium and then that an income redistribution is made. Under new income distribution, each consumer can choose the same amount of consumption as before and contribute to the public good by the original contribution level plus an increment of his income as far as his new contribution to the public good is not negative. This is the essential fact which makes the neutrality property hold. The size of possible income redistributions thus depends on the original equilibrium allocation in

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[1]It must be noted that Bernheim (1986), Andreoni (1988), Boadway-Pestieau-Wildasin (1989a) and Andreoni-Bergstrom (1996) utilized the other tools such as taxation on goods, subsidy on the contributions to public good, and the government contribution to public good.

the neutrality theorem. Otherwise, some individual's new contribution to the public good would be negative. Therefore, income redistribution must be determined *quantitatively* or *locally*. To be more precise, the income redistribution employed in the neutrality theorem satisfies the followings:

(i) the income of each non-contributor to the public good remains intact,

(ii) the total income of contributors to public good is constant, and

(iii) a contributor's income is allowed to vary by the redistribution up to the amount that the sum of original amount of his contribution and the difference between the new income and the original income is not negative.

Bergstrom-Blume-Varian (1986) presented the most general neutrality theorem based on the income redistribution satisfying (i), (ii) and (iii).

In this paper, we will offer a new neutrality theorem, Theorem 4, which depends only on properties (i) and (ii) but not on the property (iii), i.e. not on the position of original equilibrium allocation. In this sense, our neutrality theorem is *qualitative* or *global*. Saijo-Tatamitani (1995) also tried to present their own global neutrality theorem. Their main objective, however, is to reformulate the model in a setting of the theory of mechanism design. In fact, their neutrality theorem depends on the position of original equilibrium and thus it is a local result.

Our theorem has many influential implications beyond the theorem itself. By using the theorem, we can show the keen trade-off relation between the equality of income distribution and the efficiency of allocation. To show this, firstly we construct an appropriate income redistribution to obtain a new equilibrium where the number of non-contributors increases and where the neutrality property holds. Secondly, repeating the first procedure in several times, we reach the final income distribution, where the neutrality property still holds and all except one individual are non-contributors to the public good. Thirdly, we show in Theorem 5 that the strict Pareto improvement can be attained by a further income redistribution from low income individuals (i.e., non-contributors) to one high income individual (one contributor). The resulting inequality of income distribution becomes very high.

The possibilities of Pareto improvements of Nash equilibria have been scrutinized by many economists, for example, by Warr (1982) and Boadway-Pestieau-Wildasin (1989b). They, however, assumed that the government subsidized the contributions to public good by a constant rate. In short, they showed that the more efficient equilibrium could be attained by introducing distortional subsidy. And thus, their discussions have nothing to do with the equity nor with the inequality of income distribution. It is Itaya-de Meza-Myles (1997) who pointed out that a dilemma might exist between equity and efficiency in two person model. Their result is that unequal income redistribution increases social welfare although one individual becomes worse off. It is Cornes-Sandler (2000) who established the Pareto improvement can be attained by redistributing income. Their result, however, depends on the existence of non contributors and does not necessarily clarify the relation between the equity of income distribution and the efficiency of allocation. On the other hand, our result is wider in its coverage in the sense that every individual's welfare is improved even when non-contributors do not exist and stricter in showing that an extremely unequal income redistribution makes Nash equilibrium improved. In this event, our new neutrality theorem plays an important role.

Finally, we focus on the relation between the inefficiency of Nash equilibria and the population size. In this paper, we are interested in the question whether a Nash equilibrium is near to some Pareto efficient allocation as the population grows. We shall show in Theorem 6 that the ratio of the total public good in a Nash equilibrium to that in a

Lindahl equilibrium under *any* income distribution tends to zero as the population tends to infinity. In details, first we show that the total public good in a Nash equilibrium can not exceed some finite number as population increases and secondly the total public good in a Lindahl equilibrium under an arbitrarily given income distribution tends to infinity.

McGuire(1974) showed that the total amount of privately provided public good increased and converged to a finite number as the population increased under the assumption that preferences of individuals were identical and both the private and the public goods were normal. Andreoni(1988) dealt with the case where preferences of individuals were heterogeneous. His results are that the ratio of contributors to non-contributors converges to zero and the total level of public good increases and converges to a finite number as population increases under the setting that income distribution remains intact. Olson (1965) gave a conjecture that the difference between the amount of public good in Pareto efficient allocation and that of privately provided public good would increase as the population grew. It should be noted that he considered *one* particular Pareto efficient allocation. Cornes-Sandler (1996) showed in an economy consisting of identical individuals with quasi-linear preferences that the difference increased as the population grew. The assumption of quasi-linear preference brings the fact that the amount of privately provided public good is independent of the population size, whereas that of public good in Pareto efficient allocation increases.

Those results which many economists have developed so far are implications of our findings in the sense that individuals can be heterogeneous and that many kinds of efficient allocations are in our scope. Furthermore, our setting does not depend on a particular form of utility function, but assumes that the marginal rate of substitution of public good to private good tends to zero as the quantity of public good tends to infinity.

This paper is organized as follows. In section 2, we define notations and Nash equilibrium and establish some basic results on demand functions and Nash equilibria. Section 3 is devoted to neutrality theorems. Due to our neutrality theorem, a set of income distributions where the neutrality property holds is a closed connected polyhedron. We discuss the possibilities of Pareto improvements in Nash equilibria in section 4 and finally the relation between the inefficiency of Nash equilibria and population size in section 5.

2 The Model

Denote the number of individuals by n and the index set of individuals by $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ [2]. An individual i's utility function is represented by $u_i : (x_i, G) \in \mathbb{R}^2_+ \mapsto u_i(x_i, G) \in \mathbb{R}$, where x_i and G are the amount of private good of the individual i and public good respectively[3]. Let $I \stackrel{\text{def}}{=} (I_1, \ldots, I_n)$ be an *income distribution* each component of which is positive. A list $\mathfrak{E}(I) \stackrel{\text{def}}{=} (N, (u_i(\cdot, \cdot), I_i)_{i \in N})$ is an *economy*. Producing one unit of public good requires one unit of private good. The price of private good is unity. We assume:

Assumption 1 *Utility function $u_i(x_i, G)$ is continuous, increasing, quasi-concave in $\mathbb{R}^2_+$, strictly increasing and strictly quasi-concave in $\mathbb{R}^2_{++}$, for $i \in N$.*

Given a list of all the individuals' contributions to the public good $(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n) \in \mathbb{R}^n_+$, we consider a maximization problem for each i:

$$\max_{x_i, g_i} u_i\Big(x_i, \tilde{G}_{-i} + g_i\Big) \quad \text{subject to} \quad x_i + g_i = I_i, \;\; g_i \geqq 0, \tag{1}$$

where $\tilde{G}_{-i} \stackrel{\text{def}}{=} \sum_{j \in N \setminus \{i\}} \tilde{g}_j$, $i \in N$. Denote the solution to (1) by $(x_i(I_i, \tilde{G}_{-i}), g_i(I_i, \tilde{G}_{-i}))$, $i \in N$.

[2]The symbol "$\stackrel{\text{def}}{=}$" implies that the left hand side is defined by the right hand side.

[3]The sets $\mathbb{R}$, $\mathbb{R}^\ell_+$ and $\mathbb{R}^\ell_{++}$ are the set of real numbers, ℓ dimensional non-negative vectors, and ℓ dimensional strictly positive vectors respectively.

Let us consider an artificial maximization problem:

$$\max_{x_i, G} u_i\Big(x_i, G\Big) \text{ subject to } x_i + G = Y_i, \tag{2}$$

where Y_i is a positive constant. This problem turns out to be very useful afterwards when we study the properties of solution to problem (1). The solution to problem (2) is denoted by $(\xi_i(Y_i), \phi_i(Y_i))$, where the values of ξ_i and ϕ_i correspond to the amount of consumption good and the total public good respectively. $(x_i(I_i, \tilde{G}_{-i}), g_i(I_i, \tilde{G}_{-i}))_{i \in N}$ and $(\xi_i(Y_i), \phi_i(Y_i))_{i \in N}$ are continuous demand functions under Assumption 1 if the solution is unique.

Now, we shall establish useful relations on the solutions to two problems (1) and (2).

Lemma 1 *Let* $\tilde{Y}_i = I_i + \tilde{G}_{-i}$, $\tilde{G}_{-i} \stackrel{\text{def}}{=} \sum_{j \neq i} \tilde{g}_j$, $i \in N$ *for a given* n*-tuple* $(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n)$, *and we obtain the following relations under Assumption* 1:

(a) $u_i(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i)) \geqq u_i(x_i(I_i, \tilde{G}_{-i}), \tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i}))$,

(b) *strict inequality holds in* (a) *if* $(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i)) \neq (x_i(I_i, \tilde{G}_{-i}), \tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i}))$,

(c) $g_i(I_i, \tilde{G}_{-i}) = 0$ *if* $(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i)) \neq (x_i(I_i, \tilde{G}_{-i}), \tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i}))$,

(d) $(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i)) = (x_i(I_i, \tilde{G}_{-i}), \tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i}))$ *if* $g_i(I_i, \tilde{G}_{-i}) > 0$,

(e) $\xi_i(\tilde{Y}_i) \geqq x_i(I_i, \tilde{G}_{-i})$ *and* $\tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i}) \geqq \phi_i(\tilde{Y}_i)$.

Proof. (a) The pair $(x_i(I_i, \tilde{G}_{-i}), \tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i}))$ satisfies

$$x_i(I_i, \tilde{G}_{-i}) + \{\tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i})\} = I_i + \tilde{G}_{-i} = \tilde{Y}_i.$$

This completes the proof of (a).

(b) Due to Assumption 1 and the positivity of I_i, for any λ satisfying $0 < \lambda < 1$ we obtain,

$$u_i(\lambda\xi_i(\tilde{Y}_i) + (1-\lambda)x_i(I_i, \tilde{G}_{-i}), \lambda\phi_i(\tilde{Y}_i) + (1-\lambda)(\tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i}))) > u_i(x_i(I_i, \tilde{G}_{-i}), \tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i})). \tag{3}$$

The pair $(\lambda\xi_i(\tilde{Y}_i) + (1-\lambda)x_i(I_i, \tilde{G}_{-i}), \lambda\phi_i(\tilde{Y}_i) + (1-\lambda)(\tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i})))$ satisfies

$$\lambda\xi_i(\tilde{Y}_i) + (1-\lambda)x_i(I_i, \tilde{G}_{-i}) + \lambda\phi_i(\tilde{Y}_i) + (1-\lambda)(\tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i})) = \tilde{Y}_i.$$

This implies that

$$u_i(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i)) \geqq u_i(\lambda\xi_i(\tilde{Y}_i) + (1-\lambda)x_i(I_i, \tilde{G}_{-i}), \lambda\phi_i(\tilde{Y}_i) + (1-\lambda)(\tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i})).$$

Therefore we have the inequality.

(c) Suppose that $\phi_i(\tilde{Y}_i) \geqq \tilde{G}_{-i}$ then the pair $(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i) - \tilde{G}_{-i})$ satisfies the budget constraint in the problem (1). This contradicts the result in (b). Therefore we have $\phi_i(\tilde{Y}_i) < \tilde{G}_{-i}$. Hence define λ^* as

$$\lambda^*\phi_i(\tilde{Y}_i) + (1-\lambda^*)(\tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i})) = \tilde{G}_{-i}.$$

Suppose that $g_i(I_i, \tilde{G}_{-i}) > 0$, then obviously $0 < \lambda^* < 1$. By simple calculation, we have

$$\begin{aligned}\lambda^*\xi_i(\tilde{Y}_i) + (1-\lambda^*)x_i(I_i, \tilde{G}_{-i}) &= \lambda^*(I_i + \tilde{G}_{-i} - \phi_i(\tilde{Y}_i)) + (1-\lambda^*)(I_i - g_i(I_i, \tilde{G}_{-i})) \\ &= I_i - \lambda^*\phi_i(\tilde{Y}_i) - (1-\lambda^*)(g_i(I_i, \tilde{G}_{-i}) + \tilde{G}_{-i}) + \tilde{G}_{-i} \\ &= I_i.\end{aligned}$$

Hence substituting λ^* into the inequality (3), we have

$$u_i(I_i, \tilde{G}_{-i} + 0) > u_i(x_i(I_i, \tilde{G}_{-i}), \tilde{G}_{-i} + g_i(I_i, \tilde{G}_{-i})).$$

This contradicts the fact that the pair $(x_i(I_i, \tilde{G}_{-i}), g_i(I_i, \tilde{G}_{-i}))$ is the solution to problem (1). And thus we have $g_i(I_i, \tilde{G}_{-i}) = 0$.

(d) This is straightforward from (c).

(e) By the constraints of two problems, we see

$$\xi_i(\tilde{Y}_i) + \phi_i(\tilde{Y}_i) = x_i(I_i, \tilde{G}_{-i}) + g_i(I_i, \tilde{G}_{-i}) + \tilde{G}_{-i} = \tilde{Y}_i > 0,\ \ g_i(I_i, \tilde{G}_{-i}) \geqq 0.$$

Consequently, one of the inequalities in (e) is true when the other holds. Hence, it suffices for us to show $\xi_i(\tilde{Y}_i) \geqq x_i(I_i, \tilde{G}_{-i})$. Now suppose that $\xi_i(\tilde{Y}_i) < x_i(I_i, \tilde{G}_{-i})$ were true. We have

$$\begin{aligned} \phi_i(\tilde{Y}_i) - \tilde{G}_{-i} &= x_i(I_i, \tilde{G}_{-i}) - \xi_i(\tilde{Y}_i) + g_i(I_i, \tilde{G}_{-i}) \\ &\geqq x_i(I_i, \tilde{G}_{-i}) - \xi_i(\tilde{Y}_i) > 0. \end{aligned}$$

Furthermore the equality $\xi_i(\tilde{Y}_i) + \phi_i(\tilde{Y}_i) - \tilde{G}_{-i} = I_i$ holds. The individual i can choose the pair $(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i) - \tilde{G}_{-i})$ in the problem (1). By (b), it holds that

$$u_i(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i)) > u_i(x_i(I_i, \tilde{G}_{-i}), g_i(I_i, \tilde{G}_{-i}) + \tilde{G}_{-i}) \geqq u_i(\xi_i(\tilde{Y}_i), \phi_i(\tilde{Y}_i)).$$

This is a contradiction. ∎

Now we define the equilibrium of private provision of public good in terms of Nash equilibrium.

Definition 1 [**Nash Equilibrium**]
An allocation $(x_i^*, g_i^*)_{i \in N}$ is *a Nash equilibrium in an economy* $\mathfrak{E}(I)$ *when*

$$G_{-i}^* \overset{\text{def}}{=} \sum_{j \neq i} g_j^*, g_i^* = g_i(I_i, G_{-i}^*), x_i^* = x_i(I_i, G_{-i}^*),\ \text{for each } i \in N.$$

Let $(x_i^*, g_i^*)_{i\in N}$ be a Nash equilibrium in an economy $\mathfrak{E}(I)$. Define a set of individuals contributing to the public good as $J \stackrel{\text{def}}{=} \{i \in N|\ \phi_i(Y_i^*) > G_{-i}^*\}$ and define $Y_i^* = I_i + G_{-i}^*, i = 1, 2, \ldots, n$. Due to (a)–(e) above, we have

$$x_i^* = x_i(I_i, G_{-i}^*) = \xi_i(Y_i^*) \text{ and } g_i^* = g_i(I_i, G_{-i}^*) = \phi_i(Y_i^*) - G_{-i}^*, \quad \forall i \in J,$$

$$x_i^* = x_i(I_i, G_{-i}^*) = I_i, \qquad \text{and } g_i^* = g_i(I_i, G_{-i}^*) = 0, \qquad \forall i \in N \setminus J$$

$$J = \{i \in N|\ g_i^* > 0\}.$$

We introduce a new assumption:

Assumption 2 *Both the private and the public goods are normal goods for each individual $i \in N$, i.e. $\xi_i(Y_i)$ and $\phi_i(Y_i)$ are strictly increasing functions with respect to Y_i.*

The existence and the uniqueness of Nash equilibrium were shown under the Assumptions 1 and 2 by Bergstrom-Blume-Varian(1986). They are stated as follows:

Theorem 1 [Unique Existence of Nash Equilibrium]
A Nash equilibrium $(x_i^*, g_i^*)_{i\in N}$ exists in an economy $\mathfrak{E}(I)$ under Assumption 1. The Nash equilibrium is unique under Assumption 2.

Here we show that the Nash equilibrium is continuous with respect to the income distribution.

Theorem 2 [Continuity of Nash Equilibrium]
Suppose that a Nash equilibrium $(x_i^*, g_i^*)_{i\in N}$ exists uniquely in an economy $\mathfrak{E}(I)$, $I \in \mathbb{R}^n_{++}$ and that utility functions $(u_i)_{i\in N}$ satisfy Assumption 1, then the function $I \mapsto (x_i^*, g_i^*)_{i\in N}$ is continuous with respect to income distribution $I \in \mathbb{R}^n_{++}$.

Proof. Let $I^\nu \stackrel{\text{def}}{=} (I_1^\nu, \ldots, I_n^\nu) \in \mathbb{R}^n_{++},\ \nu = 1, 2, \ldots$ be a sequence of income distributions converging to $I \stackrel{\text{def}}{=} (I_1, \ldots, I_n) \in \mathbb{R}^n_{++}$. Denote by $(x_i^\nu, g_i^\nu)_{i\in N}$ the Nash equilibrium corresponding to an economy $\mathfrak{E}(I^\nu)$. Since the sequence of Nash equilibria is in a compact set $([0,s] \times [0,ns])^n$, $s \stackrel{\text{def}}{=} \sup\{I_i^\nu \mid \nu = 1,2,\ldots,\ i \in N\ \}$, an accumulation point of the sequence exists. Let $(\bar{x}_i, \bar{g}_i)_{i\in N}$ be an accumulation point. Then there exists a subsequence $(x_i^{\nu'}, g_i^{\nu'})_{i\in N}$ of $(x_i^\nu, g_i^\nu)_{i\in N}$ which is convergent to $(\bar{x}_i, \bar{g}_i)_{i\in N}$. Let (x_i, g_i) be an element of the budget set defined as:

$$B_i = \left\{(x_i, g_i) \in \mathbb{R}_+^2 \mid x_i + g_i = I_i\ \right\}.$$

Define $\theta \stackrel{\text{def}}{=} x_i/(x_i + g_i)$ and we can see that for sufficiently large ν'

$$\begin{aligned}
x_i &= \theta(x_i + g_i) = \theta(I_i - I_i^{\nu'}) = 0, && \text{if } \theta = 0,\\
x_i &= \theta(x_i + g_i) > \theta(I_i - I_i^{\nu'}), && \text{if } \theta > 0,\\
g_i &= (1-\theta)(x_i + g_i) > (1-\theta)(I_i - I_i^{\nu'}), && \text{if } \theta < 1,\\
g_i &= (1-\theta)(x_i + g_i) = (1-\theta)(I_i - I_i^{\nu'}) = 0, && \text{if } \theta = 1,
\end{aligned}$$

since $I_i^{\nu'}$ converges to I_i. Therefore, we have

$$\tilde{x}_i^{\nu'} \stackrel{\text{def}}{=} x_i + \theta(-I_i + I_i^{\nu'}) \geqq 0, \quad \tilde{g}_i^{\nu'} \stackrel{\text{def}}{=} g_i + (1-\theta)(-I_i + I_i^{\nu'}) \geqq 0.$$

The pair $(\tilde{x}_i^{\nu'}, \tilde{g}_i^{\nu'})$ satisfies $\tilde{x}_i^{\nu'} + \tilde{g}_i^{\nu'} = I_i^{\nu'}$. And thus, we have

$$u_i\Big(x_i^{\nu'}, \sum_{j\in N\setminus\{i\}} g_j^{\nu'} + g_i^{\nu'}\Big) \geqq u_i\Big(\tilde{x}_i^{\nu'}, \sum_{j\in N\setminus\{i\}} g_j^{\nu'} + \tilde{g}_i^{\nu'}\Big).$$

Due to the continuity of $u_i(\cdot,\cdot)$, the above inequality implies

$$u_i\Big(\bar{x}_i, \sum_{j\in N\setminus\{i\}} \bar{g}_j + \bar{g}_i\Big) \geqq u_i\Big(x_i, \sum_{j\in N\setminus\{i\}} \bar{g}_j + g_i\Big).$$

This inequality holds for any $(x_i, g_i) \in B_i$ and for any individual $i \in N$. Then the accumulation point $(\bar{x}_i, \bar{g}_i)_{i\in N}$ is a Nash equilibrium in an economy $\mathfrak{E}(I)$. We can apply the same discussion as above to all the accumulation points in the sequence $(x_i^\nu, g_i^\nu)_{i\in N}$, $\nu = 1,2,\ldots$. This together with the uniqueness of Nash equilibrium implies that the accumulation point exists uniquely. Then the sequence $(x_i^\nu, g_i^\nu)_{i\in N}$, $\nu = 1,2,\ldots$ converges to $(\bar{x}_i, \bar{g}_i)_{i\in N}$. This establishes the continuity of Nash equilibrium. ∎

3 Neutrality Theorems

The purpose of this section is to establish a new neutrality theorem. There are well known neutrality theorems in this field. The most general theorem was given by Bergstrom-Blume-Varian (1986) which could be rewritten here as:

Theorem 3 [**Quantitative Neutrality Theorem**]
Suppose that Assumption 1 *holds. Assume that the set of contributors* $J \stackrel{\text{def}}{=} \{j \in N \mid \hat{g}_j > 0\}$ *is not empty in a Nash equilibrium* $(\hat{x}_i, \hat{g}_i)_{i\in N}$ *of an economy* $\mathfrak{E}(\hat{I})$, $\hat{I} \in \mathbb{R}^n_{++}$. *Let* ε *be a positive real and let a set of income distributions be:*

$$E_\varepsilon(\hat{I}) \stackrel{\text{def}}{=} \left\{ I \in \mathbb{R}^n_{++} \;\middle|\; |I_j - \hat{I}_j| < \varepsilon, \forall j \in J, \sum_{j\in J} \hat{I}_j = \sum_{j\in J} I_j, I_i = \hat{I}_i, \forall i \in N \setminus J \right\}.$$

Then an allocation of private goods and total supply of public good in a Nash equilibrium remain intact for any economy $\mathfrak{E}(I)$, $I \in E_\varepsilon(\hat{I})$, *when* ε *is sufficiently small and when the Nash equilibrium is unique for any income distributions.*

It is noteworthy that the set $E_\varepsilon(\hat{I})$ depends on the position of original Nash equilibrium $(\hat{x}_i, \hat{g}_i)_{i\in N}$. In fact, the fact that $I_j < \hat{I}_j - \hat{g}_j$ for some $j \in J$ and for $I \in E_\varepsilon(\hat{I})$ implies $I_j < \hat{x}_j$ and thus the neutrality does not hold. This indicates that the theorem holds quantitatively or locally.

Let $(\hat{x}_i, \hat{g}_i)_{i\in N}$ and $(\tilde{x}_i, \tilde{g}_i)_{i\in N}$ be Nash equilibria for two economies $\mathfrak{E}(\hat{I})$ and $\mathfrak{E}(\tilde{I})$ respectively. Define two sets of contributors as

$$\hat{J} \stackrel{\text{def}}{=} \{\, i \in N \mid \hat{g}_i > 0 \,\} \text{ and } \tilde{J} \stackrel{\text{def}}{=} \{\, i \in N \mid \tilde{g}_i > 0 \,\}.$$

And denote by $(x_i(I), g_i(I))_{i\in N}$ a Nash equilibrium in an economy $\mathfrak{E}(I)$, $I \stackrel{\text{def}}{=} (I_1, \ldots, I_n)$. Define $G(I) \stackrel{\text{def}}{=} \sum_{i\in N} g_i(I)$. We need two lemmas.

Lemma 2 *Suppose that Assumptions* 1 *and* 2 *hold. And assume that* $\hat{J} \supset \tilde{J}$ *for two income distributions* $\hat{I} \stackrel{\text{def}}{=} (\hat{I}_1, \ldots, \hat{I}_n) \in \mathbb{R}^n_{++}$ *and* $\tilde{I} \stackrel{\text{def}}{=} (\tilde{I}_1, \ldots, \tilde{I}_n) \in \mathbb{R}^n_{++}$. *Then we have that*

$$\text{if} \ \ \sum_{i \in \hat{J}} \hat{I}_i \leqq \sum_{i \in \tilde{J}} \tilde{I}_i \ \ \text{then} \ \ G(\hat{I}) \leqq G(\tilde{I}). \tag{4}$$

Furthermore, we have

$$\text{if} \ \ \sum_{i \in \hat{J}} \hat{I}_i < \sum_{i \in \tilde{J}} \tilde{I}_i \ \ \text{then} \ \ G(\hat{I}) < G(\tilde{I}). \tag{5}$$

Proof. Define

$$\hat{Y}_i \stackrel{\text{def}}{=} \hat{I}_i + \sum_{j \neq i} g_j(\hat{I}), \ \tilde{Y}_i \stackrel{\text{def}}{=} \tilde{I}_i + \sum_{j \neq i} g_j(\tilde{I}), \ i \in N.$$

By Lemma 1, it follows that

$$\begin{aligned} \phi_j(\hat{Y}_j) = G(\hat{I}) &= \sum_{i \in \hat{J}} \hat{I}_i - \sum_{i \in \hat{J}} \xi_i(\hat{Y}_i), \ \xi_j(\hat{Y}_j) = x_j(\hat{I}), \quad j \in \hat{J} \\ \phi_j(\tilde{Y}_j) = G(\tilde{I}) &= \sum_{i \in \tilde{J}} \tilde{I}_i - \sum_{i \in \tilde{J}} \xi_i(\tilde{Y}_i), \ \xi_j(\tilde{Y}_j) = x_j(\tilde{I}), \quad j \in \tilde{J}. \end{aligned}$$

First assume that $\sum_{i \in \hat{J}} \hat{I}_i \leqq \sum_{i \in \tilde{J}} \tilde{I}_i$ and $G(\hat{I}) > G(\tilde{I})$ were true. It holds that $\phi_j(\hat{Y}_j) > \phi_j(\tilde{Y}_j)$ for any $j \in \tilde{J}$ since $\tilde{J} \subset \hat{J}$. The normality of public good implies that $\phi_j(Y)$ is a strictly increasing function with respect to Y. This implies that $\hat{Y}_j > \tilde{Y}_j$ for any $j \in \tilde{J}$. Furthermore, $\xi_j(Y)$ is a strictly increasing function with respect to Y. Then we obtain

$$\begin{aligned} 0 &< \sum_{i \in \tilde{J}} \tilde{I}_i - \sum_{i \in \hat{J}} \hat{I}_i + \sum_{i \in \tilde{J}} \Big(\xi_i(\hat{Y}_i) - \xi_i(\tilde{Y}_i) \Big) \\ &\leqq \sum_{i \in \tilde{J}} \tilde{I}_i - \sum_{i \in \tilde{J}} \xi_i(\tilde{Y}_i) - \sum_{i \in \hat{J}} \hat{I}_i + \sum_{i \in \hat{J}} \xi_i(\hat{Y}_i) = \phi_j(\tilde{Y}_j) - \phi_j(\hat{Y}_j) < 0, \ j \in \tilde{J}. \end{aligned}$$

This is a contradiction. Therefore we have $G(\hat{I}) \leqq G(\tilde{I})$.

Obviously, almost the same discussion made in proving (4) can be applied to establish (5). ∎

Lemma 3 *Suppose that Assumptions* 1 *and* 2 *hold. Let* (x_i^k, g_i^k) *be a solution to a problem*

$$\max_{x_i, g_i} u_i(x_i, G_{-i}^k + g_i) \text{ subject to } x_i + g_i = I_i^k, g_i \geqq 0, x_i \geqq 0, \quad k = 1, 2.$$

Assume that $g_i^1 = 0$, $I_i^2 \leqq I_i^1$ *and* $G_{-i}^2 \geqq G_{-i}^1$. *Then we have* $g_i^2 = 0$.

Proof. Define $Y^k \overset{\text{def}}{=} I_i^k + G_{-i}^k,\ k = 1, 2$. We distinguish two cases. One case is that $Y^2 > Y^1$ and the other is that $Y^2 \leqq Y^1$.

[Case 1: $Y^2 > Y^1$] Contrarily assume that $g_i^2 > 0$. Then we have

$$I_i^2 \leqq I_i^1 = x_i^1 \leqq \xi_i(Y^1) < \xi_i(Y^2) = x_i^2 < I_i^2.$$

The second inequality follows from the inequalities (e) of Lemma 1 and the third from the assumption of normal goods. This is a contradiction.

[Case 2: $Y^2 \leqq Y^1$] Due to the assumption of normal good, we have $\phi_i(Y^2) \leqq \phi_i(Y^1)$. Assume that $g_i^2 > 0$ and we have

$$\phi_i(Y^2) > G_{-i}^2 \geqq G_{-i}^1 \geqq \phi_i(Y^1).$$

The third inequality follows from $g_i^1 = 0$ and (e) in Lemma 1. This is a contradiction. ∎

We are now fully equipped to develop a new theorem.

Theorem 4 [Qualitative Neutrality Theorem]

Suppose that Assumptions 1 *and* 2 *hold. Assume that in economies* $\mathfrak{E}(\hat{I})$ *and* $\mathfrak{E}(\tilde{I})$ *qualitative relations*:

$$J \overset{\text{def}}{=} \hat{J} = \tilde{J},\ \sum_{i \in J} \hat{I}_i = \sum_{i \in J} \tilde{I}_i > 0,\ \textit{and } \hat{I}_j = \tilde{I}_j,\ \forall j \in N \setminus J,$$

hold. Then the allocation of private good and the amount of public good of Nash equilibrium in the economy $\mathfrak{E}(\hat{I})$ is identical with those in $\mathfrak{E}(\tilde{I})$ and furthermore with those in an economy $\mathfrak{E}(\lambda\hat{I} + (1-\lambda)\tilde{I})$ for any λ satisfying $0 \leqq \lambda \leqq 1$.

We give some explanations for calling the theorem to be *qualitative* before proving it. There are two equilibria corresponding to two income distributions $\tilde{I}$ and $\hat{I}$. We assume that the set of contributors in the equilibria is identical, that each income of non-contributors is identical and that the two kinds of total income of contributors are identical. The result obtained is qualitative since these properties do not depend on the quantitative size of income redistribution. Furthermore, the theorem shows that every economy between $\mathfrak{E}(\hat{I})$ and $\mathfrak{E}(\tilde{I})$ attains the same allocation. This is also a qualitative result.

Proof. The sufficient condition of Lemma 2 is satisfied, that is,

$$\tilde{J} \subset \hat{J} \text{ and } \sum_{i \in \hat{J}} \hat{I}_i \leqq \sum_{i \in \tilde{J}} \tilde{I}_i.$$

Therefore $G(\hat{I}) \leqq G(\tilde{I})$. Similarly we have $G(\tilde{I}) \leqq G(\hat{I})$. These imply that $G(\tilde{I}) = G(\hat{I})$. In other words, we have $\phi_i(\hat{Y}_i) = \phi_i(\tilde{Y}_i)$ for any $i \in J$. Due to the normality of public goods, it implies that $\hat{Y}_i = \tilde{Y}_i$. Lemma 1 leads us to $x_i(\hat{I}) = \xi_i(\hat{Y}_i) = x_i(\tilde{I}) = \xi_i(\tilde{Y}_i)$. In short, we have

$$\hat{Y}_i = \tilde{Y}_i,\ G(\hat{I}) = G(\tilde{I}),\ x_i(\hat{I}) = x_i(\tilde{I}), \forall i \in J. \tag{6}$$

Furthermore, it is easy to see

$$x_i(\hat{I}) = x_i(\tilde{I}) = \tilde{I}_i,\ g_i(\hat{I}) = g_i(\tilde{I}) = 0, \quad \forall i \in N \setminus J. \tag{7}$$

This completes the first assertion.

Let λ be a real satisfying $0 \leqq \lambda \leqq 1$. Consider an economy $\mathfrak{E}(\lambda \hat{I} + (1-\lambda)\tilde{I})$. Define

$$\begin{aligned} x_i^\lambda &\stackrel{\text{def}}{=} x_i(\hat{I}), \\ I_i^\lambda &\stackrel{\text{def}}{=} \lambda \hat{I}_i + (1-\lambda)\tilde{I}_i, \\ g_i^\lambda &\stackrel{\text{def}}{=} \lambda g_i(\hat{I}) + (1-\lambda) g_i(\tilde{I}), \ \ i \in N. \end{aligned}$$

We shall show that the allocation $(x_i^\lambda, g_i^\lambda)_{i\in N}$ is a Nash equilibrium in an economy $\mathfrak{E}(\lambda \hat{I} + (1-\lambda)\tilde{I})$. For any $i \in N$, it follows that

$$\begin{aligned} Y_i^\lambda &\stackrel{\text{def}}{=} I_i^\lambda + \sum_{j\neq i} g_j^\lambda \\ &= \lambda(\hat{I}_i + \sum_{j\neq i} g_j(\hat{I})) + (1-\lambda)(\tilde{I}_i + \sum_{j\neq i} g_j(\tilde{I})) \\ &= \lambda \hat{Y}_i + (1-\lambda)\tilde{Y}_i = \hat{Y}_i. \end{aligned}$$

Hence, by Lemma 1 we have for any $i \in J$

$$\begin{aligned} \phi_i(Y_i^\lambda) &= \phi_i(\hat{Y}_i) = G(\hat{I}), \\ \xi_i(Y_i^\lambda) &= \xi_i(\hat{Y}_i) = x_i(\hat{I}). \end{aligned}$$

Furthermore, by (6) and (7) it holds that for $i \in J$

$$\begin{aligned} \phi_i(Y_i^\lambda) - \sum_{j\neq i} g_j^\lambda &= G(\hat{I}) - \Big(\lambda(G(\hat{I}) - g_i(\hat{I})) + (1-\lambda)(G(\tilde{I}) - g_i(\tilde{I}))\Big) \\ &= \lambda g_i(\hat{I}) + (1-\lambda) g_i(\tilde{I}) = g_i^\lambda. \end{aligned}$$

This implies that the pair $(x_i(\hat{I}), g_i^\lambda)$ is the optimal choice for $i \in J$ when a list $(g_j^\lambda)_{j\in N}$ of the provision of public good is given.

Let us consider the individual $i \notin J$. Since his income is $I_i^\lambda = \hat{I}_i$ and the sum of contributions to the public good by all other individuals is equal to $G(\hat{I})$, Lemma 3 implies that his optimal choice is $(I_i^\lambda, 0)$.

The above discussions show that the allocation $(x_i^\lambda, g_i^\lambda)_{i\in N}$ is a Nash equilibrium in an economy $\mathfrak{E}(\lambda\hat{I} + (1-\lambda)\tilde{I})$. Furthermore, it is clear that

$$x_i^\lambda = x_i(\hat{I}) \ \forall i \in N, \quad \sum_{i\in N} g_i^\lambda = G(\hat{I}).$$

∎

Let us examine implications of the theorem. Consider economies $\mathfrak{E}(I)$'s where an income distribution I is in the set:

$$S \stackrel{\text{def}}{=} \left\{ I \in \mathbb{R}_+^n \;\middle|\; \sum_{i\in N} I_i = c \right\}, \ c \text{ is a positive constant.}$$

Let $(x_i(I), g_i(I))_{i\in N}$ be a Nash equilibrium in an economy $\mathfrak{E}(I)$, $I \in S$. Let $J(I) \stackrel{\text{def}}{=} \{ j \in N \mid g_j(I) > 0 \}$ be the set of contributors. Now define a set of income distributions where all the individuals are contributors as

$$Z \stackrel{\text{def}}{=} \{ I \in S \mid J(I) = N \}.$$

The qualitative neutrality theorem establishes the convexity of the set Z. Let $\hat{I}$ be an income distribution in the boundary of Z relative to S. The continuity of Nash equilibrium with respect to income distribution implies that the allocation of private good and the total amount of public good at $\hat{I}$ are the same as in Z and that the set $J(\hat{I})$ of contributors is a proper subset of N. And thus we have

$$\text{closure of } Z = \left\{ I \in S \mid (x_i(I))_{i\in N} = (x_i(\hat{I}))_{i\in N}, \sum_{i\in N} g_i(I) = \sum_{i\in N} g_i(\hat{I}) \right\}.$$

Again by the new theorem, closure of Z is a closed convex polyhedron.

Here we have to stress that the assumption of normal goods is indispensable for the qualitative neutrality theorem. Let us demonstrate this by an example. Figure illustrates two person economy where the public good is normal for the individual 1 but not normal for the individual 2. Lines AA and BB are original budget lines of type (2) of the individuals 1 and 2 when an income distribution is (I_1, I_2). Points E_1 and E_2 exhibit a Nash equilibrium. At the Nash equilibrium only the first individual is a contributor to the public good. Let $(\tilde{I}_1, \tilde{I}_2)$ be an income distribution near (I_1, I_2) satisfying $\tilde{I}_1 < I_1$, $\tilde{I}_2 > I_2$ and $\tilde{I}_1 + \tilde{I}_2 = I_1 + I_2$. Points E_1 and E_2 constitute a Nash equilibrium if the difference of two income distribution is sufficiently small. Under every income distribution $\alpha(\tilde{I}_1, \tilde{I}_2) + (1 - \alpha)(I_1, I_2)$, $0 < \alpha < 1$, a Nash equilibrium is attained at E_1 and E_2, where all the consumers contribute to the public good. Let us redistribute income (I_1, I_2) from the individual 2 to the individual 1 to obtain a new income distribution $(I'_1, I'_2) \stackrel{\text{def}}{=} (I_1 + \Delta I, I_2 - \Delta I)$ where a new equilibrium is attained at F_1 and F_2. In this equilibrium, the individual 2 is a non-contributor to the public good. Let an income distribution $(\hat{I}_1, \hat{I}_2)$ be in a vicinity of (I'_1, I'_2) and satisfy $\hat{I}_1 < I'_1$, $\hat{I}_2 > I'_2$ and $\hat{I}_1 + \hat{I}_2 = I'_1 + I'_2$. At the income distribution $(\hat{I}_1, \hat{I}_2)$, a Nash equilibrium is attained at F_1 and F_2 where all the individuals contribute the public good. Two Nash equilibria $\{\mathrm{E}_1, \mathrm{E}_2\}$ in $\mathfrak{E}(\tilde{I})$ and $\{\mathrm{F}_1, \mathrm{F}_2\}$ in $\mathfrak{E}(\hat{I})$ satisfy the sufficient condition in Theorem 4 except the normality and attain the different amounts of public good. Therefore the neutrality theorem does not hold. This implies that the assumption of normal goods is indispensable for Theorem 4.

FIGURE IS AROUND HERE.

---

4 Efficiency and Equity

We show in this section that the Pareto improvement of Nash equilibrium in the strict sense is possible by making the income distribution more unequal. Throughout the section, we assume that Assumptions 1 and 2 are true.

Let $\hat{I} \stackrel{\text{def}}{=} (\hat{I}_1, \ldots, \hat{I}_n) \in \mathbb{R}^n_{++}$ be an income distribution. Assume that $J(\hat{I}) = N$. Denote a Nash equilibrium in an economy $\mathfrak{E}(\hat{I})$ by $(\hat{x}_i, \hat{g}_i)_{i \in N}$. Let the total amount of public good be $\hat{G} \stackrel{\text{def}}{=} \sum_{i \in N} \hat{g}_i$.

Let us consider income redistributions based on a given income perturbation ΔI:

$$\Delta I \stackrel{\text{def}}{=} (\Delta I_1, \Delta I_2, \ldots, \Delta I_n), \quad \Delta I_1 + \Delta I_2 + \cdots + \Delta I_n = 0,\ \Delta I_1 > 0. \tag{8}$$

Due to the quantitative neutrality theorem, a half line $H \stackrel{\text{def}}{=} \{\ I^\mu \stackrel{\text{def}}{=} \hat{I} + \mu \Delta I \mid \mu > 0\ \}$ meets the set $Z \stackrel{\text{def}}{=} \{\ I \in \mathbb{R}^n_+ \mid \sum_{i \in N} I_i = \sum_{i \in N} \hat{I}_i,\ J(I) = N\ \}$, i.e. $Z \cap H \neq \emptyset$. Furthermore we know that an equilibrium allocation of the private good and the total supply of public good are identical with $(\hat{x}_i)_{i \in N}$ and $\hat{G}$ in every economy $\mathfrak{E}(I)$ satisfying $I \in Z \cap H$. Define

$$s \stackrel{\text{def}}{=} \sup\{\ \mu \in \mathbb{R}_+ \mid \hat{I} + \mu \Delta I \in Z\ \}. \tag{9}$$

Denote a Nash equilibrium in an economy $\mathfrak{E}(I^s)$ by $(x_i^s, g_i^s)_{i \in N}$.

We need the following extra assumptions:

Assumption 3 *A utility function $u_i(x_i, G)$ is continuously differentiable on $\mathbb{R}^2_{++}$ and*

$$\frac{\partial u_i}{\partial x_i}(x_i, G) > 0 \ \text{ if } (x_i, G) \in \mathbb{R}^2_{++}, i = 1, \ldots, n.$$

Assumption 4 $u_i(0, G) = u_i(x_i, 0) = \inf\{u_i(\tilde{x}_i, \tilde{G}) \mid (\tilde{x}_i, \tilde{G}) \in \mathbb{R}^2_+\}$, $\forall (x_i, G) \in \mathbb{R}^2_+$, $i = 1, \ldots, n$.

We can easily see that $\hat{G} > 0$ due to Assumptions 3 and 4.

We obtain the two following lemmas:

Lemma 4 *Suppose that Assumptions* 1, 3 *and* 4 *are true and that* $G > 0$. *Then the marginal rate of substitution* $m_i(x_i, G)$ *defined by*

$$m_i(x_i, G) \stackrel{\text{def}}{=} \frac{\partial u_i(x_i, G)/\partial G}{\partial u_i(x_i, G)/\partial x_i}, \ (x_i, G) \in \mathbb{R}^2_{++},$$

satisfies

$$\lim_{x_i \to 0+} m_i(x_i, G) = 0, \ \forall i \in N.$$

The proof of Lemma 4 is trivial.

Lemma 5 *Suppose that Assumptions* 1, 3 *and* 4 *hold. Then we have*

(1) $I^s \notin Z$,

(2) $m_i(x_i^s, G^s) = 1$, $x_i^s = \hat{x}_i$, $\forall i \in N$, *and* $G^s = \hat{G}$,

(3) $I_i^s > 0$,$\forall i \in N$.

Proof. (1) Assume contrarily that $I^s \in Z$. Then it holds that

$$g_i^s > 0, \ \forall i \in N.$$

The continuity of Nash equilibria implies that for sufficiently small $\varepsilon > 0$

$$\exists \tilde{I}_i \in \mathbb{R}_{++} : \ |\tilde{I}_i - I_i^s| < \varepsilon \Longrightarrow \tilde{g}_i > 0, \ \forall i \in N,$$

where $(\tilde{g}_i)_{i \in N}$ denotes a list of contributions to the public good at the Nash equilibrium in an income distribution $\tilde{I} = (\tilde{I}_1, \ldots, \tilde{I}_n)$. This contradicts the definition of s.

(2) Let μ be any real number such that $0 < \mu < s$. Then we have

$$\exists \tilde{\mu} > 0 : \mu < \tilde{\mu} < s \text{ and } I^{\tilde{\mu}} \in Z.$$

Applying the qualitative neutrality theorem to $\hat{I}$ and $I^{\tilde{\mu}}$ leads us to

$$G^{\tilde{\mu}} = \hat{G}, \ x_i^{\mu} = \hat{x}_i, \ g_i^{\tilde{\mu}} > 0, \ m_i(x_i^{\tilde{\mu}}, G^{\tilde{\mu}}) = 1.$$

Obviously, $\tilde{\mu}$ tends to s as $\mu \to s$. And thus we obtain by the continuity of marginal rate of substitution and Nash equilibria

$$G^s = \hat{G},\ m_i(x_i^s, G^s) = 1, x_i^s = \hat{x}_i.$$

(3) It suffices for us to show $x_i^s > 0$. By (2) we can see that $G^s = \hat{G} > 0$ so that there exists some $i \in N$ satisfying $g_i^s > 0$. Therefore $J(I^s) \neq \emptyset$. Furthermore it is clear that $I_i^s > 0$ if $i \in J(I^s)$. Lemma 4 together with the fact that $m_i(x_i^s, G^s) = 1$ for all $i \in N \setminus J(I^s)$ due to (2) leads us to

$$x_i^s = I_i^s > 0\ , \forall i \in N \setminus J(I^s).$$

∎

We are ready to establish the following theorem of Pareto improvement:

Theorem 5 [Pareto Improvement of Nash Equilibrium]
Suppose that Assumptions 1, 2, 3, *and* 4 *hold. Furthermore, assume that there exists some Nash equilibrium* $(\hat{x}_j, \hat{g}_j)_{j\in N}$ *at an income distribution* $(\hat{I}_1, \ldots, \hat{I}_n) \in \mathbb{R}^n_{++}$ *in which every individual contributes to the public good, i.e.* $\hat{g}_i > 0$ *for any* $i \in N$. *Then there exists some income distribution* $\tilde{I} \stackrel{\text{def}}{=} (\tilde{I}_1, \ldots, \tilde{I}_n) \in \mathbb{R}^n_{++}$ *satisfying* (i) *that* $\tilde{I}$ *is in the boundary of the set* Z *and that* (ii) *only one individual* (*individual* 1) *contributes to the public good in the equilibrium. Moreover, the Nash equilibrium resulting from a new income distribution* $(\tilde{I}_1+(n-1)d, \tilde{I}_2-d, \ldots, \tilde{I}_n-d)$ *for sufficiently small* $d > 0$ *is Pareto superior to* $(\hat{x}_i, \hat{g}_i)_{i\in N}$ *in the strict sense, when* $\phi_1'(\tilde{I}_1) > 1/(n-1)$. [4]

[4]For the differentiability of ϕ_1 we need an additional assumption that the Jacobian determinant derived from the necessary conditions of maximization problem doesn't vanish. The possibility that $\phi_1'(\tilde{I}_1) > 1/(n-1)$, becomes very high when the number of consumers is large. In fact, $\phi_1'(\tilde{I}_1) = 1-\alpha$ when his utility function is Cobb-Douglas such as $u_1(x_1, G) = x_1^{\alpha} G^{1-\alpha}$ and $0 < \alpha < 1$.

Proof. Lemma 5 (2) implies that there exists an income redistribution ΔI, $\Delta I_1 > 0$ satisfying:

$$x_i^s = \hat{x}_i,\ G^s {=} \sum_{i \in N} g_i^s = \sum_{i \in N} \hat{g}_i = \hat{G},\ m_i(x_i^s, G^s) = 1,$$
$$u_i(x_i^s, G^s) = u_i(\hat{x}_i, \hat{G}),\ \forall i \in N.$$

We make use of further income redistribution $\Delta I'$ if the cardinality of $J(I^s)$ is larger than or equal to two. $\Delta I'$ is defined as:

$$\Delta I_i' \neq 0 \ \text{ if } i \in J(I^s),\ \Delta I_i' = 0 \text{ if } i \notin J(I^s),\ \sum_{i \in J(I^s)} \Delta I_i' = 0,\ \Delta I_1' > 0.$$

Thus we can repeat the same procedure as exploited in the proof of Lemma 5 (2) to obtain $J(\tilde{I}) = \{1\}$ in an economy $\mathfrak{E}(\tilde{I})$ with an income distribution $\tilde{I} = (\tilde{I}_1, \ldots, \tilde{I}_n)$. Denoting a Nash equilibrium at $\tilde{I}$ by $(\tilde{x}_i, \tilde{g}_i)_{i \in N}$, we have

$$\tilde{x}_i = \hat{x}_i,\ \forall i \in N,\ \tilde{I}_i = \tilde{x}_i = \hat{x}_i,\ \forall i \in N \setminus \{1\},\ \tilde{G} \stackrel{\text{def}}{=} \tilde{g}_1 = \sum_{i \in N} \tilde{g}_i = \sum_{i \in N} \hat{g}_i = \hat{G},$$
$$m_i(\tilde{x}_i, \tilde{G}) = 1,\ u_i(\tilde{x}_i, \tilde{G}) = u_i(\hat{x}_i, \hat{G}),\ \forall i \in N.$$

Therefore it is true that

$$\tilde{x}_i = \hat{x}_i = \tilde{I}_i, \forall i \in N \setminus \{1\},\ m_i(\tilde{x}_i, \tilde{G}) = 1,\ \forall i \in N,\quad \hat{G} {=} \tilde{G} {=} \tilde{g}_1.$$

Now we redistribute income further. That is, we decrease each income of every individual in $N \setminus \{1\}$ by $d > 0$ and increase individual 1's income by $(n-1)d$. We can make every individual's income be positive by a suitable choice of d since $\tilde{I}_j > 0$ for all $j \in N$. Let (x_1^*, G^*) be a solution to problem:

$$\max_{x_1, G} u_1(x_1, G) \text{ subject to } x_1 + G = \tilde{I}_1 + (n-1)d.$$

The inequality $G^* > \tilde{G}$ holds since the public good is a normal good for the individual 1. Let us consider the problem of individual $i(\neq 1)$ stated as:

$$\max_{x_i, g_i} u_i(x_i, G^* + g_i) \text{ subject to } x_i + g_i = \tilde{I}_i - d.$$

By Lemma 3, the individual i maximizes his utility at $(x_i, g_i) = (\tilde{x}_i - d, 0)\{= (\tilde{I}_i - d, 0)\}$. Therefore, an allocation $((x_1^*, G^*), (\tilde{x}_2 - d, 0), \ldots, (\tilde{x}_n - d, 0))$ is a Nash equilibrium at an income distribution $(\tilde{I}_1 + (n-1)d, \tilde{I}_2 - d, \ldots, \tilde{I}_n - d)$.

By the differentiability of $u_1(\cdot, \cdot)$, there exists a function o_1 satisfying

$$\begin{aligned}
u_1(x_1^*, G^*) - u_1(\tilde{x}_1, \tilde{g}_1) &= (x_1^* - \tilde{x}_1)\frac{\partial u_1}{\partial x_1}(\tilde{x}_1, \tilde{g}_1) + (G^* - \tilde{g}_1)\frac{\partial u_1}{\partial G}(\tilde{x}_1, \tilde{g}_1) + o_1(x_1^*, G^*, \tilde{x}_1, \tilde{g}_1) \\
&= \frac{\partial u_1}{\partial x_1}(\tilde{x}_1, \tilde{g}_1) \times ((x_1^* - \tilde{x}_1) + (G^* - \tilde{g}_1)m_1(\tilde{x}_1, \tilde{g}_1)) + o_1(x_1^*, G^*, \tilde{x}_1, \tilde{g}_1) \\
&= \frac{\partial u_1}{\partial x_1}(\tilde{x}_1, \tilde{g}_1) \times ((x_1^* - \tilde{x}_1) + (G^* - \tilde{g}_1)) + o_1(x_1^*, G^*, \tilde{x}_1, \tilde{g}_1) \\
&= \frac{\partial u_1}{\partial x_1}(\tilde{x}_1, \tilde{g}_1) \times (n-1)d + o_1(x_1^*, G^*, \tilde{x}_1, \tilde{g}_1) > 0.
\end{aligned}$$

for sufficiently small $d > 0$ [5]. Similarly, for $j \in N \setminus \{1\}$ we have

$$\begin{aligned}
&u_j(\tilde{x}_j - d, G^*) - u_j(\tilde{x}_j, \tilde{G}) \\
&\qquad = (\tilde{x}_j - d - \tilde{x}_j)\frac{\partial u_j}{\partial x_j}(\tilde{x}_j, \tilde{G}) + (G^* - \tilde{G})\frac{\partial u_j}{\partial G}(\tilde{x}_j, \tilde{G}) + o_j(\tilde{x}_j - d, G^*, \tilde{x}_j, \tilde{G}) \\
&\qquad = \frac{\partial u_j}{\partial x_j}(\tilde{x}_j, \tilde{G}) \times (G^* - \tilde{G} - d) + o_j(\tilde{x}_j - d, G^*, \tilde{x}_j, \tilde{G}).
\end{aligned}$$

Taking the fact that $\tilde{G} = \phi_1(\tilde{I}_1)$ and $G^* = \phi_1(\tilde{I}_1 + (n-1)d)$ into consideration, we have for sufficiently small positive d

$$\begin{aligned}
G^* - \tilde{G} - d &= (n-1)d \times \phi_1'(\tilde{I}_1) + o_{\phi_1}(\tilde{I}_1, (n-1)d) - d \\
&= \Big((n-1)\phi_1'(\tilde{I}_1) - 1\Big)d + o_{\phi_1}(\tilde{I}_1, (n-1)d) > 0,
\end{aligned}$$

[5] The function $o_1(x_1, G, \tilde{x}_1, \tilde{g}_1)$ satisfies a property: $\frac{o_1(x_1, G, \tilde{x}_1, \tilde{g}_1)}{\|(x_1 - \tilde{x}_1, G - \tilde{g}_1)\|} \to 0$ as $(x_1, G) \to (\tilde{x}_1, \tilde{g}_1)$, where $\|\cdot\|$ is the norm in $\mathbb{R}^2$.

since $\phi_1'(\tilde{I}_1) > 1/(n-1)$. Therefore, we have

$$u_j(\tilde{x}_j - d, G^*) - u_j(\tilde{x}_j, \tilde{G}) > 0, \ \forall j \in N \setminus \{1\}.$$

∎

The role of individual 1 in Theorem 5 can be played by an arbitrary individual, particularly by the individual who earns the highest income. This implies that a strict Pareto improvement can be attained by making income distribution more unequal. This shows a keen contrast between the efficiency of allocation and the equality of income distribution. We have to remark on Itaya-de Meza-Myles (1997) as an antecedent to our result. They show the dilemma by using a symmetric social welfare function in two persons economy. They establish a concurrence of a rise in the value of social welfare and a fall of income equality. They do not show, however, Pareto improvement of allocation which is shown in this paper. We also learn that their result is not applicable when the social welfare is Rawlsian, whereas any individualistic welfare will increase in our result.

5 Efficiency and Population

In this section, we consider how the total supply of public good in a Nash equilibrium relative to that in a Pareto efficient allocation varies as the number of individuals increases.

Let $\mathfrak{E}(\hat{I}) \stackrel{\text{def}}{=} (N, (u_j, \hat{I}_j)_{j \in N})$ be an economy with a given income distribution $\hat{I} \stackrel{\text{def}}{=} (\hat{I}_1, \ldots, \hat{I}_n) \in \mathbb{R}^n_{++}$. We define k-fold economy $\mathfrak{E}^k(\hat{I}) \stackrel{\text{def}}{=} (N^k, (u_j, \hat{I}_j)_{j \in N^k})$ of $\mathfrak{E}(\hat{I})$ as the one in which k number of economies $\mathfrak{E}(\hat{I})$ is contained, where $N^k \stackrel{\text{def}}{=} N \times K$ and $K \stackrel{\text{def}}{=} \{1, 2, \ldots, k\}$ [6]. That is, the number of individual i represented by $(u_i, \hat{I}_i)$ is k. In

[6]It is true that we ought to write an element of N^k as (i, j), $i = 1, \ldots, n$, $j = 1, 2, \ldots, k$, where $u_{(i,1)}(\cdot, \cdot) = u_{(i,j)}(\cdot, \cdot)$ and $I_{(i,1)} = I_{(i,j)}$ hold for any $j = 2, \ldots, k$ and $i = 1, \ldots, n$. We, however, simply write it as $i \in N^k$ for simple exposition.

this paper, we consider an expansion of population by an increase in k.

On the other hand, it is well known that any Pareto efficient allocation in $\mathfrak{E}(\hat{I})$ can be attained by a Lindahl equilibrium with a suitable income distribution[7]. Let $I \stackrel{\text{def}}{=} (I_1, I_2, \ldots, I_n)$ be an arbitrary fixed income distribution satisfying $\sum_{i=1}^{n} I_i = \sum_{i=1}^{n} \hat{I}_i$. We are interested in the amount of public good of a Nash equilibrium in $\mathfrak{E}^k(\hat{I})$ relative to that of Lindahl equilibrium in $\mathfrak{E}^k(I)$.

We need an additional assumption which is stated as in the following:

Assumption 5 *It holds that for an arbitrary positive income* I_i, $\lim_{G\to\infty} m_i(I_i, G) = 0,\ \forall i \in N$.

We denote the sum of contributions of public good except i by G_{-i}. A sufficient condition for an individual not to contribute to the public good is:

$$m_i(\hat{I}_i, G_{-i}) < 1.$$

By Assumption 5 there exists an infimum of a set $\{\ G_{-i} \mid m_i(\hat{I}_i, G) < 1 \text{ if } G > G_{-i}\ \}$. Let the infimum be G^u_{-i}. We can assume that $G^u_{-1} \leqq G^u_{-2} \leqq \cdots \leqq G^u_{-n}$ by renumbering the indices of individuals if necessary.

Our final theorem is stated as follows:

Theorem 6 *Let an economy* $\mathfrak{E}(\hat{I})$ *satisfy Assumptions* 1, 2, 3, 4 *and* 5. *The total supply of public good in a Nash equilibrium of* $\mathfrak{E}^k(\hat{I})$ *relative to that in a Lindahl equilibrium of* $\mathfrak{E}^k(I)$ *converges to zero as* k *tends to infinity, where* $I \in \mathbb{R}^n_{++}$ *and* $\sum_{i=1}^{n} I_i = \sum_{i=1}^{n} \hat{I}_i$. *The total amount of public good in Nash equilibrium never exceeds some finite value when the population grows. On the other hand, the total amount of public good in Lindahl equilibrium for an arbitrary fixed income distribution has no upper bounds when the population grows.*

[7] See Foley (1970) in details.

Proof. Let $(x_i^E, g_i^E)_{i\in N}$ be a Nash equilibrium in economy $\mathfrak{E}(\hat{I})$. First, we will show that the total supply of public good $G^E \stackrel{\text{def}}{=} \sum_{i\in N^k} g_i^E$ never exceeds G_{-n}^u. Let us define:

$$J^k \stackrel{\text{def}}{=} \{i \in N^k |\ g_i^E > 0\}, \qquad Y_i^E \stackrel{\text{def}}{=} \hat{I}_i + G_{-i}^E, i \in N^k,$$
$$G_{-i}^E \stackrel{\text{def}}{=} \textstyle\sum_{j\in N^k\setminus\{i\}} g_j^E, i \in N^k, \quad Y_i^u \stackrel{\text{def}}{=} \hat{I}_i + G_{-i}^u, i \in N^k.$$

It holds from the properties (d), (e) in Lemma 1 and the definition of Y_i^u that

$$x_i^E = \xi_i(Y_i^E),\ \forall i \in J^k,$$
$$x_j^E \leqq \xi_j(Y_j^E),\ \forall j \in N^k \setminus J^k,$$
$$\hat{I}_h = \xi_h(Y_h^u),\ \forall h \in N^k.$$

Assumption 2 implies that

$$Y_i^E < Y_i^u,\ \forall i \in J^k,$$
$$Y_j^E \geqq Y_j^u,\ \forall j \in N^k \setminus J^k,$$

since $x_i^E < \hat{I}_i,\ \forall i \in J^k$ and $x_j^E = \hat{I}_j,\ \forall j \in N^k \setminus J^k$. Therefore, we have

$$G^E = g_i^E + G_{-i}^E = \phi_i(Y_i^E) < \phi_i(Y_i^u) = G_{-i}^u,\ \forall i \in J^k,$$
$$G^E \geqq \phi_j(Y_j^E) \geqq \phi_j(Y_j^u) = G_{-j}^u,\ \forall j \in N^k \setminus J^k.$$

This implies that

$$G_{-j}^u \leqq G^E < G_{-i}^u \leqq G_{-n}^u,\ \forall i \in J^k, \forall j \in N^k \setminus J^k.$$

Denote Lindahl equilibrium in $\mathfrak{E}^k(I)$ by $(((p_i^k)_{i\in N})_{f=1}^k, ((x_i^k, G^k)_{i\in N})_{f=1}^k)$, where p_i^k is the Lindahl tax price for the i-th individual[8]. Now, it is clear that

$$\sum_{i=1}^n kp_i^k = 1 \ \ \text{(marginal cost of the public good is unity.)}$$

[8]Note that there exist k individuals to each $i \in N$, but their initial endowment and utility function are identical, so one doesn't need to change the allocation and tax price among them.

Hence, $p^k_{i_k} \stackrel{\text{def}}{=} \min\{p^k_i \mid i \in N\} \leqq 1/nk$. Let k tend to infinity. We obtain that one of the tax prices tends to zero. There exists an individual, say $j \in N$, whose tax price is the minimum $p^k_j = p^k_{i_k}$ in infinite times, since N is finite. The pair (x^k_j, G^k) is the solution to maximization problem:

$$\max u_j(x_j, G) \text{ subject to } x_j + p^k_j G = I_j, \text{ where } p^k_j \leqq \frac{1}{nk}.$$

for infinite number of k. By Assumptions 1 and 4, we have

$$G^k \to \infty \text{ as } k \to \infty.$$

The discussion made above leads us to

$$\lim_{k\to\infty} \frac{\text{total supply of public good at a Nash equilibrium in } k\text{-fold economy}}{\text{total supply of public good at a Lindahl equilibrium in } k\text{-fold economy}} = 0.$$

∎

This theorem implies that the amount of privately provided public good relative to that of an efficient allocation is very poor when the population is huge. We know, however, that actually there are many privately provided public goods. We can interpret these facts as implying that the models on the private provision of public good are not necessarily pertinent to describing an economy with large population.

References

Andreoni, J., 1988. Privately provided public goods in a large economy: the limits of altruism. Journal of Public Economics 35, 57-73.

Andreoni, J., Bergstrom, T., 1996. Do government subsidies increase the private supply of public goods? Public Choice 88, 295-308.

Bergstrom, T.C., Blume, L., Varian, H., 1986. On the private provision of public goods. Journal of Public Economics 29, 25-49.

Bernheim, D.B., 1986. On the voluntary and involuntary provision of public goods. American Economic Review 76, 789-793.

Boadway, R., Pestieau, P., Wildasin, D., 1989a. Non-cooperative behavior and efficient provision of public goods. Public Finance 44, 1-7.

Boadway, R., Pestieau, P., Wildasin, D., 1989b. Tax-transfers policy and the voluntary provision of public goods. Journal of Public Economics 39, 157-176.

Cornes, R., and Sandler, T., 1996. The Theory of Externalities, Public Goods, and Club Goods, 2nd Edition. New York: Cambridge University Press.

Cornes, R., and T.,Sandler 2000. Preto-improving distribution and pure public goods, *German Economic Review* 1, 169-186.

Foley, D.K., 1970. Lindahl's solution and the core of an economy with public goods. Econometrica 38, 66-72.

Itaya, J., de Meza, D., Myles, G.D., 1997. In praise of inequality: public good provision and income distribution. Economics Letters 57, 289-296.

Kemp, M.C., 1984. A note on the theory of international transfers. Economics Letters 14, 259-262.

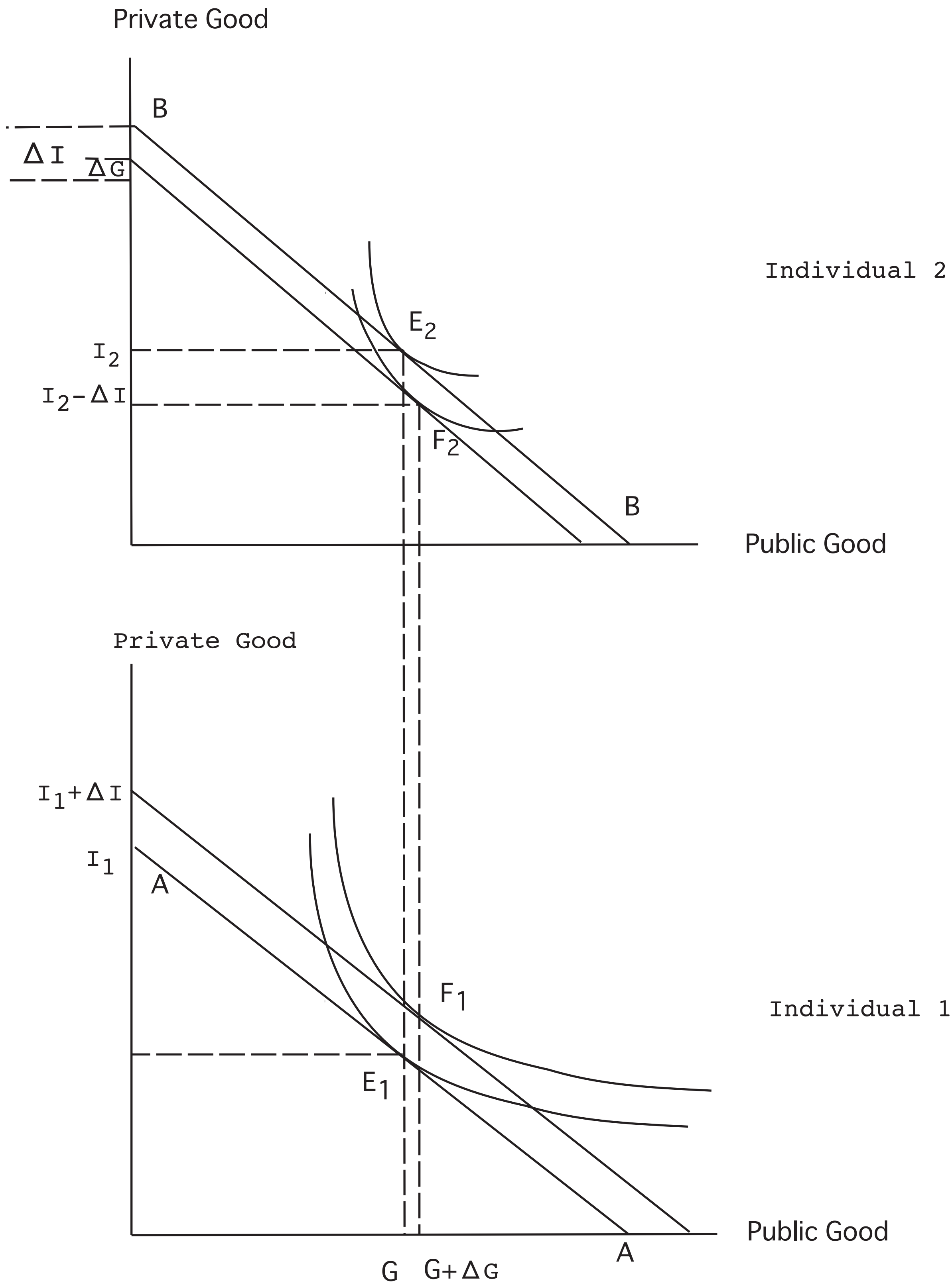
McGuire, M., 1974. Group size, group homogeneity, and the aggregate provision of a pure public good under Cournot behavior. Public Choice 18, 107-126.

Olson, M., 1965. The Logic of Collective Action: Public Goods and the Theory of Groups. Cambridge, Massachusetts.

Saijo, T., Tatamitani, Y., 1995. Characterizing neutrality in the voluntary contribution mechanism. Economic Design 1, 119-140.

Warr, P.G., 1982. Pareto optimal redistribution and private charity. Journal of Public Economics 19, 131-138.

Warr, P.G., 1983. The private provision of a pure public good is independent of the distribution of income. Economics Letters 13, 207-211.


Private Good
B
ΔI
ΔG
Individual 2
E_2
I_2
$I_2 - \Delta I$
F_2
B
Public Good
Private Good
$I_1 + \Delta I$
I_1
A
F_1
Individual 1
E_1
Public Good
G
$G + \Delta G$
A

Figure