Estimating the Nonlinear New Keynesian Model with the Zero Lower Bound for Japan

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Abstract

We estimate a small-scale macroeconomic model for Japan by taking account of nonlinearity coming from the zero lower bound (ZLB) of nominal interest rates. To this end, we apply Sequential Monte Carlo Squared developed by Chopin, Jacob, and Papaspiliopoulos (2013) and Herbst and Schorfheide (2015) to Japan, where the ZLB has constrained monetary policy for a considerably long period. Nonlinear estimation is crucial to draw implications for monetary policy. For example, the Bayesian model selection suggests that the past experience of recessions to bring the nominal interest rate down to zero is carried over to today's monetary policy. Nonlinear estimation, however, hardly changes the estimate of the natural rate of interest, which has often been negative since the mid-1990s.

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1 Introduction

In current macroeconomics, dynamic stochastic general equilibrium (DSGE) models are a workhorse tool for assessing the state of the economy and the effects of policy. However, the effectively zero lower bound (ZLB) on the nominal interest rate poses a serious technical challenge. Because of the nonlinearity the ZLB produces, solving a rational expectation equilibrium is computationally hard. Even harder is estimating the model, because it requires us to repeat the process of solving rational expectation equilibrium for a number of parameter values. Neglecting the ZLB may generate biased estimates and thus wrong policy implications, because the ZLB often changes quantitative results, as is pointed by Fernández-Villaverde et al. (2015), Boneva, Braun and Waki (2016), and Nakata (2017). The problem of the ZLB is particularly larger for Japan because the Japanese economy has been trapped at the effective ZLB for about two decades since 1995.

In this study, we estimate a DSGE (New Keynesian) model for Japan, where the model is nonlinear and stochastic. To facilitate computations, we use one of the simplest New Keynesian models that abstracts many important features in the standard DSGE models such as consumption habit, capital formations, and wage stickiness. However, we incorporate nonlinearity and stochastic shocks which make the ZLB occasionally binding. Through the estimation, we aim to investigate mainly two things. First, we analyze Japan's monetary policy. Although many central banks conducted commitment (forward guidance) policy under the ZLB in the aftermath of the financial crises, estimating such policy is difficult because we have to embed the ZLB in a model. We address a question which variable a central bank refers to when its monetary policy has an inertia and depends on a past variable. One may think that the monetary policy rule refers to the *actual* nominal interest rate in the previous period that takes either zero or above, while another possibility is that it refers to the *notional* interest rate in the previous period that can take values lower than zero. Other thins being equal, the latter type of monetary policy rule is considered to have a larger effect on the economy at the ZLB, because today's interest rate is likely to be lower due to a negative notional interest rate. Therefore, identifying a true monetary policy rule as well as estimating its parameters by fully taking account of nonlinearity coming from the ZLB is inevitable to assess the effects of monetary policy. However, most empirical studies are unable to do this because they often neglect the ZLB, and even if they consider the ZLB, their models often rest on either of the two monetary policy rules. For example, Aruoba, Cuba-Borda, and Schorfheide (2017) assume the former, while Gust et al. (2017) and Richter and Throckmorton (2016) assume the latter.[1] In addition, we investigate impulse

[1]Reifschneider and Williams (2000) propose monetary policy under the ZLB similar to the latter type of monetary policy rule. Some recent studies such as Del Negro, Giannoni, Patterson (2015) and McKay, Nakamura and Steinsson (2016) cast a doubt on the power of commitment policy, because the theoretical power of forward guidance is too strong.

response functions to shocks given various kinds of monetary policy specifications as well as the probability that the ZLB constrains monetary policy.

The second main thing we study is the natural rate of interest. The natural rate of interest is the real interest rate that would lead to price stability (Wicksell (1936)). It plays a pivotal role in New Keynesian models as in Woodford (2003). For example, when the actual real interest rate exceeds the natural rate of interest, the real economic activity is dampened and the inflation rate decreases. Krugman (1998) points out a possibility that an equilibrium real interest rate fell in Japan, although he does not use the term of the natural rate or estimate it quantitatively. By estimating the natural rate of interest, we aim to elucidate Japan's stagnant recessions, so-called the lost decades, since the early 1990s.

To estimate the model with the ZLB, we adopt mainly two novel approaches. First, we use the method developed by Richter, Throckmorton, and Walker (2014): a *time iteration method with linear interpolation* to solve a rational expectation equilibrium. This method is within the class of policy function iterations, and as they argue, it is flexible, accurate, and speedy. Second, we estimate parameters using *Sequential Monte Carlo Squared* (SMC^2), which is developed by Chopin, Jacob, and Papaspiliopoulos (2013) and applied to DSGE models by Herbst and Schorfheide (2015). There, we evaluate the likelihood of a nonlinear model given a certain parameter set by generating particles of endogenous variables (often called the particle filter). In addition, by sampling the particles of parameter sets, we draw the posterior distribution of parameters (often called the Sequential Monte Carlo or SMC).[2] As Chopin, Jacob, and Papaspiliopoulos (2013), Herbst and Schorfheide (2015) and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) argue, compared to the particle Markov chain Monte Carlo (MCMC) technique, the SMC^2 leads to a more reliable posterior inference and we do not need big measurement errors.

Our key findings are as follows. First, we show that nonlinear estimation is crucial to draw implications for monetary policy. For example, the Bayesian model selection chooses a model in which today's monetary policy depends on the notional interest rate that can take negative values in the previous period. This suggests that the past experience of recessions to bring the nominal interest rate down to zero is carried over to today's monetary policy, preventing the Bank of Japan from tightening monetary policy even if the economy picks up. Such a carry-over policy is interpreted as a forward guidance (commitment) policy. That is, the Bank of Japan has been conducting forward guidance policy at the ZLB by committing

[2]Before the development of SMC^2, the use of the SMC for parameter estimation is limited to linear state space models, where the Kalman filter can be applied to evaluate the likelihood. See, for example, Chopin (2002) and Herbst and Schorfheide (2014). Alternatively, when estimating nonlinear state space models, past studies use the particle filter by combining the Markov chain Monte Carlo (MCMC) technique (e.g., Metropolis–Hastings (MH) algorithm), which is developed by Andrieu, Doucet, and Holenstein (2010) and often called the particle MCMC technique. See also Kitagawa (1996) and Fernandez-Villaverde and Rubio-Ramirez (2005) for the particle filter.

to continuing the zero rate policy for a long time. Impulse response functions to a monetary policy shock are very different depending on the model for a monetary policy rule as well as a sign of the shock.

Second, we find that the ZLB does not produce a bias for the estimated natural rate of interest. Although the ZLB produces considerable biases for parameter estimates and thus changes policy implications, the effect of biased parameters on the natural rate of interest is canceled out by the effect of biased shocks on it. The natural rate of interest has often been negative since the mid-1990s, which is caused mainly by weak demand shocks.

We are not the first attempt to estimate a nonlinear DSGE model with the ZLB.[3] Two closest papers are Gust et al. (2017) and Richter and Throckmorton (2016). Compared with them, there are mainly three differences. First, they use data for the United States, where the ZLB matters only for several years, while the nominal interest rates have been almost zero for two decades in Japan. For the reason, the constrained linear model, in which the model is linear except for the ZLB, performs very poorly for Japan unlike the finding by Richter and Throckmorton (2016). Second, we use the SMC^2 by generating particles for not just model variables (shock processes) but also parameters, while these two papers use the MCMC to estimate the model. Like their papers, we generate particles for endogenous variables to compute the likelihood, because the model is nonlinear and thus we cannot employ the Kalman filter. In addition, in the SMC^2, we draw the posterior distribution of parameters by sampling the particles of parameter sets. As we explain in details, the SMC^2 enables us to obtain reliable posterior inference. We do not need to assume large measurement errors for feasibility, unlike Gust et al. (2017) and Richter and Throckmorton (2016). Third, Gust et al. (2017) estimate a far richer medium-sized DSGE model than Richter and Throckmorton (2016) and we do.

Studies on developments in the natural rate of interest include Krugman (1998), Laubach and Williams (2003, 2016), Neiss and Nelson (2003), Andrés, López-Salido, and Nelson (2009), Hall (2011), Barsky, Justiniano, and Melosi (2014), Ikeda and Saito (2014), Cúrdia (2015), Cúrdia et al. (2015), Bank of Japan (2016), Del Negro et al. (2017), Hirose and Sunakawa (2017), and Holston, Laubach, and Williams (2017) among many. Except for the very recent paper by Hirose and Sunakawa (2017), no study in the above rests on the DSGE model with the explicit consideration of the ZLB. Hirose and Sunakawa (2017) evaluate the natural rate of interest using the DSGE model with the ZLB for the United States, but do not estimate the model with the ZLB. Instead, they estimate the model without the ZLB for the periods before the ZLB constrains the economy and then evaluate the natural rate

[3] Hirose and Sunakawa (2015), Hirose and Inoue (2016), and Aruoba, Cuba-Borda, and Schorfheide (2017) estimate a model without the ZLB, although they generate data or make simulation considering the ZLB. By assuming that the duration of ZLB, τ_t, is perfectly foresighted in each period t, Kulish, Morley, and Robinson (2017) estimate τ_t and time-varying policy functions given estimated τ_t. Aoki and Ueno (2012) and Kim and Pruitt (2017) make use of forward rate curves and forecasters' survey, respectively.

of interest using the estimated parameters for the extended periods.

The remaining part of this paper is structured as follows. After Section 2 briefly explains our model, Section 3 outlines our estimation methods. Sections 4 and 5 discuss our estimation results regarding monetary policy and the natural rate of interest, respectively. Section 6 concludes.

2 Model

Our model is one of the simplest New Keynesian models. The economy consists of a representative household, firms, and a central bank. Firms consist of intermediate-good producers who are monopolistically competitive and final-good producers who are perfectly competitive. We consider three types of monetary policy, including one that abstracts the ZLB. The economy is subject to three types of exogenous shocks: discount factor, technology, and monetary policy.

2.1 Household

A representative household maximizes welfare:

$$E_t\left[\sum_{j=0}^{\infty}\beta^j Z^b_{t+j}\left\{\frac{C^{1-\sigma}_{t+j}}{1-\sigma}-\frac{\left(A_{t+j}\right)^{1-\sigma}\chi l^{1+\omega}_{t+j}}{1+\omega}\right\}\right], \tag{1}$$

subject to the budget constraint $C_t + B_t/P_t \leq W_t l_t + R_{t-1}B_{t-1}/P_t + T_t$, where C_t, l_t, P_t, W_t, R_t and T_t represent consumption, labor services, the aggregate price level, the real wage, the nominal rate of return, and lump-sum transfer in period t, respectively, and B_t represents the holding of one-period riskless bonds at the end of period t. Parameter $\beta \in (0,1)$ is the subjective discount factor, $\sigma > 0$ measures the inverse of the intertemporal elasticity of substitution of consumption, $\omega > 0$ is the inverse of the labor supply elasticity, and $\chi > 0$ is the scale factor. Finally, Z^b_t represents a stochastic shock to the discount factor with its unit mean and obeys the AR(1) process:

$$\log(Z^b_t) = \rho^b \log(Z^b_{t-1}) + \epsilon^b_t, \tag{2}$$

and A_t represents a stochastic shock to technology, which is specified below.

2.2 Firms

The final-good firm faces perfect competition and produces output Y_t by choosing a combination of intermediate inputs $Y_{f,t}$ ($f \in [0,1]$) so as to maximize profits subject to the Dixit-Stiglitz form of aggregations $Y_t = \left\{\int_0^1 Y_{f,t}^{\frac{\varepsilon-1}{\varepsilon}} df\right\}^{\frac{\varepsilon}{\varepsilon-1}}$, where $\varepsilon > 1$ represents the elasticity of substitution between intermediate goods.

Intermediate-good firm f produces output $Y_{f,t}$ with the production function $Y_{f,t} = A_t l_{f,t}$. The technology shock A_t obeys the I(1) process with the non-zero growth rate of γ^a, where $\mu_t^a \equiv \log(A_t/A_{t-1}) - \gamma^a$ is given by

$$\mu_t^a = \rho^a \mu_{t-1}^a + \epsilon_t^a. \tag{3}$$

Intermediate-good firm f maximizes its firm value by setting the optimal price $P_{f,t}$ in period t in the presence of Rotemberg-type price adjustment cost:

$$E_t \left[\sum_{j=0}^{\infty} \beta^j \frac{\Lambda_{t+j} Z_{t+j}^b}{\Lambda_t Z_t^b} \left(\frac{P_{f,t+j}}{P_{t+j}} - \frac{W_{t+j}}{A_{t+j}} - \frac{\phi}{2} \left(\frac{P_{f,t+j}}{P_{f,t+j-1}} - \pi^* \right)^2 \right) Y_{f,t+j} \right] \tag{4}$$

subject to downward-sloping demand, where Λ_t and π^* represent the stochastic discount factor and the target inflation rate, respectively, and ϕ captures the degree of Rotemberg-type price adjustment cost.

2.3 Central Bank

In this study, we consider three types of models regarding monetary policy. Models 1 and 2 incorporate the ZLB as

$$R_t = \max\left(1, R_t^*\right), \tag{5}$$

where R_t^* represents the notional interest rate that can take values lower than one. The actual interest rate R_t cannot be below one. The third model, Model without the ZLB, neglects the ZLB constraint. Models 1 and 2 are characterized by the following monetary policy rule

$$R_t^* = \left(R_{t-1}^*\right)^{\rho^r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_\pi} \left(\frac{Y_t/A_t}{Y_t^*/A_t}\right)^{\psi_y} \right)^{1-\rho^r} e^{\epsilon_t^r}, \tag{6}$$

and

$$R_t^* = \left(R_{t-1}\right)^{\rho^r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_\pi} \left(\frac{Y_t/A_t}{Y_t^*/A_t}\right)^{\psi_y} \right)^{1-\rho^r} e^{\epsilon_t^r}, \tag{7}$$

respectively, where ρ^r, ψ_π, and ψ_y capture monetary policy responses to the past interest rate, the inflation rate, and the output gap, respectively. Models 1 and 2 differ only in which interest rate the central bank refers to. In Model 1, it is the notional interest rate, whereas it is the actual interest rate in Model 2. We denote the steady-state natural rate of interest, the natural level of output, and the inflation rate by r^*, Y_t^*, and π_t, respectively, while ϵ_t^r represents a stochastic *i.i.d.* shock to the monetary policy with zero mean.

Compared to Model 2, Model 1 involves stronger commitment for future policy. Because R_t^* can be below zero and depends on R_{t-1}^*, the experience of adverse shocks in the past tie the hands of the central bank for long periods. In other words, the central bank compensates for its inability to lower the policy rate below zero by continuing the zero interest rate policy longer, other things being equal.

2.4 Closing the Model

The goods market is cleared as

$$Y_t = C_t + \phi \left(\pi_t - \pi^*\right)^2 Y_t/2. \tag{8}$$

In addition, the flexible-price equilibrium is defined as that without Rotemberg-type price adjustment cost.

The natural rate of interest r_t^* in the model equals the real rate of return in this flexible-price economy. Similarly, the natural level of output Y_t^* equals output in the flexible-price economy.

3 Solution, Econometric Inference, and the Advantage of the SMC^2

In this section, we outline how we solve and estimate the nonlinear DSGE model with the ZLB. See Appendix for the details. We then explain the data we use and prior specifications including the size of measurement errors. Finally, we discuss the advantage of our estimation methodology, particularly, SMC^2.

3.1 Model Solution

We solve the rational expectation equilibrium of our model using the *time iteration method with linear interpolation* (TL). This is within the class of policy function iteration methods, and Richter, Throckmorton, and Walker (2014) report that the TL provides the best balance between speed and accuracy.

To be more precise, we solve a rational expectation equilibrium or policy function given parameters θ. Note that, in our model, the policy function of any variable X_t is expressed as $X_t = X(\mu_t^a, Z_t^b, \epsilon_t^r, R_{t-1}^*)$, because there are three shocks and one state variable R_{t-1}^*. Intuitively speaking, the TL begins with making a time iteration for a policy function until intertemporal equations that describe relations between X_t and $E_t(X_{t+1})$ are satisfied at every node. Compared to the fixed-point iteration, it is costly to call a nonlinear solver on each node, but more stable exactly because the policy function is optimized on each node. We then locally approximate the policy functions with linear interpolation. Compared to global approximation methods such as the projection method using the Chebyshev polynomial basis, linear interpolation is considered to perform better in an environment where the ZLB produces kinks in the policy functions.

3.2 Estimation

We estimate the nonlinear DSGE model with the ZLB in a Bayesian manner. We estimate parameters using the SMC^2 developed by Chopin, Jacob, and Papaspiliopoulos (2013) and Herbst and Schorfheide (2015). It takes the following four steps. Step 1 is initialization. We draw N_θ particles for parameters θ. Step 2 is correction. Given θ, we compute the likelihood $\hat{p}(Y_t|\theta)$ and incremental weight w. Step 3 is selection. We resample θ and w based on w. Step 4 is mutation. We propagate θ and w using the MH algorithm. We repeat Steps 2 to 4 for N_ϕ stages.

In Step 2, we solve the model given θ using the TL. Then, after drawing N_S particles for shock processes $(\mu_t^a, Z_t^b, \epsilon_t^r)$, we generate the path of variables $\hat{Y}_t$, compare it with actual observable variables Y_t, and compute the likelihood $\hat{p}(Y_t|\theta)$ using the measurement error of Y_t. Note that, because the model is nonlinear, we cannot apply the Kalman filter. We use the particle filter, where we replace $p(Y_t|\theta)$ by $\hat{p}(Y_t|\theta)$ using a sufficiently large number of particles N_S with respect to shocks.

In our benchmark estimation, we use the particles of $N_S = 40,000$ and $N_\theta = 1,200$ and $N_\phi = 10$ stages. One estimation takes about a week using a 32-core (Intel Xeon E5-2698v3) computer.

3.3 Data

We use data for Japan from 1983:2Q to 2016:2Q. The beginning period is chosen to coincide with that for the output gap data, which we will use for a robustness analysis. In the benchmark estimation, we use three variables: the real per-capita GDP growth rate ($\Delta \log Y_t$), the CPI inflation rate (π_t), and the overnight call rate (R_t). Figure 1 shows time-series changes in these variables. In obtaining $\Delta \log Y_t$, we divide the real GDP by the population aged 15 years old or over. For π_t, we exclude the effects of consumption tax changes using X12ARIMA. These two variables are quarterly changes from the previous quarter, so R_t is divided by four to make it quarterly. As an alternative to $\Delta \log Y_t$, we later use the output gap $(\log(Y_t/Y_t^*))$ constructed by the Bank of Japan.

3.4 Prior Specifications

We choose parameter values based on Smets and Wouters (2007) and Sugo and Ueda (2008). We fix some of the parameters as $\beta = 0.995$, $\chi = 1$, and $\varepsilon = 6$. Table 1 shows the prior distribution of parameters, where κ is defined by $(\varepsilon - 1)(\omega + \sigma)/(\phi\pi^*)$. Note that, to enhance readability, we express the parameters of γ^a and $\pi^*$by $100\gamma^a$ and $100(\pi^* - 1)$, respectively. We also discuss the natural rate of interest r_t^* by deducting one.

As for the measurement errors of $\Delta \log Y_t$, π_t, and R_t, we assume that their sizes are 0.5%, 0.5%, and 0.25% of their actual standard deviations, respectively. Their sizes are far smaller than those in Gust et al. (2017) and Richter and Throckmorton (2016), where they are $\sqrt{0.25} \sim 50\%$ and $\sqrt{0.1} \sim 30\%$, respectively. We assume that the measurement error of R_t is lower than those of $\Delta \log Y_t$ and π_t, but this difference is minor and hardly changes our following results.

3.5 Advantage of the SMC^2 over the MCMC

We use the SMC^2 by generating particles for not just shock processes $(\mu_t^a, Z_t^b, \epsilon_t^r)$ but also parameters θ by N_S and N_θ, respectively, while Gust et al. (2017) and Richter and Throckmorton (2016) use the MCMC to estimate the model by generating particles only for shock processes. As Chopin, Jacob, and Papaspiliopoulos (2013), Herbst and Schorfheide (2015) and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) argue, the SMC^2 can lead to a more reliable posterior inference than the particle MCMC technique. Furthermore, we do not need big measurement errors for observable variables unlike Gust et al. (2017) and Richter and Throckmorton (2016).[4]

To understand these advantages better, let us use our estimation result (discussed in the next section) as an illustrative example. Figure 2 shows a scatter plot where each dot represents a particle for the value of parameter σ (horizontal axis) and its posterior likelihood (vertical axis). The dots are dense around $\sigma = 1.4$. Their median lies around this level, as is shown in the big circle in red. Interestingly, the likelihood becomes the largest when σ is around 1.5, as is shown in the big circle in green at the top of the graph. However, this circle seems to be an outlier for three reasons. Particles are sparse around this circle; the likelihood drops when σ slightly deviates from the value; and the computation of the likelihood is subject to errors. The last point arises because the model is nonlinear and thus the Kalman filter cannot be applied. We thus need to use particles for shock processes and introduce measurement errors for observable variables to approximate the likelihood. Figure 2 shows that our estimation is not trapped at this outlier.

What will happen if we use the MCMC in this example? In the MCMC, we compare only two parameter candidates, for example, new candidate σ_1 and previously selected σ_0. Thus, once the aforementioned circle in green is selected as either σ_1 or σ_0, our estimation is likely to be trapped at this point, because the likelihood at its neighborhood is discontinuously lower. In other words, the acceptance probability of new parameter values falls to zero. This problem becomes more serious when measurement errors for observable variables are smaller, because the likelihood becomes more sensitive to parameter changes. Therefore, we need to assume big measurement errors to keep the acceptance probability of 25% when we use the MCMC. For the same reason, the MCMC is considered to be sensitive to the shape of modes. If distribution is close to bimodal, the posterior inference becomes unstable between

[4] In addition, the SMC^2 can be easily paralleled.

the two.[5]

The SMC^2 can resolve this problem. Because the SMC^2 uses more than two particles for parameter candidates and allocates weight w on each particle corresponding to its likelihood, particles are much less likely to be stuck at the outlier. Posterior distribution becomes diverse, as is shown in the graph.[6]

4 Evaluating Japan's Monetary Policy during the Low Inflation Period

In this section, after we briefly report parameter estimates, we discuss monetary policy for Japan.

4.1 Parameter Estimates

In the following discussions, we report results mainly of Model 1, because the marginal likelihood is the highest. We hereafter call Model 1 the baseline model. The estimates of structural parameters such as σ and ω are within a range reported in earlier studies. The inflation target π^* is 0.36% quarterly, that is, 1.44% annually. This value lies between the Bank of Japan's formal target, 2%, and the mean of the actual inflation rates in the sample period, 0.44%. The trend component of the technology shock A_t, γ^a, is -0.028% quarterly, which renders the steady-state natural rate of interest $r^* - 1 = e^{\sigma\gamma^a}/\beta - 1$ to be 0.46% quarterly.

4.2 Comparing Monetary Policy Rules

In the following two subsections, we discuss differences among Model 1, Model 2, and Model without the ZLB. It is not obvious which model is better between Model 1 and Model 2. In this respect, it is important to compare the performance of two models by Bayesian estimation by fulling taking account of the ZLB and using data at the ZLB.

4.2.1 Marginal Likelihood and Posterior Probability

Table 2 shows the parameter estimates and marginal likelihood for three types of models regarding monetary policy: Model 1, Model 2, and Model without the ZLB. The marginal

[5] Herbst and Schorfheide (2015) report that they cannot estimate a Smets and Wouters-type (2007) medium-scale DSGE model by using the MCMC with the particle filter.

[6] The likelihood tempering approach in the SMC^2 is another reason why the SMC^2 performs well. In the mutation step at each stage, a scaling factor is revised to keep the acceptance probability around 25%. This enables us to obtain smooth posterior distribution of parameters around modes, as the number of stages increases.

likelihood is the highest for Model 1, supporting Model 1 with the posterior probability of one over Model 2 and Model without the ZLB. Because the nominal interest rate has been almost zero for the recent two decades, the fit of Model without the ZLB is the worst.

This suggests that the past experience of recessions such that they bring the nominal interest rate down to zero prevents the Bank of Japan from tightening monetary policy today even if the economy picks up. In other words, the Bank of Japan was conducting forward guidance policy at the ZLB by committing to continuing the zero rate policy for a long time.

4.2.2 Parameter Estimates and the Notional Interest Rate

We investigate the effects of neglecting the ZLB constraint in the estimation of the DSGE model by comparing parameter estimates between Model 1 and Model without the ZLB. A big difference is observed in the inverse of the intertemporal elasticity of substitution of consumption σ, the trend component of the technology shock γ^a, the inflation target π^*, and the inertia of the monetary policy rule ρ^r. For these four parameters, Model without the ZLB yields smaller values than Model 1. In particular, smaller γ^a and π^* suggest that the steady-state nominal interest rate is lower, because it equals $e^{\sigma\gamma^a}/\beta - 1 + \pi^*$. It is -0.38% in Model without the ZLB, while it is 0.82% in Model 1. Model without the ZLB seems to require a low and negative steady-state nominal interest rate to explain the prolonged zero interest rate that occurred in Japan.

Figure 3 shows developments in the notional nominal interest rate $R_t^* - 1$ in Models 1 and 2, whereas it coincides with actual nominal interest rate $R_t - 1$ in Model without the ZLB. In Model 1, it has been around -4% annually since 1995. This contributes to lowering future interest rates. On the other hand, the notional nominal interest rate is around -2% annually in Model 2, but it does not constrain the future monetary policy.[7]

4.3 Comparing Impulse Response Functions under Different Monetary Policy Rules

Nonlinear estimation is crucial to draw implications for monetary policy. To show this, we calculate impulse response functions (IRFs) to a monetary policy shock and demonstrate that the IRFs are very different among Model 1, Model 2, and Model without the ZLB. Because the model is nonlinear, impulse response functions differ depending on the state of the economy $(\mu_t^a, Z_t^b, \epsilon_t^r, R_{t-1}^*)$. For illustration, we thus express IRFs conditioning actual states in the two historical periods, 1985:1Q and 2010:1Q in Figure 4. The former is a period

[7] In Models 1 and 2, the notional interest rate R_t^* does not necessarily coincide with the actual interest rate R_t even when the latter became positive. This difference is explained by the measurement error we introduce when we estimate the model.

when the nominal interest rate is well over zero, while the ZLB constrains the economy in the latter period. Further, because the model is nonlinear mainly owing to the ZLB, IRFs are asymmetric to the sign of the shock. We thus show IRFs to both positive (tightening) and negative (easing) monetary policy shocks. For comparison, we give the same size of the monetary policy shocks, that is, 0.01. The left panels in the figure are IRFs in 1985:1Q, while the right panels are those in 2010:1Q. The top panels represent the IRFs of the inflation rate π_t, while the bottom panels represent those of the nominal interest rate R_t.

As for 1985:1Q, the bottom-left panel of Figure 4 shows that the IRFs of the nominal interest rate are similar among Model1, Model 2, and Model without the ZLB. However, because Model without the ZLB has the smallest inertia (ρ^r) in its policy rule, the nominal interest rate converges to zero most quickly. Therefore, as the top-left panel shows, the IRFs of inflation is the smallest. We also find that the IRFs are symmetric to positive (Pos in the graph) and negative (Neg in the graph) monetary policy shocks.

In 2010:1Q, the IRFs come to differ a lot depending on models as well as the sign of the monetary policy shock. First, in Model without the ZLB, the IRFs are almost symmetric, because the nominal interest rate can be below zero. The negative monetary policy shock increases the inflation rate and decreases the nominal interest rate, as the top- and bottom-right panels show, respectively. Second, in Model 2, the negative monetary policy shock has a small effect on the inflation and nominal interest rates, although a slight effect exists because there is a slight probability that the ZLB does not constrain the economy around 2010:1Q in our simulation. On the other hand, the positive monetary policy shock has almost the same effect in Model 2 as in Model without the ZLB. This is because, in Model 2, monetary policy in period t is influenced by history only through the actual nominal interest rate in $t-1$. Thus, the experience of prolonged recessions does not tie the hand of the central bank, and hence, the positive monetary policy shock leads to an immediate rise in the nominal interest rate, as shown in the bottom-right panel.

Third, we examine the IRFs in Model 1. The negative monetary policy shock has a bigger effect on inflation in Model 1 than in Model 2, while it is smaller than in Model without the ZLB. This result stems because Model 1 involves strong commitment for future policy. Monetary policy in period t depends on R^*_{t-1}, which can take negative values. The negative monetary policy shock in period t decreases R^*_t, which functions to lower future nominal interest rates in the future. Therefore, the nominal interest rate becomes lower than otherwise three to six quarters ahead. Such commitment increases today's inflation rate. The positive monetary policy shock, on the other hand, has a smaller effect on inflation in Model 1 compared with both Model 2. Because monetary policy in period t depends on R^*_{t-1}, the positive monetary policy shock induces a smaller increase in the nominal interest rate, as is shown in the bottom-right panel.

It is impossible to know without estimation which model is close to reality. We argued that our estimation supports Model 1, as is shown in Table 2. This suggests the commitment effect of monetary policy or the power of forward guidance.

4.4 How Often Does the ZLB Constrain Monetary Policy?

Table 3 shows the probability that the nominal interest rate equals zero after $h = 1$, 2, 4, and 8 quarters given the natural rate of interest. To calculate this, we generate the path of nominal interest rates from $t + 1$ conditioning the level of the natural rate of interest r_t^* in period t. We count the number of events by Monte Carlo simulation. The probability of the ZLB is considerably high, that is, around 60% and 40% for $h = 1$ and 8, respectively, when r_t^* is around 0%. This is significantly higher comparing with 7.1% reported by Gust et al. (2017) for the United States.[8]

Such a high probability of the ZLB is associated with a high (low) probability of deflation (inflation). We calculate the probability that the inflation rate π_{t+h} reaches 2%, which is the target of the Bank of Japan, or falls below 0% for at least a certain h ($1 \leq h \leq 8$, i.e., within two years). If r_t^* is around 0%, the inflation rate can reach 2% only with the probability of 15%. Although we do not show here, this probability is even lower for Model 2 and Model without the ZLB, because the estimated π^* is lower. On the other hand, the probability of deflation is considerably high. Even if r_t^* is as high as 3% quarterly, the probability is higher than 75%.

4.5 Validity of Using the (Un)Constrained Linear Model

Estimating nonlinear DSGE models with the ZLB consumes a lot of time. It takes almost one week for one estimation even in such a simple model as ours. In this regard, Richter and Throckmorton (2016) argue that a constrained linear model performs well, while it mitigates computational burden. To examine if their finding holds true for Japan, we estimate the log-linearized DSGE model expressed by equations (9) to (11) but continue to impose the ZLB constraint properly.[9] We call it the constrained linear model. Moreover, we also estimate the unconstrained linear model, where we estimate the log-linearized DSGE model by neglecting the ZLB. That is, the unconstrained linear model corresponds to the constrained linear model without the ZLB.

Table 4 shows estimation results. Although we are able to estimate these two models with reasonable parameter values, we find a big change for γ^a and π^*, which matters for the steady-state natural rate of interest and inflation rate. The marginal likelihoods in these models are significantly lower than that in Model 1.

[8] Richter and Throckmorton (2016) do not calculate the probability. Instead, they calculate the expected duration of the zero interest rate, whose mode and mean are 1 and 3.2 quarters, respectively.

[9] The constrained linear model in Richter and Throckmorton (2016) may be different because they explain that they "impose the constraint in the filter but not the solution."

5 Evaluating Japan's Natural Rate of Interest

Next we turn to our second objective: the natural rate of interest for Japan. After we explain the mathematical expression of the natural rate of interest in our model, we discuss how our nonlinear estimation influences the estimate of the natural rate of interest, how much the natural rate of interest has declined during Japan's lost decades, and why.

5.1 Natural Rate of Interest in the Log-Linearized Model

Our model can be expressed by three key log-linearized equations. Although we do not make use of them in our nonlinear estimation, they illustrate a role played by the natural rate of interest, r_t^*.[10] They are

$$\pi_t - \pi^* = \beta e^{(1-\sigma)\gamma^a} E_t \left[\pi_{t+1} - \pi^*\right] + \frac{(\varepsilon - 1)(\omega + \sigma)}{\phi \pi^*}(y_t - y_t^*), \tag{9}$$

$$y_t - y_t^* = E_t \left[y_{t+1} - y_{t+1}^* - \frac{1}{\sigma}\left(\frac{R_t - r^*\pi^*}{r^*\pi^*} - \frac{\pi_{t+1} - \pi^*}{\pi^*} - \frac{r_t^* - r^*}{r^*} \right) \right], \tag{10}$$

and

$$r_t^* = r^* \left[1 + \sigma \rho^a \mu_t^a + (1 - \rho^b) \log Z_t^b) \right]. \tag{11}$$

Here, we define the log-linearized variables of $\{Y_t, Y_t^*\}$ by $\{y_t, y_t^*\}$ around their non-zero trends. Note that the steady-state natural rate of interest r^* equals $e^{\sigma\gamma^a}/\beta$.

This suggests that the natural rate of interest r_t^* plays an important role. When r_t^* decreases, both output gap $y_t - y_t^*$ and the inflation rate π_t decrease, unless monetary policy is strong enough to offset this. The decrease in r_t^* also causes the nominal interest rate to decrease, so the possibility of reaching the ZLB increases.

The above equations also suggest that output gap and the inflation rate depend only on the natural rate of interest r_t^* and the actual real interest rate $(\frac{R_t - r^*\pi^*}{r^*\pi^*} - \frac{\pi_{t+1} - \pi^*}{\pi^*})$. We do not need to know μ_t^a and Z_t^b, separately. These shocks influence output gap and the inflation rate only through a change in r_t^*. Moreover, equation (11) shows that the monetary policy shock ϵ_t^r is irrelevant to the natural rate of interest.

5.2 Developments in the Natural Rate of Interest

We show in the left panel of Figure 5 the time-series path of the natural rate of interest r_t^*. The figure shows a decline in the natural rate of interest. Although it stayed positive until the late 1990s and its steady-state value is 0.46% quarterly, the natural rate of interest often

[10] Moreover, they serve as an initial value of equilibrium when we solve the model nonlinearly.

became negative in the 2000s and 2010s. It often fell to around -0.5% quarterly, and at worst, to -2%.

Behind this development, discount factor shock Z_t^b is the most important culprit. In the left panel of the figure, we show how much of the model's fit is attributable to individual shocks $(\mu_t^a, Z_t^b, \epsilon_t^r)$. Similar to Gust et al. (2017), we decompose the variable by calculating the model's dynamics by assuming only one of the three shocks is present. Because of the nonlinearity, their sum is not necessarily equal to the natural rate of interest. The right panel of Figure 5 shows the time-series paths of the three types of shocks $(\mu_t^a, Z_t^b, \epsilon_t^r)$. It shows that discount factor shock Z_t^b explains most of the changes in the natural rate of interest in our estimation periods. The technology shock μ_t^a hardly explains the change in the natural rate of interest, while the monetary policy shock ϵ_t^r does not explain it at all, which is consistent with equation (11).

This result is in line with past studies. Gust et al. (2017) find that the risk premium shock as well as the marginal efficiency of investment shock are important in explaining the Great Recession in the United States, while the technology and monetary policy shocks explain little. For Japan, Sugo and Ueda (2008) estimate the medium-scale DSGE model for Japan, although they estimate a log-linearized model without the ZLB using the sample until 1995, when the ZLB did not constrain the Japanese economy. They also find that the investment shock is the most important. Although our model is far simpler than their models, the discount factor shock in our model is considered to be in the same class with these shocks.

5.3 Comparisons of the Natural Rate of Interest

5.3.1 Different Monetary Policy Specifications

In the previous section, we showed that nonlinear estimation greatly modifies implications for monetary policy. Is this true for the natural rate of interest as well? Figure 6 shows how much the estimated natural rate of interest changes depending on the type of the models we estimate. Somewhat surprisingly, the natural rate of interest in Model 2 as well as Model without the ZLB hardly changes from that in Model 1, although Model 2 and Model without the ZLB yield biased estimates as was shown in Table 2. In particular, during the period of the ZLB since 2000, the natural rate of interest is almost identical, although it deviates in the 1980s and 1990s. This result is in sharp contrast with that of Hirose and Sunakawa (2017). Instead of estimating a DSGE model with the ZLB, they estimate the model without the ZLB for the periods before the ZLB constrains the economy and then evaluate the natural rate of interest using the estimated parameters for the extended periods. They then find that the natural rate of interest is substantially higher when considering the ZLB than when neglecting the ZLB, particularly during the ZLB period.

To understand why the estimated natural rate of interest hardly differs in our study, we separate three types of differences between Model 1 and Model without the ZLB. They are (1) the presence of the ZLB, (2) estimated parameters, and (3) estimated shocks. More concretely, we simulate the natural rate of interest in Model without the ZLB by changing one of the three differences: (1) using estimated parameters and shocks in Model without the ZLB but now explicitly taking account of the ZLB, (2) using estimated parameters in Model 1, and (3) using estimated shocks in Model 1. The above (2) is analogous to Hirose and Sunakawa (2017).

Figure 7 shows the paths of the counterfactual natural rate of interest. As for type (1), it is shown that the presence of the ZLB does not influence the natural rate of interest per se. Because the natural rate of interest rests on the flexible-price economy by definition, the ZLB per se does not matter for its movements. As for types (2) and (3), the graph suggests that the parameter difference has almost the same-sized but opposing effect on the natural rate of interest as the shock difference. Type (2) increases the natural rate of interest, as in Hirose and Sunakawa (2017). This stems from the fact that the estimate of the steady-state natural rate of interest r^* in Model 1 is higher than that in Model without the ZLB as shown in Table 2. However, type (3) decreases the natural rate of interest by the same size, because the estimated shocks of (μ_t^a, Z_t^b) are lower for Model 1. This result suggests that we should estimate parameters and shocks simultaneously.

5.3.2 Laubach-Williams (2003) and Hodrick–Prescott Filter

Next, we compare the natural rate of interest based on our model with that based on Laubach and Williams (2003) and that based on the Hodrick–Prescott (HP) filter. Laubach and Williams (2003) and Holston, Laubach, and Williams (2017) estimate the backward-looking IS and Phillips curves jointly and calculate the natural rate of interest using the Kalman filter, where they calculate the *ex ante* real interest rate by estimating the one-year ahead inflation expectation from a univariate AR(3) model. In their model, the natural rate of interest is given by

$$r_t^* = g_t + z_t, \tag{12}$$

where g_t and z_t capture the trend growth rate of the natural output and other determinants such as demand disturbances, respectively. Therefore, g_t and z_t in their model correspond to $\log(A_t/A_{t-1}) = \mu_t^a + \gamma^a$ and $\log Z_t^b$ in our model, respectively. However, we can point out one important difference. Laubach and Williams (2003) and Holston, Laubach, and Williams (2017) assume that both g_t and z_t obey an I(1) process (i.e., the natural output is I(2)), while both $\log(A_t/A_{t-1})$ and $\log Z_t^b$ in our model obey to an I(0) process.

We apply their approach to the Japanese data and report both one-sided (filtered) and two-sided (smoothed) estimates of the natural rate of interest. For the HP filter, we set the smoothing parameter λ at 1,600 and smooth the same *ex ante* real interest rate we use to calculate the natural rate of interest based on Laubach and Williams (2003).

Figure 8 shows that the one-sided estimate of the natural rate of interest based on Laubach and Williams (2003) moves very closely to that based on our model. Although Laubach and Williams's (2003) approach does not take account of the ZLB, the movements of the natural rate of interest are similar when the nominal interest rate is effectively at the ZLB. This result is consistent with our previous finding that the nonlinear estimation hardly changes the estimate of the natural rate of interest, as shown in Figures 6 and 7. Furthermore, we confirm that most of the fluctuations of the natural rate of interest based on Laubach and Williams (2003) are caused by z_t, although we do not show here. This result is again consistent with ours. The fluctuations of the other two variables, that is, the two-sided estimate of the natural rate of interest based on Laubach and Williams (2003) and the natural rate of interest based on the HP filter, are much more smooth, although their means hardly change.

5.3.3 Use of the Output Gap Data

Finally, we check the robustness of our estimation by using an alternative measure of output, that is, the output gap. We estimate the same model either using the output gap instead of the growth of the real GDP or using both the output gap and the growth of the real GDP. Table 5 shows that parameter estimates hardly differ. However, as Figure 9 shows, the path of the natural rate of interest comes to differ quantitatively. In particular, when we use the output gap instead of the growth of the real GDP, the natural rate of interest becomes more volatile. Nevertheless, qualitatively, the three lines are very similar.

6 Concluding Remarks

In this study, we estimated a nonlinear DSGE model with the ZLB using a Bayesian technique. There are several potential avenues for future research. The first would be to estimate a richer DSGE model embedding capital, wage stickiness, financial frictions, and so on. We are aware that an intrinsic persistence in our model is low, which makes the economy go back to the steady state rather quickly. This could be one reason why we succeeded in estimating the nonlinear DSGE model with the ZLB even though the duration of the ZLB is considerably long in Japan. However, embedding these features poses a computational challenge because of the curse of dimensionality. Moreover, we suspect that it becomes harder to find a determinate equilibrium for a set of parameters, because such stickiness lengthens the duration of the ZLB further and makes equilibrium indeterminate, as Aruoba, Cuba-Borda, and Schorfheide (2017) argue. Second, our method could be applied to other types of models, where nonlinearity plays an important role. Examples include currency and financial crises, where crises occur as a tail-risk event and cause tremendous impacts on the economy.

References

[1] Andrieu,Christophe, Doucet, Arnaud, and Roman Holenstein (2010) "Particle Markov Chain Monte Carlo Methods," *Journal of the Royal Statistical Society*, B, 72(3), 269–342.

[2] Andrés, Javier, J. David López-Salido, and Edward Nelson (2009) "Money and the Natural Rate of Interest: Structural Estimates for the United States and the Euro Area," *Journal of Economic Dynamics and Control*, 33(3), 758–776.

[3] Aoki, Kosuke and Yoichi Ueno (2012), "Identification of Structural Shocks under the Zero Lower Bound on Nominal Interest Rates," Bank of Japan Working Paper 12-E-8.

[4] Aruoba, S. Borağan, Pablo Cuba-Borda, and Frank Schorfheide (2017) "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries," *Review of Economic Studies*, forthcoming.

[5] Bank of Japan (2016) "Supplementary Paper Series for the "Comprehensive Assessment" (2): Developments in the Natural Rate of Interest in Japan," Bank of Japan Review, 2016-E-12.

[6] Barsky, Robert, Alejandro Justiniano, and Leonardo Melosi (2014) "The Natural Rate of Interest and Its Usefulness for Monetary Policy," *American Economic Review Papers and Proceedings*, 104(5), 37–43.

[7] Boneva, Lena Mareen, R. Anton Braun, and Yuichiro Waki (2016) "Some Unpleasant Properties of Loglinearized Solutions When the Nominal Rate is Zero," *Journal of Monetary Economics*, 84, 216–232.

[8] Chopin, Nicolas (2002) "A Sequential Particle Filter for Static Models," *Biometrika*, 89, 539–552.

[9] Chopin, Nicolas, P. E. Jacob, and O. Papaspiliopoulos, (2013) "SMC2: An Efficient Algorithm for Sequential Analysis of State Space Models," *Journal of the Royal Statistical Society*, B,75(3), 397–426.

[10] Cúrdia, Vasco (2015) "Why So Slow? A Gradual Return for Interest Rates," *Federal Reserve Bank of San Francisco Economic Letter*, 2015-32, 13 October.

[11] Cúrdia, Vasco, Andrea Ferrero, Ging Cee Ng, and Andrea Tambalotti (2015) "Has U.S. Monetary Policy Tracked the Ecient Interest Rate?" *Journal of Monetary Economics*, 70, 72–83.

[12] Del Negro, Marco, Marc P. Giannoni, and Christina Patterson (2015) "The Forward Guidance Puzzle," Working Paper.

[13] Del Negro, Marco, Domenico Giannone, Marc P. Giannoni, and Andrea Tambalotti (2017) "Safety, Liquidity, and the Natural Rate of Interest," Brooking Papers on Economic Activity Conference Drafts.

[14] Fernández-Villaverde, Jesus, and Juan Rubio-Ramírez (2005) "Estimating Dynamic Equilibrium Economies: Linear Versus Nonlinear Likelihood," *Journal of Applied Econometrics*, 20(7), 891–910.

[15] Fernández-Villaverde, Jesus, Grey Gordon, Pablo A. Guerron-Quintana, and Juan Rubio-Ramírez (2015) "Nonlinear Adventures at the Zero Lower Bound," *Journal of Economic Dynamics and Control*, 57, 182–204.

[16] Fernández-Villaverde, Jesús, Juan F. Rubio-Ramírez, and Frank Schorfheide (2016), "Solution and Estimation Methods for DSGE Models," in *Handbook of Macroeconomics*, Volume 2, edited by John B. Taylor and Harald Uhlig, Chapter 9, 527–724.

[17] Gust, Christopher, Edward Herbst, David López-Salido, and Matthew E. Smith (2017) "The Empirical Implications of the Interest-Rate Lower Bound," *American Economic Review*, 107(7), 1971–2006.

[18] Hall, Robert E. (2011) "The Long Slump," *American Economic Review*, 101, 431–469.

[19] Herbst, Edward and Frank Schorfheide (2014) "Sequential Monte Carlo Sampling for DSGE Models," *Journal of Applied Econometrics*, 29(7), 1073–1098.

[20] Herbst, Edward and Frank Schorfheide (2015) *Bayesian Estimation of DSGE Models*, Princeton University Press, Princeton, NJ.

[21] Hills, Timothy, Taisuke Nakata, and Sebastian Schmidt (2016) "The Risky Steady State and the Interest Rate Lower Bound," Federal Reserve Board, Finance and Economics Discussion Series, 2016-009.

[22] Hirose, Yasuo, and Takeki Sunakawa (2015) "Parameter Bias in an Estimated DSGE Model: Does Nonlinearity Matter?" CAMA Working Paper 46/2015.

[23] Hirose, Yasuo, and Takeki Sunakawa (2017) "The Natural Rate of Interest in a Nonlinear DSGE Model," mimeo.

[24] Hirose, Yasuo, and Atsushi Inoue (2016) "The Zero Lower Bound and Parameter Bias in an Estimated DSGE Model," *Journal of Applied Econometrics*, 31, 630–651.

[25] Holston, Kathryn, Thomas Laubach and John C. Williams (2017) "Measuring the Natural Rate of Interest: International Trends and Determinants," *Journal of International Economics*, 108, S59–75.

[26] Iiboshi, Hirokuni and Mototsugu Shintani (2016) "Zero Interest Rate Policy and Asymmetric Price Adjustment in Japan: An Empirical Analysis of a Nonlinear DSGE Model," mimeo.

[27] Ikeda, Daisuke and Masashi Saito (2014), "The Effects of Demographic Changes on the Real Interest Rate in Japan," *Japan and the World Economy*, 32, 37–48.

[28] Kim, Jinill and Seth Pruitt (2017), "Estimating Monetary Policy Rules When Nominal Interest Rates Are Stuck at Zero," *Journal of Money, Credit, and Banking*, forthcoming.

[29] Kitagawa, Genshiro (1996) "Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Space Models," *Journal of Computational and Graphical Statistics*, 5(1), 1–25.

[30] Krugman, Paul R. (1998) "It's Baaack: Japan's Slump and the Return of the Liquidity Trap," *Brookings Papers on Economic Activity*, 2, 137–187.

[31] Kulish, Mariano, James Morley, and Tim Robinson (2017) "Estimating DSGE Models with Zero Interest Rate Policy," *Journal of Monetary Economics*, forthcoming.

[32] Laubach, Thomas, and John C. Williams (2003) "Measuring the Natural Rate of Interest," *Review of Economics and Statistics*, 85(4), 1063–1070.

[33] Laubach, Thomas and John C. Williams (2016) "Measuring the Natural Rate of Interest Redux", Federal Reserve Board, Finance and Economics Discussion Series, 2016–011.

[34] McKay, Alisdair, Emi Nakamura, and Jon Steinsson (2016) "The Power of Forward Guidance Revisited," *American Economic Review*, 106(10), 3133–3158.

[35] Nakata, Taisuke (2017) "Uncertainty at the Zero Lower Bound," *American Economic Journal: Macroeconomics*, forthcoming.

[36] Neiss, Katharine S. and Edward Nelson (2003) "The Real-Interest-Rate Gap as an Inflation Indicator." *Macroeconomic Dynamics*, 7(2), 239–262.

[37] Reifschneider, David and John C. Williams (2000) "Three Lessons for Monetary Policy in a Low-Inflation Era," *Journal of Money, Credit, and Banking,* 32(4), 936–966.

[38] Richter, Alexander W., Nathaniel A. Throckmorton, and Todd B. Walker (2014), "Accuracy, Speed and Robustness of Policy Function Iteration," *Computational Economics*, 44, 445–476,

[39] Richter, Alexander W. and Nathaniel A. Throckmorton (2016), "Are Nonlinear Methods Necessary at the Zero Lower Bound?" mimeo.

[40] Sims, Christopher (2002) "Solving Linear Rational Expectations Models," *Computational Economics*, 20, 1–20.

[41] Smets, Frank and Raf Wouters (2007), "Shocks and Frictions in US Business Cycles: a Bayesian DSGE Approach." *American Economic Review*, 90, 586–606.

[42] Sugo, Tomohiro and Kozo Ueda (2008) "Estimating a Dynamic Stochastic General Equilibrium Model for Japan." *Journal of the Japanese and International Economies*, 22, 476–502.

[43] Wicksell, Knut (1936) *Interest and Prices*, Translation of 1898 edition by R. F. Kahn, Macmillan, London.

[44] Woodford, Michael (2003) *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, Princeton, NJ.

A Appendix

A.1 Model Solution

We derive the rational expectation equilibrium of our model using the *time iteration method with linear interpolation* (TL). The model's equilibrium conditions are written as a vector-valued function, $f(\cdot)$, containing the minimum state vector $(\mathbf{v}_t, \mathbf{w}_t)$, say

$$E[f(\mathbf{v}_{t+1}, \mathbf{w}_{t+1}, \mathbf{v}_t, \mathbf{w}_t)|\Omega_t] = 0,$$

where $\mathbf{v}$ is a vector of exogenous variables, $\mathbf{w}$ is a vector of endogenous variables, and Ω_t is an information set of agents, whose elements are the structural model, $\mathbf{M}$, parameters of the model, $\mathbf{P}$, and state space, $\mathbf{z}$. In our study, there are three models, $\mathbf{M} = \{\text{Model 1, Model 2, Model w/o ZLB}\}$, and $\mathbf{v} = \left\{\epsilon_t^a,\ \epsilon_t^b,\ \epsilon_t^r\right\}$, $\mathbf{w} = \left\{y_t, c_t, \pi_t, y_t^*, R_t, R_t^*, r_t^*, \mu_t^a, z_t^b\right\}$ and $\mathbf{z} = \left\{R_{t-1}^*,\ \mu_t^a,\ z_t^b,\ \epsilon_t^r\right\}$, where $y_t \equiv Y_t/A_t$, $c_t \equiv C_t/A_t$, and $y_t^* \equiv Y_t^*/A_t$.

Given the function, $f(\cdot)$, we can obtain a model's decision rules (or policy functions), $\Phi(\cdot)$, as a function of the state vector. The TL locally approximates the time-invariant policy function at each node in the state space, $\mathbf{z_t}$, i.e.,

$$\Phi(\mathbf{z}_t) \simeq \hat{\Phi}(\mathbf{z}_t).$$

We choose to iterate on $\Phi(\mathbf{z}_t)$ for $\{\ y_t,\ y_t^*,\ \pi_t\ \}$ and solve a rational expectations equilibrium by substituting $\Phi(\mathbf{z}_t)$ into future variables of the function $f(\cdot)$. We discretise nine grid points on each continuous state variables, $\left\{R_{t-1}^*,\ \mu_t^a,\ z_t^b\right\}$, and five grid points on the exogenous shock of ϵ_t^r, which implies totally $3,645 (= 9 * 9 * 9 * 5)$ nodes.

The policy function iteration algorithm takes the following steps. Let $i \in \{0, \cdots, I\}$ denote the iterations of the algorithm and $n \in \{1, \cdots, N\}$ denote the nodes of the policy function, $\Phi(\mathbf{z}_t)$.

1. For $i = 0$, we make an initial conjecture of the policy function, $\Phi^0(\mathbf{z}_t)$, from the log-linearized model without the ZLB. To do so, we use Sim's (2002) `gensys` algorithm.

2. For iteration $i \in \{1, \cdots, I\}$ and node $n \in \{1, \cdots, N\}$, we execute the following procedures.

   (a) Solve for endogenous variables $\left\{c_t, R_t, R_t^*, r^*, \mu_t^a, z_t^b\right\}$ given $y(\mathbf{z}_{t-1})$, $y^*(\mathbf{z}_{t-1})$, $\pi(\mathbf{z}_{t-1})$ under the ZLB.

   (b) Approximate future variables $\{\ \mathrm{E}(\pi_{t+1}),\ \mathrm{E}(y_{t+1}),\ \mathrm{E}(y_{t+1}^*)\}$ using piecewise linear interpolation of the policy function $\Phi^{i-1}(z_t)$. Then, substitute the future variables into $E[f(\cdot)|\Omega_t]$ conditioning the exogenous variables in the next period, $\mathbf{v}_{t+1}$, equal to zero.[11]

[11] In this respect, our estimated model is not purely stochastic. We tried to estimate the model by

(c) Use the nonlinear solver, Sims' `csolve`, to find the policy function, $\Phi^i(\mathbf{z_t})$, which minimizes the errors in intertemporal equations, that is, $E[f(\cdot)|\Omega_t] = 0$.

3. Define $\texttt{maxdist} = \texttt{max}(\,|y_n^i - y_n^{i-1}|,\, |y_n^{*,i} - y_n^{*,i-1}|,\, |\pi_n^i - \pi_n^{i-1}|)$. Repeat step 2 until the policy function converges, say $\texttt{maxdist} < 10^{-4}$, for all nodes,$n$.

A.2 Estimation

To obtain draws from the posterior distribution of parameters, θ, of a nonlinear DSGE model, we use the *Sequential Monte Carlo Squared* (SMC^2) sampler combined with *particle filter*, instead of popular methods such as MCMC sampler. Because MCMC samplers cannot be parallelized for generating the draws, they consume quite a long time. By contrast, the SMC^2 algorithm and particle filter can be easily done and, in addition, may calculate more accurate approximation of the posterior distribution than the MCMC samplers. We explain the algorithms of the SMC^2 and particle filter following Herbst and Schorfheide (2015) and Fernandez-Villaverde et al. (2016).

A.2.1 Algorithm of the Sequential Monte Carlo Squared

Suppose ϕ_n, for $n = 0, \cdots, N_\phi$, is a sequence that slowly increases from zero to one. We define a sequence of bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, that converge to the target posterior distribution for $n = N_\phi$ and $\phi_n = 1$, as

$$\pi_n(\theta) = \frac{[p(Y|\theta)]^{\phi_n} p(\theta)}{\int [p(Y|\theta)]^{\phi_n} p(\theta) d\theta}, \quad \text{for } n = 0, \cdots, N_\phi,\ \ \phi_n \uparrow 1,$$

where $p(\theta)$ and $p(Y|\theta)$ are the prior density and likelihood function, respectively. We adopt the likelihood tempering approach that generates the bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, by taking power transformation of $p(Y|\theta)$ with the parameter, ϕ_n, i.e., $[p(Y|\theta)]^{\phi_n}$.

The SMC^2 with the likelihood tempering takes the following steps. Let $i \in \{1, \cdots, N_\theta\}$ denote the particles of the parameter sets, θ^i, and $n \in \{0, \cdots, N_\phi\}$ denote the stage of the algorithm. Herbst and Schorfheide (2015) recommend a convex tempering schedule in the form of $\phi_n = (n/N_\phi)^\lambda$ with $\lambda = 2$ for a small-scale DSGE model.

1. Initialize

   (a) Set the initial stage as $n = 0$, and draw the initial particles of parameters, θ_0^i, from a prior distribution $p(\theta)$.

considering that the exogenous variables in the next period, $\mathbf{v}_{t+1}$ obey normal distribution, but could not obtain reasonable results. We suspect this is because there is no equilibrium or equilibrium becomes indeterminate or unstable in an economy like Japan where the ZLB constrains the monetary policy for a long time. See Hills, Nakata, and Schmidt (2016).

(b) Set the weight of each particle of the initial stage as $W_0^i = 1$, for $i = 1, \cdots, N_\theta$.

2. For stage $n \in \{1, \cdots, N_\phi\}$ and particle $i \in \{1, \cdots, N_\theta\}$, take the following three steps.

(a) Correction Step. Calculate the normalized weight, $\tilde{W}_n^i$, for each particle as

$$\tilde{W}_n^i = \frac{\tilde{w}_n^i W_{n-1}^i}{\frac{1}{N}\sum_{i=1}^{N} \tilde{w}_n^i W_{n-1}^i} \quad \text{for } i = 1, \cdots, N_\theta,$$

where $\tilde{w}_t^i$ is an incremental weight derived from

$$\tilde{w}_n^i = [p(Y|\theta_{n-1}^i)]^{\phi_n - \phi_{n-1}},$$

and the likelihood, $\hat{p}(Y|\theta)$, is approximated from the particle filter, which is explained in the next subsection.

We note that the correction step is a classic importance sampling step, in which particle weights are updated to reflect the stage n distribution, $\pi_n(\theta)$. Because this step does not change the particle value, we can skip this step only by calculating power transformation of $p(Y|\theta)$ with the parameter, ϕ_n.

(b) Selection (Resampling) Step.

i. Calculate an effective particle sample size, $\widehat{ESS}_n$, which is defined as

$$\widehat{ESS}_n = N_\theta / \left(\frac{1}{N_\theta} \sum_{i=1}^{N_\theta} (\tilde{W}_n^i)^2 \right).$$

ii. If $\widehat{ESS}_n < N_\theta/2$, then resample the particles, $\{\hat{\theta}_n\}_{i=1}^{N_\theta}$, via multinomial resampling and set $W_n^i = 1$.

iii. Otherwise, let $\hat{\theta}_n^i = \theta_{n-1}^i$ and $W_n^i = \tilde{W}_n^i$.

(c) Mutation Step. Propagate the particles $\{\hat{\theta}_n^i, W_n^i\}$ via the random walk MH algorithm with the proposal density,

$$\vartheta|\hat{\theta}_n^{\,i} \sim N\left(\hat{\theta}_n^{\,i}, c_n^2 \Sigma\left(\hat{\theta}_n\right)\right),$$

where $N(\cdot)$ is the nominal distribution and $\Sigma\left(\hat{\theta}_n\right)$ denotes the covariance matrix of parameter $\hat{\theta}_n$ for all particles $i \in \{1, \cdots, N_\theta\}$ at n-th stage. In order to keep the acceptance rate around 25%, we set a scaling factor c_n for $n > 2$ as

$$c_n = c_{n-1} f(A_{n-1}),$$

where A_n represents the acceptance rate in the mutation step at the n-th stage and the function $f(x)$ is given by

$$f(x) = 0.95 + 0.10 \frac{e^{16(x-0.25)}}{1 + e^{16(x-0.25)}}.$$

3. For the final stage of $n = N_\phi$, calculate the final importance sampling approximation of posterior estimator, $E_\pi[h(\theta)]$, as

$$h_{N_\phi, N_\theta} = \sum_{i=1}^{N_\theta} h(\theta^i_{N_\phi}) W^i_{N_\phi}.$$

   We note that, in the final stage, the approximated marginal likelihood of the model is also obtained as a by-product. It can be shown that

$$P_{SMC}(Y) = \prod_{n=1}^{N_\phi} \left(\frac{1}{N_\theta} \sum_{i=1}^{N_\theta} \tilde{w}^i_n W^i_{n-1} \right)$$

   converges almost surely to $p(Y)$ as the number of particles $N_\theta \to \infty$.

A.2.2 Algorithm of the Particle Filter

Suppose that a state space representation for the nonlinear DSGE model consists of

$$y^{obs}_t = \Psi(s_t,\ \theta) + u_t, \quad u_t \sim N(0,\, \Sigma_u),$$

$$s_t = \Phi(s_{t-1},\ \varepsilon_t,\ \theta), \quad \varepsilon_t \sim N(0,\, \Sigma_\varepsilon),$$

where y^{obs}_t and s_t denote observable and state variables, respectively. In our study, we set $y^{obs}_t = \{\ln(y_t/y_{t-1}),\, \pi_t,\, R_t\}$ and $\mathbf{s_t} = \left\{ y_t,\, c_t,\, \pi_t,\, y^*_t,\, R_t,\, R^*_t,\, r^*_t,\, \mu^a_t,\, z^b_t \right\}$. A measurement error vector, u_t, and an exogenous shock vector, $\varepsilon_t = \left\{\epsilon^a_t,\, \epsilon^b_t,\, \epsilon^r_t\right\}$, follow the normal distribution with covariance matrices, Σ_u and Σ_ε, respectively. The nonlinear policy functions, $\Phi(s_t,\ \varepsilon_t,\ \theta)$, are derived from Appendix A.1, while the function $\Psi(s_t, \theta)$ represents the linkage between y^{obs}_t and s_t.

The particle filter algorithm is shown as follows. Let $j \in \{0, \cdots, N_S\}$ denote the particles of the state variables and exogenous shocks.

1. For period $t = 0$, draw the N_S initial particles of the state variables at period 0, say $s^j_{0|0}$, from the distribution around the steady state derived from the policy functions, $p(s_0 \mid \Sigma_\varepsilon, \theta)$.

2. For period $t \in \{1, \cdots, T\}$ and particle $j \in \{1, \cdots, N_S\}$, take the following three steps.

   (a) Step of forecasting the state variables: $s^j_{t|t-1}$. Generate N_S particles of the shock vector ε^j_t from $N(0, \Sigma_\varepsilon)$. Using the nonlinear policy function, we obtain N_S particles of forecasts of the state variables corresponding to the shocks generated in the above:

$$s^j_{t|t-1} = \varPhi(s^j_{t-1|t-1}, \varepsilon^j_t | \theta).$$

(b) Step of forecasting the observable variables. Calculate the approximated predictive density of y_t^{obs} given by

$$p(y_t^{obs}|Y_{1:t-1}^{obs},\, \theta) \doteqdot \frac{1}{N_S}\sum_{j=1}^{N_S} w_t^j,$$

where w_t^j is the normal predictive density of the particle j measured from $\Psi(s_{t|t-1}^j, \theta)$ and the covariance matrix of the measurement error Σ_u in period t, say,

$$w_t^j = (2\pi)^{-n/2}|\Sigma_u|^{-1/2}\text{exp}\left\{ -\frac{1}{2}\,(y_t^{obs} - \Psi(s_{t|t-1}^j, \theta))'\,\Sigma_u^{-1}\,(y_t^{obs} - \Psi(s_{t|t-1}^j, \theta))\right\},$$

where n is the dimension of y_t.

(c) Step of updating the state variables: $s_{t|t}^j$. Resample N_S particles of the state variables from a multinominal distribution. That is,

$$s_{t|t}^j = \text{resample out of } (\mathrm{s}_{\mathrm{t|t-1}}^1, \cdots \mathrm{s}_{\mathrm{t|t-1}}^{\mathrm{j}}, \cdots \mathrm{s}_{\mathrm{t|t-1}}^{\mathrm{N_s}}) \text{ with probabilty } (w_t^j/\Sigma w_t^j).$$

3. For the final period of $t = T$, collect all of the predictive densities of y_t from period 1 to T, calculated in the above. Using those, the log likelihood of the model is approximated as

$$\ln p(Y_{1:t}^{obs}|\theta) \doteqdot \sum_{t=1}^{T} \ln\left(\frac{1}{N_S}\sum_{j=1}^{N_S} w_t^j\right).$$

Table 1: Prior Distribution

Parameter	Mean	S.D.	Shape	Parameter	Mean	S.D.	Shape
σ	1.5	0.3	Normal	ρ^r	0.5	0.2	Beta
γ^a	0	0.5	Normal	ψ_π	1.5	0.15	Normal
ω	3	0.5	Normal	ψ_y	0.125	0.025	Normal
κ	0.05	0.006	Normal	ω	3	0.5	Normal
π^*	0	0.5	Normal	ρ^a	0.5	0.2	Beta
				$\sigma^a, \sigma^b, \sigma^r$	$\sqrt{0.02}$	5 (d.f.)	Inv Gamma

Table 2: Posterior Distribution and Marginal Likelihood

	Model 1		Model 2		Model w/o ZLB	
Parameter	Mean	(95% low, high)	Mean	(95% low, high)	Mean	(95% low, high)
σ	1.400	(1.289, 1.55)	1.534	(1.472, 1.584)	1.037	(1.004, 1.083)
γ^a	-0.028	(-0.119, 0.05)	0.123	(0.068, 0.172)	-0.419	(-0.539, -0.31)
ω	2.477	(2.298, 2.674)	3.163	(2.908, 3.413)	3.188	(3.071, 3.287)
κ	0.055	(0.05, 0.062)	0.053	(0.05, 0.055)	0.047	(0.045, 0.049)
π^*	0.360	(-0.117, 0.619)	0.050	(-0.148, 0.274)	-0.447	(-0.728, -0.246)
r^*	0.464	(0.343, 0.574)	0.691	(0.607, 0.764)	0.067	(-0.055, 0.181)
ρ^r	0.521	(0.475, 0.611)	0.685	(0.644, 0.721)	0.214	(0.182, 0.24)
ψ_π	1.689	(1.627, 1.745)	1.776	(1.739, 1.811)	1.509	(1.453, 1.553)
ψ_y	0.105	(0.091, 0.123)	0.113	(0.098, 0.125)	0.133	(0.127, 0.141)
ρ^a	0.254	(0.129, 0.446)	0.201	(0.096, 0.292)	0.122	(0.093, 0.147)
ρ^b	0.750	(0.693, 0.802)	0.754	(0.728, 0.776)	0.740	(0.689, 0.788)
σ^a	1.175	(0.906, 1.367)	1.320	(1.146, 1.524)	1.773	(1.675, 1.859)
σ^b	1.797	(1.435, 2.318)	2.229	(2.018, 2.442)	1.354	(1.191, 1.558)
σ^r	1.439	(1.14, 1.673)	1.173	(1.022, 1.349)	0.921	(0.809, 1.012)
Likelihood	-261.753		-275.9136		-364.815	

Table 3: Probability of the ZLB, 2% Inflation, and Deflation

Natural rate (r_t^*)	Prob ($R_{t+h} = 0\%$ | r_t^*)				Prob ($\pi_{t+h} > 2\% \| r_t^*$)	Prob ($\pi_{t+h} < 0\% \| r_t^*$)
	for $h = 1$	for $h = 2$	for $h = 4$	for $h = 8$	$^\exists h$ for $1 \leq h \leq 8$	
$r_t^* \geq 3\%$	7.00%	14.35%	23.35%	31.65%	60.20%	76.60%
$2\% \leq r_t^* < 3\%$	12.64%	19.59%	26.37%	33.80%	47.16%	81.28%
$1\% \leq r_t^* < 2\%$	25.97%	28.62%	32.60%	35.61%	28.44%	88.10%
$0\% \leq r_t^* < 1\%$	52.40%	44.56%	39.94%	38.17%	15.36%	94.11%
$-1\% \leq r_t^* < 0\%$	61.86%	51.74%	43.58%	39.43%	10.40%	97.48%
$r_t^* < -1\%$	69.60%	58.83%	51.60%	41.63%	7.03%	99.70%
$r_t^* = r^*$	35.12%	36.79%	37.52%	37.29%	15.16%	94.26%

Table 4: Estimation Results of the (Un)Constrained Linear Model

	Model 1 (benchmark)		Constrained linear (Model 1)		Unconstrained linear (Model w/o ZLB)	
Parameter	Mean	(95% low, high)	Mean	(95% low, high)	Mean	(95% low, high)
σ	1.400	(1.289, 1.55)	1.287	(1.237, 1.387)	1.152	(1.039, 1.249)
γ^a	-0.028	(-0.119, 0.05)	0.140	(0.031, 0.22)	0.544	(0.44, 0.633)
ω	2.477	(2.298, 2.674)	2.702	(2.569, 2.936)	2.260	(2.125, 2.377)
κ	0.055	(0.05, 0.062)	0.050	(0.048, 0.052)	0.047	(0.046, 0.048)
π^*	0.360	(-0.117, 0.619)	-0.138	(-0.316, -0.03)	0.844	(0.69, 1.006)
r^*	0.464	(0.343, 0.574)	0.679	(0.541, 0.792)	1.126	(1.037, 1.207)
ρ^r	0.521	(0.475, 0.611)	0.386	(0.354, 0.406)	0.414	(0.377, 0.449)
ψ_π	1.689	(1.627, 1.745)	1.426	(1.379, 1.476)	1.534	(1.503, 1.563)
ψ_y	0.105	(0.091, 0.123)	0.127	(0.123, 0.13)	0.124	(0.116, 0.132)
ρ^a	0.254	(0.129, 0.446)	0.157	(0.115, 0.211)	0.206	(0.183, 0.239)
ρ^b	0.750	(0.693, 0.802)	0.659	(0.647, 0.682)	0.698	(0.666, 0.727)
σ^a	1.175	(0.906, 1.367)	1.285	(1.131, 1.412)	1.649	(1.534, 1.773)
σ^b	1.797	(1.435, 2.318)	1.483	(1.418, 1.565)	1.867	(1.617, 2.066)
σ^r	1.439	(1.14, 1.673)	1.182	(1.023, 1.41)	1.065	(0.979, 1.154)
Likelihood	-261.75		-279.29		-395.65	
Post prob	1.000		0.000		0.000	

Table 5: Estimation Results when Using the Output Gap Data

	Growth Data (benchmark)		Gap instead of growth		Gap and growth	
Parameter	Mean	(95% low, high)	Mean	(95% low, high)	Mean	(95% low, high)
σ	1.400	(1.289, 1.55)	1.596	(1.547, 1.686)	1.513	(1.468, 1.586)
γ^a	-0.028	(-0.119, 0.05)	-0.056	(-0.088, -0.003)	-0.077	(-0.132, -0.041)
ω	2.477	(2.298, 2.674)	3.030	(2.81, 3.137)	2.743	(2.585, 2.841)
κ	0.055	(0.05, 0.062)	0.044	(0.042, 0.046)	0.055	(0.054, 0.056)
π^*	0.360	(-0.117, 0.619)	0.154	(0.078, 0.186)	0.648	(0.549, 0.714)
r^*	0.464	(0.343, 0.574)	0.414	(0.364, 0.497)	0.385	(0.292, 0.441)
ρ^r	0.521	(0.475, 0.611)	0.664	(0.645, 0.675)	0.597	(0.575, 0.631)
ψ_π	1.689	(1.627, 1.745)	1.468	(1.458, 1.477)	1.431	(1.407, 1.472)
ψ_y	0.105	(0.091, 0.123)	0.113	(0.11, 0.116)	0.101	(0.098, 0.103)
ρ^a	0.254	(0.129, 0.446)	0.202	(0.153, 0.233)	0.222	(0.157, 0.262)
ρ^b	0.750	(0.693, 0.802)	0.680	(0.673, 0.687)	0.801	(0.794, 0.81)
σ^a	1.175	(0.906, 1.367)	3.473	(2.988, 4.367)	1.412	(1.202, 1.766)
σ^b	1.797	(1.435, 2.318)	2.285	(2.225, 2.383)	1.946	(1.623, 2.133)
σ^r	1.439	(1.14, 1.673)	0.823	(0.773, 0.906)	1.473	(1.424, 1.535)

Figure 1: Data

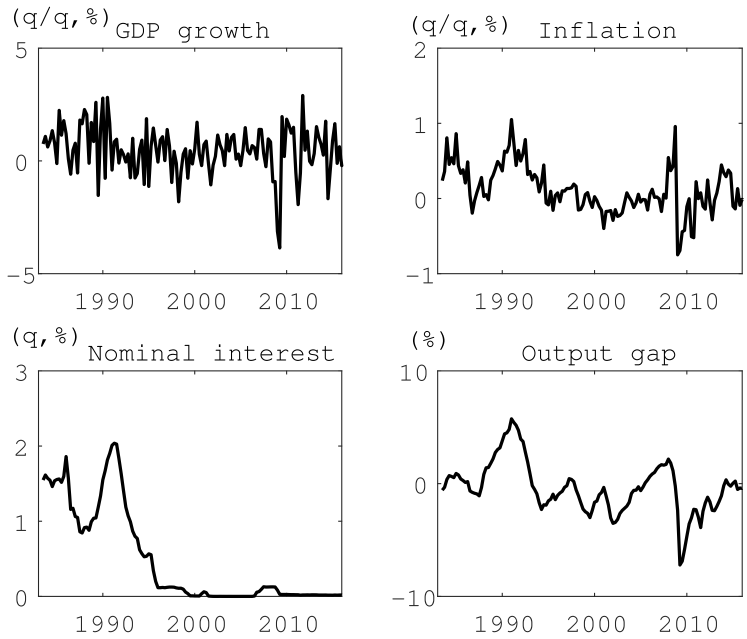
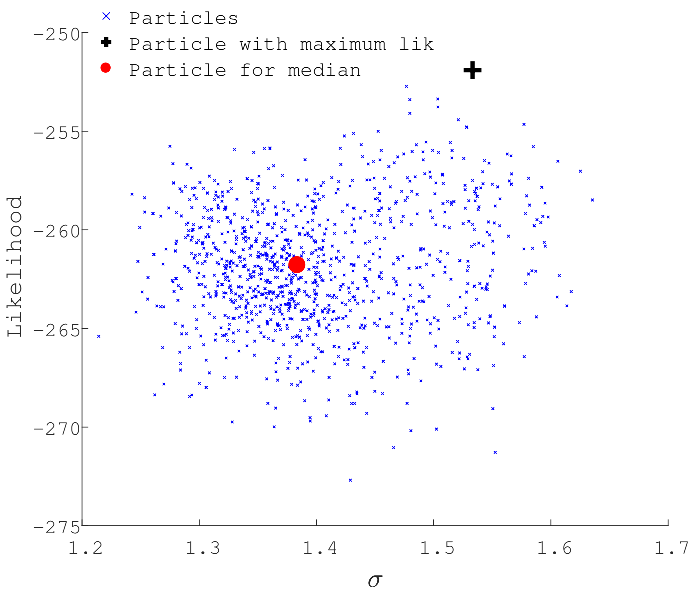


Figure 2: Particles for Parameter σ and their Likelihood



Note: Each dot represents a particle for the value of parameter σ and its posterior likelihood. The big circle in green at the top indicates the maximum of the likelihoods in all the dots.

Figure 3: Notional Nominal Interest Rate R_t^*

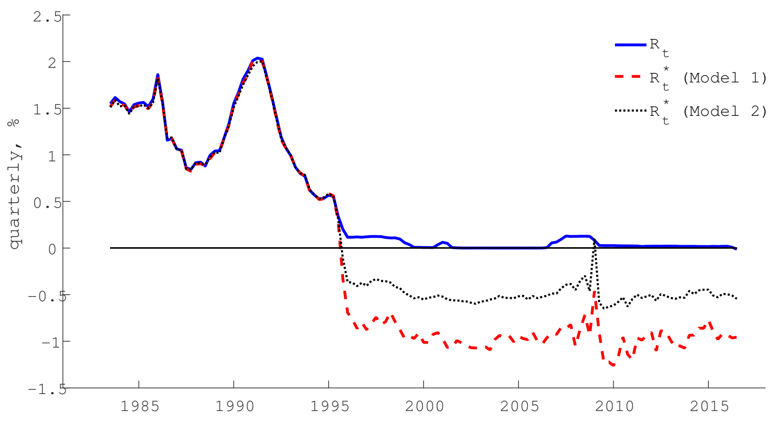
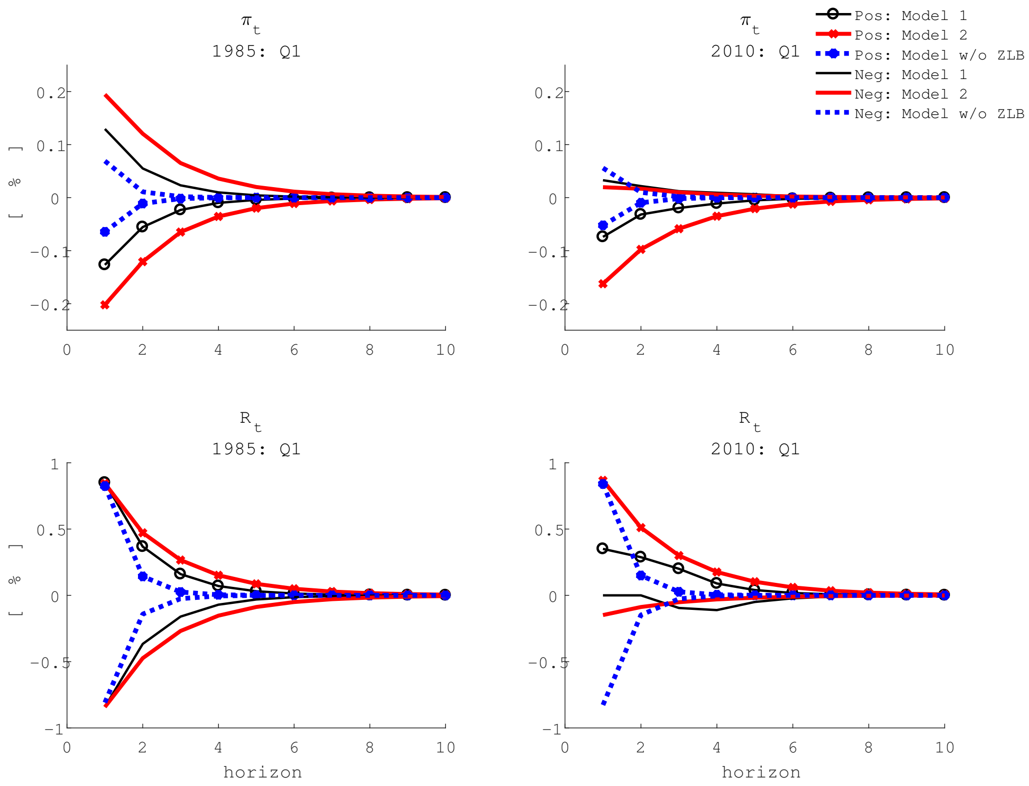


Figure 4: Impulse Responses to a Monetary Policy Shock



Note: Pos and Neg represent positive (tightening) and negative (easing) monetary policy shocks, respectively.

Figure 5: Natural Rate of Interest and the Contribution of the Estimated Shocks

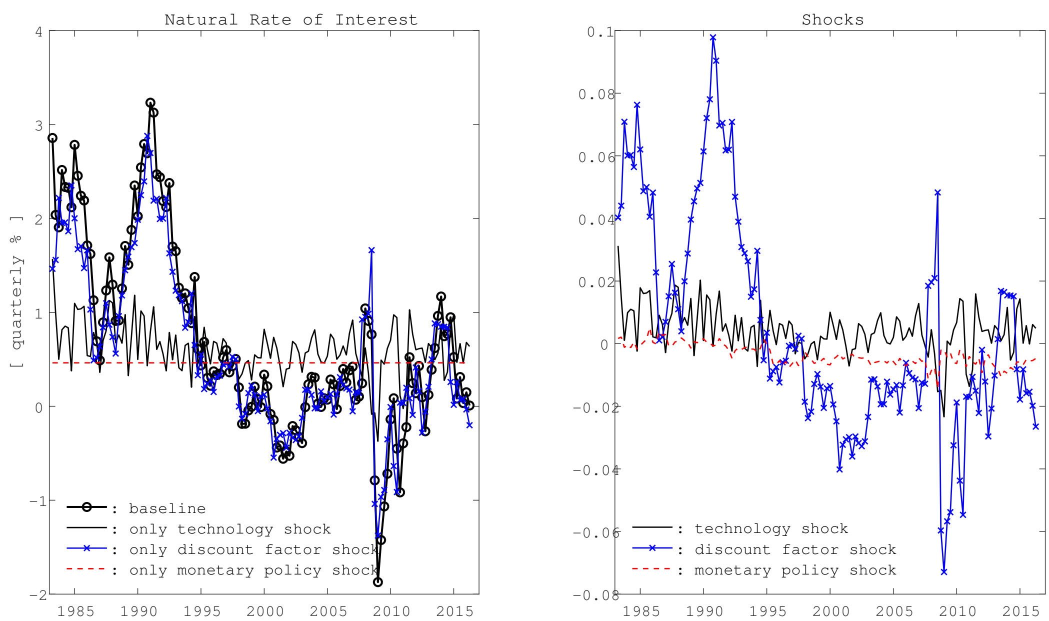


Figure 6: Natural Rate of Interest: Model Comparison (1)

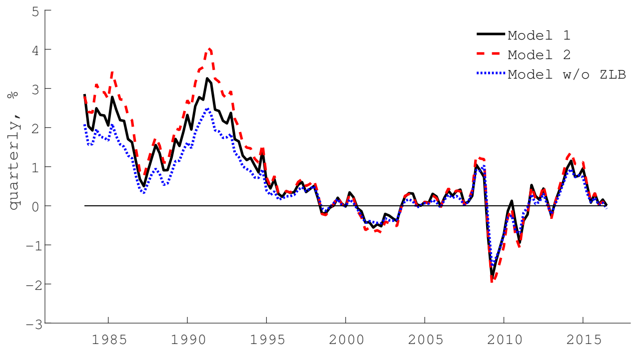
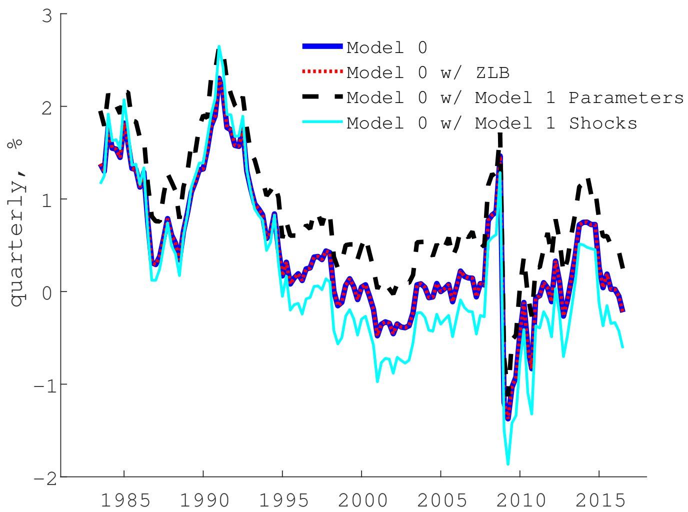


Figure 7: Natural Rate of Interest: Counterfactual Simulation



Note: "Model 0" represents the natural rate of interest based on the Model without the ZLB. (1) "Model 0 w/ ZLB," (2) "Model 0 w/ Model 1 Parameters," and (3) "Model 0 w/ Model 1 Shocks" represent the simulated natural rate of interest using (1) estimated parameters and shocks in the Model without the ZLB but now explicitly taking account of the ZLB, (2) using estimated parameters in Model 1, and (3) using estimated shocks in Model 1, respectively.

Figure 8: Natural Rate of Interest: Model Comparison (2)

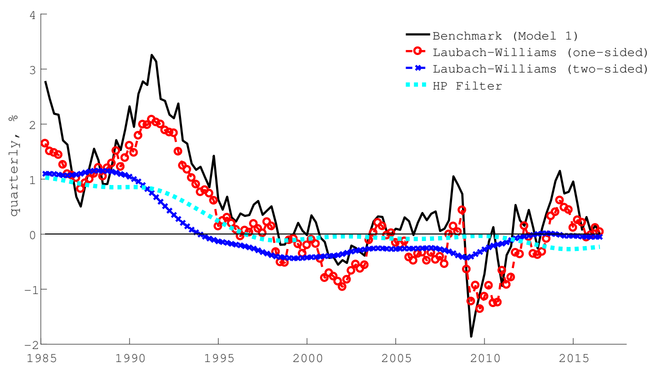


Figure 9: Natural Rate of Interest When Using the Output Gap Data