A Simple Explanation for Investment in Obsolete Technologies: Existence of Complementary Capital*†

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Abstract

This paper develops and analyzes a growth model that consists of two types of vintage-specific capital: *long-lived* and *short-lived* capital. As a result of the existence of complementary capital that is chronologically compatible but has different longevity, the model generates two distinct investment patterns: (i) if the rate of the vintage-specific technological progress is above a threshold – which is the product of long-lived capital's share and the difference in the rates of depreciation – then all new investment concentrates on the latest technology; (ii) otherwise, some investment is allocated to obsolete, short-lived capital to exploit existing excessive long-lived capital with obsolete technologies.

The result provides a new explanation for observed investment in equipment with obsolete technologies. A striking implication is that equipment price-changes do not necessarily reflect the rate of technological progress. Another implication is that acceleration in the rate of technological progress can cause an abrupt reallocation of investment towards modern capital – consistent with investment booms that are concentrated in certain "high-tech" equipment.

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1 Introduction

This paper's model has two key elements: (i) it is a vintage growth model in which a certain technology is built into each unit of capital; (ii) it has two kinds of capital which have different rates of depreciation.[1] The existence of long-lived vintage-specific complementary capital provides a simple explanation for investment in short-lived capital with obsolete technologies.

This study is motivated by a simple question; why do firms invest in obsolete technologies that are less efficient in production? Steam locomotives had long been in operation after more efficient diesel locomotive was introduced for commercial demonstration in 1924; there had been investment in steam locomotives for more than 20 years since then (Felli and Ortalo-Magne (1998), Figure 1 and 5). Table 3 in Comin and Hobijn (2004) shows that there have coexisted multiple generations of steel production technologies for decades. Typical vintage growth models with single capital type that follow Solow (1960), however, do not feature these facts. Indeed, in these models, investment should concentrate on capital with the newest vintage technology that is always more efficient than capital with obsolete technologies.

This paper provides a simple explanation for the question by assuming coexistence of two types of complementary capital that are chronologically compatible but have different longevity. The idea behind the assumption is simple; if one type of complementary capital depreciates more slowly (*long-lived*) than the other (*short-lived*) does, then investing in short-lived capital with an obsolete technology may be rationalized in order to exploit the existing stock of long-lived capital with the obsolete technology.

The existence of vintage-specific complementary capital in a vintage growth model results in two surprising implications in a steady state. First, (i) if the rate of technological progress is above a threshold – the product of the long-lived capital's share and the difference in the rates of depreciation – all new investment will concentrate on the newest two types of capital; (ii) otherwise, a part of new investment will be allocated to short-lived capital with obsolete vintages as well as both capital types with the newest vintage. In other words, the speed of diffusion of technology increases when the technological progress is fast.

Second, if the rate of technological progress is below the threshold, then the prices

[1] In this study, "depreciation" solely refers physical depreciation, and excludes obsolescence that is explicitly treated as changes in prices.

of short-lived obsolete capital remain one over time even when the rate of progress is positive. This result implies that if production involves vintage-specific complementary capital that has a longer longevity than equipment does, estimate of the rate of vintage-specific technological progress based on changes in equipment prices may be systematically biased downward.

One naturally interprets the set of two distinct types of complementary capital to the combination of tangible and intangible capital. Tangible capital typically depreciates more quickly than intangible capital does: the former, such as engine, physically wears and/or tears; the latter, such as system, knowledge, or organizational capital, does not.[2] The results of the current study can be interpreted as following. Suppose CD drive (tangible) of your PC crashes for some reason. Then, would you buy a new set of PC or merely replace the CD drive? If the change in PC model develops quickly enough, you would purchase a new PC because it has much better features. Or you would replace the CD drive to keep using the existing PC because you are used to its customized environment (intangible).[3] The decision depends on the rate of technological progress and remaining size of intangible capital.

Existing growth models that assume capital heterogeneity do not feature the investment in obsolete technologies. The model in Laitner and Stolyarov (2003) applies vintage-specific capital assumption to Shell and Stiglitz (1967)'s disembodied two types of capital model. Their model, however, assumes a single rate of depreciation of capital types. This assumption maintains the allocation of investment and stock of capital types proportional to their shares, resulting in that investment concentrates on the newest technology as in basic Solow (1960)'s model. Greenwood et al. (1997) and Chapter 2 in Aruga (2006) develop vintage growth model with structures and equipment that have different longevity. In these models, however, investment in obsolete technologies cannot be rationalized since structures are not vintage-specific, and thus they are freely reallocated across vintages technologies like labor.

A few exceptions that explain investment in obsolete capital include Chari and Hopenhayn (1991), and Parente (2000). They show that, assuming that each technology requires vintage-specific skills, investment in both obsolete and new vintages of technology can coexist. The current model predominates their models in two aspects.

[2]There is another point of view that knowledge can depreciate by leave or die of knowledgeable people, of course.

[3]You may find another beautiful interpretation of intangible capital; your existing collection of softwares that are not compatible with the newest type of PC.

First, my model explains not only the diffusion of technology but also the investment patterns in obsolete technologies. Indeed, their model does not feature the allocation of capital investment.

Second, my model has Solow (1960)'s type straightforward neo-classical vintage growth model assumptions, while their models require sets of unique and intricate vintage-specific labor assumptions. In the current model, any long-lived chronologically compatible complementary capital can be an incentive to the investment in obsolete technologies. Thus this model includes the Solow's model as a special case, and the results provide implication for growth accounting, such as measurement of the vintage-specific technological progress.

The rest of the paper is organized as follows: Section 2 presents the model's framework; Section 3 provides a characterization of a steady state; Section 4.2 discusses applications of the model; and finally, Section 5 concludes the paper.

2 Model

The model has two key elements: (i) it is a vintage growth model in which a certain technology is built into each unit of capital; (ii) it has two kinds of capital which have different rates of depreciation. Except the assumptions of the above capital heterogeneity, all assumptions are essentially identical to those of Solow (1960): Cobb-Douglas production function, fixed investment rate, competitive market, perfect foresight / rational expectation, vintage-specific technological progress, and vintage-nonspecific labor. This section lists the details of assumptions, and then develops and analyzes the model.

2.1 Setup of The Model

I assume the economy is competitive, and agents have perfect foresight and are rational. Each capital embodies a specific vintage technology; usage of a vintage technology requires capital goods that are specifically designed for the vintage technology. Let $v \geq 0$ denote a specific vintage, and assume at time t vintage $v \leq t$ technology is available for agents.

Each vintage production technology requires three types of inputs: two types of vintage-specific capital, A (*long-lived*) and B (*short-lived*), and vintage-nonspecific labor, L. Assume A and B depreciate at the rates δ^A and δ^B where $\delta^A \leq \delta^B$. Let a subscript v denote a specific vintage v technology that is embodied in each type of capital; $A_v(t)$ and $B_v(t)$ represent the number of units of A and B designed for a specific vintage technology v. $L_v(t)$ expresses the amount of labor that is employed for a vintage v, although L is not vintage-specific.

Assume each vintage-specific production function is Cobb-Douglas form,

$$Y_v(t) = q_v A_v(t)^{\alpha} B_v(t)^{\beta} L_v(t)^{1-\alpha-\beta}, \tag{1}$$

where $Y_v(t)$ is output from vintage v technology, q_v is vintage-specific technology level that is monotonically increasing in v, and α and β are constant shares of two capital types.[4] Assume that output is homogeneous and keeps a constant physical unit over time.

Each physical unit of capital can be used for investment in each physical unit of either type of capital or consumption.[5] I assume fixed portion of aggregate output is used to the investment, and investment is irreversible,

$$\begin{aligned} \sigma Y(t) &= I^A(t) + I^B(t) \qquad (2) \\ &= \int_0^t I_v^A(t)\, dv + A_t(t) + \int_0^t I_v^B(t)\, dv + B_t(t), \end{aligned}$$

where aggregate output $Y(t)$ is defined by

$$Y(t) = \int_0^t Y_v(t)\, dv. \tag{3}$$

Note that investment consists of the part for obsolete technology and the part for the frontier technology. Further, note that the prices of capital types in units of output should satisfy $P_v^A(t) \in [0,1]$ and $P_v^B(t) \in [0,1]$ since each type of capital is freely disposable, and investment in capital types with existing vintage technology is possible.

The setup of the model based on the straightforward neoclassical assumptions turns out to be crucial in applying to various levels of economic activities that are

[4]In the model presented here, I omit "neutral" technological progress that affects all vintages of production, since the omission does not change the main point of the result. Chapter 3 in Aruga (2006) exhibits the case when the neutral technological progress is considered in addition.

[5]This assumption requires that production functions of two capital types are identical, which might be an imperfect assumption in some occasions. However, I assume so as conventional growth models do so for different kinds of equipment.

discussed in Section 4.2. That is, the assumptions presented here can be used not only for the aggregate economy, but also various aggregation levels of production.

2.2 Chronological Aggregation

In this subsection, I derive the chronologically aggregated production function that summarizes the allocation of the two types of capital across vintages. The result hinges on the assumption of competitive market and is the key in characterizing the steady state in the following section.

Lemma 1 (Chronological Aggregation)**.** *(i) Define aggregate capital types,*

$$A(t) \equiv \int_0^t \frac{MPA_v(t)}{MPA_t(t)} A_v(t)\, dv, \tag{4}$$

$$B(t) \equiv \int_0^t \frac{MPB_v(t)}{MPB_t(t)} B_v(t)\, dv, \tag{5}$$

and aggregate labor,

$$L(t) \equiv \int_0^t L_v(t)\, dv, \tag{6}$$

where $MPA_v(t)$ and $MPB_v(t)$ are marginal products of A and B with vintage v at time t.

Then, the aggregate output is expressed as

$$Y(t) = q_t A(t)^\alpha B(t)^\beta L(t)^{1-\alpha-\beta}. \tag{7}$$

(ii) Furthermore, define aggregate consolidate capital,

$$J(t) \equiv \int_0^t J_v(t)\, dv,$$

where vintage consolidate capital is defined by

$$J_v(t) \equiv \left[q_v A_v(t)^\alpha B_v(t)^\beta\right]^{\frac{1}{\alpha+\beta}}. \tag{8}$$

Then, the consolidate capital is expressed as

$$J(t) = \left[q_t A(t)^\alpha B(t)^\beta\right]^{\frac{1}{\alpha+\beta}} \tag{9}$$

and the output and labor allocations across vintages are given by,

$$L_v(t) = \frac{J_v(t)}{J(t)} L(t), \tag{10}$$

$$Y_v(t) = \frac{J_v(t)}{J(t)} Y(t). \tag{11}$$

Proof: See Appendix A.1.1.

Interestingly enough, the chronologically aggregated production function is the same form as (1) with frontier technology level, q_t, and the aggregate inputs defined as (4) - (6).

Further note that if returns on capital of each type of capital are independent of vintages respectively, (4) and (5) simply show the total values of the capital types in units of frontier vintage capital types. These equations imply that the chronologically aggregated capital types per labor are identical to the frontier capital types per labor, $A_t(t)/L_t(t)$ and $B_t(t)/L_t(t)$.

Notice that I can derive the aggregate production function and determine the allocation of labor across vintages without knowing prices of capital types.[6]

3 Steady State

This section analyzes the steady state property of the model. As other economic studies, the steady state analysis as an approximation provides significant implications about the existence of the chronologically compatible complementary capital.

I define the steady state of interest as follows.

Definition 1 (Steady State)**.** In a steady state, all the quantities grow at constant rates, and the real interest rate is constant.

The statement is as usual growth models. Throughout the steady state analysis, I assume constant vintage-specific technological change, $\hat{q}_t = \hat{q}$, and constant labor growth, $\hat{L}(t) = \hat{L}$.

3.1 Investment Scheme

Solow (1960)'s vintage growth model speculates that all new investment concentrates

[6] Of course, the aggregate production function can be expressed as $Y(t) = J(t)^{\alpha+\beta} L(t)^{1-\alpha-\beta}$, which is the same form as Solow (1960). $J(t)$ stands for Solow's *Jelly Capital*.

on the capital that has the newest available vintage by model's construction. In his model, this speculation is allowed because marginal product of capital with the frontier vintage technology is always higher than any obsolete vintage capital since vintage-nonspecific labor can be freely reallocated to the frontier production technology.

This is not necessarily the case in the current model, however – the key is the existence of chronologically compatible vintage-specific capital with different longevity. Suppose initially the allocation of long-lived and short-lived capital with a specific vintage v is optimal such that the prices of two capital types are one. Then, over time, the existing stock of long-lived capital becomes relatively abundant to that of short-lived capital without investment. In this case, if long-lived capital is important enough in production and lasts much longer than short-lived capital, and the rate of the vintage-specific technological progress is slow enough, then investment in the obsolete short-lived capital may become attractive than that in new short-lived capital. The possibility of investment in obsolete vintage capital complicates the characterization of investment patterns and price distribution across vintages and capital types.

In order to overcome the complexity of the model, I now analyze four possible investment schemes of available specific vintage production, $v < t$.

Definition 2 (Investment Scheme)**.** Consider an existing vintage production, $v \leq t$. Define the four investment schemes such that if the production with the vintage technology is:

(i) in scheme (a), there is no continuous positive investment in either A_v nor B_v;

(ii) in scheme (b), there is continuous positive investment only in A_v;

(iii) in scheme (c), there is continuous positive investment only in B_v;

(iv) in scheme (d), there is continuous positive investment in both A_v and B_v.

From the definition above, we can determine the relationships of two different vintages when they have the same investment scheme.[7]

In a steady state, all $Y_v(t)$, $A_v(t)$, and $B_v(t)$ $\forall\, v$ grow at constant rates and the interest rate is a constant, $r(t) = r^*$. Thus, I can draw the following lemma.

[7]See Appendix A.2 for details.

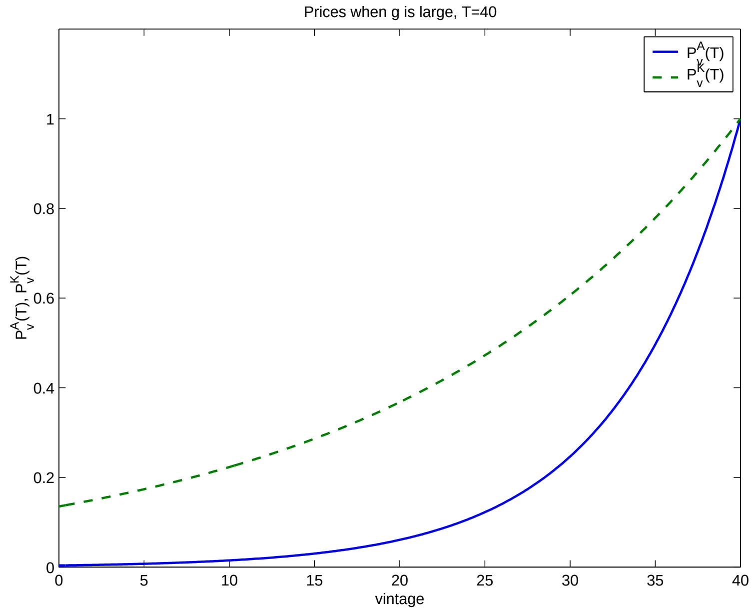

Figure 1: Prices of capital in the case $\alpha(\delta^B - \delta^A) < \hat{q}$, where the investment scheme is (a) $\forall\, v$.

Lemma 2 (Uniqueness of Investment Scheme across Vintages)**.** *In a steady state: (i) investment scheme of a vintage v does not change over time, (ii) investment scheme is unique across vintages, and (iii) the scheme is either (a), (b), or (c).*

Proof: See Appendix A.1.2.

Given the requirements of the prices in a steady state above, investment schemes with a set of parameters are characterized by the following proposition.

Proposition 1 (Investment Scheme)**.** *In a steady state:*

(i) if $\hat{q} \geq \alpha(\delta^B - \delta^A)$, then investment scheme is (a) $\forall\ v \leq t$, firms invest only in the both capital types with the frontier technology;

(ii) otherwise, investment scheme is (c) $\forall\ v \leq t$, firms invest in obsolete vintage short-lived capital in addition to the both capital types with the frontier technology.

Proof: See Appendix A.1.3.

Figure 1 and Figure 2 show the price distributions of the two capital types in a steady state since time $t = 0$ that correspond to the Proposition 1 (i) and (ii)

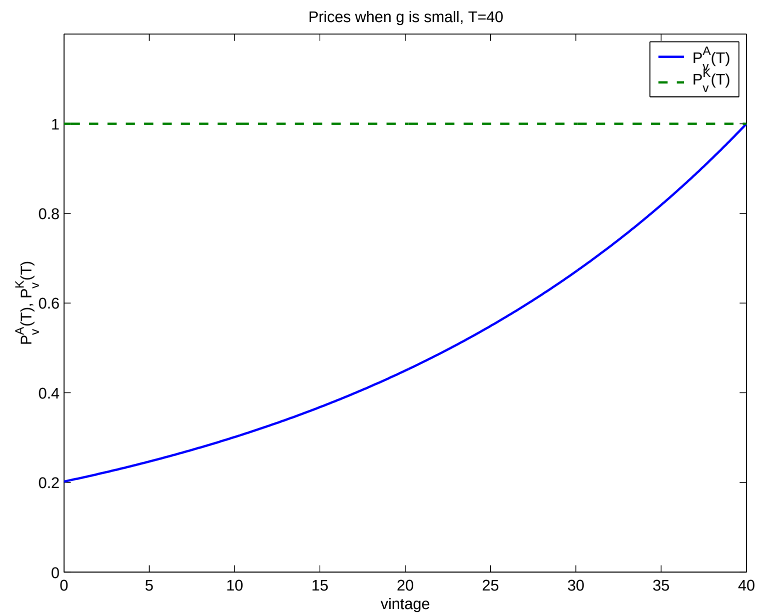

Figure 2: Prices of capital in the case $\hat{q} < \alpha(\delta^B - \delta^A)$, where the investment scheme is (c) $\forall v$.

respectively. In Figure 1, prices of both capital types of a specific vintage fall as vintage becomes obsolete because their marginal products do not exceed those of frontier capital types. On the other hand, in Figure 2, the prices of short-lived capital across vintages keep one because marginal products of obsolete short-lived capital are higher than that of newest capital types, and thus investment in obsolete vintage short-lived capital occurs.

3.2 Full Characterization

Lemma 3 (Allocation of Capital Stock)**.** *Define the aggregate effective labor,*

$$N(t) \equiv q_t^{1/(1-\alpha-\beta)} L(t), \tag{12}$$

and use lower case letters to express per effective labor amounts: $a(t) = A(t)/N(t)$, and $b(t) = B(t)/N(t)$.

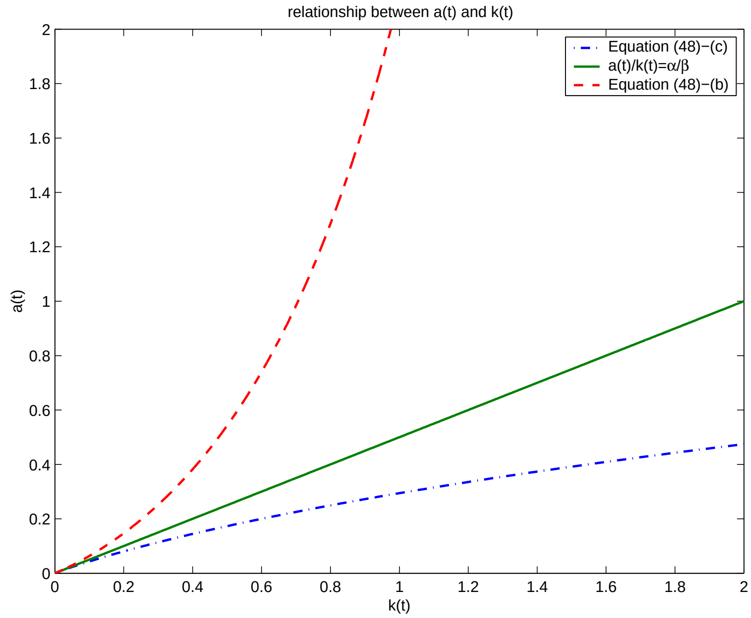

Figure 3: Possible per effective long-lived capital, $a(t)$, and short-lived capital, $b(t)$ implied by (13) when $\hat{q} > \alpha(\delta^B - \delta^A)$.

Then, in a steady state, $a(t)$ and $b(t)$ are constant and have a relationship,

$$\beta a(t)^{\alpha} b(t)^{\beta-1} - \alpha a(t)^{\alpha-1} b(t)^{\beta} = \begin{cases} 0 & (a), \\ \left[\delta^B + \frac{\hat{q}}{\beta}\right] - \delta^A & (b),\ or \\ \delta^B - \left[\delta^A + \frac{\hat{q}}{\alpha}\right] & (c), \end{cases} \tag{13}$$

depending on the investment scheme.

Proof: See Appendix A.1.4.

The result of Lemma 3 differs from the disembodied heterogeneous capital model in Chapter 2 of Aruga (2006) – in that model, right hand side of (13) is always $\delta^B - \delta^A$. In the current model, the difference in the rates of depreciation is canceled in the scheme (a), and extra term, $+\hat{q}/\beta$ for (b) and $-\hat{q}/\alpha$ for (c), show up because of the embodiment assumption. When the right hand side is positive, the curve is convex and above the straight line α/β. If the scheme is (c), the curve shifts down as the vintage specific technological progress, $\hat{q}$, goes up, which never occurs in the disembodied model. Figure 3 and Figure 4 show possible relationships of $a(t)$ and $b(t)$ implied by (13) when $\hat{q} > \alpha(\delta^B - \delta^A)$ and $\hat{q} < \alpha(\delta^B - \delta^A)$ respectively.

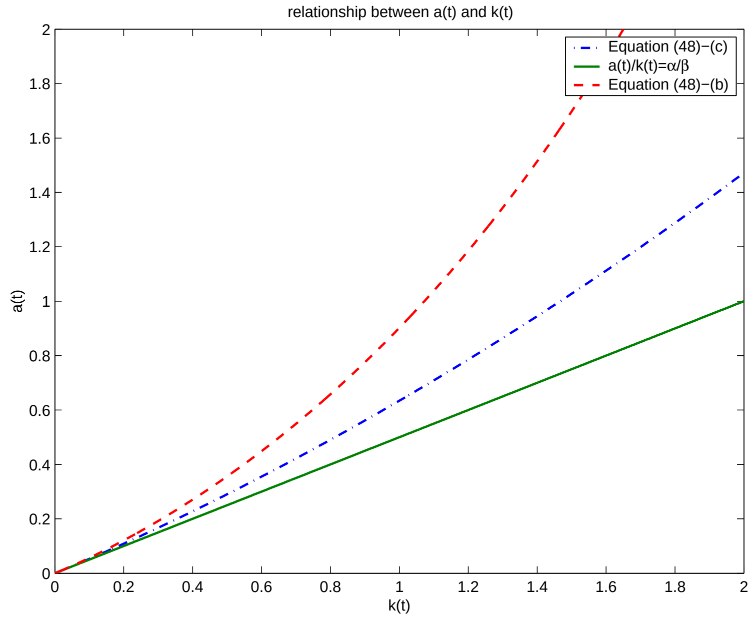

Figure 4: Possible per effective long-lived capital, $a(t)$, and short-lived capital, $b(t)$ implied by (13) when $\hat{q} < \alpha(\delta^B - \delta^A)$.

Using the result, I can obtain the steady state values of a^* and b^*, and characterize the economy. The full characterization of the cases are as follows.

Proposition 2 (Steady State)**.** *Suppose the economy started at $t = 0$ and has been in a steady state. Then, there exist steady state values of a^* and b^*, given the current $L(t)$ and q_t,*

(i) [Fast Case] if $\alpha(\delta^B - \delta^A) < \hat{q}$, the economy is characterized as following:

1. Aggregate Capital:

$$\begin{aligned} A(t) &= a^* N(t); \\ B(t) &= b^* N(t). \end{aligned}$$

2. Distribution of labor:

$$L_v(t) = \left[\frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha + \beta} + \hat{N} \right] e^{-(\frac{\hat{q}+\alpha\delta^A+\beta\delta^B}{\alpha+\beta}+\hat{N})(t-v)} L(t).$$

3. Distribution of capital:

$$\begin{aligned} A_v(t) &= \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} e^{-(\delta^A + \hat{N})(t-v)} A(t); \\ B_v(t) &= \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} e^{-(\delta^B + \hat{N})(t-v)} B(t). \end{aligned}$$

4. Allocation of investment:

$$\begin{aligned} I^A(t) &= I_t^A(t) = \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} A(t); \\ I^B(t) &= I_t^B(t) = \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} B(t). \end{aligned}$$

where $\hat{N} = \frac{\hat{q}}{1-\alpha-\beta} + \hat{L}$.

(ii) [Slow Case] otherwise, the economy is characterized as following:

1. Aggregate Capital:

$$\begin{aligned} A(t) &= a^* N(t); \\ B(t) &= b^* N(t). \end{aligned}$$

2. Distribution of labor:

$$L_v(t) = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A + \frac{\hat{q}}{\alpha} + \hat{N})(t-v)} L(t).$$

3. Distribution of capital:

$$\begin{aligned} A_v(t) &= \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A + \hat{N})(t-v)} A(t); \\ B_v(t) &= \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A + \frac{\hat{q}}{\alpha} + \hat{N})(t-v)} B(t). \end{aligned}$$

4. Allocation of investment:

$$\begin{aligned} I^A(t) &= I_t^A(t) = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] A(t); \\ I_t^B(t) &= \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] B(t); \end{aligned}$$

Table 1: Properties of two cases of steady state.

Steady state	(i) Fast	Slow
Technological progress ($\hat{q}$)	$> \alpha(\delta^B - \delta^A)$	$< \alpha(\delta^B - \delta^A)$
$P_v^B(t)$	Decline exponentially	Remains 1
Investment	Frontier only	Frontier and Obsolete B
Diffusion	Fast	Slow
$\alpha(\delta^B - \delta^A)$	Small	Large
$A(t)/B(t)$	α/β	$> \alpha/\beta$
$[P_v^A(t)A_v(t)]/[P_v^B(t)B(t)]$	α/β	$> \alpha/\beta$

$$I_v^B(t) = \left[\delta^B - \delta^A - \frac{\hat{q}}{\alpha}\right] B_v(t);$$

$$I^B(t) = I_t^B(t) + \int_0^t I_v^B(t)\, dv = \left[\delta^B + \hat{N}\right] B(t).$$

where $\hat{N} = \frac{\hat{q}}{1-\alpha-\beta} + \hat{L}$.

Proof: See Appendix A.1.5.

Table 1 summarizes the properties of the two cases of steady state.

In the fast case: the investment schemes of all the available vintages are (a); all new investment is allocated to the frontier technology capital types, $A_t(t)$ and $B_t(t)$; and the ratio of those is always the same as the ratio of capital's shares, $A_t(t)/B_t(t) = \alpha/\beta$. This case is expressed as a point on the solid line in Figure 3. In this case, both prices of two capital types of a specific vintage decline exponentially over time. The prices of short-lived capital are higher than those of long-lived capital with the same vintages because short-lived capital of that vintage becomes relatively scarce compared to long-lived capital of that vintage over time. This is because their depreciation rates differ and there is no investment in vintage capital types.

As $\hat{q}$ goes up, the allocations of two capital types and labor skew toward the newest technology. The difference in the allocations of stocks of two capital types arises from the difference in the rates of physical depreciation. The ratio of aggregate amounts of them, $a(t)/b(t)$, keeps α/β even when $\hat{q}$ changes, however. The reason is that prices of vintage capital types adjust such that they cancel the difference in their depreciation rates. Indeed, the total depreciation – the sum of obsolescence and physical depreciation – is $(\hat{q} + \alpha\delta^A + \beta\delta^B)/(\alpha + \beta)$ for both capital types in the fast case.

Laitner and Stolyarov (2003)'s model is a special case of the fast case. They assume a single rate of depreciation, $\delta^A = \delta^B$, which assures $\alpha(\delta^B - \delta^A) = 0 \leq \hat{q}$ as long as the rate of technological progress is positive. The current model shows, however, that even when depreciation rates differ, similar results to their model are observed with some sets of parameters. This is because when technological change is fast enough, the economy does not care about obsolete technology, and instead focuses on the frontier technology. This results in investment in the capital types with the frontier technology only.

The slow case is considerably different from the fast case, and thus from Laitner and Stolyarov (2003). In this case, investment is not only allocated to the frontier technology capital types, $A_t(t)$ and $B_t(t)$, but also to existing short-lived capital with obsolete vintages, $B_v(t)\,\forall\, v < t$. The ratio of investment in the frontier capital types, $A_t(t)/B_t(t)$, is identical to the aggregate amounts, $A(t)/B(t)$. This steady state is expressed as a point on the line (c) in Figure 4. The ratio $a(t)/b(t)$ is larger than α/β as in the figure.

Unlike the fast case, when $\hat{q}$ declines, the ratio $A(t)/B(t)$ rises, because a decline in $\hat{q}$ lowers interest rate r. This makes long-lived capital more attractive since long-lived capital will last relatively longer of the two. The result does not occur in the fast case since the rates of obsolescence of capital types adjust such that the sum of the rates of depreciation and of obsolescence is the same across the different capital types.

Prices of short-lived capital of all vintages are one since the marginal product of obsolete short-lived capital exceeds that of new capital types. This is because a large stock of long-lived capital raises the marginal product of short-lived capital. This attracts invest in obsolete short-lived capital, while prices of long-lived capital decline with vintage.

The investment in obsolete technology geometrically discontinuously lowers the speed of diffusion of technology. Diffusion curves – the ratio of the chronologically aggregated production after time T_0 to the whole production – are:

$$1 - e^{-(\frac{\hat{q}+\alpha\delta^A+\beta\delta^B}{\alpha+\beta}+\hat{N})(t-T_0)}$$

for the fast case; and

$$1 - e^{-(\delta^A+\frac{\hat{q}}{\alpha}+\hat{N})(t-T_0)}$$

for the slow case.

Although the allocations of those inputs skew toward newer technology as $\hat{q}$ rises, unlike in the fast case, the motion of vintage short-lived capital is affected by investment in vintage short-lived capital as well as by physical depreciation. The ratio of investment in vintage short-lived capital to the existing short-lived capital, $I_v^B(t)/B_v(t)$, rises as $\hat{q}$ falls, because a smaller $\hat{q}$ makes investment in vintage short-lived capital more attractive.

Most importantly, the model implies that estimated technological progress using changes in equipment prices over time might be under-estimated. Greenwood et al. (1997) use change in equipment prices over time provided by Gordon (1990) in order to estimate the rate of vintage-specific technological progress. But their estimates are correct only when an economy is experiencing the fast case, because in the slow case I have no obsolescence in tangible capital if there exists long-lived complementary capital; thus combinations of the fast case and slow case may result in bias in the measurement of technological progress.

These results of slow case considerably differ from those of models in existing literature, including Solow's and Laitner and Stolyarov's, which predict investment only in the frontier vintage capital at any moment. These findings can be confirmed by evaluating empirical evidence of the relationships between changes in $\hat{q}$ and: investment patterns, ratios of capital stocks, $a(t)/b(t)$, and changes in prices.

4 Discussion

4.1 Validity of the Model

In Section 3.2, the steady state analysis of the model reveals two distinct investment patterns: (i) if the rate of the progress, $\hat{q}$, is above a threshold – which is the product of long-lived capital's share and the difference in the rates of depreciation, $\alpha(\delta^B - \delta^A)$ – then all new investment concentrates on the frontier technology; (ii) otherwise, some investment is allocated to obsolete, short-lived capital to exploit existing excessive long-lived capital with obsolete technologies.

Obvious implication of this result is that if the rate of technological progress is below the threshold, then, surprisingly, the prices of obsolete short-lived capital remain one over time even when the rate of technological progress is positive. In this case, estimate of the rate of technological progress from changes in equipment prices would be systematically biased downward.

Another important implication is that acceleration in the rate of vintage-specific technological progress can cause an abrupt reallocation of investment towards modern capital – consistent with investment booms that are concentrated in certain "high-tech" equipment. There is a widely accepted observation that the economic boom in the late 1990's coincided with the diffusion of the so-called information technology (IT).[8] While typical growth models merely consider investment in IT equipment as a source of improvement in productivity, the current model considers the observation in a different point of view; the concentration of investment in IT equipment is a result of a higher rate of vintage-specific technological change.

Is it possible that an economy switches around the threshold of slow case and fast case? Suppose that production involves vintage specific long-lived intangible capital, short-lived tangible capital. Also suppose that: the labor share is 60%; the rest of the share is equally divided to the each capital share; the depreciation rate of tangible capital is 10%; and intangible capital does not depreciate. These provide an ad-hoc threshold $\alpha(\delta^B - \delta^A) = 0.02$, which is about the average labor productivity growth rate of the post-war U.S. economy. Although vintage-specific technological progress is typically smaller than this, it is possible that the economy is in the slow case at times. The economy fluctuates around the threshold and the cases would differ at times. This implies that values in Greenwood et al. may be biased downward.[9]

4.2 Applications of the Model

Although the analysis in the former sections presumes aggregate level of production, the result can be applied to broader aggregation levels of production as long as the assumptions of the model are satisfied.[10] Table 2 summarizes three possible different aggregation levels of production that can be explained by the model. I believe that the assumptions are satisfied at least to an approximation with these examples, and thus implication from the analysis should provide explanation for observed economic

[8]For example, Oliner and Sichel (2000) indicate that business investment in IT equipment rises more than fourfold from 1995 to 1999, and Chart 2 in Oliner and Sichel (2003) shows that the output shares of IT equipment have risen more than 20 % between 1994 and 2000.

[9]Bias over time might be different when average growth rate is fast enough, however. The result depends on dynamics of the model, which is unsolved yet.

[10]i.e., fixed rate of recursive investment from output, profit maximization, competitive factors market, constant vintage-specific technological progress, and Cobb-Douglas production function.

Table 2: Different aggregation levels of production.

Level	Equipment	Intangible capital	Example
Machinery	Parts	System, Skill, Environment	PC, Appliances, Cotton spinning
Factory	Machinery	Network, Process, Skill	Railroad transportation, Steel plant
Firm	Factory, Machinery	Organization, Skill, R&D	Retail firm, Manufacturing firm

activities. The three aggregation levels are machinery level, factory level, and firm level.

It is easy to interpret these levels of production in actual economic activities. For example, consider an automobile as machinery level of production. An automobile consists of various kinds of parts. In this case, purchase of parts corresponds to the investment in equipment, and assembly of automobile from parts corresponds to the investment in intangible capital (system of automobile).[11] Parts physically wear and tear, while the system of automobile does not. You can produce transportation service as output by using automobile that is set of parts and system, and then reinvest a part of revenue from output in parts or system of automobile. Now suppose the engine of an automobile breaks down. If the change in automobile model develops quickly enough, you would purchase a new automobile since it has much better features than the obsolete one. Otherwise, you would replace the malfunctioned engine with new one but designed for the obsolete model in order to keep using the existing system of obsolete automobile.[12]

Applications to the other levels – factory and firm levels – are considered as somewhere between machinery and aggregate levels. Of these, analysis that focus on the roles of intangible capital in production at firm level is well documented. Hall (2001) shows that the U.S. economy has incorporated a substantial amount of intangible capital, especially in the past decade. Other examples of this line of study include Atkeson and Kehoe (2005), and McGrattan and Prescott (2005).

There are two types of empirical relevancy of the model in this context. Table

[11] The system of automobile can be interpreted as physical layout of parts that is based on the specific design of an automobile.

[12] As presented in Introduction using the example of PC, there are broader types of candidates of intangible capital depending on the type of machine.

Table 3: Service life, warranty period, and change in prices of two types of equipment.

Equipment type	PC	Appliance
Service life[a]	7 years	10 years
Warranty period[b]	1 year	up to lifetime
Change in price[c]	0.0467	3.014

[a] From Table 3 in Fraumeni (1997).

[b] Figure on PC is from Toshiba's notebooks, and that of appliance is from Kitchen Aid's refrigerators, dishwashers, and washers.

[c] Ratios of prices of 1983 to 1947 in Gordon (1990).

3 shows service life, warranty period, and change in prices of PC and appliances. Appliances have longer warranty periods and service life than PC does.These figures are counter-intuitive if physical depreciation from wear and tear is the dominant factor of the service life, since appliances have more moving parts than PC does. These facts suggest that there be investment in obsolete technology as replacement parts for appliances, but not for PC. This observation is consistent with the implication of the model with the fact that vintage-specific technological progress of PC is faster than that of appliances as shown in Table 3.[13] An important factor of longevity of equipment is maintenance and repair cost as emphasized by McGrattan and Jr. (1999) and Mullen and Williams (2004). The decision of maintenance and repair can be explained by the current model interpreting the investment in obsolete technology as maintenance and repair.

The second type of relevancy is coexistence of different vintage technologies. At the machinery level, Saxonhouse and Wright (2000) show that mule spinning was a preferred equipment for experienced workers than ring spinning around 1900. This instance can be explained by the existence of the specific skill (intangible capital) for the mule spinning. At the factory level, Comin and Hobijn (2004) present coexistence of several steel production methods; Felli and Ortalo-Magne (1998) present the coexistence of steam and diesel locomotives for railroad transportation. In these cases,

[13]One might argue that the life of IT equipment is shorter than that of appliances because the value of IT equipment declines more quickly than that of appliances do. Indeed, this is one of the main point that are formalized in the current model.

equipment for steel production or locomotives are short-lived tangible capital, and system of specific type of steel production process or networks of railroad are long-lived intangible capital. Development of the process or the network using tangible capital is considered as investment in intangible capital, while purchase of equipment or locomotives are considered as investment in tangible capital. The coexistence of different vintage technology is accompanied by investment in obsolete short-lived capital types, which is consistent with the prediction of the current model.

Thanks to the generic assumptions of the model, the implication presented in this study can be extended to continuous aggregation levels of production.In addition to the difference in the rates of technological progress in temporal dimension, those between machinery types, factories, firms, industries, and even countries can be exploited to confirm the importance of the result of the model. The full characterization of the steady state that includes allocation and investment across vintages expands the possibility of testing the importance of complementary capital.

The model also provides a new way of thinking about investment when production involves complementary capital for managers, executives, government officials, and international officials. Business people should take into account the importance of complementary capital for maximizing the production, while public sectors should worry about the implication of the model when they analyze implication of economic policy.

5 Conclusion

The existence of heterogeneous complimentary capital yields two distinctive investment patterns: (i) if the rate of technological progress is above a threshold – the product of long-lived capital's share and the difference in the rates of depreciation – then all new investment concentrates on the capital types that embody frontier technology; (ii) otherwise, a part of the investment is allocated to obsolete short-lived capital to exploit existing obsolete long-lived capital.

The result provides a new explanation for the observed investment in equipment with obsolete technologies. An important implication is that change in prices of equipment does not necessarily reflect the rate of technological progress. Another implication is that an acceleration in the rate of technological progress can cause an abrupt reallocation of investment towards modern capital, consistent with investment booms that are concentrated in certain "high-tech" equipment.

The results from the straightforward neoclassical setup can be applied to broader types of economic activities that consist of production and investment. I presented the examples in three different aggregation levels of production: machinery, factory, and firm levels. The result of the model fit well with the evidence from other studies. Although existing literature typically focuses on the roles of intangible capital at the firm level, the current analysis suggests that intangible capital in other levels of production be important.

Avenues for future research include both empirical and theoretical. Empirically testable implications include: investment patterns across countries / industries / firms / machines that have different rates of technological progress; investment concentration in specific equipment during boom and recession; diffusion curve and technological progress; maintenance and repair of different types of equipment with different rates of technological progress; and measuring the true rate of technological change. Theoretically, generalizing production function, introduction of demands, characterizing transition, loosening vintage-specificity assumption of capital, and multi-sector model will be meaningful.

Another interesting avenue is to consider a new question: is investment in high-tech equipment source of growth, or result of higher vintage-specific technological progress? The results of these application will be critical for both business and government sectors in order to correctly evaluate economic activities with forgotten complementary capital.

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A Appendix

A.1 Proofs

A.1.1 Lemma 1 (Chronological Aggregation)

(i) Agent's profit maximization conditions are:

$$MPA_v(t) = \alpha\frac{Y_v(t)}{A_v(t)} = P_v^A(t)R_v^A(t) = P_v^A(t)\left[r(t) + \delta^A - \hat{P_v^A}(t)\right], \quad (14)$$

$$MPB_v(t) = \beta\frac{Y_v(t)}{B_v(t)} = P_v^B(t)R_v^B(t) = P_v^B(t)\left[r(t) + \delta^B - \hat{P_v^B}(t)\right], \quad (15)$$

$$MPL(t) = (1-\alpha-\beta)\frac{Y_v(t)}{L_v(t)} = W(t), \quad (16)$$

where hat ($\hat{}$) denotes the time derivative of the natural log of argument, and prices are normalized at 1 unit of output. Note that marginal product of labor, $MPL(t)$, does not have vintage subscript because labor is not vintage-specific and the wage is independent of vintages. Then, we have

$$\left[\frac{MPA_v(t)}{MPA_{v'}(t)}\right]^\alpha \left[\frac{MPB_v(t)}{MPB_{v'}(t)}\right]^\beta = \frac{q_v}{q_{v'}}. \quad (17)$$

Now, using (1), (3) - (6), and (14) - (17), the aggregate output will be

$$
\begin{aligned}
Y(t) &= \int_0^t Y_v(t)\,dv \\
&= \int_0^t q_v A_v(t)^\alpha B_v(t)^\beta L_v(t)^{1-\alpha-\beta}\,dv \\
&= \int_0^t q_v \left[\frac{A_t(t)}{L_t(t)}\frac{MPA_t(t)}{MPA_v(t)}L_v(t)\right]^\alpha \cdot \\
&\quad \left[\frac{B_t(t)}{L_t(t)}\frac{MPB_t(t)}{MPB_v(t)}L_v(t)\right]^\beta L_v(t)^{1-\alpha-\beta}\,dv \\
&= \left[\frac{A_t(t)}{L_t(t)}\right]^\alpha \left[\frac{B_t(t)}{L_t(t)}\right]^\beta \int_0^t q_v \frac{q_t}{q_v} L_v(t)\,dv \\
&= q_t A(t)^\alpha B(t)^\beta L(t)^{1-\alpha-\beta},
\end{aligned}
$$

which is equation (7).

(ii) Use (4) and (14) -(16) to show

$$
\begin{aligned}
A(t) &= \int_0^t \left[\frac{q_v}{q_t}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(t)/A_v(t)}{B_t(t)/A_t(t)}\right]^{\frac{\beta}{\alpha+\beta}} A_v(t)\,dv \\
&= q_t^{-\frac{1}{\alpha+\beta}} \left[\frac{B_t(t)}{A_t(t)}\right]^{-\frac{\beta}{\alpha+\beta}} \int_0^t \left[q_v A_v(t)^\alpha B_v(t)^\beta\right]^{\frac{1}{\alpha+\beta}}\,dv \\
&= q_t^{-\frac{1}{\alpha+\beta}} \left[\frac{B(t)}{A(t)}\right]^{-\frac{\beta}{\alpha+\beta}} \int_0^t J_v(t)\,dv \\
&= q_t^{-\frac{1}{\alpha+\beta}} \left[\frac{B(t)}{A(t)}\right]^{-\frac{\beta}{\alpha+\beta}} J(t),
\end{aligned}
$$

which is the equitation (9). From (7) - (8), and (16), we have

$$
\left[\frac{J_v(t)}{L_v(t)}\right]^{\alpha+\beta} = \frac{Y_v(t)}{L_v(t)} = \frac{Y(t)}{L(t)} = \left[\frac{J(t)}{L(t)}\right]^{\alpha+\beta},
$$

which provides (10) and (11).

■

A.1.2 Lemma 2 (Uniqueness of Investment Scheme across Vintages)

The proof has four steps. I show: (i) both $\hat{P}_v^A$ and $\hat{P}_v^B$ are constant; (ii) uniqueness of changes in prices; (iii) distribution of schemes.

The scheme (d) is not allowed across vintages, because if different vintages v and v' are in scheme (d), both capital types' prices must be one and therefore $MPA_v(s) = MPA_{v'}(s)$ and $MPB_v(s) = MPB_{v'}(s)$, which breaks the condition (14) or (15).

(i) [Growth Rate of Prices of Capital] In a steady state, (14) and (15) imply that both $MPA_v(t)$ and $MPB_v(t)$ grow at constant rates since $Y_v(t), A_v(t)$, and $B_v(t)$ all grow at constant rates. Now, suppose $\hat{P}_v^A(t) > M\hat{P}A_v(t)$. Then, $MPA_v(t)/P_v^A(t) = r^* + \delta^A - \hat{P}_v^A(t)$ declines and therefore the growth rate of $P_v^A(t)$ should accelerate over time. Then, $P_v^A(t)$ should reach one with positive growth rate in a finite time, which breaks the condition, $P_v^A(t) \in [0, 1]$.

Next, suppose $\hat{P}_v^A(t) < M\hat{P}A_v(t)$. Then, $P_v^A(t)$ should reach zero in a finite time with negative growth rate and either it breaks the condition, $P_v^A(t) \in [0, 1]$, or firms want to get rid of the capital, which breaks the constant growth.

Therefore, I need $\hat{P}_v^A(t) = M\hat{P}A_v(t)$ and thus $P_v^A(t)$ grows at a constant rate. Then, $\hat{P}_v^A \leq 0$ since otherwise P_v^A exceeds one in a finite time, which is impossible in a steady state. Similar arguments apply to the prices of tangible capital, $P_v^B(t)$.

Thus, both $P_v^A(t)$ and $P_v^B(t)$ must grow at constant rates, and

$$M\hat{P}A_v = \hat{P}_v^A \leq 0, \tag{18}$$

$$M\hat{P}B_v = \hat{P}_v^B \leq 0. \tag{19}$$

(ii) [Uniqueness of Change in Prices] Next, consider the vintage v and v' that are in scheme (a). Since there is no continuous positive investment, $\hat{J_v}(t) = \hat{J_{v'}}(t)$, which implies $\hat{Y_v}(t) = \hat{Y_{v'}}(t)$ from (11). Therefore, (14) and (15) imply $P_v^{\hat{A}}(t) = MP\hat{A}_v(t) = MP\hat{A}_{v'}(t) = P_{v'}^{\hat{A}}(t)$. Argument for $P_v^B(t)$ and $P_{v'}^B(t)$ is the same.

For the vintage v and v' that are in scheme (b), I have $\hat{P}_v^A(t) = \hat{P}_{v'}^A(t) = 0$, and thus (17) implies $MPB_v(t)/MPB_{v'}(t) = P_v^B(t)/P_{v'}^B(t)$ is constant. The similar argument applies to the scheme (c).

Thus, if vintages v and v' are in a same investment scheme, then

$$\hat{P}_v^A(t) = \hat{P}_{v'}^A(t), \tag{20}$$

$$\hat{P_v^B}(t) = \hat{P_{v'}^B}(t). \tag{21}$$

(iii) [Scheme across Vintages] Relationships of prices across vintages of capital types in a same investment scheme are:

$$P_v^A(t) = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(t)/A_v(t)}{B_{v'}(t)/A_{v'}(t)}\right]^{\frac{\beta}{\alpha+\beta}} P_{v'}^A(t); \tag{22}$$

$$P_v^B(t) = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(t)/A_v(t)}{B_{v'}(t)/A_{v'}(t)}\right]^{-\frac{\alpha}{\alpha+\beta}} P_{v'}^B(t). \tag{23}$$

In a steady state, both $\hat{A}_v$ and $\hat{B}_v$ are constant. Changes in investment scheme is not allowed because they require change in either $\hat{A}_v$ or $\hat{B}_v$, which breaks the definition of a steady state.

Now, suppose vintage v is scheme (a) or (c) and vintage v' is scheme (b). Then, since $[B_v\hat{/}A_v] > [B_{v'}\hat{/}A_{v'}]$, (22) implies

$$\hat{P_v^A} > \hat{P_{v'}^A} = 0,$$

which cannot be held in a steady state because price of capital A_v exceeds one in a finite time, and I cannot change the scheme. I have similar arguments against combinations of schemes (a) and (c) with (23).

Thus, in a steady state: both $P_v^A(t)$ and $P_v^B(t)$ must grow at constant rates, and $M\hat{P}A_v = \hat{P_v^A} \leq 0$ and $M\hat{P}B_v = \hat{P_v^B} \leq 0$; and if vintages v and v' are in a same investment scheme, then $\hat{P_v^A}(t) = \hat{P_{v'}^A}(t)$ and $\hat{P_v^B}(t) = \hat{P_{v'}^B}(t)$.

Therefore, investment scheme of a vintage v cannot change over time, and investment scheme is unique across vintages, and is either (a), (b), or (c). ■

A.1.3 Proposition 1 (Investment Scheme)

(i) Suppose investment scheme is (b) $\forall\, v$. Then, $P_v^A = 1$ and $\hat{P_v^A} = 0$. Since $\left[\frac{MPA_v}{MPB_v}\right]\hat{} = \left[\frac{B_v}{A_v}\right]\hat{} < -[\delta^B - \delta^A]$, and $M\hat{P}A_v = \hat{P_v^A} = 0$, I have $M\hat{P}B_v = \hat{P_v^B} > \delta^B - \delta^A > 0$, which cannot be true in a steady state because price of B_v exceeds one eventually.

Next, suppose investment scheme is (c) $\forall\, v$. Then, $P_v^B = 1$, $\hat{P_v^B} = 0$, and (22) implies $\hat{P_v^A(t)} = -\hat{q}_t/\alpha$. Following the same argument above, I know

$$-\frac{\hat{q}_t}{\alpha} = \hat{P_v^A} > -[\delta^B - \delta^A].$$

Therefore, in a steady state with $\hat{q} > \alpha(\delta^B - \delta^A)$, investment scheme must be (a) $\forall\, v$.

(ii) Investment scheme (b) is impossible as in (i). Now, suppose investment scheme is (a) $\forall v$. We know that there is no investment and thus $\hat{Y(t)} = \hat{I^A(t)} = \hat{I^B(t)} = \hat{A_t(t)} = \hat{B_t(t)}$ from (2). But this is impossible because (23) implies that $P_v^B(t)$ exceeds one when $\hat{q} < \alpha(\delta^B - \delta^A)$. Therefore, investment scheme must be (c) when $\hat{q} < \alpha(\delta^B - \delta^A)$.

■

A.1.4 Proposition 3 (Allocation of Capital Stock)

The laws of motion of the capital types of each vintage are

$$\begin{aligned}
\dot{A}_v(t) &= I_v^A(t) - \delta^A A_v(t), \\
\dot{B}_v(t) &= I_v^B(t) - \delta^B B_v(t).
\end{aligned}$$

In a steady state, $P_t^A(t) = P_t^B(t) = 1$ from Lemma 2. Then, I now know from (20), (21), (4) and (5) can be rewritten as

$$A(t) = \int_0^t P_v^A(t) A_v(t)\, dv, \tag{24}$$

$$B(t) = \int_0^t P_v^B(t) B_v(t)\, dv. \tag{25}$$

By differentiating (24) and (25), I can obtain the laws of motion of aggregate capital:

$$\begin{aligned}
\dot{A}(t) &= \frac{\partial}{\partial t}\int_0^t P_v^A(t) A_v(t)\, dv \qquad (26)\\
&= \int_0^t [P_v^A(t) A_v(t)][\hat{P_v^A}(t) + \hat{A}_v(t)]\, dv + A_t(t) \\
&= \left[\hat{P_v^A}(t) - \delta^A\right] A(t) + \int_0^t I_v^A(t)\, dv + I_t^A(t)
\end{aligned}$$

$$= \left[\hat{P_v^A}(t) - \delta^A\right] A(t) + I^A(t);$$

and

$$\dot{B}(t) = \left[\hat{P_v^B}(t) - \delta^B\right] B(t) + I^B(t). \tag{27}$$

Since $A(t)$ grows at a constant rate, (26) implies $\hat{I^A} = \hat{A}$. Similarly, I have $\hat{I^B} = \hat{B}$. Therefore, from (2), $\hat{Y} = \hat{I^A} = \hat{I^B}$. These imply $\hat{Y} = \hat{A} = \hat{B}$ and thus (7) gives $\hat{Y} = \hat{N} = \frac{\hat{z}+\hat{q}}{1-\alpha-\beta} + \hat{L}$ in a steady state.

These growth rate is the same as those of the aggregate output and the aggregate amounts of two capital types in a steady state.

Now, by canceling $r(t)$ from (14) and (15), I have

$$\left[\frac{\beta}{P_v^B(t)B_v(t)} - \frac{\alpha}{P_v^A(t)A_v(t)}\right] Y_v(t) = [\delta^B - \hat{P_v^B}(t)] - [\delta^A - \hat{P_v^A}(t)]. \tag{28}$$

Since again $Y(t)/L(t) = Y_t(t)/L_t(t)$, $A(t)/L(t) = A_t(t)/L_t(t)$, and $B(t)/L(t) = B_t(t)/L_t(t)$ from (1), (4), (5), and (7), by applying $v \to t$ and with per effective capita amounts, I can rewrite (28) as

$$\frac{\beta a(t)^\alpha b(t)^{\beta-1}}{P_t^B(t)} - \frac{\alpha a(t)^{\alpha-1} b(t)^\beta}{P_t^A(t)} = [\tilde{\delta^B} - \hat{P_v^B}(t)] - [\delta^A - \hat{P_v^A}(t)].$$

■

A.1.5 Proposition 2 (Steady State)

[Laws of Motion] In per effective labor expressions, the sum of the laws of motion, (26) and (27), becomes

$$\dot{a}(t) + \dot{b}(t) =$$
$$\sigma a(t)^\alpha b(t)^\beta - [\delta^A - \hat{P_v^A}(t) + \hat{N}(t)]a(t) - [\delta^B - \hat{P_v^B}(t) + \hat{N}(t)]b(t).$$

Now, consider the case of Proposition 1 (i). In a steady state, the right hand side of (13) is zero and therefore $A_t(t)/B_t(t) = \alpha/\beta$. Note that all the investment is spent on two capital types with frontier technology because only P_T^A and P_T^B can be one.

Applying (22) and (23) to $v \to t$, I have

$$\dot{a}(t) + \dot{b}(t) =$$

$$= \quad \sigma a(t)^{\alpha} b(t)^{\beta} - \left[\frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} + \hat{N}\right][a(t)+b(t)].$$

In a steady state, using the condition $A(t)/B(t) = \alpha/\beta$, this can be simplified to

$$\dot{b}(t) = \frac{\sigma\beta}{\alpha+\beta}\left[\frac{\alpha}{\beta}\right]^{\alpha} b(t)^{\alpha+\beta} - \left[\frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} + \hat{N}\right] b(t). \tag{29}$$

Now, consider the case of Proposition 1 (ii). Using the prices of capital types from (22) and (23), I have

$$\begin{aligned} &\dot{a}(t) + \dot{b}(t) = \\ &\quad \sigma a(t)^{\alpha} b(t)^{\beta} - \left[\delta^A + \frac{\hat{q}}{\alpha}\right] a(t) - \delta^B b(t) - \hat{N}[a(t)+b(t)]. \end{aligned} \tag{30}$$

[Full Characterization] Now, I focus on the proof of (ii) [Slow Case] because the proof of (i) [Fast Case] is an easier case of that of the former one.

First, aggregate capital can be specified by

$$\frac{A(t)}{N(t)} = a^*,$$
$$\frac{B(t)}{N(t)} = b^*,$$

which are from the per effective labor definition.

Then, consider the investment allocation between aggregate long-lived capital and aggregate tangible capital. Since at the steady state $\dot{a(t)} = \dot{b(t)} = 0$, from (26), (27) and the proof of Lemma 2, I have

$$\frac{I^A(t)}{N(t)} = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] a^*, \tag{31}$$

$$\frac{I^B(t)}{N(t)} = \left[\delta^B + \hat{N}\right] b^*, \tag{32}$$

where $\hat{N} = \frac{\hat{q}}{1-\alpha-\beta} + n$.

(31) and (32) also imply the investment allocation between them is constant,

$$\frac{I^A(t)}{I^A(t)+I^B(t)} = \frac{[\delta^A + g/\alpha + \hat{N}](a^*/b^*)}{[\delta^A + g/\alpha + \hat{N}](a^*/b^*) + [\delta^B + \hat{N}]} = \theta.$$

Since the investment in long-lived capital is only for the frontier vintage, I have

$$A_v(t) = A_v(v)e^{-\delta^A(t-v)} = \theta\sigma Y(v)e^{-\delta^A(t-v)}. \tag{33}$$

Since $\hat{Y}(t) = \hat{N}(t) = \hat{N}$,

$$Y(v) = Y(t)e^{-\hat{N}(t-v)}.$$

Thus, (33) can be written as

$$A_v(t) = \theta\sigma Y(t)e^{-(\delta^A+\hat{N})(t-v)}. \tag{34}$$

Now, find the optimal allocation of labor, $L_v(t)$. From (14) - (16), the proof of Lemma 2, and (34) with per effective labor notation, read

$$L_v(t) = P_v^A(t)\frac{A_v(t)}{A_t(t)}L_t(t) = e^{-(\delta^A+\frac{\hat{q}}{\alpha}+\hat{N})(t-v)}L_t(t). \tag{35}$$

So, since the total amount of labor is $L(t) = \int_0^t L_v(t)\,dv$, I have

$$L_t(t) = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right]L(t). \tag{36}$$

Therefore, (35) and (36) can determine the distribution of the labor, $L_v(t)$.

(35) combined with (31) provides distribution of long-lived capital,

$$A_v(t) = \frac{L_v(t)}{L_t(t)}\frac{A_t(t)}{P_v^A(t)} = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right]e^{-(\delta^A+\hat{N})(t-v)}A(t). \tag{37}$$

Now consider vintage tangible capital. Since $MPB_v(t) = MPB_t(t)$ and thus from (14) (16)

$$\frac{B_v(t)}{L_v(t)} = \frac{B_t(t)}{L_t(t)}. \tag{38}$$

Using (37), I have

$$B_t(t) = A_t(t)\frac{B(t)}{A(t)} = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right]B(t).$$

Therefore,

$$B_v(t) = B_t(t)\frac{L_v(t)}{L_t(t)} = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A+\frac{\hat{q}}{\alpha}+\hat{N})(t-v)}B(t).$$

On the other hand, in the per effective labor notation, (38) is

$$b_v(t) = b_t(t)\left[\frac{q_t}{q_v}\right]^{\frac{1}{1-\alpha-\beta}} = b^* e^{\frac{\hat{q}}{1-\alpha-\beta}(t-v)}.$$

So, I have

$$\hat{B}_v(t) = \frac{\hat{q}}{1-\alpha-\beta} + \hat{L_v}(t) = -\left[\delta^A + \frac{\hat{q}}{\alpha}\right].$$

Therefore,

$$\frac{I_v^B(t)}{B_v(t)} = \hat{B}_v(t) + \delta^B = \delta^B - \delta^A - \frac{\hat{q}}{\alpha},$$

and thus

$$I_v^B(t) = \left[\delta^B - \delta^A - \frac{\hat{q}}{\alpha}\right] B_v(t).$$

■

A.2 Prices

A.2.1 Relationship Across Vintages

From (14) and (15), I have

$$\frac{MPA_v(t)/MPB_v(t)}{MPA_{v'}(t)/MPB_{v'}(t)} = \frac{B_v(t)/A_v(t)}{B_{v'}(t)/A_{v'}(t)},$$

which implies with (17),

$$\frac{MPA_v(t)}{MPA_{v'}(t)} = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(t)/A_v(t)}{B_{v'}(t)/A_{v'}(t)}\right]^{\frac{\beta}{\alpha+\beta}}, \quad (39)$$

$$\frac{MPB_v(t)}{MPB_{v'}(t)} = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(t)/A_v(t)}{B_{v'}(t)/A_{v'}(t)}\right]^{-\frac{\alpha}{\alpha+\beta}}. \quad (40)$$

Now, consider the relationship of prices across vintages that have a same investment scheme over time.

First, suppose vintages v and v' are in scheme (a) since time T_0 such that $v \leq$

$v' \leq T_0 \leq s \leq t$ where t is the current time and s is some time between T_0 and t. Then, I can rewrite (39) as

$$\frac{MPA_v(s)}{MPA_{v'}(s)} = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(T_0)/A_v(T_0)}{B_{v'}(T_0)/A_{v'}(T_0)}\right]^{\frac{\beta}{\alpha+\beta}} = \varphi^A(v, v', T_0),$$

since change in capital stock is only from physical depreciation. Note that $\varphi^A(v, v', T_0)$ is time independent in this case. Substitution of (14) gives

$$P_v^A(s)\{r(s) + \delta^A\} - \dot{P}_v^A(s) = \varphi^A(v, v', T_0) \left[P_{v'}^A(s)\{r(s) + \delta^A\} - \dot{P}_{v'}^A(s)\right], \tag{41}$$

and similarly,

$$P_v^B(s)\{r(s) + \delta^B\} - \dot{P}_v^B(s) = \varphi^B(v, v', T_0) \left[P_{v'}^B(s)\{r(s) + \delta^B\} - \dot{P}_{v'}^B(s)\right],$$

where

$$\varphi^B(v, v', T_0) = \frac{MPB_v(s)}{MPB_{v'}(s)} = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(T_0)/A_v(T_0)}{B_{v'}(T_0)/A_{v'}(T_0)}\right]^{-\frac{\alpha}{\alpha+\beta}}.$$

So, given the initial distribution of prices and stocks of capital types and path of interest rate $\{r(s)\}$, I can determine the relative prices of two types of capital when there is no investment in the types of capital over time, which is in scheme (a).[14]

Now, consider the scheme (b), where there is investment in vintage capital v and v' since T_0 such that $v \leq v' \leq T_0 \leq s \leq t$. In this case, I know $P_v^A(s) = P_{v'}^A(s) = 1$ and thus $MPA_v(s) = MPA_{v'}(s)$. So, (17) becomes

$$\left[\frac{MPB_v(s)}{MPB_{v'}(s)}\right]^{\beta} = \frac{q_v}{q_{v'}}, \tag{42}$$

and, again I have

$$P_v^B(s)\{r(s) + \delta^B\} - \dot{P}_v^B(s) = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\beta}} \left[P_{v'}^B(s)\{r(s) + \delta^B\} - \dot{P}_{v'}^B(s)\right].$$

The next scheme is (c), where investment is only in the short-lived capital. Simi-

[14] Details of dynamics is in Appendix A.2.2.

larly I have $P_v^B(s) = P_{v'}^B(s) = 1$, and

$$P_v^A(s)\{r(s) + \delta^A\} - \dot{P}_v^A(s) = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha}} \left[P_{v'}^A(s)\{r(s) + \delta^A\} - \dot{P}_{v'}^A(s)\right].$$

In this way, I can determine relationship of prices of capital types in the same investment scheme given the investment scheme and the initial distributions of stocks and prices of capital types across vintages, as well as history of real interest rate. Which investment scheme dominates in vintage v in what conditions? How are the relative prices of obsolete capital types to the new investment prices determined? What are the relationships of prices between different investment schemes? These answers depend on distribution of the vintage long-lived and short-lived capital, rate of vintage-specific technological change, and steady state which an economy reaches eventually. In the following section, I consider the distributions of prices in a steady state that is sensible for an economy.

A.2.2 Dynamics of Prices

Investment Scheme (a) Multiply both sides of (41) by $-e^{-\{\int_{T_0}^{s} r(u)\,du + \delta^A (s-T_0)\}}$, and integrate over s from T_0 to the current time t. Then, read

$$\int_{T_0}^{t} [e^{-\{\int_{T_0}^{s} r(u)\,du + \dot{\delta}^A (s-T_0)\}} P_v^A(s)]\,ds = \varphi^A(v, v', T_0) \int_{T_0}^{t} [e^{-\{\int_{T_0}^{s} r(u)\,du + \dot{\delta}^A (s-T_0)\}} P_{v'}^A(s)]\,ds$$

which is, given the initial prices $P_v^A(T_0)$ and $P_{v'}^A(T_0)$, equivalent to

$$\begin{aligned} & P_v^A(t) - \varphi^A(v, v', T_0) P_{v'}^A(t) \\ = \; & e^{\int_{T_0}^{t} r(u)\,du + \delta^A (t-T_0)} [P_v^A(T_0) - \varphi^A(v, v', T_0) P_{v'}^A(T_0)]. \end{aligned} \tag{43}$$

(43) shows the relationship of intangible capital prices across vintage. Similarly, the relationship of prices of tangible capital across vintage is, given the initial stocks and the initial prices $P_v^B(T_0)$ and $P_{v'}^B(T_0)$,

$$\begin{aligned} & P_v^B(t) - \varphi^B(v, v', T_0) P_{v'}^B(t) \\ = \; & e^{\int_{T_0}^{t} r(u)\,du + \delta^B (t-T_0)} [P_v^B(T_0) - \varphi^B(v, v', T_0) P_{v'}^B(T_0)], \end{aligned} \tag{44}$$

where

$$\varphi^B(v, v', T_0) = \frac{MPB_v(s)}{MPB_{v'}(s)} = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v(T_0)/A_v(T_0)}{B_{v'}(T_0)/A_{v'}(T_0)}\right]^{-\frac{\alpha}{\alpha+\beta}}.$$

Another useful characterization of prices of capital is, the relationship between the capital types with the same vintage from (14) and (15) when there is no investment in vintage v at time $s \in [T_0, t]$. Now, from (14) and (15), I have

$$\frac{MPA_v(s)}{MPB_v(s)} = \frac{\alpha}{\beta}\frac{B_v(s)}{A_v(s)}, \tag{45}$$

and further,

$$P_v^A(s)\{r(s) + \delta^A\} - \dot{P_v^A}(s) = \frac{\alpha}{\beta}\frac{B_v(s)}{A_v(s)}\left[P_v^B(s)\{r(s) + \delta^B\} - \dot{P_v^B}(s)\right]$$

Then, I also have

$$\begin{aligned} & e^{-\left\{\int_{T_0}^t r(u)\,du + \delta^A(t-T_0)\right\}} P_v^A(t) - P_v^A(T_0) \\ = \ & \frac{\alpha}{\beta}\frac{B_v(T_0)}{A_v(T_0)}\left[e^{-\left\{\int_{T_0}^t r(u)\,du + \delta^B(t-T_0)\right\}} P_v^B(t) - P_v^B(T_0)\right]. \end{aligned}$$

Investment Scheme (b) Similarly,

$$\begin{aligned} P_v^B(t) = & \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\beta}} P_{v'}^B(t) \\ + \ & e^{\int_{T_0}^T r(u)\,du + \delta^B(T-T_0)}\left[P_v^B(T_0) - \left\{\frac{q_v}{q_{v'}}\right\}^{\frac{1}{\beta}} P_{v'}^B(T_0)\right]. \end{aligned}$$

Investment Scheme (c) Similarly,

$$\begin{aligned} P_v^A(t) = & \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha}} P_{v'}^A(t) \\ + \ & e^{\int_{T_0}^T r(u)\,du + \delta^A(T-T_0)}\left[P_v^A(T_0) - \left\{\frac{q_v}{q_{v'}}\right\}^{\frac{1}{\alpha}} P_{v'}^A(T_0)\right]. \end{aligned}$$