# Optimal Sustainable Monetary Policy*

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**Abstract**

Many recent monetary policy analyses have studied optimal commitment policy or Woodford's (1999) timeless perspective policy, with emphasis devoted to welfare gains from policy commitment, regardless of the time consistency problem called the stabilization bias. In this paper we adopt Chari and Kehoe's (1990) sustainable equilibrium to a dynamic general equilibrium model used in the recent analyses and examine optimal sustainable policy, the authority's policy strategy in the best sustainable equilibrium. The paper shows with realistic calibrations of model parameters that the plausible benchmark for policy conduct with no commitment technology is neither optimal commitment policy nor the timeless perspective policy but is optimal sustainable policy. Once we deliberate on the actual policy conduct, however, we cannot but consider optimal sustainable policy unreliable as the guidepost for policy, since it involves some issues due to its dependence on the entire history of Lagrange multipliers on the sustainability constraint. The paper suggests as a reliable guidepost a sustainable policy that contains no such issues and attains higher social welfare than discretionary policy does. This sustainable policy is conducted by following a policy rule that achieves the best Markov equilibrium.

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## 1 Introduction

Since Kydland and Prescott's (1977) seminal paper, the time consistency problem of optimal commitment policy has been well known. In the absence of commitment technologies that compel policy makers to keep their previous commitment on current-period policy, they have temptations to deviate from the commitment by reconsidering policy in the current period. Consequently, the optimal policy involves an issue of its credibility under rational expectations.[1] In the real world there is no central bank with the commitment technologies. Nevertheless, many recent monetary policy analyses have studied optimal commitment policy or Woodford's (1999) timeless perspective policy,[2] with emphasis on welfare gains from policy commitment in rational expectations models with a *stabilization bias*, the time consistency problem due to an inefficient trade-off in monetary policy-making with private agents' *forward-looking* behavior.[3] Clarida, Galí, and Gertler (1999), McCallum and Nelson (2004), and Woodford (2003, Ch. 7), for instance, stress that optimal commitment policy or the timeless perspective policy exhibits superior performance relative to discretionary policy, which makes no policy commitment.

In this paper we adopt Chari and Kehoe's (1990) sustainable equilibrium to a simple dynamic general equilibrium model with the stabilization bias, which has been used in recent literature such as Woodford (2003, Ch. 7, Sec. 2.1),[4] and examine optimal sustainable policy, the authority's policy strategy in the best sustainable equilibrium, which has not been studied in the recent literature. Chari and Kehoe proposed sustainable equilibrium as a concept of equilibrium for policy games played between the government and competitive private agents in an infinite horizon economy model with fiscal policy. These authors addressed the question of

[1] As Chari, Kehoe, and Prescott (1989) pointed out, if no commitment technology is available, any policy design problem requires that policy be sequentially rational: each period policy maximize a social welfare function given that private agents behave rationally.

[2] The timeless perspective policy is one that leads the equilibrium evolution from any period on to be optimal subject to an *additional* constraint that the evolution in the initial period of the policy design problem be one associated with the policy. See also Woodford (2003, Ch. 8, Sec. 1.1). This policy, however, is not generally time consistent in the *original* policy design problem.

[3] The stabilization bias differs from the well-known inflation bias, which has been studied in traditional literature starting from Kydland and Prescott (1977) and Barro and Gordon (1983a). In this literature it is presumed that the monetary authority has a desire to push output above its natural level because of some distortions and that some private agents make their decisions (e.g. price setting) before current monetary policy is set. Under these assumptions, optimal commitment policy is not generally time consistent, since the authority has a temptation to induce surprise inflation after private agents make their decisions. As Clarida, Galí, and Gertler (1999, Sec. 4.1) point out, how important the inflation bias is in practice, however, is a matter of controversy.

[4] Although this simple model has been criticized for failing to match the actual data of inflation and output that display significant persistence, it is a promising first step toward developing empirically plausible models.

how the concept of Kydland and Prescott's (1977) time consistent equilibrium in finite horizon economy models can be extended to infinite horizon ones.[5] By adapting Abreu's (1988) optimal penal codes to the policy games,[6] Chari and Kehoe obtained three important findings: (i) the entire set of outcomes of sustainable equilibria can be represented completely by private agents' optimality conditions, the government budget constraint, and a set of inequality constraints called the *sustainability constraint*; (ii) any outcome in the entire set can be implemented by the government strategy that specifies continuation with that outcome as long as it has been chosen in the past; otherwise, the strategy specifies to adopt a policy that induces the worst sustainable equilibrium outcome in the subsequent economy; and (iii) even in the absence of the commitment technologies, an outcome of equilibrium with optimal commitment policy is attainable if private agents are sufficiently patient.[7] Their first two findings suggest that once we can feature the entire set of sustainable equilibrium outcomes, optimal sustainable policy can be found as the authority's policy strategy that specifies continuation with a policy that yields the best sustainable equilibrium outcome as long as this policy has been adopted in the past; otherwise, the strategy specifies to adopt a policy that induces the worst sustainable equilibrium outcome in the subsequent economy. Their last finding provides the condition whereby we can examine whether optimal commitment policy or the timeless perspective policy is the plausible benchmark for policy conduct with no commitment technology, as discussed later.

This paper shows first that the entire set of sustainable equilibrium outcomes of the model can be fully represented by private agents' optimality condition and a sustainability constraint.[8] This constraint requires that for every possible history the expected present discounted value of social welfare attained by an outcome be at least as high as that by the worst sustainable equilibrium outcome, which is induced by discretionary policy in the model, as shown below.

[5]This question was also examined by Stokey (1991).

[6]Because Abreu's method involves finding the worst equilibrium, so does that of Chari and Kehoe. It may be more difficult to find the worst sustainable equilibrium in a more general model. To overcome this difficulty, Chang (1998) and Phelan and Stacchetti (2001) adapt the approach of Abreu, Pearce, and Stachetti (1990) to their policy games to characterize the entire set of sustainable equilibrium outcomes. Judd, Yeltekin, and Conklin (2003) provide a computational method for this approach. Sleet (2001) uses it to analyze a dynamic general equilibrium model with the inflation bias in which the monetary authority has private information.

[7]Ireland (1997) applies Chari and Kehoe's method to a dynamic general equilibrium model with the inflation bias and shows that even in the absence of the commitment technologies an outcome of equilibrium with optimal commitment policy (i.e. Friedman's rule) is achievable with realistic calibrations of model parameters.

[8]This paper follows the recently common practice in monetary policy analyses, of leaving hidden fiscal policy and the government budget constraint. This would be the case if fiscal policy is "Ricardian": fiscal policy appropriately accommodates the consequences of monetary policy for the government budget constraint.

In obtaining optimal sustainable policy, we use a Lagrange method adopted from Marcet and Marimon (1998), Benhabib and Rustichini (1997), and Kehoe and Perri (2002), who all develop Kydland and Prescott's (1980) pioneering work, to derive a policy that yields the best sustainable equilibrium outcome. This paper refers to such a policy as *optimal quasi-sustainable policy*. Specifically, it can be derived by maximizing the social welfare function subject to private agents' optimality condition and the sustainability constraint, since these two constraints fully represent the entire set of sustainable equilibrium outcomes. Optimal sustainable policy is then conducted by following optimal quasi-sustainable policy and thus implements the best sustainable equilibrium outcome on the rational expectations equilibrium path.[9] This is because private agents are policy takers and because optimal sustainable policy leads the monetary authority to have no temptation to deviate from optimal quasi-sustainable policy.[10] Thus, by examining the latter policy, we can obtain the following three features of optimal sustainable policy. First, the best sustainable equilibrium outcome implemented by optimal sustainable policy is intermediate between outcomes of equilibria with optimal commitment policy and with discretionary policy, since optimal quasi-sustainable policy is featured as an intermediate one between the latter two policies. Second, the form of optimal quasi-sustainable policy approaches that of optimal commitment policy in future periods, which in turn implies that in sufficiently later periods, optimal sustainable policy achieves the outcome of equilibrium with optimal commitment policy.[11] Last but not least, if the sustainability constraint is never binding, optimal quasi-sustainable policy is consistent with optimal commitment policy and hence the time consistency problem does not matter in that optimal sustainable policy achieves the outcome of equilibrium with optimal commitment policy. This is the case if private

[9]To be precise, optimal sustainable policy is the authority's policy strategy that specifies continuation with adoption of optimal quasi-sustainable policy, which yields the best sustainable equilibrium outcome, as long as this policy has been adopted in the past; otherwise, the strategy specifies to adopt discretionary policy forever, which induces the worst sustainable equilibrium outcome in the subsequent economy.

[10]The authority has no such temptation for the following three reasons. First, optimal sustainable policy specifies to adopt discretionary policy forever once the authority deviates from optimal quasi-sustainable policy. Second, the sustainability constraint ensures that for every possible history, optimal quasi-sustainable policy attains at least as high an expected present discounted value of social welfare as discretionary policy does. Last, adopting discretionary policy from any period on, together with associated decision rule of private agents, constitutes a sustainable equilibrium in the subsequent economy.

[11]This finding seems to be related to basic ideas of Woodford's (1999) timeless perspective policy and of Jensen's (2003) delay in implementation of discretionary policy. Woodford regards the timeless perspective policy as one to which the authority would have wished to commit at a date far in the past. Jensen shows that the performance of discretionary policy improves by introducing a delay between the publication and implementation of it and approaches that of optimal commitment policy as the period of the delay lengthens.

agents are sufficiently patient, as shown below. Indeed, sticking to optimal commitment policy yields such a large expected present discounted value of future welfare that it never pays for the monetary authority to deviate from that policy as long as the discount factor is sufficiently close to one. But, this does not hold for a range of realistic calibrations of model parameters: a certain lower bound on the discount factor that can sustain the outcome of equilibrium with optimal commitment policy is extremely close to one.[12] Almost the same result is obtained for the timeless perspective policy. Therefore, the plausible benchmark for policy conduct with no commitment technology in the model is neither optimal commitment policy nor the timeless perspective policy but is optimal sustainable policy.

Once we deliberate on the actual policy conduct, however, optimal sustainable policy involves some problems. This policy is conducted by following optimal quasi-sustainable policy in the sense noted above. The latter policy, as shown below, depends on the current sum of Lagrange multipliers on the sustainability constraint, which implies that the monetary authority is required to trace the history of the multipliers from the initial period of the policy design problem. This fact raises at least three issues regarding actual implementation of optimal sustainable policy. The first issue is about the initial period: when does the monetary authority set the initial period? The second is how actual central banks precisely know values of the Lagrange multipliers that are neither actual economic variables, such as inflation and output, nor variables that have explicit relationships with these actual variables. The last, but not least, issue is whether the best sustainable equilibrium outcome generated by optimal quasi-sustainable policy is unique. Taking account of these problems, we cannot but consider optimal sustainable policy unreliable as the guidepost for monetary policy. This paper then suggests as a reliable guidepost a sustainable policy that contains no such issues and attains higher social welfare than discretionary policy does. Specifically, this sustainable policy is conducted by following a policy rule that achieves the best Markov equilibrium.[13]

[12] This result is in stark contrast with Ireland (1997) and Albanesi, Chari, and Christiano (2003), who use dynamic general equilibrium models with the inflation bias to show that the time consistency problem is unlikely to matter.

[13] A similar exposition to that for optimal sustainable policy in Footnote 9 describes this sustainable policy in detail. The relationship between the sustainable policy and the policy rule is the same as that between optimal sustainable policy and optimal quasi-sustainable policy. The policy rule is also investigated by Clarida et al. (1999, Sec. 4.2.1) and is consistent, in the model, with what Woodford (2003, Ch. 7, Sec. 3.1) calls the "optimal non-inertial plan". One crucial point of this paper is that the sustainable policy requires no commitment technology but implements the same best Markov equilibrium outcome as the policy rule with such technologies does.

The remainder of the paper is organized as follows. Section 2 describes a monetary policy design problem and reviews optimal commitment policy, Woodford's timeless perspective policy, and discretionary policy. Section 3 examines optimal sustainable policy. Section 4 asks whether optimal commitment policy or the timeless perspective policy is the plausible benchmark for policy conduct with no commitment technology. Section 5 discusses the actual conduct of optimal sustainable policy. Finally, some concluding remarks are offered in Section 6.

# 2 A monetary policy design problem

This section describes a policy design problem that has been used in recent monetary policy analyses. The model is a simple dynamic general equilibrium model with the stabilization bias.

## 2.1 A dynamic general equilibrium model with the stabilization bias

The economy contains a continuum of infinitely lived private agents and the monetary authority. The behavior of private agents is represented by the following inflation equation, which describes the optimality condition for private agents' problem.[14]

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t, \tag{1}$$

where $\pi_t$ is the inflation rate, $x_t$ is the output gap, $u_t$ is an exogenous disturbance, $E_t$ is the expectations operator conditional on the period $t$ information set of private agents, $\beta \in (0,1)$ is the discount factor, and $\kappa > 0$ is the output-gap elasticity of inflation.

The policy objective of the monetary authority is assumed to be to maximize a social welfare function of the form[15]

$$-E_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \lambda x_t^2\right), \tag{2}$$

where $\lambda > 0$ is a weight of output-gap stabilization relative to inflation stabilization. As shown in Woodford (2003, Ch. 6, Sec. 2), (2) can be derived as a second-order approximation of the representative household's utility function with Calvo (1983) style staggered pricing of

[14] Eq. (1) represents optimal pricing behavior under price rigidity. As Roberts (1995) showed, (1) can be derived from a variety of price rigidity models, e.g., a Calvo (1983) style staggered pricing model and a Rotemberg (1982) style price adjustment cost model. Optimal spending behavior of private agents is described by a consumption Euler equation, which can be found in recent monetary policy analyses. This Euler equation is not needed in this analysis, since the inflation rate is assumed to be the monetary policy instrument, as noted later.

[15] Period zero denotes the initial period of the policy design problem, but not the beginning of the economy.

monopolistically competitive firms, whose behavior is described by (1).[16] Also, the relative weight $\lambda$ can be determined by fundamental parameters of such pricing behavior.

The presence of the disturbance to (1), $u_t$, is of all importance for this analysis.[17] This disturbance induces a trade-off in monetary policy-making between stabilization of inflation and the output gap, thereby causing the time consistency problem of optimal commitment policy in the presence of private agents' forward-looking behavior described by (1). Svensson and Woodford (2005) refer to this time consistency problem as the *stabilization bias*, which differs from the well-known inflation bias that has been studied in traditional literature starting from Kydland and Prescott (1977) and Barro and Gordon (1983a). The inflation bias is absent in the model for simplicity. The shock $u_t$ is assumed to be observable in the beginning of period $t$ and to follow a stationary first-order autoregressive process

$$u_t = \rho u_{t-1} + \varepsilon_t, \tag{3}$$

where $\rho \in (-1, 1)$ is an autoregression parameter and $\varepsilon_t$ is a white noise with its variance $\sigma_\varepsilon^2 > 0$. Also, the shock $u_t$ is assumed to be bounded, i.e. there is $B > 0$ such that $|u_t| < B \ \forall t$, which implies that $|\varepsilon_t| < B(1 + |\rho|) \ \forall t$.

The monetary authority is assumed to be able to directly control the current inflation rate rather than the current nominal interest rate, which recent analyses employ as the monetary policy instrument to reflect those of actual central banks in most industrialized countries. It does not matter whether the policy instrument is either the nominal interest rate or the inflation rate as long as there is no uncertainty about the transmission of monetary policy, since one can obtain identical results by controlling either of them.[18] Private agents form expectations about future inflation rates (i.e. future monetary policy) and determine the current output gap.[19] As Chari and Kehoe (1990) emphasized, it is important to stress that private agents behave competitively and hence do not collude to "punish" the monetary authority while choosing

[16] It is also assumed that fiscal policy eliminates from (2) the inflation bias due to monopolistic competition.

[17] Clarida et al. (1999) call this disturbance the "cost-push shock", while Woodford (2003, Ch. 6, Sec. 4.5) refers to it as the "inefficient supply shock".

[18] Jensen (2003) shows how one can use the consumption Euler equation (see Footnote 14) to derive an interest rate rule that attains the same equilibrium as a given inflation rate rule does. But, once the authority faces uncertainty about parameters or current disturbances of the Euler equation, the policy instrument choice matters.

[19] This setting is somewhat misleading in the interpretation of an original model in which private agents determine current prices of their products and hence the current inflation rate via the inflation equation (1), while the monetary authority controls the current nominal interest rate and thereby the current output gap via the consumption Euler equation. However, we can obtain identical results using either of these settings.

their current decisions, i.e. private agents are policy takers. In the model the authority has strategic power while competitive private agents do not. Therefore, the policy game played between them differs totally from games consisting of a finite number of players, each of whom has strategic power. Although private agents' expectations about future policy have an effect on current policy-making in equilibrium, this is not due to strategic behavior of private agents but is due to rational expectations. Timing of decision-making of all the agents is assumed as follows. In each period, after the current shocks are observed, the monetary authority and private agents make their decisions simultaneously and instantaneously in that all of them are able to observe their past and current choices each other. This timing assumption is the same as that assumed in recent literature with the stabilization bias, but it is in stark contrast to traditional literature with the inflation bias, in which some private agents make their decisions before current monetary policy is set.

In the rest of this section, we use the model presented above to review three policies, each of which some recent analyses have examined as desirable monetary policy: optimal commitment policy, Woodford's (1999) timeless perspective policy, and discretionary policy.

### 2.2 Optimal commitment and Woodford's timeless perspective policies

In the situation in which there is a commitment technology whereby the monetary authority keeps its previous commitment on current-period policy, the authority can credibly adopt optimal commitment policy, which makes policy commitment for the entire future in period zero, the initial period of the policy design problem. This policy maximizes (2) subject to (1) from period zero on. Hence, in the presence of the commitment technology, the optimal policy is the most desirable in period zero. The associated Lagrangian $\mathcal{L}^c$ can be written as

$$\mathcal{L}^c \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left[ - \left( \pi_t^2 + \lambda x_t^2 \right) + 2\phi_t(\pi_t - \beta \pi_{t+1} - \kappa x_t - u_t) \right] ,$$

where $\phi_t$ is the Lagrange multiplier on (1) in period $t \geq 0$. Then, the first-order condition for the optimal policy is given by

$$\pi_t - \phi_t + \phi_{t-1} = 0, \quad \lambda x_t + \kappa \phi_t = 0 \quad \forall t \geq 0$$

together with the initial condition that $\phi_{-1} = 0$, which indicates no previous commitment in period zero. Substituting out the Lagrange multipliers yields an inflation rate rule that implements the optimal policy

$$\pi_0 = -(\lambda/\kappa)\, x_0, \quad \pi_t = -(\lambda/\kappa)\,(x_t - x_{t-1}) \quad \forall t > 0. \tag{4}$$

A rational expectations equilibrium (REE) with optimal commitment policy, which is the so-called Ramsey equilibrium, is a pair of stochastic processes of inflation and the output gap such that the pair is a non-explosive solution to the system consisting of (1) and (4). Hence it follows that the determinate REE is given by

$$\pi_t^c = a_\pi u_t + b_\pi x_{t-1}^c, \quad x_t^c = a_x u_t + b_x x_{t-1}^c \quad \forall t \geq 0 \tag{5}$$

together with $x_{-1}^c = 0$,[20] where $a_\pi \equiv 1/[\beta(b^+ - \rho)] > 0$, $b_\pi \equiv (\lambda/\kappa)(1 - b_x) > 0$, $a_x \equiv -(\kappa/\lambda)\, a_\pi < 0$, $b_x \equiv b^- \in (0,1)$, and $b^\pm \equiv [\,1 + \beta + \kappa^2/\lambda \pm \sqrt{(1+\beta+\kappa^2/\lambda)^2 - 4\beta}\,]/(2\beta)$ with $b^+ > 1$. As shown in Appendix A, optimal commitment policy attains the expected present discounted value of social welfare from the current period $t \geq 0$ on, given by

$$\begin{aligned} V^c(u_t, x_{t-1}^c) \;&\equiv\; -E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[ (\pi_s^c)^2 + \lambda\,(x_s^c)^2 \right] \\ &=\; -\frac{1}{1-\beta\rho^2} \left[ a_\pi^2 + \lambda a_x^2 + \frac{2\beta\rho a_x}{1-\beta\rho b_x}\,(a_\pi b_\pi + \lambda a_x b_x) + \frac{\beta a_x^2}{1-\beta b_x^2}\,\left(b_\pi^2 + \lambda b_x^2\right) \right] \left( u_t^2 + \frac{\beta\sigma_\varepsilon^2}{1-\beta} \right) \\ &\quad - \left\{ \frac{b_\pi^2 + \lambda b_x^2}{1-\beta b_x^2}\, x_{t-1}^c + 2 \left[ \frac{a_\pi b_\pi + \lambda a_x b_x}{1-\beta\rho b_x} + \frac{\beta a_x b_x \left(b_\pi^2 + \lambda b_x^2\right)}{(1-\beta\rho b_x)\,(1-\beta b_x^2)} \right] u_t \right\} x_{t-1}^c. \end{aligned} \tag{6}$$

This consists of welfare losses due to the current and future shocks, $u_t^2 + \beta\sigma_\varepsilon^2/(1-\beta)$, and current-period welfare gains or losses due to previous commitment, $x_{t-1}^c$.

Woodford (1999) argues that optimal commitment policy lacks continuity in its form and thereby "fails to be time consistent *only* if the central bank considers 'optimality' at each point in time" (p. 293). He thus proposes the timeless perspective policy, which leads the equilibrium evolution from any period on to be optimal subject to an additional constraint that the evolution in period zero be one associated with the policy. Indeed, this policy of the model is implemented by an inflation rate rule of the time invariant form

$$\pi_t = -(\lambda/\kappa)\,(x_t - x_{t-1}) \quad \forall t \geq 0. \tag{7}$$

Note that the timeless perspective policy generally involves the time consistency problem and hence a commitment technology is required for adopting this policy. The inflation rate rule (7),

[20] This means no previous commitment in period zero, but not the output gap of zero in period $-1$.

together with (1), brings about the determinate REE with the timeless perspective policy in the same form as (5)

$$\pi_t^w = a_\pi u_t + b_\pi x_{t-1}^w, \quad x_t^w = a_x u_t + b_x x_{t-1}^w \quad \forall t \geq 0, \tag{8}$$

given some level of the output gap in period $-1$, $x_{-1}^w = x_{-1}$, which is assumed to be bounded. Hence the policy attains the expected period $t\,(\geq 0)$ discounted value of social welfare of the same form as (6)

$$V^w(u_t, x_{t-1}^w) = V^c(u_t, x_{t-1}^w). \tag{9}$$

### 2.3 Discretionary policy

In the absence of the commitment technologies, optimal commitment policy has a problem with its credibility, since private agents know the monetary authority's temptation to deviate from the optimal policy after period zero. The lack of such technologies thus leads the authority to choose policy sequentially as opposed to once and for all in period zero. Discretionary policy is then one of policy choices for the authority. Under this policy, the authority determines policy in every period so as to maximize the expected present discounted value of (2) subject to (1) from the current period on, taking private agents' expectations as given. As Clarida et al. (1999) indicate, this policy design problem can be reduced to a sequence of static problems in which the authority chooses the current inflation rate so as to maximize current-period welfare in (2), $-(\pi_t^2 + \lambda x_t^2)$, subject to (1) with private agents' expectations exogenously given. Then, an inflation rate rule that implements discretionary policy can be given by

$$\pi_t = -(\lambda/\kappa)\, x_t \quad \forall t \geq 0. \tag{10}$$

Discretionary optimization yields (10), which differs from (4) in all periods except period zero. This shows that in the absence of the commitment technologies, rational expectations lead optimal commitment policy to involve an issue of its credibility and the associated equilibrium to contain an issue of its implementability. The inflation rate rule (10), together with (1), generates the determinate REE with discretionary policy, given by

$$\pi_t^d = c_\pi u_t, \quad x_t^d = c_x u_t \quad \forall t \geq 0, \tag{11}$$

where $c_\pi \equiv 1/(1-\beta\rho+\kappa^2/\lambda) > 0$ and $c_x \equiv -(\kappa/\lambda)\, c_\pi < 0$, and hence it attains, as shown in Appendix A, the expected period $t\,(\geq 0)$ discounted value of social welfare given by

$$V^d(u_t) \equiv \, -E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[ (\pi_s^d)^2 + \lambda (x_s^d)^2 \right] = \, -\frac{c_\pi^2 + \lambda c_x^2}{1-\beta\rho^2} \left( u_t^2 + \frac{\beta \sigma_\varepsilon^2}{1-\beta} \right). \tag{12}$$

Because discretionary policy makes no policy commitment, (12) represents welfare losses due only to current and future shocks, $u_t^2 + \beta\sigma_\varepsilon^2/(1-\beta)$.

We have so far reviewed the three policies, each of which some recent analyses have studied as desirable monetary policy. Optimal commitment policy, if available, is the most desirable in the initial period of the policy design problem but involves an issue of its credibility under rational expectations in the absence of the commitment technologies, which no central bank possesses in the real world. The same issue applies to Woodford's timeless perspective policy, although the continuity in its form brings some advantages in the actual policy conduct. In contrast, discretionary policy is credible but attains relatively low social welfare as stressed in the recent analyses such as Clarida et al. (1999), McCallum and Nelson (2004) and Woodford (2003, Ch. 7).[21] Then, it seems quite natural to ask the question of whether there is any concept of optimal policy that is credible with no commitment technology and achieves higher social welfare than discretionary policy does. According to traditional literature with the inflation bias, such as Barro and Gordon (1983b) and Rogoff (1989), even in the absence of the commitment technologies, infinite horizon economy models may involve reputational equilibria, which are REE that attain higher welfare in terms of the monetary authority's objective function than REE with discretionary policy does. More generally, Chari and Kehoe (1990) proposed sustainable equilibrium as a concept of equilibrium for policy games played between the government and competitive private agents, employing an infinite horizon economy model with fiscal policy. These authors addressed the question of how the concept of Kydland and Prescott's (1977) time consistent equilibrium in finite horizon economy models can be extended to infinite horizon ones. To this end Chari and Kehoe adapted Abreu's (1988) optimal penal codes to the policy games. Specifically, they featured the entire set of outcomes of sustainable equilibria in the infinite horizon economy model by finding its worst sustainable equilibrium. They also showed that any outcome in the entire set can be implemented by

[21]Section 4 uses realistic calibrations of model parameters to illustrate welfare gains from optimal commitment policy relative to discretionary policy.

the government strategy that specifies continuation with that outcome as long as it has been chosen in the past; otherwise, the strategy specifies to adopt a policy that induces the worst sustainable equilibrium outcome in the subsequent economy. These results of Chari and Kehoe lead us to study the monetary authority's policy strategy in the best sustainable equilibrium of the model here as a promising alternative to the three policies reviewed above. This paper refers to such a policy strategy of the authority as *optimal sustainable policy*. We investigate this optimal policy in the following sections.

# 3 Optimal sustainable policy

In this section we adopt Chari and Kehoe's (1990) sustainable equilibrium to the monetary policy design problem presented in the previous section and examine optimal sustainable policy, the authority's policy strategy in the best sustainable equilibrium.

## 3.1 Characterization of entire set of sustainable equilibrium outcomes

We begin by defining sustainable equilibrium of the model. Let $h_t$ denote the history in the beginning of period $t$, defined recursively by $h_0 = u_0$, $h_t = (h_{t-1}, \pi_{t-1}, u_t)\,\forall t > 0$.[22] Then, the assumption on timing of decision-making can be formulated as follows. After the current shocks $u_t$ are observed, the monetary authority chooses the current inflation rate $\pi_t$ as a function of the history $h_t$, $\pi_t = \sigma_t(h_t)$, together with a contingency plan $(\sigma_s)_{s\geq t+1}$ for setting future inflation rates for all possible future histories.[23] Simultaneously and instantaneously in the sense noted above, private agents form expectations about future inflation rates and determine the current output gap $x_t$ as a function of the history $(h_t, \pi_t)$, $x_t = f_t(h_t, \pi_t)$, together with a contingency plan $(f_s)_{s\geq t+1}$ for choosing future output gaps for all possible future histories. Note that given some current history $h_t$, the monetary policy strategy for inflation rates, $\sigma = (\sigma_t)_{t\geq 0}$, induces future histories in the recursive way such that $h_{t+1} = (h_t, \sigma_t(h_t), u_{t+1})\,\forall t \geq 0$.

A monetary policy strategy $\sigma$ and private agents' decision rule for output gaps, $f = (f_t)_{t\geq 0}$, are chosen as follows. Given a monetary policy strategy $\sigma$, private agents' decision rule $f$ must

[22] The reader may wonder why histories $h_t$ do not include private agents' choices. For a discussion of this point, see Chari and Kehoe (1990).

[23] Note that the function $\sigma_t$ depends on period $t$ because the dimension of $h_t$ is $2t + 1$, which also applies to the function $f_t$. Note also that under the timing assumption, the monetary authority can observe the current output gap determined by private agents when it sets the current inflation rate.

satisfy (1) for every possible history induced by $\sigma$:

$$\sigma_t(h_t) = \beta E_t[\sigma_{t+1}(h_{t+1})] + \kappa f_t(h_t, \sigma_t(h_t)) + u_t. \tag{13}$$

This is because (1) is the optimality condition for private agents' problem. A monetary policy strategy $\sigma$ is chosen such that given a decision rule $f$ of private agents and given a current history $h_t$ induced by $(\sigma_s)_{s=0}^{t-1}$, the continuation $(\sigma_s)_{s\geq t}$ of $\sigma$ solves the authority's problem:

$$\begin{aligned} \max_{(\tilde{\sigma}_s)_{s\geq t}} \quad & -E_t \sum_{s=t}^{\infty} \beta^{s-t}\{[\tilde{\sigma}_s(h_s)]^2 + \lambda [f_s(h_s, \tilde{\sigma}_s(h_s))]^2\} \\ \text{s.t.} \quad & \tilde{\sigma}_s(h_s) = \beta E_s[\tilde{\sigma}_{s+1}(h_{s+1})] + \kappa f_s(h_s, \tilde{\sigma}_s(h_s)) + u_s \quad \forall s \geq t. \end{aligned} \tag{14}$$

We can now define sustainable equilibrium of the model.

**Definition 1** *A sustainable equilibrium of the model is defined as a pair* $(\sigma, f)$ *of the monetary policy strategy for inflation rates and private agents' decision rule for output gaps such that*

*(D1) given the policy strategy* $\sigma$*, private agents' decision rule* $f$ *satisfies the optimality condition for their problem,* (13)*, for every possible history induced by* $\sigma$ *and*

*(D2) given the decision rule* $f$*, each continuation* $(\sigma_s)_{s\geq t}$ *of the monetary policy strategy* $\sigma$ *solves the authority's problem* (14) *for every possible history induced by* $\sigma$*.*

The following two points of this definition are noteworthy. First, as similar to time consistent equilibrium of Kydland and Prescott (1977), sustainable equilibrium requires both the monetary policy strategy and private agents' decision rule to be sequentially rational. Second, as Barro and Gordon (1983a) stressed, the monetary authority chooses, in equilibrium, to confirm private agents' expectations of future inflation rates, which is a requirement for REE but not a prior constraint in the authority's problem. With Definition 1 we have the next result about the REE with discretionary policy.

**Proposition 1** *The REE with discretionary policy is the worst sustainable equilibrium of the model.*

*Proof.* See Appendix B. ■

The economic intuition for this proposition is as follows. Because private agents are policy takers, the monetary authority always has an option of adopting discretionary policy and hence of implementing the associated REE (11). Therefore, the authority never takes any policies that result in inferior social welfare relative to the REE (11). Then, discretionary policy and associated decision rule of private agents are sequentially rational, and hence it follows that the REE (11) is the worst sustainable equilibrium of the model.

Like Chari and Kehoe (1990), we can use Proposition 1 to feature the entire set of outcomes generated by sustainable equilibria of the model. Recall that a sustainable equilibrium $(\sigma, f) = \{(\sigma_t), (f_t)\}_{t\geq 0}$ is a sequence of functions that specify inflation rates and output gaps for all possible histories induced by $\sigma$. Hence, a sustainable equilibrium yields a particular pair of contingent sequences of inflation rates and output gaps, $(\pi, x) = \{(\pi_t), (x_t)\}_{t\geq 0}$. This pair is indeed the outcome of the sustainable equilibrium. For an arbitrary pair of contingent sequences of inflation rates and output gaps, the next proposition provides the necessary and sufficient condition for the existence of a sustainable equilibrium whose outcome is such a pair.

**Proposition 2** *An arbitrary pair $(\pi, x)$ of contingent sequences of inflation rates and output gaps is an outcome of a sustainable equilibrium if and only if*

*(S1) the pair $(\pi, x)$ satisfies* (1) *for every possible history induced by $\pi$ and*

*(S2) the next inequality holds for every possible history induced by $\pi$:*

$$-E_t \sum_{s=t}^{\infty} \beta^{s-t}\left(\pi_s^2 + \lambda x_s^2\right) \geq V^d(u_t). \tag{15}$$

*Proof.* See Appendix C. ■

In the literature the condition (S2) is referred to as the *sustainability constraint*.[24] It is important to emphasize that it would be a gross mis-reading of Proposition 2 to think that this proposition only features the set of outcomes sustainable by particular trigger-like behavior. The traditional literature with the inflation bias, such as Barro and Gordon (1983b) and Rogoff (1989), uses some sorts of trigger-like behavior to seek reputational equilibria. On the contrary, Proposition 2 characterizes the entire set of outcomes that can be supported by any conceivable sustainable equilibria.

[24] Ireland (1997) uses a dynamic general equilibrium model with the inflation bias to obtain a similar result to Proposition 2. In his model it is presumed that monopolistically competitive firms set prices of their products before current monetary policy is determined, which induces the time consistency problem. Hence, in the sustainability constraint of his model, the right-hand side of the counterpart to (15) consists of current-period welfare gained by the monetary authority's "cheating" plus the expected present discounted value of future welfare induced by discretionary policy. In the model of this paper, the right-hand side of (15) is the expected present discounted value of current and future welfare attained by discretionary policy.

### 3.2 Features of optimal sustainable policy

With Proposition 1 and 2, we can find optimal sustainable policy, the monetary policy strategy in the best sustainable equilibrium. Indeed, from the argument in the proof of Proposition 2, we can see that this optimal policy is the policy strategy that specifies continuation with the best sustainable equilibrium outcome as long as this outcome has been chosen in the past; otherwise, it specifies to adopt discretionary policy forever, which induces the worst sustainable equilibrium outcome in the subsequent economy as shown in Proposition 1. Thus, to examine optimal sustainable policy, we need to derive a policy that generates the best sustainable equilibrium outcome. This paper refers to such a policy as *optimal quasi-sustainable policy*.

Optimal quasi-sustainable policy can be obtained by maximizing (2) subject to the set of (1) and (15) for every possible history, since this set completely represents the entire set of sustainable equilibrium outcomes as shown in Proposition 2. To solve this maximization problem, we use a Lagrange method adopted from Marcet and Marimon (1998), Benhabib and Rustichini (1997), and Kehoe and Perri (2002), who all develop Kydland and Prescott's (1980) pioneering work.[25] The associated Lagrangian $\mathcal{L}^s$ can be written as

$$\begin{aligned}\mathcal{L}^s \equiv\ & E_0\sum_{t=0}^{\infty}\beta^t\left\{-(\pi_t^2+\lambda x_t^2)+2\phi_t(\pi_t-\beta\pi_{t+1}-\kappa x_t-u_t)-\psi_t\left[\sum_{s=t}^{\infty}\beta^{s-t}(\pi_s^2+\lambda x_s^2)+V^d(u_t)\right]\right\}\\ =\ & E_0\sum_{t=0}^{\infty}\beta^t\left[-\Psi_t(\pi_t^2+\lambda x_t^2)+2\phi_t\left(\pi_t-\beta\pi_{t+1}-\kappa x_t-u_t\right)-\psi_t V^d(u_t)\right],\end{aligned}$$

where $\phi_t$ and $\psi_t \geq 0$ are Lagrange multipliers on, respectively, (1) and (15) in period $t \geq 0$, and $\Psi_t \geq 1$ is defined recursively by $\Psi_{-1} = 1$, $\Psi_t = \Psi_{t-1} + \psi_t$ $\forall t \geq 0$ or sequentially by $\Psi_t = 1 + \sum_{s=0}^{t}\psi_s$ $\forall t \geq 0$, so $\Psi_t$ contains the current sum of Lagrange multipliers $\{\psi_s\}_{s=0}^{t}$ on the sustainability constraint (S2). The first-order condition is then given by

$$\Psi_t\pi_t - \phi_t + \phi_{t-1} = 0, \quad \Psi_t\lambda x_t + \kappa\phi_t = 0 \quad \forall t \geq 0 \tag{16}$$

[25] The use of the Lagrange method adopted from the recursive contract theory of Marcet and Marimon (1998) implies that optimal quasi-sustainable policy can be interpreted as an optimal contract between private agents and the monetary authority in which private agents intend the authority to attain higher social welfare than discretionary policy does. Interestingly, this interpretation of optimal quasi-sustainable policy is in stark contrast to that of optimal sustainable policy, which is that the authority itself willingly adopts optimal sustainable policy to seek the highest social welfare in the absence of the commitment technologies. Although the optimal contract and optimal sustainable policy are based on the exactly opposite perspectives on the design of monetary policy, these two implement the same equilibrium outcome.

together with $\phi_{-1} = 0$ and the complementary slackness condition. Substituting $\phi_t$ out from (16) yields an inflation rate rule that would implement optimal quasi-sustainable policy

$$\pi_0 = -(\lambda/\kappa)\, x_0, \quad \pi_t = -(\lambda/\kappa)\left[x_t - (\Psi_{t-1}/\Psi_t)\, x_{t-1}\right] \quad \forall t > 0. \tag{17}$$

The next proposition summarizes three important properties of the inflation rate rule (17).

**Proposition 3** *In the inflation rate rule* (17)*, the following three hold.*

*(a)* $0 < \Psi_{t-1}/\Psi_t \leq 1\ \forall t \geq 0$*;*

*(b)* $\Psi_{t-1}/\Psi_t \to 1$ *as* $t \to \infty$*;*

*(c) If* $\Psi_t = 1\ \forall t \geq 0$ *(i.e.* $\psi_t = 0\ \forall t \geq 0$*), then* (17) *is consistent with* (4)*.*

*Proof.* Property (a) immediately follows from the fact that $1 \leq \Psi_{t-1} \leq \Psi_t$ for all $t \geq 0$. This fact also implies that for some histories such that $\lim_{t\to\infty} \Psi_t < \infty$, property (b) immediately follows, and for other histories (i.e. $\lim_{t\to\infty} \Psi_t = \infty$) we have that $\Psi_{t-1}/\Psi_t = 1 - \psi_t/\Psi_t \to 1$ as $t \to \infty$. Property (c) holds since $\Psi_{t-1}/\Psi_t = 1\ \forall t \geq 0$. ∎

Now we can say simply that optimal sustainable policy is conducted by following optimal quasi-sustainable policy. This is because private agents are policy takers and because optimal sustainable policy leads the monetary authority to have no temptation to deviate from optimal quasi-sustainable policy, which yields the best sustainable equilibrium outcome, for the following three reasons. First, optimal sustainable policy specifies to adopt discretionary policy forever once the authority deviates from optimal quasi-sustainable policy. Second, the sustainability constraint ensures that for every possible history, optimal quasi-sustainable policy attains at least as high an expected present discounted value of social welfare as discretionary policy does. Last, adopting discretionary policy from any period on, together with associated decision rule of private agents, constitutes a sustainable equilibrium in the subsequent economy. Thus, optimal sustainable policy implements the best sustainable equilibrium outcome on the REE path.

From Proposition 3, we can see the following three features of optimal sustainable policy. First, property (a) implies that the inflation rate rule (17), which would implement optimal quasi-sustainable policy, is featured as an intermediate one between inflation rate rules (4) and (10), which implement, respectively, optimal commitment and discretionary policies. Hence, the best sustainable equilibrium outcome implemented by optimal sustainable policy turns out to be an intermediate one between outcomes of equilibria with optimal commitment policy and with discretionary policy.

Second, from property (b) we have that the form of the inflation rate rule (17) approaches that of (4) in future periods. This in turn implies that in sufficiently later periods, optimal sustainable policy achieves the outcome of equilibrium with optimal commitment policy. This finding seems to be related to Woodford's (1999) timeless perspective policy and Jensen's (2003) delay in implementation of discretionary policy. Woodford regards the timeless perspective policy as one to which the monetary authority would have wished to commit at a date far in the past. Then, if we set the initial period of the monetary policy design problem at the date (i.e. $t = -\infty$), the basic idea of Woodford seems very close to that behind property (b). Also, we can see a similar idea in Jensen, who shows that the performance of discretionary policy approaches that of optimal commitment policy as the period of the delay between the publication and implementation of discretionary policy lengthens.

Last but not least, property (c) implies that if $\psi_t = 0 \; \forall t \geq 0$, which means that the sustainability constraint (S2) is never binding, then the time consistency problem does not matter in that optimal sustainable policy achieves the outcome of equilibrium with optimal commitment policy. The question we then address is under what condition this is the case. Because the shocks $u_t$ induce the stabilization bias, optimal commitment policy attains higher social welfare in period zero than discretionary policy does, i.e. $V^c(u_0, 0) > V^d(u_0)$, and hence (15) holds in period zero. Then, if for every possible history $h_t$, $t > 0$, optimal commitment policy also achieves a greater expected period $t$ discounted value of social welfare than discretionary policy does, i.e. $V^c(u_t, x^c_{t-1}) > V^d(u_t)$, then the sustainability constraint (S2) is never binding. The next proposition provides a sufficient condition for that.

**Proposition 4** *There exists $\underline{\beta} \in (0, 1)$ such that for any discount factor $\beta \in \left(\underline{\beta}, 1\right)$, optimal sustainable policy attains the outcome of equilibrium with optimal commitment policy.*

*Proof.* See Appendix D. ■

Proposition 4 suggests that if private agents are sufficiently patient (i.e. $\underline{\beta} < \beta < 1$), the outcome of equilibrium with optimal commitment policy is attainable even in the absence of the commitment technologies. Indeed, sticking to optimal commitment policy yields such a large expected present discounted value of future welfare that it never pays for the monetary authority to deviate from that policy if the discount factor is sufficiently close to one.[26] A similar result to Proposition 4 holds for Woodford's (1999) timeless perspective policy, since this policy differs from optimal commitment policy only in period zero. We states this result in the next corollary.

**Corollary 1** *If $|x_{-1}| < B|a_x|/(1-b_x)$, where $B$ is the assumed bound for the shocks $u_t$ and where $a_x$ and $b_x$ are given in* (5)*, then $\underline{\beta}$ given in Proposition 4 ensures that for any discount factor $\beta \in \left(\underline{\beta}, 1\right)$, the outcome of equilibrium with the timeless perspective policy is attainable. Otherwise, there exists $\beta_w \in \left[\underline{\beta}, 1\right)$ such that for any discount factor $\beta \in (\beta_w, 1)$, that outcome is attainable.*

*Proof.* This corollary immediately follows from a similar proof to that in Appendix D. ■

From Proposition 4 and Corollary 1, we can consider a test that asks whether optimal commitment policy or the timeless perspective policy is the plausible benchmark for policy conduct with no commitment technology in the model. If the actual value of the discount factor $\beta$ is greater than $\underline{\beta}$ given in Proposition 4, optimal sustainable policy is conducted by following optimal commitment policy in the sense noted above and hence optimal commitment policy can be regarded as the plausible benchmark. Otherwise, there would be no grounds for supporting optimal commitment policy. The same argument can apply to the timeless perspective policy. In the next section, only optimal commitment policy is investigated, since the condition for $x_{-1}$ in Corollary 1 is relevant with a sufficiently large $B$ and hence the result obtained below also holds for the timeless perspective policy.

# 4 Is optimal commitment policy the plausible benchmark?

This section addresses the question of whether optimal commitment policy is the plausible benchmark for monetary policy conduct with no commitment technology in the model.

From the proof of Proposition 4, we have a lower bound $\underline{\beta}$ on the discount factor required for sustaining the outcome of equilibrium with optimal commitment policy, given by

$$\underline{\beta} = \inf\{\beta \in (0,1) \,:\, F(\tilde{\beta}) \geq 0 \;\forall \tilde{\beta} \in (\beta, 1)\}, \tag{18}$$

[26]This is an analogy to the reason why the folk theorem holds in infinitely repeated games.

where[27]

$$F(\beta) = \frac{\beta\sigma_\varepsilon^2\Delta(\beta)}{(1-\beta)(1-\beta\rho^2)} - \frac{B^2a_x^2(b_\pi^2+\lambda b_x^2)}{(1-\beta b_x^2)(1-b_x)^2} + \frac{2B^2a_x}{1-b_x}\left[\frac{a_\pi b_\pi - \lambda a_x b_x}{1-\beta\rho b_x} - \frac{\beta a_x b_x(b_\pi^2+\lambda b_x^2)}{(1-\beta\rho b_x)(1-\beta b_x^2)}\right],$$

$$\Delta(\beta) = \frac{c_\pi^2+\lambda c_x^2}{1-\beta\rho^2} - (a_\pi^2+\lambda a_x^2) - \frac{2\beta\rho a_x}{1-\beta\rho b_x}(a_\pi b_\pi + \lambda a_x b_x) - \frac{\beta a_x^2}{1-\beta b_x^2}(b_\pi^2+\lambda b_x^2).$$

Note first that $\Delta(\beta) > 0$ for all $\beta \in (0, 1]$, since the function $\Delta$ takes a positive value that is proportional to the difference between welfare losses, due only to the current and future shocks, induced by optimal commitment policy and by discretionary policy, i.e. $V^c(u_t, 0) - V^d(u_t)$. Note also that the function $F$ satisfies that for every $\beta \in (0, 1]$, $V^c(u_t, x^c_{t-1}) - V^d(u_t) > F(\beta)$ for all possible histories $h_t$. Then, $F$ never depends on histories $h_t$, so that for $\beta \in (0, 1)$ such that $F(\beta) \geq 0$, we have that $V^c(u_t, x^c_{t-1}) > V^d(u_t)$ for all possible histories $h_t$, which implies that the sustainability constraint (S2) is never binding and hence the time consistency problem does not matter in that optimal sustainable policy achieves the outcome of equilibrium with optimal commitment policy.

As mentioned above, if the actual value of the discount factor $\beta$ is greater than $\underline{\beta}$ given by (18), optimal sustainable policy is conducted by following optimal commitment policy in the sense noted above. This provides a justification for regarding optimal commitment policy as the plausible benchmark for monetary policy conduct with no commitment technology in the model. We now compute $\underline{\beta}$ to examine whether $\beta > \underline{\beta}$.

## 4.1 Calibrations of model parameters

To carry out the test, we need realistic calibrations of model parameters. The model contains six parameters: $\beta, \kappa, \lambda, \rho, \sigma_\varepsilon, B$. Table 1 summarizes the calibrations of the first four parameters. These values are taken from McCallum and Nelson (2004). Note that these calibrations are for the quarterly model with the annualized inflation rate and that in terms of this inflation rate the value of $\lambda = (1/4)^2 = .0625$ means equal weights on (2). The bound $B$ for the shocks $u_t$ is set such that $B = k\sigma_\varepsilon/(1+|\rho|)$, where $k = 4, 5$. This yields that $|\varepsilon_t| < B(1+|\rho|) = k\sigma_\varepsilon$. Note that if $\varepsilon_t$ is normally distributed, then $\text{Prob}\{|\varepsilon_t| > 4\sigma_\varepsilon\} = .0000634$ and $\text{Prob}\{|\varepsilon_t| > 5\sigma_\varepsilon\} = .000000574$. Thus, the choice of $B$ seems reasonable. Also, this choice allows us to compute $\underline{\beta}$ given by (18) without specifying $\sigma_\varepsilon$. This is because we can then write $F(\beta) = \sigma_\varepsilon^2\tilde{F}(\beta)$, where

[27] Note that $a_\pi, b_\pi, a_x, b_x$ are functions of $\beta$ and satisfy that for each $\beta \in (0, 1]$, $0 < a_x < 1/(1-\beta\rho)$, $0 < b_x < \lambda/\kappa$, $-(\kappa/\lambda)/(1-\beta\rho) < a_x < 0$, and $0 < b_x < 1$.

$\tilde{F}(\beta)$ is independent of $\sigma_\varepsilon$, and hence we have that $F(\beta) \geq 0$ if and only if $\tilde{F}(\beta) \geq 0$.

### 4.2 Test results

Before presenting results of the test, we confirm the recent literature's emphasis on welfare gains from policy commitment, or equivalently, how severe the time consistency problem, the stabilization bias, is in the model. Table 2 shows welfare gains from optimal commitment policy relative to discretionary policy, i.e. $[V^c(u_0, 0) - V^d(u_0)]/|V^d(u_0)|$.[28] We can see that the welfare gains vary from 3.32% to 83.35%, depending on the model parameters, $\kappa$, $\lambda$, and $\rho$. One feature of this table is that the welfare gains enlarge as $\rho$ increases, i.e. the shocks $u_t$, which are the source of the time consistency problem, become more persistent. Discretionary policy performs relatively well when the persistence of the shocks is sufficiently low. Otherwise, the time consistency problem is very severe in the model. This result is in stark contrast with the traditional literature with the inflation bias, such as Ireland (1997) and Albanesi et al. (2003), who demonstrate that the time consistency problem, the inflation bias, is unlikely to matter, respectively, in Ireland's model and in variants of a cash credit good model of Lucas and Stokey (1983) and of a limited participation model of Christiano, Eichenbaum, and Evans (1997).

We now discuss whether optimal commitment policy is the plausible benchmark for policy conduct. The value of $\underline{\beta}$ determined by (18) in each calibration case is presented in Table 3.[29] In this table the values without shadows represent calibration cases in which $\underline{\beta}$ is less than .99, so that in these cases optimal sustainable policy is conducted by following optimal commitment policy in the sense noted above. In more than half of all the calibration cases, however, $\underline{\beta}$ is greater than .99. This suggests that there would be no grounds for regarding optimal commitment policy as the plausible benchmark for monetary policy conduct with no commitment technology in the model. One feature of Table 3 is that $\underline{\beta}$ is more likely to be greater than .99 when the output-gap elasticity of inflation, $\kappa$, decreases.[30] As McCallum and Nelson (2004) point out, the actual value of $\kappa$ lies between .01 and .05, so that it is likely $\underline{\beta}$ is greater than .99 and optimal sustainable policy implements an equilibrium outcome that differs from the one with optimal commitment policy. Another feature of the table is that $\underline{\beta}$ is more likely

[28] Note that this measure of welfare gains depends on niether the shocks $u_0$ in period zero nor the variance $\sigma_\varepsilon^2$ of the white noise $\varepsilon_t$ in the process of shocks $u_t$.

[29] In each calibration case the function $F$ is strictly increasing in $\beta$, so there is no $\beta < \underline{\beta}$ such that $F(\beta) \geq 0$.

[30] Note also that $\underline{\beta}$ is more likely to be greater than .99 as $\lambda$ or $B$ increases.

to be less than .99 as the persistence of the shocks, $\rho$, increases. Thus, when the shocks become more persistent, optimal sustainable policy is more likely to be conducted by following optimal commitment policy. Note that the latter feature does not necessarily arise from the above-mentioned fact that the welfare gains from optimal commitment policy relative to discretionary policy enlarge with more persistent shocks. We can see this from the result that when $\rho = .8$, Table 2 shows that the welfare gains are larger in the case of $(\kappa, \lambda) = (.01, .0625)$, i.e., 40.14%, than in the case of $(\kappa, \lambda) = (.10, .0010)$, i.e., 9.69%, whereas Table 3 shows that in the former case $\underline{\beta}$ takes a greater value of .9930 for $k = 4$ or .9953 for $k = 5$ than it does in the latter case, i.e., .9076 for $k = 4$ or .9342 for $k = 5$. Thus, whether high persistence of the shocks leads optimal commitment policy to be the plausible benchmark for monetary policy conduct depends on the other parameters, such as the output-gap elasticity of inflation, $\kappa$, and the policy weight of output-gap stabilization relative to inflation stabilization, $\lambda$.

# 5 On actual conduct of optimal sustainable policy

We have shown with realistic calibrations of model parameters that the plausible benchmark for monetary policy conduct without the commitment technologies, which no central bank possesses in the real world, is not optimal commitment policy (and neither is Woodford's timeless perspective policy) but is optimal sustainable policy. This policy is conducted by following optimal quasi-sustainable policy in the sense noted above. In such policy conduct, however, there are some issues about actual implementation of it. As shown above, optimal quasi-sustainable policy depends on the current sum $\Psi_t (= 1 + \sum_{s=0}^{t} \psi_s)$ of Lagrange multipliers $\{\psi_s\}_{s=0}^{t}$ on the sustainability constraint (S2). This implies that the monetary authority is required to trace the history of the Lagrange multipliers from the initial period of the policy design problem. We can then raise the following at least three problems with the actual conduct of optimal sustainable policy.

## 5.1 Issues on actual conduct of optimal sustainable policy

The first issue is about the initial period of the monetary policy design problem. That is, when does the monetary authority set the initial period? This problem is indeed the same as that faced by optimal commitment policy, which recent literature considers one of the most critical defects of optimal commitment policy and hence Woodford (1999) proposes the timeless perspective policy. Optimal sustainable policy has additional difficulty with this problem. Even if the output gap happens to be zero in some period,[31] we cannot necessarily obtain continuation of optimal quasi-sustainable policy adopted before by re-optimizing the associated problem in the next period, due to the dependence on the history of the Lagrange multipliers from some initial period. Although optimal sustainable policy is sequentially rational, its optimality is guaranteed only in the initial period.

The second issue is how actual central banks precisely know values of Lagrange multipliers on the sustainability constraint (S2). These multipliers are neither actual economic variables, such as inflation and output, nor variables that have explicit relationships with these actual variables. Moreover, the sustainability constraint (S2) is a sequence of inequality constraints, so we cannot analytically compute an outcome of equilibrium with optimal quasi-sustainable policy, nor can we apply the usual Blanchard-Kahn method for numerically solving linear rational expectations models. Then, the existing literature contains two numerical methods for solving such a nonlinear model. One is the method of Christiano and Fisher (2000), who use first-order conditions for optimal policy with inequality constraints. Another is that of Marcet and Marimon (1998), who use a Bellman equation called the "recursive saddle point functional equation". In either method, we need some techniques to approximate expectation functions in Christiano and Fisher or a value function in Marcet and Marimon. This implies that we cannot obtain the exact values of the Lagrange multipliers and we only know their approximate values implicitly. Such lack of knowledge about the precise values of the Lagrange multipliers in turn would cause lack of transparency in monetary policy conduct.

The last, but not least, issue is whether the best sustainable equilibrium outcome generated by optimal quasi-sustainable policy is unique. If so, private agents and the monetary authority could coordinate on that unique equilibrium outcome. Otherwise, we face a critical problem of how a particular outcome can be chosen from multiple equilibrium outcomes with the same level of social welfare. The above-mentioned two numerical methods for solving the nonlinear model give no guarantee of a unique solution to the problem associated with optimal quasi-sustainable policy. To the best of my knowledge there is no way to investigate whether the nonlinear model contains a unique solution.

[31] In this case, we can see from (4) that the continuation of optimal commitment policy of the model can be obtained by re-optimizing the associated problem in the next period.

### 5.2 A sustainable policy reliable as the guidepost for policy

Taking account of the problems raised above, we cannot but consider optimal sustainable policy unreliable as the guidepost for monetary policy. The question we then address is whether there is a sustainable policy that contains no such problems and attains higher social welfare than discretionary policy does. To answer this question, it is reasonable to focus on monetary policy strategies played in Markov equilibria of the model. This is because the sustainability constraint (S2) given in Proposition 2 requires that for every possible history, social welfare attained by the policy in question be comparable with that by discretionary policy, which yields a Markov equilibrium.[32] Thus, if a proposed policy generates a Markov equilibrium, it is easy to examine whether an outcome of equilibrium with this policy is sustainable. Note that discretionary policy does not necessarily bring about the best Markov equilibrium, so it may be possible to find a sustainable policy that achieves higher social welfare than discretionary policy does. Indeed, we can obtain the following sustainable policy that achieves the highest social welfare among all Markov equilibria of the model including that induced by discretionary policy. The sustainable policy specifies continuation with adoption of a policy rule of the form

$$\pi_t = -(\lambda/\kappa)(1-\beta\rho)\, x_t \tag{19}$$

as long as this rule has been adopted in the past; otherwise, it specifies to adopt discretionary policy forever. This sustainable policy implements, on the REE path, the best Markov equilibrium outcome of the form

$$\pi_t^s = d_\pi u_t, \quad x_t^s = d_x u_t \quad \forall t \geq 0, \tag{20}$$

where $d_\pi = (1-\beta\rho)/[(1-\beta\rho)^2 + \kappa^2/\lambda]$ and $d_x = -(\kappa/\lambda)/(1-\beta\rho)d_\pi$, and hence it attains the expected period $t\,(\geq 0)$ discounted value of social welfare given by

$$V^s(u_t) = -\frac{d_\pi^2 + \lambda d_x^2}{1-\beta\rho^2}\left(u_t^2 + \frac{\beta\sigma_\varepsilon^2}{1-\beta}\right). \tag{21}$$

[32] Another reason for the focus on policy strategies played in Markov equilibria is that the entire set of sustainable policies in this class is not sensitive to model parameters as long as $\rho \neq 0$. We can alternatively consider a more general class of policy strategies that specify continuation with adoption of a policy rule of an intermediate form between (7) and (10), $\pi_t = -(\lambda/\kappa)\,(x_t - c\,x_{t-1})$, where $0 \leq c \leq 1$, as long as this rule has been adopted in the past; otherwise, they specify to adopt discretionary policy forever. The entire set of sustainable policies in this class, however, would be very sensitive to model parameters, particularly the discount factor $\beta$, as similar to optimal commitment and Woodford's timeless perspective policies.

Clarida et al. (1999, Sec. 4.2.1) have established that in the presence of a commitment technology, the policy rule (19) achieves the best Markov equilibrium of the model. One crucial point of this paper is that the sustainable policy requires no commitment technology but implements the same best Markov equilibrium outcome as the policy rule (19) with such a technology does.

The sustainable policy with (19) contains no such issues that optimal sustainable policy faces, since there are no problems with the initial period and with the history dependence and since the best Markov equilibrium (outcome) is unique in the model. Table 4 shows the performance of the sustainable policy with (19). Note that in the case of no persistence of the shocks, i.e. $\rho = 0$, the sustainable policy is consistent with discretionary policy, so this case is omitted from Table 4. One point of this table is that as the shocks $u_t$ become more persistent, welfare gains from the sustainable policy relative to discretionary policy enlarge and the performance of the sustainable policy relative to optimal commitment policy is ameliorated. As noted above, when the persistence of the shock is sufficiently low, discretionary policy performs relatively well. Therefore, the sustainable policy exhibits a very good performance. This paper thus suggests the sustainable policy as a reliable guidepost for monetary policy in the model.

# 6 Concluding remarks

In this paper we have examined optimal sustainable policy, employing a simple dynamic general equilibrium model with the stabilization bias that has been used in recent monetary policy analyses. The paper has shown with realistic calibrations of model parameters that the plausible benchmark for monetary policy conduct without the commitment technologies, which no actual central bank possesses, is neither optimal commitment policy nor Woodford's (1999) timeless perspective policy but is optimal sustainable policy. Once we deliberate on the actual conduct of optimal sustainable policy, however, there are some problems with implementation of it and hence we cannot but consider optimal sustainable policy unreliable as the guidepost for monetary policy. The paper has suggested as a reliable guidepost the sustainable policy that implements the best Markov equilibrium outcome. This sustainable policy contains no such problems faced by optimal sustainable policy and achieves higher social welfare than discretionary policy does.

Although optimal sustainable policy involves some issues regarding the actual conduct of it, there is still great interest in examining how this policy responds to shocks, particularly in the case in which the sustainability constraint is binding, i.e. optimal quasi-sustainable policy differs from optimal commitment policy. This is because we have no idea about the exact responses of optimal quasi-sustainable policy to shocks and because optimal sustainable policy is the best monetary policy strategy in the absence of the commitment technologies. Thus, the numerical investigation of optimal (quasi-)sustainable policy is one future research topic.

Another topic of future research is to examine sustainability of outcomes of equilibria with particular monetary policy rules. Since Taylor's (1993) pioneering work, simple policy rules have received much attention in monetary policy analyses. Many recent analyses assume that the monetary authority can credibly commit itself to a proposed policy rule. It is far from clear, however, exactly how or whether such credibility would arise. To address this question, this paper suggests examining whether the proposed policy rule leads to an outcome of a sustainable equilibrium. If the policy rule does so, the authority's commitment to it can be supported by the sustainable equilibrium in which the authority takes the policy strategy that specifies continuation with the policy rule as long as it has been adopted in the past; otherwise, the strategy specifies to adopt a policy that induces the worst sustainable equilibrium outcome in the subsequent economy. Hence the commitment is credible. Otherwise, private agents know the authority's temptation to deviate from that policy rule, so such commitment cannot be credible. Thus, the sustainability of equilibrium outcomes seems to be a requirement that policy rules must meet. A companion paper by Kurozumi (2005) investigates sustainability of outcomes of equilibria with monetary policy rules of the sort proposed by Taylor (1993).

# Appendix

## A Calculation of (6) and (12)

In each period $t \geq 0$ we have

$$V^c(u_t, x_{t-1}^c) = -E_t \sum_{s=t}^{\infty} \beta^{s-t} \Big[ (E_t \pi_s^c)^2 + V_t \pi_s^c + \lambda \left( E_t x_s^c \right)^2 + \lambda V_t x_s^c \Big], \tag{22}$$

where $x_{-1}^c = 0$. Using (3) and (5) we have for $z = \pi, x$,

$$\begin{aligned} E_t z_s^c &= b_z b_x^{s-t} x_{t-1}^c + \rho^{s-t-1} \left[ a_z \rho + a_x b_z \frac{1-(b_x/\rho)^{s-t}}{1-b_x/\rho} \right] u_t \quad \forall s \geq t, \\ V_t z_s^c &= \sigma_\varepsilon^2 \left\{ a_z^2 + (a_z \rho + a_x b_\pi)^2 + \cdots + \rho^{2(s-t-2)} \left[ a_z \rho + a_x b_z \frac{1-(b_x/\rho)^{s-t-1}}{1-b_x/\rho} \right]^2 \right\} \ \forall s > t. \end{aligned}$$

and $V_t z_t^c = 0$. Substituting these into (22) yields (6).

Similarly, in each period $t \geq 0$ substituting (11) into $V^d(u_t)$ yields

$$V^d(u_t) = -(c_\pi^2 + \lambda c_x^2) \sum_{s=t}^{\infty} \beta^{s-t} \left[ (E_t u_s)^2 + V_t u_s \right]. \tag{23}$$

From (3) we have $E_t u_s = \rho^{s-t} u_t$, $V_t u_s = \sigma_\varepsilon^2 [1 - \rho^{2(s-t)}]/(1-\rho^2)$ $\forall s \geq t$. Substituting these into (23) yields (12).

## B Proof of Proposition 1

Let $(\sigma^d, f^d)$ denote the REE with discretionary policy. From (11) we have for every possible history $h_t$ induced by $\sigma^d$,

$$\sigma_t^d(h_t) = c_\pi u_t, \quad f_t^d(h_t, \sigma_t^d(h_t)) = c_x u_t. \tag{24}$$

First let us show that $(\sigma^d, f^d)$ is a sustainable equilibrium. Clearly, $(\sigma^d, f^d)$ satisfies (D1). To show that $(\sigma^d, f^d)$ satisfies (D2), consider the monetary authority's problem (14), where $\sigma$ and $f$ are replaced with $\sigma^d$ and $f^d$. From (24) we have

$$\frac{\partial E_s[\sigma_{s+j}^d(h_{s+j})]}{\partial \sigma_s^d(h_s)} = 0, \quad j > 0, \quad s \geq t$$

for all possible histories induced by $\sigma^d$, so each continuation $(\sigma_s^d)_{s \geq t}$ of $\sigma^d$ satisfies the optimality condition for the authority's problem, which turns out to be

$$\sigma_s^d(h_s) = -(\lambda/\kappa) f_s^d(h_s, \sigma_s^d(h_s))$$

for all possible histories $h_s, s \geq t$ induced by $\sigma^d$. This is indeed the optimality condition for the discretionary policy design problem. Hence $(\sigma^d, f^d)$ is a sustainable equilibrium.

We next show that $(\sigma^d, f^d)$ is the worst sustainable equilibrium. Suppose to the contrary that there is the worst sustainable equilibrium $(\sigma^*, f^*) \neq (\sigma^d, f^d)$. Then we have

$$-E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^d(h_t)]^2 + \lambda[f_t^d(h_t, \sigma_t^d(h_t))]^2\} \; > \; -E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^*(h_t)]^2 + \lambda[f_t^*(h_t, \sigma_t^*(h_t))]^2\}. \quad (25)$$

If $\sigma^* = \sigma^d$, from an argument analogous to that above it follows that for every possible history $h_t$ induced by $\sigma^d$,

$$\sigma_t^d(h_t) = -(\lambda/\kappa) f_t^*(h_t, \sigma_t^d(h_t)), \quad (26)$$

and hence

$$f_t^*(h_t, \sigma_t^d(h_t)) = -(\kappa/\lambda)\sigma_t^d(h_t) = f_t^d(h_t, \sigma_t^d(h_t)). \quad (27)$$

Consequently, we have

$$\begin{aligned} -E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^*(h_t)]^2 + \lambda[f_t^*(h_t, \sigma_t^*(h_t))]^2\} \;&=\; -E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^d(h_t)]^2 + \lambda[f_t^*(h_t, \sigma_t^d(h_t))]^2\} \\ &=\; -E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^d(h_t)]^2 + \lambda[f_t^d(h_t, \sigma_t^d(h_t))]^2\}, \end{aligned}$$

which contradicts (25). Suppose next that $\sigma^* \neq \sigma^d$. In the sustainable equilibrium $(\sigma^*, f^*)$, consider the authority's deviation from $(\sigma_t^*)_{t\geq t^*}$ to $(\sigma_t^d)_{t\geq t^*}$ in period $t^* = \inf\{t \geq 0 : \sigma_t^*(h_t) \neq \sigma_t^d(h_t)\}$. Under the assumption on timing of decision-making, private agents know this deviation in period $t^*$. Therefore, it follows that $(\sigma^d, f^*)$ satisfies (26) and hence (27) for every possible history $h_t$ induced by $\sigma^d$. Then, because $(\sigma^*, f^*)$ is a sustainable equilibrium, we have

$$\begin{aligned} -E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^*(h_t)]^2 + \lambda[f_t^*(h_t, \sigma_t^*(h_t))]^2\} \;&\geq\; -E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^d(h_t)]^2 + \lambda[f_t^*(h_t, \sigma_t^d(h_t))]^2\} \\ &=\; -E_0\sum_{t=0}^{\infty}\beta^t\{[\sigma_t^d(h_t)]^2 + \lambda[f_t^d(h_t, \sigma_t^d(h_t))]^2\}, \end{aligned}$$

which contradicts (25). Thus, discretionary policy induces the worst sustainable equilibrium.

## C Proof of Proposition 2

We begin by showing that if $(\pi, x)$ is an outcome of a sustainable equilibrium $(\sigma, f)$, then (S1) and (S2) are satisfied. Clearly, (S1) holds because any sustainable equilibrium satisfies (13). To show that (S2) is satisfied, suppose to the contrary that there is some history $h_{t^*}$ induced by $(\sigma_s)_{s=0}^{t^*-1}$ such that

$$-E_{t^*}\sum_{s=t^*}^{\infty}\beta^{s-t^*}\{[\sigma_s(h_s)]^2+\lambda[f_s(h_s,\sigma_s(h_s))]^2\} \;<\; V^d(u_{t^*}). \tag{28}$$

Consider the authority's deviation from $(\sigma_s)_{s\geq t^*}$ to $(\tilde{\sigma}_s)_{s\geq t^*}$ with $\tilde{\sigma}_s = c_\pi u_s$, which is discretionary policy from period $t^*$ on, at the history $h_{t^*}$. Then, under the assumption on timing of decision-making, private agents know this deviation in period $t^*$ and hence it follows from (13) that $f_s(h_s, \tilde{\sigma}_s(h_s)) = c_x u_s$ for all possible histories $h_s$, $s \geq t^*$. Consequently, we have

$$-E_{t^*}\sum_{s=t^*}^{\infty}\beta^{s-t^*}\{[\tilde{\sigma}_s(h_s)]^2+\lambda[f_s(h_s,\tilde{\sigma}_s(h_s))]^2\} \;=\; V^d(u_{t^*}), \tag{29}$$

which contradicts the fact that $(\sigma, f)$ is a sustainable equilibrium, since (28) and (29) imply that given $f$ and given a history induced by $(\sigma_s)_{s=0}^{t^*-1}$, the deviation $(\tilde{\sigma}_s)_{s\geq t^*}$ attains higher social welfare from period $t^*$ on than the continuation $(\sigma_s)_{s\geq t^*}$ does.

We next show that if a pair $(\pi, x)$ of contingent sequences of inflation rates and output gaps satisfies (S1) and (S2), then this pair is an outcome of a sustainable equilibrium. To show this, we define "revert-to-discretion" plans, which will turn out to be an analogue to Abreu's (1988) optimal penal codes. For an arbitrary pair $(\tilde{\pi}, \tilde{x})$, the revert-to-discretion plans specify continuation with the candidate pair $(\tilde{\pi}, \tilde{x})$ as long as the contingent sequences in $(\tilde{\pi}, \tilde{x})$ have been chosen in the past; otherwise, the plans specify reversion to the discretion plans $(\sigma^d, f^d)$ given by (24) in Appendix B. We then show that for the pair $(\pi, x)$, the associated revert-to-discretion plans constitute a sustainable equilibrium. First consider histories $h_t$ under which there have been no deviation from $\pi$ up until period $t$. Then, from (S1), the continuation of $x$ is optimal for private agents in period $t$ when they are faced with the continuation of $\pi$. If the monetary authority deviates from $\pi$ in period $t$, private agents revert to $f^d$ from period $t$ on. By construction, for the authority who is confronted with $f^d$, it is optimal to choose $\sigma^d$ from period $t$ on. Hence, the most a deviation by the authority in period $t$ can attain is the right-hand side of (15). Then, from (S2) it follows that it is optimal for the authority to stick to $\pi$. Next consider histories $h_t$ in which there have been deviations from $\pi$ before period $t$. The revert-to-discretion plans then specify that the monetary authority and private agents choose, respectively, $\sigma^d$ and $f^d$ from period $t$ on. Clearly, such plans are sustainable.

# D Proof of Proposition 4

To clarify that the expected period $t\,(\geq 0)$ discounted values of social welfare attained by optimal commitment policy and by discretionary policy, (6) and (12), depend on the discount factor $\beta$, let these be denoted by $V^c(u_t, x^c_{t-1};\beta)$ and $V^d(u_t;\beta)$. Also, let the coefficients of REE (5) and (11) be represented as $e_j = e_j(\beta)$ for $e = a, b, c$ and for $j = \pi, x$. Note that for each $\beta \in (0,1\,]$, $0 < a_\pi(\beta) < 1/(1-\beta\rho)$, $0 < b_\pi(\beta) < \lambda/\kappa$, $-(\kappa/\lambda)/(1-\beta\rho) < a_x(\beta) < 0$, and $0 < b_x(\beta) < 1$. To prove Proposition 4, it suffices to show that the difference between the expected period $t$ discounted values of social welfare, $V^c(u_t, x^c_{t-1};\beta) - V^d(u_t;\beta)$, goes to an infinity uniformly in $h_t$ as the discount factor $\beta$ approaches one. From (6) and (12), we have that for each $\beta \in (0,1\,]$ and for every possible history $h_t$,

$$\begin{aligned}
&V^c(u_t, x^c_{t-1};\beta) - V^d(u_t;\beta)\\
&= \frac{\Delta(\beta)}{1-\beta\rho^2}\left(u_t^2 + \frac{\beta\sigma_\varepsilon^2}{1-\beta}\right) - \frac{b_\pi^2(\beta) + \lambda\, b_x^2(\beta)}{1-\beta\, b_x^2(\beta)}(x^c_{t-1})^2\\
&\quad - 2\left\{\frac{a_\pi(\beta)\, b_\pi(\beta) + \lambda\, a_x(\beta)\, b_x(\beta)}{1-\beta\rho\, b_x(\beta)} + \frac{\beta\, a_x(\beta)\, b_x(\beta)[b_\pi^2(\beta) + \lambda\, b_x^2(\beta)]}{[1-\beta\rho\, b_x(\beta)][1-\beta\, b_x^2(\beta)]}\right\} u_t x^c_{t-1},
\end{aligned}$$

where

$$\begin{aligned}
\Delta(\beta) &= [c_\pi^2(\beta) + \lambda\, c_x^2(\beta)] - [a_\pi^2(\beta) + \lambda\, a_x^2(\beta)]\\
&\quad - \frac{2\beta\rho\, a_x(\beta)}{1-\beta\rho\, b_x(\beta)}[a_\pi(\beta)\, b_\pi(\beta) + \lambda\, a_x(\beta)\, b_x(\beta)] - \frac{\beta\, a_x^2(\beta)}{1-\beta\, b_x^2(\beta)}[b_\pi^2(\beta) + \lambda\, b_x^2(\beta)]\\
&= \frac{\beta(b^-)^2(1-b^-)[1-\beta b^- + \beta(b^-)^2(1-\beta\rho^2)]}{[(b^-)(1-\beta\rho b^-)(1-\beta\rho + \kappa^2/\lambda)]^2} + \frac{2\beta^2\rho(b^-)^3(1-b^-)}{(1-\beta\rho b^-)^3} \;>\; 0,
\end{aligned}$$

where the last equality follows from the coefficients of REE (5) and (11). Because $u_t$ is bounded, i.e. there is $B > 0$ such that $|\,u_t| < B\;\forall t$, we have from (5) that $|\,x^c_t| < B|\,a_x(\beta)|/[1-b_x(\beta)]\;\forall t \geq 0$ and hence that for each $\beta \in (0,1\,]$ and for every possible history $h_t$,

$$\begin{aligned}
&V^c(u_t, x^c_{t-1};\beta) - V^d(u_t;\beta)\\
&> \frac{\beta\sigma_\varepsilon^2\,\Delta(\beta)}{(1-\beta)(1-\beta\rho^2)} - \frac{b_\pi^2(\beta) + \lambda\, b_x^2(\beta)}{1-\beta\, b_x^2(\beta)}\left[\frac{B\, a_x(\beta)}{1-b_x(\beta)}\right]^2\\
&\quad - 2\left\{\frac{a_\pi(\beta)\, b_\pi(\beta) + \lambda\,|\,a_x(\beta)|\, b_x(\beta)}{1-\beta\rho\, b_x(\beta)} + \frac{\beta\,|\,a_x(\beta)|\, b_x(\beta)\,[b_\pi^2(\beta) + \lambda\, b_x^2(\beta)]}{[1-\beta\rho\, b_x(\beta)][1-\beta\, b_x^2(\beta)]}\right\}\frac{B^2|\,a_x(\beta)|}{1-b_x(\beta)}.
\end{aligned} \tag{30}$$

Then, as the discount factor $\beta$ approaches one, the first term in the right-hand side of (30) goes to an infinity but the other two terms converge to finite numbers, so the right-hand side of (30) goes to an infinity. Therefore, there is $\underline{\beta} \in (0, 1)$ such that for any $\beta \in (\underline{\beta}, 1]$, $V^c(u_t, x^c_{t-1}; \beta) - V^d(u_t; \beta) > 0$ for all possible histories $h_t$. For each $\beta \in (\underline{\beta}, 1)$, we have that $V^c(u_t, x^c_{t-1}; \beta) < \infty$ and $V^d(u_t; \beta) < \infty$ and hence $V^c(u_t, x^c_{t-1}; \beta) > V^d(u_t; \beta)$ for all possible histories $h_t$.

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Table 1:
Calibrations of model parameters, quarterly

| | | |
|---|---|---|
| $\beta$ | discount factor | .99 |
| $\kappa$ | output-gap elasticity of inflation | .01, .05, or .10 |
| $\lambda$ | relative weight on output-gap stabilization | .0010, .0100, or .0625 |
| $\rho$ | autoregression parameter for $u_t$ | .00, .35, or .80 |

Table 2:
Welfare gains from optimal commitment policy
relative to discretionary policy, %

| | | $\lambda$ | | |
|---|---|---|---|---|
| | $\kappa$ | .0010 | .0100 | .0625 |
| $\rho = .00$ | .01 | 19.38 | 8.18 | 3.32 |
| | .05 | 17.79 | 23.51 | 14.44 |
| | .10 | 7.62 | 23.40 | 21.73 |
| | | $\lambda$ | | |
| | $\kappa$ | .0010 | .0100 | .0625 |
| $\rho = .35$ | .01 | 41.00 | 21.91 | 9.75 |
| | .05 | 23.41 | 43.78 | 34.14 |
| | .10 | 8.49 | 35.37 | 43.12 |
| | | $\lambda$ | | |
| | $\kappa$ | .0010 | .0100 | .0625 |
| $\rho = .80$ | .01 | 83.35 | 68.40 | 40.14 |
| | .05 | 31.76 | 78.43 | 81.57 |
| | .10 | 9.69 | 54.15 | 81.78 |

Table 3:
Values of $\underline{\beta}$ given by (18)

i. $k = 4$

| | $\kappa$ | $\lambda$ = .0010 | .0100 | .0625 |
|---|---|---|---|---|
| $\rho = .00$ | .01 | .9973 | .9996 | .9999 |
| | .05 | .9893 | .9952 | .9987 |
| | .10 | .9882 | .9912 | .9964 |
| $\rho = .35$ | .01 | .9885 | .9977 | .9995 |
| | .05 | .9728 | .9820 | .9935 |
| | .10 | .9740 | .9740 | .9853 |
| $\rho = .80$ | .01 | .9238 | .9756 | .9930 |
| | .05 | .8950 | .9020 | .9479 |
| | .10 | .9076 | .8895 | .9118 |

ii. $k = 5$

| | $\kappa$ | $\lambda$ = .0010 | .0100 | .0625 |
|---|---|---|---|---|
| $\rho = .00$ | .01 | .9983 | .9998 | .9999 |
| | .05 | .9931 | .9970 | .9992 |
| | .10 | .9924 | .9944 | .9977 |
| $\rho = .35$ | .01 | .9925 | .9985 | .9997 |
| | .05 | .9821 | .9882 | .9958 |
| | .10 | .9830 | .9829 | .9904 |
| $\rho = .80$ | .01 | .9432 | .9828 | .9953 |
| | .05 | .9237 | .9265 | .9618 |
| | .10 | .9342 | .9184 | .9339 |

Table 4:
Welfare gains from sustainable policy with (19)
relative to discretionary policy, %

| | $\kappa$ | $\lambda$ = .0010 | .0100 | .0625 |
|---|---|---|---|---|
| $\rho = .35$ | .01 | 2.07 (5.05) | .27 (1.24) | .05 (.46) |
| | .05 | 2.93 (12.51) | 3.55 (8.10) | .99 (2.90) |
| | .10 | 1.05 (12.33) | 4.21 (11.89) | 2.82 (6.54) |
| $\rho = .80$ | .01 | 39.80 (47.76) | 11.66 (17.05) | 2.23 (5.56) |
| | .05 | 17.62 (55.46) | 42.78 (54.54) | 28.97 (35.52) |
| | .10 | 5.68 (58.60) | 30.06 (55.52) | 42.56 (52.05) |

Note: Numbers in parentheses show ratios of welfare gains from sustainable policy with (19) to those from optimal commitment policy, $[V^s(u_0) - V^d(u_0)]/[V^c(u_0, 0) - V^d(u_0)]$, %.