A Theory of Growth and Cycles

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Abstract

This paper investigates the role of endogenous labor-saving technological choice in the generation and propagation of business cycles. We emphasize the importance of endogenously varying relative factor prices as a force behind the introduction of new technologies and the scrapping of existing ones. The interaction between labor-saving innovations and changes in the relative price of labor gives rise to endogenous growth and cycles that match, quali- and quantitatively, observed ones.

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1. Introduction

This paper investigates the role of endogenous technological change in the generation and propagation of economic growth and business cycles. We emphasize the importance of endogenously varying factor prices, hence: of income distribution, as a force behind the introduction of new technologies and the scrapping of existing ones. Put differently, we study endogenous technological progress that is “biased” by the relative price of inputs, and derive a model in which persistent growth, persistent business fluctuations, and persistent movements in the share of income going to, respectively, labor and capital are simultaneously determined. We claim this makes for a model of growth and the business cycle that fits the facts better than the existing alternatives.

In our model, growth in total factor productivity is endogenous and results from the adoption of new technologies - capital deepening - their subsequent expansion - capital widening - and their eventual replacement with with better ones - capital scrapping. The duration of each phase is endogenous, and determined by the equilibrium movements in the relative prices of labor and (different kinds of) capital. Recessions occurr when capital widening has reached its upper limit and scrapping, followed by deepening, becomes economically beneficial; expansions set in when capital deepening is successful and widening may be undertaken.

Apart from the general goal of building a quantitative-theoretical model in which growth and cycles are equilibrium outcomes and not preset exogenous phenomena, our point of departure consists of three important, but often ignored, observations. First, the shares of income accruing to capital and labor move in a systematic way with the business cycle; second, profits and the growth rate of labor productivity, beside being correlated, are procyclical, but peak substantially earlier than the cycle does, i.e. when recessions set in, profits have been decreasing and labor productivity has stopped growing for a few quarters already; third, while there is little short term substitutability between capital and labor, most technological improvements appear to be labor saving. The empirical relevance of these three observations, and the extent to which they relate to each other, is carefully discussed in the ”Stylized Facts” section; here we focus on the first, which is the one standard models have the hardest time to capture.

Start by plotting, Figure 1, the capital income share for the U.S. during the period 1947-2005. The figure reveals that it ranged between a minimum of 28 percent in 1970 to a maximum of 34.5 percent in 1997. Moreover, as documented by Young (2005), nearly all those changes are due to movements in capital share within industries and not because of variations in the weight of different industries. Figure 2, reproduced from Young (2005), clearly makes this point.[1]

That such movements are systematically related to the growth cycle can be shown by noting that, during the first years of an expansion, the capital income share raises, together with a substantial increase in the profitability of capital. However, after a peak around the center of the expansion, the capital income share falls, reaching a minimum during the recession.

The long expansion of the 1990s is paradigmatic. During the 1990s, the share of the corporate output counted as Net Operating Surplus, a concept that includes payments to both share and bond-holders, increased from 16 percent to 20 percent. Similarly, corporate profits (the Net Operating Surplus less net interests) went from 11 percent to 17 percent. Given that during the 1990s, the assets of corporations were roughly one and half times the value of corporate output, we are talking about changes in profitability of around 2.66 percent (from 10.6 to 13.3) as measured by the Net Operating Surplus and 4 percent (7.33 to 11.33 percent) as measured by corporate profits. These movements are important. For example, if sustained, the change in corporate profits that took place between 1992 and 1998 would imply, by itself, a raise in the market value of the corporate sector of the U.S. of about 50 percent.

Standard business cycle models have a difficult time explaining the behavior of factor shares. The reason why this is so can be understood by looking at the formula for the capital income share α_t at time t:

$$\alpha_t = 1 - \frac{w_t l_t}{y_t} = 1 - \frac{w_t}{lp_t}$$

[1]Not only the change in capital share induced by reallocation across sectors are small, but they may also be the endogenous response to the changes in factor prices that we emphasize in this paper. We elaborate on this below.

where w_t is the wage, l_t is the amount of labor, y_t is output, and lp_t is labor productivity. An approximation of this expression, using log-deviations of the variables with respect to their steady-state value, is:

$$\widehat{\alpha}_t = -\widehat{w}_t - \widehat{l}_t + \widehat{y}_t = -\widehat{w}_t + \widehat{lp}_t \tag{1}$$

It is plain evident that capital income moves away from its long-run average only to the extent that labor productivity and wages are less than perfectly correlated. If we further assume a Cobb-Douglas production function $y_t = e^{z_t} k_t^{\theta} l_t^{1-\theta}$, where z_t is the level of technology, we have:

$$\alpha_t = 1 - \frac{w_t}{e^{z_t} k_t^{\theta} l_t^{-\theta}}$$

or

$$\widehat{\alpha}_t = z_t + \theta \left(\widehat{k}_t - \widehat{l}_t \right) - \widehat{w}_t \tag{2}$$

We apply expressions (**??**) and (**??**) to several models. The baseline Real Business Cycle model cannot deliver any of the changes in capital share by construction: the combination of Cobb-Douglas production function and competitive input markets makes capital income share trivially equal to the (constant) elasticity of output with respect to capital. In (**??**), we will have that, for all t, $\widehat{w}_t = \widehat{lp}_t$ and in (**??**), $\widehat{w}_t = z_t + \theta \left(\widehat{k}_t - \widehat{l}_t \right)$. We could modify the basic model to make θ stochastic (as, e.g., Young, 2004), but this alternative, beyond generating counterfactually high correlation between α_t and output, is dangerously close to assuming a trivial answer (exogenous movements) to the empirical puzzle.

Similarly, we could abandon the Cobb-Douglas production function in favor of a more general CES production function, case in which the factor shares can change over time. However, if we compute a standard RBC model with CES production function and an elasticity of substitution similar to the ones reported in the literature (for example, around 0.8 as in Antràs, 2004 or Young, 2005b), we find that such models predict movements in factor shares quite smaller than the observed ones. Moreover, if we assume that technological progress is Harrod-neutral (as required to have a stationary capital-output ratio), wages become strongly countercyclical, contrary to all the empirical evidence.

Models with sticky prices and/or sticky wages do not have a much easier time at capturing the facts. In response to a monetary shock, wages will go up because of a higher demand for labor, and labor productivity will go down as labor increases faster than capital. Since the two elements in the right-hand side of (**??**) are negative, the capital income share will go down during an expansion driven by a positive monetary shock. Only in the, empirically unlikely, event that nominal wages are completely rigid and prices adjust very rapidly to the monetary shock, would real wages decrease. In the, even more unlikely, event that they decrease more than labor productivity does as demand for labor increases ,will profit display a procyclical tendency, but to achieve this we need wage rigidity to last many quarters in the face of continuous monetary supply surprises and raising prices, an improbability to say the least. The same argument applies, without the latter caveat, if the expansion is driven by some non-ricardian fiscal "stimulus."

Expression (**??**), beyond clarifying the problems of existing business cycle models, also tells us the element that a successful theory requires: a mechanism to increase labor productivity faster than wages at the beginning and slower at the end of the expansion. Our paper focuses on one possible channel: the endogenous adoption of labor saving technology. At the beginning of the expansion, firms will pick new technologies that are labor-saving relatively to previous ones. As the latter are scrapped and the new capital that embodies the new technology is accumulated, labor moves accordingly and its productivity increases faster than wages, hence the capital share and output increase rapidly. However, as the replacement process completes and more and more labor is employed, wages will eventually go up, drying the corporate profits, reducing investment, and finishing with it the expansion. Only at the bottom of the recession, after old and inefficient productive capacity has been scrapped, a new technology is introduced and the whole cycle starts again.

The paper is organized as follows. Section 2 discusses the closely related literature. Section 3 presents some stylized facts. Section 4 outlines our main theoretical model, and the results obtained so far. Sections 5 - calibration - and 6 - numerical simulations and discussion, are missing together with the conclusions.

2. Related Literature

We are not the first to deal with some of the issues discussed above. Let us leave aside, for the time being, the very vast literature concerned with the endogenous determination of both growth and cycles; it suffices here to say that nowhere in that literature one can find a quantitative-theoretical model that can be, even less: is, brought to face the data. We will also spare the reader a long survey of the century-long debate on the nature of technological progress, its biasedness in one direction or another and the extent to which Harrod-neutral exogenous producvity does or does not mimic the data in a satisfactory form. To us, that technological progress *must be* labor - more generally: natural resource - saving is almost tautological beside being blatantly evident. The relevant issues are how to best model this fact, and if the pace at which technological change advances should or should not be made responsive to movements in factor prices. We refer to XXX and YYY for recent discussions of this issues, and survey of the literature.

Coming next to our first observation - that factor shares are strongly cyclical - we begin by distinguish three branches of the literature interested in the evolution of the input income shares and the business cycle. First, there have been papers that focused on the distribution of risk over the cycle. Boldrin and Horvath (1995) present a model of contractual arrangements between employees and employers when employees are prevented from accessing capital markets and they are more risk-adverse than employers. The paper characterizes an optimal contract that maps the aggregate states of the economy into wages and labor market outcomes. Similarly, Gomme and Greenwood (1995) build a model where workers purchase insurance from the entrepreneurs through optimal contracts. Since our model assumes complete markets, none of the considerations used in those papers is directly pertinent to the mechanism explored here, even if, as it will be clear later, the introduction of risk-sharing contractual arrangements would reinforce some of the conclusions.

The second branch of the literature has focused on explanations based on models with imperfect competition and/or increasing returns to scale. Hornstein (1993) developed a model of monopolistic competition where capital income share is procyclical. However, the correlation between output and capital share is perfect, hence the cyclical "hump-shape" pattern for profits cannot be replicated. Other examples include Ambler and Cardia (1998), Bils (1987), and the models surveyed in the Rotemberg and Woodford (1999). Hansen and Prescott [2005] is an additional contribution along the same lines, which does not make use of monopolistic competition but, instead, of fixed capacity at the plant level.

Finally, and the most relevant for us, is the third branch of the literature, spearheaded by Blanchard (1997) and Caballero and Hammour (1998). These papers have explored the dynamics over the middle-run induced by exogenous changes in real wages. After an initial increase in wages, due for example to an exogenous strengthening of the bargaining power of workers, the capital share goes down. What happens over time depends on the long-run elasticity of substitution, either with a permanent fall on capital share or with a return to the initial level. Blanchard (1997) suggests that changes in efficiency induced by the original increase in wages may even increase the long-run share of capital income. Some of the intuitive arguments given by Blanchard, inspired by the European experience in the 1970s and 1980s, are close in spirit to the model we suggest here, in particular the idea that, facing a persistently high exogenous wage, firms may strive to adopt technologies that reduce the labor input per unit of output, thereby leading to an eventual decrease in the share of labor income.

The key differences in our investigations is that we do not begin with an initial, exogenously given shock to wages (due to a change in technology, bargaining power or mark-up) and explore also the aggregate dynamics after such shock. Further, we view the changes in capital income share participation as a systematic and recurrent feature of the economy, and a main driving force behind the introduction of new technology and, therefore, of sustained growth. To put it plainly, we posit that growth must come through oscillations in the rate of technology adoption, that such oscillations are endogenous, and that their main source is the hidden - not always, sometime it becomes very explicit - ”conflict” over the share of income going to production factor versus another.

Finally, we note the similarities between some points of our model and the literature on directed technological change surveyed by Acemoglu (2002). A more careful comparison of intuitions, models, and results will be added in future versions of this paper. In any case, three macroscopic differences are that (i) we claim business cycles are "caused" by labor-saving technological change, (ii) we focus on the fundamental bias (labor vs capital) in a perfectly competitive environment, and, (iii) we make the bias endogenous and not exogenous.

3. Stylized Facts

In this section we discuss some of the U.S. evidence pertinent to our model of the business cycle.[2] As most business cycle features of aggregate quantities are well know, our basic object of observation is the variation of the capital share of input over the business cycle. We contend that expansions begin with increases in the capital participation share, that this participation share peaks in the middle of the cycle, and that the last phase of the expansion is correlated with a fall in the capital share.

We illustrate our assertion by computing the capital share of the U.S. economy in three different ways. First, we compute the capital share for the whole economy. This measure has the advantage of comprehensiveness but the drawback that it includes the household and government sectors, many of whose goods and services are not sold in markets and which have a fixed capital share (respectively, one and almost zero) by construction. Moreover, we need to handle the distribution of proprietors income between (imputed) wages and capital income. To overcome some of these difficulties, we compute the capital share for the corporate sector. Finally, we compute the capital share for the nonfinancial corporate sector.

3.1. Overall Economy

Our first take at evaluating the capital share in the U.S. economy uses aggregate data from the whole economy. As explained before, following this route faces the basic difficulty of how to divide Proprietors Income between labor and capital. A common solution to this problem (Cooley and Prescott, 1995) is to allocate the Proprietors Income according to the share

[2] Abundant evidence is also available for pretty much each and every EU country, that we will summarize in subsequent versions. The styilzed facts reported here are, if possible, even more clearly visible in the European post-WWII data, which is what motivated Blanchard and Caballero-Hammour initial work. Further, a cross country comparison may be useful in assessing the empirical relevance of the model's main predictions.

of capital income observed in the non-proprietors part of the economy. To do so, we can subtract from our measure of output the Proprietors Income and include as capital income only the unambiguous capital income.

This strategy implies, first, that capital income includes income coming from two different sources:

1. Unambiguous capital income, equal to Rental Income of persons, Corporate Profits, Net Interest and miscellaneous payments, and the current surplus of government enterprises. We diverge from Cooley and Prescott (1995) in our definition of unambiguous capital income only in our inclusion of the current surplus of government enterprises as capital income. This surplus can be considered an income of the capital used by those firms. However, the quantitative importance of this number is quite small, less than 0.05% of output.

2. The consumption of fixed capital by the non-proprietors private sector and the government. We do not include the consumption of fixed capital by proprietors to be consistent with our exclusion of their income from our computations.

Second, we define output as Gross Nation Product less Proprietors Income. In addition, we subtract the difference between Net National Product and National Income (since this statistical discrepancy is also of difficult imputation between capital and labor) and Net Taxes on Production and Imports, since again this item cannot be divided between capital and labor.

As a consequence, our capital share α is defined:

$$\alpha = \frac{\text{Unambiguous Capital Income+Depreciation}}{\left(\begin{array}{c} GNP - Proprietors \text{ Income-Net National Product+National Income} \\ \text{-Net Taxes on Production and Imports} \end{array}\right)}$$

Our different measures are taken directly from NIPA, Table 1.7.5 (Relation of Gross Domestic Product, Gross National Product, Net National Product, National Income, and Personal Income) and Table 1.12. (National Income by Type of Income). Since we only need percentages, we take nominal quantities that avoid distortions induced by price indexes. Our sample, of quarterly data, goes from 1947:1 to 2005:2.

Our findings are plotted in figure 1. Clearly, capital share fluctuates quite a bit. Figure 2 presents the cyclical component obtained with a Hodrick-Prescott filter with $\lambda = 1600$ and (as vertical lines) the NBER dating of recessions.

But the fluctuations are even clearer when we decompose α in its different components, as plot in figure 3. For comparison purposes, the plot also draws the trend of the series obtained with the Hodrick-Prescott filter and NBER recessions. The top line in the plot corresponds to α. Even if it stays around a value of 0.32, as usually discussed, it fluctuates over time between 0.28 and 0.35, a change of 23%.

The second line corresponds to the share of Unambiguous Capital Income. This series illuminates more than α the cyclical variations of capital income. By construction, depreciation (the third line, with a small positive trend[3] will be a relatively smooth series. We can see how the Unambiguous Capital Income clearly, almost deterministically, fluctuates with the cycle: it tends to go up at the beginning of the expansion, pick at the middle, and drop in the second half of the expansion.

The fourth and fifth line correspond to corporate profits and net interest. We will discuss corporate profits in detail in the next subsection. Net interest are relatively acyclical and we only need to notice the big increase in the 80's associated with the high real rates of interest of the time. Finally, the last line corresponds to the rental income, which is also relatively smooth over time.

3.2. Corporate Sector and Nonfinancial Corporate Sector

Our next measure of the capital share uses data from the corporate sector. These measurement is closer to the main theoretical thrust of the paper.

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[3]It is relevant to find out the origin of this positive trend in the data. Hypotheses: More capital-intensity? Quicker depreciation and obsolescence rates? Inconsistencies in the definition of real versus nominal prices of capital? Is there any relation between this trend and the Stock Market evolution? Is this a temporary, possible compositional, effect?

We define the output of the corporate sector to be equal to the Gross value added of corporate business less the Taxes on production and imports net of subsidies. As capital income we add the Net operating surplus plus the Consumption of fixed capital. We repeat the exercise with the same concepts for the Nonfinancial Corporate Sector.

Our different measures are taken directly from NIPA, Table 1.14. Gross Value Added of Domestic Corporate Business in Current Dollars and Gross Value Added of Nonfinancial Domestic Corporate Business in Current and Chained Dollars. As in the previous subsection, we employ nominal quantities.

Figure 4 plots the capital share in the corporate sector. Again we can see how, even if the capital share fluctuates around a mean, these fluctuations are not trivial. The next line is the Net Operating Surplus series that shows a strong cyclicality. The third line is corporate profits, also quite procyclical, and the last line is depreciation. Figure 5 reproduces the same series for the nonfinancial corporate sector.

An interesting pattern appears if we study the three longest expansions in the U.S. after the second world war. We will call these expansions the 60s expansion, the 80s expansion, and the 90s expansions. These three episodes can be thought as particularly interesting because the length of the expansion allows one to identify more clearly the type of phenomena we are concerned with. We plot in figure 6, the Net Operating Surplus during the three expansions and, in figure 7, the Corporate Profits. We observe a common structure: both measures of capital income go up at the beginning of the expansion by a considerable amount, peaks roughly at the middle of the expansion, and decreases afterwards.

Table 1 summarizes the information. The first column corresponds to the value of the Net Operating Surplus at the beginning of the expansion, the second column to the maximum value in the expansion, the third column to the percentage increase, the fourth column is the final value at the end of the expansion, and finally, the fifth column reports the percentage decrease.

Table 1: Evolution of the Net Operating Surplus

	Initial Value	Maximum Value	Increase	Final Value	Decrease
Expansion 60s	18.4%	23.6%	28.4%	18.0%	30.6%
Expansion 80s	15.4%	19.0%	23.5%	16.6%	14.5%
Expansion 90s	16.6%	19.9%	19.8%	15.8%	25.9%

The main message of this table is the sizable changes in the Net Operating Surplus over the business cycle. If we take a benchmark capital-output ratio of 3, we can transform this numbers into rates of return dividing them by 3. With this back-of-the-envolope calculation, we see, for example, that the rate of return of the Corporate sector in the 60s went from 6.1% to 7.9% and then fell to 6.0% at the start of the recession.

Table 2 reports the same information as table 1, except that now we measure the evolution of corporate profits. The increases in Corporate Profits are of an even more substantial magnitude, especially for the 80s and 90s cycles. For example in the 80s, Corporate Profits went up by a 43% and in the 90s by a 42.2%; while this is not the topic of the present paper, one may want to consider how much these dramatic increases in profitability account for the large stock market rallies witnessed during those two expansions, and for the early 1970s and 2000 crashes as well.

Table 2: Evolution of Corporate Profits

	Initial Value	Maximum Value	Increase	Final Value	Decrease
Expansion 60s	17.6%	22.4%	27.0%	15.4%	44.9%
Expansion 80s	9.6%	13.7%	43.0%	11.5%	18.8.%
Expansion 90s	11.9%	16.9%	42.2%	10.9%	54.6%

We can also repeat the back-of-the-envolope calculation of our previous paragraph, considering now the leverage implied by debt. We can see then how the profitability rate of the Corporate sector went in the 90's from 4% to 5.6% and then fell again to 3.6%.

Another way to think about this: corporate profits are highly procyclical and five times more volatile than output (Boldrin and Horvath, 1995).[4]

4. A Model of the Business Cycle

We present now a simple business cycle model to account for the previous observations.

4.1. Preferences

There is a representative agents whose preferences are represented by the standard expected utility

$$\max E_t \left\{ \sum_{t=0}^{\infty} \delta^t \left[u(C_t) + v(1 - L_t) \right] \right\}$$

where both utility functions satisfy standard monotonicity, concavity and differentiability restrictions.

4.2. Production

Production takes places in two (three) different sectors, each of which should be conceived as composed of a continuum of heterogeneous firms (or plants); the latter are affected by idiosyncratic technology shocks, to be discussed later. Description of the three sectors follow.

The first sector uses active technologies (embodied in technology-specific machines) to produce aggregate output through labor (L) and productive capacity (Π), and it is represented by a neoclassical production function $Y = F(\Pi, L)$. The second sector uses aggregate output (and labor, or active machines and labor, depending on the specific version being considered) to introduce new technologies by producing the prototypes of the new machines.

[4] A similar analysis of "representative" recessions should be added in later versions, for two purposes. The labor share tends to rise later in expansions and to fall, sometime substantially, late in recessions. The timing of these movements, and their relation with the investment rate and the exit/entry of firms (failure rate, etc.) is a crucial test of the causation mechanism we are pursuing. Further, one needs to disentangle how much of the drop in labor share is due to a lower wage rate driven by an increase in unemployment, and how much is due to capital/labor substitution driven by the adoption of more efficient technologies. This is also relevant for a proper calibration of our model.

Once a technology is introduced, i.e. a prototype of its machine is created, we label that technology *active* and assume it contributes to productive capacity at par with all other previously active technologies. When a third sector is introduced we assume that accumulation of capital stock (machines) for active technologies requires a production function different from the one generating aggregate output; in particular: that investment in each kind of machine can only come from output generated by that kind of machine, and labor.

Hence, in general, there will always be an aggregate sector producing aggregate output, which is consumed and, in the baseline case with two sectors, also invested in the capital stock of active technologies; there is always also an innovation sector, which creates new technologies using aggregate output, and labor in some specifications. All sectors have neoclassical constant return to scale production functions with machines and labor as inputs.

4.2.1. Fixed Coefficients Technology

There exists a countable number of potential technologies, indexed by the superscript $j = 0, 1, ...$ We say that a technology j is *active* in period t if $K_t^j > 0$, i.e. a positive amount of capital stock of type j was produced in some previous period and has not yet been either depreciated or scrapped. Denote with $J_t = \{0, 1, 2, ..., \widehat{j}_t\}$ the set of all technologies that are active at time t.

Output of technology $j = 0, 1, ...,$ in period t, is

$$Y_t^j = \min\{A^j K_t^j, \gamma^j L_t^j\} + \nu^j S_t^j,$$

where $0 \leq S_t^j \leq K_t^j$ denotes "scrapping" of the capital stock j, and $0 \leq \nu^j < A^j$; scrapping takes place right after the capital stock has been used, for the last time, to produce aggregate output. From the equality

$$A^j K_t^j = \gamma^j L_t^j$$

we get

$$\frac{A^j K_t^j}{\gamma^j} = a_j K_t^j = L_t^j$$

So the symbol a_j denotes the efficient *Labor/Kapital* ratio in sector j.[5] Marginal productivities in technology j are

$$\frac{\partial Y_t}{\partial L_t^j} = \gamma^j, \text{ for } L_t^j \leq a_j K_t^j, \text{ and zero otherwise;}$$

$$\frac{\partial Y_t}{\partial K_t^j} = A^j, \text{ for } K_t^j \leq a_j^{-1} L_t^j, \text{ and zero otherwise.}$$

In the baseline deterministic version, as long as a technology is active, adding to its capital stock costs as much as adding to the capital stock of any other active technology; one unit of homogeneous output yields one unit of capital of any active technology. Because a unit of capital of type j costs the same as a unit of capital $j-1$ but uses less inputs to produce a unit of homogeneous output, among the active technologies the only one with positive gross investment will be the *best available technology* $\widehat{j}_t$.[6] We also define the *marginal technology* $\underline{j}_t$, in period t, as that technology $\underline{j}_t \in J_t$ for which $L_t^{\underline{j}} > 0$ and $L_t^j = 0$ for all $j < \underline{j}_t$.

We introduce a few additional concepts. First, *potential labor demand* Λ_t in period t is

$$\Lambda_t = \sum_{j \in J_t} a_j K_t^j.$$

Second, *productive capacity* Π_t is

$$\Pi_t = \sum_{j \in J_t} (A^j + \nu^j) K_t^j.$$

Aggregate output is bounded by productive capacity

$$Y_t = \sum_{j \in J_t} Y_t^j \leq \Pi_t.$$

[5]When uncertainty is introduced, the parameter a^j is the source of uncertainty. In general, we will assume it to follow the process $a_{t+1}^j = a^j + \epsilon_t$, with ϵ_t and AR(1) process with (possibly very) low persistence.

[6]In the stochastic version, a technology is the best available only in an expected value sense, and positive investment in active technologies other than the best one is an equilibrium outcome when shocks have some degree of persistence.

The resource constraints are, respectively

$$C_t + \sum_{j \in J_t} I_t^j + D_t \leq Y_t,$$

for aggregate output, and

$$L_t^d = \sum_{j \in J_t} L_t^j \leq \Lambda_t,$$

for aggregate labor demand. The law of motion for the active capital stocks $j \in J_t$ is

$$K_{t+1}^j = (1-\mu)K_t^j + I_t^j - S_t^j,$$

together with the non-negativity constraints

$$K_t^j \geq 0, I_t^{j_t} \geq 0, D_t \geq 0, S_t^j \geq 0, C_t \geq 0, L_t^j \geq 0.$$

Next, let the sequences $\{X_t = \sum_{j \in J_t} I_t^{j_t} + D_t\}_{t=0}^{\infty}$ and $\{S_t^1, ..., S_t^{\widehat{j}_t}\}_{t=0}^{\infty}$ be given. The *marginal technology* $\underline{j}_t$ satisfies the following two conditions. For all $\underline{j}_t < j \leq \widehat{j}_t$

$$v'\left(1 - \sum_{i=j}^{\widehat{j}_t} a^i K_t^i\right) < u'\left(\sum_{i=j}^{\widehat{j}_t} A^i K_t^i + \sum_{i=j}^{\widehat{j}_t} \nu^i S_t^i - X_t\right)\gamma^{j-1}$$

while

$$v'\left(1 - \sum_{i=\underline{j}_t}^{\widehat{j}_t} a^i K_t^i\right) > u'\left(\sum_{i=\underline{j}_t}^{\widehat{j}_t} A^i K_t^i + \sum_{i=\underline{j}_t}^{\widehat{j}_t} \nu^i S_t^i - X_t\right)\gamma^{\underline{j}_t - 1}.$$

Then, *employment* at time t is uniquely determined by the condition

$$v'\left(1 - \sum_{j=\underline{j}_t+1}^{\widehat{j}_t} a^j K_t^j + \phi a^{\underline{j}_t} K_t^{\underline{j}_t}\right) = u'\left(\sum_{j=\underline{j}_t+1}^{\widehat{j}_t} A^j K_t^j + \phi A^{\underline{j}_t} K_t^{\underline{j}_t} + \sum_{j \in J_t} \nu^i S_t^j - X_t\right)\gamma^{\underline{j}_t}.$$

for some $0 \leq \phi \leq 1$.

Example with two technologies. Assume the stocks of two machines and investment x are given. Employment is found by solving the static problem

$$\max u(c) + v(1 - L)$$

subject to

$$c + x = \min\{A^1K^1, \gamma^1L^1\} + \min\{A^2K^2, \gamma^2L^2\}$$
$$L = L^1 + L^2$$

Assume

$$u'(A^2K^2 - x)\gamma^1 > v'(1 - a^2K^2)$$

then labor supply L^* solves

$$u'\left(A^2K^2 + \gamma^1(L^* - a^2K^2) - x\right)\gamma^1 = v'(1 - L^*).$$

Notice that, because $\gamma^2 - \gamma^1 \geq \varepsilon > 0$, the following is possible

$$u'\left(A^2K^2 - x\right)\gamma^1 < v'(1 - a^2K^2) < u'\left(A^2K^2 - x\right)\gamma^2,$$

which implies that an "interior" solution satisfying FOCs exactly may not exist. A unique solutio, still exists, though, which may imply either $L^* = a^2K^2$ or $L^* = a^2K^2 + L^1$, for L^1 small enough; to compute it one needs to compare utility levels directly.

4.2.2. Linear Innovation Technology

In our baseline model we assume a linear technology for production of the new machines, with aggregate output as the sole required input

$$K_{t+1}^{\widehat{j}_t+1} = \zeta D_t,\ \zeta < 1.$$

Notice that, because we assume that $\zeta < 1$, introducing a new technology is not always profitable and depends on relative prices; as the latter change over the cycle, this generates irregular innovation patterns. The first machine of a new technology costs more that one additional machine of an already active technology; this extra cost is, in the appropriate circumstances, compensated by smaller input requirements. Let us check when this is the case, i.e. let's figure out the circumstances under which, with the linear innovation technology, it is convenient to innovate rather than accumulate old capital.

Conditions for innovation. Because, in the deterministic case, only $I_t^{\widehat{j}_t} > 0$, it suffices to compare investment in the best available technology $\widehat{j}_t$ with investment in the new technology $\widehat{j}_t + 1$. The latter is more profitable than the former if

$$\frac{1-\zeta}{\zeta} < \frac{P_{t+1}}{P_t}\left[A^{\widehat{j}_t+1} - A^{\widehat{j}_t}\right] - \frac{w_{t+1}}{P_t}\left[a^{\widehat{j}_t+1} - a^{\widehat{j}_t}\right].$$

In the exponential case, this simplifies to

$$\frac{(1-\zeta)}{\zeta} < A^{\widehat{j}_t}\frac{P_{t+1}}{P_t}\left[A-1\right] - a^{\widehat{j}_t}\frac{w_{t+1}}{P_t}\left[a-1\right].$$

Interestingly, this shows that it matters, in general, if the new technology is relatively more capital or more labor saving, not just if they are overall "better".

When they are *more capital saving*, we have $a = A/\gamma > 1$, hence the higher is the expected wage rate the less profitable it is to invest in the new machinery - alternatively, the higher is the relative price of tomorrow's consumption, the more valuable is the new technology.

When the new machine is *more labor saving*, i.e. $a = A/\gamma < 1$, a high expected wage next period makes the new technology profitable. This is our benchmark case.

Consider *the important special case* in which $A^{\widehat{j}_t+1} = A^{\widehat{j}_t} = A$, then innovating is more profitable than accumulating active technologies if

$$\zeta\left[A^{\widehat{j}_t+1}P_{t+1} - a_{\widehat{j}_t+1}w_{t+1}\right] > \left[A^{\widehat{j}_t}P_{t+1} - a_{\widehat{j}_t}w_{t+1}\right].$$

$$[a_{\widehat{j}_t} - a_{\widehat{j}_t+1}]w_{t+1} > \left[A^{\widehat{j}_t} - \zeta A^{\widehat{j}_t+1}\right]P_{t+1}.$$

$$\frac{1-\zeta}{\zeta} < \frac{w_{t+1}}{P_t}\left[\frac{\gamma - 1}{\gamma^{\widehat{j}_t+1}}\right].$$

Notice that, with a constant factor $\gamma > 1$, the denominator of the right hand side goes to zero, hence the wage rate (in unit of current consumption) must grow, on average, at a rate of $\gamma - 1$ per period to maintain a stable pattern over time. When the real wage grows at a rate lower than $\gamma - 1$, technological innovation is delayed; the reverse when the real wage grows at a rate higher than $\gamma - 1$, then new machines are introduced more rapidly.[7]

Minimum Size Constraint Later on we will also consider the impact that a minimum size constraint in the innovation technology may have on the model's dynamics. This is because of the following intuitive reason: without minimum size the "pure innovation problem" is badly defined, at least in the limit case of no discouting. Because $\zeta < 1$, and any new technology can be accumulated at a fixed marginal cost of one once an infinitesimal prototype has been introduced, it is optimal to let $D_t \to 0$, and accumulate next period, when the technology has become active. The minimum size restriction has the following form

$$K_{t+1}^{\widehat{j}_t+1} = \zeta D_t, \text{ for } D_t \geq \underline{D}^{\widehat{j}_t+1}.$$

$$K_{t+1}^{\widehat{j}_t+1} = 0, \text{ otherwise.}$$

An alternative solution to this problem, which preserves convexity of the technology set, is to assume that, for each active technology j, only output used by technology j can be used to accumulate additional machines of that kind. In this case, starting with a prototype of an infinitesimal size makes the accumulation of additional capital very costly, and a well defined solution exists for the "pure innovation problem." In this case one can still write

[7]This formalizes the intuition according to which in Europe labor saving innovations are more often adopted, labor productivity is higher and the capital/output ratio is also higher, because wage rates are "artificially" kept high by union power, political considerations, and, more generally, labor and product market regulations.

$$K_{t+1}^{\widehat{j}_t+1} = \zeta D_t,$$

for the innovation sector, while the law of motion of each active technology becomes

$$K_{t+1}^j = (1-\mu)K_t^j + I_t^j - S_t^j,$$

with

$$I_t^j \leq Y_t^j + \nu^j S_t^j.$$

The aggregate resource constraint on consumption and innovation expenditure has to be rewritten as

$$C_t + D_t \leq \sum_{j \in J_t} (Y_t^j - I_t^j),$$

The question here is one of modeling techniques, i.e. is the non-convexity needed to capture the quantitative pattern of recessions and aggregate innovation, or not?

Other Innovation Technologies Alternative formulations to be considered include, the Cobb-Douglas case

$$K_{t+1}^{\widehat{j}_t+1} = BD_t^\beta (L_t^D)^{1-\beta}, \; B < 1$$

paired with the technology-specific accumulation rule.

Of some relevance, at least in principle and again as a modeling technique curiosity, is the probabilistic case in which the innovation technology takes aggregate output and labor as inputs, and generates a level of capital stock $K_{t+1}^{\widehat{j}_t+1} \in [0, B]$ according to a probability distribution with mass shifting toward B as the amount of resources invested increases.

4.2.3. CES Technology

Definition of *active technologies*, *best available technology* and *marginal technology* is as in the fixed coefficient case. The production functions now are

$$Y_t^j = A^j \left[\eta (K_t^j)^\rho + \gamma^j (L_t^j)^\rho\right]^{1/\rho} + \nu^j S_t^j,$$

$0 \leq \nu < A^j < \infty$, $-\infty < \rho < 1$, $\eta > 0$, $\gamma > 1$. For each technology $j = 0, 1, ...$, we assume that there exist $\underline{x}^j$ and $\overline{x}^j$ such that $\underline{x}^j \leq \frac{K_t^j}{L_t^j} \leq \overline{x}^j$ must hold; i.e. for each technology there exists a minimum and a maximum admissible capital/labor ratio. Rewrite the production functions in term of capital intensities $x = K/L$.

$$Y_t^j = A^j L_t^j \left[\eta (x_t^j)^\rho + \gamma^j\right]^{1/\rho} + \nu^j S_t^j,$$

Marginal productivities, for technology j, are

$$\frac{\partial Y_t}{\partial L_t^j} = A^j \gamma^j \left[\eta (x_t^j)^\rho + \gamma^j\right]^{(1-\rho)/\rho}, \text{ for } \underline{x}^j \leq x_t^j \leq \overline{x}^j, \text{ and zero otherwise;}$$

$$\frac{\partial Y_t}{\partial K_t^j} = A^j \eta (x_t^j)^{\rho - 1} \left[\eta (x_t^j)^\rho + \gamma^j\right]^{(1-\rho)/\rho}, \text{ for } \underline{x}^j \leq x_t^j \leq \overline{x}^j, \text{ and zero otherwise.}$$

Potential labor demand Λ_t is the sum, over technologies, of maximum achievable employment given installed capacity

$$\Lambda_t = \sum_{j \in J_t} \overline{L}_t^j = \sum_{j \in J_t} \frac{K_t^j}{\underline{x}^j},$$

while *productive capacity* Π_t is

$$\Pi_t = \sum_{j \in J_t} \left\{ A^j \left[\eta + \frac{\gamma^j}{(\underline{x}^j)^\rho}\right]^{1/\rho} + \nu^j \right\} K_t^j = \sum_{j \in J_t} (\kappa^j + \nu^j) K_t^j.$$

As before, the aggregate constraints are

$$Y_t = \sum_{j \in J_t} Y_t^j \leq \Pi_t,$$

$$C_t + \sum_{j \in J_t} I_t^j + D_t \leq Y_t,$$

$$L_t^d = \sum_{j \in J_t} L_t^j \leq \sum_{j \in J_t} \overline{L}_t^j = \Lambda_t,$$

$$K^j_{t+1} = (1-\mu)K^j_t + I^j_t - S^j_t,$$

and

$$K^j_t \geq 0, I^{j_t}_t \geq 0, D_t \geq 0, S^j_t \geq 0, C_t \geq 0, L^j_t \geq 0.$$

Let the sequences $\{X_t = \sum_{j \in J_t} I^j_t + D_t\}^\infty_{t=0}$ and $\{S^1_t, ..., S^{\widehat{j}_t}_t\}^\infty_{t=0}$ be given, and characterize the period by period allocation of total labor supply across sectors, for given productive capacity installed. As before, the planner will

$$\max_{\{L^1,...,L^{\widehat{j}_t}\}} u(C) + v(1-L)$$

subject to

$$C + X = \sum_{j \in J_t} \left\{ A^j L^j_t \left[\eta(x^j_t)^\rho + \gamma^j\right]^{1/\rho} + \nu^j S^j_t \right\},$$

$$L = \sum_{j \in J_t} L^j_t$$

First order conditions

$$A^j \gamma^j \left[\eta(x^j_t)^\rho + \gamma^j\right]^{(1-\rho)/\rho} = w_t, \text{ for all } j \in J_t \text{ such that } L^j_t > 0.$$

In the exponential case the latter implies that, for all pairs $j, i \in J_t$, $j > i$, for which $L^j_t, L^i_t > 0$

$$(A\gamma)^{\frac{\rho(j-i)}{(1-\rho)}} \left[\eta(x^j_t)^\rho + \gamma^j\right] = \eta(x^i_t)^\rho + \gamma^i.$$

This requires $x^j_t < x^i_t$, independently from the sign of ρ; hence, we should always observe a lower capital intensity in the most advanced sectors. This holds also in the special, but important, case in which technological progress is purely labor saving, i.e. when $A^j = A^i = A$.

Then, *employment* at time t is uniquely determined by the $\underline{x}^{\underline{j}_t} \le x_t^{\underline{j}_t} \le \overline{x}^{\underline{j}_t}$ solving

$$v'\left(1 - \sum_{j=\underline{j}_t+1}^{\widehat{j}_t} \frac{K_t^j}{\underline{x}^j} + \frac{K_t^{\underline{j}_t}}{x_t^{\underline{j}_t}}\right) =$$

$$u'\left(\sum_{j=\underline{j}_t+1}^{\widehat{j}_t} \kappa^j K_t^j + A^{\underline{j}_t}\left[\eta(K_t^{\underline{j}_t})^\rho + \gamma^{\underline{j}_t}\left(\frac{K_t^{\underline{j}_t}}{x_t^{\underline{j}_t}}\right)^\rho\right]^{1/\rho} + \sum_{j \in J_t} \nu^j S_t^j - X_t\right) A^{\underline{j}_t}\gamma^{\underline{j}_t}\left[\eta\left(x^{\underline{j}_t}\right)^\rho + \gamma^{\underline{j}_t}\right]^{(1-\rho)/\rho}.$$

Example with two technologies. Assume the stocks K^1, K^2, investment X and scrapping S^1 and S^2 are given. Employment is found by

$$\max_{L^1,L^2} u(C) + v(1-L)$$

subject to

$$C + X = A^1\left[\eta(K^1)^\rho + \gamma^1(L^1)^\rho\right]^{1/\rho} + A^2\left[\eta(K^2)^\rho + \gamma^2(L^2)^\rho\right]^{1/\rho} + \nu^1 S^1 + \nu^2 S^2$$

$$L = L^1 + L^2$$

Reduced form objective function is

$$\max_{L^1,L^2} u\left[A^1\left[\eta(K^1)^\rho + \gamma^1(L^1)^\rho\right]^{1/\rho} + A^2\left[\eta(K^2)^\rho + \gamma^2(L^2)^\rho\right]^{1/\rho} + \nu^1 S^1 + \nu^2 S^2 - X\right] + v(1-L^1-L^2)$$

Interior solutions when

$$(L^1)^{\rho-1}\gamma^1 A^1\left[\eta(K^1)^\rho + \gamma^1(L^1)^\rho\right]^{(1-\rho)/\rho} = (L^2)^{\rho-1}\gamma^2 A^2\left[\eta(K^2)^\rho + \gamma^2(L^2)^\rho\right]^{(1-\rho)/\rho} = w$$

$$u'\left[A^1\left[\eta(K^1)^\rho + \gamma^1(L^1)^\rho\right]^{1/\rho} + A^2\left[\eta(K^2)^\rho + \gamma^2(L^2)^\rho\right]^{1/\rho} + \nu^1 S^1 + \nu^2 S^2 - X\right] w = v'(1-L)$$

Use first, under the assumption of exponential productivity parameters, to find capital intensities in the two sectors

$$\left[\eta(x^1)^\rho + \gamma^1\right]^{(1-\rho)/\rho} = \gamma A \left[\eta(x^2)^\rho + \gamma^2\right]^{(1-\rho)/\rho}$$

and rewrite second as

$$u' \left[A^1 \frac{K^1}{x^1} \left[\eta(x^1)^\rho + \gamma^1\right]^{1/\rho} + A^2 \frac{K^2}{x^2} \left[\eta(x^2)^\rho + \gamma^2\right]^{1/\rho} + \nu^1 S^1 + \nu^2 S^2 - X\right] \gamma^2 A^2 \left[\eta(x^2)^\rho + \gamma^2\right]^{(1-\rho)/\rho}$$
$$= v' \left(1 - \frac{K^1}{x^1} + \frac{K^2}{x^2}\right)$$

Solve the two equations for x^1 and x^2.

Also, use

$$\left[\eta(x^1)^\rho + \gamma^1\right]^{(1-\rho)/\rho} = \gamma A \left[\eta(x^2)^\rho + \gamma^2\right]^{(1-\rho)/\rho}$$

to write

$$\frac{\left[\eta(x^1)^\rho + \gamma^1\right]}{\left[\eta(x^2)^\rho + \gamma^2\right]} = (\gamma A)^{\rho/(1-\rho)} .$$

As long as capital intensities satisfy the previous restriction, and the two ratios x^1 and x^2 are in the technologically admissible cone, both sectors are being used in production, and positive investment takes place in both sectors. Next we look at the case in which the less efficient sector is abandoned, and only the second sector has positive employment.

This amounts to finding *corner solutions*. In particular, when do we set $L^1 = 0$ and only use the stock of capital K^2? When

$$\left[\eta(\overline{x}^1)^\rho + \gamma^1\right]^{(1-\rho)/\rho} < \gamma A \left[\eta(\underline{x}^2)^\rho + \gamma^2\right]^{(1-\rho)/\rho} .$$

the minimum productivity of labor in sector 2 is higher than the maximum productivity of labor in sector 1.

Efficient Dynamic Allocations Begin with simple case in which only capital of type 1 is active, and capital of type 2 may or may not be introduced. We have

$$\max \sum_{t=0}^{\infty} \left[u(C_t) + v(1 - L_t) \right] \delta^t$$

subject to

$$C_t + I_t + D_t = A^1 \left[\eta (K_t^1)^\rho + \gamma^1 (L)^\rho \right]^{1/\rho} + \nu^1 S_t^1$$

$$K_{t+1}^1 = (1 - \mu) K_t^1 + I_t$$

$$K_{t+1}^2 = \zeta D_t$$

FOC for labor supply

$$u'(C_t) \frac{\partial C_t}{\partial L_t} = v'(1 - L_t)$$

$$A^1 \gamma^1 \left[\eta \left(\frac{K_t^1}{L_t} \right)^\rho + \gamma^1 \right]^{(1-\rho)/\rho} = \frac{v'(1 - L_t)}{u'(Y_t - I_t - D_t)}$$

which has a unique solution $L(K_t^1, K_{t+1}^1, D_t, S_t^1)$.

FOCs for the two investments (Note, $I_t > 0 \Leftrightarrow S_t^1 = 0$ and viceversa)

$$u'(C_t) = \delta \left[u'(C_{t+1}) \frac{\partial Y_{t+1}^1}{\partial K_{t+1}^1} - v'(1 - L_{t+1}) \frac{\partial L_{t+1}}{\partial K_{t+1}^1} \right]$$

$$u'(C_t) = \delta \zeta \left[u'(C_{t+1}) \frac{\partial Y_{t+1}^2}{\partial K_{t+1}^2} - v'(1 - L_{t+1}) \frac{\partial L_{t+1}}{\partial K_{t+1}^2} \right]$$

Notice that, contrary to the Fixed Coefficients case, BOTH these FOCs may be satisfied with equality as long as x_{t+1}^1 and x_{t+1}^2 are chosen to satisfy the equality of rate of returns condition, given in the appendix.

5. Planner Problem

5.1. Fixed Coefficient, Linear Innovation

We do not assume a minimum size constraint here, hence all competitive equilibria are Pareto Optima, and viceversa because the technology set is a convex cone, preferences are regular, and markets are complete.

$$\max_{\{C_t, L_t^j, I_t^j, S_t^j, D_t\}} E_t \left\{ \sum_{t=0}^{\infty} \delta^t \left[u(C_t) + v(1 - L_t) \right] \right\}$$

subject to

$$C_t + \sum_{j \in J_t} I_t^j + D_t \leq \sum_{j \in J_t} \left[\min\{A^j K_t^j, \gamma^j L_t^j\} + \upsilon^j S_t^j \right],$$

$$L_t = \sum_{j \in J_t} L_t^j,$$

$$K_{t+1}^j = (1 - \mu) K_t^j + I_t^j - S_t^j,$$

$$K_{t+1}^{\widehat{j}_t + 1} = \zeta D_t.$$

$$K_t^j \geq 0, I_t^{j_t} \geq 0, D_t \geq 0, S_t^j \geq 0, C_t \geq 0, L_t^j \geq 0.$$

First Order Conditions Begin with the static FOC for L_t and L_t^j (this is the three-step procedure detailed earlier)

$$u' \left(\sum_{j=\underline{j}_t+1}^{\widehat{j}_t} A^j K_t^j + \phi A^{\underline{j}_t} K_t^{\underline{j}_t} + \sum_{j \in J_t} \upsilon^j S_t^j - X_t \right) \gamma^{\underline{j}_t} = v' \left(1 - \sum_{j=\underline{j}_t+1}^{\widehat{j}_t} a^j K_t^j + \phi a^{\underline{j}_t} K_t^{\underline{j}_t} \right)$$

Notice what this conditions says: it says that, in period t, the marginal utility of leisure, i.e. the real wage rate, is equal to the marginal utility of consumption, i.e. the current price of the homogeneous output, times the marginal productivity of labor in the marginal technology. This implies that the capital stock invested in the marginal technology breaks even, and makes zero profit, while all other sectors receive competitive rents, which are reflected in the higher equilibrium price of their capital. Recall that, in the stochastic case, the marginal technology in period t may well be one of those introduced recently.

Given the vectors of K_t^j, S_t^j and X_t we solve the above for $L_t = L(K_t, S_t, X_t)$ and for the sectorial L_t^j. Next, we use the aggregate resource constraint to write

$$C_t = \sum_{j \in J_t} \left[\min\{A^j K_t^j, \gamma^j L_t^j\} + v^j S_t^j \right] - \sum_{j \in J_t} I_t^j - D_t$$

and reformulate the planning problem as

$$\max_{\{I_t^{\widehat{j}}, S_t^j, D_t\}} E_t \left\{ \sum_{t=0}^{\infty} \delta^t \left[u \left(\sum_{j \in J_t} \left[\min\{A^j K_t^j, \gamma^j L_t^j\} + v^j S_t^j \right] - \sum_{j \in J_t} I_t^j - D_t \right) + v(1 - L(K_t, S_t, X_t)) \right] \right\}$$

subject to

$$K_{t+1}^j = (1-\mu) K_t^j + I_t^j - S_t^j,$$

$$K_{t+1}^{\widehat{j}_t+1} = \zeta D_t.$$

$$K_t^j \geq 0, I_t^{j_t} \geq 0, D_t \geq 0, S_t^j \geq 0.$$

6. Implications of the Zero Profit Conditions

Putty Clay and Linear Innovation Technologies The price at time t of additional machines for an active technology is equal to the price of output/consumption, P_t, as the two are perfectly substitutable in the aggregate resource constraint; the price of a machine for the new technology is P_t/ρ. On the other hand, because installed capital is "technology-specific" (the model is "putty-clay") each existing machine has its own price, q_t^j. Here we look at the equilibrium relations among these prices (these are present value prices), as they are determined by the zero profit conditions. The present value of output/consumption in period t is $P_t = \delta^t u'(C_t)$.

Zero profits for active technology j in the production of aggregate output, gives

$$P_t = \frac{q_t^j}{A^j} + \frac{w_t}{\gamma^j} \Longrightarrow q_t^j = A^j P_t - a_j w_t$$

for $j = 1, ..., \widehat{j}_t$. Zero profit for the innovation technology gives

$$q_{t+1}^{\widehat{j}_t+1} = \frac{P_t}{\zeta}.$$

Notice that

$$q_t^j = A^j \left[P_t - \frac{w_t}{\gamma^j}\right] > A^{j-1} \left[P_t - \frac{w_t}{\gamma^{j-1}}\right] = q_t^{j-1}$$

for $j = 1, ..., \widehat{j}_t$. So, assuming that $A^j = (A)^j$ $\gamma^j = (\gamma)^j$, the prices of machines embodying active technologies, when they are active, satisfy

$$\frac{q_t^j}{q_t^{j-1}} = \frac{A\left[P_t - \frac{w_t}{\gamma^j}\right]}{\left[P_t - \frac{w_t}{\gamma^{j-1}}\right]} = A\left[\frac{\gamma^j P_t - w_t}{\gamma^j P_t - \gamma w_t}\right] > 1.$$

Clearly, for some $j \in \{1, ..., \widehat{j}_t\}$ we may have $\gamma^j P_t \leq w_t$, and then $q_t^j = 0$, meaning that technology j is not used in period t (Question, may it come back in later periods? NO: once the condition $q_t^j = 0$ is realized, technology j is scrapped for ever in this economy.) Relate this to first order conditions above determining employment and capacity utilization.

Notice also that, as long as $I_t^{\widehat{j}_t} > 0$, the price of investment in machine $\widehat{j}_t$ today must equal the market value of the machine tomorrow

$$q_{t+1}^{\widehat{j}_t} = P_t, \text{ if } I_t^{\widehat{j}_t} > 0, \text{ and}$$

$$q_{t+1}^{\widehat{j}_t} < P_t, \text{ if } I_t^{\widehat{j}_t} = 0,$$

meaning that Tobin Q is always less or equal to one in this version of the model. This implies that $I_t^j = 0$ for all $j = 1, ..., \widehat{j}_t - 1$, and only $I_t^{\widehat{j}_t} \geq 0$ (this is also be set equal to zero, in periods when a new machine is produced.) From these considerations and the zero profit condition for technology $\widehat{j}_t$ in period t we conclude that, *when* $I_t^{\widehat{j}_t} > 0$

$$q_{t+1}^{\widehat{j}_t} = \frac{q_t^{\widehat{j}_t}}{A^{\widehat{j}_t}} + \frac{w_t}{\gamma^{\widehat{j}_t}} = P_t,$$

which gives the first order process followed by the prices of the best installed machines, as long as it does not drop to zero (we should figure out when is it that this takes place.) For the other active technologies $j = 1, ..., \widehat{j}_t - 1$, we know that $q_t^j < q_t^{\widehat{j}_t}$ and that

$$q_t^j = A^j P_t - a_j w_t$$

Manipulate the latter under the assumption of exponential productivity parameters, to find the marginal technology in period t. This is the lowest index j for which $q_t^j \geq 0$; from the zero profit condition we have that

$$w_t \geq \gamma^{\gamma^{\underline{j}_t}} P_t.$$

Next, consider the hypothetical case in which a new machine gets introduced, i.e. $D_t > 0$, and there is also positive investment in the best available technology, i.e. $I_t^{\widehat{j}_t} > 0$. Then it must be true that, in this particular circumstances,

$$\frac{q_{t+1}^{\widehat{j}_t+1}}{q_{t+1}^{\widehat{j}_t}} = \zeta^{-1}.$$

Clearly this is not a generic case. In general, you either innovate (and then you do not invest anything in any of the active technology) or you do not, and then you invest only in the most efficient among the active technologies. You innovate when

$$\zeta q_{t+1}^{\widehat{j}_t+1} > q_{t+1}^{\widehat{j}_t}$$

or, more properly, we innovate in period t, i.e. $D_t > 0$ and $I_t^{\widehat{j}_t} = 0$, when

$$\zeta\left[A^{\widehat{j}_t+1}P_{t+1} - a_{\widehat{j}_t+1}w_{t+1}\right] > \left[A^{\widehat{j}_t}P_{t+1} - a_{\widehat{j}_t}w_{t+1}\right].$$

7. Appendix

Here we collect some algebra that is useful to understand the subtleties of the general CES case.

First, in what sense are we modeling *labor saving technological progress*? Assume exponential productivity parameters, and compare two technologies, $j > i$, along isoquants at which they produce the same amount of aggregate output and use one unit of capital stock.

$$A^j\left[\eta + \gamma^j (L_t^j)^\rho\right]^{1/\rho} = A^i\left[\eta + \gamma^i (L_t^i)^\rho\right]^{1/\rho}$$

$$\left(A^{j-i}\right)^\rho\left[\eta + \gamma^j (L_t^j)^\rho\right] = \eta + \gamma^i (L_t^i)^\rho$$

$$\left(A^\rho\gamma\right)^{j-i}(L_t^j)^\rho = (L_t^i)^\rho - \frac{\eta}{\gamma^i}[\left(A^{j-i}\right)^\rho - 1]$$

Capital is immobile, and investment flows only to the best available technology (in the deterministic case), hence it is not obvious that the marginal productivity of capital should ever be equalized across sectors. In any case, let's compute the conditions under which *capital productivity is equalized across sectors.*

$$\left(A^{j-i}\right)^{\rho/(1-\rho)}\eta(x_t^j)^{-\rho}\left[\eta(x_t^j)^\rho + \gamma^j\right] = \eta(x_t^i)^{-\rho}\left[\eta(x_t^i)^\rho + \gamma^i\right]$$

$$\left(A^{j-i}\right)^{\rho/(1-\rho)}\gamma^{j-i}(x_t^j)^{-\rho} = (x_t^i)^{-\rho} - \frac{\eta}{\gamma^i}[\left(A^{j-i}\right)^{\rho/(1-\rho)} - 1]$$

Special case, in which $A^j = A^i = A$ gives

$$(x_t^j)^{-\rho}\left[\eta(x_t^j)^\rho + \gamma^j\right] = (x_t^i)^{-\rho}\left[\eta(x_t^i)^\rho + \gamma^i\right]$$

$$\gamma^j(x_t^j)^{-\rho} = \gamma^i(x_t^i)^{-\rho}$$

$$\frac{\gamma^j}{\gamma^i} = \left(\frac{x_t^j}{x_t^i}\right)^\rho$$

Notice that when $\rho > 0$ the capital labor ratio is higher in the most efficient sector, but the opposite is true when $\rho < 0$.

Zero profit in investment activity has no additional implications. A unit of investment, no matter where it is allocated, always costs a unit of consumption today, i.e. p_t. Its payoff in sector j is $p_{t+1}\frac{\partial Y_{t+1}}{\partial K_{t+1}^j}$.

Movements in labor shares.

$$\frac{wL^j}{Y^j} = \frac{L^j A^j \gamma^j \left[\eta(x^j)^\rho + \gamma^j\right]^{(1-\rho)/\rho}}{A^j L^j \left[\eta(x^j)^\rho + \gamma^j\right]^{1/\rho}} =$$

$$= \frac{\gamma^j}{\eta(x^j)^\rho + \gamma^j} = \frac{1}{1 + \frac{\eta(x^j)^\rho}{\gamma^j}}$$

Hence, we have the two following cases

$\rho > 0$

$$\frac{1}{1 + \frac{\eta(x^j)^\rho}{\gamma^j}}$$

is a decreasing function of x^j and an increasing function of γ^j, hence, for a given technology, the labor share decreases as the capital intensity increases. Recall that, at least in principle, along an expansion the capital intensity decreases in all sectors, hence the labor share should increase during an expansion, when $\rho > 0$.

When there is an innovation, i.e. a technology with higher index is adopted, then, ceteris paribus, the labor share would increase. This suggests that, if our intuition works, right after an innovation the capital intensity of all sectors, and of the most recent ones in particular, should increase more than proportionally, thereby lowering the labor share of income.

$\rho < 0$

The opposite is true. Notice that, in this case, in order for the labor share to increase along an expansion we would need the capital intensity to increase along an expansion. That is to say, employment increases but investment increases more than proportionally driving up the labor share of income.

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