Mittag-Leffler Distributions and Long-run Behavior of Macro-economic models

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Some questions and answers

Questions:

what are Mittag-Leffler Distributions?

How are they related to long-run analysis of macro-models?

Answers:

They are generic $\cdots$ they generically characterize long-run behavior; tail of M-L distributions are power laws

Examples

Feng-Hoppe analysis of branching model;

Extension of the one parameter Poisson-Dirichlet (Ewens) model to two-parameter version by J. Pitman

Long-run behavior of both classes of models have M-L distributions

Some Facts and Applications

- Mittag-Leffler distributions are uniquely determined by their moments.$\cdots$ Method of moments applies:
- g_α has Γ(p+1)/ $\Gamma(\alpha$ p +1) as its p-th moment, $0<\alpha<1$
- Fractional master equations $\cdots$ mean-first passage times, waiting distributions in finance, and possibly others.

Two-parameter Extensions

- $g_{\alpha,\theta} = B\, g_{\alpha}$

where

$B=\Gamma(\theta+1)/\Gamma(\theta/\alpha+1)$

It is known that as $n \to \infty$

$E(K_n/n^{\alpha}) \to \theta\,\Gamma(\theta)/[\alpha\,\Gamma(\theta+\alpha)]$.

This is the same as the mean of $Bx^{\theta/\alpha} g_{\alpha}(x)$.

Feng-Hoppe Model

- A simple branching process due to Karlin and McGregor: Let I(t) be a stream of new types of agents (resources) arriving stochastically.
- Arrival rate $=\theta + k\,\alpha$, where $k=|I(t)|$, $\theta=\beta-\alpha$
- Each new arrival (innovation) starts its own group that grow stochastically.
- Let N(t) be the total size of the economy
- Its growth rate $=\alpha\,(k-1)+\beta+\sum_{j=1}^{k}(n_j-\alpha)=n+\theta$ where the i-th arrival (innovation) grows at rate $n_i-\alpha$.

- I(t)/N(t) converges to a ratio of two dependent Gamma random variables, which is M-L distributed

Two-parameter Poisson-Dirichlet distribution (extension of the Ewens distribution)

- Let K_n denote the number of clusters formed by n agents.
- Suppose
- $P(K_{n+1}=k|K_n =k)=(n-k\alpha)/(n+\theta)$
- $P(K_{n+1}=k+1|K_n=k)=(\theta +k\alpha)/(n+\theta)$
- Then
- K_n/n^{α} converges a.s. to M-L distrbiution

Analysis of Long-run Behavior: Simple Cases

- Let F(s) be the Laplace transforms of some f(t).
- In control or system theory, the final value theorem says
- $\text{Lim}_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$.
- Tauberian theorm of Karamata slowly varying function

Darling-Kac Theorem

- Under a set of conditions, Mittag-Leffler distributions are the only possible limit laws. (Regular Variations, Bingham, Goldie, Teugels, CUP 1998, pp.388)

Some Asymptotic Differences in one- and two-parameter Ewens models

- K_n / n^{α} in the two-parameter Ewens model is not self-averaging.

Question

- Does simulations of models with non-self averaging properties yield estimate of α and θ ?
- Moments of M-L distributions given earlier can be used for that purpose?