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Volatility and Quantile Forecasts by Realized Stochastic Volatility Models with Generalized Hyperbolic Distribution

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Abstract

The predictive performance of the realized stochastic volatility model of Takahashi, Omori, and Watanabe (2009), which incorporates the asymmetric stochastic volatility model with the realized volatility, is investigated. Considering well known characteristics of financial returns, heavy tail and negative skewness, the model is extended by employing a wider class distribution, the generalized hyperbolic skew Student's t-distribution, for financial returns. With the Bayesian estimation scheme via Markov chain Monte Carlo method, the model enables us to estimate the parameters in the return distribution and in the model jointly. It also makes it possible to forecast volatility and return quantiles by sampling from their posterior distributions jointly. The model is applied to quantile forecasts of financial returns such as value-at-risk and expected shortfall as well as volatility forecasts and those forecasts are evaluated by various tests and performance measures. Empirical results with the US and Japanese stock indices, Dow Jones Industrial Average and Nikkei 225, show that the extended model improves the volatility and quantile forecasts especially in some volatile periods.

Key words: Backtesting; Expected shortfall; Generalized hyperbolic skew Student's t-distribution; Markov chain Monte Carlo; Realized volatility; Stochastic volatility; Value-at-risk.

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1 Introduction

This paper proposes a general volatility model designed for predictions of volatility and quantiles of financial returns. The volatility and quantile forecasts are important to assess the financial risk. For example, the value-at-risk (VaR) and expected shortfall (ES), computed from the quantile forecasts, have been widely known as measures of the financial tail risk.

The proposed model incorporates two important aspects for the volatility and quantile forecasts: the distribution of financial returns and the estimation of the volatility. First, the unconditional distribution of financial returns is known to be leptokurtic. This leptokurtosis can fully or partly be captured by time-varying volatility, but the distribution conditional on volatility may still be leptokurtic. Moreover, the return distribution may also be skewed. To incorporate the important properties in the return distribution, we employ the general distribution class, called generalized hyperbolic (GH) distribution, introduced by Aas and Haff (2006). The GH distribution takes a flexible form to fit the return characteristics such as skewness and leptokurtosis.

Second, the volatility is unobservable and thus needed to be estimated from the available data. In the early literature, autoregressive conditional heteroskedasticity (ARCH) type models and stochastic volatility (SV) type models have been developed to capture the stylized volatility properties such as volatility clustering and volatility asymmetry.[1] Recently, thanks to the availability of high frequency data containing the price and other asset characteristics sampled at a time horizon shorter than one day, it becomes possible to measure the latent volatility quite accurately. Andersen and Bollerslev (1998) propose the so-called realized volatility (RV) as an accurate volatility measure computed from 5-minute returns. Under some assumptions, the RV is a consistent estimator of the true volatility.[2] The proposed model incorporates the RV measure via the so-called realized stochastic volatility (RSV) model.

The RSV model is a contemporaneous modeling of financial returns and the RV estimators. Takahashi, Omori, and Watanabe (2009) propose to model daily returns and the RV

[1] ARCH type models include the ARCH and GARCH models proposed by Engle (1982) and Bollerslev (1986), respectively, and their extensions. See, for example, Andersen, Bollerslev, Christoffersen, and Diebold (2013) for other ARCH type models. The SV type models, developed by Taylor (1986), are reviewed in Shephard (1996).

[2] More detail properties of the RV can be found in Andersen, Bollerslev, and Diebold (2010) and references therein.

estimator simultaneously under the framework of the SV model. Additionally, Dobrev and Szerszen (2010) and Koopman and Scharth (2013) propose the models in a similar manner. These models are referred to as the RSV models. On the other hand, Hansen, Huang, and Shek (2011) propose to extend GARCH models incorporating them with the RV, which is called the realized GARCH model. The contemporaneous models can adjust a possible bias in the RV estimator within the models.[3]

In this paper, we investigate the predictive performance of the RSV model which has not been fully applied to quantile forecasts.[4] Considering the skewness and leptokurtosis in the return distribution, we extend the RSV model of Takahashi, Omori, and Watanabe (2009) by employing the GH skew Student's t-distribution which includes normal and Student's t-distributions as special cases. Bayesian estimation scheme via Markov chain Monte Carlo (MCMC) technique enables us to estimate the parameters in the return distribution and in the model jointly, which also makes it possible to adjust the bias in the RV estimator simultaneously. The MCMC technique samples the future volatility and return jointly from their posterior distributions. Using the samples of the future volatility and return, we can easily compute the volatility and quantile forecasts such as the VaR and ES.

We apply the model to daily returns and RKs of the US and Japanese stock indices, Dow Jones Industrial Average (DJIA) and Nikkei 225, respectively. The prediction results show that the extended model improves both volatility and quantile forecasts especially in some volatile periods such as late 2008. Therefore, the extended model is suited for conservative risk management necessary for commercial banks and pension funds.

The rest of this paper is organized as follows. In Section 2, we present the basic and extended RSV models with a brief description of the SV model and RV estimators. Then, we explain the estimation and prediction scheme to estimate the parameters, volatility and quantile forecasts jointly via the MCMC technique in Section 3. Further, we introduce

[3]The RV has two practical problems in the real market, non-trading hours and market microstructure noise, which results in a bias in the RV estimator. O'Hara (1995) and Hasbrouck (2007) provide a comprehensive review of the market microstructure theory and its applications. We defer the details to Section 2.2

[4]Other RV models have been applied to quantile forecasts. For example, Giot and Laurent (2004) and Clements, Galvão, and Kim (2008) investigate the quantile forecast performance of GARCH models with the RV estimator although they are not fully contemporaneous models. Recently, Watanabe (2012) applies the realized GARCH model to quantile forecasts and show that the RV estimator improves the forecast performance and that the realized GARCH model can adjust the bias in the RV estimator. Dobrev and Szerszen (2010) apply their model to the VaR forecasts but do not examine its performance formally.

several methods to evaluate the volatility and quantile forecasts in Section 4. We present the empirical results using the DJIA and Nikkei 225 data in Section 5. Finally, we conclude the paper in Section 6.

2 Realized Stochastic Volatility Model

In this section, we describe the RSV model, which incorporates the asymmetric SV model with the RV estimator. In Section 2.1, we introduce the SV model and then briefly describe the RV estimator in Section 2.2. We introduce the basic RSV model proposed by Takahashi, Omori, and Watanabe (2009) and present its extension in Section 2.3.

2.1 Stochastic Volatility Model

The asymmetric SV model is written as

$$r_t = \exp(h_t/2)\epsilon_t, \quad t = 1, \ldots, n, \tag{1}$$

$$h_{t+1} = \mu + \phi(h_t - \mu) + \eta_t, \quad t = 0, \ldots, n-1, \tag{2}$$

where r_t is a daily asset return and h_t is an unobserved log-volatility. It is common to assume that $|\phi| < 1$ for a stationarity of the log-volatility process. For the moment, we assume the normality for the return and volatility innovations as follows,

$$\begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix} \sim N(0, \Sigma), \quad \Sigma = \begin{bmatrix} 1 & \rho\sigma_\eta \\ \rho\sigma_\eta & \sigma_\eta^2 \end{bmatrix}. \tag{3}$$

The parameter ρ in (3) represents the correlation between ϵ_t and η_t, which captures the correlation between r_t and h_{t+1}. A negative value of ρ implies a negative correlation between today's return and tomorrow's volatility, which is a well known phenomenon in stock markets and referred to as a volatility asymmetry.[5] Additionally, we assume the following initial conditions,

$$h_0 = \mu, \quad \eta_0 \sim N\left(0, \frac{\sigma_\eta^2}{1-\phi^2}\right). \tag{4}$$

2.2 Realized Volatility

We first consider a simple continuous time process,

$$dp(s) = \sigma(s)dw(s), \tag{5}$$

[5] See, for example, Black (1976) and Christie (1982).

where $p(s)$ denotes the log price of a financial asset at time s, and $\sigma^2(s)$ is the instantaneous or spot volatility, which is assumed to be stochastically independent of the Wiener process $w(s)$. Then, the true volatility for a day t is defined as

$$\sigma_t^2 = \int_t^{t+1} \sigma^2(s)ds, \tag{6}$$

which is called an integrated volatility.

Andersen and Bollerslev (1998) propose a model-free estimator of the true volatility σ_t^2, which is called a RV estimator. Suppose that we have m intraday returns during the day t, $\{r_{t,i}\}_{i=1}^m$, then a simple RV estimator is defined as a sum of squared returns,

$$RV_t = \sum_{i=1}^{m} r_{t,i}^2, \tag{7}$$

which converges to the true volatility σ_t^2 as $m \to \infty$. That is, RV_t is a consistent estimator of σ_t^2 and thus may provide a precise estimate of the true volatility when there are sufficient number of intraday returns.

There are, however, some problems in computing the RV estimator using the high frequency data. First, the high frequency asset price contains the market microstructure noise (MMN) such as a bid-ask bounce and non-synchronous trading.[6] With the presence of the MMN, the RV estimator is biased and is not a consistent estimator of the true volatility. Hansen and Lunde (2006) and Ubukata and Oya (2008) study the MMN effects on the RV estimator. In general, the MMN effect becomes larger at the higher sampling frequency while the information loss becomes larger at the lower frequency.

There are several methods available for mitigating the MMN effects on the RV estimators.[7] Among them, Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008) propose a realized kernel (RK) estimator,

$$RK_t = \sum_{q=-Q}^{Q} k\left(\frac{q}{Q+1}\right)\gamma_q, \quad \gamma_q = \sum_{i=|q|+1}^{m} r_{t,i} r_{t,i-|q|}, \tag{8}$$

where $k(\cdot) \in [0,1]$ is a non-stochastic weight function. As for the choice of $k(\cdot)$, Barndorff-

[6]See, for example, Campbell, Lo, and MacKinlay (1997) for details.

[7]For example, Aït-Sahalia, Mykland, and Zhang (2005) and Bandi and Russell (2006, 2008) derive an optimal sampling frequency to balance the trade off between the MMN effect and the information loss. Additionally, Zhang, Mykland, and Aït-Sahalia (2005) propose a two (multi) scale estimator, which combines two (multiple) RV estimators calculated from returns with different sampling frequencies.

Nielsen, Hansen, Lunde, and Shephard (2009) suggests the Parzen kernel given by

$$k(x) = \begin{cases} 1 - 6x^2 + 6x^3 & 0 \leq x \leq 1/2 \\ 2(1-x)^3 & 1/2 \leq x \leq 1 \\ 0 & x > 1, \end{cases} \tag{9}$$

which satisfies the smoothness conditions, $k'(0) = k'(1) = 0$, and is guaranteed to produce a non-negative estimate.

The second problem in computing the RV estimator is the presence of the non-trading hours. For example, New York Stock Exchange is open only for six and a half hours from 9:30 a.m. to 4 p.m. (in Eastern Time). If we calculate the RV estimator using the intraday returns only in the market open period, it may underestimate the true volatility σ_t^2. To avoid this underestimation, Hansen and Lunde (2005) propose to scale the RV calculated from returns for the market open period as

$$RV_t^{scale} = cRV_t, \quad c = \frac{\sum_{t=1}^n (r_t - \bar{r})^2}{\sum_{t=1}^n RV_t}, \tag{10}$$

where r_t is the daily return and $\bar{r} = \sum_{t=1}^n r_t/n$. This ensures that the mean of the scaled RV (RV_t^{scale}) is equal to the variance of daily returns.[8]

2.3 Realized Stochastic Volatility Model

Takahashi, Omori, and Watanabe (2009) propose modeling daily returns and the RV estimator simultaneously as follows,

$$r_t = \exp(h_t/2)\epsilon_t, \quad t = 1, \ldots, n, \tag{11}$$

$$x_t = \xi + h_t + u_t, \quad t = 1, \ldots, n, \tag{12}$$

$$h_{t+1} = \mu + \phi(h_t - \mu) + \eta_t, \quad t = 0, \ldots, n-1, \tag{13}$$

where x_t is a logarithm of the RV estimator. The parameter ξ in (12) is designed to correct the bias due to the MMN and non-trading hours. If ξ is positive, the RV estimator has an upward bias, which implies that the effect of the MMN dominates that of non-trading hours, and vice versa as long as the MMN causes a positive bias in the RV estimator. We

[8]One may consider including returns for the non-trading hours (overnight interval) but this can make the RV estimator less precise since such returns contain much discretization noise.

assume that the disturbance u_t and other disturbances (ϵ_t, η_t) are not correlated, that is,

$$\begin{bmatrix} \epsilon_t \\ u_t \\ \eta_t \end{bmatrix} \sim N(0, \Sigma), \quad \Sigma = \begin{bmatrix} 1 & 0 & \rho\sigma_\eta \\ 0 & \sigma_u^2 & 0 \\ \rho\sigma_\eta & 0 & \sigma_\eta^2 \end{bmatrix}. \tag{14}$$

Dobrev and Szerszen (2010) and Koopman and Scharth (2013) also propose the joint modeling of daily returns and the realized volatility based on the SV model. Following Koopman and Scharth (2013), we refer the model consisting of (11)-(14) as a realized stochastic volatility (RSV) model.

We extend the RSV model in (11)-(14) with more generalized distribution for daily returns. Following Nakajima and Omori (2012), we employ the general hyperbolic (GH) skew Student's t-distribution for the return distribution.[9] Specifically, the return equation (11) is extended as follows,

$$r_t = \frac{\beta(z_t - \mu_z) + \sqrt{z_t}\epsilon_t}{\sqrt{\beta^2\sigma_z^2 + \mu_z}} \exp(h_t/2), \quad t = 1, \ldots, n, \tag{15}$$

where

$$z_t \sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}\right), \quad \mu_z = \mathrm{E}[z_t] = \frac{\nu}{\nu - 2}, \quad \sigma_z^2 = \mathrm{Var}[z_t] = \frac{2\nu^2}{(\nu-2)^2(\nu-4)}, \tag{16}$$

and $IG(\cdot, \cdot)$ denotes the inverse gamma distribution. We assume that $\nu > 4$ for the existence of the variance of z_t. The term $\sqrt{\beta^2\sigma_z^2 + \mu_z}$ standardizes the return so that the variance of the return remains $\exp(h_t)$. This specification includes the Student's t-distribution as a special case when $\beta = 0$ as well as the normal distribution when $\beta = 0$ and $\nu \to \infty$ (that is, $z_t = 1$ for all t). Following Nakajima and Omori (2012), we refer to the RSV model with the GH skew Student's t-distribution as the RSVskt model, hereafter. Similarly, the RSV models with the Student's t and normal distributions are referred to as the RSVt and RSVn models, respectively.[10]

[9]The GH skew Student's t-distribution is a subclass of the GH distribution. The GH distribution has a wider class of distribution but the parameters of the GH distribution are difficult to estimate as pointed out by Prause (1999) and Aas and Haff (2006). Nakajima and Omori (2012) also show that a wider class of the GH distribution could lead to either the inefficient MCMC sampling or the over-parametrization. Thus, we focus on the GH skew Student's t-distribution throughout the paper.

[10]We can extend the RV equation (12) as follows,

$$x_t = \xi + \psi h_t + u_t, \quad t = 1, \ldots, n.$$

Hansen, Huang, and Shek (2011) first consider this type of specification in their realized GARCH framework which is the joint modeling of daily returns and the RV estimator based on the GARCH type models. We

3 Estimation and Prediction Scheme

In this section, we describe the estimation and prediction scheme for the RSVskt model. In Section 3.1, we present a Bayesian estimation procedure via Markov chain Monte Carlo method. Then, we explain how to obtain the volatility and quantile forecasts within the Bayesian estimation procedure in Section 3.2.

3.1 Bayesian Estimation Procedure

The RSVskt model is written as

$$r_t = \frac{\beta(z_t - \mu_z) + \sqrt{z_t}\epsilon_t}{\sqrt{\beta^2\sigma_z^2 + \mu_z}} \exp(h_t/2), \quad t = 1, \ldots, n, \tag{17}$$

$$x_t = \xi + h_t + u_t, \quad t = 1, \ldots, n, \tag{18}$$

$$h_{t+1} = \mu + \phi(h_t - \mu) + \eta_t, \quad t = 0, \ldots, n-1, \tag{19}$$

where

$$z_t \sim IG\left(\frac{\nu}{2}, \frac{\nu}{2}\right), \quad \mu_z = \mathrm{E}[z_t] = \frac{\nu}{\nu - 2}, \quad \sigma_z^2 = \mathrm{Var}[z_t] = \frac{2\nu^2}{(\nu-2)^2(\nu-4)}, \tag{20}$$

and

$$\begin{bmatrix} \epsilon_t \\ u_t \\ \eta_t \end{bmatrix} \sim N(0, \Sigma), \quad \Sigma = \begin{bmatrix} 1 & 0 & \rho\sigma_\eta \\ 0 & \sigma_u^2 & 0 \\ \rho\sigma_\eta & 0 & \sigma_\eta^2 \end{bmatrix}. \tag{21}$$

To estimate the RSVskt model, we combine the MCMC algorithms for Bayesian estimation scheme of the SVskt model proposed by Nakajima and Omori (2012) and the RSV model by Takahashi, Omori, and Watanabe (2009). Let $\theta = (\phi, \sigma_\eta, \rho, \mu, \beta, \nu, \xi, \sigma_u)$, $y = \{r_t, x_t\}_{t=1}^n$, $h = \{h_t\}_{t=1}^n$, and $z = \{z_t\}_{t=1}^n$. Then, we draw random samples from the posterior distributions of (θ, h, z) given y for the RSVskt model using the MCMC method as follows:

0. Initialize θ, h, and z.

1. Generate $\phi|\sigma_\eta, \rho, \mu, \beta, \nu, \xi, \sigma_u, h, z, y$.

2. Generate $(\sigma_\eta, \rho)|\phi, \mu, \beta, \nu, \xi, \sigma_u, h, z, y$.

3. Generate $\mu|\phi, \sigma_\eta, \rho, \beta, \nu, \xi, \sigma_u, h, z, y$.

estimate the RSV models with this specification but it turns out that this extension does not improve the volatility forecasts nor quantile forecasts. Therefore, we focus on the RSV models with $\psi = 1$ in this paper.

4. Generate $\beta|\phi, \sigma_\eta, \rho, \mu, \nu, \xi, \sigma_u, h, z, y$.

5. Generate $\nu|\phi, \sigma_\eta, \rho, \mu, \beta, \xi, \sigma_u, h, z, y$.

6. Generate $\xi|\phi, \sigma_\eta, \rho, \mu, \beta, \nu, \sigma_u, h, z, y$.

7. Generate $\sigma_u|\phi, \sigma_\eta, \rho, \mu, \beta, \nu, \xi, h, z, y$.

8. Generate $z|\theta, h, y$.

9. Generate $h|\theta, z, y$.

10. Go to 1.

Since u_t is independently and identically distributed, we can implement the same sampling scheme proposed by Nakajima and Omori (2012) for steps 1-5 and 8. We can also easily modify the sampling scheme by Takahashi, Omori, and Watanabe (2009) for steps 6, 7, and 9. The detail procedures are given in Appendix A.

3.2 Volatility and Quantile Forecasts

To obtain the one-day-ahead log-volatility and daily return, we implement the following sampling scheme for each sample of (θ, h, z) generated from the MCMC algorithm described above.

i. Generate $h_{n+1}|\theta, h, z, y \sim N(\mu_{n+1}, \sigma^2_{n+1})$, where

$$\mu_{n+1} = \mu + \phi(h_n - \mu) + \rho\sigma_\eta \frac{\sqrt{\beta^2\sigma_z^2 + \mu_z} r_n - \beta(z_n - \mu_z)\exp(h_n/2)}{\sqrt{z}_n \exp(h_n/2)}, \quad (22)$$

$$\sigma^2_{n+1} = (1 - \rho^2)\sigma^2_\eta. \quad (23)$$

ii. Generate $z_{n+1} \sim IG(\nu/2, \nu/2)$.

iii. Generate $r_{n+1}|\theta, h_{n+1}, z_{n+1} \sim N(\hat{\mu}_{n+1}, \hat{\sigma}^2_{n+1})$, where

$$\hat{\mu}_{n+1} = \frac{\beta(z_{n+1} - \mu_z)\exp(h_{n+1}/2)}{\sqrt{\beta^2\sigma_z^2 + \mu_z}}, \quad (24)$$

$$\hat{\sigma}^2_{n+1} = \frac{z_{n+1}\exp(h_{n+1})}{\beta^2\sigma_z^2 + \mu_z}. \quad (25)$$

The quantile forecasts, VaR and ES, can easily be computed from the predictive distribution of financial returns obtained above. Let $\text{VaR}_t(\alpha)$ be the one-day-ahead forecast for the VaR of the daily return r_t with probability α. Then, assuming the long position, the VaR forecast satisfies

$$\Pr[r_t < \mathrm{VaR}_t(\alpha)|\mathcal{I}_{t-1}] = \alpha, \tag{26}$$

where $\mathcal{I}_{t-1}$ is the available information up to $t-1$.

Although the VaR has been widely used to evaluate the quantile forecast of financial returns, it only measures a quantile of the distribution and ignores the important information of the tail beyond the quantile. To evaluate the quantile forecast with the tail information, we compute the ES, which is defined as the conditional expectation of the return given that it violates the VaR. The one-day-ahead forecast of the ES with probability α, $\mathrm{ES}_t(\alpha)$, satisfies

$$\mathrm{ES}_t(\alpha) = \mathrm{E}[r_t|r_t < \mathrm{VaR}_t(\alpha), \mathcal{I}_{t-1}]. \tag{27}$$

Let n and T be the number of samples for the estimation and prediction, respectively. Then, the one-day-ahead forecasts of the VaR $(\mathrm{VaR}_{n+1}(\alpha), \ldots, \mathrm{VaR}_{n+T}(\alpha))$ and the ES $(\mathrm{ES}_{n+1}(\alpha), \ldots, \mathrm{ES}_{n+T}(\alpha))$ are computed repeatedly in the following way.

1. Set $i = 1$.
2. Generate the MCMC sample of the model parameters and one-day-ahead return r_{n+i} using the sample of $(y_i, \ldots, y_{n+i-1})$.
3. Compute $\mathrm{VaR}_{n+i}(\alpha)$ as the α-percentile of the MCMC sample of r_{n+i}.
4. Compute $\mathrm{ES}_{n+i}(\alpha)$ as a sample average of r_{n+i} conditional on $r_{n+i} < \mathrm{VaR}_{n+i}(\alpha)$.
5. Set $i = i + 1$ and return to 1 while $i < T$.

4 Evaluation of Volatility and Quantile Forecasts

In this section, we describe how to evaluate the predictive ability of the RSV models with different specifications. Since there is no single measure which ranks the models thoroughly, we compare the model performance from various perspectives. In Section 4.1, we introduce two loss functions for the volatility forecasts and a predictive ability test. In Section 4.2, we describe various evaluation methods for the VaR forecasts. In Section 4.3, we present a backtesting measure of the ES forecasts.

4.1 Evaluating Volatility Forecasts

To evaluate the volatility forecasts of different models, we use two loss functions, mean squared error (MSE) and quasi-likelihood (QLIKE) up to additive and multiplicative constants. Let $\hat{\sigma}_t^2$ and h_t be a volatility proxy and volatility forecast, respectively and consider the two loss functions,[11]

$$L_t^{MSE} = \frac{(\hat{\sigma}_t^2 - h_t)^2}{2}, \quad L_t^{QLIKE} = \frac{\hat{\sigma}_t^2}{h_t} - \log \frac{\hat{\sigma}_t^2}{h_t} - 1. \tag{28}$$

Recall that n and T are the number of samples for the estimation and prediction, respectively. Then, MSE and QLIKE are defined as the means of the corresponding loss functions, that is,

$$MSE = \bar{L}^{MSE} = \frac{1}{T}\sum_{t=1}^{T} L_t^{MSE}, \quad QLIKE = \bar{L}^{QLIKE} = \frac{1}{T}\sum_{t=1}^{T} L_t^{QLIKE}. \tag{29}$$

Since the true volatility is unobservable, the loss functions are computed using an imperfect volatility proxy, $\hat{\sigma}_t^2$. However, Patton (2011) shows that some class of loss functions including the above two provides a ranking consistent with the one using the true volatility as long as the volatility proxy is a conditionally unbiased estimator of the volatility, that is, $\mathrm{E}[\hat{\sigma}_t^2 | \mathcal{I}_{t-1}] = \sigma_t^2$.

Although the above loss functions provide a consistent ranking of the competing models, it is necessary to check whether the loss difference is statistically significant. To this end, we employ a predictive ability test based on Giacomini and White (2006). Let $L_t(m_1)$ and $L_t(m_2)$ be loss functions of models m_1 and m_2, respectively. Further, denote a $q \times 1$ $\mathcal{I}_t$-measurable vector by g_t, which we refer to as the test function. Then, in the case of one-step ahead forecasts, the null hypothesis of equal conditional predictive ability of models m_1 and m_2 is

$$\mathrm{E}[g_{n+t-1}\{L_{n+t}(m_1) - L_{n+t}(m_2)\}|\mathcal{I}_{n+t-1}] = \mathrm{E}[g_{n+t-1}\Delta L_{n+t}(m_1, m_2)|\mathcal{I}_{n+t-1}] = 0, \tag{30}$$

for $t = 1, 2, \ldots, T$. To test the null hypothesis, we use a Wald-type test statistic of the form

$$W_T = T\bar{Z}_T\hat{\Omega}_T^{-1}\bar{Z}_T, \tag{31}$$

where $\bar{Z}_T = T^{-1}\sum_{t=1}^{T} Z_{n+t}$, $Z_{n+t} = g_{n+t-1}\Delta L_{n+t}(m_1, m_2)$ and $\hat{\Omega}_T = T^{-1}\sum_{t=1}^{T} Z_{n+t}Z'_{n+t}$. By standard asymptotic normality arguments, the statistic W_T is asymptotically distributed

[11] Both loss functions are normalized to be the robust and homogeneous loss functions proposed by Patton (2011). For instance, L_t^{QLIKE} is normalized to yield a distance of zero when $\hat{\sigma}_t^2 = h_t$.

as a chi-square distribution with q degrees of freedom, denoted by $\chi^2(q)$. Thus, we reject the null of equal conditional predictive ability when $W_T > \chi^2_{1-p}(q)$, where p is a probability level of the test and $\chi^2_{1-p}(q)$ is the $(1-p)$ quantile of the $\chi^2(q)$ distribution.[12]

4.2 Evaluating Value-at-Risk

4.2.1 Likelihood ratio tests

To describe various likelihood ratio tests for the VaR forecasts, recall that T is the number of VaR forecasts and let T_1 be the number of times when the VaR is violated, that is, $r_t < \text{VaR}_t(\alpha)$. Then the empirical failure rate is defined as $\hat{\pi}_1 = T_1/T$. Kupiec (1995) proposes the likelihood ratio (LR) test for the null hypothesis of $\pi_1 = \alpha$, where π_1 is the true failure rate. Since this is a test that on average the coverage is correct, Christoffersen (1998) refers to this as the correct unconditional coverage test. Let $L(p)$ be the likelihood function for an i.i.d. Bernoulli with probability p, that is,

$$L(p) = p^{T_1}(1-p)^{T-T_1}. \tag{32}$$

The LR statistic of the unconditional coverage test is then

$$LR_{uc} = 2\{\ln L(\hat{\pi}_1) - \ln L(\alpha)\}, \tag{33}$$

which is asymptotically distributed as a $\chi^2(1)$ under the null hypothesis of $\pi_1 = \alpha$. Note that this test implicitly assumes that the violations are independent, which is not guaranteed in practice.

To test the independence hypothesis explicitly, Christoffersen (1998) considers the alternative of the first-order Markov process with the switching probability matrix

$$\Pi = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix}, \tag{34}$$

where π_{ij} is the probability of an $i \in \{0,1\}$ on day $t-1$ being followed by a $j \in \{0,1\}$ on day t (1 represents a violation and 0 not). The likelihood under the alternative hypothesis is

$$L(\pi_{01}, \pi_{11}) = (1-\pi_{01})^{T_0 - T_{01}} \pi_{01}^{T_{01}} (1-\pi_{11})^{T_1 - T_{11}} \pi_{11}^{T_{11}}, \tag{35}$$

[12] See Theorem 1 of Giacomini and White (2006) for the asymptotic justification of the test.

where $T_0 = T - T_1$ and T_{ij} denotes the number of observations with a j following an i. The maximum likelihood estimates of π_{i1} are $\hat{f}_{i1} = T_{i1}/T_i$ for all i. The LR statistic for the null hypothesis of independence, $\pi_{01} = \pi_{11}$, is then

$$LR_{ind} = 2\{\ln L(\hat{\pi}_{01}, \hat{\pi}_{11}) - \ln L(\hat{\pi}_1)\}, \tag{36}$$

which is again asymptotically distributed as a $\chi^2(1)$ under the null hypothesis.[13]

The two tests for the unconditional coverage and independence can be combined in one test with the null hypothesis of $\pi_{01} = \pi_{11} = \alpha$. Christoffersen (1998) refers to this test as the test of conditional coverage. The LR statistic of the conditional coverage is

$$LR_{cc} = LR_{uc} + LR_{ind} = 2\{\ln L(\hat{\pi}_{01}, \hat{\pi}_{11}) - \ln L(\alpha)\}, \tag{37}$$

which is asymptotically distributed as a $\chi^2(2)$ under the null hypothesis of $\pi_{01} = \pi_{11} = \alpha$. Although the above test considers the clustered violations, which is an important signal of risk model misspecification, the first-order Markov alternative represents a limited form of clustering.

The implicit assumption of the independent VaR violations in Kupiec (1995)'s LR test and the restrictive first order Markov alternative in the independence and conditional coverage tests are not usually satisfied in practice.[14] Consequently, these tests may not be suited for the model evaluation and we need more general tests to evaluate the VaR forecasts.

Christoffersen and Pelletier (2004) propose more general tests for the clustering based on the duration of days between the violations of the VaR. Define the duration of time (the number of days) between two VaR violations as

$$D_i = t_i - t_{i-1}, \tag{38}$$

where t_i denotes the day of the i-th violation. Under the null hypothesis of independent VaR violations, the duration has no memory and its mean of $1/\alpha$ days. The exponential distribution is the only continuous distribution with these properties. Under the null hypothesis, the likelihood of the durations is then

$$f_{\exp}(D; \alpha) = \alpha \exp(-\alpha D). \tag{39}$$

[13]If the sample has $T_{11} = 0$, which may happen in small samples with small α, the likelihood is computed as $L(\pi_{01}, \pi_{11}) = (1 - \pi_{01})^{T_0 - T_{01}} \pi_{01}^{T_{01}}$.

[14]We thank the editor, Esther Ruiz, and the anonymous associate editor, for pointing out this concern.

As a simple alternative of dependent durations, we consider the Weibull distribution which includes the null of exponential distribution as a special case. Under the Weibull alternative, the distribution of the duration is

$$f_W(D; a, b) = a^b b D^{b-1} \exp\{-(aD)^b\}, \tag{40}$$

which becomes the exponential one with probability parameter a when $b = 1$. The null hypothesis is then $b = 1$ in this case. This test can capture the higher-order dependence in the VaR violations by testing the unconditional distribution of the durations.

To test the conditional dependence of the VaR violations, we consider the exponential autoregressive conditional duration (EACD) framework of Engle and Russell (1998). The simple EACD(1,0) model characterizes the conditional expected duration, ψ_i, as

$$\psi_i = \mathrm{E}[D_i] = c + dD_{i-1}, \tag{41}$$

where $d \in [0, 1)$. Assuming the exponential distribution with mean one for the error term, $D_i - \psi_i$, the conditional distribution of the duration is

$$f_{EACD}(D_i|\psi_i) = \frac{1}{\psi_i} \exp\left(-\frac{D_i}{\psi_i}\right). \tag{42}$$

The null hypothesis of the independent durations is then $d = 0$ against the alternative of the conditional durations.

To implement the (un)conditional duration tests, we need to compute the likelihood of the durations with a different treatment for the first and last durations. Let C_i indicate if a duration is censored ($C_i = 1$) or not ($C_i = 0$). For the first observation, if the violation does not occur, then D_1 is the number of days until the first violation occurs and $C_1 = 1$ because the observed duration is left-censored. If instead the violation occurs at the first day, then D_1 is the number of days until the second violation and $C_1 = 0$. The similar procedure is applied to the last duration, $D_{N(T)}$. If the violation does not occur for the last observation, then $D_{N(T)}$ is the number of days after the last violation and $C_{N(T)} = 1$ because the observed duration is right-censored. If instead the violation occurs at the last day, then $D_{N(T)} = t_{N(T)} - t_{N(T)-1}$ and $C_{N(t)} = 0$. For the rest of observations, D_i is the number of days between each violation and $C_i = 0$.

The log-likelihood under the distribution, f, is then

$$\begin{aligned} \ln L(D; \Theta) = C_1 \ln S(D_1) + (1 - C_1) \ln f(D_1) + \sum_{i=2}^{N(T)-1} \ln f(D_i) \\ + C_{N(T)} \ln S(D_{N(T)}) + (1 - C_{N(T)}) \ln f(D_{N(T)}), \end{aligned} \tag{43}$$

where we use the survival function $S(D_i) = 1 - F(D_i)$ for a censored observation since it is unknown whether the process lasts at least D_i days. The parameters of the likelihood under the alternative specifications (a and b of the Weibull distribution and c and d of the EACD(1,0) model) need to be estimated numerically since the maximum likelihood estimates has no closed form solutions.

Because the sample size is not large and EACD(1,0) model has a potential difficulty to obtain the asymptotic distribution, we take the Monte Carlo testing technique of Dufour (2006) and follow the specific testing procedure of the LR tests by Christoffersen and Pelletier (2004).

4.2.2 Predictive ability test

As pointed out by the anonymous associate editor, the likelihood ratio tests described above are designed to see how a model performs for specific nominal theoretical values of duration and VaR. This means that these tests are not useful to state that one model is more accurate than the others. Therefore, we also evaluate the VaR forecasts using the predictive ability test described in Section 4.1.

Following Clements, Galvão, and Kim (2008), we define a loss function of model m as

$$L_t^\alpha(m) = [\alpha - 1\{r_t < \mathrm{VaR}_t(\alpha)\}][r_t - \mathrm{VaR}_t(\alpha)], \tag{44}$$

where $1\{\cdot\}$ denotes an indicator function. Then, the loss difference between models m_1 and m_2 is given by

$$\Delta L_t^\alpha(m_1, m_2) = L_t^\alpha(m_1) - L_t^\alpha(m_2). \tag{45}$$

Using the loss difference, we can compute the Wald-type statistic in (31) and test the null of equal coditional predictive ability of models m_1 and m_2.

4.3 Evaluating Expected Shortfall

To evaluate the ES forecasts with probability α, we use the measure proposed by Embrechts, Kaufmann, and Patie (2005). Define $\delta_t(\alpha) = r_t - ES_t(\alpha)$ and $\kappa(\alpha)$ as a set of time points for which a violation of the VaR occurs. Further, define $\tau(\alpha)$ as a set of time points for which $\delta_t(\alpha) < q(\alpha)$ occurs, where $q(\alpha)$ is the empirical α-quantile of $\delta_t(\alpha)$. The measure is then defined as

$$V(\alpha) = \frac{|V_1(\alpha)| + |V_2(\alpha)|}{2}, \tag{46}$$

where

$$V_1(\alpha) = \frac{1}{T_1} \sum_{t \in \kappa(\alpha)} \delta_t(\alpha), \quad V_2(\alpha) = \frac{1}{T_2} \sum_{t \in \tau(\alpha)} \delta_t(\alpha), \tag{47}$$

and T_1 and T_2 are the numbers of time points in $\kappa(\alpha)$ and $\tau(\alpha)$, respectively.

$V_1(\alpha)$ provides the standard backtesting measure using the VaR estimates. Since only the values with the violations are considered, this measure strongly depends on the VaR estimates without adequately reflecting the correctness of these values. To correct this weakness, a penalty term $V_2(\alpha)$, which evaluates the values which should happen once every $1/\alpha$ days, is combined with $V_1(\alpha)$. Finally, note that better ES estimates provide lower values of both $|V_1(\alpha)|$ and $|V_2(\alpha)|$ and so for $V(\alpha)$.

5 Empirical Studies

We apply the RSV model to daily (close-to-close) returns and RKs of the U.S. and Japanese stock indices, DJIA and Nikkei 225, respectively. The DJIA data is obtained from Oxford-Man Institute and the Nikkei 225 data is constructed from Nikkei NEEDS-TICK data.[15] The DJIA sample contains 2,884 trading days from January 4, 2000 through July 29, 2011 whereas the Nikkei 225 sample contains 3,336 trading days from June 5, 1997 to December 30, 2010. Figure 1 shows the time-series plot of the daily returns and logarithms of the RKs for both series.

Table 1 shows the descriptive statistics of the daily returns (r) and logarithms of RKs ($\ln RK$). For both DJIA and Nikkei 225, the mean of r is not statistically significant from zero and its Ljung-Box (LB) statistic does not reject the null hypothesis of no autocorrelation up to 10 lags, which allows us to estimate the RSV models using the daily returns without adjustment of mean and autocorrelation. The kurtosis of r shows that its distribution is leptokurtic as commonly observed in the financial returns and the Jacque-Bera (JB) statistic rejects its normality. The skewness of r is not statistically significant from zero for DJIA whereas it is significantly negative for Nikkei 225. In the RSVskt model, the leptokurtosis of r_t may be explained by stochastic volatility but the distribution of $\beta(z_t - \mu_z) + \sqrt{z_t}\epsilon_t$ may also be leptokurtic and skewed.

For both DJIA and Nikkei 225, the LB statistic of $\ln RK$ rejects the null of no autocorrelation, which is consistent with the high persistence of volatility known as the volatility

[15]The RKs for Nikkei 225 are calculated from 1-minute returns with the Parzen kernel in (9). See, for example, Ubukata and Watanabe (2014) for details.

clustering. The skewness of $\ln RK$ is significantly positive and its kurtosis shows that the distribution of $\ln RK$ is leptokurtic. Consequently, the JB statistic rejects the normality of $\ln RK$. This contradicts the normality assumption for u_t and η_t in (21) but we stick to the normality assumption in this paper and leave alternative specifications for future research.

In the following sections, we present the estimation and prediction results. In Section 5.1, we show the estimation results of the RSV models using all samples and compare the models by the marginal likelihood. In Section 5.2, we show the results of the volatility and quantile forecasts obtained by the rolling window estimation.

5.1 Estimation Results

Using the full sample of daily returns and RKs of DJIA and Nikkei 225, respectively, we estimate the RSV models with the priors for the parameters as follows,

$$\mu \sim N(0,10), \quad \beta \sim N(0,1), \quad \nu \sim Gamma(5,0.5)I(\nu > 4), \tag{48}$$

$$\xi \sim N(0,1), \quad \sigma_u^{-2} \sim Gamma(2.5,0.1), \tag{49}$$

$$\frac{\phi+1}{2} \sim Beta(20,1.5), \quad \sigma_\eta^{-2} \sim Gamma(2.5,0.025), \quad \frac{\rho+1}{2} \sim Beta(1,2). \tag{50}$$

Table 2 summarizes the MCMC estimation results of the RSV models with normal, Student's t, and skew t distributions obtained by 20,000 samples recorded after discarding 5,000 samples from MCMC iterations.[16] CD is the p-value of the convergence diagnostic test by Geweke (1992). All values indicate that the convergence of the posterior samples is not rejected at 5% level. The inefficiency factor measures how well the MCMC chain mixes.[17] Its values show that the chain is reasonably efficient and the 20,000 posterior samples are large enough to give a statistical inference.

For both DJIA and Nikkei 225, the parameters in the latent volatility equation (19) are consistent with the stylized features in the volatility literature. The posterior mean of ϕ is close to one for all models, which indicates the high persistence of volatility. Additionally, the posterior mean of ρ is negative and the 95% credible interval does not contain zero for

[16] All calculations in this paper are done by using Ox of Doornik (2009).

[17] The inefficiency factor is defined as $1 + 2\sum_{s=1}^{\infty} \rho_s$, where ρ_s is the sample autocorrelation at lag s. It is the ratio of the numerical variance of the posterior sample mean to the variance of the posterior sample mean from uncorrelated draws. The inverse of the inefficiency factor is also known as relative numerical efficiency (See, for example, Chib (2001)). When the inefficiency factor is equal to x, we need to draw MCMC samples x times as many as uncorrelated samples to obtain the same accuracy.

all models, which confirms the volatility asymmetry. The posterior mean of μ is similar among models using the same data.

For DJIA, the posterior mean of β is negative and the 95% credible interval does not contain zero, which appears to contradict the insignificant skewness of the daily returns for DJIA shown in Table 1. We attribute this seemingly contradictory result to the difference between unconditional and conditional distribution of the daily returns. In Figure 1, large negative returns followed by large positive returns are observed in some periods such as the Lehman crisis in 2008, which results in the (almost) symmetric unconditional distribution of the returns, that is, the insignificant skewness. On the other hand, the initial negative return (r_t) may not be fully explained by the stochastic volatility component (h_t) and the remaining part may be explained by the return shock component ($\beta(z_t - \mu_z) + \sqrt{z_t}\epsilon_t$). The negative return increases the subsequent volatilities (h_{t+1}, $h_{t+2}, \ldots$), which may explain the most part of the subsequent large positive returns. As a result, the large positive returns are mostly explained by the stochastic volatility component whereas the large negative returns are largely explained by the return shock component. Consequently, the conditional distribution of returns becomes negatively skewed, which is captured by the negative value of β.

For Nikkei 225, the posterior mean of β is negative but the 95% credible interval contains zero. Again, this result seemingly contradicts the significant negative skewness of the daily returns for Nikkei 225 shown in Table 1. From the argument above, this result implies that the stochastic volatility component explains the most part of the return variation irrespective of its sign. We argue that such an opposite result is due to the data characteristics. For DJIA and Nikkei 225, the means of $\ln RK$ are -0.361 and -0.076, respectively, whereas the standard deviations are 0.982 and 0.840. That is, Nikkei 225 shows larger volatility with less variation than DJIA, which is also clear from the posterior mean of μ in Table 2. Thus, for Nikkei 225, the stochastic volatility component takes larger value and explains the most of return variation even in the volatile period. Consequently, the conditional distribution of returns becomes less skewed and β becomes closer to zero.

The posterior mean of ν is around 23 for DJIA and it is around 30 for Nikkei 225, which implies that the fat tail is mostly explained by the stochastic volatility component. The large value of ν is not consistent with the previous studies. For example, Nakajima and Omori (2012) estimate the SV model with the GH skew Student's t-distribution and report that the posterior mean of ν is around 13 for the S&P500 returns from January 1970 to December 2003. We attribute such a difference to the persistence of the return shock in the data. The data in Nakajima and Omori (2012) contains a quite large but temporal shock, Black Monday shock in 1987, whereas our dataset contains the Lehman crisis, which persists for relatively longer period as shown in Figure 1. As a result, the temporal shock in the former data is explained by the small value of ν while the persistent shock in the latter is explained by the stochastic volatility component.

The parameters in the RV equation (18), ξ and σ_η, are almost same among models using the same data. The posterior means of ξ are negative and the credible intervals do not contain zero for all models, showing the downward bias of the RK mainly due to the non-trading hours.

For model comparisons, we compute the marginal likelihoods of the RSV models by the method of Chib (2001).[18] Table 3 shows the marginal likelihood estimates. For DJIA, the RSVskt model provides the highest marginal likelihood whereas the RSVt model does not improve the model fit compared to the RSVn model. This is consistent with the negative estimate of β for the RSVskt model and larger values of ν for the RSVt and RSVskt models. On the other hand, for Nikkei 225, neither the RSVskt model nor the RSVt model improves the model fit, which is again consistent with the credible interval of β containing zero and larger values of ν for the models. Overall, incorporating the negative skewness and leptokurtosis in the return distribution improves the model fit depending on the data characteristics.

5.2 Prediction Results

We estimate the volatility and quantile forecasts using a rolling window estimation scheme with the window size fixed. For DJIA, the fixed window size is 1,989 and the last observation dates vary from December 31, 2007 to July 28, 2011. For Nikkei 225, the window size is 1,985 and the last observation dates vary from June 30, 2005 to December 29, 2011. For each estimation, we compute one-day-ahead forecasts of volatility, VaR, and ES from 15,000 posterior samples.[19] Eventually, we obtain 895 prediction samples from January 2, 2008 to July 29, 2011 for DJIA and 1,350 samples from July 1, 2005 to December 30 for Nikkei 225.

[18]See Appendix B for a brief description of the procedure to calculate the marginal likelihood.

[19]From the second estimation, we use the posterior means obtained from the previous period as the initial values and generate 15,000 posterior samples after discarding 1,500 burn-in samples.

5.2.1 Volatility forecasts

Table 4 shows the MSE and QLIKE of the volatility forecasts with the RK as a proxy of the latent volatility. Following Hansen and Lunde (2005), we adjust the effect of the non-trading hours on the RK as in (10). The volatility forecasts and the adjusted RKs are shown in Figure 2. The RSVskt and RSVt models provide the least MSE and QLIKE for DJIA and Nikkei 225, respectively.

To test if the difference is statistically significant, we implement the predictive ability test, described in Section 4.1, using constant and lagged loss difference as test functions. The Wald-type statistics W_T in (31), which tests the null of equal predictive ability of the RSVn model against the RSVt and RSVskt models, are given in Table 4. For DJIA, the test reveals that predictive abilities, measured by QLIKE, of the RSVt and RSVskt models significantly outperform the RSVn model at significance level 10% and 1%, respectively. However, we cannot see the significant difference between other pairs of models in QLIKE and between all pairs of models in MSE.

On the other hand, for Nikkei 225, the RSVt model significantly outperform the RSVn model at significance level 5% for both MSE and QLIKE. Although the statistic is not shown in Table 4, we confirm that the RSVt model also outperform the RSVskt model at significance level 10 %. Overall, either the RSVt or RSVskt model improves volatility forecasts for both DJIA and Nikkei 225.

5.2.2 Quantile forecasts

Table 5 shows the empirical failure rates of VaR forecasts for target probabilities $\alpha \in \{0.01, 0.05\}$. $\hat{\pi}_1$ is an empirical probability of VaR violations. $\hat{\pi}_{01}$ is the empirical probability of VaR violations conditional on no VaR violation on previous day while $\hat{\pi}_{11}$ is the one conditional on VaR violation on previous day. For both DJIA and Nikkei 225, the empirical failure rates are higher than the target probabilities (α) due to the VaR violations in a volatile period from 2008 through 2009 as depicted in Figure 3. However, the failure rates of the RSVskt model are closer to the target probabilities than those of the RSVn and RSVt models.

Table 5 also shows the finite sample p-values of the likelihood ratio tests described in Section 4.2.[20] Column UC shows the p-values of the LR statistic for the unconditional coverage test, LR_{uc} in (33), with the null of $\pi_1 = \alpha$. Reflecting that the failure rates ($\hat{\pi}_1$)

[20]We compute the finite sample p-values based on the Monte Carlo testing technique of Dufour (2006).

of the RSVskt model are closer to α, the p-values of the model are slightly higher than those of the RSVn and RSVt models for DJIA. Column IND shows the p-values of the LR statistic for the independence test, LR_{ind} in (36), with the null of $\pi_{01} = \pi_{11}$. The null of independence is not rejected at 5% except the RSVn model for Nikkei 225. Column CC shows the p-values of the LR statistic of the conditional coverage test, LR_{cc} in (37). The null hypothesis, $\pi_{01} = \pi_{11} = \alpha$, is rejected for all cases except the RSVskt model with $\alpha = 0.01$ for DJIA.

The results of the duration-based tests are in Columns W and EACD, which shows the p-values of the LR statistic of the duration-based tests with the null of independent VaR violations under which the likelihood of the durations is given by (39). W denotes the alternative of the Weibull distribution for the unconditional durations, which results in the likelihood in (40), whereas EACD denotes the alternative of the EACD(1,0) model for the conditional expected duration in (41), which results in the conditional distribution of the duration in (42). The p-values of the Weibull and EACD tests exceed 5% for all cases except the RSVn and RSVt models at $\alpha = 0.05$ for Nikkei 225.

To test if the predictive performance is significantly different, we implement the predictive ability test, described in Section 4.2, using constant and lagged loss difference as test functions. The p-values of the Wald-type statistic W_T in (31), which tests the null of equal predictive ability of the RSVn model against the RSVt and RSVskt models, are given in Column W_T of Table 5. For DJIA, the test reveals that predictive ability of the RSVskt model significantly outperforms the RSVn model at significance level 5% for $\alpha = 0.01$ whereas the RSVt model significantly outperforms the RSVn model at level at 10% for $\alpha = 0.05$. However, we cannot see the significant difference between other paris of models for all $\alpha \in \{0.01, 0.05\}$.

On the other hand, for Nikkei 225, the RSVskt model outperforms the RSVn model at significance level 1% for all $\alpha \in \{0.01, 0.05\}$ and the RSVt model does the RSVn model at 1% for $\alpha = 0.01$. Moreover, although the statistic is not shown in Table 5, we confirm that the RSVskt model outperform the RSVt model at significance level 1% for $\alpha = 0.05$ and at 5% for $\alpha = 0.01$.

Table 6 shows the backtesting measures of the ES forecasts proposed by Embrechts, Kaufmann, and Patie (2005). The RSVskt model shows the best performance, followed by the RSVt model, for all null probabilities $\alpha \in \{1\%, 5\%\}$. This indicates the importance of the fat tail and skewness in the return distribution. That is, the extended model also improves the ES forecasts.

These results show that the RSVskt model provides better VaR and ES forecasts. This implies that the skewed and heavy-tailed error distribution is important when estimating the return quantiles. Overall, the extended model, either the RSVt or RSVskt model, improves the quantile forecasts for both DJIA and Nikkei 225.

5.2.3 Cumulative loss difference

As pointed out by the anonymous associate editor, it is worthwhile to see when the assumption of normality of the returns makes the volatility and quantile predictions worse. To this end, we compute the cumulative loss difference,

$$CLD_s = \sum_{t=1}^{s} \Delta L_{n+t}, \quad s = 1, \dots, T, \tag{51}$$

where n and T denote the last observation of the estimation and prediction samples, respectively. We calculate CLD for the volatility forecasts using MSE and QLIKE in (28) as well as for the VaR forecasts using the loss difference in (45).

Figure 5 shows the CLD for the volatility forecasts of the RSVn model against the RSVt model (red line) and RSVskt model (blue line). The top panels show the CLD for MSE as well as RKs with the adjustment of Hansen and Lunde (2005) whereas the bottom panels show the CLD for QLIKE as well as logarithms of the RKs. For DJIA, the CLD of MSE shows a notable leap in late 2008, which indicates that the volatility forecasts of the RSVn model are worse than those of the RSVt and RSVskt models. The leaps clearly coincide with a rise of RK.

Additionally, the CLD of MSE and QLIKE show a jump in mid 2010, associated with a jump of RK due to the flash crash on May 6. At this point, the blue line rises but the red line drops, which indicates that the RSVskt model provides better volatility forecast whereas the RSVt model does worse. In contrast to the rise of RK in late 2008, the rise of RK in mid 2010 is quite temporal, which implies that the RSVskt model may be less sensitive to such a temporal volatility jump. Moreover, the differences are relatively flat in the less volatile period such as mid 2008 and late 2010. Therefore, we argue that the extended model provides better volatility forecasts especially in the volatile period.

For Nikkei 225, the CLD of MSE also shows a notable jump in late 2008. The rise of the red line indicates that the RSVt model significantly outperforms the RSVn model whereas the drop of the blue line indicates that the RSVskt model underperforms the RSVn model. Although the blue line leaps right after the drop, the one time poor prediction associated with the drop deteriorates the overall volatility forecasting ability as shown in Table 4. the CLD is relatively flat or slightly decreasing in the less volatile period from mid 2005 to 2007. The qualitatively similar result applies to the CLD of QLIKE. These results confirm that the extended model provides better volatility forecasts especially in the volatile period.

Figure 6 shows the CLD for the VaR forecasts of the RSVn models against the RSVt and RSVskt models. The top and bottom panels show the CLD for VaR(1%) and VaR(5%), respectively. For DJIA, both panels show notable leaps during late 2008 to early 2009 as well as mid 2010, which indicates that the normality assumption on returns makes the VaR forecasts worse. The leaps again occur in the highly volatile period. On the other hand, the differences are sometimes decreasing, which indicates that the RSVn model provides better VaR forecasts, in the less volatile period such as late 2009. Thus, we also argue that the extended model is useful for the VaR forecasts especially in the volatile period.

For Nikkei 225, the CLD of VaR(1%) shows a notable leap in late 2008, which indicates the poor VaR forecasts of the RSVn model compared to the RSVt and RSVskt models. On the other hand, the CLD for VaR(5%) against the RSVskt model (blue line) is increasing from late 2008 whereas the CLD against the RSVt model (red line) does not show such a monotonic increase over the prediction period. This result indicates that the RSVskt model provides better VaR forecasts especially in the volatile period.

Overall, the normality assumption on the returns makes both volatility and quantile predictions worse in the volatile period. In other words, the extended model, either the RSVt or RSVskt, or both, is useful for both volatility and quantile forecasts especially in the volatile period. That is, the extended model is suited for conservative risk management which is important for large financial institutions such as commercial banks and pension funds.

6 Conclusion

The RSV model of Takahashi, Omori, and Watanabe (2009), which incorporates the asymmetric SV model with the RV estimator, is extended with the GH skew Student's t-distribution for financial returns. The extension makes it possible to consider the heavy tail and skewness in financial returns. With the Bayesian estimation scheme via Markov chain Monte Carlo method, the model enables us to estimate the parameters in the return distribution and in the model simultaneously. It also makes it possible to forecast the volatility and return quantiles by sampling from their posterior distributions jointly.

We apply the model to daily returns and RKs of the U.S. and Japanese stock indices, DJIA and Nikkei 225, and investigate its performance in volatility and quantile forecasts. The estimation results show that the extended model improves the model fit evaluated by the marginal likelihood for DJIA but not for Nikkei 225. Moreover, the prediction results show that the extended model improves both volatility and quantile forecasts especially in some volatile periods such as late 2008. Therefore, the extended model is suited for conservative risk management necessary for commercial banks and pension funds.

The RSV model can be extended further in several directions. Recently, Trojan (2013) proposes a regime switching RSVskt model and confirms several regimes in the S&P 500 index data. We can also consider different types of skew Student's t-distribution such as Fernández and Steel (1998) and Azzalini and Capitanio (2003). Additionally, extending the univariate RSV model to the multivariate model enables the portfolio risk management and optimal portfolio selection. Moreover, modeling multiple RV estimators with different frequencies and/or different computational methods may improve the volatility and quantile prediction as well as the model fit. In fact, Hansen and Huang (2012) introduce the realized exponential GARCH model, which can utilize multiple RV estimators, and show that the model with multiple RV estimators dominates the one with a single RV estimator. We leave these extensions for future research.

Appendices

A MCMC Sampling Procedure

Consider the RSVskt model in (17)-(21). Let $\theta = (\phi, \sigma_\eta, \rho, \mu, \beta, \nu, \xi, \sigma_u)$, $y = \{r_t, x_t\}_{t=1}^n$, $h = \{h_t\}_{t=1}^n$, $z = \{z_t\}_{t=1}^n$, and $\Theta = (\theta, h, z)$. We denote a prior distribution of an arbitrary variable w as $f(w)$ and its (conditional) posterior as $f(w|\cdot)$. Given y, the full posterior density is

$$
\begin{aligned}
f(\Theta|y) &\propto f(r|\Theta) \times f(x|\theta, h) \times f(h|z, \theta) \times f(z|\theta) \times f(\theta) && (52)\\
&= \prod_{t=1}^{n-1} f(r_t|\theta, h_t, h_{t+1}, z_t) f(r_n|\theta, h_n) \times \prod_{t=1}^{n} f(x_t|\theta, h_t) && (53)\\
&\quad \times f(h_1|\theta) \prod_{t=1}^{n-1} f(h_{t+1}|\theta, h_t) \times \prod_{t=1}^{n} f(z_t|\theta) \times f(\theta) && (54)\\
&= \prod_{t=1}^{n-1} f(h_{t+1}|\theta, h_t, z_t, r_t) \times f(h_1|\theta) \times \prod_{t=1}^{n} f(r_t|\theta, h_t, z_t) \times \prod_{t=1}^{n} f(x_t|\theta, h_t) && (55)\\
&\quad \times \prod_{t=1}^{n} f(z_t|\theta) \times f(\theta) && (56)\\
&\propto (1-\rho^2)^{-(n-1)/2} \sigma_\eta^{-(n-1)} \prod_{t=1}^{n-1} \exp\left\{ -\frac{(\bar{h}_{t+1} - \phi \bar{h}_t - \bar{r}_t)^2}{2(1-\rho^2)\sigma_\eta^2} \right\} && (57)\\
&\quad \times (1-\phi^2)^{1/2} \sigma_\eta^{-1} \exp\left\{ -\frac{(1-\phi^2)\bar{h}_1^2}{2\sigma_\eta^2} \right\} && (58)\\
&\quad \times (\beta^2 \sigma_z^2 + \mu_z)^{n/2} \prod_{t=1}^{n} z_t^{-1/2} \exp\left(-\frac{h_t}{2} - \frac{\tilde{r}_t^2}{2} \right) && (59)\\
&\quad \times \sigma_u^{-n} \prod_{t=1}^{n} \exp\left\{ -\frac{(x_t - \xi - h_t)^2}{2\sigma_u^2} \right\} && (60)\\
&\quad \times \left(\frac{\nu}{2} \right)^{n\nu/2} \Gamma\left(\frac{\nu}{2} \right)^{-n} \prod_{t=1}^{n} z_t^{-\nu/2+1} \exp\left(-\frac{\nu}{2z_t} \right) \times f(\theta), && (61)
\end{aligned}
$$

where

$$
\tilde{r}_t = \frac{\sqrt{\beta^2 \sigma_z^2 + \mu_z}\, r_t \exp(-h_t/2) - \beta \bar{z}_t}{\sqrt{z_t}}, \quad \bar{h}_t = h_t - \mu, \quad \bar{r}_t = \rho \sigma_\eta \tilde{r}_t, \quad \bar{z}_t = z_t - \mu_z \qquad (62)
$$

We can sample $w \in \Theta$ from the posterior density given other parameters and variables Θ_{-w}. Let $\theta_1 = (\phi, \sigma_\eta, \rho, \mu)$, $\theta_2 = (\beta, \nu)$, and $\theta_3 = (\xi, \sigma_u)$. We describe how to sample θ_1, θ_2, θ_3, z, and h in the following subsections.

A.1 Generation of θ_1

Given θ_2, h, and z, the full conditional density of θ_1 is

$$
\begin{aligned}
f(\theta_1|\theta_2, h, z, y) &\propto (1-\rho^2)^{-(n-1)/2} \sigma_\eta^{-(n-1)} \prod_{t=1}^{n-1} \exp\left\{ -\frac{(\bar{h}_{t+1} - \phi \bar{h}_t - \bar{r}_t)^2}{2(1-\rho^2)\sigma_\eta^2} \right\} && (63)\\
&\quad \times (1-\phi^2)^{1/2} \sigma_\eta^{-1} \exp\left\{ -\frac{(1-\phi^2)\bar{h}_1^2}{2\sigma_\eta^2} \right\} \times f(\theta_1), && (64)
\end{aligned}
$$

which is similar to the one for the SVskt model of Nakajima and Omori (2012). Thus, we follow the same sampling procedure described in Nakajima and Omori (2012) with different specifications of $\bar{r}_t$ defined in (62).

A.2 Generation of θ_2

Given θ_1, h, and z, the full conditional density of θ_2 is

$$f(\theta_2|\theta_1, h, z, y) \propto \prod_{t=1}^{n-1} \exp\left\{-\frac{(\bar{h}_{t+1} - \phi\bar{h}_t - \bar{r}_t)^2}{2(1-\rho^2)\sigma_\eta^2}\right\} \times (\beta^2\sigma_z^2 + \mu_z)^{n/2} \prod_{t=1}^{n} \exp\left(-\frac{\tilde{r}_t^2}{2}\right) \quad (65)$$

$$\times \left(\frac{\nu}{2}\right)^{n\nu/2} \Gamma\left(\frac{\nu}{2}\right)^{-n} \prod_{t+1}^{n} z_t^{-\nu/2+1} \exp\left(-\frac{\nu}{2z_t}\right) \times f(\theta_2). \quad (66)$$

Since it is not easy to sample from this density, we apply the Metropolis-Hastings (MH) algorithm based on a normal approximation of the density around the mode. We implement the MH sampling for β and ν separately.

A.3 Generation of θ_3

Given θ_1, θ_2, h, and z, the full conditional density of θ_3 is

$$f(\theta_3|\theta_1, \theta_2, h, z, y) \propto \sigma_u^{-n} \prod_{t=1}^{n} \exp\left\{-\frac{(x_t - \xi - h_t)^2}{2\sigma_u^2}\right\} \times f(\theta_3). \quad (67)$$

Let the prior distributions of parameters in θ_3 be

$$\xi \sim N(m_\xi, s_\xi^2), \quad \sigma_u^{-2} \sim Gamma(n_u, S_u). \quad (68)$$

Then, we can sample the parameters in θ_3 from the following posterior distributions,

$$\xi|\sigma_u, h, y \sim N(\tilde{m}_\xi, \tilde{s}_\xi^2), \quad (69)$$

$$\sigma_u^2|\xi, h, y \sim Gamma(\tilde{n}_u, \tilde{S}_u), \quad (70)$$

where

$$\tilde{m}_\xi = \frac{s_\xi^2 \sum_{t=1}^{n}(x_t - h_t) + \sigma_u^2 m_\xi}{ns_\xi^2 + \sigma_u^2}, \quad \tilde{s}_\xi^2 = \frac{\sigma_u^2 s_\xi^2}{ns_\xi^2 + \sigma_u^2}, \quad (71)$$

$$\tilde{n}_u = \frac{n}{2} + n_u, \quad \tilde{S}_u = \frac{1}{2}\sum_{t=1}^{n}(x_t - \xi - h_t)^2 + S_u. \quad (72)$$

A.4 Generation of z

Given θ_1, θ_2, θ_3, and h, the full conditional density of z_t is

$$f(z_t|\theta_1, \theta_2, \theta_3, h, y) \propto g(z_t) \times z_t^{-\frac{\nu+1}{2}+1} \exp\left(-\frac{\nu}{2z_t}\right), \tag{73}$$

where

$$g(z_t) = \exp\left\{-\frac{\tilde{r}_t^2}{2} - \frac{(\bar{h}_{t+1} - \phi\bar{h}_t - \bar{r}_t)^2}{2(1-\rho^2)\sigma_\eta^2} I(t < n)\right\}, \tag{74}$$

and $I(\cdot)$ is an indicator function. Following Nakajima and Omori (2012), we use the MH algorithm. Specifically, we generate a candidate $z_t^* \sim IG((\nu+1)/2, \nu/2)$ and accept it with probability $\min\{g(z_t^*)/g(z_t), 1\}$.

A.5 Generation of h

We first rewrite the RSVskt model in (17)-(19) as

$$r_t = \{\beta(z_t - \mu_z) + \sqrt{z_t}\epsilon_t\} \exp(\alpha_t/2)\gamma, \quad t = 1, \ldots, n \tag{75}$$

$$x_t = c + \alpha_t + u_t, \quad t = 1, \ldots, n \tag{76}$$

$$\alpha_{t+1} = \phi\alpha_t + \eta_t, \quad t = 0, \ldots, n-1 \tag{77}$$

where $\alpha_t = h_t - \mu$, $\gamma = \exp(\mu/2)/\sqrt{\beta^2\sigma_z^2 + \mu_z}$, and $c = \xi + \mu$.

To sample the latent variable $(\alpha_1, \ldots, \alpha_n)$ efficiently, we use the block sampler by Shephard and Pitt (1997) and Omori and Watanabe (2008). First, we divide $(\alpha_1, \ldots, \alpha_n)$ into $K+1$ blocks $(\alpha_{k_{j-1}+1}, \ldots, \alpha_{k_j})$ for $j = 1, \ldots, K+1$ with $k_0 = 0$ and $k_{K+1} = n$, where $k_j - k_{j-1} \geq 2$. We select K knots $(k_1, \ldots, k_K)$ randomly and sample the error term $(\eta_{k_{j-1}}, \ldots, \eta_{k_{j-1}})$, instead of $(\alpha_{k_{j-1}+1}, \ldots, \alpha_{k_j})$, simultaneously from their full conditional distribution.

Suppose that $k_{j-1} = s$ and $k_j = s+m$ for the jth block and let $y_t = (r_t, x_t)$. Then $(\eta_s, \ldots, \eta_{s+m-1})$ are sampled simultaneously from the following full conditional distribution,

$$f(\eta_s, \ldots, \eta_{s+m-1}|\alpha_s, \alpha_{s+m+1}, y_s, \ldots, y_{s+m}) \propto \prod_{t=s}^{s+m} f(y_t|\alpha_t, \alpha_{t+1}) \prod_{t=s}^{s+m-1} f(\eta_t), \tag{78}$$

for $s + m < n$, and

$$f(\eta_s, \ldots, \eta_{s+m-1}|\alpha_s, y_s, \ldots, y_{s+m}) \propto \prod_{t=s}^{s+m-1} f(y_t|\alpha_t, \alpha_{t+1}) f(y_n|\alpha_n) \prod_{t=s}^{s+m-1} f(\eta_t), \tag{79}$$

for $s+m=n$. The logarithm of $f(y_t|\alpha_t,\alpha_{t+1})$ or $f(y_n|\alpha_n)$ in (78) and (79) (excluding constant term) is given by

$$l_t = -\frac{\alpha_t}{2} - \frac{(r_t-\mu_t)^2}{2\sigma_t^2} - \frac{(x_t-c-\alpha_t)^2}{2\sigma_u^2}, \tag{80}$$

where

$$\mu_t = \begin{cases} \{\beta\bar{z}_t + \sqrt{z_t}\rho\sigma_\eta^{-1}(\alpha_{t+1}-\phi\alpha_t)\}\exp(\alpha_t/2)\gamma, & t<n, \\ \beta\bar{z}_n\exp(\alpha_n/2)\gamma, & t=n, \end{cases} \tag{81}$$

and

$$\sigma_t^2 = \begin{cases} (1-\rho^2)z_t\exp(\alpha_t)\gamma^2, & t<n, \\ z_n\exp(\alpha_n)\gamma^2, & t=n. \end{cases} \tag{82}$$

Then the logarithm of the conditional density in (78) and (79) is given by (excluding a constant term)

$$\sum_{t=s}^{s+m-1}\log f(\eta_t) + L, \tag{83}$$

where

$$L = \begin{cases} \sum_{t=s}^{s+m} l_t - \dfrac{(\alpha_{s+m+1}-\phi\alpha_{s+m})^2}{2\sigma_\eta^2}, & s+m<n, \\ \sum_{t=s}^{s+m} l_t, & s+m=n. \end{cases} \tag{84}$$

Further, for $s \geq 0$, we define

$$\alpha = (\alpha_{s+1},\ldots,\alpha_{s+m})', \tag{85}$$

$$d = (d_{s+1},\ldots,d_{s+m})', \quad d_t = \frac{\partial L}{\partial \alpha_t}, \quad t=s+1,\ldots,s+m, \tag{86}$$

$$Q = -E\left[\frac{\partial^2 L}{\partial\alpha\partial\alpha'}\right] = \begin{bmatrix} A_{s+1} & B_{s+2} & 0 & \cdots & 0 \\ B_{s+2} & A_{s+2} & B_{s+3} & \cdots & 0 \\ 0 & B_{s+3} & A_{s+3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & B_{s+m} \\ 0 & \cdots & 0 & B_{s+m} & A_{s+m} \end{bmatrix}, \tag{87}$$

$$A_t = -E\left[\frac{\partial^2 L}{\partial\alpha_t^2}\right], \quad t=s+1,\ldots,s+m, \tag{88}$$

$$B_t = -E\left[\frac{\partial^2 L}{\partial\alpha_t\partial\alpha_{t-1}}\right], \quad t=s+2,\ldots,s+m, \quad B_{s+1}=0. \tag{89}$$

The first derivative of L with respect to α_t is given by

$$d_t = -\frac{1}{2} + \frac{(r_t - \mu_t)^2}{2\sigma_t^2} + \frac{r_t - \mu_t}{\sigma_t^2}\frac{\partial \mu_t}{\partial \alpha_t} + \frac{r_{t-1} - \mu_{t-1}}{\sigma_{t-1}^2}\frac{\partial \mu_{t-1}}{\partial \alpha_t} + \frac{(x_t - c - \alpha_t)}{\sigma_u^2} + \kappa(\alpha_t), \quad (90)$$

where

$$\frac{\partial \mu_t}{\partial \alpha_t} = \begin{cases} \left\{\frac{\beta \bar{z}_t}{2} + \sqrt{z_t}\rho\sigma_\eta^{-1}\left(-\phi + \frac{\alpha_{t+1} - \phi\alpha_t}{2}\right)\right\}\exp(\alpha_t/2)\gamma, & t < n, \\ \frac{\beta \bar{z}_n \exp(\alpha_n/2)\gamma}{2}, & t = n, \end{cases} \quad (91)$$

$$\frac{\partial \mu_{t-1}}{\partial \alpha_t} = \begin{cases} 0, & t = 1, \\ \sqrt{z_{t-1}}\rho\sigma_\eta^{-1}\exp(\alpha_{t-1}/2)\gamma, & t = 2, \ldots, T. \end{cases} \quad (92)$$

$$\kappa(\alpha_t) = \begin{cases} \frac{\phi(\alpha_{t+1} - \phi\alpha_t)}{\sigma_\eta^2}, & t = s + m < n, \\ 0, & \text{otherwise.} \end{cases} \quad (93)$$

Taking expectations of second derivatives multiplied by -1 with respect to y_t's, we obtain the A_t's and B_t's as follows,

$$A_t = \frac{1}{2} + \sigma_t^{-2}\left(\frac{\partial \mu_t}{\partial \alpha_t}\right)^2 + \sigma_{t-1}^{-2}\left(\frac{\partial \mu_{t-1}}{\partial \alpha_t}\right)^2 + \frac{1}{\sigma_u^2} + \kappa'(\alpha_t), \quad (94)$$

$$B_t = \sigma_{t-1}^{-2}\frac{\partial \mu_{t-1}}{\partial \alpha_{t-1}}\frac{\partial \mu_{t-1}}{\partial \alpha_t}, \quad (95)$$

where

$$\kappa'(\alpha_t) = \begin{cases} \frac{\phi^2}{\sigma_\eta^2}, & t = s + m < n, \\ 0, & \text{otherwise.} \end{cases} \quad (96)$$

Applying the second order Taylor expansion to the conditional density (78) will produce the approximating normal density $f^*(\eta_s, \ldots, \eta_{s+m-1}|\alpha_s, \alpha_{s+m+1}, y_s, \ldots, y_{s+m})$ as follows (see Omori and Watanabe (2008) for details),

$$\log f(\eta_s, \ldots, \eta_{s+m-1}|\alpha_s, \alpha_{s+m+1}, y_s, \ldots, y_{s+m}) \quad (97)$$

$$\approx \text{const} - \frac{1}{2}\sum_{t=s}^{s+m-1}\eta_t^2 + \hat{L} + \left.\frac{\partial L}{\partial \eta}\right|_{\eta=\hat{\eta}}(\eta - \hat{\eta}) + \frac{1}{2}(\eta - \hat{\eta})' \left. E\left[\frac{\partial^2 L}{\partial \eta \partial \eta'}\right]\right|_{\eta=\hat{\eta}}(\eta - \hat{\eta}) \quad (98)$$

$$= \text{const} - \frac{1}{2}\sum_{t=s}^{s+m-1}\eta_t^2 + \hat{L} + \hat{d}'(\alpha - \hat{\alpha}) - \frac{1}{2}(\alpha - \hat{\alpha})'\hat{Q}(\alpha - \hat{\alpha}) \quad (99)$$

$$= \text{const} + \log f^*(\eta_s, \ldots, \eta_{s+m-1}|\alpha_s, \alpha_{s+m+1}, y_s, \ldots, y_{s+m}), \quad (100)$$

where $\eta = (\eta_s, \ldots, \eta_{s+m-1})'$, and $\hat{d}$, $\hat{L}$, and $\hat{Q}$ denote d, L, and Q evaluated at $\alpha = \hat{\alpha}$ (or, equivalently, at $\eta = \hat{\eta}$), respectively. The expectations are taken with respect to y_t's conditional on α_t's. Similarly, we can obtain the normal density which approximates the conditional density (79).

To make the linear Gaussian state-space model corresponding to the approximating density, we first compute the following D_t, K_t, J_t, and b_t for $t = s+2, \ldots, s+m$ recursively,

$$D_t = \hat{A}_t - D_{t-1}^{-1}\hat{B}_t^2, \quad D_{s+1} = \hat{A}_{s+1}, \tag{101}$$

$$K_t = \sqrt{D_t}, \tag{102}$$

$$J_t = \hat{B}_t K_{t-1}^{-1}, \quad J_{s+1} = 0, \quad J_{s+m+1} = 0, \tag{103}$$

$$b_t = \hat{d}_t - J_t K_{t-1}^{-1} b_{t-1}, \quad b_{s+1} = \hat{d}_{s+1}. \tag{104}$$

Second, we define auxiliary variables $\hat{y}_t = \hat{\gamma}_t + D_t^{-1} b_t$ where

$$\hat{\gamma}_t = \hat{\alpha}_t + K_t^{-1} J_{t+1} \hat{\alpha}_{t+1}, \quad t = s+1, \ldots, s+m. \tag{105}$$

Then the approximating density corresponds to the density of the linear Gaussian state-space model given by

$$\hat{y}_t = Z_t \alpha_t + G_t \zeta_t, \quad t = s+1, \ldots, s+m, \tag{106}$$

$$\alpha_{t+1} = \phi \alpha_t + H_t \zeta_t, \quad t = s, s+1, \ldots, s+m, \quad \zeta_t \sim \mathrm{N}(0, I), \tag{107}$$

where

$$Z_t = 1 + K_t^{-1} J_{t+1} \phi, \quad G_t = K_t^{-1}(1,\, J_{t+1}\sigma_\eta), \quad H_t = (0,\, \sigma_\eta). \tag{108}$$

We can sample $(\eta_s, \ldots, \eta_{s+m-1})$ from the full posterior distribution in (78) and (79) by applying the simulation smoother[21] to this state-space model and using Acceptance-Rejection (AR) MH algorithm proposed by Tierney (1994). See Omori and Watanabe (2008) and Takahashi, Omori, and Watanabe (2009) for the details of the ARMH algorithm.

B Marginal Likelihood

The marginal likelihood $m(y)$ is defined as the integral of the likelihood with respect to the prior density of the parameter,

$$m(y) = \int_\Theta f(y|\Theta) f(\Theta) d\Theta = \frac{f(y|\Theta) f(\Theta)}{f(\Theta|y)}, \tag{109}$$

[21] See, for example, de Jong and Shephard (1995) and Durbin and Koopman (2002).

where Θ is a parameter, $f(y|\Theta)$ is a likelihood, $f(\Theta)$ is a prior probability density, and $f(\Theta|y)$ is a posterior probability density.[22] Following Chib (1995), we estimate the logarithm of the marginal likelihood as

$$\log m(y) = \log f(y|\Theta) + \log f(\Theta) - \log f(\Theta|y). \tag{110}$$

Although the equality holds for any values of Θ, we use the posterior mean of Θ to obtain a stable estimate of $m(y)$.

Given the posterior sample of Θ, the prior density $f(\Theta)$ is easily calculated. However, the likelihood and posterior components must be evaluated by simulation. The likelihood is estimated using the auxiliary particle filter with 10,000 particles, which provides an unbiassed estimator at a particular ordinate Θ for $f(y|\Theta)$.[23] The likelihood estimate and its standard error are obtained as the sample mean and standard deviation of the likelihoods from 10 iterations. The posterior probability density and its numerical standard error are evaluated by the method of Chib and Greenberg (1995) and Chib and Jeliazkov (2001) with 50,000 reduced MCMC samples.

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[22]The last equality is obtained by the Bayes' rule,

$$f(\Theta|y) = \frac{f(y|\Theta)f(\Theta)}{\int_\Theta f(y|\Theta)f(\Theta)d\Theta}.$$

[23]See, for example, Pitt and Shephard (1999) and Omori, Chib, Shephard, and Nakajima (2007) for the details.

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Tables

Table 1: Descriptive statistics of the daily return (r) and logarithm of the realized kernel ($\ln RK$) for DJIA from January 4, 2000 to July 29, 2011 (2,884 samples) and for Nikkei 225 from June 5, 1997 to December 30, 2010 (3,336 samples). The DJIA data is obtained from Oxford-Man Institute and Nikkei 225 data is constructed from Nikkei NEEDS-TICK data. Standard errors of skewness and kurtosis for DJIA are 0.0456 and 0.0911, respectively, whereas those for Nikkei 225 are 0.0424 and 0.08476, respectively. JB is the p-value of the Jaque-Bera statistic to test the null hypothesis of normality. LB is the p-value of the Ljung-Box statistic adjusted for heteroskedasticity following Diebold (1988) to test the null hypothesis of no autocorrelation up to 10 lags.

DJIA

Variable	Mean	SE	SD	Skew	Kurt	Min	Max	JB	LB
r	0.002	0.024	1.264	0.008	10.414	-8.615	10.532	0.00	0.14
$\ln RK$	-0.361	0.018	0.982	0.640	3.860	-2.958	4.514	0.00	0.00

Nikkei 225

Variable	Mean	SE	SD	Skew	Kurt	Min	Max	JB	LB
r	-0.021	0.028	1.610	-0.217	8.319	-12.111	13.235	0.00	0.57
$\ln RK$	-0.076	0.015	0.840	0.158	3.655	-2.700	3.560	0.00	0.00

Table 2: MCMC estimation results of RSV models with normal, student's t, and skewed t distributions, using the full sample of daily returns and RKs for DJIA and Nikkei 225. The results are obtained by 20,000 samples recorded after discarding 5,000 samples from MCMC iterations (all calculations in this paper are done by using Ox of Doornik (2009)). 95%L and 95%U are the lower and upper quantiles of 95% credible interval, respectively. The last two columns are the p-value of the convergence diagnostic test by Geweke (1992) and the inefficiency factor by Chib (2001). Priors are set as $\mu \sim N(0, 10)$, $(\phi+1)/2 \sim Beta(20, 1.5)$, $\sigma_\eta^{-2} \sim Gamma(2.5, 0.025)$, $(\rho + 1)/2 \sim Beta(1, 2)$, $\beta \sim N(0, 1)$, $\nu \sim Gamma(5, 0.5)I(\nu > 4)$, $\xi \sim N(0, 1)$, $\sigma_u^{-2} \sim Gamma(2.5, 0.1)$.

DJIA (2,884 samples from January 4, 2000 to July 29, 2011)

Model		Mean	SD	95%L	Median	95%U	CD	Inef.
RSVn	ϕ	0.9650	0.0044	0.9560	0.9652	0.9736	0.480	4.84
	σ_η	0.2186	0.0079	0.2036	0.2183	0.2345	0.331	16.50
	ρ	-0.4825	0.0300	-0.5401	-0.4832	-0.4218	0.731	10.45
	μ	-0.0764	0.1100	-0.2908	-0.0778	0.1441	0.826	4.46
	ξ	-0.2032	0.0327	-0.2688	-0.2028	-0.1404	0.489	46.68
	σ_u	0.3964	0.0077	0.3817	0.3962	0.4120	0.812	5.20
RSVt	ϕ	0.9655	0.0045	0.9565	0.9656	0.9740	0.725	3.43
	σ_η	0.2165	0.0082	0.2010	0.2163	0.2330	0.242	15.42
	ρ	-0.5010	0.0314	-0.5617	-0.5015	-0.4376	0.761	12.19
	μ	-0.0579	0.1087	-0.2668	-0.0589	0.1563	0.348	6.27
	ν	22.5909	4.5608	14.7718	22.2752	32.1559	0.705	173.51
	ξ	-0.2139	0.0329	-0.2798	-0.2140	-0.1500	0.066	48.66
	σ_u	0.3980	0.0076	0.3832	0.3979	0.4132	0.950	5.57
RSVskt	ϕ	0.9660	0.0043	0.9573	0.9660	0.9742	0.145	6.62
	σ_η	0.2181	0.0078	0.2031	0.2179	0.2339	0.370	25.92
	ρ	-0.5248	0.0306	-0.5831	-0.5253	-0.4637	0.057	20.33
	μ	-0.0668	0.1080	-0.2761	-0.0684	0.1474	0.480	3.32
	β	-0.6292	0.2101	-1.0870	-0.6147	-0.2691	0.429	87.91
	ν	23.0995	4.2464	15.8961	22.7920	32.4364	0.142	221.19
	ξ	-0.2160	0.0346	-0.2847	-0.2155	-0.1501	0.826	42.33
	σ_u	0.3980	0.0075	0.3838	0.3979	0.4131	0.057	6.63

Table 2: MCMC estimation results of RSV models – Continued

Nikkei 225 (3,336 samples from June 5, 1997 to December 30, 2010)								
Model		Mean	SD	95%L	Median	95%U	CD	Inef.
RSVn	ϕ	0.9540	0.0049	0.9441	0.9540	0.9634	0.472	15.13
	σ_η	0.2158	0.0079	0.2004	0.2158	0.2316	0.670	34.79
	ρ	-0.3391	0.0282	-0.3943	-0.3392	-0.2840	0.758	17.99
	μ	0.5326	0.0703	0.3955	0.5319	0.6735	0.439	5.08
	ξ	-0.7236	0.0262	-0.7755	-0.7230	-0.6724	0.363	37.15
	σ_u	0.4062	0.0066	0.3933	0.4062	0.4192	0.418	14.71
RSVt	ϕ	0.9554	0.0047	0.9459	0.9555	0.9646	0.254	14.53
	σ_η	0.2120	0.0077	0.1968	0.2118	0.2273	0.197	41.72
	ρ	-0.3648	0.0288	-0.4216	-0.3648	-0.3083	0.649	24.01
	μ	0.5414	0.0707	0.4041	0.5406	0.6819	0.177	5.20
	ν	29.5778	5.7965	19.3176	29.2167	41.8819	0.163	244.39
	ξ	-0.7313	0.0259	-0.7829	-0.7312	-0.6815	0.343	31.24
	σ_u	0.4088	0.0065	0.3960	0.4087	0.4216	0.423	18.03
RSVskt	ϕ	0.9558	0.0048	0.9461	0.9558	0.9649	0.798	13.57
	σ_η	0.2110	0.0077	0.1964	0.2109	0.2264	0.553	37.09
	ρ	-0.3757	0.0292	-0.4323	-0.3760	-0.3174	0.429	15.24
	μ	0.5412	0.0707	0.4023	0.5405	0.6823	0.700	3.62
	β	-0.3146	0.2148	-0.7708	-0.3070	0.0773	0.236	50.71
	ν	31.4847	6.3439	21.3406	30.6827	46.3899	0.504	238.13
	ξ	-0.7334	0.0261	-0.7857	-0.7330	-0.6836	0.384	23.41
	σ_u	0.4096	0.0065	0.3970	0.4096	0.4227	0.694	13.60

Table 3: Marginal likelihood and its components, likelihood, prior, and posterior (in logarithm), for the RSV models estimated by using the full sample of daily returns and RKs of DJIA index (2,884 samples from January 4, 2000 through July 29, 2011) and Nikkei 225 index (3,336 samples from June 5, 1997 to December 30, 2010). The likelihood is estimated using the auxiliary particle filter of Pitt and Shephard (1999) with 10,000 particles. The likelihood estimate and its standard error are computed as the sample mean and standard deviation of the likelihoods from 10 iterations. The posterior probability density and its numerical standard error are evaluated by the method of Chib and Greenberg (1995) and Chib and Jeliazkov (2001) with 50,000 reduced MCMC samples. The numbers in the parentheses show the standard errors.

DJIA

Model	Likelihood		Prior	Posterior		Marginal	
RSVn	-5911.83	(0.69)	-1.18	20.30	(0.01)	-5933.30	(0.69)
RSVt	-5914.83	(0.64)	-6.69	15.99	(0.05)	-5937.50	(0.64)
RSVskt	-5906.16	(0.82)	-8.10	15.25	(0.05)	-5929.51	(0.82)

Nikkei 225

Model	Likelihood		Prior	Posterior		Marginal	
RSVn	-11030.86	(2.74)	-1.71	21.18	(0.04)	-11053.75	(2.74)
RSVt	-11037.17	(2.41)	-9.46	16.45	(0.25)	-11063.07	(2.43)
RSVskt	-11035.21	(2.26)	-11.11	15.92	(0.19)	-11062.24	(2.27)

Table 4: Mean squared error (MSE) and quasi-likelihood (QLIKE) of volatility forecasts for DJIA (895 prediction samples from January 2, 2008 to July 29, 2011) and NIkkei 225 (1,350 prediction samples from July 1, 2005 to December 30, 2010). Realized kernel with the adjustment of Hansen and Lunde (2005) are used as a proxy of latent volatility. Numbers in square brackets denote the Wald-type statistics W_T in (31) with the null of equal predictive ability of the RSVn model and a model specified in the first column. * and *** indicate that the predictive ability test rejects the null hypothesis at the significance level 10% and 1%, respectively. We implement the predictive ability test based on Giacomini and White (2006) using constant and lagged loss difference as test functions.

DJIA

Model	MSE		QLIKE	
RSVn	9.050		0.212	
RSVt	8.949	[3.035]	0.210	[4.960]*
RSVskt	8.911	[1.506]	0.208	[11.328]***

Nikke 225

Model	MSE		QLIKE	
RSVn	4.839		0.156	
RSVt	4.783	[7.697]**	0.155	[6.582]**
RSVskt	4.812	[1.198]	0.156	[3.819]

Table 5: Evaluation results of the VaR forecasts for DJIA and Nikkei225. α and $\hat{\pi}_1$ denote target and empirical probabilities of VaR violations, respectively. $\hat{\pi}_{01}$ is the empirical probability of VaR violations conditional on no VaR violation on previous day while $\hat{\pi}_{11}$ is the one conditional on VaR violation on previous day. UC denotes the unconditional coverage test with the null hypothesis of $\pi_1 = \alpha$. IND denotes the independence test which tests the null of $\pi_{01} = \pi_{11}$ against the alternative of the first-order Markov process. CC denotes the conditional coverage test which combines the UC and IND tests with the null of $\pi_{01} = \pi_{11} = \alpha$. W denotes the unconditional duration test which tests the null of independent violations against the alternative of the Weibull distribution for the distribution of the duration. EACD denotes the conditional duration test which tests the null of the independent durations against the alternative of the EACD(1,0) model for the conditional durations. We compute the finite sample p-values of the likelihood ratios of these tests based on the Monte Carlo testing technique of Dufour (2006). Numbers in the column W_T denote p-values of the Wald-type statistic W_T in (31) with the null of equal predictive ability of the RSVn model and a model specified in the first column.

DJIA (895 prediction samples from January 2, 2008 to July 29, 2011)

Model	α	$\hat{\pi}_1$	$\hat{\pi}_{01}$	$\hat{\pi}_{11}$	UC	IND	CC	W	EACD	W_T
RSVn	0.01	0.027	0.028	0.000	0.000	0.095	0.001	0.441	0.983	
RSVt	0.01	0.022	0.023	0.000	0.002	0.084	0.005	0.156	0.911	0.168
RSVskt	0.01	0.019	0.019	0.000	0.015	0.097	0.030	0.192	0.575	0.031
RSVn	0.05	0.077	0.077	0.058	0.001	0.069	0.000	0.064	0.066	
RSVt	0.05	0.078	0.079	0.057	0.000	0.074	0.000	0.075	0.084	0.060
RSVskt	0.05	0.074	0.074	0.061	0.003	0.075	0.001	0.057	0.101	0.113

Nikkei 225 (1,350 prediction samples from July 1, 2005 to December 30, 2010)

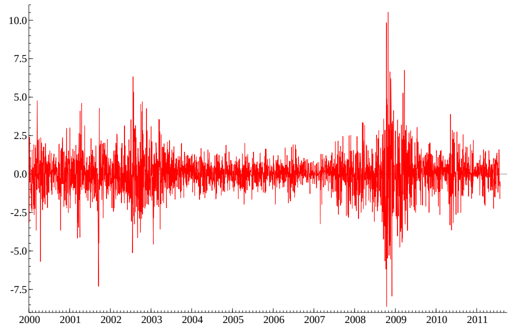
Model	α	$\hat{\pi}_1$	$\hat{\pi}_{01}$	$\hat{\pi}_{11}$	UC	IND	CC	W	EACD	W_T
RSVn	0.01	0.031	0.032	0.000	0.000	0.048	0.000	0.454	0.383	
RSVt	0.01	0.028	0.029	0.000	0.000	0.069	0.000	0.326	0.325	0.007
RSVskt	0.01	0.026	0.027	0.000	0.000	0.081	0.000	0.522	0.054	0.001
RSVn	0.05	0.101	0.105	0.073	0.000	0.276	0.000	0.179	0.041	
RSVt	0.05	0.102	0.105	0.080	0.000	0.390	0.000	0.204	0.040	0.663
RSVskt	0.05	0.098	0.100	0.076	0.000	0.385	0.000	0.258	0.051	0.001

Table 6: Backtesting measure of Embrechts, Kaufmann, and Patie (2005) for the expected shortfall forecasts of DJIA (895 prediction samples from January 2, 2008 to July 29, 2011) and NIkkei 225 (1,350 prediction samples from July 1, 2005 to December 30, 2010).

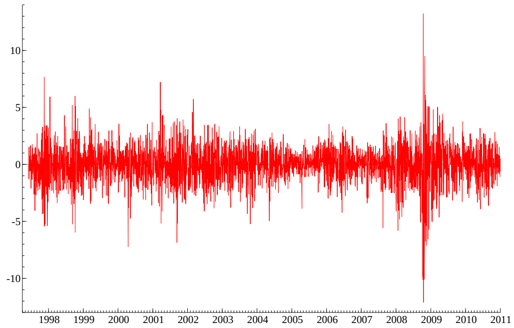
	DJIA		Nikkei 225	
Model	1%	5%	1%	5%
RSVn	0.2621	0.2721	0.4364	0.4806
RSVt	0.1743	0.2010	0.3285	0.4180
RSVskt	0.1437	0.1712	0.3072	0.3812

Figures

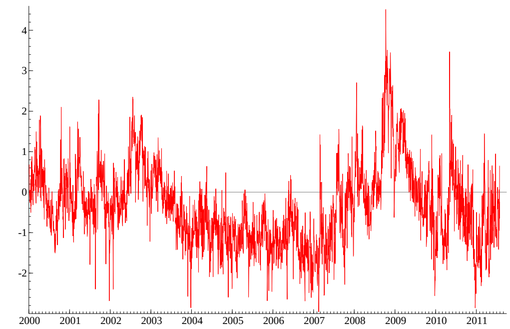
Figure 1: Time series plot of daily returns (r) and logarithms of realized kernels ($\ln RK$) for DJIA from January 4, 2000 though July 29, 2011 (2,884 samples) and Nikkei 225 from June 5, 1997 to December 30, 2010 (3,336 samples).



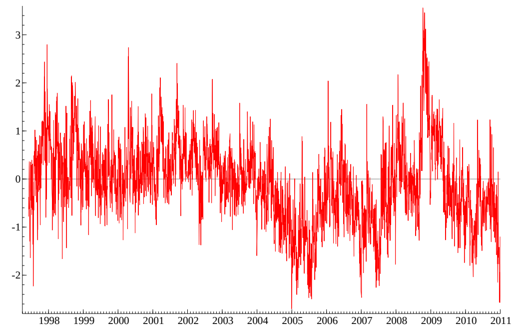
r for DJIA



r for Nikkei 225

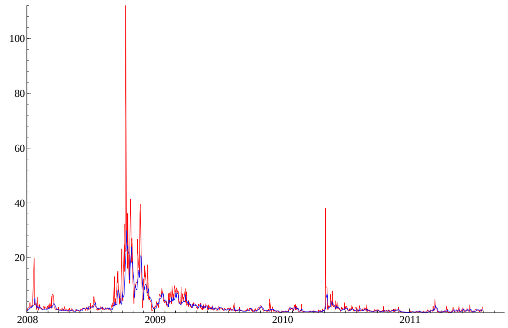


$\ln RK$ for DJIA

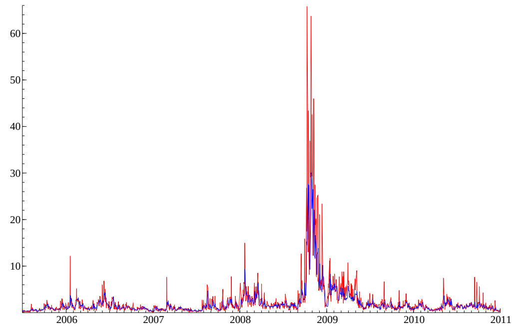


$\ln RK$ for Nikkei 225

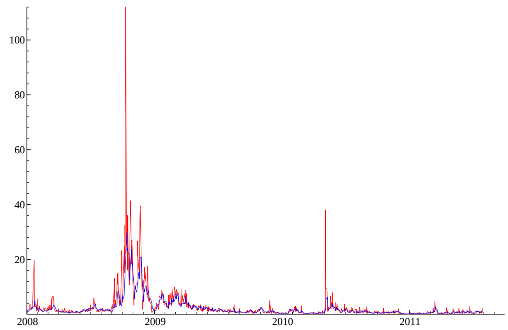
Figure 2: Volatility forecasts (blue line) and realized kernel with the adjustment of Hansen and Lunde (2005) (red line) for DJIA (895 prediction samples from January 2, 2008 to July 29, 2011) and NIkkei 225 (1,350 prediction samples from July 1, 2005 to December 30, 2010).



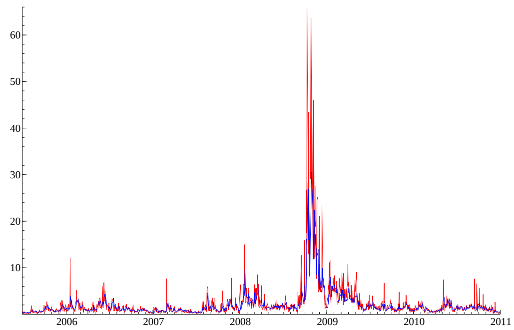
RSVn for DJIA



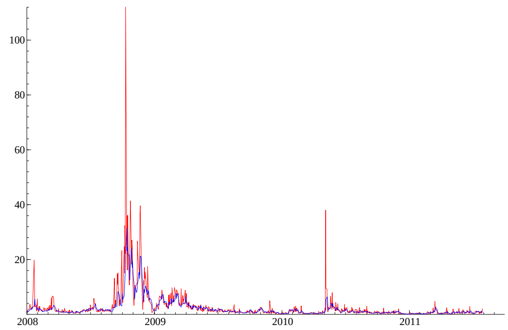
RSVn for NIkkei 225



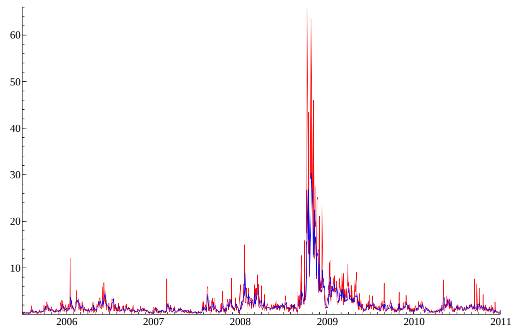
RSVt for DJIA



RSVt for NIkkei 225

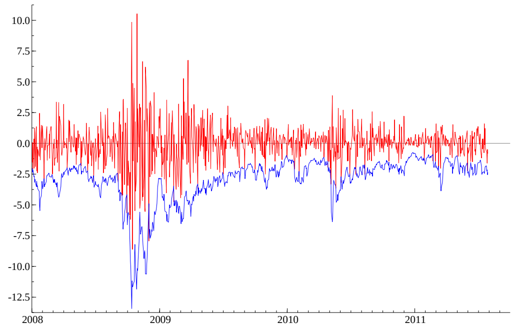


RSVskt for DJIA

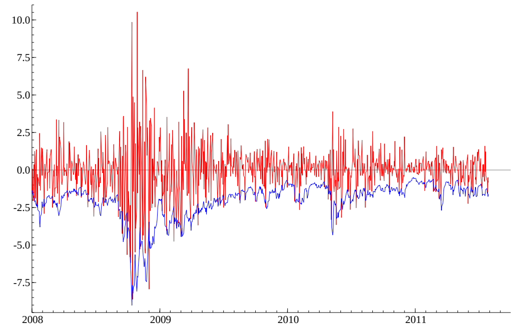


RSVskt for NIkkei 225

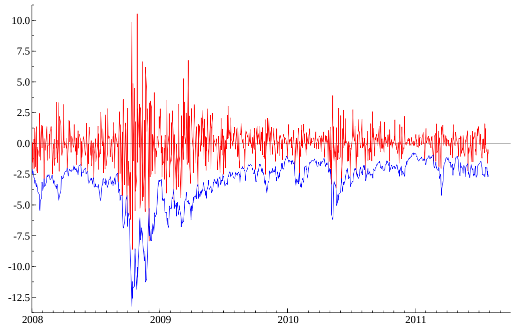
Figure 3: VaR forecasts (blue line) and daily returns (red line) for DJIA (895 prediction samples from January 2, 2008 to July 29, 2011).



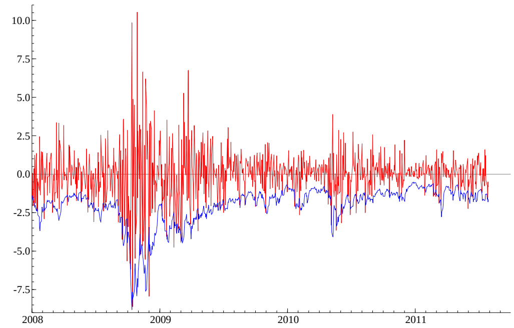
RSVn (1%)



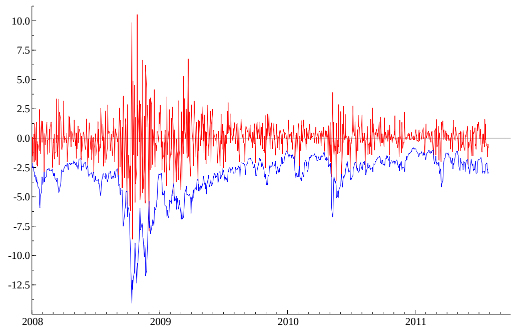
RSVn (5%)



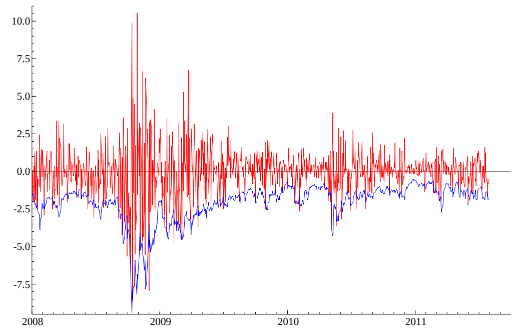
RSVt (1%)



RSVt (5%)



RSVskt (1%)



RSVskt (5%)

Figure 4: VaR forecasts (blue line) and daily returns (red line) for Nikkei 225 (1350 prediction samples from July 1, 2005 to December 30, 2010).

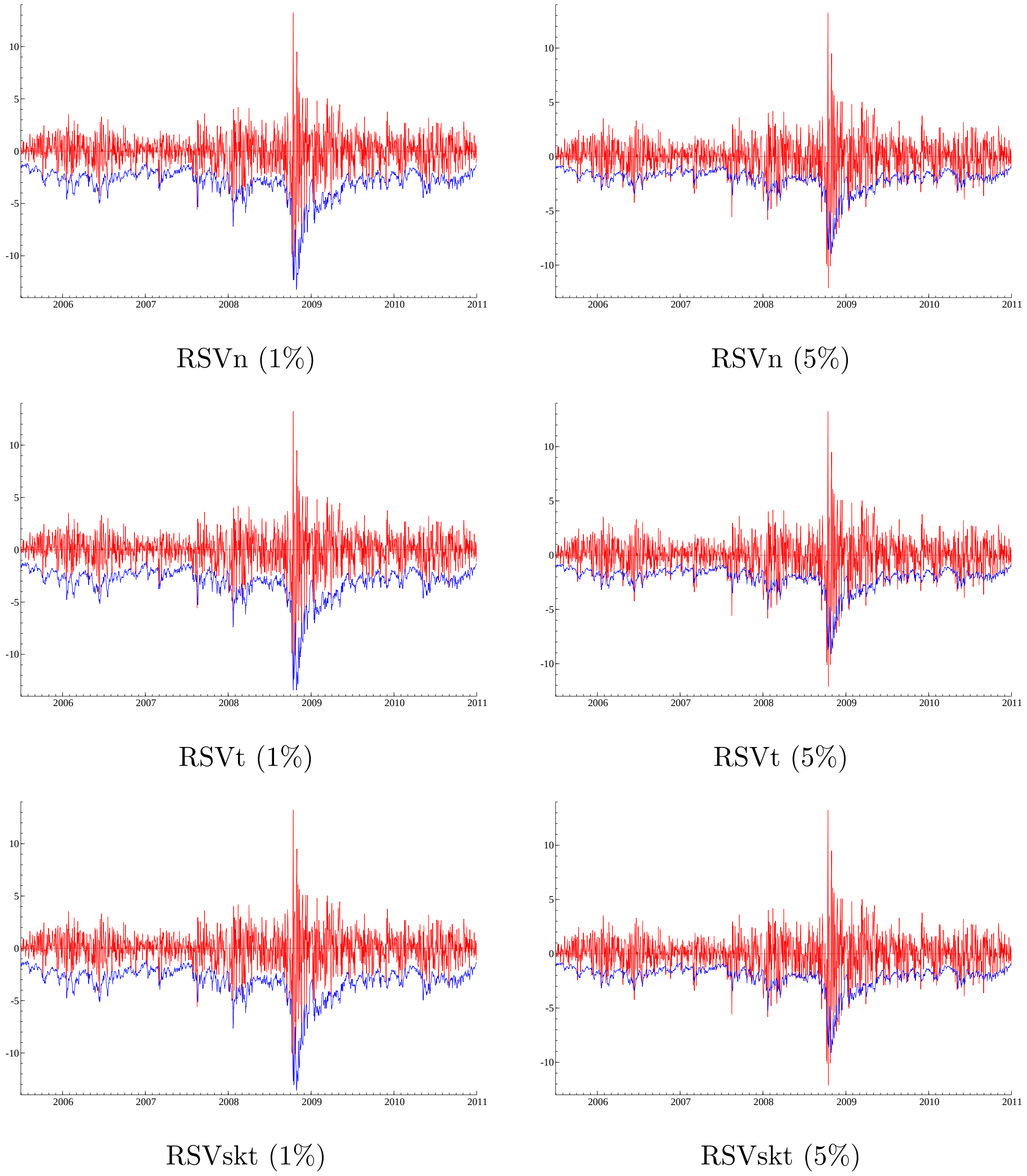
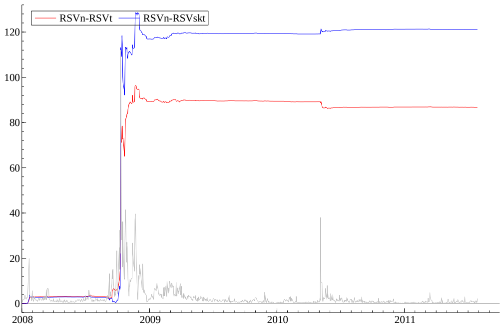
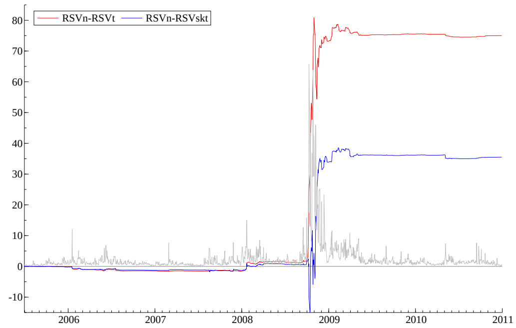


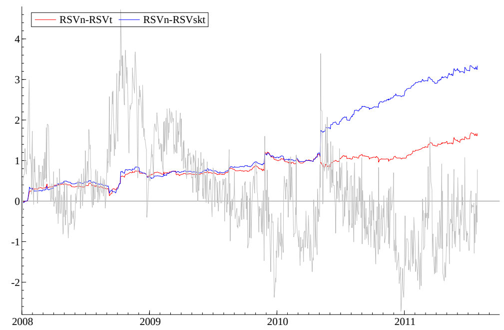
Figure 5: Cumulative loss differences for volatility forecasts of the RSVn models against the RSVt (red line) and RSVskt (blue line) models. The prediction period for DJIA is from January 2, 2008 to July 29, 2011 (895 samples) and for NIkkei 225 is from July 1, 2005 to December 30, 2010 (1350 samples). The top panels show the cumulative differences for MSE as well as realized kernels with the adjustment of Hansen and Lunde (2005) (gray line) whereas the bottom panels show the cumulative differences for QLIKE as well as logarithms of the realized kernels (gray line).



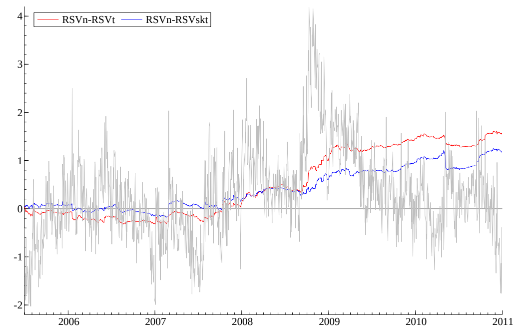
MSE for DJIA



MSE for Nikkei 225

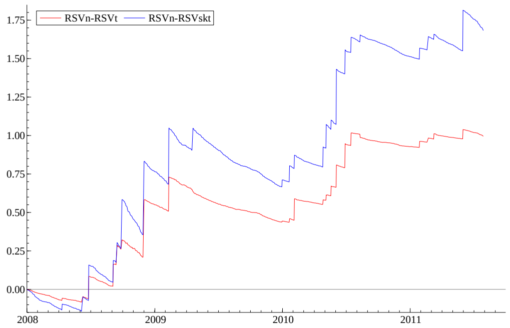


QLIKE for DJIA

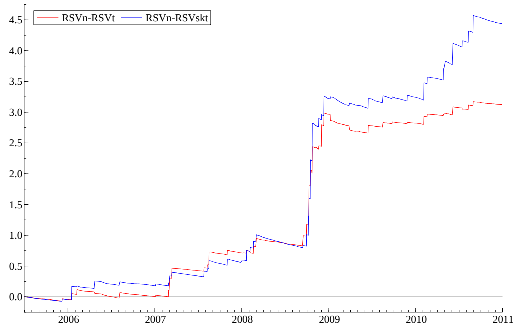


QLIKE for Nikkei 225

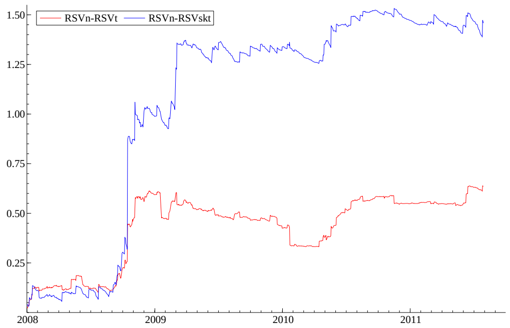
Figure 6: Cumulative loss differences for VaR forecasts of the RSVn models against the RSVt (red line) and RSVskt (blue line) models. Top and bottom panels show the cumulative differences for VaR(1%) and VaR(5%), respectively. The prediction period for DJIA is from January 2, 2008 to July 29, 2011 (895 samples) and for Nikkei 225 is from July 1, 2005 to December 30, 2010 (1350 samples).



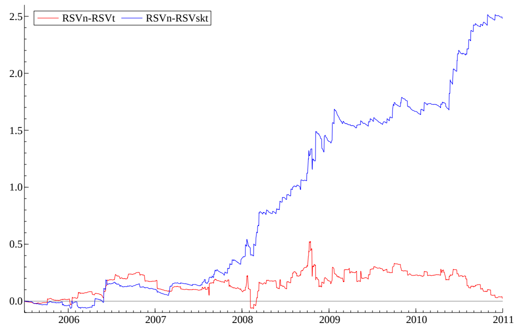
VaR(1%) for DJIA



VaR(1%) for Nikkei 225



VaR(5%) for DJIA



VaR(5%) for Nikkei 225