Parties with Policy Preferences and Uncertainty on Voter Behavior *

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Abstract

We study equilibrium redistributive policy proposals of two parties with policy preferences. Each party's ideal policy coincides with that of citizens with a particular income level and the party's utility function further embodies its attitude toward incompatible two goals: implementation of more preferred policy and electoral victory. If parties face uncertainty about citizens' abstention, diverged equilibrium proposals are derived which are more moderate than their contrasting ideal policies. Then, political equilibria under different prior beliefs on abstention are compared. We show that the lower likelihood of abstention in an income group induces both parties to make proposals catering to the group in equilibrium.

Keywords: redistribution, parties with policy preferences, voter behavior, politico-economic equilibrium

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1 Introduction

In the real politics, parties often modify their courses. The party takes an extreme stance consistently with its principles, or takes more moderate stances according to the time. Indeed, the 1960s' Democratic administration in the U. S. implemented positive fiscal policies with the slogan "guns and butter" and gave up balanced finance, while the Clinton administration of the 1990s succeeded in cutting Medicare and Medicaid expenses through major legislation and reforms. It appears natural to suppose that these different stances on redistribution by the identical party might be affected by its assessment of voter concern and behavior.

The literature is rich with models of two-party political competition. Since Hotelling's (1929) pioneering work, the archetypal conclusion in rational-choice analyses on the competition where the goal of two parties is solely to seek office has been convergence of their policy positions on the median voters' ideal point. In reality, however, even if parties for office sometimes adopt similar positions, they rarely adopt exactly the same position, which contradicts the prediction of the median voter theorem. Another category of the literature has supposed that parties have ideological preferences on policies. While some works in the category hold with convergence of parties' positions on the particular citizens' ideal point [Wittman (1977) and Calvert (1985)], others produce policy divergence [Wittman (1983), Hansson and Stuart (1984), Lindbeck and Weibull (1993), Roemer (1994), and Roemer (1997)]. These models, however, aim mainly at producing equilibrium policy divergence rather than providing implications for real politics. The exceptions are Roemer (1998) and Roemer (1999). They are on two-party competition with two-dimensional policy space: the former explores how increasing importance of the non-economic issue affects parties' choice of redistributive policies, while the latter derives choice of progressive income taxation by both the leftist and the rightist parties. On the contrary, our model analysis employs one-dimensional policy space, constructs the existence theorem of politico-economic equilibrium of redistribution, and further shows how the parties' selection of courses is affected by their conjecture on abstention, or voting rate, by income group. To our knowledge, there has been no analysis on parties with policy preferences and conjecture on voter behavior.

Let's summarize the model. There are two parties with contrasting preferences on redistribution. They represent specific income groups in that their ideal policies coincide with the ideal policies of these groups. Being based on the prior on voters' income distribution, parties make a binding electoral promise to all the electorate in the form of posttax income function with two parameters, and the balanced-budget requirement reduces the policy space to one dimension. Each voter sincerely votes for the party promising him the highest posttax income. The winning party implements the policy which it announced in the election campaign. Obviously, in the game with no abstention and therefore with no uncertainty about vote shares parties receive, the unique equilibrium involves each party's choice of the median voters' ideal policy. When the parties face uncertainty about income distribution of the *voters* who will actually go to the polls, parties' equilibrium positions are divergent and located on the side of their respective ideal points. Then, under the sufficient condition for the uniqueness of equilibrium, political equilibria with different prior beliefs on abstention are compared. It is shown that parties' common prior of the high voting rate among the low-income citizens compared with the high-income ones induces both parties to take the equilibrium positions more courting the low-income group, and *vice versa*. These results provide a well-described picture of redistributive politics.

Among the works listed above, our model closely relates to Roemer's (1997). He employed one-dimensional policy space, candidates with policy preferences, uncertainty about income distribution of voters, and generated equilibrium policy divergence. The works before him [e.g., Wittman (1983) and Hansson and Stuart (1984)] presupposed that each party's probability of winning is concave in its strategy to deduce the concavity of the payoff function. On the contrary, Roemer (1997) composed the probability function from microfoundations, but he was instead induced to suppose that the probability function has decreasing hazard rate. Contrasting with him, we suppose parties' utility function to embody their attitude toward the trade-off between ideological contentment and likelihood of victory. The supposition amounts to a sufficient condition for the existence of equilibrium. Due to this formulation, as the party takes the rival's choice as given and moves its position away from the rival's position closer to its ideal point, the ratio of the marginal increase in utility from its deviation to the marginal decrease in the probability of victory decreases. We believe that our utility function construction should be as plausible as suppositions to be set up in the other economic literature to manage trade-offs.

The paper is constructed as follows. The basic model assumptions on income distribution among the citizens and citizens' and parties' preferences on policies are given in Section 2. In Section 3, political equilibrium is derived. Section 4 concludes. Proofs except the one of Lemma 3 are gathered in Appendix.

2 The Model

There is a continuum of citizens. Citizens are considered as initially homogeneous before the nature determines their endowment, which is denoted by x.[1] The probability distribution of x is defined over the real interval $X_0 = (0, 1]$. Suppose that x is distributed on X_0 according to cumulative distribution function $F_0(x) = x$ and hence the expected value of x is $\frac{1}{2}$. The uniform distribution supposition is posited because it leads us to well-defined solutions.[2]

[1] This supposition implies that citizens have no *a priori* ideological preferences.

[2] Our results still hold with any cumulative distribution function that produces single-peakedness of citizens' preferences and monotonicity of their most preferred policy in their endowment under the tax scheme (1) and parties' balanced-budget constraint (2).

Each citizen is supposed to inelastically provide a unit of labor and to earn income whose amount is identical to endowment. Let's represent a citizen's *type* by his realized value of x. Define a median voter as type m such that $F_0(m) = \frac{1}{2}$, i.e., $m = \frac{1}{2}$.

There are two parties competing for the same office: one is leftist L and the other is rightist R. An electoral competition takes the form of simultaneous offers of posttax incomes. Parties make offers to citizens on the basis of the common prior on the probability distribution of x, which corresponds to F_0. Offers are supposed to be binding. Each party is committed to implementing its offer if elected. In the case of a tie, a fair coin is flipped to determine a winner.

We define type x's posttax income offered by party i, $i \in \{L, R\}$, as

$$y_i(x) \quad = \quad \alpha_i x^{\beta_i}, \tag{1}$$

where $y_i(x) \in (0, \alpha_i]$ denotes posttax income guaranteed to type x by party i; $\alpha_i (> 0)$ and $\beta_i \in \beta = [\underline{\beta}, \overline{\beta}]$, $0 < \underline{\beta} < \overline{\beta} \leq 1$, are coefficients chosen by party i. Notably, β_i measures the elasticity of posttax income with respect to pretax income. This measure of tax progression is well-known as the "residual income progression" parameter, which was first introduced by Musgrave and Thin (1948).[3] If $\beta_i = 1$, a tax scheme is proportional. As we choose successively lower values of β_i, the more progressive the tax scheme becomes, thereby reducing inequality in the resulting posttax income distribution. According to the suppositions on the ranges of α_i and β_i, $\frac{\partial y_i}{\partial x} > 0$ and thus x is mapped into y_i in the same order.

Suppose that transfers are financed purely by taxing citizens' incomes. Then, the budget constraint for party i is given by the amount of available resources:

$$\int_0^1 \alpha_i x^{\beta_i} dx = \frac{1}{2}. \tag{2}$$

[3] Bénabou's (2000) politico-economic model also employs the constant residual progression scheme, outlining the linkage between the Lorentz curve and it.

This balanced-budget constraint can be arranged as

$$\alpha_i \;=\; \frac{1+\beta_i}{2}, \tag{3}$$

and hence we can reduce each party's strategy to choosing only a real number in the interval β. Define that $Y(x,\beta_i) \equiv y_i(x)$. By the proposal of $\beta_i = 1$, each citizen is assured of posttax income equal to his pretax income. As β_i gets closer to zero, posttax incomes of all the citizens converge more closely on the mean $\frac{1}{2}$.

Let the utility of type x from party i's proposal be identical to $Y(x,\beta_i)$. Let $\hat{\beta}_x$ denote type x's most preferred policy, i.e., type x's ideal policy. From (1) and (3), $\hat{\beta}_x$ corresponds to

$$arg \max_{z\in\beta} \frac{1+z}{2} x^z, \tag{4}$$

and (4) is solved for by examining the partial derivative of the maximand with regard to z:

$$\frac{1+z}{2} x^z \left(\frac{1}{1+z} + \log x \right). \tag{5}$$

It immediately follows from (5) that the maximand in (4) is single-peaked in $z \in \beta$, implying that $y_i(x)$ is single-peaked in $\beta_i \in \beta$. By implicit function theorem, for x such that $\underline{\beta} < \hat{\beta}_x < \overline{\beta}$, i.e., for $e^{-\frac{1}{1+\underline{\beta}}} < x < e^{-\frac{1}{1+\overline{\beta}}}$,

$$\frac{\partial \hat{\beta}_x}{\partial x} \;=\; \frac{(1+\hat{\beta}_x)^2}{x} > 0, \tag{6}$$

and thus ideal policy $\hat{\beta}_x$ is monotonous in voter type x.

Let's suppose that party i's utility depends on the preference of the type party i represents among the electorate, which is denoted by x_i, and the policy to be implemented, $\beta_h \in \{\beta_L, \beta_R\}$. The utility function of party i is defined by

$$U(x_i, \beta_h) \;=\; \tilde{u}\left(Y(x_i, \beta_h)\right), \tag{7}$$

where $\tilde{u}(\cdot)$ is continuously twice differentiable and strictly increasing. It immediately follows from (7) that party i's ideal policy coincides with that of type x_i.

Assumption 1. $x_L < m < x_R$.

It is clear from (6) that $\hat{\beta}_{x_L} < \hat{\beta}_m (= \frac{1}{\log 2} - 1) < \hat{\beta}_{x_R}$ under Assumption 1. Thus, Assumption 1 describes that party L's stance is associated with an egalitarian philosophy, preferring higher progression and less inequality. On the contrary, party R feels that individuals are more entitled to the fruits they have generated, and thus, it is less concerned about equity.

Let (β_L, β_R) signify a set of proposals by two parties. Voting is carried out after each citizen is informed of both parties' proposals and the realized value of his endowment (which has been supposed to equal his pretax income). Each one sincerely votes for one of the two parties whose offer leads to the greatest utility. Namely, type x votes for party i, $i \in \{L, R\}$, not party j, $j \in \{L, R | j \neq i\}$, if $y_i(x) > y_j(x)$. He votes for one of them randomly if two parties propose the same posttax income. Let $V_i(\beta_i, \beta_j)$ denote the fraction of citizens with x such that $y_i(x) > y_j(x)$, given β_i and β_j. It corresponds to the fraction of votes party i receives without abstention. Then, we can derive that

$$V_i(\beta_i, \beta_j) = \begin{cases} \left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j - \beta_i}}, & \text{if } \beta_i < \beta_j, \\ \frac{1}{2}, & \text{if } \beta_i = \beta_j, \\ 1 - \left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i - \beta_j}}, & \text{if } \beta_i > \beta_j. \end{cases} \tag{8}$$

For a detailed derivation of (8), see Appendix. From (8), type $\left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j - \beta_i}}$ is a marginal citizen who is indifferent to both parties' proposals if $\beta_i < \beta_j$; similarly, type $\left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i - \beta_j}}$ is a marginal citizen if $\beta_i > \beta_j$. Thus, V_i is determinate if given β_L and β_R.

The following lemmas on the change in the party's vote share caused by its deviation are intuitively comprehensible from the single-peakedness of citizens' preferences. Their formal proofs are contained in Appendix.

Lemma 1. If $\beta_i < \beta_j$, $\frac{\partial V_i}{\partial \beta_i} > 0$. If $\beta_i > \beta_j$, $\frac{\partial V_i}{\partial \beta_i} < 0$.

Lemma 2. If $\beta_j < \hat{\beta}_m$, $\lim_{\beta_i \to \beta_j -} V_i < \frac{1}{2}$ and $\lim_{\beta_i \to \beta_j +} V_i > \frac{1}{2}$. If $\beta_j > \hat{\beta}_m$, $\lim_{\beta_i \to \beta_j -} V_i > \frac{1}{2}$ and $\lim_{\beta_i \to \beta_j +} V_i < \frac{1}{2}$. If $\beta_j = \hat{\beta}_m$, $\lim_{\beta_i \to \beta_j -} V_i = \lim_{\beta_i \to \beta_j +} V_i = \frac{1}{2}$.

Lemma 1 captures the idea that each party can obtain more votes by moving its policy position closer to the rival's. It also suggests that $\frac{\partial V_i}{\partial \beta_j} > (<)0$ if $\beta_i < (>)\beta_j$ since $\frac{\partial V_i}{\partial \beta_j} = -\frac{\partial V_j}{\partial \beta_j}$. Lemma 2 shows that V_i is continuous on the line $\beta_i = \beta_j$ if and only if party j takes the median voters' ideal policy.

According to the standard approach in the literature on parties with policy preferences, party i's payoff is given by party i's expected utility, which is denoted by W_i:

$$W_i(\beta_i, \beta_j) = P_i(\beta_i, \beta_j)U(x_i, \beta_i) + (1 - P_i(\beta_i, \beta_j))U(x_i, \beta_j), \tag{9}$$

where P_i represents party i's probability of winning the election. The correspondence from V_i to P_i will be mentioned later.

Thus, the electoral game is defined as follows: the strategy set of party i, $i \in \{L, R\}$, is β and for each pair of strategies $(\beta_L, \beta_R) \in \beta^2$, party i's payoff is given by W_i. The equilibrium concept is Nash. An equilibrium is described as a pair of strategies of parties L and R.

3 Equilibrium

First, as a benchmark, suppose that every citizen casts a vote and hence parties face no uncertainty on vote shares they receive. Then, party i recognizes P_i in (9) as

$$P_i(\beta_i, \beta_j) = \begin{cases} 0, & if \ V_i(\beta_i, \beta_j) < \frac{1}{2}, \\ \frac{1}{2}, & if \ V_i(\beta_i, \beta_j) = \frac{1}{2}, \\ 1, & if \ V_i(\beta_i, \beta_j) > \frac{1}{2}. \end{cases} \tag{10}$$

Namely, the payoff for party i is taken to be $U(x_i, \beta_i)$ if it wins on policy β_i; $U(x_i, \beta_j)$ if party j wins on policy β_j; $\frac{1}{2}U(x_i, \beta_i) + \frac{1}{2}U(x_i, \beta_j)$ by a fair lottery in the case of a tie.

Theorem 1. Let Assumption 1 hold. Suppose no abstention. Then, $(\hat{\beta}_m, \hat{\beta}_m)$ is a unique equilibrium of the game where $V_L(\hat{\beta}_m, \hat{\beta}_m) = V_R(\hat{\beta}_m, \hat{\beta}_m) = P_L(\hat{\beta}_m, \hat{\beta}_m) = P_R(\hat{\beta}_m, \hat{\beta}_m) = \frac{1}{2}$.

Theorem 1 says that when there is no uncertainty on vote share the party receives, each party's equilibrium strategy involves solely the choice of the median voters' ideal policy. Thus, Theorem 1 holds the idea of Hotelling (1929). Let's outline the proof in Appendix. In the game with policy preferences, if given $(\beta_L, \beta_R) = (\hat{\beta}_m, \hat{\beta}_m)$, each party's deviation from $\hat{\beta}_m$ to its more preferred position forces it to lose the competition outright and never raises its payoff. If given $(\beta_L, \beta_R) \neq (\hat{\beta}_m, \hat{\beta}_m)$, one of two parties has a position which is preferable to the current one and guides it to certain electoral victory. Thus, $(\beta_L, \beta_R) \neq (\hat{\beta}_m, \hat{\beta}_m)$ cannot be a candidate of equilibrium. These results arise from the formulation in (10).

Then, let parties face uncertainty. Uncertainty about income distribution of the *voters* who actually cast votes originates from the possibility of some citizens' abstention driven by the weather, polls, scandals or booms of party leaders, or whims. We shall model uncertainty lead by such factors by taking a continuum of states $s \in S = [0, 1]$ and s's cumulative distribution function H defined on S. For each s, x is distributed on X_0 according to conditional cumulative distribution function $G(x|s)$. Namely, $G(x|s)$ denotes the fraction of voters with their pretax income below x in state s.

Assumption 2. (i) $H(s)$ is continuously twice differentiable and strictly increasing in $s \in S$; (ii) $G(x|s)$ is continuously twice differentiable in (x, s); (iii) for any $s \in S$, $G(x|s)$ is strictly increasing in $x \in X_0$; (iv) $G(x|s)$ is strictly increasing in $s \in S$ if $x \in (0, 1)$; $G(1|s) = 1$ for any $s \in S$; (v) $H(\cdot)$ and $G(\cdot|\cdot)$ are the common prior beliefs of L and R.

Put differently, if $s_1 > s_2$, $G(x|s_1) > G(x|s_2)$ for any $x \in (0,1)$. Namely, $G(\cdot|s_2)$ first-order stochastically dominates $G(\cdot|s_1)$. To speak intuitively and simply, the higher s is associated with income distribution with the higher ratio of the poor, given any definable boundary that divides the electorate into two income groups. One interpretation of s is the impact of campaign ads invoking the political awareness among the low-income group: with higher s, more low-income citizens turn out to vote.

Under Assumption 2, party i's vote share in state s, denoted by $V_i^s(\beta_i, \beta_j|s)$, is derived as

$$V_i^s(\beta_i, \beta_j|s) = \begin{cases} G\left(\left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j - \beta_i}} |s\right), & \text{if } \beta_i < \beta_j, \\ \frac{1}{2}, & \text{if } \beta_i = \beta_j, \\ 1 - G\left(\left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i - \beta_j}} |s\right), & \text{if } \beta_i > \beta_j. \end{cases} \quad (11)$$

See Appendix for the derivation of (11). Thus, V_i^s represents the fraction of voters whose pretax income is below $\left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j - \beta_i}}$ (above $\left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i - \beta_j}}$) in state s if $\beta_i < \beta_j$ ($\beta_i > \beta_j$). Note that $V_i^s = G(V_i|s)$ if $\beta_i < \beta_j$; $V_i^s = 1 - G(1 - V_i|s)$ if $\beta_i > \beta_j$. Thus, V_i^s is a random variable defined on S if $\beta_i \neq \beta_j$.

Let's suppose that given $\beta_L \neq \beta_R$, there is a cutpoint $\overline{s} \in (0,1)$ such that $V_L^s(\beta_L, \beta_R|\overline{s}) = G(V_L(\beta_L, \beta_R)|\overline{s}) = \frac{1}{2}$ for every $V_L(\beta_L, \beta_R)$ if $\beta_L < \beta_R$; $V_R^s(\beta_R, \beta_L|\overline{s}) = G(V_R(\beta_R, \beta_L)|\overline{s}) = \frac{1}{2}$ for every $V_R(\beta_R, \beta_L)$ if $\beta_L > \beta_R$. Under Assumption 2, V_L or V_R is mapped into $\overline{s}$ in the one-to-one manner. In state $s > (<)\overline{s}$, the party with lower proposal wins (loses) the election since its vote share strictly increases in s, given β_L and β_R. Then, we can derive the continuity and the monotonicity of P_L and P_R with regard to V_L and V_R, respectively.

Lemma 3. Let Assumption 2 hold. Then, if $\beta_L \neq \beta_R$, $P_i \in (0,1)$, $i \in \{L, R\}$, is continuously differentiable and strictly increasing in V_i.

Proof. Let $\beta_i < \beta_j$. Let function $\overline{s}(V_i)$ be defined by $G\left(V_i|\overline{s}(V_i)\right) = \frac{1}{2}$: since $V_i \in (0,1)$, V_i is mapped into $\overline{s}$ in the one-to-one manner under Assumption 2. Then, from implicit function theorem, $\frac{\partial \overline{s}}{\partial V_i} = -\frac{\partial G}{\partial V_i}/\frac{\partial G}{\partial \overline{s}} < 0$. Given V_i, $P_i = 1 - H\left(\overline{s}(V_i)\right)$ and thus P_i strictly increases in V_i since $\overline{s}' < 0$ and $H' > 0$.

Let $\beta_i > \beta_j$. Then, $1-G(1-V_i|s) = 1-G(V_j|s)$. By definition, $1-G(V_j|\overline{s}(V_j)) = \frac{1}{2}$ and hence $P_i = H\left(\overline{s}(V_j)\right)$. Thus, P_i strictly increases in V_i since $\frac{\partial V_j}{\partial V_i} < 0$.

Continuous differentiability of P_i with regard to V_i comes from continuous differentiability of $H(\cdot)$ and $G(\cdot|\cdot)$. It is clear that $0 < P_i < 1$ from the supposition that $0 < \overline{s} < 1$. $\|$

Accordingly, under Assumptions 1 and 2, party i's choice of strategy effects its payoff displayed in (9) through two channels: via a change in $P_i(\beta_i, \beta_j)$ and via a change in $U(x_i, \beta_i)$. From Lemmas 1 and 3, $P_i(\beta_i, \beta_j)$ strictly increases in $\beta_i \in [\underline{\beta}, \beta_j)$ and strictly decreases in $\beta_i \in (\beta_j, \overline{\beta}]$: it may leap up or down to $\frac{1}{2}$ at $\beta_i = \beta_j$. On the other hand, $U(x_i, \beta_i)$ strictly increases in $\beta_i \in [\underline{\beta}, \hat{\beta}_{x_i})$ and strictly decreases in $\beta_i \in (\hat{\beta}_{x_i}, \overline{\beta}]$. Thus, party i has to consider the *tactical* effect on $P_i(\beta_i, \beta_j)$ and the *ideological* effect on $U(x_i, \beta_i)$ arising from its decision.

Lemma 4. Given β_j, $\beta_i < \min\{\hat{\beta}_{x_i}, \beta_j\}$ or $\beta_i > \max\{\hat{\beta}_{x_i}, \beta_j\}$ is never a maximizer of W_i.

Lemma 4 suggests that $\beta_i \in [\min\{\hat{\beta}_{x_i}, \beta_j\}, \max\{\hat{\beta}_{x_i}, \beta_j\}]$ is a candidate of party i's best reaction. Notice that if $\beta_j \neq \hat{\beta}_{x_i}$, two effects illustrated above work in the opposite directions in the interval $(\min\{\hat{\beta}_{x_i}, \beta_j\}, \max\{\hat{\beta}_{x_i}, \beta_j\})$ and therefore the party has to consider their relative strength in making a choice. Let's measure it by the ratio of the marginal increase (decrease) in $U(x_i, \beta_i)$ to the marginal decrease (increase) in $P_i(\beta_i, \beta_j)$. Define that $\eta_i \equiv \frac{\partial U}{\partial \beta_i}(x_i, \beta_i)/\frac{\partial P_i}{\partial \beta_i}(\beta_i, \beta_j)$. Then, $\eta_i < 0$ if i's policy position is located between $\hat{\beta}_{x_i}$ and β_j. The following assumption on the curvature of function $\tilde{u}$ stipulates party i's attitude toward the trade-off between the ideological gain produced by its marginal deviation (represented by η_i's numerator) and the tactical gain produced alike (represented by η_i's denominator).

Assumption 3. Given functions F_0, Y, H, and G, function $\tilde{u}$ satisfies the following conditions: (i) the absolute value of $\eta_i = -\frac{\partial \tilde{u}}{\partial Y}\frac{\partial Y}{\partial \beta_i} / \frac{\partial H}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial V_i}\frac{\partial V_i}{\partial \beta_i}$ weakly increases in $\beta_i \in (\hat{\beta}_{x_i}, \beta_j)$; (ii) the absolute value of $\eta_i = -\frac{\partial \tilde{u}}{\partial Y}\frac{\partial Y}{\partial \beta_i} / \frac{\partial H}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial V_j}\frac{\partial V_i}{\partial \beta_i}$ weakly decreases in $\beta_i \in (\beta_j, \hat{\beta}_{x_i})$.

Intuition is the following: when i comes up against the trade-off, its ideological gain relative to its tactical loss gets smaller as its choice gets closer to its ideal policy. We are now ready to construct the existence theorem of political equilibrium under uncertainty since Assumption 3 provides a sufficient condition for i's best reaction to be single-valued.

Theorem 2. Let Assumptions 1 to 3 hold. Namely, parties L and R face uncertainty on *voters'* distribution. Then, an equilibrium (β_L^*, β_R^*) of the game exists where $\hat{\beta}_{x_L} < \beta_L^* < \beta_R^* < \hat{\beta}_{x_R}$.

Theorem 2 tells us that policy-motivated parties, facing uncertainty, propose divergent policies in equilibrium. One possible example of the pair of party L's and party R's reaction curves is illustrated in Figure 1. It displays that each party's best reaction against the rival's choice uniquely exists in the interval between its ideal policy and the rival's position. The equilibrium of the game, which is marked by the intersection of two reaction curves, is certainly involved inside the triangle drawn with the thick dotted lines where $\hat{\beta}_{x_L} < \beta_L < \beta_R < \hat{\beta}_{x_R}$.[4]

We are now interested in the linkage between parties' common prior beliefs on voters' income distribution and parties' equilibrium strategies. Redefine H as the set of cumulative distribution functions of s which satisfy Assumption 2 and take

[4]Our results are consistent with Roemer's (1997) conclusion that both policy-motivated parties and uncertainty are needed to generate equilibrium policy divergence.

H_1, $H_2 \in H$. Represent an equilibrium with H_k as $(\beta_L^{k*}, \beta_R^{k*})$, $k = 1, 2$. Note that $\hat{\beta}_{x_L} < \beta_L^{k*} < \beta_R^{k*} < \hat{\beta}_{x_R}$ from Theorem 2. Then, let's examine the movement of the equilibrium according to the shift of parties' prior from H_1 to H_2. We will concentrate on the shift such that $H_1'(\overline{s}^{1*}) = H_2'(\overline{s}^{1*})$ where $\overline{s}^{1*} \equiv \overline{s}(V_L(\beta_L^{1*}, \beta_R^{1*}))$, i.e., the density at the cutpoint $\overline{s}^{1*}$ is kept constant.[5]

For our purpose, a sufficient condition for the uniqueness of equilibrium is assumed.

Assumption 4. Let i's best reaction for every $\beta_j \in \beta$ be denoted by $b_i(\beta_j) \in \beta$. If $\beta_j \neq \hat{\beta}_{x_i}$, $0 < \frac{\partial^2 W_i}{\partial \beta_i \partial \beta_j}(b_i(\beta_j), \beta_j) < |\frac{\partial^2 W_i}{\partial \beta_i^2}(b_i(\beta_j), \beta_j)|$ with any $H_k \in H$.

It follows from the discussions so far that $\frac{\partial W_i}{\partial \beta_i}(b_i(\beta_j), \beta_j) = 0$ for $\beta_j \neq \hat{\beta}_{x_i}$. Then, under Assumption 4, $0 < \frac{\partial b_i}{\partial \beta_j} = -\frac{\partial^2 W_i}{\partial \beta_i \partial \beta_j} / \frac{\partial^2 W_i}{\partial \beta_i^2}(b_i(\beta_j), \beta_j) < 1$ for $\beta_j \neq \hat{\beta}_{x_i}$ since $\frac{\partial^2 W_i}{\partial \beta_i^2}(b_i(\beta_j), \beta_j) < 0$, and thus, L and R's reaction curves intersect only once, just as displayed in Figure 1. Note that Assumption 4 also assures that political equilibrium is stable.

Remind that the higher s is associated with income distribution with the higher ratio of voters whose income is below the certain level. Accordingly, to speak intuitively and simply, the lower (higher) value of the cumulative distribution function for fixed $\overline{s}^{1*}$ is associated with the higher likelihood of the high voting rate among the poor (rich) compared to the rich (poor). The following theorem suggests that both parties, with the prior belief of the higher likelihood of the high voting rate among the poor relative to the rich, take equilibrium positions closer to $\hat{\beta}_{x_L}$, and *vice versa*.

Theorem 3. Let Assumptions 1 to 4 hold. Then, a unique equilibrium of the game with H_1, $(\beta_L^{1*}, \beta_R^{1*})$, and a unique equilibrium of the game with H_2, $(\beta_L^{2*}, \beta_R^{2*})$, are as

[5] We will do so because without the supposition of $H_1'(\overline{s}^{1*}) = H_2'(\overline{s}^{1*})$, we have to incorporate further more suppositions to demonstrate the movement of the equilibrium, which would make the model too complicated to hold its descriptive power.

follows: (i) $(\beta_L^{1*}, \beta_R^{1*}) \gg (\beta_L^{2*}, \beta_R^{2*})$ if $H_1(\overline{s}^{1*}) > H_2(\overline{s}^{1*})$; (ii) $(\beta_L^{1*}, \beta_R^{1*}) \ll (\beta_L^{2*}, \beta_R^{2*})$ if $H_1(\overline{s}^{1*}) < H_2(\overline{s}^{1*})$.

Theorem 3 has two major practical implications. First, if we classify the electorate into two income groups and if one group is likely to vote at the higher rate than the other, then the party pro that group takes more extreme equilibrium position (closer to its ideology) than it would do otherwise, where the party can count on a decent level of the probability of winning despite its radicalization. Second, the equilibrium position of the party con that group is also shifted to the point possibly more preferred by that group.

4 Concluding Remarks

The paper has presented a model where two parties, the leftist and the rightist, propose tax-transfer schemes as their electoral promises. This is a reasonable framework since redistribution is one of the main campaign issues of interest to the electorate. Each of the leftist and the rightist parties represents a group of citizens with a particular income level in the electorate in that the party's ideal policy coincides with that of the group. It is assumed that the ideal policy of the leftist (rightist) party is more (less) redistributive than that of the median voter. Parties' policy preferences, however, are not perfectly identical to the citizens' since parties' utility function embodies their attitude toward the trade-off between implementation of more preferred policy and electoral victory. Our formulation of the utility function provides a sufficient condition for the existence of equilibrium. It leads to the decreasing ratio of the marginal increase in the party's utility from its choice of policy to the marginal decrease in its probability of victory as the party deviates away from the rival's position more closely to its ideal policy.

Political competitions with and without uncertainty have been studied. Uncertainty about income distribution of the voters who actually go to the polls and cast votes arises from the possibility of some citizens' abstention. We model uncertainty by employing parties' common prior on the probability distribution of states each of which is linked with a certain income distribution of voters. In equilibrium without uncertainty, both parties propose the median voters' ideal policy as a campaign promise. By introducing uncertainty into the model, however, equilibrium policy divergence is produced where each party chooses a more moderate stance than its ideological ideal. Then, under the sufficient condition for the uniqueness of equilibrium, political equilibria with different prior beliefs are compared. It is shown that parties' common prior belief of the high voting rate among the certain income group induces both parties to take the equilibrium positions catering to that group.

The results from our analysis of the competition with uncertainty epitomize real politics. Parties should modify their courses according to their inference on voting behavior as well as on the rival's action. For instance, Theorem 3 predicts that when the rise in the voting rate among low-income citizens is likely, not only the leftist party's appeal of egalitarianism but also the rightist party's moderation should be observed. Thus, we have shown theoretically that one party's radicalization and seemingly ideological compromise of the other party may arise from parties' changed belief on voting behavior, not from changed ideologies.

A Appendix

A.1 Formal Proofs

Proof of Lemma 1. Let $\beta_i < \beta_j$. Let i choose a higher value $\beta_i + d\beta_i$ rather than β_i, where $d\beta_i$ is sufficiently small and $\beta_i + d\beta_i < \beta_j$. Then, due to the single-peakedness of $Y(x, \cdot)$, citizens with x such that $\hat{\beta}_x \geq \beta_j$ or citizens with x such that $\hat{\beta}_x \leq \beta_i$ never change the party they vote for. Let's investigate voting behavior of the citizens with x such that $\beta_i < \hat{\beta}_x < \beta_j$. Among them, those with x such that $x < \left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j - \beta_i}}$ still vote for i since $Y(x, \beta_j) < Y(x, \beta_i) < Y(x, \beta_i + d\beta_i)$.

Furthermore, due to the continuity of $Y(\cdot, \beta_i)$, we have an interval x with strictly positive length such that $\beta_i < \hat{\beta}_x < \beta_j$ and $Y(x, \beta_i) < Y(x, \beta_j) < Y(x, \beta_i + d\beta_i)$. These discussions imply that $\frac{\partial V_i}{\partial \beta_i} > 0$ if $\beta_i < \beta_j$. We can derive that $\frac{\partial V_i}{\partial \beta_i} < 0$ if $\beta_i > \beta_j$ in the same manner. ‖

Proof of Lemma 2. Let $\beta_i < \beta_j$. The logarithm of $\left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j - \beta_i}}$ is given by

$$\frac{\log(1+\beta_i) - \log(1+\beta_j)}{\beta_j - \beta_i}. \tag{A1}$$

Then,

$$\lim_{\beta_i \to \beta_j -} \frac{\log(1+\beta_i) - \log(1+\beta_j)}{\beta_j - \beta_i} = \frac{d\left(-\log(1+\beta_i)\right)}{d\beta_i}(\beta_j) = -\frac{1}{1+\beta_j}. \tag{A2}$$

From (A2), the limit of i's vote share when it moves its policy position toward j's position from below is given by

$$\lim_{\beta_i \to \beta_j -} V_i = e^{-\frac{1}{1+\beta_j}}. \tag{A3}$$

Since $\hat{\beta}_m = \frac{1}{\log 2} - 1$ and $e^{-\frac{1}{1+\hat{\beta}_m}} = \frac{1}{2}$, $e^{-\frac{1}{1+\beta_j}} < \frac{1}{2}$ if $\beta_j < \hat{\beta}_m$; $e^{-\frac{1}{1+\beta_j}} = \frac{1}{2}$ if $\beta_j = \hat{\beta}_m$; $e^{-\frac{1}{1+\beta_j}} > \frac{1}{2}$ if $\beta_j > \hat{\beta}_m$. The similar logic can be applied to the case where $\beta_i > \beta_j$. These discussions lead to Lemma 2. ‖

Proof of Theorem 1. Let $(\beta_L, \beta_R) = (\hat{\beta}_m, \hat{\beta}_m)$. Then, $V_i(\hat{\beta}_m, \hat{\beta}_m) = \frac{1}{2}$ and $W_i(\hat{\beta}_m, \hat{\beta}_m) = U(x_i, \hat{\beta}_m)$, $i \in \{L, R\}$. If i deviates from $\hat{\beta}_m$ to $\hat{\beta}_m - \varepsilon$, $\varepsilon > 0$, $V_i(\hat{\beta}_m - \varepsilon, \hat{\beta}_m) < \frac{1}{2}$ from Lemmas 1 and 2 and hence $W_i(\hat{\beta}_m - \varepsilon, \hat{\beta}_m) = U(x_i, \hat{\beta}_m) = W_i(\hat{\beta}_m, \hat{\beta}_m)$. If i deviates from $\hat{\beta}_m$ to $\hat{\beta}_m + \varepsilon$, $\varepsilon > 0$, $V_i(\hat{\beta}_m + \varepsilon, \hat{\beta}_m) < \frac{1}{2}$ and hence $W_i(\hat{\beta}_m + \varepsilon, \hat{\beta}_m) = U(x_i, \hat{\beta}_m) = W_i(\hat{\beta}_m, \hat{\beta}_m)$. Thus, i's deviation never raises its payoff and $(\hat{\beta}_m, \hat{\beta}_m)$ is indeed an equilibrium of the game.

Next we shall demonstrate that any pair of $(\beta_L, \beta_R) \neq (\hat{\beta}_m, \hat{\beta}_m)$ is not an equilibrium.

Let $\max\{\beta_L, \beta_R\} < \hat{\beta}_m$. Then, R can increase its payoff by choosing $\hat{\beta}_m$ instead of β_R since $W_R(\beta_R, \beta_L) \leq \max\{U(x_R, \beta_L), U(x_R, \beta_R)\} < U(x_R, \hat{\beta}_m) = W_R(\hat{\beta}_m, \beta_L)$

from Assumption 1 and this deviation thus enables R to win the election outright and to implement its more preferred policy. The case where $\min\{\beta_L, \beta_R\} > \hat{\beta}_m$ is similarly examined.

Let $\beta_R = \hat{\beta}_m$ and $\beta_L \neq \hat{\beta}_m$. Then, $V_R(\beta_R, \beta_L) > \frac{1}{2}$. Since $\hat{\beta}_{x_R} > \hat{\beta}_m$ under Assumption 1 and $V_R(\cdot, \beta_L)$ is continuous if not on the line $\beta_L = \beta_R$, R can choose an $\varepsilon > 0$ such that $V_R(\beta_R + \varepsilon, \beta_L) > \frac{1}{2}$ and $U(x_R, \beta_R + \varepsilon) > U(x_R, \beta_R)$. Then, $W_R(\beta_R + \varepsilon, \beta_L) = U(x_R, \beta_R + \varepsilon) > U(x_R, \beta_R) = W_R(\beta_R, \beta_L)$ and thus R can increase its payoff by deviating from $\hat{\beta}_m$ to its more preferred policy. The case where $\beta_L = \hat{\beta}_m$ and $\beta_R \neq \hat{\beta}_m$ is similarly examined.

Let $\beta_R < \hat{\beta}_m < \beta_L$ and $V_L(\beta_L, \beta_R) > \frac{1}{2}$. Then, $V_L(\hat{\beta}_m, \beta_R) > \frac{1}{2}$, $U(x_L, \hat{\beta}_m) > U(x_L, \beta_L)$, and hence $W_L(\hat{\beta}_m, \beta_R) = U(x_L, \hat{\beta}_m) > U(x_L, \beta_L) = W_L(\beta_L, \beta_R)$. Thus, L's deviation to $\hat{\beta}_m$ sustains L's victory and moreover leads to implementation of its more preferred policy. The case where $\beta_R < \hat{\beta}_m < \beta_L$ and $V_R(\beta_R, \beta_L) > \frac{1}{2}$ is similarly examined.

Let $\beta_R < \hat{\beta}_m < \beta_L$ and $V_L(\beta_L, \beta_R) = V_R(\beta_R, \beta_L) = \frac{1}{2}$. Then, R can choose an $\varepsilon > 0$ such that $V_R(\beta_L - \varepsilon, \beta_L) > \frac{1}{2}$, $U(x_R, \beta_L - \varepsilon) > \frac{1}{2}U(x_R, \beta_R) + \frac{1}{2}U(x_R, \beta_L)$, and therefore $W_R(\beta_L - \varepsilon, \beta_L) = U(x_R, \beta_L - \varepsilon) > \frac{1}{2}U(x_R, \beta_R) + \frac{1}{2}U(x_R, \beta_L) = W_R(\beta_R, \beta_L)$. Thus, R can increase its payoff by choosing the policy which is closer to its ideal policy and enables R's victory.

Let $\beta_L < \hat{\beta}_m < \beta_R$ and $V_L(\beta_L, \beta_R) > \frac{1}{2}$. Then, $V_R(\hat{\beta}_m, \beta_L) > \frac{1}{2}$, $U(x_R, \hat{\beta}_m) > U(x_R, \beta_L)$, and hence $W_R(\hat{\beta}_m, \beta_L) = U(x_R, \hat{\beta}_m) > U(x_R, \beta_L) = W_R(\beta_R, \beta_L)$. Thus, R's deviation to $\hat{\beta}_m$ enables R to win the election outright and to implement its more preferred policy. The case where $\beta_L < \hat{\beta}_m < \beta_R$ and $V_R(\beta_R, \beta_L) > \frac{1}{2}$ is similarly examined.

Let $\beta_L < \hat{\beta}_m < \beta_R$ and $V_L(\beta_L, \beta_R) = V_R(\beta_R, \beta_L) = \frac{1}{2}$. Then, R can choose an $\varepsilon > 0$ such that $V_R(\beta_R - \varepsilon, \beta_L) > \frac{1}{2}$, $U(x_R, \beta_R - \varepsilon) > \frac{1}{2}U(x_R, \beta_R) + \frac{1}{2}U(x_R, \beta_L)$, and therefore $W_R(\beta_R - \varepsilon, \beta_L) = U(x_R, \beta_R - \varepsilon) > \frac{1}{2}U(x_R, \beta_R) + \frac{1}{2}U(x_R, \beta_L) = W_R(\beta_R, \beta_L)$. Thus, R can increase its payoff by choosing the policy which enables R's victory.

These results prove that $(\beta_L, \beta_R) \neq (\hat{\beta}_m, \hat{\beta}_m)$ is not an equilibrium of the game. The assertion on each party's vote share and its probability of winning is immediately derived from (8) and (10). ‖

Proof of Lemma 4. Define that $I = [\min\{\hat{\beta}_{x_i}, \beta_j\}, \max\{\hat{\beta}_{x_i}, \beta_j\}]$. It is clear that if $\beta_j = \hat{\beta}_{x_i}$, $\beta_i \in \beta/I$ is not a best strategy since $U(x_i, \beta_i) < U(x_i, \hat{\beta}_{x_i}) \, (= U(x_i, \beta_j))$ and i can increase its payoff by taking its ideal policy.

Next, let $\beta_j \neq \hat{\beta}_{x_i}$. Let $\beta_i \in \beta/I$ and $|\beta_i - \beta_j| < |\beta_i - \hat{\beta}_{x_i}|$, i.e., let β_i be located on the side of β_j. Then, β_i is not a best strategy since $U(x_i, \beta_i) < U(x_i, \beta_j)$ and i can increase its payoff by choosing the same policy as j's. Let $\beta_i \in \beta/I$ and $|\beta_i - \beta_j| > |\beta_i - \hat{\beta}_{x_i}|$, i.e., let β_i be located on the side of $\hat{\beta}_{x_i}$. Then, β_i is not a best strategy since $P_i(\hat{\beta}_{x_i}, \beta_j) > P_i(\beta_i, \beta_j)$, $U(x_i, \hat{\beta}_{x_i}) > U(x_i, \beta_i)$, $U(x_i, \hat{\beta}_{x_i}) > U(x_i, \beta_j)$, and hence $W_i(\hat{\beta}_{x_i}, \beta_j) = P_i(\hat{\beta}_{x_i}, \beta_j)\left(U(x_i, \hat{\beta}_{x_i}) - U(x_i, \beta_j)\right) + U(x_i, \beta_j) > P_i(\beta_i, \beta_j)\left(U(x_i, \beta_i) - U(x_i, \beta_j)\right) + U(x_i, \beta_j) = W_i(\beta_i, \beta_j)$. Thus, i can increase its payoff by taking its ideal policy. ‖

Proof of Theorem 2. We shall prove the existence of a Nash equilibrium in the domain $D = \{(\beta_L, \beta_R) \in \beta^2 | \hat{\beta}_{x_L} < \beta_L < \beta_R < \hat{\beta}_{x_R}\}$ (corresponding to the area inside the triangle drawn with thick dotted lines in Figure 1), which leads us to Theorem 2. For this purpose, our strategy that follows involves showing that (i) each party has a single-valued and continuous best reaction in β against any strategy by the rival in β, that (ii) by Brouwer's fixed point theorem, a pair of strategies which constitutes a Nash equilibrium exists in β^2, but that (iii) any pair of strategies contained in β^2/D is not a Nash equilibrium.

(i) We have shown in Lemma 4 that W_i does not have a maximizer in β/I against any $\beta_j \in \beta$. Particularly, it has been demonstrated in the proof of Lemma 4 that given $\beta_j = \hat{\beta}_{x_i}$, W_i is maximized solely by $\hat{\beta}_{x_i}$. Then, we shall show that given $\beta_j \neq \hat{\beta}_{x_i}$, W_i has a single-valued maximizer within $(\min\{\hat{\beta}_{x_i}, \beta_j\}, \max\{\hat{\beta}_{x_i}, \beta_j\}) \subset I$.

It is clear from the continuity of $P_i(\cdot,\cdot)$ and $U(x_i,\cdot)$ where $\beta_i \neq \beta_j$ that $W_i(\cdot,\cdot)$ is continuous for all points (β_L, β_R) where $\beta_L \neq \beta_R$. We will adapt Lemma 1 in Roemer (1997) to our framework and assure the continuity of $W_i(\cdot,\cdot)$ for all points $(\beta_L, \beta_R) \in \beta^2$ and the existence of the maximizer of $W_i(\cdot, \beta_j)$ in the interval I.

Lemma A1 [Roemer (1997)]. $\lim_{\beta_i \to \beta_j} W_i(\beta_i, \beta_j) = W_i(\beta_j, \beta_j)$ and $\lim_{\beta_j \to \beta_i} W_i(\beta_i, \beta_j) = W_i(\beta_i, \beta_i)$.

Proof. Remind that $W_i(\beta_i, \beta_j) = P_i(\beta_i, \beta_j)\,(U(x_i, \beta_i) - U(x_i, \beta_j)) + U(x_i, \beta_j)$. Then, $\lim_{\beta_i \to \beta_j} W_i(\beta_i, \beta_j) = U(x_i, \beta_j) = W_i(\beta_j, \beta_j)$ since $P_i(\beta_i, \beta_j)$ is bounded and $\lim_{\beta_i \to \beta_j} (U(x_i, \beta_i) - U(x_i, \beta_j)) = 0$. Similarly, $\lim_{\beta_j \to \beta_i} W_i = U(x_i, \beta_i) = W_i(\beta_i, \beta_i)$. ‖

As the second step, the following lemmas set forth that $\beta_i = \beta_j$ or $\beta_i = \hat{\beta}_{x_i}$ is not i's best strategy for β_j such that $\beta_j \neq \hat{\beta}_{x_i}$.

Lemma A2. Given $\beta_j \neq \hat{\beta}_{x_i}$, β_j is never a maximizer of W_i.

Proof. By definition, any convex combination of $U(x_i, \hat{\beta}_{x_i})$ and $U(x_i, \beta_j)$ with positive weights takes a higher value than $U(x_i, \beta_j)$. Therefore, $W_i(\hat{\beta}_{x_i}, \beta_j) > W_i(\beta_j, \beta_j)$ and i can increase its payoff by taking its ideal policy. ‖

Lemma A3. Given $\beta_j \neq \hat{\beta}_{x_i}$, $\hat{\beta}_{x_i}$ is never a maximizer of W_i.

Proof. If $\hat{\beta}_{x_i} > \beta_j$,

$$\frac{\partial W_i}{\partial \beta_i}|_{\beta_i = \hat{\beta}_{x_i}} = \frac{\partial P_i}{\partial \beta_i}|_{\beta_i = \hat{\beta}_{x_i}} \left(U(x_i, \hat{\beta}_{x_i}) - U(x_i, \beta_j) \right) < 0, \tag{A4}$$

from lemmas 1 and 3 and $\frac{\partial U}{\partial \beta_i}(x_i, \hat{\beta}_{x_i}) = 0$, and thus i can increase its payoff by choosing a value closer to β_j instead of $\hat{\beta}_{x_i}$. The case where $\hat{\beta}_{x_i} < \beta_j$ is similarly examined. ‖

From Lemma 4 and Lemmas A1 to A3, we have a candidate of the maximizer of $W_i(\cdot, \beta_j)$, i.e., i's best reaction, in the interval $(\min\{\hat{\beta}_{x_i}, \beta_j\}, \max\{\hat{\beta}_{x_i}, \beta_j\})$ given $\beta_j \neq \hat{\beta}_{x_i}$. It suffices to show that the maximizer is convex-valued in the interval in order to construct the existence theorem. See that in the interval $(\beta_j, \hat{\beta}_{x_i})$ or $(\hat{\beta}_{x_i}, \beta_j)$, the partial derivative of W_i with regard to β_i is given by

$$\begin{aligned}\frac{\partial W_i}{\partial \beta_i} &= \frac{\partial P_i}{\partial \beta_i}\left(U(x_i, \beta_i) - U(x_i, \beta_j)\right) + P_i \frac{\partial U}{\partial \beta_i} \\ &= \frac{\partial P_i}{\partial \beta_i}\left(U(x_i, \beta_i) - U(x_i, \beta_j)\right)\left(1 + \frac{P_i}{U(x_i, \beta_i) - U(x_i, \beta_j)}\eta_i\right). \qquad \text{(A5)}\end{aligned}$$

Let $\beta_j < \hat{\beta}_{x_i}$. Then, for $\beta_i \in (\beta_j, \hat{\beta}_{x_i})$, $\frac{\partial P_i}{\partial \beta_i} < 0$ from Lemmas 1 and 3, $U(x_i, \beta_i) - U(x_i, \beta_j) > 0$, and the sign of (A5) depends on the sign of $1 + \frac{P_i}{U(x_i,\beta_i) - U(x_i,\beta_j)}\eta_i$. See that $\frac{P_i}{U(x_i,\beta_i) - U(x_i,\beta_j)}(> 0)$ strictly decreases in β_i. Furthermore, under Assumption 3, $|\eta_i|$ weakly decreases in β_i, thus implying that $1 + \frac{P_i}{U(x_i,\beta_i) - U(x_i,\beta_j)}\eta_i$ strictly increases in β_i. Thus, $W_i(\cdot, \beta_j)$ is strictly quasiconcave in the interval $(\beta_j, \hat{\beta}_{x_i})$ and indeed i has a unique best reaction in the interval.

Let $\beta_j > \hat{\beta}_{x_i}$. Then, for $\beta_i \in (\hat{\beta}_{x_i}, \beta_j)$, $\frac{\partial P_i}{\partial \beta_i} > 0$, $U(x_i, \beta_i) - U(x_i, \beta_j) > 0$, and the sign of (A5) depends on the sign of $1 + \frac{P_i}{U(x_i,\beta_i) - U(x_i,\beta_j)}\eta_i$. See that $\frac{P_i}{U(x_i,\beta_i) - U(x_i,\beta_j)}(> 0)$ strictly increases in β_i, $|\eta_i|$ weakly increases in β_i, and hence $1 + \frac{P_i}{U(x_i,\beta_i) - U(x_i,\beta_j)}\eta_i$ strictly decreases in β_i. Thus, $W_i(\cdot, \beta_j)$ is strictly quasiconcave in the interval $(\hat{\beta}_{x_i}, \beta_j)$ and indeed i has a unique best reaction in the interval.

To summarize, i has a unique best reaction in β for any $\beta_j \in \beta$: it exists in $(\beta_j, \hat{\beta}_{x_i})$ if $\beta_j \in [\underline{\beta}, \hat{\beta}_{x_i})$; it corresponds to $\hat{\beta}_{x_i}$ if $\beta_j = \hat{\beta}_{x_i}$; it exists in $(\hat{\beta}_{x_i}, \beta_j)$ if $\beta_j \in (\hat{\beta}_{x_i}, \overline{\beta}]$.

(ii) As in Assumption 4, let's denote party i's best reaction function for every $\beta_j \in \beta$ by $b_i(\beta_j)$. Continuity of $b_i(\cdot)$ comes from continuity of $W_i(\cdot, \cdot)$, which has been proved in Lemma A1.

Let's define the function $b(\beta_L, \beta_R) = b_L(\beta_R) \times b_R(\beta_L)$. This function is a map from the nonempty, compact, and convex set β^2 into itself. Functions $\beta_L(\cdot)$ and $\beta_R(\cdot)$ are continuous and so $b(\cdot, \cdot)$ is. Thus, all the conditions of Brouwer's fixed point theorem are satisfied and hence there exists a fixed point $(\beta_L^*, \beta_R^*) \in \beta^2$ such that $(\beta_L^*, \beta_R^*) = b(\beta_L^*, \beta_R^*)$, which constitutes a Nash equilibrium of the game.

(iii) Define that $D_{R1} = \{(\beta_L, \beta_R) | \beta_L \in [\underline{\beta}, \hat{\beta}_{x_R}),\ \beta_R \in (\beta_L, \hat{\beta}_{x_R})\}$, $D_{R2} = (\hat{\beta}_{x_R}, \hat{\beta}_{x_R})$, $D_{R3} = \{(\beta_L, \beta_R) | \beta_L \in (\hat{\beta}_{x_R}, \overline{\beta}],\ \beta_R \in (\hat{\beta}_{x_R}, \beta_L)\}$, and $D_R = \cup_{k=1}^{3} D_{Rk}$. Thus, the pair of strategies $(\beta_L, b_R(\beta_L))$ exists in D_R for every β_L. Similarly, define that $D_{L1} = \{(\beta_L, \beta_R) | \beta_R \in [\underline{\beta}, \hat{\beta}_{x_L}),\ \beta_L \in (\beta_R, \hat{\beta}_{x_L})\}$, $D_{L2} = (\hat{\beta}_{x_L}, \hat{\beta}_{x_L})$, $D_{L3} = \{(\beta_L, \beta_R) | \beta_R \in (\hat{\beta}_{x_L}, \overline{\beta}],\ \beta_L \in (\hat{\beta}_{x_L}, \beta_R)\}$, and $D_L = \cup_{k=1}^{3} D_{Lk}$. Then, it is clear that there exists an equilibrium (β_L^*, β_R^*) in $D = D_L \cap D_R$. $\|$

Proof of Theorem 3. It follows immediately from Assumption 4 and Theorem 2 that the equilibrium $(\beta_L^{k*}, \beta_R^{k*})$, $k = 1, 2$, uniquely exists and $\hat{\beta}_{x_L} < \beta_L^{k*} < \beta_R^{k*} < \hat{\beta}_{x_R}$. Signify party i's probability of winning with H_k by P_i^k. Then, the equilibrium $(\beta_L^{k*}, \beta_R^{k*})$ satisfies the following condition:

$$\frac{\partial P_i^k}{\partial \beta_i}\left(\beta_i^{k*}, \beta_j^{k*}\right)\left(U(x_i, \beta_i^{k*}) - U(x_i, \beta_j^{k*})\right) + P_i^k\left(\beta_i^{k*}, \beta_j^{k*}\right)\frac{\partial U}{\partial \beta_i}(x_i, \beta_i^{k*}) = 0, \quad \text{(A6)}$$

where $i \in \{L, R\}$ and $j \in \{L, R | j \neq i\}$.

Now let (A6) hold with $k = 1$. Suppose that $H_1(\overline{s}^{1*}) > H_2(\overline{s}^{1*})$. Then, $P_L^1(\beta_L^{1*}, \beta_R^{1*}) = 1 - H_1(\overline{s}^{1*}) < 1 - H_2(\overline{s}^{1*}) = P_L^2(\beta_L^{1*}, \beta_R^{1*})$ from the proof of Lemma 3. Similarly, $P_R^1(\beta_R^{1*}, \beta_L^{1*}) > P_R^2(\beta_R^{1*}, \beta_L^{1*})$. Note that $\frac{\partial P_i^1}{\partial \beta_i}\left(\beta_i^{1*}, \beta_j^{1*}\right) = \frac{\partial P_i^2}{\partial \beta_i}\left(\beta_i^{1*}, \beta_j^{1*}\right)$ since $\frac{\partial P_L^k}{\partial \beta_L}(\beta_L^{1*}, \beta_R^{1*}) = -\frac{\partial H_k}{\partial \overline{s}}\frac{\partial \overline{s}}{\partial V_L}\frac{\partial V_L}{\partial \beta_L}(\beta_L^{1*}, \beta_R^{1*})$ and $\frac{\partial P_R^k}{\partial \beta_R}(\beta_R^{1*}, \beta_L^{1*}) = -\frac{\partial H_k}{\partial \overline{s}}\frac{\partial \overline{s}}{\partial V_L}\frac{\partial V_R}{\partial \beta_R}(\beta_R^{1*}, \beta_L^{1*})$ are constant for $k = 1, 2$ from the model composition.

Let $i = L$ and $j = R$. Given $(\beta_L^{1*}, \beta_R^{1*})$, with the prior of H_2,

$$\frac{\partial P_L^2}{\partial \beta_L}\left(\beta_L^{1*}, \beta_R^{1*}\right)\left(U(x_L, \beta_L^{1*}) - U(x_L, \beta_R^{1*})\right) + P_L^2\left(\beta_L^{1*}, \beta_R^{1*}\right)\frac{\partial U}{\partial \beta_L}(x_L, \beta_L^{1*}) < 0, \quad \text{(A7)}$$

since $\frac{\partial U}{\partial \beta_L}(x_L, \beta_L^{1*}) < 0$ and $P_L^1(\beta_L^{1*}, \beta_R^{1*}) < P_L^2(\beta_L^{1*}, \beta_R^{1*})$. To recover the equality, we have to have β_L' such that $\beta_L' < \beta_L^{1*}$, instead of β_L^{1*}, for given β_R^{1*} due to the strict quasiconcavity of $W_L(\cdot, \beta_R^{1*})$.

Let $i = R$ and $j = L$. Given $(\beta_L^{1*}, \beta_R^{1*})$, with the prior of H_2,

$$\frac{\partial P_R^2}{\partial \beta_R}\left(\beta_R^{1*}, \beta_L^{1*}\right)\left(U(x_R, \beta_R^{1*}) - U(x_R, \beta_L^{1*})\right) + P_R^2\left(\beta_R^{1*}, \beta_L^{1*}\right)\frac{\partial U}{\partial \beta_R}(x_R, \beta_R^{1*}) < 0, \quad \text{(A8)}$$

since $\frac{\partial U}{\partial \beta_R}(x_R, \beta_R^{1*}) > 0$ and $P_R^1(\beta_R^{1*}, \beta_L^{1*}) > P_R^2(\beta_R^{1*}, \beta_L^{1*})$. To recover the equality, we have to have β'_R such that $\beta'_R < \beta_R^{1*}$, instead of β_R^{1*}, for given β_L^{1*} due to the strict quasiconcavity of $W_R(\cdot, \beta_L^{1*})$.

From these discussions, with the common prior of H_2, L's and R's reaction curves pass through (β'_L, β_R^{1*}) and (β_L^{1*}, β'_R), respectively. See Figure 2 and consider four quadrants around $(\beta_L^{1*}, \beta_R^{1*})$. It follows from Assumption 4 that we have no equilibrium with H_2 in the southeast, northwest, and northeast quadrants since two parties' reaction curves under H_2 cannot intersect there. Therefore, there exists $(\beta_L^{2*}, \beta_R^{2*})$ in the southwest quadrant of $(\beta_L^{1*}, \beta_R^{1*})$, which leads to $(\beta_L^{1*}, \beta_R^{1*}) \gg (\beta_L^{2*}, \beta_R^{2*})$. The case where $H_1(\bar{s}^{1*}) < H_2(\bar{s}^{1*})$ is examined in the similar manner. ∥

A.2 Parties' Vote Share without Uncertainty (Derivation of (8))

From (1) and (3), type x prefers β_i to β_j if and only if

$$\frac{1+\beta_i}{2} x^{\beta_i} \quad > \quad \frac{1+\beta_j}{2} x^{\beta_j}, \tag{A9}$$

that is, iff

$$x^{\beta_i - \beta_j} \quad > \quad \frac{1+\beta_j}{1+\beta_i}. \tag{A10}$$

If $\beta_i < \beta_j$, (A10) is arranged as

$$x \quad < \quad \left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j - \beta_i}}, \tag{A11}$$

and the probability of (A11) is given by $F_0\left(\left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j-\beta_i}}\right) = \left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j-\beta_i}}$, which corresponds to party i's vote share.

If $\beta_i > \beta_j$, (A10) is arranged as

$$x \quad > \quad \left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i - \beta_j}}, \tag{A12}$$

and the probability of (A12) is given by $1-F_0\left(\left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i-\beta_j}}\right)=1-\left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i-\beta_j}}$.

Finally, if $\beta_i=\beta_j$, type x votes for party i with the probability of $\frac{1}{2}$.

A.3 Parties' Vote Share with Uncertainty (Derivation of (11))

If $\beta_i<\beta_j$, from (A11), $V_i^s(\beta_i,\beta_j|s)$ is given by

$$G\left(\left(\frac{1+\beta_i}{1+\beta_j}\right)^{\frac{1}{\beta_j-\beta_i}}|s\right)=G\left(V_i(\beta_i,\beta_j)|s\right). \tag{A13}$$

If $\beta_i>\beta_j$, from (A12), $V_i^s(\beta_i,\beta_j|s)$ is given by

$$1-G\left(\left(\frac{1+\beta_j}{1+\beta_i}\right)^{\frac{1}{\beta_i-\beta_j}}|s\right)=1-G\left(1-V_i(\beta_i,\beta_j)|s\right). \tag{A14}$$

If $\beta_i=\beta_j$, the voter votes for one of i and j randomly and hence $V_i^s(\beta_i,\beta_j|s)=\frac{1}{2}$.

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Figure 1: Parties' Reaction Curves

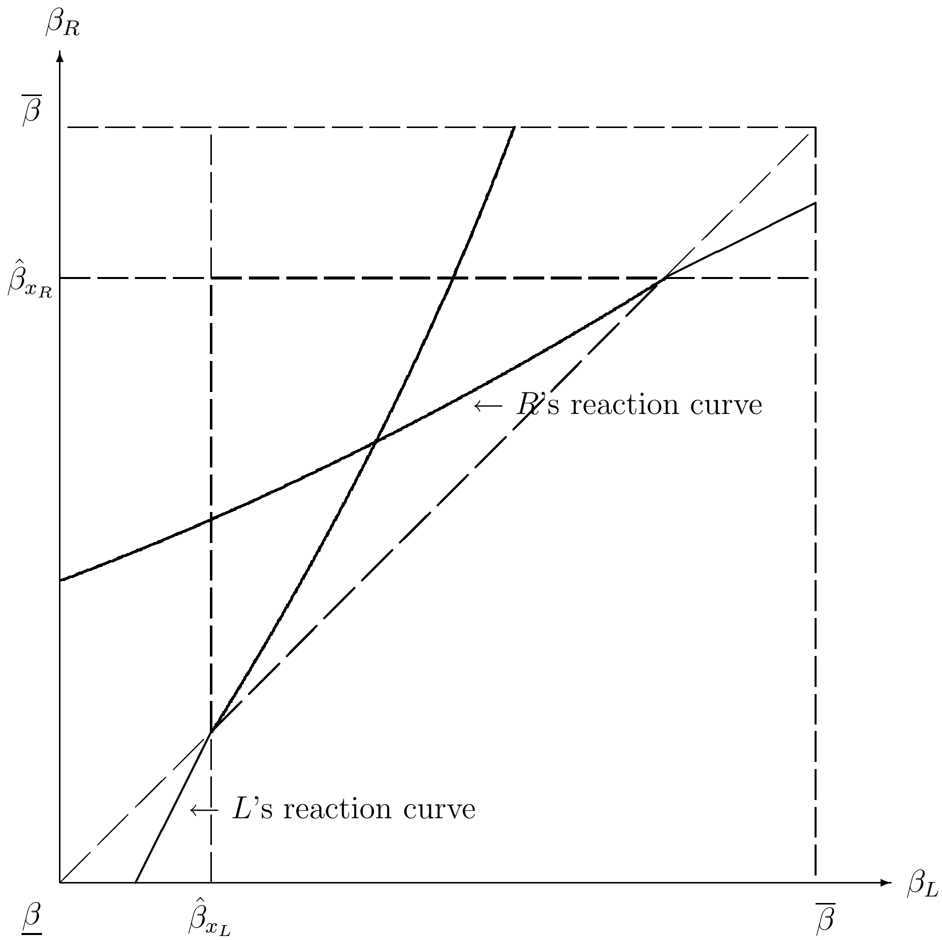

Figure 2: Shifts of Reaction Curves with H_2