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# On Validity of the Asymptotic Expansion Approach in Contingent Claim Analysis

Naoto Kunitomo
University of Tokyo

Akihiko Takahashi
The Industrial Bank of Japan

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Naoto Kunitomo[†]
and
Akihiko Takahashi[‡]

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### Abstract

Kunitomo and Takahashi (1995), and Takahashi (1997) have proposed a new methodology, called *Small Disturbance Asymptotics*, for the valuation problem of financial contingent claims when the underlying asset prices follow a general class of continuous Itô processes. It can be applicable to a wide range of valuation problems including complicated contingent claims associated with the Black-Scholes model and the term structure model of interest rates in the Heath-Jarrow-Morton framework. Our approach can be rigorously justified by an infinite dimensional analysis called the Watanabe-Yoshida theory on the Malliavin Calculus recently developed in stochastic analysis.

### Key Words

Valuation of Financial Contingent Claims, Asymptotic Expansion, Small Disturbance Asymptotics, Validity, Watanabe—Yoshida theory, Malliavin Calculus

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*This note contains some portions of Kunitomo and Takahashi (1995), and Takahashi (1997). This is due to our intention that it should be complete and readable for the possible audience. We thank Professor N. Yoshida for some discussions on the related technical issues. However, we are responsible for the remaining errors in this note.

[†]Faculty of Economics, University of Tokyo, Bunkyo-ku, Hongo 7-3-1, Tokyo 113, JAPAN.

[‡]Financial Engineering Department, The Industrial Bank of Japan, Marunouchi 1-3-3, Chiyoda-ku, Tokyo 100, Japan.

# 1 Introduction

In the past decade various contingent claims including futures, options, swaps, and other derivative securities have been introduced and actively traded in financial markets. Except some simple cases such as the original Black-Scholes model in which the underlying assets follow the geometric Brownian motions and the risk free rate is constant, however, it has been difficult to derive the explicit formulae for the fair market values of these financial contingent claims. Then several numerical methods for the valuation of financial contingent claims in more general situations have been proposed and used in finance.

Meanwhile, Kunitomo and Takahashi (1995), and Takahashi (1997) have presented a new methodology called *Small Disturbance Asymptotic Theory* which is applicable to the valuation problem of financial contingent claims including such as various futures, options, and other derivatives on stock, exchange rates, interest rates, and others when the underlying asset prices follow the general class of continuous Itô processes. They have shown that the asymptotic expansion method in the small disturbance asymptotics can be effectively applicable to various valuation problems of contingent claims in financial economics and gives rather simple formulae when the underlying asset prices follow the general class of continuous Itô processes. The asymptotic expansion approach is very simple, but gives an unified method to the valuation problem of interest rate based contingent claims. However, they have pointed out that some economic considerations of theoretical restrictions on the structure of stochastic processes should be indispensable when we apply the asymptotic expansion method to the valuation problem of financial contingent claims. We need strong conditions on the form of their drift functions because of the no-arbitrage theory, which has been standard in financial economics. It implies, for instance, that the continuous stochastic processes for spot interest rate and forward rates are not necessarily Markovian or diffusion processes in the usual sense.

More specifically, for the term structure model of interest rates in the HJM framwork, let $P(t,T)$ denote the price of the discount bond at $t$ with maturity date $T$ $(0 \leq t \leq T \leq \bar{T} < +\infty)$. We use the notational convention that $P(T,T) = 1$ at maturity date $t = T$ for normalization. Let also $P(t,T)$ be continuously differentiable with respect to $T$ and $P(t,T) > 0$ for $0 \leq t \leq T \leq \bar{T}$. Then the instantaneous forward rate at $s$ for future date $t$ $(0 \leq s \leq t \leq T)$ is defined by

$$f(s,t) = -\frac{\partial \log P(s,t)}{\partial t} \quad .$$

We consider the situation when a family of forward rates processes $\{f(s,t)\}$ for

$0 \leq s \leq t \leq T$ follow the stochastic integral equation :

$$
\begin{aligned}
f(s,t) = f(0,t) \quad &+ \quad \int_0^s \left[ \sum_{i=1}^m \sigma_i^*(f(v,t),v,t) \int_v^t \sigma_i^*(f(v,y),v,y)dy \right] dv \\
&+ \quad \int_0^s \sum_{i=1}^m \sigma_i^*(f(v,t),v,t)dw_i(v) ,
\end{aligned}
\tag{1.1}
$$

where $f(0,t)$ are non-random initial forward rates, $\{w_i(v), i = 1,\cdots,m\}$ are $m$ independent Brownian motions, and $\{\sigma_i^*(f(v,t),v,t), i = 1,\cdots,m\}$ are the volatility functions [1]. The initial forward rates are observable and hence regarded as fixed. Because $f(s,t)$ is continuous at $s = t$ for $0 \leq s \leq t \leq T$, the spot interest rate at $t$ can be defined by $r(t) = f(t,t)$ . In this framework of stochastic interest rate economy, Kunitomo and Takahashi (1995) have investigated the valuation of contingent claims when a family of forward rate processes obey the stochastic integral equation :

(1.2)

$$
\begin{aligned}
f^{(\varepsilon)}(s,t) = f(0,t) \quad &+ \quad \varepsilon^2 \int_0^s \left[ \sum_{i=1}^m \sigma_i(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy \right] dv \\
&+ \quad \varepsilon \int_0^s \sum_{i=1}^m \sigma_i(f^{(\varepsilon)}(v,t),v,t)dw_i(v) ,
\end{aligned}
$$

where $0 < \varepsilon \leq 1$ and $0 \leq s \leq t \leq T \leq \bar{T}$. The volatility functions $\sigma_i(f^{(\varepsilon)}(s,t),s,t)$ depend not only on s and t, but also on $f^{(\varepsilon)}(s,t)$ in the general case. Let $f^{(\varepsilon)}(s,t)$ be continuous at $s = t$ for $0 \leq s \leq t \leq T \leq \bar{T}$. Then the spot interest rate process can be defined by

$$
r^{(\varepsilon)}(t) = f^{(\varepsilon)}(t,t) . \tag{1.3}
$$

Then a ***small disturbance*** asymptotic theory can be constructed in this case by considering the situation when $\epsilon \to 0$ and we can develop the valuation method of contingent claims based on $\{S^{(\epsilon)}(t)\}$ with the no-arbitrage theory.

On the other hand, for the Black-Scholes economy, Takahashi (1997) has systematically investigated the valuation problem of various contingent claims when $d \times 1$ asset prices $S(t) = (S_i(t))$ $(i = 1,\cdots,d)$ follow the general class of diffusion processes :

$$
S_i(t) = S_i(0) + \int_0^t \mu_i(S(v),v)dv + \int_0^t \sum_{j=1}^m \sigma_{ij}^*(S(v),v)dw_j(v) , \tag{1.4}
$$

[1]The restrictions in (1.1) on the drift functions we are imposing in this arbitrage-free formulation have been derived by Heath, Jarrow, and Morton (1992).

where $d \times 1$ vector $\mu(S(v), v)$ and $d \times m$ matrix $\sigma^*(S(v), v) = (\sigma^*_{ij}(S(v), v))$ are the instantaneous mean and the volatility functions, and $w(v) = (w_i(v))$ is the $m \times 1$ standard Brownian motion. It is evident that the Black-Scholes economy and the Cox-Ingersol-Ross model on the spot interest rate are special cases of this framework. In the simplest Black-Scholes economy, we have to change the underlying measure and some restrictions on the drift terms are imposed because of the no-arbitrage theory in finance. (See Chapter 6 of Duffie (1996), for instance.) Then we consider the situation when $S^{(\epsilon)}(t)$ satisfies the integral equation of diffusion type:

$$S^{(\epsilon)}(t) = S(0) + \int_0^t r(S^{(\epsilon)}(v), v) S^{(\epsilon)}(v) dv + \epsilon \int_0^t \sigma(S^{(\epsilon)}(v), v) dw(v) \ , \tag{1.5}$$

where $\sigma(S^{(\epsilon)}(v), v)$ $(d \times m)$ is the volatility term with $0 < \epsilon \leq 1$ and $r(\cdot, \cdot)$ is the risk free (positive) interest rate. Then a ***small disturbance*** asymptotic theory can be constructed in this case by considering the situation when $\epsilon \to 0$ and we can develop the valuation method of contingent claims based on $\{S^{(\epsilon)}(t)\}$ with the no-arbitrage theory.

The main purpose of this note is to give the validity of the asymptotic expansion approach along the line called the Watanabe-Yoshida theory on the Malliavin Calculus recently developed in stochastic analysis. The Malliavin Calculus has been developed as an infinite dimensional analysis of Wiener functional by several probablists in the last two decades. We are intending to apply this powerful calculus on continuous stochastic processes to the valuation problem of financial contingent claims along the line developed by Watanabe (1987) and subsequently by Yoshida (1992a,b). However, we should mention that the continuous time stochastic processes appeared in financial economics are not necessarily time-homogeneous Markovian in the usual sense while the existing asymptotic expansion methods initiated by Watanabe (1987) and refined by Yoshida (1992a,b) in stochastic analysis have been treated for the case of time homogeneous Markovian processes. Hence we need to extend the existing results on the validity of the asymptotic expansion approach. Also the mathematical devices used in the Watanabe-Yoshida theory have not been standard for finance as well as in many applied fields, and there could be many mathematical refinements for researchers except probablists. Hence we are intending a complete discussion on the validity of the asymptotic expansion apprach, although some of our discussions could be regarded as rather straightforward applications of the existing results in the Watanabe-Yoshida theory from the view of stochastic analysis. We think that our developments are quite new to many researchers and practitioners in finance.

Some of the following derivations have been already reported in Kunitomo and Takahashi (1995), and Takahashi (1997). However, these papers have cut some important proofs on the validity of our asymptotic theory due to the lack of space problems. This note tries to make the complete discussion of the validity of our approach in a unified way.

In Section 2, we give some preliminary mathematical devices, which shall be needed in the following derivations. Section 3 is on the validity of our approach for the continuous Markovian setting, while Section 4 is on the validity of our approach for the HJM setting of the interest rates models. Some concluding remarks will be given in Section 5.

## 2. Preliminary Mathematics

We shall first prepare the fundamental results including *Theorem 2.2* of Yoshida (1992b), which is in turn a truncated version of *Theorem 2.3* of Watanabe (1987). The theory by Watanabe (1987) on the Malliavin Calculus and Theorem 2.2 of Yoshida (1992b) are the fundamental ingredients and the key results to show the validity of our asymptotic expansion method in this note from the view of mathematics. This is the reason why we call them as ***the Watanabe– Yoshida theory*** on ***the Malliavin Calculus***. For our purpose, we shall freely use the notations by Ikeda and Watanabe (1989) as a standard textbook. The results in this section are given without any proof. The interested readers may see Watanabe (1984), Watanabe (1987), Ikeda and Watanabe (1989), Yoshida (1992b, 1995), and other related works in stochastic analysis.

### 2.1 Some Notations

Let $\boldsymbol{W}$ be the $m$–dimensional Wiener space, which is a Banach space consisting of the totality of continuous functions $w : [0,T] \to R^m$ $(w(0) = 0)$ with the topology induced by countable system of norms

$$\| w \|_n = \max_{0 \leq t \leq n} |w(t)| \ (n = 1, 2, \cdots).$$

Let also $\boldsymbol{H}$ be the Cameron-Martin subspace of $\boldsymbol{W}$, where $h(t) = (h^j(t)) \in \boldsymbol{H}$ is in $\boldsymbol{W}$ and is absolutely continuous on $[0,T]$ with square integrable derivative $\dot{h}(t)$ endowed with an inner product defined by

$$< h_1, h_2 >_H = \int_0^T \dot{h}_1(s) \cdot \dot{h}_2(s) ds \ .$$

A function $f : \boldsymbol{W} \mapsto \boldsymbol{R}$ is called a polynomial functional if there exist $n \in \boldsymbol{N}$, $h_1, h_2, \cdots, h_n \in \boldsymbol{H}$ and a real polynomial $p(x_1, x_2, \cdots, x_n)$ of n-variables such that

$$f(w) = p([h_1](w), [h_2](w), \cdots, [h_n](w)) ,$$

where $h_i \in \boldsymbol{H}$ and

$$[h_i](w) = \sum_{j=1}^{m} \int_0^T \dot{h}_i^j dw_j$$

are defined in the sense of stochastic integrals.

The standard $L_p$-norm of $\boldsymbol{R}-$valued Wiener functional $F$ is defined by

$$\|F\|_p = (\int_W |F|^p P(dw))^{1/p} .$$

Also a sequence of the norm of $\boldsymbol{R}$-valued Wiener functional $F$ for any $s \in \boldsymbol{R}$, and $p \in (1, \infty)$ is defined by

$$\|F\|_{p,s} = \|(I - \mathcal{L})^{s/2} F\|_p ,$$

where $\mathcal{L}$ is the Ornstein-Uhlenbeck operator and $\| \cdot \|_p$ is the $L_p$-norm in the stochastic analysis. In this notation

$$(I - \mathcal{L})^{s/2} F = \sum_{n=0}^{\infty} (1 + n)^{s/2} J_n F ,$$

where $J_n$ are the projection operators in the Wiener's homogeneous chaos decomposition in $L_2$. They are constructed by the totality of $\boldsymbol{R}-$valued polynomials of degree at most $n$ denoted by $\boldsymbol{P}_n$ .

Let $\boldsymbol{P}(\boldsymbol{R})$ denote the totality of $\boldsymbol{R}-$valued polynomials on the Wiener space $(\boldsymbol{W}, P)$. Then $\boldsymbol{P}(\boldsymbol{R})$ is dense in $L_p(\boldsymbol{R})$ and can be extended to $\boldsymbol{S}$, which is the totality of smooth functionals. Then we construct the Banach space $\boldsymbol{D}_p^s(\boldsymbol{R})$ as the completion of $\boldsymbol{P}(\boldsymbol{R})$ with respect to $\| \cdot \|_{p,s}$. The dual space of $\boldsymbol{D}_p^s(\boldsymbol{R})$ is the $\boldsymbol{D}_q^{-s}(\boldsymbol{R})$, where $s \in \boldsymbol{R}, p > 1$, and $1/p + 1/q = 1$. The space $\boldsymbol{D}^\infty(\boldsymbol{R}) = \cap_{s>0} \cap_{1<p<+\infty} \boldsymbol{D}_p^s(\boldsymbol{R})$ is the set of Wiener functionals and $\tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{R}) = \cup_{s>0} \cap_{1<p<+\infty}$ $\boldsymbol{D}_p^{-s}(\boldsymbol{R})$ is a space of generalized Wiener functionals. For $F \in \boldsymbol{P}(\boldsymbol{R})$ and $h \in \boldsymbol{H}$, the derivative of $F$ in the direction of $h$ is defined by

$$D_h F(w) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{F(w + \varepsilon h) - F(w)\} .$$

Then for $F \in \boldsymbol{P}(\boldsymbol{R})$ and $h \in \boldsymbol{H}$ there exists $DF \in \boldsymbol{P}(\boldsymbol{H} \otimes \boldsymbol{R})$ such that $D_h F(w) =< DF(w), h >_H$, where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$ and $DF$ is called the $H-$derivative of $F$. Then there exists a unique $DF \in \boldsymbol{S}(\boldsymbol{H} \otimes \boldsymbol{R})$ .

It is known that the norm $\| \cdot \|_{p,s}$ is equivalent to the norm $\sum_{k=0}^{s} \|D^k \cdot \|_p$. For $F \in \boldsymbol{D}^{\infty}(\boldsymbol{R})$, we can define the Malliavin-covariance by

$$\sigma(F) =< DF(w), DF(w) >_H ,$$

where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$. It is known that the operator $D$ can be well-defined in $\boldsymbol{D}^{\infty}(\boldsymbol{R})$.

More generally, for a separable Hilbert space $\boldsymbol{E}$, a function $f : \boldsymbol{W} \mapsto \boldsymbol{E}$ is called a polynomial functional if there exist $n \in \boldsymbol{N}$, $h_1, h_2, \cdots, h_n \in \boldsymbol{H}$ and a real polynomial $p(x_1, x_2, \cdots, x_n)$ of n-variables such that

$$f(w) = \sum_{i=1}^{d} p_i([h_1](w), [h_2](w), \cdots, [h_n](w))e_i$$

for some $d \in \boldsymbol{N}$, where $e_1, \cdots, e_d \in \boldsymbol{E}$ and $p_1, \cdots, p_d$ are real polynomials. The totality of $\boldsymbol{E}-$valued polynomial functions and the totality of $\boldsymbol{E}-$valued smooth functionals are denoted by $\boldsymbol{P}(\boldsymbol{E})$ and $\boldsymbol{S}(\boldsymbol{E})$, respectively. By extending the above construction for $\boldsymbol{P}(\boldsymbol{R})$ to $\boldsymbol{S}(\boldsymbol{E})$, there exists $DF \in \boldsymbol{S}(\boldsymbol{H} \otimes \boldsymbol{E})$ such that $D_h F(w) =< DF(w), h >_H$, where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$. By repeating this procedure, we can define the $k-$th order $H-$derivative $D^k F \in \boldsymbol{S}(\boldsymbol{H}^{\otimes^k} \otimes \boldsymbol{R}^d)$ for $k \geq 1$ and $d \geq 1$. (See Chapter V of Ikeda and Watanabe (1989) or Yoshida (1995) for the mathematical details.)

## 2.2 Asymptotic Expansions

Let $X^{(\varepsilon)}(w), \varepsilon \in (0, 1]$ be a Wiener functional with a parameter $\varepsilon$. Then we need to define the asymptotic expansion of $X^{(\varepsilon)}(w)$ in the proper mathematical sense. For $k > 0$ , $X^{(\varepsilon)}(w) = O(\varepsilon^k)$ in $\boldsymbol{D}_p^s$ as $\varepsilon \downarrow 0$ means that

$$\limsup_{\varepsilon \downarrow 0} \frac{\| X^{(\varepsilon)} \|_{p,s}}{\varepsilon^k} < +\infty ,$$

where we use the notation $\boldsymbol{D}_p^s = \boldsymbol{D}_p^s(\boldsymbol{R}^d)$. If for all $p > 1, s > 0$ and every $k = 1, 2, \cdots$,

$$X^{(\varepsilon)}(w) - (g_1 + \varepsilon g_2 + \cdots + \varepsilon^{k-1} g_k) = O(\varepsilon^k)$$

in $\boldsymbol{D}_p^s$ as $\varepsilon \downarrow 0$, then we say that $X^{(\varepsilon)}(w)$ has an asymptotic expansion :

$$X^{(\varepsilon)}(w) \sim g_1 + \varepsilon g_2 + \cdots$$

in $\boldsymbol{D}^{\infty}$ as $\varepsilon \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^{\infty}$.
Also if for every $k = 1, 2, \cdots$, there exists $s > 0$ such that, for all $p > 1$, $X^{(\varepsilon)}(w), g_1, g_2, \cdots \in \boldsymbol{D}_p^{-s}$ and

$$X^{(\varepsilon)}(w) - (g_1 + \varepsilon g_2 + \cdots + \varepsilon^{k-1} g_k) = O(\varepsilon^k)$$

in $\boldsymbol{D}_p^{-s}$ as $\varepsilon \downarrow 0$, then we say that $X^{(\varepsilon)}(w) \in \tilde{\boldsymbol{D}}^{-\infty}$ has an asymptotic expansion:

$$X^{(\varepsilon)}(w) \sim g_1 + \varepsilon g_2 + \cdots$$

in $\tilde{\boldsymbol{D}}^{-\infty}$ as $\varepsilon \downarrow 0$ with $g_1, g_2, \cdots \in \tilde{\boldsymbol{D}}^{-\infty}$.

With these notations we are ready to state a simplified version of *Theorem 2.2* of Yoshida (1992b), which is a truncated version of *Theorem 2.3* of Watanabe (1987). The validity of the asymptotic expansion in this note is obtained by showing that the conditions of this theorem are met. We shall apply the following theorem when $d = 1$ and often use the convention $\boldsymbol{D}^\infty(\boldsymbol{R}^d) = \boldsymbol{D}^\infty$ for instance.

**Theorem 2. 1** [ *Yoshida (1992b)* ]
*Let $\psi(y)$ be a smooth function such that $0 \le \psi(y) \le 1$ for $y \in \boldsymbol{R}$, $\psi(y) = 1$ for $|y| \le 1/2$ and $\psi = 0$ for $|y| \ge 1$. Suppose a set of sufficient conditions given below are satisfied.*
*(1) $X^{(\varepsilon)}(w) = (X_i^{(\varepsilon)}(w)) \in \boldsymbol{D}^\infty(\boldsymbol{R}^d)$ .*
*(2) $X^{(\varepsilon)}(w)$ has the asymptotic expansion :*

$$X^{(\varepsilon)}(w) \sim g_1 + \varepsilon g_2 + \cdots$$

*in $\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ as $\varepsilon \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^\infty(\boldsymbol{R}^d)$.*
*(3) $\{\eta_c^\varepsilon(w); \varepsilon \in (0,1]\}$ is $O(1)$ in $\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ as $\varepsilon \downarrow 0$ where $c > 0$ .*
*(4)* [2] *There exists $c_0 > 0$ such that for $c > c_0$ and any $p > 1$,*

$$\sup_{\varepsilon \in (0,1]} \mathbf{E}[1_{\{|\eta_c^\varepsilon| \le 1\}} \left(\det \sigma(X^{(\varepsilon)})\right)^{-p}] < \infty ,$$

*where $\sigma(X^{(\varepsilon)}) = (\sigma_{ij}(X^{(\varepsilon)})) = (< DX_i^{(\varepsilon)}(w), DX_j^{(\varepsilon)}(w) >_H)$.*
*(5) For any $k \ge 1$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{-k} P\{|\eta_c^\varepsilon| > \frac{1}{2}\} = 0 .$$

*(6) Let $\phi^{(\varepsilon)}(x)$ be a smooth function in $(x, \varepsilon)$ on $\boldsymbol{R}^d \times (0,1]$ with all derivatives of polynomial growth order in $x$ uniformly in $\varepsilon$.*

*Then, $\psi(\eta_c^\varepsilon)\phi^{(\varepsilon)}(X^{(\varepsilon)}) I_B(X^{(\varepsilon)})$ has an asymptotic expansion:*

$$\psi(\eta_c^\varepsilon)\phi^{(\varepsilon)}(X^{(\varepsilon)}) I_B(X^{(\varepsilon)}) \sim \Phi_0 + \varepsilon \Phi_1 + \cdots \tag{2.1}$$

*in $\tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{R}^d)$ as $\varepsilon \downarrow 0$, where $\mathcal{B}$ is a Borel set and $\Phi_0, \Phi_1, \cdots$ are determined by the formal Taylor expansion.*

[2]This is the key condition that the Malliavin covariance of $X^\varepsilon(w)$ is uniformly non-degenerate with a truncation.

As a remark of this section, we have to mention an intuitive meaning of the asymptotic expansion (2.1) in the above theorem. It has been known in stochastic analysis that for $s \leq s'$ and $1 < p \leq p'$ we have $\|F\|_{p,s} \leq \|F\|_{p',s'}$ for $F \in \mathbf{P}(\mathbf{R}^d)$ and $\mathbf{D}_p^0(\mathbf{R}^d) = \mathbf{L}_p(\mathbf{R}^d)$. (See Ikeda and Watanabe (1989) p.362.) Then for any integer $k \geq 1$, the asymptotic expansion in (2.1) implies

$$\begin{aligned}(2.2) \qquad \limsup_{\varepsilon \downarrow 0} \mathbf{E}[\frac{1}{\varepsilon^k}|\psi(\eta_c^\varepsilon)\phi^\varepsilon(X^{(\varepsilon)})I_B(X^{(\varepsilon)}) \\ -(\Phi_0 + \varepsilon\Phi_1 + \cdots + \varepsilon^{k-1}\Phi_{k-1})|] < +\infty\end{aligned}$$

if we use the expectation operation in the proper mathematical sense. The precise mathematical meanings of (2.1)-(2.2), the generalized expectation operations for generalized Wiener functionals, and the related issues will be discussed at the end of Section 3. (See Chapter V of Ikeda and Watanabe (1989).) Kunitomo and Takahashi (1995), and Takahashi (1997) did use more intuitive methods for deriving asymptotic expansions of random variables and their expected values in continuous stochastic processes, which are based on the characteristic functions of random variables. Although their methods on asymptotic expansions are intuitive and formal, the resulting formulae can be rigorously justified by the arguments based on *Theorem 2.1.*

## 3. The Validity in the Black-Scholes Economy

Now we give the proof of validity of our method in the Black-Sholes economy. Without loss of generality, we consider the case when $d = m = 1$ because more complicated notations are needed in the general case [3] . For the fixed $T > 0$ and $\varepsilon \in (0, 1]$,

$$(3.1) \qquad S_T^{(\varepsilon)} = S_0 + \int_0^T \mu(S_s^{(\varepsilon)}, s)ds + \int_0^T \varepsilon\sigma(S_s^{(\varepsilon)}, s)d\tilde{w}_s ,$$

where

$$\mu(S_s^{(\varepsilon)}, s) = r(S_s^{(\varepsilon)}, s)S_s^{(\varepsilon)}$$

and $\sigma(S_s^{(\varepsilon)}, s)$ are $[0, T] \times R \to R$ and Borel measurable in $(S_s^{(\varepsilon)}, s)$. For the notational convenience, we shall use this formulation of the drift function in this section. We further assume that the drift and the volatility functions are $C^\infty(R \to R)$ for $s \in [0, T]$ with bounded derivatives of any order in the first argument. That is, for the first argument there exist $M_i > 0$ $(i = 1, 2)$ such that

[3] In the general case, we need some notations and the corresponding assumptions to the ones we use in Section 3. On this issue we shall give only some remarks in the text for illustrations.

$$\sup_{S\in R,0\leq s\leq T}|\frac{\partial^k\mu(S_s,s)}{\partial S_s^k}| < M_1 \,, \tag{3.2}$$
$$\sup_{S\in R,0\leq s\leq T}|\frac{\partial^k\sigma(S_s,s)}{\partial S_s^k}| < M_2$$

for any $k = 1,2,3,\cdots$. We further assume that there exists a positve $M_3 > 0$ such that

$$\sup_{0\leq s\leq T}[|\mu(0,s)|+|\sigma(0,s)|] < M_3 \,. \tag{3.3}$$

These conditions imply that there exist some positve $K_i > 0$ $(i = 1,2)$ such that for all $s \in [0,T]$,

$$|\mu(S_s^{(\varepsilon)},s)|+|\sigma(S_s^{(\varepsilon)},s)| < K_1(1+|S_s^{(\varepsilon)}|) \tag{3.4}$$

and

$$|\mu(S_{1s}^{(\varepsilon)},s)-\mu(S_{2s}^{(\varepsilon)},s)|+|\sigma(S_{1s}^{(\varepsilon)},s)-\sigma(S_{2s}^{(\varepsilon)},s)| < K_2|S_{1s}^{(\varepsilon)}-S_{2s}^{(\varepsilon)}| \,. \tag{3.5}$$

Hence the standard argument (i.e. Ikeda and Watanabe (1989)) shows the existence of the unique strong solution which has continuous sample paths and is in $L_p$ for any $1 \leq p < \infty$. In the remaining of the section, we will discuss the validity of the asymptotic expansion of $\phi(X_T^{(\varepsilon)})I_{\mathcal{B}}(X_T^{(\varepsilon)})$, where $X_T^{(\varepsilon)}$ is defined by $X_T^{(\varepsilon)} = \frac{S_T^{(\varepsilon)}-S_T^0}{\varepsilon}$ and $\mathcal{B}$ is a Borel set.

In the typical example of *European call options*, we take $\phi(x) = (x+y)$ and $I_{\mathcal{B}}(x) = \{x \geq -y\}$, where $y$ is a constant. Another simple application in this section is *the Average Options*, which is sometimes called Asian options. For this example we shall discuss the validity of the asymptotic expansion of $\phi(Z_T^{(\varepsilon)})I_{\mathcal{B}}(Z_T^{(\varepsilon)})$, where $Z_T^{(\varepsilon)}$ is defined by

$$Z_T^{(\varepsilon)} = \frac{1}{\varepsilon}\left[\int_0^T f(S_s^{(\varepsilon)})ds - \int_0^T f(S_s^{(0)})ds\right] \tag{3.6}$$

and $f(x)$ is a $C^\infty(R \to R)$ function. If we take $f(x) = x$ and $I_{\mathcal{B}}(x) = \{x \geq -y\}$, we have the standard Average options case, which has been called *Asian Options.*

First, we shall show that the $S_T^{(\varepsilon)}$ in the general case is a smooth Wiener functional in the sense of Malliavin.

**Theorem 3. 1** *Under the assumptions we have made, $S_T^\varepsilon$ is in $\boldsymbol{D}^\infty$ and has an asymptotic expansion :*

$$S_T^{(\varepsilon)} \sim S_T^0 + \varepsilon g_{1T} + \varepsilon^2 g_{2T} + \cdots \tag{3.7}$$

*is in $\boldsymbol{D}^\infty$ as $\varepsilon \downarrow 0$ with $g_{1T}, g_{2T}, \cdots \in \boldsymbol{D}^\infty$.*

**Proof** : The first part of our proof is to show that $S_T^{(\varepsilon)}$ in $\boldsymbol{D}^\infty$. For this purpose, let us define a stochastic process $\{Y_t^{(\varepsilon)}\}$ by

$$dY_t^{(\varepsilon)} = \partial\mu(S_t^{(\varepsilon)}, t)Y_t^{(\varepsilon)}dt + \varepsilon\partial\sigma(S_t^{(\varepsilon)}, t)Y_t^{(\varepsilon)}dw_t \ , \tag{3.8}$$

where $Y_0^{(\varepsilon)} = 1$, and $\partial\mu$ and $\partial\sigma$ denote the $\frac{\partial\mu}{\partial S^{(\varepsilon)}}$ and $\frac{\partial\sigma}{\partial S^{(\varepsilon)}}$, respectively. The we see $Y^{(\varepsilon)}$ has the unique strong solution and $Y^{(\varepsilon)} \in L_p$. Let $W_t^{(\varepsilon)} = Y_t^{(\varepsilon)-1}$. Then $W_t^{(\varepsilon)}$ satisfies the stochastic differential equation :

$$dW_t^{(\varepsilon)} = -\{\partial\mu(S_t^{(\varepsilon)}, t) - \varepsilon^2\partial\sigma(S_t^{(\varepsilon)}, t)^2\}W_t^{(\varepsilon)}dt - \varepsilon\partial\sigma(S_t^{(\varepsilon)}, t)W_t^{(\varepsilon)}dw_t,$$

$W_0^{(\varepsilon)} = 1$, and $W_t^{(\varepsilon)}$ has also the unique strong solution and $Y_t^{(\varepsilon)-1} \in L_p$.

As the first step, we calculate the first order H-derivative of $S_T^{(\varepsilon)}$. For any $h \in \boldsymbol{H}$, we note that $D_h S_T^{(\varepsilon)}$ satisfies

$$D_h S_T^{(\varepsilon)} = \int_0^T \varepsilon\partial\sigma(S_s^{(\varepsilon)}, s)D_h S_s^{(\varepsilon)}dw(s) + \int_0^T \partial\mu(S_s^{(\varepsilon)}, s)D_h S_s^{(\varepsilon)}ds + \int_0^T \varepsilon\sigma(S_s^{(\varepsilon)}, s)\dot{h}_s ds \ .$$

Then by using *Lemma 3.3* below, we have that for $h \in \boldsymbol{H}$,

$$D_h S_T^{(\varepsilon)} = \int_0^T Y_T^{(\varepsilon)}Y_s^{(\varepsilon)-1}\varepsilon\sigma(S_s^{(\varepsilon)}, s)\dot{h}_s ds \ .$$

We note that this relation is a result of the application of Lemma 3.2 below. Then we have that for the first order H-derivative

$$|DS_T^{(\varepsilon)}|_H^2 = \int_0^T |Y_T^{(\varepsilon)}Y_s^{(\varepsilon)-1}\varepsilon\sigma(S_s^{(\varepsilon)}, s)|^2 ds.$$

We note

$$|DS_T^{(\varepsilon)}|_H^2 \leq \varepsilon^2|Y_T^{(\varepsilon)}|^2\left[\int_0^T |Y_s^{(\varepsilon)-1}|^2K^2(1 + |S_s^{(\varepsilon)}|)^2 ds\right].$$

Then,

$$\mathbf{E}\left[|DS_T^{(\varepsilon)}|_H^2\right] \leq \varepsilon^2\mathbf{E}\left[|Y_T^{(\varepsilon)}|^2\{\int_0^T |Y_s^{(\varepsilon)-1}|^2K^2(1 + |S_s^{(\varepsilon)}|)^2 ds\}\right] \ .$$

Likewise for any $2 < p < \infty$, we can show

$$\mathbf{E}\left[|DS_T^{(\varepsilon)}|_H^p\right] \leq (\varepsilon K)^p T^{(\frac{p-2}{2})}\mathbf{E}\left[|Y_T^{(\varepsilon)}|^p\{\int_0^T |Y_s^{(\varepsilon)-1}|^p(1 + |S_s^{(\varepsilon)}|)^p ds\}\right].$$

In our evaluation of expectations, we repeatedly use the Hölder inequality :

$$\mathbf{E}[|x_s y_s|] \leq \mathbf{E}[|x_s|^p]^{\frac{1}{p}}\mathbf{E}[|y_s|^q]^{\frac{1}{q}}$$

where $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, and the inequality,

$$(|x| + |y|)^p \leq 2^{(p-1)}(|x|^p + |y|^p)$$

for $p \geq 1$. By using these inequalities and Fubini's theorem, we can evaluate the right hand side of the last equation as

$$\begin{aligned}
&\mathbf{E}\left[|Y_T^{(\varepsilon)}|^p \{\int_0^T |Y_s^{(\varepsilon)-1}|^p (1+|S_s^{(\varepsilon)}|)^p ds\}\right] \\
\leq\ & \mathbf{E}\left[|Y_T^{(\varepsilon)}|^{2p}\right]^{\frac{1}{2}} \mathbf{E}\left[(\int_0^T \{|Y_s^{(\varepsilon)-1}|(1+|S_s^{(\varepsilon)}|)\}^p ds)^2\}\right]^{\frac{1}{2}} \\
\leq\ & \mathbf{E}\left[|Y_T^{(\varepsilon)}|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}} \mathbf{E}\left[\int_0^T |Y_s^{(\varepsilon)-1}|^{2p}(1+|S_s^{(\varepsilon)}|)^{2p} ds\right]^{\frac{1}{2}} \\
\leq\ & \mathbf{E}\left[|Y_T^{(\varepsilon)}|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}} \{\int_0^T \mathbf{E}\left[|Y_s^{(\varepsilon)-1}|^{4p}\right]^{\frac{1}{2}} \mathbf{E}\left[(1+|S_s^{(\varepsilon)}|)^{4p}\right]^{\frac{1}{2}} ds\}^{\frac{1}{2}} \\
\leq\ & \mathbf{E}\left[|Y_T^{(\varepsilon)}|^{2p}\right]^{\frac{1}{2}} T^{\frac{1}{2}} \{\int_0^T \mathbf{E}\left[\{|Y_s^{(\varepsilon)-1}|^{4p}\right]^{\frac{1}{2}} \mathbf{E}\left[2^{(4p-1)}(1+|S_s^{(\varepsilon)}|^{4p})\right]^{\frac{1}{2}} ds\}^{\frac{1}{2}}.
\end{aligned}$$

Hence, by $S_s^{(\varepsilon)}, Y_s^{(\varepsilon)}, Y_s^{(\varepsilon)-1} \in L_p$ for $s \in [0,T]$ and any $1 < p < \infty$, we have $\mathbf{E}\left[|DS_T^{(\varepsilon)}|_H^p\right] < \infty$ for any $p > 1$.

The rigorous proof for the existence of the sets of random variables $\{Y_t^{(\varepsilon)}\}$ and $\{D_h S_T^{(\varepsilon)}\}$ is a result of the approximations by the corresponding discretized random variables and $L_p$-convergence arguments for $p > 1$. Let define $\phi_n(s) = \frac{k}{2^n}$ if $s \in [\frac{k}{2^n}, \frac{(k+1)}{2^n})$ for $k = 0, \cdots, (2^n - 1)T$. We define a sequence of random variables $D_h^n S_T^{(\varepsilon)}$, and $Y_t^{(\varepsilon)n}$, which satisfy the stochastic integral equations :

$$Y_t^{(\varepsilon)n} = Y_0^{(\varepsilon)} + \int_0^t \partial\mu(S_{\phi_n(s)}^{(\varepsilon)}, s) Y_{\phi_n(s)}^{(\varepsilon)n} ds + \int_0^t \varepsilon\partial\sigma(S_{\phi_n(s)}^{(\varepsilon)}, s) Y_{\phi_n(s)}^{(\varepsilon)n} dw_s \ ,$$

and

$$\begin{aligned}
D_h^n S_T^{(\varepsilon)} &= \int_0^T \varepsilon\partial\sigma(S_{\phi_n(s)}^{(\varepsilon)}, s) D_h^n S_s^{(\varepsilon)} dw(s) + \int_0^T \partial\mu(S_{\phi_n(s)}^{(\varepsilon)}, s) D_h^n S_s^{(\varepsilon)} ds \\
&\quad + \int_0^T \varepsilon\sigma(S_{\phi_n(s)}^{(\varepsilon)}, s)\dot{h}_s ds \ ,
\end{aligned}$$

respectively. By considering the differences of $Y_t^{(\varepsilon)n} - Y_t^{(\varepsilon)}$ and $D_h^n S_T^{(\varepsilon)} - D_h S_T^{(\varepsilon)}$, and evaluating the expectations as *Lemma 4.1* and *Lemma 4.3* in the next section [4], we can show the following result.

[4]The arguments based on the discrete approximations to the continuous stochastic processes have been standard in stochastic analysis. We shall illustrate these aspects in the proofs of *Lemma 4.1* and *Lemma 4.2* in Section 4.

**Lemma 3. 1** *Under the assumptions in this section, for any $p > 1$ and $n > 1$ the random variables $Y_t^{(\varepsilon)n}$, $Y_t^{(\varepsilon)}$, $Y_t^{(\varepsilon)-1}$, $|D^n S_T^{(\varepsilon)}|_H$, and $|DS_T^{(\varepsilon)}|_H$ are in $L_p$. Also for any $p > 1$*

$$\boldsymbol{E}[\sup_{0 \le t \le T} |Y_t^{(\varepsilon)n} - Y_t^{(\varepsilon)}|^p] \to 0$$

*and*

$$\boldsymbol{E}[\sup_{0 < t \le T} |D^n S_t^{(\varepsilon)} - DS_t^{(\varepsilon)}|_H^p] \to 0$$

*as $n \to +\infty$.*

From these considerations for the first order $H-$derivative we conclude that $S_T^{(\varepsilon)} \in \cap_{1<p<\infty} \boldsymbol{D}_p^1$ .

As the second step, we consider the second order H-derivative in the direction of $(h_1, h_2)$, $D^2_{h_1,h_2} S_T^{(\varepsilon)}$, which satisfies a stochastic integral equation :

$$\begin{aligned} D^2_{h_1,h_2} S_T^{(\varepsilon)} &= \int_0^T \varepsilon \partial\sigma(S_s^{(\varepsilon)}, s) D^2_{h_1,h_2} S_s^{(\varepsilon)} dw_s + \int_0^T \partial\mu(S_s^{(\varepsilon)}, s) D^2_{h_1,h_2} S_s^{(\varepsilon)} ds \\ &+ \Bigg[ \int_0^T \varepsilon \partial^2\sigma(S_s^{(\varepsilon)}, s) D_{h_1} S_s^{(\varepsilon)} D_{h_2} S_s^{(\varepsilon)} dw_s \\ &+ \int_0^T \partial^2\mu(S_s^{(\varepsilon)}, s) D_{h_1} S_s^{(\varepsilon)} D_{h_2} S_s^{(\varepsilon)} ds \\ &+ \int_0^T \varepsilon \partial\sigma(S_s^{(\varepsilon)}, s) D_{h_1} S_s^{(\varepsilon)} \dot{h}_{2s} ds + \int_0^T \varepsilon \partial\sigma(S_s^{(\varepsilon)}, s) D_{h_2} S_s^{(\varepsilon)} \dot{h}_{1s} ds \Bigg] . \end{aligned}$$

Then, by using $\{Y_t^{(\varepsilon)}\}$, we can obtain a representation of the second order H-derivative in the direction of $(h_1, h_2)$ as

$$\begin{aligned} D^2_{h_1,h_2} S_T^{(\varepsilon)} &= \int_0^T Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1} [\partial^2\mu(S_s^{(\varepsilon)}, s) D_{h1} S_s^{(\varepsilon)} D_{h_2} S_s^{(\varepsilon)} ds \\ &+ \varepsilon \partial^2\sigma(S_s^{(\varepsilon)}, s) D_{h_1} S_s^{(\varepsilon)} D_{h_2} S_s^{(\varepsilon)} dw_s \\ &+ \varepsilon \partial\sigma(S_s^{(\varepsilon)}, s) D_{h_1} S_s^{(\varepsilon)} \dot{h}_{2s} ds + \varepsilon \partial\sigma(S_s^{(\varepsilon)}, s) D_{h_2} S_s^{(\varepsilon)} \dot{h}_{1s} ds]. \end{aligned}$$

In order to show the $L_p$-boundedness of the second order H-derivative, we need three lemmas because the integrands in the stochastic integral equations are defined in a Hilbert space. We give the first lemma as *Lemma 3.1* for convenience, which is *Lemma 7* of Yoshida (1995).

**Lemma 3. 2** [ *Yoshida (1995)* ]
*Let $\boldsymbol{H}$ be the Cameron-Martin subspace of an m-dimensional Wiener space. Let*

$F \in L^2(\boldsymbol{R}_+ \to \boldsymbol{E} \otimes \boldsymbol{R}^d)$. *Suppose that the linear operator* $L : \boldsymbol{H} \to \boldsymbol{E}$ *is defined by*

$$L[h] = \int_0^\infty F_s \dot{h}_s ds, \ h \in \boldsymbol{H}.$$

*Then,* $L : \boldsymbol{H} \to \boldsymbol{E}$ *is a continuous linear operator, and if* $L : \boldsymbol{H} \times \boldsymbol{E} \to \boldsymbol{R}$ *denotes the corresponding continuous bilinear form, then* $L \in \boldsymbol{H} \otimes \boldsymbol{E}$ *and*

$$|L|_{H \otimes E} = \sqrt{\int_0^\infty |F_s|^2_{E \otimes R^d} ds}.$$

We set $d = m = 1$ and will apply this lemma to $D_h[DS_T^{(\varepsilon)}]$, where $DS_T^{(\varepsilon)}$ is in $\boldsymbol{H} \otimes \boldsymbol{R}$. In our case, we notice that

$$\begin{aligned} D_h[DS_T^{(\varepsilon)}] &= \int_0^T \varepsilon\partial\sigma(S_s^{(\varepsilon)}, s) D_h[DS_s^{(\varepsilon)}] dw_s + \int_0^T \partial\mu(S_s^{(\varepsilon)}, s) D_h[DS_s^{(\varepsilon)}] ds \\ &+ \left[ \int_0^T \varepsilon\partial^2\sigma(S_s^{(\varepsilon)}, s) D_h S_s^{(\varepsilon)} DS_s^{(\varepsilon)} dw_s \right. \\ &+ \int_0^T \partial^2\mu(S_s^{(\varepsilon)}, s) D_h S_s^{(\varepsilon)} DS_s^{(\varepsilon)} ds \\ &+ \left. \int_0^T \varepsilon\partial\sigma(S_s^{(\varepsilon)}, s) DS_s^{(\varepsilon)} \dot{h}_s ds + \int_0^T \varepsilon\partial\sigma(S_s^{(\varepsilon)}, s) D_h S_s^{(\varepsilon)} \dot{h}_s ds \right]. \end{aligned}$$

In order to apply *Lemma 3.2* to the present case, we need to express $D_h[DS_T^{(\varepsilon)}]$ by the process $\{Y_t^{(\varepsilon)}\}$ in (3.8) and other terms, which can be easily handled. For this ourpose, we prepare the following lemma.

**Lemma 3. 3** *Define* $U_T^{(\varepsilon)}$ *by*

$$U_T^{(\varepsilon)} = \int_0^T Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1} [a_s ds + b_s dw_s] \,,$$

*where* $Y_t^{(\varepsilon)}$ *is the solution of the stochastic differential equation (3.8). Then we have a representation of* $U_T^{(\varepsilon)}$ *:*

$$\begin{aligned} U_T^{(\varepsilon)} &= U_0^{(\varepsilon)} + \int_0^T U_s^{(\varepsilon)} [\partial\mu(S_s^{(\varepsilon)}, s) ds + \varepsilon\partial\sigma(S_s^{(\varepsilon)}, s)] dw_s \\ &+ \int_0^T [a_s ds + b_s dw_s] \,. \end{aligned} \tag{3.9}$$

**Proof of Lemma 3.3** : It is enough to verify that $U_T^{(\varepsilon)}$ satisfies (3.9). By the rule of stochastic differentiation, we know that

$$\begin{aligned} dU_T^{(\varepsilon)} &= d[Y_T^{(\varepsilon)} \int_0^T Y_s^{(\varepsilon)-1} (a_s ds + b_s dw_s)] \\ &= dY_T^{(\varepsilon)} \cdot \int_0^T Y_s^{(\varepsilon)-1} (a_s ds + b_s dw_s) + Y_T^{(\varepsilon)} \cdot Y_T^{(\varepsilon)-1} (a_T dT + b_T dw_T) \\ &= dY_T^{(\varepsilon)} \cdot U_T^{(\varepsilon)} Y_T^{(\varepsilon)-1} + [a_T dT + b_T dw_T] \,. \end{aligned}$$

Then by substituting (3.8) into the above equation we have the desired result (3.9). ***Q.E.D.***

By using Lemma 3.2 and Lemma 3.3, we can use a simple representation for the derivative of the first order $H$–derivative in the direction of $h$ as

$$\begin{aligned} D_h[DS_T^{(\varepsilon)}] &= \int_0^T Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1}[\partial^2\mu(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}DS_s^{(\varepsilon)}ds \\ &+ \varepsilon\partial^2\sigma(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}DS_s^{(\varepsilon)}dw_s \\ &+ 2\varepsilon\partial\sigma(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}\dot{h}_s ds] \,. \end{aligned}$$

Thus we have for any $p > 1$

$$\begin{aligned} |D^2 S_T^{(\varepsilon)}|_{H\otimes H}^p &\leq 4^{p-1}\Big[|\int_0^T Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1}\partial^2\mu(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}\otimes DS_s^{(\varepsilon)}ds|_{H\otimes H}^p \\ &+ |\int_0^T Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1}\varepsilon\partial^2\sigma(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}\otimes DS_s^{(\varepsilon)}dw_s|_{H\otimes H}^p \\ &+ 2|\int_0^T |Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1}\varepsilon\partial\sigma(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}|_H^2 ds|^{\frac{p}{2}}\Big] \,. \end{aligned}$$

By the boundedness of $\partial^2\mu(S_s^{(\varepsilon)},s)$, and $\partial\sigma(S_s^{(\varepsilon)},s)$, and $L_p$-boundedness of $Y_T^{(\varepsilon)}$, $Y_s^{(\varepsilon)-1}$, and $DS_s^{(\varepsilon)}$, the similar evaluations as for the first order derivative show the $L_p$-boundedness of the first and third term in the last equation.

As for the term involving the stochastic integral $\{dw\}$, by using the boundedness of random variables we have

$$\begin{aligned} &\mathbf{E}\Big[|\int_0^T Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1}\varepsilon\partial^2\sigma(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}\otimes DS_s^{(\varepsilon)}dw_s|_{H\otimes H}^p\Big] \\ \leq\ &\mathbf{E}\left[|Y_T^{(\varepsilon)}|^{2p}\right]^{\frac{1}{2}}\mathbf{E}\Big[|\int_0^T Y_s^{(\varepsilon)-1}\varepsilon\partial^2\sigma(S_s^{(\varepsilon)},s)DS_s^{(\varepsilon)}\otimes DS_s^{(\varepsilon)}dw_s|_{H\otimes H}^{2p}\Big]^{\frac{1}{2}} \,. \end{aligned}$$

Clearly, the first part is bounded for any $p > 1$. As for the second part, we need a version of Burkholder's inequality for Hilbert space valued stochastic integrals. For this purpose, we need *Lemma 2.1* of Kusuoka and Stroock (1982), which is stated as the next lemma for convenience.

**Lemma 3. 4** [ *Kusuoka and Stroock (1982)* ]
*Let* $\boldsymbol{E}$ *be a separable real Hilbert space and suppose that* $f : [0,\infty) \times \boldsymbol{W}^m \to \boldsymbol{R}^d \otimes \boldsymbol{E}$ *is a progressively measurable process. Suppose also that for some* $p \geq 2$, $\mathbf{E}\left[\int_0^T |f_s|_{R^d\otimes E}^p ds\right] < \infty$. *Then there exists a constant* $c_p$ *depending only on* $p$ *such that*

$$\mathbf{E}\left[sup_{0\leq t\leq T}|\int_0^t f_u d\tilde{w}_t|_E^p\right] \leq c_p T^{\frac{p-2}{2}}\mathbf{E}\left[\int_0^T |f_s|_{R^d\otimes E}^p ds\right] .$$

Because $DS_s^{(\varepsilon)} \otimes DS_s^{(\varepsilon)}$ is in $\boldsymbol{H}^{\otimes^2} \otimes \boldsymbol{R}$, we can apply Lemma 3.3 when $d = m = 1$. By using the above inequality, we can evaluate the last part as

$$\mathbf{E}\left[ | \int_0^T Y_s^{(\varepsilon)-1} \varepsilon \partial^2 \sigma(S_s^{(\varepsilon)}, s) DS_s^{(\varepsilon)} \otimes DS_s^{(\varepsilon)} dw_s |_{H\otimes H}^{2p} \right]^{\frac{1}{2}}$$

$$\leq (c_p)^{\frac{1}{2}} T^{\frac{p-2}{2}} \mathbf{E}\left[ \int_0^T |Y_s^{(\varepsilon)-1} \varepsilon \partial^2 \sigma(S_s^{(\varepsilon)}, s)|^{2p} |DS_s^{(\varepsilon)}|_H^{2p} |DS_s^{(\varepsilon)}|_H^{2p} ds \right]^{\frac{1}{2}}.$$

Hence, we can see this part is also bounded for any $p > 1$ by using the boundedness of $\partial^2 \sigma(S_s^{(\varepsilon)}, s)$, and $L_p$-boundedness of $Y_s^{(\varepsilon)-1}$, and most importantly $DS_s^{(\varepsilon)}$. By applying the same arguments for the first order $H-$derivative to the second order $H-$derivative, we can prove that $S_T^{(\varepsilon)} \in \cap_{1<p<\infty} \boldsymbol{D}_p^2$.

We can repeat this method and use an induction argument because we have a series of recursive stochastic differential equations. Then we can show the boundedness of higher order H-derivatives with $L_p$ estimates of $S_T^{\delta}$. By applying the same arguments for the first and second order $H-$derivatives to any order $H-$derivative, we conclude that $S_T^{(\varepsilon)} \in \boldsymbol{D}^{\infty}$.

Next, we shall prove the second part of our proof. The coefficients appeared in the asymptotic expansion of $S_T^{(\varepsilon)}$ are given by the Taylor formula. For instance,

$$g_{1T} = \int_0^T Y_T Y_s^{-1} \sigma(S_s^{(0)}, s) dw_s$$

$$g_{2T} = \int_0^T \frac{1}{2} Y_T Y_s^{-1} \{\partial^2 \mu(S_s^{(0)}, s) g_{1s}^2 ds + \sigma(S_s^{(0)}, s) g_{1s} dw_s\}$$

*and*

$$g_{3T} = \int_0^T Y_T Y_s^{-1} \{\partial^2 \mu(S_s^{(0)}, s) g_{1s} g_{2s} ds + \frac{1}{6} \partial^3 \mu(S_s^{(0)}, s) g_{1s}^3 dw_s + \partial \sigma(S_s^{(0)}, s) g_{2s} dw_s\},$$

where $Y_t = Y_t^{(0)}$ is the solution of the deterministic differential equation

$$dY = \partial \mu(S^{(0)}, t) Y dt,$$

where $Y_0 = 1$. The solution of this equation is $Y_t = exp(\int_0^t \mu(S_s^{(0)}, s) ds)$. By the boundedness of $Y_T, Y_s^{-1}, \sigma(S_s^{(0)}, s)$ on $[0, T]$, we see that $\mathbf{E}[|g_{1s}|^p] < \infty, s \in [0, T]$ for any $1 < p < \infty$. Given $g_{1s} \in L_p$, by using Burkholder's inequality (or local martingale inequality in *Theorem III-3.1* of Ikeda and Watanabe (1989)), we have $\mathbf{E}[|g_{2s}|^p] < \infty$ for any $1 < p < \infty$. By the same token, the relation $g_{ks} \in L_p$ can be obtainable recursively given $g_{js} \in L_p, j = 1, 2, \cdots k-1$. Hence $g_{1T}, g_{2T}, \cdots \in \cap_{1<p<\infty} \boldsymbol{D}_p^1$. By noting that $D_h g_{1T} = Y_T \int_0^T Y_s^{-1} \sigma(S_s^{(0)}, s) \dot{h}_s ds$ and

$D^k_{h_1,\cdots,h_k} g_1 = 0$ for $k = 2, 3, \cdots$, we see that $g_{1T} \in \boldsymbol{D}^\infty$. Also we have

$$
\begin{aligned}
D_h g_{2T} &= Y_T \int_0^T Y_s^{-1} \partial^2 \mu(S_s^{(0)}, s) g_{1s} D_h g_{1s} ds + \int_0^T \partial\sigma(S_s^{(0)}, s) D_h g_{1s} dw_s \\
&\quad + \int_0^T \partial\sigma(S_s^{(0)}, s) \dot{h}_s ds ,
\end{aligned}
$$

$$
D^2_{h_1,h_2} g_{2T} = \int_0^T Y_T Y_s^{-1} \partial^2 \mu(S_s^{(0)}, s) D_{h_1} g_{1s} D_{h_2} g_{1s} ds + \int_0^T \partial\sigma(S_s^{(0)}, s) D_{h_1} g_{1s} \dot{h}_{2s} ds.
$$

and $D^k_{h_1,\cdots,h_k} g_{2T} = 0$ for $k = 3, 4, \cdots$. Then, given $g_{1s} \in \boldsymbol{D}^\infty$ for any $s \in [0, T]$, we can conclude that $g_{2T} \in \boldsymbol{D}^\infty$.

Again, recursively we can show the $L_p$-boundedness of any order H-derivatives of $g_{kT}, k = 3, 4, \cdots$. Therefore, we have proven the last part. **Q.E.D.**

Next, we define the normalized random variable $X^{(\varepsilon)} a_T$ by

$$
X_T^{(\varepsilon)} = \frac{S_T^{(\varepsilon)} - S_T^0}{\varepsilon} .
$$

By using Theorem 3.1, we see $X_T^{(\varepsilon)}$ is in $\boldsymbol{D}^\infty$ and has a proper asymptotic expansion

$$
X_T^{(\varepsilon)} \sim g_{1T} + \varepsilon g_{2T} + \cdots
$$

is in $\boldsymbol{D}^\infty$ as $\varepsilon \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^\infty$. We also have the first order H-derivative of $\{X_T^{(\varepsilon)}\}$ as

$$
D_h X_T^{(\varepsilon)} = \int_0^T Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1} \sigma(S_s^{(\varepsilon)}, s) \dot{h}_s ds .
$$

Then, the Malliavin covariance $\sigma(X_T^{(\varepsilon)}) = < DX_T^{(\varepsilon)}, DX_T^{(\varepsilon)} >_H$ is given by

$$
\sigma(X_T^{(\varepsilon)}) = \int_0^T \{Y_T^{(\varepsilon)} Y_s^{(\varepsilon)-1} \sigma(S_s^{(\varepsilon)}, s)\}^2 ds . \tag{3.10}
$$

We notice that

$$
\sigma(X_T^{(\varepsilon)}) \to \Sigma_{g_1} = \int_0^T \{Y_T Y_s^{-1} \sigma(S_s^{(0)}, s)\}^2 ds \tag{3.11}
$$

as $\varepsilon \downarrow 0$, where $\Sigma_{g_1}$ denotes the variance of $g_1$.

We shall next consider the uniform non-degeneracy of the Malliavin covariance, which is the important step of the application of *Theorem 2.1.* For this purpose, we need the following assumption.

**Assumption I** : For any $T > 0$,

$$
\Sigma_{g_1} = \int_0^T \{Y_T Y_s^{-1} \sigma(S_s^{(0)}, s)\}^2 ds > 0 . \tag{3.12}
$$

This assumption assures the non-degeneracy of the limiting distribution of random variables, which can be easily checked in many applications. The condition can be extended to the more general case when $d \geq 1$ and $m \geq 1$ in a straightforward manner. We define $\eta_c^\varepsilon$ by for any $c > 0$,

$$\eta_c^\varepsilon = c \int_0^T |Y_T^{(\varepsilon)}(Y_s^{(\varepsilon)})^{-1}\sigma(S_s^{(\varepsilon)}, s) - Y_T Y_s^{-1}\sigma(S_s^{(0)}, s)|^2 ds \, .$$

Then, we have the following resut on the uniform non-degeneracy of the Malliavin-covariance.

**Theorem 3. 2** *Under the assumptions we have made and Assumption I, the Malliavin covariance $\sigma(X_T^{(\varepsilon)})$ is uniformly non-degenerate. That is, there exists $c_0 > 0$ such that for $c > c_0$ and any $p > 1$,*

$$sup_{\varepsilon \in (0,1]} \mathbf{E}\left[1_{\{\eta_c^\varepsilon \leq 1\}}\{det \; \sigma(X_T^{(\varepsilon)})\}^{-p}\right] < \infty. \tag{3.13}$$

**Proof** : Let

$$\xi_{s,t}^{(\varepsilon)} = Y_t^{(\varepsilon)}(Y_s^\varepsilon)^{-1}\sigma(S_s^{(\varepsilon)}, s)$$

and $\xi_{s,t} = Y_t Y_s^{-1}\sigma(S_s^{(0)}, s)$. Then, the condition $|\eta_c^\delta| \leq 1$ implies

$$\int_0^T |\xi_{s,T}^{(\varepsilon)} - \xi_{s,T}|^2 ds \leq \frac{2}{c}.$$

We have an inequality

$$\begin{aligned} |\sigma(X_T^{(\varepsilon)}) - \Sigma_{g_1}| &= |\int_0^T (\xi_{s,T}^{(\varepsilon)})^2 - (\xi_{s,T})^2 ds| \\ &\leq \int_0^T |\xi_{s,T}^{(\varepsilon)} - \xi_{s,T}|^2 ds + 2\int_0^T |\xi_{s,T}||\xi_{s,T}^{(\varepsilon)} - \xi_{s,T}| ds \\ &\leq \frac{2}{c} + 2\Sigma_{g_1}^{\frac{1}{2}}(\frac{2}{c})^{\frac{1}{2}} \, . \end{aligned}$$

Hence we can take $c_0 > 0$ such that for any $c > c_0$,

$$0 < \Sigma_{g_1} - |\sigma(X_T^{(\varepsilon)}) - \Sigma_{g_1}| < \sigma(X_T^{(\varepsilon)})$$

holds uniformly for $\varepsilon \in (0, 1]$. Thus, (3.13) is concluded. *Q.E.D.*

Next, we present two inequalities which are useful to show that the truncation by $\eta_c^\varepsilon$ is negligible in probability when we derive the asymptotic expansion.

**Lemma 3.5** *(1) There exist positive constants $a_i$ $(i = 1,2)$ independent of $\varepsilon$ such that*

$$(3.14) \quad P(sup_{0\le s\le T}|S_s^{(\varepsilon)} - S_s^0| > a_0) \le \frac{a_1}{a_0}(a_0 + C)exp(-\frac{a_2 a_0^2}{(a_0+C)^2}\varepsilon^{-2})$$

*for all $a_0 > 0$.*
*(2) There exist positive constants $a_i$ $(i = 1,2)$ independent of $\varepsilon$ such that*

$$(3.15) \quad P(sup_{0\le s\le T}|Y_s^{(\varepsilon)} - Y_s| > a_0) \le \frac{a_1}{a_0}(a_0 + C)exp(-\frac{a_2 a_0^2}{(a_0+C)^2}\varepsilon^{-2})$$

*for all $a_0 > 0$.*

**Proof** : (1) Let

$$S_T^{(\varepsilon)} = S_0 + \int_0^T \mu(S_s^{(\varepsilon)}, s)ds + \int_0^T \varepsilon\sigma(S_s^{(\varepsilon)}, s)dw_s$$

and

$$S_T^{(0)} = S_0 + \int_0^T \mu(S_s^{(0)}, s)ds \ .$$

Using the Lipschitz continuity of $\mu(S^{(\varepsilon)}, t)$ in the first argument,

$$|S_t^{(\varepsilon)} - S_t^{(0)}| \le K\int_0^t |S_s^{(\varepsilon)} - S_s^{(0)}|ds + sup_{0\le s\le t}|\int_0^s \varepsilon\sigma(S_u^{(\varepsilon)}, u)dw_u|.$$

In order to evaluate this inequality we recall the useful Gronwall's inequality (i.e. Elliott (1982) p.192, for instance). Suppose $\alpha(t)$ is a Lebesgue integrable function on $[a,b]$, and that $C$ and $D$ are constants such that

$$\alpha(t) \le C + D\int_a^t \alpha(s)ds$$

for all $t \in [a,b]$. Then

$$\alpha(t) \le Ce^{D(t-a)} \ .$$

By using this Gronwall's inequality,

$$sup_{0\le s\le T}|S_s^{(\varepsilon)} - S_s^{(0)}| \le sup_{0\le s\le T}|\int_0^s \varepsilon\sigma(S_u^{(\varepsilon)}, u)dw_u|e^{KT}.$$

We next use the method of time change (i.e. Ikeda and Watanabe (1989) p.197, for example) in stochastic analysis. There exists a Brownian motion $B(t)$ such that

$$B(A_t) = \int_0^t \varepsilon\sigma(S_s^{(\varepsilon)}, s)dw_s \ ,$$

where $A_t = \int_0^t \varepsilon^2 \sigma(S_s^{(\varepsilon)}, s)^2 ds$ .
Then,

$$sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| \leq sup_{0\leq s\leq T}|B(A_s)|e^{KT} .$$

Let a stopping time be $\tau = inf\{s; |S_s^{(\varepsilon)} - S_s^{(0)}| > a_0\}$ . Then by using the fact that the event $\{\tau < T\}$ implies $\{sup_{0\leq s\leq \tau<T}|B(A_s)|e^{KT} > a_0\}$, we have

$$P(\{sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| > a_0\}) = P(\{\tau < T, sup_{0\leq s\leq \tau}|B(A_s)|e^{KT} > a_0\}).$$

Also we note that for $s \leq \tau$, $|S_s^{(\varepsilon)}| - |S_s^{(0)}| \leq |S_s^{(\varepsilon)} - S_s^{(0)}| \leq a_0$. Then, we have

$$|S_s^{(\varepsilon)}| \leq a_0 + |S_s^{(0)}| \leq a_0 + sup_{0\leq s\leq T}|S_s^{(0)}|$$

and

$$|\sigma(S_s^{(\varepsilon)}, s)| \leq (1 + |S_s^{(\varepsilon)}|) \leq (a_0 + C) ,$$

where $C \equiv 1 + sup_{0\leq s\leq T}|S_s^{(0)}|$. Thus $A_s$ for $s \in [0, \tau]$ is evaluated as

$$A_s = \int_0^s \varepsilon^2 \sigma(S_u^{(\varepsilon)}, u)^2 du \leq \varepsilon^2 T(a_0 + C)^2.$$

Therefore, we have an inequality

$$\begin{aligned} & P(\{\tau < T, sup_{0\leq s\leq \tau}|B(A_s)|e^{KT} > a_0\}) \\ \leq\ & P(\{sup_{0\leq u\leq \varepsilon^2 T(a_0+C)^2}|B(u)|e^{KT} > a_0\}). \end{aligned}$$

By using the reflection principle and the inequality

$$\frac{1}{\sqrt{2\pi}}\int_{a_0}^{\infty} e^{\frac{-x^2}{2}} dx < \frac{1}{\sqrt{2\pi}} a_0^{-1} e^{\frac{-a_0^2}{2}},$$

we obtain

$$\begin{aligned} P(\{sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| > a_0\}) &\leq 2P(\{\max B(\varepsilon^2 T(a_0 + C)^2) > a_0 e^{-KT}\}) \\ &= 4P(\{B(\varepsilon^2 T(a_0 + C)^2) > a_0 e^{-KT}\}) \\ &\leq \frac{4e^{KT}\varepsilon\sqrt{T}}{\sqrt{2\pi}} \frac{(a_0 + C)}{a_0} \times \\ &\quad exp(-\frac{e^{-2KT}a_0^2}{2T(a_0 + C)^2}\varepsilon^{-2}) . \end{aligned}$$

Finally, by defining two constants $a_1$ and $a_2$ in an appropriate way, we conclude the result. This can be done by taking

$$a_1 = \frac{4e^{KT}\delta\sqrt{T}}{\sqrt{2\pi}}$$

and

$$a_2 = \frac{e^{-2KT}}{2T} \ .$$

(2) Let a stochastic differential equation be

$$dY_t^{(\varepsilon)} = \partial\mu(S_t^{(\varepsilon)}, t)Y_t^{(\varepsilon)}dt + \partial\sigma(S_t^{(\varepsilon)}, t)Y_t^{(\varepsilon)}dw_t \ .$$

The corresponding deterministic differential equation is given by

$$dY_t = \partial\mu(S^{(0)}, t)Y_t dt \ .$$

By using the smoothness and the boundedness of derivatives of $\mu(S_s^{(\varepsilon)}, s)$ in the first argument and the boundedness of $Y_s$ on $[0, T]$, there exist positive $M_1$ and $M_2$ such that

$$\begin{aligned} & |\partial\mu(S_s^{(\varepsilon)}, s)Y_s^{(\varepsilon)} - \partial\mu(S_s^{(0)}, s)Y_s| \\ \leq \ & |\partial\mu(S_s^{(\varepsilon)}, s)||Y_s^{(\varepsilon)} - Y_s| + |Y_s||\partial\mu(S_s^{(\varepsilon)}, s) - \partial\mu(S_s^{(0)}, s)| \\ \leq \ & M_1|Y_s^{(\varepsilon)} - Y_s| + M_2|S_s^{(\varepsilon)} - S_s^{(0)}| \ . \end{aligned}$$

Then we have

$$\begin{aligned} |Y_s^{(\varepsilon)} - Y_s| \ \leq \ & \left[M_2 T sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| + sup_{0\leq s\leq T}\left|\int_0^s \varepsilon\partial\sigma(S_u^{(\varepsilon)}, u)Y_u^{(\varepsilon)}dw_u\right|\right] \\ + \ & M_1 \int_0^s |Y_u^{(\varepsilon)} - Y_u|du \ . \end{aligned}$$

Again, by using the Gronwall's inequality, it is possible to take a positive constant $M_3$ such that

$$sup_{0\leq s\leq T}|Y_s^{(\varepsilon)} - Y_s| \leq M_3\left[sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| + sup_{0\leq s\leq T}\left|\int_0^s \varepsilon\partial\sigma(S_u^{(\varepsilon)}, u)Y_u^{(\varepsilon)}dw_u\right|\right] .$$

We take a constant $\delta = \frac{a_0}{2M_3}$ and a stopping time

$$\tau = inf\{s; |S_s^{(\varepsilon)} - S_s^{(0)}| > \delta \text{ or } |Y_s^{(\varepsilon)} - Y_s| > a_0\} \ .$$

Then we have

$$\begin{aligned} & P(\{sup_{0\leq s\leq T}|Y_s^{(\varepsilon)} - Y_s| > a_0\}) \\ = \ & P(\{sup_{0\leq s\leq T}|Y_s^{(\varepsilon)} - Y_s| > a_0, \ sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| \leq \delta\}) \\ & +P(\{sup_{0\leq s\leq T}|Y_s^{(\varepsilon)} - Y_s| > a_0, \ sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| > \delta\}) \\ \leq \ & P(\{sup_{0\leq s\leq T}|Y_s^{(\varepsilon)} - Y_s| > a_0, \ sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| \leq \delta\}) \\ & +P(\{sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| > \delta\}) \\ \leq \ & P(\{\tau < T, \ sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| \leq \delta, \end{aligned}$$

$$
\begin{aligned}
&(a_0 - M_3\varepsilon) \le M_3 sup_{0\le s\le\tau}|\int_0^s \partial\sigma(S_u^{(\varepsilon)},u)Y_u^{(\varepsilon)}dw_u|\}) \\
&+P(\{sup_{0\le s\le T}|S_s^{(\varepsilon)} - S_s^{(0)}| > \delta\}) \\
= \;& P(\{\tau < T,\ sup_{0\le s\le T}|S_s^{\delta} - S_s^{(0)}| \le \varepsilon, \\
&\frac{a_0}{2} \le M_3 sup_{0\le s\le\tau}|B(\varepsilon^2\int_0^s \partial\sigma(S_u^{(\varepsilon)},u)^2 Y_u^{(\varepsilon)2}du)|\}) \\
&+P(\{sup_{0\le s\le T}|S_s^{(\varepsilon)} - S_s^{(0)}| > \delta\}) .
\end{aligned}
$$

The second term in the last equation is equivalent to the first part of Lemma. Hence, what we have to do is to evaluate the first term. We note that the condition $0 \le s \le \tau(< T)$ implies $|Y_s^{(\varepsilon)} - Y_s| \le a_0$ for $s \in [0,\tau]$. Then for $s \in [0,\tau]$, we have $|Y_s^{(\varepsilon)}| \le a_0 + C$ , where we take $C \equiv sup_{0\le s\le T}Y_s$. Together with $|\partial\sigma(S_u^{(\varepsilon)},u)| \le M_1$, we can show an inequality :

$$
\varepsilon^2\int_0^s \partial\sigma(S_u^{(\varepsilon)},u)^2 Y_u^{(\varepsilon)2}du \le \varepsilon^2 T M_1^2(a_0+C)^2 .
$$

Then

$$
\begin{aligned}
&P(\{\tau < T,\ sup_{0\le s\le T}|S_s^{(\varepsilon)} - S_s^{(0)}| \le \delta, \\
&\qquad \frac{a_0}{2} \le M_3 sup_{0\le s\le\tau}|B(\varepsilon^2\int_0^s \partial\sigma(S_u^{(\varepsilon)},u)^2 Y_u^{(\varepsilon)2}du)|\}) \\
\le \;& P(\{\frac{a_0}{2M_2} \le sup_{[0\le u\le\varepsilon^2TM_1^2(a_0+C)^2]}|B(u)|\}) .
\end{aligned}
$$

Therefore, repeating the similar arguments as we used in (1), we can conclude that there exist positive constants $a_{i1}$ $(i = 1,2)$ independent of $\varepsilon$ such that

$$
\begin{aligned}
&P(\{\tau < T,\ sup_{0\le s\le T}|S_s^{(\varepsilon)} - S_s^{(0)}| \le \delta, \\
&\qquad \frac{a_0}{2} \le M_3 sup_{0\le s\le\tau}|B(\varepsilon^2\int_0^s \partial\sigma(S^{(\varepsilon)},u)^2 Y_u^{(\varepsilon)2}du)|\}) \\
\le \;& \frac{a_{11}}{a_0}(a_0+C)exp(-\frac{a_{21}a_0^2}{(a_0+C)^2}\varepsilon^{-2}) .
\end{aligned}
$$

*Q.E.D.*

We now can show that the truncation by the bounded random variable $\eta_c^{\varepsilon}$ is negligible in probability by utilizing the above large deviation inequalities. We summarize this result as the next lemma.

**Lemma 3. 6** *For $c > 0$, $\eta_c^{\varepsilon}$ is $O(1)$ in $\boldsymbol{D}^{\infty}$ and for $c_0 > 0$, there exist some constants $c_i$ $(i = 1,2,3)$, such that*

$$
P(\{|\eta_c^{\varepsilon}| > c_0\}) \le c_1 exp(-c_2\varepsilon^{-c_3}) . \tag{3.16}
$$

**Proof** : The result follows from the inequalities (1) and (2) in the previous lemma. First, we note

$$\begin{aligned}|\eta_c^\varepsilon| &= c\int_0^T |Y_T^{(\varepsilon)}Y_s^{(\varepsilon)-1}\sigma(S_s^{(\varepsilon)},s) - Y_TY_s^{-1}\sigma(S_s^{(0)},s)|^2 ds \\ &\leq cT sup_{0\leq s\leq T}\left[Y_T^{(\varepsilon)}Y_s^{(\varepsilon)-1}\sigma(S_s^{(\varepsilon)},s) - Y_TY_s^{-1}\sigma(S_s^{(0)},s)\right]^2 .\end{aligned}$$

Then the condition $|\eta_c^\varepsilon| > c_0$ implies

$$\{|\eta_c^\varepsilon| > c_0\} \subset \{sup_{0\leq s\leq T}\left[Y_T^{(\varepsilon)}Y_s^{(\varepsilon)-1}\sigma(S_s^{(\varepsilon)},s) - Y_TY_s^{-1}\sigma(S_s^{(0)},s)\right] > (\frac{c_0}{cT})^{\frac{1}{2}}\} .$$

We set a constant $c_4 = (\frac{c_0}{cT})^{\frac{1}{2}}$. Then

$$\text{(3.17)}\quad \begin{aligned}\{|\eta_c^\varepsilon| > c_0\} &\subset \{sup_{0\leq s\leq T}|Y_TY_s^{-1}||\sigma(S^{(\varepsilon)},s) - \sigma(S_s^{(0)},s)| > \frac{c_4}{3}\} \\ &\cup \{sup_{0\leq s\leq T}|Y_s^{(\varepsilon)-1}||\sigma(S_s^{(\varepsilon)},s)||Y_T^{(\varepsilon)} - Y_T| > \frac{c_4}{3}\} \\ &\cup \{sup_{0\leq s\leq T}|Y_T||\sigma(S_s^{(\varepsilon)},s)||Y_s^{(\varepsilon)-1} - Y_s^{-1}| > \frac{c_4}{3}\} .\end{aligned}$$

By using the boundedness of $|Y_TY_s^{-1}|$, the Lipschitz continuity of $\sigma(S^{(\varepsilon)},s)$ in the first argument, and *Lemma 3.5*, the first term of the right hand side implies that there exist positive $c_{11}, c_{21}$, and $c_{31}$ such that

$$\begin{aligned}&P(\{sup_{0\leq s\leq T}|Y_TY_s^{-1}||\sigma(S^{(\varepsilon)},s) - \sigma(S^0,s)| > \frac{c_4}{3}\}) \\ \leq\ &P(\{sup_{0\leq s\leq T}|S_s^{(\varepsilon)} - S_s^{(0)}| > c_5\}) \leq c_{11}exp(-c_{21}\delta^{-c_{31}}) ,\end{aligned}$$

where we take

$$c_5 = \frac{c_4}{3K sup_{0\leq s\leq T}|Y_TY_s^{-1}|} .$$

We note that for any $c_{03} > 0$, the condition $|Y_s^{(\varepsilon)-1} - Y_s^{-1}| > c_{03}$ is implied by $|Y^{(\varepsilon)-1}| > max(|c_{03} + Y_s^{-1}|, |c_{03} - Y_s^{-1}|)$. Then by $Y_s^{-1} > 0$ for $0 \leq s \leq T$,

$$\{|Y_s^{(\varepsilon)-1}| > m_1\} \subset \{|Y_s^{(\varepsilon)-1} - Y_s^{-1}| > c_{03}\}$$

where we set $m_1 = c_{03} + sup_{0\leq s\leq T}Y_s^{-1}$. We also note that for any $c_{01} > 0$, the condition $|S_s^{(\varepsilon)} - S_s^{(0)}| > c_{01}$ is implied by $|S_s^{(\varepsilon)}| > max(|c_{01} + S_s^{(0)}|, |c_{01} - S_s^{(0)}|)$ . Then it also implied by $|S_s^{(\varepsilon)}| > c_{01} + sup_{0\leq s\leq T}|S_s^{(0)}|$. Hence by $|\sigma(S^{(\varepsilon)},s)| \leq M_1(1 + |S_s^{(\varepsilon)}|)$,

$$\{|\sigma(S_s^{(\varepsilon)},s)| > m_2\} \subset \{|S_s^{(\varepsilon)} - S_s^{(0)}| > c_{01}\}.$$

where we set $m_2 = K(1+c_{01}+sup_{0\leq s\leq T}|S_s^{(0)}|)$. Therefore, as for the second term,

$$\{sup_{0\leq s\leq T}|Y_s^{(\varepsilon)-1}||\sigma(S_s^{(\varepsilon)},s)||Y_T^{(\varepsilon)} - Y_T| > \frac{c_4}{3}\}$$

$$
\begin{aligned}
\subset\ & \{|Y_t^{(\varepsilon)} - Y_t| > \frac{c_4}{3m_1m_2}, sup_{0\le s\le T}|Y_s^{(\varepsilon)-1}| \le m_1, sup_{0\le s\le T}|\sigma(S_s^{(\varepsilon)}, s)| \le m_2\} \\
\cup\ & \{sup_{0\le s\le T}|Y_s^{(\varepsilon)-1}||\sigma(S_s^{(\varepsilon)}, s)||Y_T^{(\varepsilon)} - Y_T| > \frac{c_4}{3}, sup_{0\le s\le T}|Y_s^{(\varepsilon)-1}| > m_1\} \\
\cup\ & \{sup_{0\le s\le T}|Y_s^{(\varepsilon)-1}||\sigma(S_s^{(\varepsilon)}, s)||Y_T^{(\varepsilon)} - Y_T| > \frac{c_4}{3}, sup_{0\le s\le T}|\sigma(S_s^{(\varepsilon)}, s)| > m_2\} \\
\subset\ & \{|Y_T^{(\varepsilon)} - Y_T| > \frac{c_4}{3m_1m_2}\} \cup \{sup_{0\le s\le T}|Y_s^{(\varepsilon)-1} - Y_s^{-1}| > c_{03}\} \\
\cup\ & \{sup_{0\le s\le T}|S_s^{(\varepsilon)} - S_s^0| > c_{01}\} .
\end{aligned}
$$

Then, by using the inequalities in *Lemma 3.5* , we can take positive constants $c_{12}, c_{22}, c_{32} > 0$ such that

$$
P(\{sup_{0\le s\le t}|Y_s^{(\varepsilon)-1}||\sigma(S_s^{(\varepsilon)}, s)||Y_t^{(\varepsilon)} - Y_t| > \frac{c_4}{3}\}) < c_{12}exp(-c_{22}\varepsilon^{-c_{32}}) .
$$

The similar arguments can be applied to the third term. Then by summarizing all terms and the result is concluded. *Q.E.D.*

By summarizing Theorem 3.3, Theorem 3.3, and Lemma 3.5, we have shown that the conditions of *Theorem 2.1* are satisfied. Then we immediately obtain the next result.

**Theorem 3. 3** *Under the assumptions with Assumption I we have made in this section, for a smooth function* $\phi^{(\varepsilon)}(x)$ *with all derivatives of polynomial growth orders,* $\psi(\eta_c^{\varepsilon})\phi^{(\varepsilon)}(X_T^{(\varepsilon)})I_{\mathcal{B}}(X_T^{(\varepsilon)})$ *has an asymptotic expansion :*

$$
\psi(\eta_c^{\varepsilon})\phi^{(\varepsilon)}(X_T^{(\varepsilon)})I_{\mathcal{B}}(X_T^{(\varepsilon)}) \sim \Phi_0 + \varepsilon\Phi_1 + \cdots \tag{3.18}
$$

*in* $\tilde{\boldsymbol{D}}^{-\infty}$ *as* $\varepsilon \downarrow 0$, *where* $\mathcal{B}$ *is a Borel set,* $\psi(x)$ *is a smooth function such that* $0 \le \psi(x) \le 1$ *for* $x \in R$, $\psi(x) = 1$ *for* $|x| \le 1/2$ *and* $\psi = 0$ *for* $|x| \ge 1$, *and* $\Phi_0, \Phi_1, \cdots$ *are determined by the formal Taylor expansion.*

Finally, we obtain an asymptotic expansion of the expectation of $\phi^{(\varepsilon)}(X_T^{(\varepsilon)})I_{\mathcal{B}}(X_T^{(\varepsilon)})$, which is the main result in this section. It is the direct consequence of the uniform integrability of $\{|\phi^{(\varepsilon)}(x)|^p\}(p \ge 1)$, *Theorem 3.3*, and *Lemma 3.6*, which has shown that the effects of truncation is negligible in probability.

**Theorem 3. 4** *Under the assumptions with Assumption I we have made in this section, an asymptotic expansion of* $\mathbf{E}[\phi^{(\varepsilon)}(X^{(\varepsilon)})I_{\mathcal{B}}(X^{(\varepsilon)})]$ *is given by*

$$
\begin{aligned}
\mathbf{E}[\phi^{(\varepsilon)}(X^{(\varepsilon)})I_{\mathcal{B}}(X^{(\varepsilon)})] &\sim \mathbf{E}[\psi(\eta_c^{\varepsilon})\phi^{(\varepsilon)}(X^{(\varepsilon)})I_{\mathcal{B}}(X^{(\varepsilon)})] \\
&\sim \mathbf{E}[\Phi_0] + \varepsilon\mathbf{E}[\Phi_1] + \cdots
\end{aligned} \tag{3.19}
$$

*as* $\varepsilon \downarrow 0$, *where* $\phi^{(\varepsilon)}(\cdot)$, $\psi(\cdot)$, *and* $\mathcal{B}$ *are defined as Theorem 3.4.*

As another application, we consider the validity of the asymptotic expansion of

$$Z^{(\varepsilon)} \equiv \frac{(Z_T^{(\varepsilon)} - Z_T^{(0)})}{\varepsilon} = \frac{1}{\varepsilon}\left[\int_0^T f(S_s^{(\varepsilon)})ds - \int_0^T f(S_s^{(0)})ds\right] ,$$

where $f(x)$ is a smooth function, which is in $\boldsymbol{C}^{\infty}(R \to R)$. Then, an asymptotic expansion of the random variable $Z_T^{(\varepsilon)}$ is formally given by

$$\begin{aligned} Z_T^{(\varepsilon)} &\sim \int_0^T f(S_s^{(0)})ds + \varepsilon \int_0^T \partial f(S_s^0) g_{1s} ds \\ &+\varepsilon^2 \int_0^T \{\frac{1}{2}\partial^2 f(S_s^{(0)}) g_{1s}^2 + \partial f(S_s^{(0)}) g_{2s}\} ds \\ &+\varepsilon^3 \int_0^T \{\frac{1}{6}\partial^3 f(S_s^{(0)}) g_{1s}^3 + \partial^2 f(S_s^{(0)}) g_{1s} g_{2s} + \partial f(S_s^{(0)}) g_{3s}\} ds + \cdots \\ &\equiv Z_T^{(0)} + \varepsilon g_{1T}^Z + \varepsilon^2 g_{2T}^Z + \varepsilon^3 g_{3T}^Z + \cdots . \end{aligned}$$

By using the smoothness of $f(x)$, $S_T^{(\varepsilon)} \in \boldsymbol{D}^{\infty}$, and $g_{1s}, g_{2s}, g_{3s}, \cdots \in \boldsymbol{D}^{\infty}$, we see that $Z_T \in \boldsymbol{D}^{\infty}$ and $Z_T^{(\varepsilon)}$ has an asymptotic expansion, which is in $\boldsymbol{D}^{\infty}$ as $\varepsilon \downarrow 0$ with $g_{kT}^Z, k = 1, 2, \cdots$.

The Malliavin covariance of $Z_T^{(\varepsilon)}$, which is denoted as $\sigma(Z_T^{(\varepsilon)})$, is given by

$$\sigma(Z_T^{(\varepsilon)}) = \int_0^T \left[\{\int_u^T \partial f(S_s^{(\varepsilon)}) Y_s^{(\varepsilon)} ds\} Y_u^{(\varepsilon)-1} \sigma(S_u^{(\varepsilon)}, u)\right]^2 du .$$

In this particular case, $\sigma(Z_T^{(\varepsilon)}) \to \Sigma_{g_{1T}^Z}$ as $\varepsilon \downarrow 0$, where

$$\Sigma_{g_{1T}^Z} = \int_0^T \left[\{\int_u^T \partial f(S_s^{(0)}) Y_s ds\} Y_u^{-1} \sigma(S_u^{(0)}, u)\right]^2 du.$$

If we define $\eta_c^{\varepsilon}(Z)$ as before by

$$\begin{aligned} \eta_c^{\varepsilon}(Z) &= c\int_0^T \left[\{\int_u^T \partial f(S_s^{(\varepsilon)}) Y_s^{(\varepsilon)} ds\} Y_u^{(\varepsilon)-1} \sigma(S_u^{(\varepsilon)}, u) \right. \\ &\left. -\{\int_u^T \partial f(S_s^{(0)}) Y_s ds\} Y_u^{-1} \sigma(S_u^{(0)}, u)\right]^2 du , \end{aligned}$$

then we can have the corresponding results as *Lemma 3.2* and *Lemma 3.6* for $\eta_c^{\varepsilon}(Z)$ instead of $\eta_c^{\varepsilon}$ in the same way. As before, in the process we need to make a use of *Lemma 3.5* and the smoothness of $f(x)$. Consequently, we can apply *Theorem 2.1* to $\psi(\eta_c^{\varepsilon}(Z))\phi(Z_T^{(\varepsilon)}) I_B$, and the same results as in *Theorem 3.4* and *Theorem 3.5* hold for $Z_T^{(\varepsilon)}$ as follows under the following assumption instead of Assumption I.

**Assumption I'** : For any $T > 0$,

$$(3.20)\qquad \Sigma_{g_{1T}^Z} = \int_0^T \left[ \{ \int_u^T \partial f(S_s^{(0)}) Y_s ds \} Y_u^{-1} \sigma(S_u^{(0)}, u) \right]^2 du > 0 \,.$$

**Theorem 3. 5** *Under the assumptions with Assumption I' instead of Assumption I we have made in this section, for a smooth function $\phi^{(\varepsilon)}(x)$ with all derivatives of polynomial growth orders, $\psi(\eta_c^\varepsilon(Z))\phi^{(\varepsilon)}(Z_T^{(\varepsilon)})I_{\mathcal{B}}(Z_T^{(\varepsilon)})$ has an asymptotic expansion :*

$$(3.21)\qquad \psi(\eta_c^\varepsilon(Z))\phi^{(\varepsilon)}(Z_T^{(\varepsilon)})I_{\mathcal{B}}(Z^{(\varepsilon)}) \sim \Phi_0 + \varepsilon\Phi_1 + \cdots$$

*in $\tilde{D}^{-\infty}$ as $\varepsilon \downarrow 0$, where $\mathcal{B}$ is a Borel set, $\psi(x)$ is a smooth function such that $0 \le \psi(x) \le 1$ for $x \in R, \psi(x) = 1$ for $|x| \le 1/2$ and $\psi = 0$ for $|x| \ge 1$, and $\Phi_0, \Phi_1, \cdots$ are determined by the formal Taylor expansion.*

**Theorem 3. 6** *Under the assumptions with Assumption I' instead of Assumption I we have made in this section, an asymptotic expansion of $\mathbf{E}[\phi^{(\varepsilon)}(Z^{(\varepsilon)})I_{\mathcal{B}}(Z^{(\varepsilon)})]$ is given by*

$$(3.22)\qquad \begin{aligned} \mathbf{E}[\phi^{(\varepsilon)}(Z_T^{(\varepsilon)})I_{\mathcal{B}}(Z_T^{(\varepsilon)})] &\sim \mathbf{E}[\psi(\eta_c^\varepsilon(Z))\phi^{(\varepsilon)}(Z^{(\varepsilon)})I_{\mathcal{B}}(Z^{(\varepsilon)})] \\ &\sim \mathbf{E}[\Phi_0] + \varepsilon\mathbf{E}[\Phi_1] + \cdots \end{aligned}$$

*as $\varepsilon \downarrow 0$, where $\phi^{(\varepsilon)}(\cdot)$, $\psi(\cdot)$, and $\mathcal{B}$ are defined as Theorem 3.6.*

Our next objective is to show that the resulting formulae of asymptotic expansion are equivalent to those from our method which is based on the simple inversion technique for the characteristic function, which have been used by Kunitomo and Takahashi (1995), and Takahashi (1997). For this purpose, we only discuss the case of $X_T^{(\varepsilon)}$ because the same argument holds for the asymptotic expansion of $Z_T^{(\varepsilon)}$. In our method we shall explicitly derive the formulas of the asymptotic distribution function and the density function, and also that of the expectation of the random variable $X_T^{(\varepsilon)}$ in a certain range. Then we shall show that they are equivalent to those by our simple method, which is based on a formal inversion technique known in the standard asymptotic theory of mathematical statistics. We start with the explicit evaluation of the expectations appeared in *Theorem 3.5.* We observe that in *Theorem 3.5*, the terms of $\Phi_0, \Phi_1$, and $\Phi_2$ are given by

$$\begin{aligned} \Phi_0 &= \phi^{(0)}(g_1)I_{\mathcal{B}}(g_1)\,, \\ \Phi_1 &= \left[ \frac{\partial\phi^{(\varepsilon)}}{\partial\varepsilon}|_{\varepsilon=0}(g_1) + \partial\phi^{(\varepsilon)}(g_1)g_2 \right] I_{\mathcal{B}}(g_1) + \phi^{(0)}(g_1)\partial I_{\mathcal{B}}(g_1)g_2\,, \end{aligned}$$

and

$$
\begin{aligned}
\Phi_2 &= \left[\frac{\partial\phi^{(\varepsilon)}}{\partial\varepsilon}|_{\varepsilon=0}(g_1) + \partial\phi^{(\varepsilon)}(g_1)g_2\right]\partial I_B(g_1)g_2 \\
&+ \left[\frac{1}{2}\frac{\partial^2\phi^{(\varepsilon)}}{\partial\varepsilon^2}|_{\varepsilon=0}(g_1) + \{\frac{\partial^2\phi^{(\varepsilon)}(x)}{\partial x\partial\varepsilon}|_{\varepsilon=0,x=g_1}\}g_2\right. \\
&\left. + \partial\phi^{(0)}(g_1)g_3 + \frac{1}{2}\partial^2\phi^{(0)}(g_1)g_2^2\right] I_B(g_1) \\
&+ \phi^{(0)}(g_1)\{\frac{1}{2}\partial^2 I_B(g_1)g_2^2 + \partial I_B(g_1)g_3\} , \qquad (3.23)
\end{aligned}
$$

which can be derived by applying a formal Taylor expansion and we use the notation $g_i$ for $g_{iT}$ $(i = 1, 2, 3)$ for convenience.

In the above expressions we have used the notation $\partial\phi^{(\varepsilon)}(x)y$ and $\partial^2\phi^{(\varepsilon)}(x)y^2$ when $d = m = 1$ for differentiation because they become $\sum_{i=1}^{d}\partial_i\phi^{(\varepsilon)}(x)y_i$ and $\sum_{i=1}^{d}\sum_{j=1}^{d}\partial_{ij}^2\phi^{(\varepsilon)}(x)y_iy_j$ for $x = (x_i)$ and $y = (y_i)$ in the general case. We shall also use the notation $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ even when $d = m = 1$ because it is straightforward to derive the corresponding results with the notation $\frac{\partial}{\partial x_i}$ and $\frac{\partial^2}{\partial x_i \partial x_j}$ in the general case when $d \geq 1$ without any confusion. The differentiation of the indicator function $I_B$ has a proper mathematical meaning as the generalized Wiener functional. The rigorous mathematical foundation of differentiation has been given in Chapter V of Ikeda and Watanabe (1989). The next result summarizes the explicit expressions for the asymptotic expansion of expectations of the above random variables based on the Gaussian density function.

**Theorem 3. 7** *Each terms in the asymptotic expansion of (3.21)* $\mathbf{E}[\Phi_i]$ $(i = 0, 1, 2)$ *are given by*

$$
\begin{aligned}
\mathbf{E}[\Phi_0] &= \int_B \phi^{(0)}(x)n[x|0,\Sigma_{g_1}]dx, \qquad (3.24) \\
\mathbf{E}[\Phi_1] &= \int_B \{\frac{\partial\phi^{(\varepsilon)}}{\partial\varepsilon}|_{\varepsilon=0}(x)n[x|0,\Sigma_{g_1}] \\
&+ \phi^{(0)}(x)\left[\frac{-\partial\mathbf{E}[g_2|g_1 = x]n[x|0,\Sigma_{g_1}]}{\partial x}\right]\}dx \\
\mathbf{E}[\Phi_2] &= \int_B -\frac{\partial\phi^{(\varepsilon)}}{\partial\varepsilon}|_{\varepsilon=0}(x)\frac{\partial}{\partial x}\{\mathbf{E}[g_2|g_1 = x]n[x|0,\Sigma_{g_1}]\} \\
&+ \frac{1}{2}\frac{\partial^2\phi^{(\varepsilon)}}{\partial\varepsilon^2}|_{\varepsilon=0}(x)n[x|0,\Sigma_{g_1}] \\
&+ \frac{1}{2}\phi^{(0)}(x)\frac{\partial^2}{\partial x^2}\{\mathbf{E}[g_2^2|g_1 = x]n[x|0,\Sigma_{g_1}]\} \\
&+ \phi^{(0)}(x)\frac{\partial}{\partial x}\{-\mathbf{E}[g_3|g_1 = x]n[x|0,\Sigma_{g_1}]\} ,
\end{aligned}
$$

*where $n[x|0,\Sigma]$ is the density function of the Gaussian distribution with zero mean and the variance $\Sigma$.*

**Proof** : The essential part of the present proof is in the fact that we can use the integration by parts operation repeatedly. The use of integration by parts formula for Wiener functional has been explained by Chapter V of Ikeda and Watanabe (1989), and extensively used by Yoshida (1992a,b).

First, the formula for $\mathbf{E}[\Phi_0]$ is the direct result of calculation. Second, the expectation of the first term of $\Phi_1$ is given by

$$\begin{aligned}
&\mathbf{E}\left[\{\frac{\partial\phi^{(\varepsilon)}}{\partial\varepsilon}|_{\varepsilon=0}(g_1)+\partial\phi(g_1)g_2\}I_{\mathcal{B}}(g_1)\right] \\
&= \int_{\mathcal{B}}\{\frac{\partial\phi^{(\varepsilon)}}{\partial\varepsilon}|_{\varepsilon=0}(x)+\partial\phi^{(\varepsilon)}(x)\mathbf{E}[g_2|g_1=x]\}n[x|0,\Sigma_{g_1}]dx\,.
\end{aligned}$$

As for the expectation of $\phi^{(0)}(g_1)\partial I_{\mathcal{B}}(g_1)g_2$, we notice that $\phi^{(0)}(g_1)g_2\in\boldsymbol{D}^{\infty}$. Then by using the integration by parts formula for Wiener functional, we have

$$\begin{aligned}
\mathbf{E}\left[\phi^{(0)}(g_1)\partial I_{\mathcal{B}}(g_1)g_2\right] &= \mathbf{E}\left[\phi^{(0)}(g_1)g_2\partial I_{\mathcal{B}}(g_1)\right] \\
&= \mathbf{E}\left[G(w)I_{\mathcal{B}}(g_1)\right] \\
&= \mathbf{E}\left[\mathbf{E}[G(w)|g_1=x]I_{\mathcal{B}}(g_1)\right] \\
&= \int_{\mathcal{B}}\mathbf{E}[G(w)|g_1=x]n[x|0,\Sigma_{g_1}]dx \\
&\equiv \int_{\mathcal{B}}p_1(x)dx
\end{aligned}$$

for a smooth Wiener functional $G(w)$. In order to obtain an explicit representation of $p_1(x)$, we set $\mathcal{B}_x=(-\infty,x]$ . Then we have

$$\begin{aligned}
\mathbf{E}\left[\phi^{(0)}(g_1)\partial I_{\mathcal{B}_x}(g_1)g_2\right] &= \int_{-\infty}^{\infty}\phi^{(0)}(y)E[g_2|g_1=y]\partial I_{\mathcal{B}_x}(y)n[y|0,\Sigma_{g_1}]dy \\
&= -\int_{-\infty}^{\infty}\phi^{(0)}(y)E[g_2|g_1=y]\delta_x(y)n[y|0,\Sigma_{g_1}]dy \\
&= -\phi^{(0)}(x)E[g_2|g_1=x]n[x|0,\Sigma_{g_1}]\,,
\end{aligned}$$

where $\delta_x(y)$ denotes the delta function of $y$ at $x$. By differentiating the above term with respect to $x$, we have

$$p_1(x)=\frac{\partial}{\partial x}\left[-\phi^{(0)}(g_1)\mathbf{E}[g_2|g_1=x]n[x|0,\Sigma_{g_1}]\right]\,.$$

By adding two terms, we have the explicit formula for $\mathbf{E}[\Phi_1]$ as

$$\mathbf{E}[\Phi_1]=\int_{\mathcal{B}}\{\frac{\partial\phi^{(\varepsilon)}}{\partial\varepsilon}|_{\varepsilon=0}(x)n[x|0,\Sigma_{g_1}]+\phi^{(0)}(x)\left[\frac{-\partial\mathbf{E}[g_2|g_1=x]n[x|0,\Sigma_{g_1}]}{\partial x}\right]\}dx\,.$$

Third, we shall derive an explicit representation for $\mathbf{E}[\Phi_2]$, which is more complicated. For this purpose, we write it as

$$\begin{aligned}\mathbf{E}[\Phi_2] &= \int_{\mathcal{B}} p_2(x)dx \\ &= \int_{\mathcal{B}} p_{21}(x)dx + \int_{\mathcal{B}} p_{22}(x)dx + \int_{\mathcal{B}} p_{23}(x)dx ,\end{aligned}$$

where $p_{2i}$ $(i = 1,2,3)$ corresponds to each line of $\Phi_2$ in (3.21). The first term $p_{21}(x)$ can be calculated directly as $\mathbf{E}[\Phi_1]$ by using the integration by parts formula and is given by

$$p_{21}(x) = \frac{\partial}{\partial x}\left[-\{\frac{\partial \phi^{(\varepsilon)}}{\partial \varepsilon}|_{\varepsilon=0}(x)\mathbf{E}[g_2|g_1 = x] + \partial\phi^{(\varepsilon)}(x)\mathbf{E}[g_2^2|g_1 = x]\}n[x:0,\Sigma_{g_1}]\right] .$$

For the second term, we only need the standard differentiation and $p_{22}(x)$ is given by

$$\begin{aligned}p_{22}(x) &= \left[\frac{1}{2}\frac{\partial^2 \phi^{(\varepsilon)}}{\partial \varepsilon^2}|_{\varepsilon=0}(x) + \{\frac{\partial^2 \phi^{(\varepsilon)}(x)}{\partial x \partial \varepsilon}|_{\varepsilon=0}\}\mathbf{E}[g_2|g_1 = x] + \partial\phi^{(0)}(x)\mathbf{E}[g_3|g_1 = x]\right. \\ &+ \left.\frac{1}{2}\partial^2\phi^{(0)}(x)\mathbf{E}[g_2^2|g_1 = x]\right] n[x|0,\Sigma_{g_1}] .\end{aligned}$$

In order to derive $p_{23}(x)$, first we need an expression of the second order generalized derivatives of Winer functional $\mathbf{E}\left[\frac{1}{2}\phi^{(0)}(g_1)\partial^2 I_{\mathcal{B}}(g_1)g_2^2\right]$ . By taking $\mathcal{B} = \mathcal{B}_x = (-\infty, x]$ and using the integration by parts formulas for Wiener functionals, we have

$$\begin{aligned}&\mathbf{E}\left[\frac{1}{2}\phi^{(0)}(g_1)\partial^2 I_{\mathcal{B}_x}(g_1)g_2^2\right] \\ &= \int_{-\infty}^{\infty} \partial^2 I_{\mathcal{B}_x}(y)\{\frac{1}{2}\phi^{(0)}(y)\mathbf{E}[g_2^2|g_1 = y]n[y|0,\Sigma_{g_1}]\}dy \\ &= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \delta_x(y)\{\frac{1}{2}\phi^{(0)}(y)\mathbf{E}[g_2^2|g_1 = y]n[y|0,\Sigma_{g_1}]\}dy \\ &= \frac{\partial}{\partial x}\{\frac{1}{2}\phi^{(0)}(x)\mathbf{E}[g_2^2|g_1 = x]n[x|0,\Sigma_{g_1}]\} \\ &= \int_{-\infty}^{x} \frac{\partial^2}{\partial y^2}\{\frac{1}{2}\phi^{(0)}(y)\mathbf{E}[g_2^2|g_1 = y]n[y|0,\Sigma_{g_1}]\}dy .\end{aligned}$$

For the term of $\mathbf{E}\left[\phi^{(0)}(g_1)\partial I_{\mathcal{B}}(g_1)g_3\right]$, we obtain

$$\mathbf{E}\left[\phi^{(0)}(g_1)\partial I_{\mathcal{B}_x}g_3\right] = \int_{-\infty}^{x} \frac{\partial}{\partial y}\{-\phi^{(0)}(y)\mathbf{E}[g_3|g_1 = y]n[y|0,\Sigma_{g_1}]\}dy .$$

Hence, $p_{23}(x)$ is given by

$$\begin{aligned}p_{23}(x) &= \frac{\partial^2}{\partial x^2}\{\frac{1}{2}\phi^{(0)}(x)\mathbf{E}[g_2^2|g_1 = x]n[x|0,\Sigma_{g_1}]\} \\ &+ \frac{\partial}{\partial x}\{-\phi^{(0)}(x)\mathbf{E}[g_3|g_1 = x]n[x|0,\Sigma_{g_1}]\} .\end{aligned}$$

Finally, by collecting and rearranging each term of $p_{21}(x), p_{22}(x)$, and $p_{23}(x)$, we conclude

$$
\begin{aligned}
p_2(x) &= -\frac{\partial \phi^{(\varepsilon)}}{\partial \varepsilon}|_{\varepsilon=0}(x)\frac{\partial}{\partial x}\{\mathbf{E}[g_2|g_1 = x]n[x:0,\Sigma_{g_1}]\} \\
&+ \frac{1}{2}\frac{\partial^2 \phi^{(\varepsilon)}}{\partial \varepsilon^2}|_{\varepsilon=0}(x)n[x|0,\Sigma_{g_1}] \\
&+ \frac{1}{2}\phi^{(0)}(x)\frac{\partial^2}{\partial x^2}\{\mathbf{E}[g_2^2|g_1 = x]n[x|0,\Sigma_{g_1}]\} \\
&+ \phi^{(0)}(x)\frac{\partial}{\partial x}\{-\mathbf{E}[g_3|g_1 = x]n[x|0,\Sigma_{g_1}]\}.
\end{aligned}
$$

**Q.E.D.**

If we take a particular function $\phi^{(\varepsilon)}(x)$, we can derive the corresponding formulae in the asymptotic expansion. We shall give two examples for an illustration. If we take $\phi^{(\varepsilon)}(x) \equiv 1$ and $\mathcal{B} = (-\infty, x]$, then we have an asymptotic expansion of the distribution function, which is given by

$$
\begin{aligned}
P(\{X_T^{(\varepsilon)} \leq x\}) &\sim \int_{-\infty}^{x} n[y|0,\Sigma_{g_1}]dy + \varepsilon\int_{-\infty}^{x}\frac{-\partial \mathbf{E}[g_2|g_1 = y]n[y|0,\Sigma_{g_1}]}{\partial y}dy \\
&+ \varepsilon^2\left[\int_{-\infty}^{x}\frac{1}{2}\frac{\partial^2}{\partial y^2}\{\mathbf{E}[g_2^2|g_1 = y]n[y|0,\Sigma_{g_1}]\}\right. \\
&+ \left.\frac{\partial}{\partial y}\{-\mathbf{E}[g_3|g_1 = y]n[y|0,\Sigma_{g_1}]\}\right]dy + \cdots \\
&= N\left[\frac{x}{\Sigma_{g_1}^{\frac{1}{2}}}\right] + \varepsilon\{-\mathbf{E}[g_2|g_1 = x]n[x|0,\Sigma_{g_1}]\} \\
&+ \varepsilon^2\left[\frac{1}{2}\frac{\partial}{\partial x}\mathbf{E}[g_2^2|g_1 = x]n[x|0,\Sigma_{g_1}] - \mathbf{E}[g_3|g_1 = x]n[x|0,\Sigma_{g_1}]\right] + \cdots .
\end{aligned}
$$

As the second example, we take the payoff function of European call options. For this purpose, we set $\phi^{\varepsilon}(x) = x + y$ for a constant $y$ and $\mathcal{B} = [-y, \infty)$. Then we have

$$
\begin{aligned}
\mathbf{E}\left[(x+y)^+\right] &\sim \int_{-y}^{\infty}(y+x)\ n[x|0,\Sigma_{g_1}]dx \\
&+ \varepsilon\int_{-y}^{\infty} x\ \frac{-\partial \mathbf{E}[g_2|g_1 = x]n[x|0,\Sigma_{g_1}]}{\partial x}dx \\
&+ \varepsilon^2\int_{-y}^{\infty} x\left[\frac{\partial}{\partial x}\{-\mathbf{E}[g_3|g_1 = x]n[x|0,\Sigma_{g_1}]\}\right. \\
&+ \left.\frac{1}{2}\frac{\partial^2}{\partial x^2}\{\mathbf{E}[g_2^2|g_1 = x]n[x|0,\Sigma_{g_1}]\}\right]dx + \cdots .
\end{aligned}
$$

These formulas we have obtained are equivalent to the formulae by using the inversion technique [5] for the characteristic function of the corresponding random variable for the European call options under an additional assumption, which have been reported by Takahashi (1997).

The valuation problem of financial contingent claims in the Black Sholes economy can be simply defined as to find its "fair" value at financial markets. Let $V(T)$ be the payoff of a contingent claim at the terminal period $T$. Then the standard martingale theory in financial economics predicts that the fair price of $V(T)$ at time $t$ $(0 \leq t < T)$ should be given by

$$V_t(T) = \mathbf{E}_t \left[ e^{-\int_t^T r(S_v^{(\varepsilon)},v)dv} V(T) \right] ,$$

where $\mathbf{E}_t [\cdot]$ stands for the conditional expectation operator given the information available at $t$ with respect to the equivalent martingale measure.

In particular, Takahashi (1997) has given many asymptotic expansion formulas for the examples we have mentioned in this section when $r$ is a positive constant in details. In this case, the conditions in (3.4) and (3.5) do not have any restriction on the drift term.

# 4 The Validity in the Term Structure Model of Interest Rates

We shall show the validity of our method in an arbitrage-free pricing model based on a family of the instantaneous forward rates processes in the HJM framework. For the sake of completeness, we repeat (1.2) and assume the forward rates processes obey the stochastic integral equation :

$$\begin{aligned} f^{(\varepsilon)}(s,t) = f(0,t) \quad &+ \quad \varepsilon^2 \int_0^s \left[ \sum_{i=1}^m \sigma_i(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy \right] dv \\ &+ \quad \varepsilon \int_0^s \sum_{i=1}^m \sigma_i(f^{(\varepsilon)}(v,t),v,t)dw_i(v), \end{aligned}$$

where $0 < \varepsilon \leq 1$ and $0 \leq s \leq t \leq T \leq \bar{T}$. The volatility function $\sigma_i(f^{(\varepsilon)}(s,t),s,t)$ depends not only on s and t, but also on $f^{(\varepsilon)}(s,t)$ in the general case.

In this section we make the following two assumptions.

[5] Fujikoshi et. al. (1982) have given useful inversion formulas, which have been extensively used by Kunitomo and Takahashi (1995), and Takahashi (1997) to derive explicit results of asymptotic expansions.

**Assumption II** : The volatility functions $\{\sigma_i(f^{(\varepsilon)}(s,t),s,t)\}$ are non-negative, bounded, Lipschitz continuous, and smooth in its first argument, and all derivatives are bounded uniformly in $\varepsilon$, where $f^{(\varepsilon)}(s,t)$ are properly defined in $(\varepsilon, s, t, f^{(\varepsilon)}(s,t))$ $\in (0,1] \times \{0 < s \leq t \leq T\} \times R^1$. The initial forward rates $f(0,t)$ are also Lipschitz continuous with respect to $t$.

**Assumption III** : For any $0 \leq s < t \leq T$,

$$\Sigma(s,t) = \int_0^s \sum_{i=1}^m \sigma_i^{(0)}(v,t)^2 dv > 0, \tag{4.25}$$

where

$$\sigma_i^{(0)}(v,t) = \sigma_i(f^{(\varepsilon)}(v,t),v,t)|_{\varepsilon=0}.$$

The conditions we have made in Assumption II can exclude the possiblity of explosions for the solution of (1.2) [6]. They are quite strong and could be relaxed considerably, which may be interesting from the view of stochastic analysis. For practical purposes, however, we can often use the truncation arguments as an example given by Heath, Jarrow, and Morton (1992). Assumption III ensures the key condition of non-degeneracy of the Malliavin-covariance in our problem, which is essential for the validity of the asymptotic expansion approach as we shall see in the following derivations for the forward rate processes. Under these assumptions we can get the stochastic expansions of the forward rates and spot interest rates processes. We show this in several steps.

Without loss of generality, we consider the case when $m = 1$ because more complicated notations are needed in the general case. We set $m = 1$ and $\varepsilon = 1$ in (1.2) in the first step. The starting point of our discussion is the result by Morton (1989) on the existence and uniqueness of the solution of the stochastic integral equation (1.2) for forward rate processes.

**Theorem 4. 1** [ *Morton (1989)* ]
*Under Assumption II, there exists a jointly continuous process $\{f^{(\varepsilon)}(s,t), 0 \leq s \leq t \leq T\}$ satisfying (1.2) with $\varepsilon = 1$. There is at most one solution of (1.2) with $\varepsilon = 1$.*

We shall consider the $H$-derivatives of the forward rate processes $\{f^{(1)}(s,t)\}$. For any $h \in H$, we successively define a sequence of random variables $\{\xi^{(n)}(s,t)\}$ by the integral equation:

[6] For example, Morton (1989) has shown that there does not exist any meaningful solution when the volatility function is proportional to the forward rate process.

$$\xi^{(n+1)}(s,t) = \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy\xi^{(n)}(v,t)\right]dv$$
$$+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi^{(n)}(v,y)dy\right]dv$$
$$+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)\xi^{(n)}(v,t)dw(v)$$
$$(4.26) \qquad + \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}_v dv$$

where the initial condition is given by $\xi^{(0)}(s,t) = 0$. Then we have the next result by using the standard method in stochastic analysis.

**Lemma 4. 1** *: For any $p > 1$ and $0 \le s \le t \le T$,*

$$(4.27) \qquad \mathbf{E}[|\xi^{(n)}(s,t)|^p] < \infty ,$$

*and as $n \to \infty$*

$$(4.28) \qquad \sup_{0\le s\le t\le T} \mathbf{E}[\sup_{0\le u\le s} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)|^2] \to 0 .$$

**Proof** [i] We use the induction argument for $n$. We have (4.27) when $n = 1$ because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is a square-integrable function in (4.26). Suppose (4.27) hold for $n = k$. Then there exist positive constants $M_i (i = 1, \cdots, 4)$ such that

$$|\xi^{(k+1)}(s,t)|^p \le M_1 \int_0^s |\xi^{(k)}(v,t)|^p dv + M_2[\sup_{0\le u\le s} |\int_0^u \xi^{(k)}(v,t)dw(v)|]^p$$
$$(4.29) \qquad +M_3 \int_0^s \int_v^t |\xi^{(k)}(v,y)|^p dy dv + M_4[\int_0^s |\dot{h}_v|^2 dv]^{p/2} .$$

By a martingale inequality [7], the expectation of the second term on the right hand side of (4.29) is less than

$$M_3'\mathbf{E}[\{\int_0^s |\xi^{(k)}(v,t)|^2 dv\}^{p/2}] \le M_3'' \int_0^s \mathbf{E}[|\xi^{(k)}(v,t)|^p]dv ,$$

where $M_3'$ and $M_3''$ are positive constants. Because $\dot{h}_v$ is square-integrable, we have (4.27) when $n = k+1$.

[7] The maritingale inequality in the standard case has been given as *Theorem III-3.1* of Ikeda and Watanabe (1989), for instance, which could be regarded as a special case of *Lemma 3.4* in Section 3.

**[ii]** From (4.26), there exist positive constants $M_i(i = 5, 6, 7)$ such that for $0 \le s \le t$,

$$
\begin{aligned}
|\xi^{(n+1)}(s,t) - \xi^{(n)}(s,t)|^2 &\le M_5[\int_0^s |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dv]^2 \\
\text{(4.30)} \qquad &+ M_6[\int_0^s \int_v^t |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dydv]^2 \\
&+ M_7[\int_0^s \partial\sigma(f^{(1)}(v,t), v, t)|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dw(v)]^2 \\
&\equiv \sum_{i=1}^{3} I_i^{(n)}(s,t) \, ,
\end{aligned}
$$

where we have defined $I_i^{(n)}(s,t)$ by the last equality. By using the Cauchy-Schwartz inequality,

$$
\mathbf{E}[\sup_{0 \le u \le s} I_1^{(n)}(u,t)] \le M_5 s \int_0^s \mathbf{E}[|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]dv \, .
$$

By repeating the above argument to the second term of (4.30), we have

$$
\begin{aligned}
\text{(4.31)} \qquad I_2^{(n)}(u,t) &\le M_6 u \int_0^u [\int_v^t |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|dy]^2 dv \\
&\le M_6 ut \int_0^u \int_v^t |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2 dydv \, .
\end{aligned}
$$

Then

$$
\text{(4.32)} \quad \mathbf{E}[\sup_{0 \le u \le s} I_2^{(n)}(u,t)] \le M_6 st \int_0^s \int_v^t \mathbf{E}[|\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2]dydv \, .
$$

For the third therm of (4.30), we have

$$
\text{(4.33)} \qquad \mathbf{E}[\sup_{0 \le u \le s} I_3^{(n)}(u,t)] \le M_7' \int_0^s \mathbf{E}[|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]dv
$$

because of the boundedness of $\partial\sigma(\cdot)$, where $M_7'$ is a positive constant. By using (4.31), (4.32), and (4.33), we have

$$
\begin{aligned}
\mathbf{E}[\sup_{0 \le u \le s} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)|^2] &\le M_8 \Big( \int_0^s \mathbf{E}[\sup_{0 \le v \le u} |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]du \\
&+ \int_0^s \int_u^t \mathbf{E}[\sup_{0 \le v \le u} |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2]dydu \Big)
\end{aligned}
$$

where $M_8$ is a positive constant. By defining a sequence of $\{u^{(n)}(s,t)\}$ by

$$
u^{(n+1)}(s,t) = \mathbf{E}[\sup_{0 \le u \le s} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)|^2] \, ,
$$

we have the relation

$$u^{(n+1)}(s,t) \leq M_8 \int_0^s [\int_u^t u^{(n)}(u,y)dy + u^{(n)}(u,t)]du \ ,$$

If we have an inequality

$$u^{(n+1)}(s,t) \leq \frac{1}{(n+1)!}[M_9(t+1)s]^{n+1} \ , \tag{4.34}$$

we can show (4.28) as $n \to +\infty$. We use the induction argument for $n \geq 1$. When $n = 1$, there exists a positive constant $M_9$ such that

$$\begin{aligned} u^{(1)}(s,t) &= \mathbf{E}[\sup_{0\leq u\leq s} |\xi^{(1)}(u,t) - \xi^{(0)}(u,t)|^2] \\ &= \mathbf{E}[\sup_{0\leq u\leq s} |\int_0^s \sigma(f^{(1)}(u,t),u,t)\dot{h}_v du|^2] \\ &\leq M_8(1+t)s \end{aligned}$$

because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is square–integrable. Suppose (4.34) hold for $n = k$. Then

$$\begin{aligned} u^{(k+1)}(s,t) &\leq M_8 \int_0^s [\int_u^t u^{(k)}(u,y)dy + u^{(k)}(u,t)]du \\ &\leq M_8 \int_0^s [\int_u^t M_8^k(t+1)^k \frac{s^k}{k!} dy + M_8^k(t+1)^k \frac{s^k}{k!}]du \\ &\leq M_8^{k+1}(t+1)^{k+1} \frac{s^{k+1}}{(k+1)!} \ . \end{aligned}$$

*Q.E.D.*

Because of (4.34) and the Chebyshev's inequality, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} P\{\sup_{0\leq u\leq s\leq t} |\xi^{(n+1)}(u,s) - \xi^{(n)}(u,s)| > \frac{1}{2^n}\} \\ \leq &\sum_{n=1}^{\infty} \frac{1}{n!}[4M_8(T+1)T]^n < +\infty \ . \end{aligned}$$

Then by the Borel-Cantelli lemma, the sequence of random variables $\{\xi^{(n)}(s,t)\}$ converges uniformly on $0 \leq u \leq s \leq t$ $(\leq T)$. Hence we have established the existence of the $H$–derivative of $f^{(1)}(s,t)$, which is given by the solution of the stochastic integral equation :

$$\begin{aligned} D_h f^{(1)}(s,t) &= \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dyD_hf^{(1)}(v,t)\right]dv \\ &+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)D_hf^{(1)}(v,y)dy\right]dv \\ &+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)D_hf^{(1)}(v,t)dw(v) \\ &+ \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}_v dv . \end{aligned} \tag{4.35}$$

We note that for the spot rate process $\{r^{(\varepsilon)}(t)\}$ the H-derivative can be well-defined by

$$D_h r^{(\varepsilon)}(t) = \lim_{s\to t} D_h f^{(\varepsilon)}(s,t).$$

Now we define the random variables $\{\xi_{s,t}^{(1,1)}(u)\}$ for $0 \le u \le s \le t \le T$ by a stochastic integral equation :

$$\begin{aligned} \xi_{s,t}^{(1,1)}(u) &= \int_u^s \partial\sigma(f^{(1)}(v,t),v,t)\xi_{v,t}^{(1,1)}(u)\int_v^t \sigma(f^{(1)}(v,y),v,y)dydv \\ &+ \int_u^s \sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi_{v,y}^{(1,1)}(u)dydv \\ &+ \int_u^s \partial\sigma(f^{(1)}(v,t),v,t)\xi_{v,t}^{(1,1)}(u)dw(v) \\ &+ \sigma(f^{(1)}(u,t),u,t). \end{aligned} \tag{4.36}$$

Then we can show that

$$\int_0^s \xi_{s,t}^{(1,1)}(u)\dot{h}_u du = D_h f^{(1)}(s,t).$$

The rigorous proof for the existence of the sets of random variables $\{D_h f^{(1)}(s,t)\}$ and $\{\xi_{s,t}^{(1,1)}(u)\}$ in $L_p$−norm can be given by using the approximations by discretized stochastic processes and their convergence in $L_p$norm. For this purpose, we define $\phi_n(s) = \frac{k}{2^n}$ if $s \in [\frac{k}{2^n}, \frac{(k+1)}{2^n})$, and $\psi_n(s) = \frac{k}{2^n}$ if $s \in (\frac{k-1}{2^n}, \frac{k}{2^n}]$ for $k = 1, \cdots, (2^n - 1)T$. Then we can define a sequence of random variables $\xi_{s,t}^{(1,1)n}(u)$ by the solution of a stochastic integral equation :

$$\begin{aligned} \xi_{s,t}^{(1,1)n}(u) &= \int_{\psi_n(u)\vee s}^s \partial\sigma(f^{(1)}(\phi_n(v),t),v,t)\xi_{\phi_n(v),t}^{(1,1)n}(u) \\ &\qquad \times \int_v^t \sigma(f^{(1)}(\phi_n(v),y),v,t)dydv \\ &+ \int_{\psi_n(u)\vee s}^s \sigma(f^{(1)}(\phi_n(v),t),v,t) \end{aligned}$$

$$\times \int_v^t \partial\sigma(f^{(1)}(\phi_n(v),t),v,t)\xi^{(1,1)n}_{\phi_n(v),y}(u)dydv$$
$$+ \int_{\psi_n(u)\vee s}^{s} \partial\sigma(f^{(1)}(\phi_n(v),t),v,t)\xi^{(1,1)n}_{\phi_n(v),t}(u)dw(v)$$
$$+ \sigma(f^{(1)}(\phi_n(u),t),u,t) .$$

Then we can define a sequence of random variables $\{D^n_h f^{(1)}(s,t)\}$ by the solution of a stochastic integral equation :

$$\begin{aligned}
D^n_h f^{(1)}(s,t) &= \int_0^s \int_{\psi_n(u)\vee s}^{s} \partial\sigma(f^{(1)}(\phi_n(v),t),v,t)\xi^{(1,1)n}_{\phi_n(v),t}(u) \\
&\qquad \times \int_v^t \sigma(f^{(1)}(\phi_n(v),y),v,t)dydv\dot{h}_u du \\
&+ \int_0^s \int_{\psi_n(u)\vee s}^{s} \sigma(f^{(1)}(\phi_n(v),t),v,t) \\
&\qquad \times \int_v^t \partial\sigma(f^{(1)}(\phi_n(v),t),v,t)\xi^{(1,1)n}_{\phi_n(v),y}(u)dydv\dot{h}_u du \\
&+ \int_0^s \int_{\psi_n(u)\vee s}^{s} \partial\sigma(f^{(1)}(\phi_n(v),t),v,t)\xi^{(1,1)n}_{\phi_n(v),t}(u)dw(v)\dot{h}_u du \\
&+ \int_0^s \sigma(f^{(1)}(\phi_n(u),t),u,t)\dot{h}_u du .
\end{aligned}$$

We also consider a sequence of the corresponding random variables $\{f^{(1)n}(s,t)\}$, which are defined by

$$\begin{aligned}
f^{(1)n}(s,t) &= f(0,t) + \int_0^s \sigma(f^{(1)n}(\phi_n(v),t),(v,t) \int_v^t \sigma(f^{(1)n}(\phi_n(v),y),v,y)dydv \\
&\quad + \int_0^s \sigma(f^{(1)n}(\phi_n(v),t),v,t)dw(v) .
\end{aligned}$$

In order to examine the existence of higher order moments of $\{\xi^{(1,1)n}_{s,t}(u)\}$, $\{\xi^{(1,1)}_{s,t}(u)\}$, and other related random variables, we prepare the following inequality.

**Lemma 4. 2** : *Suppose for $k_0 > 0, k_1 > 0, A_N > 0$ and $0 < s \le t \le T$, a function $w_N(u,s,t)$ satisfies (i) $0 < w_N(u,s,t) \le A_N$ and (ii)*

$$w_N(u,s,t) \le k_0 + k_1 \left[\int_u^s w_N(u,v,t)dv + \int_u^s \int_v^t w_N(u,v,y)dydv\right] . \tag{4.37}$$

*Then,*

$$w_N(u,s,t) \le k_0 e^{k_1(1+t)s} . \tag{4.38}$$

**Proof** : By substituting (i) into the right hand side of (4.37), we have

$$(4.39) \quad \begin{aligned} w_N(u,s,t) &\leq k_0 + A_N k_1 \left[ \int_u^s ds + \int_u^s \int_v^t dydv \right] \\ &\leq k_0 + A_N k_1 (1+t)s \,. \end{aligned}$$

By repeating the substitution of (4.39) into the right hand side of (4.37), we have

$$w_N(u,s,t) \leq k_0 \sum_{k=0}^{n} \frac{1}{k!} [k_1(1+t)s]^k + \frac{1}{(n+1)!} A_N [k_1(1+t)s]^{n+1} \,.$$

Then we have (4.38) by taking $n \to +\infty$. *Q.E.D.*

We consider the truncated random variable :

$$(4.40) \qquad \zeta_{s,t}^N(u) = \left[\xi_{s,t}^{(1,1)}(u)\right] I_N(s,t) \,,$$

where $I_N(s,t) = 1$ if

$$\sup_{0 \leq v \leq s, v \leq y \leq t} |\xi_{v,y}(u)| \leq N$$

and $I_N(s,t) = 0$ otherwise. By using the boundedness conditions in Assumption II and $\dot{h}_s$ being square-integrable, we can show that there exist positive constants $M_i$ $(i = 10, 11, 12)$ such that

$$(4.41) \quad \begin{aligned} |\zeta_{s,t}^N(u)|^p &\leq M_9 \int_u^s |\zeta_{v,t}^N(u)|^p dv + M_{10} | \int_u^s \zeta_{v,t}^N(u) dw(v)|^p \\ &+ M_{11} \int_u^s \int_v^t |\zeta_{v,y}^N(u)|^p dydv + M_{12} |\sigma(f^{(1)}(u,t),u,t)|^p \,. \\ &\equiv \sum_{i=1}^{4} J_i^N(u,s,t) \,, \end{aligned}$$

where we have defined $J_i^N(u,s,t)(i = 1, \cdots, 4)$ by the last equality. By using the martingale inequality, we have

$$(4.42) \quad \begin{aligned} \mathbf{E}[J_2^N(u,s,t)] &\leq M_{10}^{'} \mathbf{E}[\int_u^s |\zeta_{v,t}^N(u)|^2 dv|^{p/2} \\ &\leq M_{10}^{''} \mathbf{E}[\int_u^s |\zeta_{v,t}^N(u)|^p dv] \,, \end{aligned}$$

where $M_{10}^{'}$ and $M_{10}^{''}$ are positive constants. Also $J_4^N(s,t)$ is bounded because $\sigma(\cdot)$ is bounded. If we set $w_N(u,s,t) = \mathbf{E}[|\zeta_{s,t}(u)|^p]$, then we can directly apply Lemma 4.2. By taking the limit of the expectation function $w_N(u,s,t)$ as $N \to \infty$, we have the following lemma.

**Lemma 4. 3** : *Under the assumption II, for any* $p > 1$,

$$(4.43) \qquad \mathbf{E}[|\xi_{s,t}^{(1,1)}(u)|^p] < +\infty \,.$$

By modifying slightly this method for the existence of moments, we can also show that for any $p > 1$,

$$\mathbf{E}[\sup_{0\le u\le s} |\xi_{s,t}^{(1,1)n}(u)|^p] < +\infty .$$

Because of Assumption II, for any $p > 1$ we have

$$\mathbf{E}[\sup_{0\le u\le s} |f^{(1)n}(u,t)|^p] < +\infty$$

and

$$\mathbf{E}[\sup_{0\le u\le s} |f^{(1)}(u,t)|^p] < \infty .$$

Then we can develop the method similar to the one used in the proof of Lemma 4.1 for proving the convergence of sequences of the discretized random variables to the corresponding continuous processes in $L_p$–norm. Because the method has been standard in stochastic analysis (see *Lemma 2.1* in Chapter V of Ikeda and Watanabe (1989), for instance) and it needs lengthy evaluations, we omit the details. As the result of these arguments, we have the next result.

**Lemma 4. 4** : *Under Asumption II, for any* $p > 1$ *we have*

$$\sup_{0\le s\le t\le T} \mathbf{E}[\sup_{0\le u\le s} |f^{(1)n}(u,t) - f^{(1)}(u,t)|^p] \to 0$$

*as* $n \to +\infty$. *For any* $p > 1$ *we can show that*

$$\sup_{0\le s\le t\le T} \mathbf{E}[\sup_{0\le u\le s} |\xi_{s,t}^{(1,1)n}(u) - \xi_{s,t}^{(1,1)}(u)|^p] \to 0$$

*as* $n \to +\infty$. *Also for any* $p > 1$ *we have*

$$\sup_{0\le s\le t\le T} \mathbf{E}[\sup_{0\le u\le s} |Df^{(1)n}(u,t) - Df^{(1)}(u,t)|_H^p] \to 0$$

*as* $n \to +\infty$.

By using Lemma 4.3, Lemma 4.4, and the equivalence of two norms stated in Section 2.1, we now have established the following property of the first order $H$–derivative :

$$f^{(1)}(s,t) \in \cap_{1<p<+\infty} \boldsymbol{D}_p^1 .$$

Since we have completed the investigation of the first order $H$– derivative, our next task is to investigate some properties of the second order $H$–derivative of $f^{(1)}(s,t)$. As for the second order $H$– derivative, we have

$$
\begin{aligned}
& D_h[Df^{(1)}(s,t)] \\
= & \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dyD_h[Df^{(1)}(v,t)]\right]dv \\
+ & \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)D_h[Df^{(1)}(v,y)]dy\right]dv \\
+ & \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)D_h[Df^{(1)}(v,t)]dw(v) \\
+ & \int_0^s \left[\partial^2\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dyD_hf^{(1)}(v,t)Df^{(1)}(v,t)\right]dv \\
+ & \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)Df^{(1)}(v,t)D_hf^{(1)}(v,y)dy\right]dv \\
+ & \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)D_hf^{(1)}(v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)Df^{(1)}(v,y)dy\right]dv \\
+ & \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial^2\sigma(f^{(1)}(v,y),v,y)D_hf^{(1)}(v,y)Df^{(1)}(v,y)dy\right]dv \\
+ & \int_0^s \partial^2\sigma(f^{(1)}(v,t),v,t)D_hf^{(1)}(v,t)Df^{(1)}(v,t)dw(v) \\
+ & \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)[D_hf^{(1)}(v,t)\dot{h}_v + Df^{(1)}(v,t)]dv\,.
\end{aligned}
$$

Although the above stochastic differential equation has many terms, the basic structure is the same as the first order $H$–derivative of $f^{(1)}(s,t)$. Now we define the random variables $\{\xi_{s,t}^{(1,2)}(u)\}$ for $0 \le u \le s \le t \le T$ by

$$
\begin{aligned}
& \xi_{s,t}^{(1,2)}(u) \\
= & \int_u^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy\xi_{v,t}^{(1,2)}(u)\right]dv \\
+ & \int_u^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi_{v,y}^{(1,2)}(u)dy\right]dv \\
+ & \int_u^s \partial\sigma(f^{(1)}(v,t),v,t)\xi_{v,t}^{(1,2)}(u)dw(v) \\
+ & \int_u^s \left[\partial^2\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dyD_hf^{(1)}(v,t)\xi_{v,t}^{(1,1)}(u)\right]dv \\
+ & \int_u^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)\xi_{v,t}^{(1,1)}(u)D_hf^{(1)}(v,y)dy\right]dv \\
+ & \int_u^s \left[\partial\sigma(f^{(1)}(v,t),v,t)D_hf^{(1)}(v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi_{v,y}^{(1,1)}(u)dy\right]dv \\
+ & \int_u^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial^2\sigma(f^{(1)}(v,y),v,y)D_hf^{(1)}(v,y)\xi_{v,y}^{(1,1)}(u)(v,y)dy\right]dv \\
+ & \int_u^s \partial^2\sigma(f^{(1)}(v,t),v,t)D_hf^{(1)}(v,t)\xi_{v,t}^{(1,1)}(u)dw(v) \\
+ & \int_u^s \partial\sigma(f^{(1)}(v,t),v,t)[\xi_{v,t}^{(1,1)}(u) + Df^{(1)}(v,t)]\dot{h}_vdv\,.
\end{aligned}
$$

Then we can show that

$$\int_0^s \xi_{s,t}^{(1,2)}(u)\dot{h}_u du = D_h[Df^{(1)}(s,t)] \; .$$

The rigorous proof for the existence of random variables $\{D_h[Df^{(1)}(s,t)]\}$ and $\{\xi_{s,t}^{(1,2)}(u)\}$ can be again given by modifying the same method as $\{D_h f^{(1)}(s,t)\}$ and $\{\xi_{s,t}^{(1,1)}(u)\}$. From the above representation, we have

$$|D^2 f^{(1)}(s,t)|^2_{H^{\otimes 2}\otimes R} = \int_0^s |\xi_{s,t}^{(1,2)}(u)|^2_{H\otimes R} du \; . \tag{4.44}$$

Then by applying Lemma 4.2, repeating the procedure as Lemma 4.3, and using induction with respect to $j$ ($j \geq 1$), we have the next lemma.

**Lemma 4. 5** : *Under the assumption II, for any integer $j \geq 2$ and $p > 1$,*

$$\mathbf{E}[|\xi_{s,t}^{(1,j)}(u)|^p_{H^{\otimes j-1}\otimes R}] < +\infty \; . \tag{4.45}$$

Then by using this lemma, we can derive the higher order $H$−derivatives of $f^{(1)}(s,t)$, which are in $L_p$ for any $p > 1$. By the same constructions and induction arguments, for positive integers $j$ ($\geq 2$) we can define a sequence of random variables : $\{f^{(\varepsilon)}(s,t)\}$, $\{\xi_{s,t}^{(\varepsilon,j)}(u)\}$ $\{D_h^j f^{(\varepsilon)}(s,t)\}$, and $\{D^j f^{(\varepsilon)}(s,t)\}$. The rigorous proof is again the results of convergence arguments of the corresponding discretized random variables. Hence we can obtain the following result.

**Theorem 4. 2** : *Suppose Assumption II hold for the forward rate processes. Then for any $\varepsilon \in (0,1]$ and $0 < s \leq t \leq T$,*

$$f^{(\varepsilon)}(s,t) \in \boldsymbol{D}^{\infty} \; . \tag{4.46}$$

Let a stochastic process $\{Y^{(\varepsilon)}(s,t), 0 \leq s \leq t \leq T\}$ be the solution of the stochastic integral equation :

$$\begin{aligned} Y^{(\epsilon)}(s,t) = 1 \; &+ \; \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy\right] Y^{(\varepsilon)}(v,t)dv \\ &+ \; \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t) Y^{(\varepsilon)}(v,t)dw(v) \; . \end{aligned} \tag{4.47}$$

Since the coefficients of $Y^{(\varepsilon)}(s,t)$ on the right hand side of (4.47) are bounded by Assumption I, we can obtain the next result.

**Lemma 4. 6** : *For any $1 < p < +\infty, 0 < \varepsilon \leq 1$, and $0 < s \leq t \leq T$,*

$$\mathbf{E}[|Y^{(\varepsilon)}(s,t)|^p] + \mathbf{E}[|Y^{(\varepsilon)-1}(s,t)|^p] < +\infty \; . \tag{4.48}$$

**Proof** : We define a sequence of random variables $\{Y_n^{(\varepsilon)}(s,t)\}$ by

$$Y_{n+1}^{(\epsilon)}(s,t) = 1 + \varepsilon^2 \int_0^s \left[ \partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy \right] Y_n^{(\varepsilon)}(v,t)dv$$
$$+ \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t) Y_n^{(\varepsilon)}(v,t)dw(v) ,$$

where the initial condition is given by $Y_0^{(\varepsilon)}(s,t) = 1$. Then by the same argument as in the proof of *Lemma 4.1*, we have

$$\mathbf{E}[|Y_n^{(\varepsilon)}(s,t)|^p] < \infty ,$$

and as $n \to \infty$

$$\mathbf{E}[\sup_{0 \le s \le t \le T} |Y_{n+1}^{(\varepsilon)}(s,t) - Y_n^{(\varepsilon)}(s,t)|^2] \to 0 .$$

Hence we can establish the existence of the random variables $\{Y^{(\varepsilon)}(s,t)\}$ satisfying (4.47). Then by the same argument as (4.40)-(4.43), we have

$$\mathbf{E}[|Y^{(\varepsilon)}(s,t)|^p] < \infty$$

for any $p > 1$. Let $Z^{(\varepsilon)}(s,t) = Y^{(\varepsilon)-1}(s,t)$. Then we can show that

$$d[Z^{(\varepsilon)}(s,t)Y^{(\varepsilon)}(s,t)] = 0$$

and

$$Z^{(\epsilon)}(s,t) = 1 - \varepsilon^2 \int_0^s \left[ \partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy \right] Z^{(\varepsilon)}(v,t)dv$$
$$- \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t) Z^{(\varepsilon)}(v,t)dw(v)$$

by using Itô's formula and $Z^{(\varepsilon)}(0,t) = 1$. Hence by the similar argument as on $Y^{(\varepsilon)}(s,t)$, we can establish

$$\mathbf{E}[|Z^{(\varepsilon)}(s,t)|^p] < \infty$$

for any $p > 1$. *Q.E.D.*

Now we consider the asymptotic behavior of a functional

$$F^{(\varepsilon)}(s,t) = \frac{1}{\varepsilon}[f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)] \tag{4.49}$$

as $\varepsilon \to 0$. By using the stochastic process $\{Y^{(\varepsilon)}(s,t)\}$, the $H$–derivative of $F^{(\varepsilon)}(s,t)$ can be represented as

$$D_h F^{(\varepsilon)}(s,t) = \int_0^s Y^{(\varepsilon)}(s,t) Y^{(\varepsilon)-1}(v,t) C^{(\varepsilon)}(v,t)dv , \tag{4.50}$$

where

$$C^{(\varepsilon)}(v,t) = \sigma(f^{(\varepsilon)}(v,t),v,t)\dot{h}_v + \varepsilon\sigma(f^{(\varepsilon)}(v,t),v,t) \times \int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)D_h f^{(\varepsilon)}(v,y)dy\ .$$

Then by re-arranging terms in the integrands of (4.50), we have the representation

$$D_h F^{(\varepsilon)}(s,t) = \int_0^s \nu_{s,t}^{(\varepsilon,1)}(u)\dot{h}_u du,$$

where

$$\begin{aligned} \nu_{s,t}^{(\varepsilon,1)}(u) &= Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(u,t)\sigma(f^{(\varepsilon)}(u,t),u,t) \\ &+ \varepsilon\int_u^s Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t) \\ &\times \left(\int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)\xi_{v,y}^{(\varepsilon,1)}(u)dy\right)dv, \end{aligned} \tag{4.51}$$

and $\{\xi_{v,y}^{(\varepsilon,1)}(u)\}$ are defined by $\{\xi_{v,y}^{(1,1)}(u)\}$ by replacing $(1,1)$ with $(\varepsilon,1)$. Hence the Malliavin covariance of $F^{(\varepsilon)}(s,t)$, $\sigma(F^{(\varepsilon)}(s,t))$ is obtained by

$$\begin{aligned} \sigma(F^{(\varepsilon)}(s,t)) &= < DF^{(\varepsilon)}, DF^{(\varepsilon)} >_H \\ &= \int_0^s |\nu_{s,t}^{(\varepsilon,1)}(u)|^2 du. \end{aligned}$$

Let

$$\begin{aligned} \eta_c^{(\varepsilon)}(s,t) &= c\int_0^s |\varepsilon\left(\int_u^s Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t)\right. \\ &\times \left.\int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)\xi_{v,y}^{(\varepsilon,1)}(u)dydv\right)|^2 du \\ &+ c\int_0^s |Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(u,t)\sigma(f^{(\varepsilon)}(u,t),u,t) - \sigma(f^{(0)}(u,t),u,t)|^2 du, \end{aligned}$$

for a positive constatnt $c > 0$. Then the condition in Assumption III is equivalent to the non-degeneracy condition :

$$\Sigma(s,t) = \int_0^s \sigma\left(f^{(0)}(v,t),v,t\right)^2 dv > 0$$

because $Y^{(0)}(v,t) = 1$ for $0 \le v \le s \le t$. The next lemma shows that the truncation by $\eta_c^{(\varepsilon)}(s,t)$ is negligible in probability.

**Lemma 4. 7** : *For* $0 < s \le t \le T$ *and any* $k \ge 1$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-k} P\{|\eta_c^{(\varepsilon)}(s,t)| > \frac{1}{2}\} = 0\ . \tag{4.52}$$

**Proof :** We re-write (4.52) as $\eta_c^{(\varepsilon)}(s,t) = \eta_1^{(\varepsilon)} + \eta_2^{(\varepsilon)}$. By using Assumption I, *Lemma 4.3*, *Lemma 4.6*, and the Markov inequality, it is straightforward to show that for any $p > 1$ and $c_1 > 0$ there exists a positive constant $c_2$ such that

$$P\{|\eta_1^{(\varepsilon)}| > c_1\} \leq c_2\varepsilon^{2p} . \tag{4.53}$$

By the Lipschitz continuity of the volatility function $\sigma(\cdot)$, there exist positive constants $M_{13}$ and $M_{14}$ such that

$$|\eta_2^{(\varepsilon)}| \leq M_{14}|f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| + M_{15}|Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t) - 1| . \tag{4.54}$$

Then by Lemma 3.3 and Lemma 3.4, for a positive $c_3$ and sufficiently small $\varepsilon > 0$, there exist positive constants $c_4$ and $c_5$ such that

$$P\{\sup_{0\leq s\leq t\leq T} |f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| > c_3\} \leq c_4 exp(-c_5\varepsilon^{-2}) . \tag{4.55}$$

For the second term of the right hand side of (4.54) for $\eta_2^{\varepsilon}$, we re-write

$$\eta_{22}^{(\varepsilon)} = M_{14}Y^{(\varepsilon)}(v,t)^{-1}|Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t)| ,$$

where

$$\begin{aligned} Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t) &= \varepsilon^2 \int_v^s \left[\partial\sigma(f^{(\varepsilon)}(u,t),u,t) \int_u^t \sigma(f^{(\varepsilon)}(u,y),u,y)dy\right] Y^{(\varepsilon)}(u,t)du \\ &+ \varepsilon \int_v^s \partial\sigma(f^{(\varepsilon)}(u,t),u,t)Y^{(\varepsilon)}(u,t)dw(u) . \end{aligned}$$

Then by *Lemma 4.6*, for any $p \geq 1$ and $c_6 > 0$ there exists a positive constant $c_7$ such that

$$P\{|\eta_{22}^{(\varepsilon)}| > c_6\} \leq c_7\varepsilon^{2p} . \tag{4.56}$$

By using (4.53), (4.55), and (4.56), we have (4.52). *Q.E.D.*

By a similar argument as Theorem 3.3 in the Black-Sholes economy, we shall obtain a truncated version of the non-degeneracy condition of the Malliavin-covariance for the spot interest rates and forward rates processes, which is the key step to show the validity of the asymptotic expansion approach.

**Theorem 4. 3** : *Under Assumptions II and III, the Malliavin-covariance $\sigma(F^{(\varepsilon)}(s,t))$ of $F^{(\varepsilon)}(s,t)$ is uniformly non-degenerate in the sense that there exists $c_0 > 0$ such that for any $c > c_0$ and any $p > 1$*

$$\sup_{0<\varepsilon\leq 1} \mathbf{E}[I(|\eta_c^{(\varepsilon)}| \leq 1)\sigma(F^{(\varepsilon)}(s,t))^{-p}] < +\infty , \tag{4.57}$$

*where $I(\cdot)$ is the indicator function.*

Hence the validity of the asymptotic expansions of the distribution function or the density function of spot rate and instantaneous forward rates can be obtained under Assumption II and Assumption III because we have proved that a set of conditions in *Theorem 2.1* are satisfied. By expanding the Wiener functional $F^{(\varepsilon)}(s,t)$ as

$$F^{(\varepsilon)}(s,t) \sim A_1(s,t) + \varepsilon A_2(s,t) + \cdots ,$$

where the coefficients in the asymptotic expansion can be obtained by a formal Taylor expansion. For instance, $A_1(s,t)$ and $A_2(s,t)$ are given by

$$A_1(s,t) = \int_0^s \sigma(f(0,t),v,t)dw_v ,$$

and

$$\begin{aligned} A_2(s,t) &= \int_0^s \sigma(f(0,t),v,t) \int_v^t \sigma(f(0,y),v,y)dydv \\ &\quad + \int_0^s A_1(v,t)\partial\sigma(f(0,t),v,t)dw(v) . \end{aligned}$$

These formulas have been previously obtained by Kunitomo and Takahashi (1995) in the general case. Then by applying similar arguments, which are quite tedious, we can show that the $L_p$-boundedness of any order H-derivatives of $A_j(s,t)$ for any $0 \leq s \leq t \leq T$ and integers $j \geq 1$. Hence we conclude that for any $j \geq 1$

$$A_j(s,t) \in \boldsymbol{D}^{\infty} .$$

We summarize our result as the next theorem.

**Theorem 4. 4** *Under Assumption II,* $F^{(\varepsilon)}(s,t)$ *is in* $\boldsymbol{D}^{\infty}$ *and has an asymptotic expansion :*

$$F^{(\varepsilon)}(s,t) \sim A_1(s,t) + \varepsilon A_2(s,t) + \cdots \tag{4.58}$$

*as* $\varepsilon \downarrow 0$ *and* $A_1(s,t), A_2(s,t), \cdots \in \boldsymbol{D}^{\infty}$.

We notice that the limit of the Malliavin-covariance as $\varepsilon \to 0$ is given by $\Sigma(s,t)$, which is in turn the variance of $A_1(s,t)$. Hence we have the Gaussian random variable as the leading term in (4.55) and we can use the same method as in Section 3 to derive the asymptotic expansion of the expected values of random variables. Let $\psi : R \to R$ be a smooth function such that $0 \leq \psi(x) \leq 1, \psi(x) = 1$ for $|x| \leq \frac{1}{2}$, and $\psi(x) = 0$ for $|x| \geq 1$ as before. Then the composite functional $\psi(\eta^{(\varepsilon)})I_A(F^{(\varepsilon)})$ is well-defined for any $A \in \mathcal{B}$ in the sense that it is in $\tilde{\boldsymbol{D}}^{-\infty}$, where $\mathcal{B}$ is the Borel $\sigma-$field in $\boldsymbol{R}$ and $I_A(\cdot)$ is the indicator function. By using *Theorem 2.1*, it has a proper asymptotic expansion as $\varepsilon \to 0$ uniformly in $\tilde{\boldsymbol{D}}^{-\infty}$. Hence we can obtain an asymptotic expansion of the expectation of $\phi^{(\varepsilon)}(F^{(\varepsilon)}(s,t))I_B(F^{(\varepsilon)}(s,t))$, which is the main result in this section.

**Theorem 4. 5** *Under Assumptions II and III in this section, an asymptotic expansion of* $\mathbf{E}[\phi^{(\varepsilon)}(F^{(\varepsilon)})I_B(F^{(\varepsilon)})]$ *is given by*

$$\begin{aligned}(4.59) \qquad \mathbf{E}[\phi^{(\varepsilon)}(F^{(\varepsilon)})I_B(F^{(\varepsilon)})] &\sim \mathbf{E}[\psi(\eta_c^{\varepsilon})\phi^{(\varepsilon)}(F^{(\varepsilon)})I_B(F^{(\varepsilon)})] \\ &\sim \mathbf{E}[\Phi_0] + \varepsilon\mathbf{E}[\Phi_1] + \cdots\end{aligned}$$

*as* $\varepsilon \downarrow 0$, *where* $\Phi_j$ $(j \geq 0)$ *are obtained by a formal Taylor expansion of the left-hand side in the expectation operator, and* $\psi(\eta_c^{\varepsilon})$, $\phi^{(\varepsilon)}(\cdot)$, *and* $I_B(\cdot)$ *are defined as in Theorem 3.3.*

Also it is straightforward to obtain the similar non-degeneracy conditions of the Malliavin covariance for the discounted coupon bond price process and the average interest rate process in the general case, i.e., when $m \geq 1$. We note that the discount bond price process is given by

$$P^{(\varepsilon)}(t,T) = exp\left[-\int_t^T f^{(\varepsilon)}(t,u)du\right] .$$

Then using (1.2) and the Itô's lemma, we can consider the stochastic process :

$$G^{(\varepsilon)}(t,T,p) = [P^{(\varepsilon)}(t,T)]^p$$

for any integer $p > 1$, which is the solution of the stochastic integral equation :

$$\begin{aligned} G^{(\varepsilon)}(t,T,p) &= G^{(\varepsilon)}(0,T,p) \\ &+ \int_0^t \left[pr^{(\varepsilon)}(v) + \frac{p^2-p}{2}\varepsilon^2\sum_{i=1}^m(\int_t^T \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy)^2\right]G^{(\varepsilon)}(v,T,p)dv \\ &+ \int_0^t(-p\varepsilon)\sum_{i=1}^m\int_t^T \sigma_i(f^{(\varepsilon)}(v,y),v,y)dyG^{(\varepsilon)}(v,T,p)dw_i(v) .\end{aligned}$$

Hence by using the fact that $E[|r^{(\varepsilon)}(t)|^p] < +\infty$ for any $p > 1$ under Assumption II and the standard arguments in stochastic analysis, we have the following result.

**Lemma 4. 8** *Under Assumption II, for any* $p > 1$ *and* $0 \leq s \leq t \leq T$,

$$\mathbf{E}\left[|P^{(\varepsilon)}(s,t)|^p\right] < +\infty .$$

Similarly, we can investigate the properties of the $H-$derivatives on the set of discount bond price processes as for the forward rate and spot rate processes we have discussed. Because the essential arguments are the same, we only report the result.

**Theorem 4. 6** *Under Assumption II for the forward rate processes, for any* $\varepsilon \in (0,1]$ *and* $0 \le t \le T$, $P^{(\varepsilon)}(t,T)$ *is in* $\mathbf{D}^{\infty}$ *and has an asymptotic expansion :*

$$P^{(\varepsilon)}(t,T) \sim B_1(t,T) + \varepsilon B_2(t,T) + \cdots \tag{4.60}$$

*as* $\varepsilon \downarrow 0$ *and* $B_1(t,T), B_2(t,T), \cdots \in \mathbf{D}^{\infty}$, *where* $B_j(t,T)$ $(j \ge 1)$ *are obtained by a formal Taylor expansion of* $P^{(\varepsilon)}(t,T)$ .

More generally, most interest rate based contingent claims can be regarded as functionals of bond prices with different maturities. Let $\{c_j, j = 1, \cdots, k\}$ be a sequence of non-negative coupon payments and $\{T_j, j = 1, \cdots, k\}$ be a sequence of payment periods satisfying the condition $0 \le t \le T_1 \le \cdots \le T_k \le T$. Then the price of the coupon bond with coupon payments $\{c_j, j = 1, \cdots, k\}$ at $t$ is given by

$$P^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t) = \sum_{j=1}^{k} c_j P^{(\varepsilon)}(t,T_j),$$

where $\{P^{(\varepsilon)}(t,T_j), j = 1, \cdots, k\}$ are the prices of zero-coupon bonds with different maturities. The corresponding normalized random variable $Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t)$ is defined by

$$Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t) = \frac{1}{\varepsilon}\sum_{j=1}^{k} c_j [P^{(\varepsilon)}(t,T_j) - \frac{P(0,T_j)}{P(0,t)}] .$$

By using (4.49), the first order $H$-derivative of $Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t)$ can be represented as

$$\begin{aligned} D_h[Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t)] &= (-1)\sum_{j=1}^{k} c_j P^{(\varepsilon)}(t,T_j) \int_t^{T_j} D_h[F^{(\varepsilon)}(t,u)]du \\ &= (-1)\sum_{j=1}^{k} c_j P^{(\varepsilon)}(t,T_j) \int_0^t [\int_t^{T_j} \nu^{(\varepsilon,1)}_{t,u}(v)du]\dot{h}_v dv , \end{aligned}$$

where $F^{(\varepsilon)}(t,u)$ and $\nu^{(\varepsilon,1)}_{t,u}(v)$ are defined by (4.49) and (4.51).
Then as for the previous case we immediately obtain the next result under the following condition when $m \ge 1$ by using *Theorem 4.6*. When $m = 1$ in particular, it is equivalent to Assumption III.

**Assumption III'** : For any $0 \le s \le t \le T_1 \le \cdots \le T_k$,

$$\Sigma_{g_1}(s,t,k) = \int_0^s \sum_{i=1}^{m} [\sum_{j=1}^{k} c_j \frac{P(0,T_j)}{P(0,t)} \int_t^{T_j} \sigma_i^{(0)}(v,u)du]^2 dv > 0 . \tag{4.61}$$

**Theorem 4. 7** *Under Assumptions II and III' in this section, an asymptotic expansion of* $\mathbf{E}[\phi^{(\varepsilon)}(Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t))I_{\mathcal{B}}(Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t))]$ *is given by*

$$
\begin{aligned}
& \mathbf{E}[\phi^{(\varepsilon)}(Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t))I_{\mathcal{B}}(Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t))] \\
\sim\ & \mathbf{E}[\psi(\eta^{\varepsilon}_c)\phi^{(\varepsilon)}(Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t))I_{\mathcal{B}}(Q^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(t))] \qquad (4.62) \\
\sim\ & \mathbf{E}[\Phi^*_0] + \varepsilon\mathbf{E}[\Phi^*_1] + \cdots
\end{aligned}
$$

*as* $\varepsilon \downarrow 0$, *where* $\Phi^*_j$ $(j \geq 0)$ *are obtained by a formal Taylor expansion of the left-hand side in the expectation operator, and* $\psi(\eta^{\varepsilon}_c)$, $\phi^{(\varepsilon)}(\cdot)$ *and* $I_{\mathcal{B}}(\cdot)$ *are defined as in Theorem 3.3.*

As a simple example, the payoff function of call options on the coupon bond with coupon payments $\{c_j, j = 1, \cdots, k\}$ at $\{T_j, j = 1, \cdots, k\}$ can be written as $V^{(1)}(T) = \left[P^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(T) - K\right]^+$, where $K$ is a fixed strike price. We note that the call options of swap contracts at expiry date $T$ $(0 < T \leq T_k)$ has the same payoff function. In this case we can take $\phi^{(\varepsilon)}(x) = x + y$ for a constant $y$ and $\mathcal{B} = [-y, \infty)$, then we apply the asymptotic expansion method under an additional assumption. Also as another type of interest rates based contingent claims, we should mention the payoff function of the call options on average interest rates, which is given by $V^{(3)}(T) = \left[\frac{1}{T}\int_0^T L^{\tau}(t)dt - K\right]^+$, where $K$ is a fixed strike price and the yield of a zero coupon bond at $t$ with time to maturity of $\tau$ $(0 < t < t + \tau < T_k)$ years is given by $L^{\tau}(t) = \left[\frac{1}{P^{(\varepsilon)}(t,t+\tau)} - 1\right]\frac{1}{\tau}$ . For the rigorous validity of an asymptotic expansion in this case, however, we need to modify the condition of Assumption III by another assumption, which ensures the non-degeneracy of Malliavin covariance.

The valuation problem of many interest rates based contingent claims in complete market can be simply defined as to find the "fair" value of a function of a series of bond prices at financial markets. Then the fair price of $V(T)$ at time $t$ $(0 \leq t < T)$ should be given by

$$
V_t(T) = \mathbf{E}_t\left[e^{-\int_t^T r^{(\varepsilon)}(s)ds}V(T)\right] ,
$$

where $V(T)$ be the payoff of a contingent claim at the terminal period $T$ and $\mathbf{E}_t[\,\cdot\,]$ stands for the conditional expectation operator given the information available at $t$ with respect to the equivalent martingale measure. Because we can derive an asymptotic expansion of the spot interest rate $r^{(\varepsilon)}(s)$, it is straightforward to obtain the fair value of interest based contingent claims. For instance, the normalized random variable for the call options on the discounted coupon bond at the initial period $t = 0$ is given by

$$R_{k,\{T_j\},\{c_j\}}^{(\varepsilon)}(T) = \frac{1}{\varepsilon}\{e^{-\int_0^T r^{(\varepsilon)}(s)ds}[P_{k,\{T_j\},\{c_j\}}^{(\varepsilon)}(T) - K] - [\sum_{j=1}^{k} c_j P(0,T_j) - KP(0,T)]\} ,$$

where $0 \leq T \leq T_1 \leq \cdots \leq T_k$. This random variable is considerably different from the normalized random variable $Q_{k,\{T_j\},\{c_j\}}^{(\varepsilon)}(t)$ on the coupon bond itself [8] . By using (4.49), the first order $H$-derivative of $R_{k,\{T_j\},\{c_j\}}^{(\varepsilon)}(T)$ can be represented as

$$D_h[R_{k,\{T_j\},\{c_j\}}^{(\varepsilon)}(T)] = -e^{-\int_0^T r^{(\varepsilon)}(s)ds}[P_{k,\{T_j\},\{c_j\}}^{(\varepsilon)}(T) - K]\int_0^T D_h[F^{(\varepsilon)}(s,s)]ds - e^{-\int_0^T r^{(\varepsilon)}(s)ds}[\sum_{j=1}^{k} c_j P^{(\varepsilon)}(T,T_j)\int_T^{T_j} D_h[F^{(\varepsilon)}(T,u)]du] ,$$

where $F^{(\varepsilon)}(t,u)$ is defined by (4.50).
From this expression we can obtain a simplified representation of the first order $H$-derivative in this case as before. Hence we can obtain the asymptotic expansion of the expected payoff value of discounted coupon bond as *Theorem 4.7* if we use the following condition instead of Assumption III, which ensures the non-degeneracy of the Malliavin covariance.

**Assumption IV** : For any $0 \leq T \leq T_1 \leq \cdots \leq T_k$,

(4.63)
$$\Sigma_{g_1}(k) = \int_0^T \boldsymbol{\sigma}_{g_1}^*(v)\boldsymbol{\sigma}_{g_1}^{*'}(v)dv > 0 ,$$

where

$$\boldsymbol{\sigma}_{g_1}^*(v) = -[\sum_{j=1}^{k} c_j P(0,T_j) - KP(0,T)]\boldsymbol{\sigma}_T^{(0)}(v) - \sum_{i=1}^{k} c_j P(0,T_j)\boldsymbol{\sigma}_{T,T_j}^{(0)}(v) ,$$

and $\boldsymbol{\sigma}_T^{(0)}(v)$ and $\boldsymbol{\sigma}_{T,T_j}^{(0)}(v)$ are $1 \times m$ vectors such that

$$\boldsymbol{\sigma}_T^{(0)}(v) = \left[\int_v^T \sigma_i^{(0)}(v,t)dt\right] , \quad \boldsymbol{\sigma}_{T,T_j}^{(0)}(v) = \left[\int_T^{T_j} \sigma_i^{(0)}(v,u)du\right] .$$

[8]It has an intuitive interpretation in financial applications. Its meaning and the related additional assumptions for practical applications have been discussed in Section 3 of Kunitomo and Takahashi (1995).

**Theorem 4. 8** *Under Assumptions II and IV in this section, an asymptotic expansion of* $\mathbf{E}[\phi^{(\varepsilon)}(R^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(T))I_B(R^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(T))]$ *is given by*

$$
\begin{aligned}
& \mathbf{E}[\phi^{(\varepsilon)}(R^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(T))I_B(R^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(T))] \\
\sim\ & \mathbf{E}[\psi(\eta^{\varepsilon}_c)\phi^{(\varepsilon)}(R^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(T))I_B(R^{(\varepsilon)}_{k,\{T_j\},\{c_j\}}(T))] \\
\sim\ & \mathbf{E}[\Phi^{**}_0]+\varepsilon\mathbf{E}[\Phi^{**}_1]+\cdots
\end{aligned}
\tag{4.64}
$$

*as* $\varepsilon \downarrow 0$, *where* $\Phi^{**}_j$ $(j \geq 0)$ *are obtained by a formal Taylor expansion of the left-hand side in the expectation operator, and* $\psi(\eta^{\varepsilon}_c)$, $\phi^{(\varepsilon)}(\cdot)$ *and* $I_B(\cdot)$ *are defined as in Theorem 3.3.*

The proof of this theorem is similar to those in the previous results but quite tedious and thus we omit the details. By the same token, for the average interest rates options, we can also obtain the corresponding result under another assumption for the non-degeneracy of the Malliavin covariance instead of Assumption III. For these examples we have mentioned in this section, Kunitomo and Takahashi (1995) have already derived more explicit formulae of the asymptotic expansions in details.

As the final remark of this section, we should mention that we can use the equivalence between the formulae by the Schwartz's type distribution theory for the generalized Wiener functionals and the formulae by the simple inversion technique for the characteristic functions of random variables as we have discussed in Section 3. Kunitomo and Takahashi (1995) have used the latter method because it is rather simple. Hence the results of Kunitomo and Takahashi (1995) can be justified in the proper mathematical sense by the arguments we have developed in this section.

# 5 Concluding Remarks

This note gives the mathematical validity of the asymptotic expansion approach for the valuation problem of financial contingent claims when the underlying forward rates follow a general class of continuous Itô processes in the HJM term structure of interest rates model and the underlying asset prices follow a general class of diffusion processes in the Black Scholes economy. Our method called the ***small disturbance asymptotic theory*** can be applicable to a wide range of valuation problems of financial contingent claims. It is evident from our discussions in this note that our approach can be extended and applied to more general situations than those treated by Kunitomo and Takahashi (1995), and Takahashi (1997).

For instance, the diffusion processes for asset prices with stochastic interest rates in the multi-countries framework are simple examples.

Since the asymptotic expansion approach can be justified rigorously by the Watanabe-Yoshida theory on the Malliavin calculus in stochastic analysis, it is not an ad-hoc method to give numerical approximations. In our previous papers (Kunitomo and Takahashi (1995), and Takahashi (1997)), we have illustrated that the approximations we have obtained via the asymptotic expansion method can be satisfactory in many cases for practical purposes as well.

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