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Strong Currency and Weak Currency

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Abstract

This paper presents a two-country model in which two currencies compete with each other. There exists an equilibrium in which the two currencies with different rates of inflation circulate as media of exchange despite that neither currency is forced to be used for transactions. Taxes payable in local currency and asymmetric injection of fiat money by the government through purchases of a certain good generate demands even for the currency with a higher inflation rate. In such an equilibrium, the government that issues the currency with a lower rate of inflation collects seigniorage not only from its own residents but from the residents of the other country provided that the rate of inflation is positive. The strong currency in the sense of a low inflation rate becomes an international medium of exchange.

Policy games, in which the two governments simultaneously choose and commit to tax rates and inflation rates, are also examined. We show, among other things, that the equilibrium rate of inflation is zero in this policy game. In other words, unlike a common argument, the rate of inflation does not go below zero. This result is due to the fact that a negative rate of inflation induces a negative amount of seigniorage and *vice versa.*

Some alternative currency regimes are examined. Even for a country with a weak currency, abandonment of its currency leads to a lower level of welfare. Monetary unions are briefly discussed as well.

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1 Introduction

In the modern world, economic activities often require the use of money. International commerce is not an exception. Rather, it has an additional concern as to which currency traders agree upon to use. When, for example, a U.S. trader meets with a Japanese trader, they have to make a decision as to which currency they use as a medium of exchange.[1] This task is non-trivial as it is affected by many factors. Determination of the currency they use is important not only to themselves but also to the economists who are interested in questions such as: which characteristics of a country makes its currency used in international transactions and hence become an international medium of exchange?; what would the pattern of exchange be when one currency constantly appreciates against another?; what would be benefits and costs of having its own currency as an international medium of exchange?; how would inflation in an international currency affect the world economy?; how would fiscal and monetary policies in various countries be related to each other?; what kind of macroeconomic differences are caused by different currency regimes?; which currency regime would each country prefer? The significance of these questions has attracted a number of scholars, including Hume (1752) as one of the earliest, and, more recently, Swoboda (1969), Cohen (1971), McKinnon (1979), Kindleberger (1981), and Krugman (1992). Yet, there has been little in the way of formal modelling. For example, most models do not have sufficient micro foundations to compare various currency regimes in a single framework. The purpose of the present paper is to construct a model with micro foundations which is capable of addressing some of these issues.

Unarguably, one of the major factors to determine which currency to use in trade is the relative strength of currencies. The stronger a currency is, the more often is it used in trade; the strongest currency tends to become an international medium of exchange. But in what sense is a currency stronger than another? We first clarify what we mean by the strength of currencies. Among several criteria, we focus on

[1] The present model has nothing to say about the role of an international currency as a unit of account. It is known that this role is sometimes played by a currency different from the one used as a medium of exchange. For example, during the period when Venice was dominant in Mediterranean commerce, *ducato* was used as the unit of account in Florence even though Florentines were still using *florino* as a primary medium of exchange in their territory, Toscana (Lane & Mueller).

the following two.[2] The first is the range of circulation. When the value of a coin is closely related to the weight of precious metal it contains, acceptability of currency, which determines the range of circulation, is the most important criterion under which the strength of currencies is determined. For example, in the thirteenth century when Florence was strong in the Mediterranian commerce, its currency *florino* was used as the medium of exchange in Mediterranean commerce. Once its power was taken over by Venice, so was *florino* taken over by *ducato* of Venice. Throughout this period, the exchange rate was stable (Lane & Mueller, 1985). More recently, under fixed exchange rate regimes, Brittish pound and then U.S. dollar enjoyed the status of the vehicle currency for the same reason. This phenomenon is captured by Matsuyama, Kiyotaki and Matsui (MKM henceforth; 1993), who consider issues on acceptability of currencies in the framework of random matching. There, strategic complementarity is the main source. That is, the more people use a currency, the higher its value becomes.

The second criterion is the direction and degree of change in exchange rate, which is measured by the purchasing power of currencies in the present paper: one currency becomes stronger than another if the former appreciates relative to the latter. In the last few years, U.S. dollar became stronger than Japanese yen in this respect. This criterion has become more important than ever as the world currency regime is shifted from the fixed exchange rate regime to the floating exchange rate one. The present paper focuses on this criterion. In the present model, the strength based on this criterion determines the range of circulation of currencies as well. That is, people use the currency with a lower inflation rate. The second criterion determines the pattern of exchange.

The present model is roughly described as follows. In the model are two identical countries, each of which consists of two types of *producers cum consumers*. Each country has its own government which is a sole issuer of the currency of the country. Each government issues its own fiat money by way of purchasing goods from certain types of producers. The good obtained by the government is used to provide public goods. Each agent may be imposed taxes by the government of the country in which

[2] Another important criterion is uncertainty. The less uncertain the rate of return of a currency is, the more likely people are to use it. The present paper precludes this aspect, focusing on deterministic outcomes.

he resides. Taxes, if imposed, have to be paid in its own currency.[3] Other than their relation to their respective governments through tax payment, agents in both countries are completely identical. Every exchange is made in a bilateral market. For simplicity, we assume there is no barter, though it is not difficult to introduce it without changing the basic structure of the model. No credit is allowed. That is, every exchange is made through money. As a result, a cash-in-advance constraint appears, but agents can choose which money to use. In this economy, there exists an equilibrium in which the exchange rate of the two active currencies changes over time. We show that in such an equilibrium, one currency is stronger than the other in the two respects mentioned above. First, the former circulates more widely than the latter as in MKM. Second, the former appreciates over time with respect to the latter. Unlike MKM, where agglomeration effect drives the main result, the range of circulation is determined by the relative rate of inflation. The lower the rate of inflation of a currency is, the more is it used in transactions.

Using this framework, the effects of different inflation and tax rates are examined. We also examine two policy games. The first is the one in which the two governments choose their respective inflation and tax rates at the beginning of the economy. The second game is the one in which the governments can choose only tax rates. In both games, the objective of each government is to maximize the welfare of its own country. In the first game, we show, among other things, that the rate of inflation goes down to zero. This result should be contrasted with the Friedman rule according to which the optimal policy is to set the rate of nominal interest, not the rate of inflation, at zero.

This difference is due to the fact that the two governments are fighting over seigniorage. One country has an incentive to set the rate of inflation just below that of the other country if the rate of inflation of the other country is positive. This is because this country can obtain seigniorage from the agents of the other country. On the other hand, if the opponent's rate of inflation is nonpositive, then setting its own rate below the opponent's will induce a negative amount of seigniorage, paying subsidy to the agents of the other country, and therefore, it has no incentive to cut the rate of inflation below the opponent's; rather, it tries to increase the rate to avoid this burden.

[3] This assumption is modified later when we consider different monetary regimes.

We also discuss three alternative monetary regimes: the first is to abandon the weak currency by allowing agents to pay taxes in the currency they prefer; the second is an autarchic policy; and the third is a common currency regime. The first alternative regime is dominated by the original regime for the country with the weak currency since its government loses seigniorage. Under the autarchic policy, the welfare improves if the government can control the rate of inflation below a certain threshold. This is because the agents do not pay the inflation tax to the other country. If the government cannot control inflation and it remains above the threshold, however, the country is worse off due to very high inflation taxes, which crowd out consumption. Issues on monetary unions are discussed when we consider a common currency regime. It is shown that some of the arguments that have been made in the literature do not hold.

Our model is the same as a cash-in-advance economy (see, e.g., Grandmont and Younes (1972) and Lucas (1980)) except that we do not specify *which* fiat money is needed for transactions. In this respect, the present paper is different from other models of multiple currencies such as Lucas (1982) and King, Wallace and Weber (1991), who assume that certain people *have to* use their own currency for transactions besides their motivation is to show an endogenous fluctuation of exchange rate. The setup of the present model is also related to trading post games *à la* Shapley and Shubik (1969). Their main difference from market economies is that a certain trading post can be endogenously inactive even if there is double coincidence of wants. Using this property, Alonso (1994) constructs a sunspot equilibrium in which there are some states with multiple active currencies. Key to the result of that paper is that no exchange is made through a currency when its potential value becomes the highest. This phenomenon is precluded in our approach since we assume an auctioneer in each market for bilateral transaction, and therefore, transaction always takes place whenever there is double coincidence of wants. The spirit of the present paper is similar to Engineer and Bernhardt (1992) and Hayashi and Matsui (1994) in the sense that no single good is required to be used for transactions. In these papers, barter competes with fiat money, while in the present paper two fiat monies compete with each other. In this respect, search-theoretic models of money (see Iwai (1988), Kiyotaki & Wright (1989), Oh (1989) for models with a single currency, and MKM and Zhou (1993) for models with multiple currencies[4]) are related to the present paper, too. However, since in the present model is an auctioneer in each bilateral market, and agents are free to go to as many such markets as they want, we can consider a representative agent for each type, and there is no problem of "tracking inventories," which makes a model with divisible money tractable.

The rest of the paper is organized as follows. Section 2 presents a single country model as a benchmark. Section 3 introduces the second country and finds stationary symmetric equilibria. Section 4 examines the effects of inflation and various tax rates and policy games. This section considers two classes of policy games. In the first class of games, it is assumed that the governments choose and commit to both tax rates and inflation rates. While in the second, only tax rates are choice variables. Section 5 considers three different monetary regimes and examines their welfare implications. Section 6 concludes the paper.

2 One Country Model

As a benchmark, this section presents a single country model, which exhibits a cash-in-advance constraint. Time is discrete and the horizon is infinite. There are two types of agents, 1, 2, with equal size, and the government. We consider a representative agent for each type. They are both price takers. There are two types of commodities. In addition to these commodities, there exists good 0 called fiat money, which is exclusively supplied by the government. At the beginning of each period, the kth agent ($k = 1, 2$) decides how many units of good k to produce. All goods are perishable at the end of the period in which they are produced except fiat money, which is durable forever. Agent k derives utility only from good $k + 1\,(modulo\,2)$. In each period, there simultaneously open pairwise markets $(0, 1)$ and (0,2). In market $(0, k)$ $(k = 1, 2)$, goods 0 and k are transacted. There is no barter market (1,2).[5] An agent can go to both markets and post as much amount as he wants, but no short sales are allowed. Markets clear simultaneously,

[4] Trejos & Wright (1994) provides another model of multiple currencies where commodities are divisible so that exchange rates can be discussed in terms of purchasing power of two currencies.

[5] This is equivalent to the situation where barter is allowed, but there are, say, four agents in the economy, and agent k consumes only good $k + 1(modulo\ 4)$, which generates a lack of double coincidence of wants.

and the agent cannot sell goods that are acquired in that period.

The two agents are asymmetric in their relation to the government. In period t, the government purchases $G(t)$ (> 0) units of good 1, which may be determined endogenously. When purchasing the goods, the government has to pay the same price as other agents. In the sequel, we use modulo notation, i.e., $k+1 = 1$ if $k = 2$. Let $y_k(t)$ be the output of good k and $x_{k+1}(t)$ be the amount of good $k+1$ in the tth period ($t = 1, 2, \cdots$), respectively. The utility of the kth agent is given by

$$U(\{x_{k+1}(t), y_k(t); G(t)\}_{t=1}^{\infty}) = \sum_{t=1}^{\infty} \beta^t \left[\ln x_{k+1}(t) - y_k(t) + \frac{\alpha}{2} \ln G(t)\right], \quad (\alpha > 0)$$

where β is a common discount factor strictly between zero and one.

The government imposes a proportional income tax with rate τ on every agent payable in each period after transaction. Each agent faces a series of liquidity constraints, one for each period. Let $p_k(t)$ be the price of good k at time t. Given a sequence of prices $\{(p_k(t))_{k=1,2}\}_{t=1}^{\infty}$ and a tax rate τ, the problem that the kth agent is faced with is:

$$\begin{aligned}
&\max_{x_{k+1}(t), y_k(t), m_k(t)} \textstyle\sum_{t=1}^{\infty} \beta^t \left[\ln x_{k+1}(t) - y_k(t) + \frac{\alpha}{2} \ln G(t)\right] \\
s.t. \quad & p_{k+1}(t) x_{k+1}(t) + m_k(t+1) \leq (1-\tau) p_k(t-1) y_k(t-1) + m_k(t) \quad (\lambda_k(t)) \\
& x_{k+1}(t),\ y_k(t),\ m_k(t) \geq 0, \quad (t = 1, 2, \cdots)
\end{aligned} \tag{1}$$

where $\lambda_k(t)$ is the Lagrange multiplier associated with the constraint at time t, and $m_k(t)$ is the amount of the currency which is acquired in the past periods but not used in the tth period. The Kuhn-Tucker condition for this problem is:

$$\begin{aligned}
&\beta^t \tfrac{1}{x_{k+1}(t)} - \lambda_k(t) p_{k+1}(t) \leq 0, && \text{“=” holds if } x_{k+1}(t) > 0, \\
&(1-\tau)\lambda_k(t+1) p_k(t) - \beta^t \leq 0, && \text{“=” holds if } y_k(t) > 0, \text{ and} \\
&-\lambda_k(t-1) + \lambda_k(t) \leq 0, && \text{“=” holds if } m_k(t) > 0.
\end{aligned}$$

Through the rest of the paper, we focus on stationary equilibrium in the sense that every real variable is constant across time, every nominal variable has a constant rate of change, denoted by π. Using stationarity, we write, for example, $x_1(t) = x_1$

and $p_2(t) = (1+\pi)^t p_2$. The above conditions for $k = 1, 2$ and the following market clearing conditions will determine an equilibrium of this economy:

$$x_1 + G = y_1, \quad \text{market (0,1)} \tag{2}$$

and

$$x_2 = y_2, \quad \text{market (0,2)}. \tag{3}$$

Given $\tau \geq 0$ and $\pi \geq 0$, a *stationary equilibrium* is a tuple $\{(x_k, y_k, m_k; p_k)_{k=1}^2; G\}_{t=1}^{\infty}$ such that given (p_1, p_2) and π, (x_{k+1}, y_k, m_k) satisfies the Kuhn-Tucker condition with the constraint in (1) for $k = 1, 2$, and (2) and (3) hold. There always exists an equilibrium with no economic activity, i.e., $x_k = y_k = 0$ $(k = 1, 2)$. We do not consider this as a flaw of the model; rather, we believe any sound model of fiat money ought to have this type of equilibrium under appropriate assumptions.[6] We focus on active equilibria in which some agents produce goods. If an agent produces goods, then $x_k > 0$ and $y_k > 0$ hold for $k = 1, 2$ since no agent produces goods unless he can consume the goods provided by the other agent.

It is verified that the Kuhn-Tucker condition together with stationarity implies that agents do not hold money more than one period, i.e., $m_k = 0$ unless $1 + \pi = \beta$. Through the rest of the analysis, we assume $1 + \pi > \beta$.

Since we have $x_k,\ y_k > 0$, the first two inequalities of the Kuhn-Tucker condition hold with equalities. Thus, we have

$$x_{k+1} = \frac{1-\tau}{1+\pi} \frac{p_k}{p_{k+1}} \beta, \; k = 1, 2. \tag{4}$$

Using (2), (3), and (4), we obtain

$$x_1 = \frac{(1-\tau)^2}{(1+\pi)^2} \beta = \beta - G,$$

$$x_2 = y_1 = y_2 = \beta,$$

and

$$\frac{p_1}{p_2} = \frac{1+\pi}{1-\tau}.$$

[6] This equilibrium is eliminated if there is a lump sum tax.

From these equations, the real government spending is written as

$$G = \beta \left\{ 1 - \frac{(1-\tau)^2}{(1+\pi)^2} \right\}. \tag{5}$$

The real tax revenues measured in good 1 in the equilibrium are given by

$$T = \tau Y \tag{6}$$

where

$$Y \equiv \frac{p_1 y_1 + p_2 y_2}{p_1(1+\pi)} = \beta \frac{(1+\pi)+(1-\tau)}{(1+\pi)^2}$$

is the value of total output measured in terms of good 1 divided by $1+\pi$. Subtracting (6) from (5), we obtain the (real) inflation tax of πY. Thus, the government spending is decomposed into two parts, depending on the sources of revenues, i.e.,

$$G = \tau Y + \pi Y.$$

The trading pattern of this equilibrium is shown in Figure 1.

Since Pareto efficiency requires the marginal rates of substitution to be consistent across agents, i.e., $u'(x_2)u'(x_1) = 1/[\beta(\beta - G)] = 1$, $G \geq 0$ implies that any equilibrium is inefficient. Note that inflation itself does not affect the economy at all if it is offset by a change in tax rate. That is, proportional income tax and inflation tax are perfect substitutes.

If the objective of the government is to maximize the sum of utilities of agents, $W = u_1 + u_2$, then the optimal policy is a solution to:

$$\max \ u_1 + u_2 = \ln[\beta(\beta - G)] - 2\beta + \alpha \ln G. \tag{7}$$

Solving this, we get

$$G_{opt} = \frac{\alpha}{1+\alpha}\beta,$$

or

$$\frac{1-\tau_{opt}}{1+\pi_{opt}} = \frac{1}{\sqrt{1+\alpha}},$$

where the subscript "*opt*" stands for "optimal".

3 Two Country Model

We now introduce another country called the Foreign country or simply F country. It is physically identical with the first country, which we call H country. Good k in H country and good k in F country ($k = 1, 2$) are identical. We put asterisks to denote the variables of F country which correspond to those of H country. H country government issues H currency or good 0, while F country government issues F currency or good 0^*. They are both fiat monies and clearly distinguishable. Each government is a sole issuer of her own currency. As in the one country model, G denotes the government spending of H country measured in units of good 1. Similarly, G^* units of good 1 is the government spending of F country. They are transformed into public goods, which benefit only their respective residents. There are now four potentially active markets, $(0, k)$ and $(0^*, k)$ ($k = 1, 2$). Every agent of either country can access to every such market without costs. Each government may impose taxes only on its own residents, which are payable only in its currency. Taxes on production goods sold for the other currency are levied using the price in its own currency. In this economy, each agent is faced with two sequences of liquidity constraints, one is for his country's currency, and the other for the currency of the other country. Any decision variables for the second liquidity constraint is expressed with "hat" such as $\hat{x}_k(t)$. Given sequences of prices and tax rates in both countries, $\{((p_k(t))_{k=1,2}), ((p_k^*(t))_{k=1,2})\}_{t=1}^{\infty}$ and (τ, τ^*), the kth agent of H country ($k = 1, 2$) is faced with the following problem:

$$\max \sum_{t=1}^{\infty} \beta^t \left[\ln(x_{k+1}(t) + \hat{x}_{k+1}(t)) - (y_k(t) + \hat{y}_k(t)) + \tfrac{\alpha}{2} \ln G\right]$$

$$s.t. \quad \begin{aligned} p_{k+1}(t)x_{k+1}(t) + \tau p_k(t-1)\hat{y}_k(t) + m_k(t+1) \\ \leq (1-\tau)p_k(t-1)y_k(t-1) + m_k(t) \end{aligned} \qquad (\lambda_k(t))$$

$$p_{k+1}^*(t)\hat{x}_{k+1}(t) + \hat{m}_k(t+1) \leq p_k^*(t-1)\hat{y}_k(t-1) + \hat{m}_k(t) \qquad (\hat{\lambda}_k(t))$$

$$x_{k+1}(t),\ \hat{x}_{k+1}(t),\ y_k(t),\ \hat{y}_k(t),\ m_k(t),\ \hat{m}_k(t) \geq 0, \ (t = 1, 2, \cdots). \tag{8}$$

where $\lambda_k(t)$'s and $\hat{\lambda}_k(t)$'s are the associated Lagrange multipliers. The first constraint is for H currency, while the second one is for F currency. Together with the constraints in (8), the Kuhn-Tucker condition for this problem is:

$$\beta^t \frac{1}{x_{k+1}(t)+\hat{x}_{k+1}(t)} - \lambda_k(t) p_{k+1}(t) \le 0, \quad \text{"=" holds if } x_{k+1}(t) > 0,$$

$$\beta^t \frac{1}{x_{k+1}(t)+\hat{x}_{k+1}(t)} - \hat{\lambda}_k(t) p^*_{k+1}(t) \le 0, \quad \text{"=" holds if } \hat{x}_{k+1}(t) > 0,$$

$$(1-\tau)\lambda_k(t+1)p_k(t) - \beta^t \le 0, \quad \text{"=" holds if } y_k(t) > 0,$$

$$-\tau\lambda_k(t+1)p_k(t) + \hat{\lambda}_k(t+1)p^*_k(t) - \beta^t \le 0, \quad \text{"=" holds if } \hat{y}_k(t) > 0,$$

$$-\lambda_k(t-1) + \lambda_k(t) \le 0, \quad \text{"=" holds if } m_k(t) > 0,$$

$$-\hat{\lambda}_k(t-1) + \hat{\lambda}_k(t) \le 0, \quad \text{"=" holds if } \hat{m}_k(t) > 0.$$

The problem for agent k of F country is obtained by attaching asterisks to and removing them from relevant variables. We restrict our attention to stationary equilibria where we do have the same requirement as before for the two countries. The real variables are constant across time, the nominal variables for country H changes at rate π, and the nominal variables for country F changes at rate π^*. We do not require symmetry between the countries, i.e., $\pi \neq \pi^*$ is allowed to happen. We omit t from the variables as in the previous section. Then the above conditions for $k = 1, 2$ and the following market clearing conditions will determine an equilibrium of this economy:

$$x_1 + \hat{x}^*_1 + G = y_1 + \hat{y}^*_1, \quad \text{market } (0, 1) \tag{9}$$

$$\hat{x}_1 + x^*_1 + G^* = \hat{y}_1 + y^*_1, \quad \text{market } (0^*, 1) \tag{10}$$

$$x_2 + \hat{x}^*_2 = y_2 + \hat{y}^*_2, \quad \text{market } (0, 2) \tag{11}$$

and

$$\hat{x}_2 + x^*_2 = \hat{y}_2 + y^*_2, \quad \text{market } (0^*, 2). \tag{12}$$

If the rates of inflation of the two countries, π and π^* are the same, then there exists a stationary equilibrium which is a replication of the equilibrium discussed in the previous section.[7]

[7] There are other equilibria some of which will be obtained by setting $\pi = \pi^*$ in the following analysis. A degenerate equilibrium where agents do not trade at all also exists.

We now turn to the analysis of equilibria with different rates of inflation between the two currencies. We look for equilibrium in which both currencies are active. In this section, we assume that $\pi < \pi^*$ holds until we consider policy games. If $\tau^* = 0$, then $\pi < \pi^*$ implies that it is better to use H currency since the cost of holding H currency is less than that of holding F currency, and therefore there will be no demand for F currency, i.e., no equilibrium with two active currencies exists. If taxes are payable in their respective currencies, this is not the case since each agent in F country has to get some F currency to pay the taxes.

First of all, like in the single country model, $m_k = m_k^* = 0$ for all $k = 1, 2$. We also know $x_k + \hat{x}_k > 0$ and $x_k^* + \hat{x}_k^* > 0$. Then, from the first constraint in (8) and $\tau > 0$, we have $y_k > 0$ $(k = 1, 2)$. Similarly, for F country, we have $y_k^* > 0$ $(k = 1, 2)$. Suppose now that $x_k^* > 0$. Then from the Kuhn-Tucker condition for F country, we obtain

$$\frac{1}{1+\pi^*}\frac{p_k^*}{p_{k+1}^*} \geq \frac{1}{1+\pi}\frac{p_k}{p_{k+1}}, k = 1, 2. \tag{13}$$

If (13) holds for $k = 1$, then it does not hold for $k = 2$, and vice versa since we have $\pi^* > \pi$. Therefore, we have either $x_1^* = 0$ or $x_2^* = 0$. In fact, similar reasoning assures that either $x_1^* = \hat{x}_1 = 0$ or $x_2^* = \hat{x}_2 = 0$ holds. But $y_2^* > 0$ and (12) imply that either $x_2^* > 0$ or $\hat{x}_2 > 0$ holds. Therefore, we have $x_1^* = \hat{x}_1 = 0$. Type 2 agents irrespective of nationality do not purchase good 1 with F currency for consumption. In particular, agent 2 of F country necessarily go to H currency market for good 1. This implies $\hat{y}_2^* > 0$ since they constantly need H currency (see the second constraint in (8) with asterisks). We also know $x_1 > 0$ and $\hat{y}_2 > 0$. This implies that type 2 agents of H country never use F currency.

Next, note that (11) implies either $x_2 > 0$ or $\hat{x}_2^* > 0$. Then either the Kuhn-Tucker conditions for $x_2 > 0$ and $y_1 > 0$ or those for $\hat{x}_2^* > 0$ and $\hat{y}_1^* > 0$ imply that (19) holds with the reverse inequality for $k = 1$. Thus, we have

$$\frac{1}{1+\pi^*}\frac{p_1^*}{p_2^*} = \frac{1}{1+\pi}\frac{p_1}{p_2}, k = 1, 2. \tag{14}$$

Hence, the relative price of goods demanded by the government is higher in F country than that in H country. Equation (14) holds since agent 1 of F country ought to be indifferent between the two currencies in order to supply their domestic currency to agent 2 who need it to pay taxes. Thus, agent 1 in both countries are indifferent between two pairs of markets, (0,1) and $(0^*, 1)$, and (0,2) and $(0^*, 2)$. In solving the problem, we momentarily assume that either $\hat{x}_1 = \hat{y}_1 = 0$ or $\hat{x}_1^* = \hat{y}_1^* = 0$ holds. Any other equilibrium will be obtained by changing the values of these variables, keeping $\hat{x}_1 - \hat{x}_1^*$ and $\hat{y}_1 - \hat{y}_1^*$ constant. After tedious calculation, which is given in Appendix, we obtain first $y_1 + \hat{y}_1 = y_1 = y_2 + \hat{y}_2 = y_2 = y_1^* + \hat{y}_1^* = y_2^* + \hat{y}_2^* = \beta$, and $y_2^* = \tau^*\beta$, that is, the amount of output is always the same, and then

$$x_1 + \hat{x}_1 = x_1 = \frac{(1-\tau)(1-\bar{\tau})}{(1+\pi)^2}\beta, \tag{15}$$

$$x_2 + \hat{x}_2 = x_2 = \frac{1-\tau}{1-\bar{\tau}}\beta, \tag{16}$$

$$x_1^* + \hat{x}_1^* = \hat{x}_1^* = \frac{(1-\tau^*)(1-\bar{\tau})}{(1+\pi)^2}\beta, \tag{17}$$

$$x_2^* + \hat{x}_2^* = \frac{1-\tau^*}{1-\bar{\tau}}\beta, \tag{18}$$

and

$$\frac{p_1}{p_2} = \frac{1+\pi}{1-\bar{\tau}}, \quad \frac{p_1^*}{p_2^*} = \frac{1+\pi^*}{1-\bar{\tau}}, \tag{19}$$

where

$$\bar{\tau} \equiv \frac{\tau + \tau^*}{2}$$

is the average tax rate of the two countries. For the governments' spendings, we have

$$G = 2\beta\left[1 - \tau^* - \frac{(1-\bar{\tau})^2}{(1+\pi)^2} + \frac{1}{2}\tau^*\bar{\tau}\right] = \tau Y + \pi Y + \pi \hat{Y}^*, \tag{20}$$

and

$$G^* = \beta\tau^*(2-\bar{\tau}) = \tau^*(Y^* + \hat{Y}^{**}) + \pi^* Y^*, \tag{21}$$

where

$$Y \equiv \frac{p_1 y_1 + p_2 y_2}{p_1(1+\pi)},$$

$$Y^* \equiv \frac{p_1^* y_1^* + p_2^* y_2^*}{p_1^*(1+\pi^*)},$$

$$\hat{Y}^* \equiv \frac{p_1 \hat{y}_1^* + p_2 \hat{y}_2^*}{p_1(1+\pi)},$$

and

$$\hat{Y}^{**} \equiv \frac{p_1^* \hat{y}_1^* + p_2^* \hat{y}_2^*}{p_1^*(1+\pi^*)}.$$

The trading pattern of this equilibrium is shown in Figure 2. In this equilibrium, H agents have no need to use F currency, while F agents necessarily use both currencies.[8] In particular, when two agents in different countries trade, they use H currency. It circulates as an international medium of exchange. Note that demand for F currency by agent 1 in F country exceeds the amount of taxes he has to pay. Taxes imposed upon agent 2 create a positive value for F currency for those who do not need it. As a result, F currency is used as a medium of exchange in F country. Like in the single country model, the government spending can be decomposed into a few terms depending on the sources of revenues. The expenditures of H government are financed by three sources, tax revenues from H agents, the seigniorage collected from H agents' activities, and the seigniorage collected from F agents' activities made through H currency. In other words, F agents pay inflation tax, $\pi\hat{Y}^*$, to H country for liquidity service where $\hat{Y}^*$ is the real economic activities of F agents made through H currency.

4 Comparative Statics and Policy Games

4.1 Comparative Statics

This section analyzes the effects of inflation, including its effect on the welfare of the countries. Using stationarity, we let the welfare of country H, denoted by W, be the sum of the single period utilities of the two agents in the country, i.e.,

$$W = \ln(x_1 + \hat{x}_1) + \ln(x_2 + \hat{x}_2) - (y_1 + \hat{y}_1) - (y_2 + \hat{y}_2) + \alpha \ln G.$$

Substituting the solution of the previous section into the above expression, we obtain

$$W = \ln\left(\frac{1-\tau}{1+\pi}\beta\right)^2 - 2\beta + \alpha \ln\left[2\beta(1-\tau^*) - 2\frac{(1-\bar{\tau})^2}{(1+\pi)^2}\beta + \beta\tau^*\bar{\tau}\right]. \tag{22}$$

Likewise, the welfare of country F is given by

$$W^* = \ln\left(\frac{1-\tau^*}{1+\pi}\beta\right)^2 - 2\beta + \alpha \ln\left[\tau^*\beta(2-\bar{\tau})\right]. \tag{23}$$

[8] Since Agent 1 in each country is indifferent between the two currencies for transaction purpose, Agent 1 in H country may use F currency in other equilibria, which has no effect on real outcomes except that different people hold the currency.

Note that the rate of inflation of F country affects neither country's welfare as long as $\pi^* > \pi$ holds since π^* appears neither in (22) nor in (23). The larger π^* becomes, the more inflation taxes are collected, but the less the actual tax revenues are. Moreover, they offset each other. If $\pi = \pi^*$, the same equilibrium still exists although now there exist many other equilibria, including the one in which F country collects seigniorage from H agents' activities. Therefore, a fixed exchange rate regime may or may not improve F country's welfare.

On the other hand, π, the inflation rate of the strong currency, affects the real variables. For example, if H government increases π, adjusting τ at the same time to keep $(1-\tau)/(1+\pi)$ constant, we obtain

$$\left.\frac{dW}{d\pi}\right|_{\frac{1-\tau}{1+\pi}=const.} = \frac{\alpha}{G}\left[\frac{4(1-\bar{\tau})^2-(1-\bar{\tau})(1-\tau)}{(1+\pi)^3}\beta + \beta\tau^*\frac{1-\tau}{2(1+\pi)}\right] > 0. \tag{24}$$

Therefore, the inflation of H country is not neutral any longer even if the consumption is compensated by a lower tax rate. Note that in a single country model, a higher inflation rate compensated by a lower tax rate has no effect on the real variables. Here, an increase in π in the above manner increases the welfare of H country since it increases the inflation tax on F agents. On the other hand, the effect on the welfare of F country is ambiguous. An increase in π decreases the utility from private goods consumption in F country, while it increases their government expenditures. An increase in the rate of inflation may increase the welfare of both countries simply because it improves allocation between private goods and public goods when the tax rate is zero, i.e., at its lowest level. In order to examine the true effects of inflation, each government adjusts its tax rate to the optimal level for each rate of inflation. It turns out that we have to consider a policy game since the optimal rates are interrelated between the two countries.

Before we analyze a situation in which each government cares only about its country's welfare, we examine a benchmark case where the two governments cooperate with each other to maximize the sum of the welfare, $W + W^*$, by transferring public goods from one country to the other if necessary. It is assumed that no transaction cost is incurred. Then the optimal transfer amount is the one that equates G with G^*. Therefore, the problem we face is:

$$\max_{\tau,\tau^*} W + W^* = \ln\left(\frac{1-\tau}{1+\pi}\beta\right)^2 + \ln\left(\frac{1-\tau^*}{1+\pi}\beta\right)^2 - 4\beta + 2\alpha\ln\beta\left[1-\left(\frac{1-\bar{\tau}}{1+\pi}\right)^2\right].$$

Differentiating it with respect to τ and τ^* and equating the expressions with zero, we get

$$(1+\pi)^2 = \{(1-\bar{\tau}) + \alpha(1-\tau)\}(1-\bar{\tau}),$$

and

$$(1+\pi)^2 = \{(1-\bar{\tau}) + \alpha(1-\tau^*)\}(1-\bar{\tau}).$$

Solving these equations, we obtain

$$\frac{1-\tau_{opt}}{1+\pi_{opt}} = \frac{1-\tau^*_{opt}}{1+\pi_{opt}} = \frac{1}{\sqrt{1+\alpha}}.$$

Like in the single country model, the tax rates and the rate of inflation are perfect substitutes for interior solutions.

4.2 Policy Games

This subsection considers policy games. By a *policy game* we mean a game in which the two governments commit, before the economy starts, to tax rates and, if possible, rates of inflation. At the same time, there is a once-and-for-all adjustment of the initial distribution of monies to assure the existence of a stationary equilibrium with two active currencies. Given τ, τ^*, π, and π^*, an outcome of this game is assumed to be a stationary equilibrium given the commitment of the governments. If $\pi \neq \pi^*$, there is essentially a unique stationary equilibrium. If $\pi = \pi^*$, then we choose the duplication of the equilibrium of the single country model. Each government tries to maximize the welfare of its own country. We consider two situations. The first is the case in which each government can control the rate of inflation of its own currency. The second is the case in which it cannot.

4.2.1 Controllable Inflation

Here we consider the policy game in which the governments choose their respective rates of inflation as well as tax rates. First of all, we show that $\pi = \pi^*$ and $\tau = \tau^*$ hold in any equilibrium. Suppose the contrary, *e.g.,* that $\pi < \pi^*$ holds. Then from (24), W increases as π increases, keeping $\pi < \pi^*$. Therefore, no such equilibrium exists. The case of $\pi > \pi^*$ is similarly taken care of. As we assumed above, if $\pi = \pi^*$, we have two separate economies, *i.e.,* H currency circulates only in H country, and F currency circulates only in F country. Given π and π^*, the outcome coincides with a duplication of the single currency model. In this case, the optimal tax rate is uniquely chosen, which are the same for the both countries. This proves the above claim, *i.e.*, $\pi = \pi^*$ and $\tau = \tau^*$.

Next, we check whether or not either government has an incentive to change its rate of inflation from a certain level $\pi = \pi^*$. Using symmetry, we check only for H government. Note that from (7), at $\pi = \pi^*$, the welfare level of H country is given by

$$W_0 = (2+\alpha)\ln\beta + 2\ln\left[\frac{1-\tau}{1+\pi}\right] - 2\beta + \alpha\ln\left[1 - \frac{1-\tau}{1+\pi}\right]. \tag{25}$$

Evaluating (22) and (25) at $\tau = \tau^*$, we obtain that (25) is larger than (22) if and only if

$$\ln\left[2(1-\tau) - 2\frac{(1-\tau)^2}{(1+\pi)^2} + \tau^2\right] > \ln\left[1 - \frac{(1-\tau)^2}{(1+\pi)^2}\right]. \tag{26}$$

Summarizing this inequality, H country has an incentive to lower π a little if and only if $\pi > 0$. Similarly, evaluating (23) and (25) at $\tau = \tau^*$, H country has an incentive to increase π a little if and only if $\pi < 0$. Thus, the equilibrium condition requires $\pi = \pi^* = 0$. In this case, it is verified that no country has an incentive to change its rate of inflation, nor the tax rate once it is set at

$$\tau_{opt} = 1 - \frac{1}{\sqrt{1+\alpha}}.$$

The unique equilibrium of the policy game prescribes zero rate of inflation and the optimal tax rate calculated for the single country model.

The logic behind this result is straightforward. It is always better for a government to set its rate of inflation a little below that of the other country as long as the rate of inflation of the opponent is positive. On the other hand, if the opponent's rate is negative, the government has an incentive to set the rate of inflation above the opponent's rate. This is because the circulation of its own currency at a negative rate as an international medium of exchange generates negative seigniorage. "Beggar-thy-neighbor disinflation" occurs but only up to zero.

This result should be contrasted with a common argument that the Friedman rule of setting the nominal interest rate being zero is optimal and hence chosen by the government if possible. Here, because of the negative seigniorage, or a subsidy given to the other country's agents, neither government would be willing to bear an additional burden to drive the rate of inflation down to a negative value.

4.2.2 Uncontrollable Inflation

The second policy game is the one in which the governments cannot control the rate of inflation. In this case, a different equilibrium emerges. Technically, we analyze the situation under exogenously fixed rates of inflation. We analyze situations with $\pi < \pi^*$. The case $\pi > \pi^*$ is its mirror image, and the case $\pi = \pi^*$ is the duplication of the single country model. In an equilibrium, we have

$$\frac{\partial W}{\partial \tau} = -\frac{2}{1-\tau} + \frac{\alpha\beta}{G}\left[2\frac{1-\bar{\tau}}{(1+\pi)^2} + \frac{1}{2}\tau^*\right] \leq 0, \quad \text{"="} \text{ if } \tau > 0, \tag{27}$$

and

$$\frac{\partial W^*}{\partial \tau^*} = -\frac{2}{1-\tau^*} + \frac{\alpha\beta}{G^*}\left[2 - \bar{\tau} - \frac{1}{2}\tau^*\right] = 0. \tag{28}$$

Equation (28) holds since τ^* always holds in order to finance the government spending. After tedious calculation, partially given in Appendix 2, it is proven that there is a unique solution to the policy game if $\alpha \leq 4$, which we confine our attention to. Evaluating $\partial W/\partial \tau$ and $\partial W^*/\partial \tau^*$ at $\tau = \tau^*$, we obtain

$$\frac{\partial W}{\partial \tau} = -\frac{2}{1-\tau} + \frac{\alpha\beta}{G}\left[2\frac{1-\tau}{(1+\pi)^2} + \frac{1}{2}\tau\right],$$

and

$$\frac{\partial W^*}{\partial \tau^*} = -\frac{2}{1-\tau} + \frac{\alpha\beta}{G^*}\left[2 - \frac{3}{2}\tau\right],$$

respectively. Since we know $G \geq G^*$ where equality holds if and only if $\pi = 0$, we have

$$\frac{\partial W^*}{\partial \tau^*} \geq \frac{\partial W}{\partial \tau}$$

where equality holds if and only if $\pi = 0$. This implies that the optimal tax rate is greater for F country than H country. This is because H country can raise extra revenues in the form of inflation taxes if the rate of inflation in H currency is positive. Tables 1 and 2 give some numerical examples in equilibrium of the policy game for each exogenous rate of inflation. Note that as π increases with $\pi < \pi^*$, the welfare of F country monotonically decreases, while the welfare of H country first increases and then decreases. F country's welfare decreases and H country's welfare increases mainly due to inflation tax paid to H country by F country's agents. On the other hand, H country's welfare decreases after a certain threshold since too high a rate of inflation crowds out private sector's consumption. Therefore, if π^* is sufficiently high, there is an optimal rate of inflation π for H country that is strictly below π^*.

5 Alternative Regimes

This section considers three alternative regimes. One is abandonment of the weak currency. The other is an autarchic policy. Monetary unions are also briefly discussed.

5.1 Single Currency Regime

F government, whose currency is assumed to be weak, may give up its own currency and allow its people to pay taxes in the currency they prefer. If the government takes this course, it cannot circulate F currency any longer unless the rate of inflation is at least as low as the other currency. A disadvantage of the policy, in the present framework, is that the government cannot collect seigniorage. Each agent now faces a single budget constraint in each period. In period t, the budget constraint of agent k in H country ($k = 1, 2$) is given by

$$p_{k+1}(t)x_{k+1}(t) + m_k(t+1) \leq (1-\tau)p_k(t-1)y_k(t-1) + m_k(t),$$

while that of agent k in F country is given by

$$p_{k+1}(t)\hat{x}^*_{k+1}(t) + \hat{m}^*_k(t+1) \leq (1-\tau^*)p_k(t-1)\hat{y}^*_k(t-1) + \hat{m}^*_k(t).$$

In addition to these changes, F government now faces a budget constraint since, by assumption, it cannot issue H currency. Under stationarity, it is given by

$$p_1 G^* \leq \frac{\tau^*}{1+\pi}\left(p_1\hat{y}^*_1 + p_2\hat{y}^*_2\right).$$

Solving this problem, we obtain (15) through (19) together with

$$G = \beta\left[2 - \frac{\tau^*}{1+\pi} - \frac{(1-\tau)(1-\bar{\tau})}{(1+\pi)^2}\right], \tag{29}$$

and

$$G^* = \beta\tau^* \left[\frac{1}{1+\pi} + \frac{1-\bar{\tau}}{(1+\pi)^2}\right]. \tag{30}$$

Comparing (20) with (29), and (21) with (30), it is verified that H country is better off, while F country is worse off by this policy unless $\pi = 0$. Therefore, F government, if its objective is to maximize either its welfare or its seigniorage, has a strict incentive to have its own currency circulated by imposing taxes in this currency.

5.2 Autarchy

Another policy possibly available to F government is to close its door to H country. Suppose that F government can successfully prohibit F agents from using H currency. Then we have a single country situation in both countries. If F government can also control the rate of inflation, then F country is strictly better off by choosing a proper inflation rate. On the other hand, this may not be the case if the government fails to control inflation. To see this, observe that the welfare of F country is given by (7) with appropriate variables starred: call it W_A^*. If $\pi^* = \pi$, then from (7) and (23) we have

$$W_A^* - W^* = (1-\tau^*)^2 \left[1 - \frac{1}{(1+\pi^*)^2}\right] > 0.$$

Therefore, F country is better off under the autarchic policy. If, on the other hand, π^* tends to infinity, W^* stays constant, while W_A^* monotonically goes to negative infinity. Thus, if its inflation rate is sufficiently high, then F government should keep the original policy.

5.3 Monetary Unions

There are a variety of forms that a monetary union can take. Here, we restrict our attention to comparison between a common currency regime and the original model (which includes a fixed exchange rate regime by setting $\pi = \pi^*$). A common currency usually requires explicit designation of a single central bank. Otherwise, we would have excessive issuance of the currency. To see this point, we let H and F countries choose the rates π and π^*, respectively, of which average is the rate of inflation with respect to the currency, $\bar{\pi} = (\pi + \pi^*)/2$. The welfare of H country is given by

$$W_c = \ln\left(\frac{1-\tau}{1+\bar{\pi}}\beta\right)^2 - 2\beta + \alpha \ln \pi Y_c,$$

and the welfare of F country is given by

$$W_c^* = \ln\left(\frac{1-\tau^*}{1+\bar{\pi}}\beta\right)^2 - 2\beta + \alpha \ln \pi^* Y_c.$$

where

$$Y_c \equiv \frac{p_1\beta + p_2\beta}{p_1(1+\bar{\pi})} = \frac{(1+\bar{\pi}) + (1-\bar{\tau})}{(1+\bar{\pi})^2}\beta$$

We consider the policy game in which H and F countries independently choose (τ, π) and (τ^*, π^*), respectively. First, it is easily verified that $\tau = \tau^* = 0$ holds in an equilibrium. Differentiating W_c with respect to τ and equating it with zero, we obtain

$$\frac{\alpha}{\pi} + \frac{\alpha}{2(2+\bar{\pi})} = \frac{1+\alpha}{1+\bar{\pi}}.$$

We have the same expression for F country with π^* in place of π. Solving the simultaneous equations, we obtain

$$\pi = \pi^* = \frac{4 - 3\alpha + \sqrt{16 + 8\alpha - 7\alpha^2}}{2(2-\alpha)}.$$

This has a solution only when $\alpha < 2$. Even when $\alpha < 2$, this inflation rate far exceeds the optimal rate $\sqrt{1+\alpha}$ for the case $\tau = \tau^* = 0$. A central bank is needed for a common currency regime to work well.

Even if a central bank is established, however, a question remains as to which country should collect how much seigniorage. In a common currency regime, a political process and/or bargaining determines it. Whereas in a two currency regime, even under a fixed exchange regime, it is a market force that determines the ratio.

This analysis reveals an important difference between a common currency regime and a regime/equilibrium in which two competing currencies circulate side by side. For example, Krugman (1992) argues, "If the currencies really were accepted equally, we would have the competitive seigniorage problem discussed [above]." In the present framework, however, if there are two currencies circulating side by side with $\pi = \pi^*$, then neither country can increase its seigniorage as it causes higher inflation with respect to its currency, which results in a loss of seigniorage. We obtain an opposite result from the regime with a common currency. Note that this result would not be derived if we required, as in a cash-in-advance economy that agents use their own currency.

6 Concluding Remarks

The present paper offers a model of multiple currencies where the currency with a lower rate of inflation becomes an international medium of exchange, and the currency with a higher rate of inflation circulates as a local medium of exchange despite that nobody is required to use it for transaction. Asymmetric injection of the currency into the economy and imposition of taxes payable in this currency are sufficient for its circulation. Some other results are obtained. Here we mention four of them.

First, while the proportional income tax and the inflation tax are perfect substitutes in the single country model, this is no longer true in the two country model. The welfare of, say, H country increases as its inflation rate up to the opponent's rate provided that this rate is positive.

Second, in the policy game in which the two governments choose their respective inflation and tax rates at the beginning of the economy, we show, among other things, that the rate of inflation goes down to zero. This result should be contrasted with the Friedman rule according to which the optimal policy is to set the rate of nominal interest, not the rate of inflation, at zero. This is due to the fact that the negative inflation rate induces a negative seigniorage, paying subsidy to the agents of the other country.

Third, although the rate of inflation of the weak currency does not affect the real economy, the currency is by no means redundant; if the government allows its residents to pay taxes in the other currency, then this country is necessarily worse off on condition that the rate of inflation with respect to the strong currency is positive.

Fourth, a situation where two currencies circulate side by side has a completely different implication from a common currency regime. In the former, tight monetary policy is attained since each government tries to collect seigniorage from the agents of the other country by making its own currency strong, while in the latter, excess supply of money is inevitable unless there is a designated central bank.

It goes without saying that the present model is far from complete. There are several directions of extensions that one can immediately think of. First, the present paper has focused on stationary and deterministic equilibria. There may be other types of equilibria such as non-stationary equilibria and stochastic equilibria. The problem of modelling expectation formation become acute if we analyze non-stationary situations. How people form expectations in a non-stationary environment is a fascinating but hard question to answer. Stochastic equilibrium with a stationary, say, Markov process may be examined, too. Unlike a deterministic equilibrium, such an equilibrium may have active currency exchange between people with different attitudes toward risk.

Introducing barter in the model is relatively easy, though calculation would become messier. Substitutability between inflation and tax may no longer hold. On the other hand, adding credit and other financial assets to the model requires more fundamental changes in assumptions. One possibility is to assume that a creditor and a debtor come to know each other after they make a credit arrangement. However, discussing these possibilities in a concrete way would go beyond the scope of the paper.

It would be an interesting extension to consider the world economy with more than two countries. In such a model, several currencies compete with each other as international media of exchange, each of which is used by residents of its satelite countries— the picture that is close to the current world economy.

The present paper assumes physically identical countries. In the real world, different countries have different economic fundamentals, which gives rise to the standard benefit of international trade. Its immediate implication is that autarchy may not be as desirable as in the present model. It may be interesting to see how trading patterns affect circulation of international currencies, especially when there are more than two countries.

In the present model, there is no friction in each bilateral market: every agent has free access to any market and has no problem in finding a trading partner. This course of modelling is chosen since one of the motivations of this paper is to show that a weak currency circulates even if there is no friction in using a strong one. But this was not done without a cost. Besides the fact that it is more realistic to assume certain friction, we cannot distinguish a "thick" market from a "thin" market. Consequently, the model fails to incorporate an agglomeration effect, one of the important aspects in the analysis of money. Introducing some friction will definitely enrich the analysis, though the model may become more complicated.

7 Appendices

7.1 Equilibrium in the Two Country Model

This appendix derives a solution to the two-country model. From what we obtained and assumed, we have $y_1 > 0$, and either $x_2 > 0$ or $\hat{x}_2 > 0$ and $\hat{y}_1 > 0$. Therefore, from the Kuhn-Tucker condition for type 1 agent in H country and (14),

$$x_2 + \hat{x}_2 = \frac{p_1}{p_2}\frac{1-\tau}{1+\pi}\beta. \tag{31}$$

From the two budget constraints of the same agent and (14), we obtain

$$x_2 + \hat{x}_2 = \frac{p_1}{p_2}\frac{1-\tau}{1+\pi}(y_1 + \hat{y}_1). \tag{32}$$

Therefore, we have

$$y_1 + \hat{y}_1 = \beta.$$

For agent 1 in F country, we get the same expressions as (31) and (32), with appropriate variables starred, and then $y_1^* + \hat{y}_1^* = \beta$. Likewise, for agent 2 in H country, we obtain

$$x_1 + \hat{x}_1 = \frac{p_2}{p_1}\frac{1-\tau}{1+\pi}\beta, \tag{33}$$

and

$$x_1 + \hat{x}_1 = \frac{p_2}{p_1}\frac{1-\tau}{1+\pi}(y_2 + \hat{y}_2). \tag{34}$$

Thus, we get $y_2 + \hat{y}_2 = \beta$. Starring appropriate variables, we get the similar equations for agent 2 in F country. From the first budget constraint for agent 2 in F country and $x_1^* = 0$, we have

$$(1-\tau^*)y_2^* = \tau^*\hat{y}_2^*.$$

This equation and $y_2^* + \hat{y}_2^* = \beta$ imply $y_2^* = \tau^*\beta$.

Next, from (11), (12), (14), (31), and their counterparts for agent 1 in F country, we obtain

$$\frac{p_1}{p_2}\frac{1-\bar{\tau}}{1+\pi} = 1, \tag{35}$$

where $\bar{\tau} = (\tau + \tau^*)/2$. Equations (35) and (14) imply

$$\frac{p_1^*}{p_2^*} \frac{1-\bar{\tau}}{1+\pi^*} = 1. \tag{36}$$

Using (35) and (36) among others, we obtain (15) through (21).

7.2 Uniqueness of Equilibrium in a Policy Game

This appendix shows that there is a unique solution to a policy game with fixed inflation rates $\pi < \pi^*$ if $\alpha \leq 4$. Equating (27) with zero, we have

$$A\tau^2 + B\tau + C = 0, \tag{37}$$

where

$$A \equiv \frac{1+\alpha}{(1+\pi)^2}$$

$$B \equiv -(\alpha+2)\left[\frac{2-\tau^*}{(1+\pi)^2} + \frac{1}{2}\tau^*\right] - \frac{\alpha}{(1+\pi)^2},$$

and

$$C \equiv -(2-\tau^*)^2 + \frac{(2-\tau^*)^2}{(1+\pi)^2} + \alpha\left[\frac{2-\tau^*}{(1+\pi)^2} + \frac{1}{2}\tau^*\right].$$

It is easy to check that the left hand side of (37) is negative at $\tau = 1$. Moreover, $A > 0$ holds. Therefore, there is a unique optimal value of τ. When it is an interior solution, it is given by

$$\tau_{opt} = \frac{1}{2A}\left[-B - \sqrt{B^2 - 4AC}\right].$$

Differentiating this expression with respect to τ^*, we obtain

$$\frac{\partial \tau_{opt}}{\partial \tau^*} = \frac{1}{2A}\left[-\frac{\partial B}{\partial \tau^*} - \frac{1}{2\sqrt{B^2 - 4AC}}\left\{2B\frac{\partial B}{\partial \tau^*} - 4A\frac{\partial C}{\partial \tau^*}\right\}\right].$$

This is greater than

$$-\frac{\alpha+2}{2(1+\alpha)} + \frac{1}{4} + \frac{\alpha}{5\alpha/2+3} > -1.$$

Next, equating (28) with zero and using similar technique as in the previous paragraph, we obtain

$$\tau^*_{opt.} = \frac{1}{2(1+\alpha)}\left[-D - \sqrt{D^2 - 4(1+\alpha)\alpha(2-\frac{\tau}{2})}\right]$$

where

$$D \equiv -4 + \tau - 3\alpha + \frac{\alpha\tau}{2}.$$

Differentiating $\tau^*_{opt.}$ with respect to τ, we obtain

$$\frac{\partial \tau^*_{opt}}{\partial \tau} = \frac{1}{2(1+\alpha)}\left[-(1+\frac{\alpha}{2}) - \frac{1}{\sqrt{D^2 - 4(1+\alpha)\alpha(2-\frac{\tau}{2})}}\left\{D(1+\frac{\alpha}{2}) + (1+\alpha)\alpha\right\}\right].$$

The first term in the bracket is less than $2(1+\alpha)$ in absolute value, while the second term is positive. Therefore, $\partial \tau^*_{opt}/\partial \tau$ is greater than -1. We now assume $\alpha \leq 4$ and show that the derivative is negative. It is negative if and only if

$$D + \frac{1+\alpha}{1+\alpha/2}\alpha + \sqrt{D^2 - 4(1+\alpha)\alpha(2-\frac{\tau}{2})} > 0.$$

This inequality holds whenever

$$3(2+\alpha) > \frac{2\alpha^2}{1+\alpha/2},$$

which in turn holds if $\alpha \leq 4$. Thus, there is a unique equilibrium $(\tau_{opt}, \tau^*_{opt})$ for any $\pi < \pi^*$.

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π	τ	τ^*	G/β	G^*/β	x_1/β	x_2/β	X_1^*/β	X_2^*/β	W_0	W_0^*
0	.3121	.3121	.5268	.5268	.4732	1.0000	.4732	1.0000	-1.3892	-1.3892
.01	.3003	.3122	.5275	.5288	.4759	1.0086	.4678	.9914	-1.3735	-1.4056
.02	.2885	.3123	.5284	.5307	.4784	1.0170	.4625	.9830	-1.3584	-1.4218
.05	.2540	.3125	.5315	.5365	.4850	1.0408	.4470	.9592	-1.3155	-1.4697
.1	.1985	.3128	.5385	.5456	.4931	1.0768	.4228	.9232	-1.2520	-1.5467
.5	.0000	.3139	.7897	.5785	.3747	1.1861	.2571	.8139	-1.0470	-2.1117
1	.0000	.3139	1.0662	.5785	.2108	1.1861	.1446	.8139	-1.3222	-2.6870

Table 1

$$\alpha = 1, \quad X_1^* \equiv x_1^* + \hat{x}_1^*, \quad X_2^* \equiv x_2^* + \hat{x}_2^*$$

$$W_0 \equiv W - (2+\alpha)\ln\beta + 2\beta$$

$$W_0^* \equiv W^* - (2+\alpha)\ln\beta + 2\beta$$

π	τ	τ^*	G/β	G^*/β	x_1/β	x_2/β	X_1^*/β	X_2^*/β	W_0	W_0^*
0	.4597	.4597	.7081	.7081	.2919	1.0000	.2919	1.0000	-1.9217	-1.9217
.01	.4495	.4598	.7063	.7106	.2943	1.0095	.2888	.9905	-1.9090	-1.9350
.02	.4395	.4599	.7048	.7131	.2965	1.0186	.2857	.9814	-1.8970	-1.9482
.05	.4104	.4603	.7014	.7202	.3020	1.0442	.2764	.9558	-1.8635	-1.9875
.1	.3651	.4609	.6990	.7314	.3080	1.0816	.2616	.9184	-1.8153	-2.0518
.5	.0937	.4639	.7392	.7984	.2905	1.2566	.1719	.7434	-1.6120	-2.5079
1	.0000	.4648	.8878	.8216	.1919	1.3028	.1027	.6972	-1.6333	-3.0296

Table 2

$$\alpha = 2, \quad X_1^* \equiv x_1^* + \hat{x}_1^*, \quad X_2^* \equiv x_2^* + \hat{x}_2^*$$

$$W_0 \equiv W - (2+\alpha)\ln\beta + 2\beta$$

$$W_0^* \equiv W^* - (2+\alpha)\ln\beta + 2\beta$$

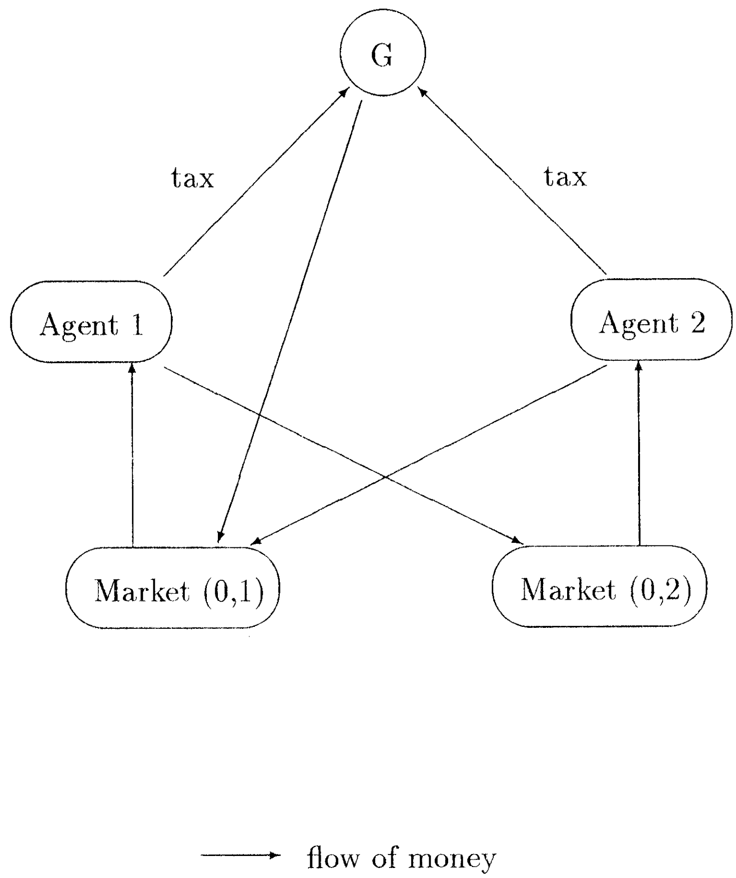

Figure 1: One Country Model

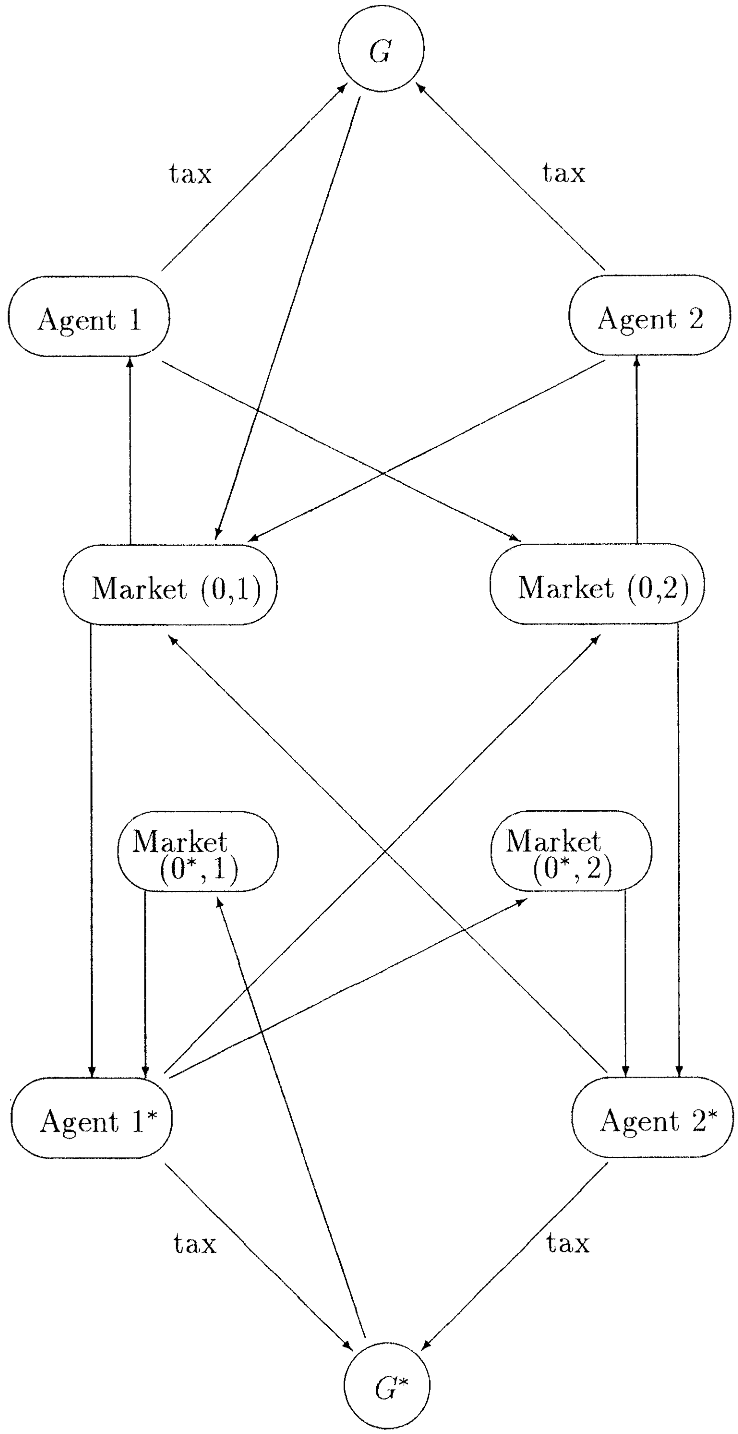

Figure 2: Two Country Model