CIRJE-F-13

Efficient Entrepreneurship

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July 1998

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July 28, 1998
(The First Version: October 8, 1997)

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* An earlier version of this paper is entitled "Procedural Rationality and Inductive Learning II: Efficient Entrepreneurship" (Matsushima (1997b)). The current version has lots of new insights which has not been discussed in the elder versions. This paper was presented at the University of Tokyo Microeconomics Workshop, Tokyo, October 1997, and the University of Osaka ISER seminar, Osaka, January 1998.

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Abstract

This paper considers a situation in which a decision maker chooses between the *safe* action and the *uncertain* action infinitely many times. The decision maker knows the payoff for the safe action, but does *not* know the payoff for the uncertain action which is determined by an *unknown* probability function. The decision may be influenced by the *payoff-irrelevant context* in which the current decision problem is to be considered. The context fluctuates according to another *unknown* probability function. The decision maker is modeled by a *Markov learning rule with reflecting barriers* which determines a state of mind in every period on the basis of past experiences.

We argue that the context-dependence of decision making plays an important role in finding out the *efficient* action in the long run, because it causes the decision maker to gather *unbiased* information at any time. We show that there exists a learning rule according to which the decision maker succeeds to choose the efficient action in the long run irrespective of how *payoff-uncertainty* and *context-uncertainty* are specified. We also characterize the class of such efficient learning rules, and argue that it is necessary that the upper reflecting barrier, regarded as *the maximal strength of confidence* that the uncertain action is more profitable than the safe action, greatly surpasses the negative of the lower reflecting barrier regarded as the *minimal* strength of confidence.

JEL Classification Numbers: D80, D81, D83.
Key Words: Uncertainty, Context, Efficient Learning, Unbiased Information Gathering, Strength of Confidence.

1. Introduction

This paper considers a situation in which a decision maker chooses between two actions, i.e., the *safe* action and the *uncertain* action, infinitely many times. The decision maker knows that she obtains payoff *zero* by choosing the safe action, whereas she does *not* know the payoff she obtains by choosing the uncertain action. For example, an *entrepreneur* repeatedly decides whether she should devote herself to only a routine work, or do a new enterprise the consequence of which may be influenced by unforeseen contingencies.

In such a situation, a real individual makes her decision dependent on the surrounding circumstances, or the *context*, in which the current decision problem is to be considered. For example, an entrepreneur's decision may be influenced by an atmosphere in the society which emerges from the mass psychology of other individuals as being outside her control: A gloomy atmosphere may discourage the entrepreneur to choose the uncertain action, whereas a merry atmosphere may encourage her to do it positively. However, the context does *not* necessarily contain some objective information about the profitability of the uncertain action in the current decision problem, and may fluctuate suddenly due to some payoff-irrelevant factors.[1]

The question to be answered is why a decision maker is willing to make her decision dependent on the context, even though it may be *payoff-irrelevant*.[2] We will argue that the dependence of her decision on the context plays an important role in leading the decision maker to gather *unbiased* information on the payoffs for the uncertain action at any time, and to come to choose the *efficient* action in the long run.

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[1] Keynes (1936) presented a related argument —— "A conventional valuation which is established as the outcome of the mass psychology of a large number of ignorant individuals is liable to change violently as the result of a sudden fluctuation of opinion due to factors which do not really make much difference to the prospective yield . . . the market will be subject to waves of optimistic and pessimistic sentiment, which are unreasoning and yet in a sense legitimate where no solid basis exists for a reasonable calculation" (1936, Chapter 12, p.154).

[2] Several experimental results showed that a decision maker may be influenced by factors irrelevant to the physical structure of the decision problem such as the difference of *framing* (Tversky and Kahneman (1981)) and the difference of *response mode* (Tversky, Sattath and Slovic (1988)). Schelling (1960) proposed a principle of equilibrium selection in game theory that player's public knowledge about some contextual labels makes an action profile salient as a *focal point*.

Uncertainty with which the decision maker is confronted is divided into two categories, i.e., *payoff-uncertainty* and *context-uncertainty*. At the end of every period after the choice of the uncertain action, the payoff is *randomly* determined according to a probability function on a closed interval $V \equiv [\underline{v}, \overline{v}]$. At the beginning of every period, the context is also *randomly* determined according to another probability function on the set of real numbers $C \equiv R$. The larger the context in the current problem is, the more gloomy the atmosphere in the society is. We assume that the decision maker does *not* know both of these probability functions. The purpose of this paper is to clarify the possibility that the decision maker succeeds to choose the efficient action in the long run *irrespective* of how payoff-uncertainty and context-uncertainty, i.e., these probability functions, are specified.

The decision maker is modeled as an *adaptive learning rule* which translates past experiences, i.e., past histories of the contexts, her decisions, and the resulting payoffs, into a set of contexts in which the decision maker will choose the uncertain action.[3] We shall confine our attention to learning rules which are *finite Markov chains with reflecting barriers*: The decision maker's *state of mind* is represented by an integer x in a finite set $\{\underline{x}, \ldots, \overline{x}\}$, where $\underline{x} < 0 < \overline{x}$. A state of mind x has a *boundary of context* $\mu(x) \in C$ such that the decision maker chooses the uncertain action if and only if the current context c is more merry than this boundary, i.e., if and only if $c < \mu(x)$. We assume that $\mu(x)$ is *increasing*: The higher the state of mind is, the wider the set of possible contexts in which the decision maker will choose the uncertain action is. Of particular importance, we assume that whenever the decision maker chooses the safe action, the state of mind remains *unchanged* in the next period. On the contrary, if she chooses the uncertain action, the state of mind may change *only slightly*: Given that the current state of mind is x, it changes from x into $x+1$ (into $x-1$) in the next period, only if the resulting payoff is sufficiently high (low, respectively).

As the main result of this paper, we presents *necessary and sufficient conditions* on learning rules according to which the decision maker succeeds to choose the efficient action in the long run irrespective of how payoff-uncertainty and context-uncertainty are specified. Especially, it is necessary that the negative of the ratio between the lower and upper reflecting barriers, $-\underline{x}/\overline{x}$, is *close to zero*. This necessary condition expresses the following remarkable characteristic of psychological aspects about the *strength of*

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[3] A classical reference is the model of fictitious play by Brown (1951). Adaptive learning has recently been receiving considerable attention (Fudenberg and Levine (1997), for example).

confidence: A state of mind expresses the degree to which the decision maker is *confident* that the uncertain action is *more* profitable than the safe action. The higher the state of mind is, the stronger this confidence is, and therefore, the less probable it is that the decision maker changes her mind and comes to choose the safe action in a wide class of contexts in the near future. The upper reflecting barrier $\bar{x}$ is regarded as the *maximal* strength of confidence that the uncertain action is more profitable, because the decision maker, once reaching $\bar{x}$, no more strengthens this confidence. Similarly, the negative of the lower reflecting barrier $-\underline{x}$ is regarded as the maximal strength of confidence that the uncertain action is *less* profitable than the safe action. This necessary condition implies that *the latter maximal strength of confidence is surpassed by the former*. At the overwhelming majority of states of mind x, the decision maker will choose the uncertain action with almost certainty.

The above necessary condition is very restrictive, but there *do* exist learning rules which satisfy the necessary and sufficient conditions. The existence of such efficient learning rules is in contrast to the previous works such as Sarin and Vahid (1997a) and Matsushima (1997a, 1998a) which showed that the decision maker always comes to choose only the safe action in the long run, maybe inefficiently, irrespective of how payoff-uncertainty is specified: By assuming that there exists a possible state of mind at which the decision maker is confident that the uncertain action is unprofitable enough to stop choosing it, Sarin and Vahid, and Matsushima showed that the decision maker always becomes stuck in some of these states of mind in the long run.[4] [5] In the same way as these works, the present paper allows that every context has a possible state of mind at which the decision maker is confident that the uncertain action is unprofitable enough to stop choosing it in this context. The distinction between these works and the present paper is that Sarin and Vahid, and Matsushima assumed that the decision maker meets only a *single* context, whereas the present paper assumes that she meets various contexts and makes a decision according to *reasoning by analogy* in the following way: At a state of mind, the decision maker may stop choosing the uncertain action in some

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[4] See also Sarin and Vahid (1997b), Sarin (1997), and Cattaraj (1997). Multi-armed bandit models have been studied by several authors such as Rothschild (1974) and Gittins (1989) which showed related properties in a situation when individuals are Bayesian.

[5] Matsushima (1997a, 1998a) considered a decision maker who observes the state of the world with some noise and estimates the subjective probability on the set of the state of the world. Another paper by Matsushima (1998b) investigated a game-theoretic situation in which players fails to equalize their subjective games to the objective game.

contexts but still be willing to choose it in the other contexts. The decision maker will utilize the experiences of the latter contexts for her evaluation for the profitability of the uncertain action in the former contexts, which leads the decision maker to gather unbiased information at any time.

We argue that under the necessary and sufficient conditions, the difference of strength of confidence may *not* be translated into the difference of subjective expected payoff for the uncertain action: The subjective expected payoff must be equal to zero at *almost every* state of mind and in *almost every* context, provided the decision maker always updates the subjective expected payoff in a coherent way. This point is in contrast to the standard Bayesian approach by Savage (1954).

We argue also that the decision maker is liable to increase the subjective expected payoff for the uncertain action even though she obtains a payoff which is strictly *less* than the current subjective expected payoff, provided the decision maker may *not* update the subjective expected payoff in a coherent way. This point is closely related to the explanation by Keynes (1936) about the psychology of vital entrepreneurs.[6]

In the situation considered in the present paper, the decision maker may not reliably estimate the probabilities of the possible payoffs, because, as Sarin and Vahid, and Matsushima have explained, she might have failed to gather unbiased information up to the present. According to Knight (1921), this situation is classified as "radical" uncertainty which is distinguished from "risk" in cases of which probabilities are well-estimated and insurance is possible. A real entrepreneur faces radical uncertainty when she starts a new enterprise. Probably most of the situations which require decision making of the entrepreneur fall in the category of radical uncertainty because the failure of the enterprise is not insurable.

When this sort of uncertainty is present, the rational basis for action is greatly weakened. Several experimental studies such as Ellsberg (1961) showed that ordinal people prefer to act on a known rather than unknown probabilities. However, irrational, psychological forces such as *animal spirits*, which Keynes (1936) conceived as "a spontaneous urge to action rather than inaction" (Chapter 12, p.161), prevent uncertainty from stopping positive action, even though the resulting payoffs are disappointing. The characterization theorem of the present paper clarifies what kind of learning rule brings about objective efficiency through adjustments of the animal spirits.

Several works have investigated decision makers who are so irrational to randomize, or *experiment* with, undesirable actions. Cross (1983) and Börgers and Sarin (1996)

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[6] See Footnote 7.

have studied models of reinforcement learning in which a decision maker chooses according to the suggestions of randomly selected stimuli. Sarin and Vahid (1997a) have studied a model of adaptive learning in which a decision maker's choice is affected by a randomly determined "mood". In these works, however, psychological factors which cause a decision maker to experiment are left unmodelled. This point is crucial to the distinction between these works and the present paper. The present paper explicitly models how the animal spirits are adjusted in a way that the animal spirits are dimmed when a state of mind is pessimistic and the current context has a gloomy atmosphere, whereas the animal spirits are activated when a state of mind is optimistic and the current context has a merry atmosphere.

The organization of this paper is as follows. Section 2 defines a long-run decision problem, a learning rule, an infinite sequence of learning rules, and efficiency of a sequence of learning rules with payoff-uncertainty and context-uncertainty. We assume that a decision maker's learning rule is approximated by the limit of a sequence of learning rules. Section 3 gives the main theorem which characterizes the class of sequences of learning rules which are efficient. Section 4 presents the proof of this theorem. Finally, Section 5 gives concluding remarks.

2. The Model

2.1. Long-Run Decision Problems

We consider *a long-run decision problem* $D \equiv (A, C, V, H, p, f)$ which is defined in the following way. $A = \{a^s, a^u\}$ is the set of actions, a^s is called *the safe action,* and a^u is called *the uncertain action.* A *decision maker* repeatedly chooses between the safe action a^s and the uncertain action a^u infinitely many times. The set of possible *contexts* is defined by the set of real numbers $C \equiv R$. The less context in the current problem is, the more merry the atmosphere in the society is. The set of possible payoffs is defined by a closed interval $V \equiv [\underline{v}, \bar{v}]$, where $\underline{v} < 0 < \bar{v}$.

At the beginning of every period $t \geq 1$, the decision maker observes a context $c(t) \in C$ and then chooses an action $a(t) \in A$. At the end of this period, she obtains a payoff $v(t) \in V$. Let h^0 be the null history, $H^0 \equiv \{h^0\}$, and let $h^t \equiv (c(\tau), a(\tau), v(\tau))_{\tau=1}^t$ denote a history up to period t. Let $H(t)$ denote the set of possible h^t, and $H \equiv \bigcup_{t=0}^{\infty} H(t)$.

Let $p: V \to R_+$ be a probability function on V. When the decision maker chooses the uncertain action in a period t, the payoff $v(t) = v$ is randomly determined according to p. The payoff for the uncertain action is drawn independently of the context in the current problem. The *objective* expected payoff for the uncertain action is defined by

$$v(p) \equiv \int_{v \in V} v p(v) dv .$$

We define the associated cumulative distribution function by $P(v') \equiv \int_{\underline{v}}^{v'} p(v) dv$.

When the decision maker chooses the safe action, she certainly obtains payoff zero. The decision maker a priori knows that she obtains payoff zero by choosing the safe action.

Let $f: C \to R_+$ be a probability function on C. At the beginning of every period t, the context $c(t) = c$ is randomly determined according to f the realization of which is observed by the decision maker. We define the associated cumulative distribution function by $F(c') \equiv \int_{-\infty}^{c'} f(c) dc$.

We assume *payoff-uncertainty* and *context-uncertainty,* that is, we assume that the decision maker has no knowledge about these probability functions p and f. We denoted by Φ the set of all probability functions on V. We denoted by Ξ the set of all probability functions on C.

Throughout this paper, we will fix A, C and V. Hence, a long-run decision problem is simply denoted by $D=(p,f)$.

2.2. Markov Learning Rules

A decision maker is modeled by a *Markov learning rule with reflecting barriers*, or a *learning rule*, $\Gamma \equiv (X,\mu,w,\theta)$. $X \equiv \{\underline{x},\ldots,\overline{x}\}$ is the set of *states of mind*, $\underline{x} < 0 < \overline{x}$, Z is the set of all integers, $\mu: Z \to C$, $\theta: V \times Z \times C \to [0,1]$, and $w: Z \times C \to V$. We assume that $\mu(x)$ is *increasing* with respect to x.

Suppose that the decision maker has a state of mind $x \in X$ and observes a context $c \in C$ in the current decision problem. Then, the decision maker chooses the uncertain action if and only if $c < \mu(x)$, i.e., if and only if context c is more merry than *the boundary of context* $\mu(x)$ *associated with state of mind* x.

The state of mind changes as time goes on in the following way. If she chooses the safe action, the state of mind remains *unchanged* in the next period. If she chooses the uncertain action and obtains payoff $v \in V$, then she changes the state of mind *only slightly*, i.e.,

> the state of mind remains unchanged with probability $\theta(v,x,c)$,
>
> it changes upwards from x into $x+1$ with probability $1-\theta(v,x,c)$ if $v \geq w(x,c)$,

and

> it changes downwards from x into $x-1$ with probability $1-\theta(v,x,c)$ if $v < w(x,c)$.

A state of mind expresses the degree to which the decision maker is confident that the uncertain action is *more* profitable than the safe action. The higher the state of mind is, the wider the set of contexts in which the decision maker will choose the uncertain action is, and the less probable it is that the decision maker changes her mind and comes to choose the safe action in a wide class of contexts in the near future.

The upper reflecting barrier $\overline{x}$ expresses the *maximal strength of confidence* that the uncertain action is more profitable than the safe action. The decision maker never strengthens this confidence more than $\overline{x}$. On the other hand, the negative of the lower reflecting barrier $-\underline{x}$ expresses the *maximal* strength of confidence that the uncertain action is *less* profitable than the safe action. The decision maker never strengthens this latter confidence more than $-\underline{x}$. We define *the relative maximal strength of confidence* by the negative of the ratio between the lower and upper reflecting barriers, i.e., $-\underline{x}/\bar{x}$.

The probability that the state of mind changes from x into $x+1$ given that the current state of mind is $x \neq \bar{x}$, is defined by

$$\xi^{+}(x,D,\Gamma)=\xi^{+}(x)\equiv\int_{-\infty}^{\mu(x)}\left\{\int_{w(x,c)}^{\bar{v}}(1-\theta(v,x,c))p(v)dv\right\}f(c)dc.$$

The probability that the state of mind changes from x into $x-1$ given that the current state of mind is $x \neq \underline{x}$, is defined by

$$\xi^{-}(x,D,\Gamma)=\xi^{-}(x)\equiv\int_{-\infty}^{\mu(x)}\left\{\int_{\underline{v}}^{w(x,c)}(1-\theta(v,x,c))p(v)dv\right\}f(c)dc.$$

The stationary probabilities of the occurrence of state of mind, $g(x,D,\Gamma)=g(x)$, are defined by these inequalities:

$$\xi^{+}(\underline{x})g(\underline{x})=\xi^{-}(\underline{x}+1)g(\underline{x}+1),$$
$$\xi^{-}(\bar{x})g(\bar{x})=\xi^{+}(\bar{x}-1)g(\bar{x}-1),$$

and for every $x\in\{\underline{x}+1,\ldots,\bar{x}-1\}$, (1)

$$(\xi^{+}(x)+\xi^{-}(x))g(x)=\xi^{+}(x-1)g(x-1)+\xi^{-}(x+1)g(x+1).$$

Finally, the stationary probability that the decision maker chooses the uncertain action is defined by

$$\Lambda(D,\Gamma)=\Lambda\equiv\sum_{x=\underline{x}}^{\bar{x}}F(\mu(x))g(x).$$

Throughout this paper, we will fix (μ,w,θ). A learning rule $\Gamma=(X,\mu,w,\theta)$ is simply denoted by $\Gamma=(\underline{x},\bar{x})$. For convenience of the arguments, we assume the following technical assumptions:

1) there exists $\bar{\theta}: V\times C\to[0,1)$ such that $\lim_{x\to+\infty}\theta(v,x,c)=\bar{\theta}(v,c)$,

2) there exists $\underline{\theta}: V\to[0,1)$ such that $\lim_{x\to-\infty,\, c\to-\infty}\theta(v,x,c)=\underline{\theta}(v)$,

3) there exists $\bar{w}: C\to V$ such that $\lim_{x\to+\infty}w(x,c)=\bar{w}(c)$,

and

4) there exists $\underline{w}\in V$ such that $\lim_{x\to-\infty,\, c\to-\infty}w(x,c)=\underline{w}$.

2.3. Sequence of Learning Rules and Efficiency

We shall confine our attention to a learning rule whose upper reflecting barrier $\bar{x}$

and lower reflecting barrier $\underline{x}$ are sufficiently large and sufficiently small respectively. Such a learning rule will be approximated by the limit of a *sequence of learning rules* defined by

$$\Gamma^{\infty} \equiv (\Gamma_m)_{m=1}^{+\infty},$$

where

$$\Gamma_m = (\underline{x}_m, \overline{x}_m), \quad \lim_{m \to +\infty} \underline{x}_m = -\infty, \text{ and } \lim_{m \to +\infty} \overline{x}_m = +\infty.$$

We define *the limit relative maximal strength of confidence associated with* Γ^{∞} by

$$\gamma \equiv \lim_{m \to +\infty} \left(-\frac{\underline{x}_m}{\overline{x}_m}\right).$$

Define $\Xi(\mu) \subset \Xi$ by the set of all probability functions f on C such that

$$f(c) > 0 \text{ for all } c \in C,$$

and

$$\lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} > 0.$$

REMARK 1: We must note that $\Xi(\mu)$ includes a wide variety of probability functions on C, and, for every $b \in (0,1)$, there exists $f \in \Xi(\mu)$ such that

$$\lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} = b.$$

We can construct such a f as follows. Choose a real number $q \in (0,1)$ arbitrarily. There exists an increasing and differentiable function F on C such that $F(\mu(x)) = b^{-x}$ for all $x \leq 0$, and $\lim_{c \to +\infty} F(x) = 1$. Let $f(x) \equiv \frac{\partial}{\partial x} F(x)$. We must note that $f(\cdot)$ is included in $\Xi(\mu)$ because $\lim_{c \to -\infty} F(c) = \lim_{x \to -\infty} F(\mu(x)) = 0$.

REMARK 2: We must note also that $\Xi(\mu)$ is *independent* of μ in the following sense: Consider μ and $\tilde{\mu}$ such that

$$\lim_{x \to -\infty} (\mu(x) - \mu(x-1)) < +\infty \text{ and } \lim_{x \to -\infty} (\tilde{\mu}(x) - \tilde{\mu}(x-1)) < +\infty.$$

We can show that $[f \in \Xi(\mu)] \Rightarrow [[f \in \Xi(\tilde{\mu})]$ below: Define $\overline{z}: Z \to Z$ and $\underline{z}: Z \to Z$ such that for every $x \in Z$

$$\mu(\overline{z}(x)) \geq \tilde{\mu}(x) \geq \mu(\overline{z}(x)-1) \text{ and } \mu(\underline{z}(x)+1) \geq \tilde{\mu}(x) \geq \mu(\underline{z}(x)).$$

We must note that $\overline{z}(x) \geq \underline{z}(x)$, and there exists $\rho > 0$ such that

$$\lim_{x \to -\infty} (\overline{z}(x) - \underline{z}(x)) \equiv \rho < +\infty.$$

Let $f \in \Xi(\mu)$. Since there exists $b \in (0,1)$ such that $b = \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))}$, one gets

$$\lim_{x\to-\infty}\frac{F(\widetilde{\mu}(x-1))}{F(\widetilde{\mu}(x))}\geq\lim_{x\to-\infty}\prod_{z=\underline{z}(x)+1}^{\bar{z}(x)}\frac{F(\mu(z-1))}{F(\mu(z))}=b^{\rho}>0,$$

which implies $f\in\Xi(\widetilde{\mu})$.

The main purpose of this paper is to characterize sequences of learning rules which induce the decision maker to choose the efficient action in the long run, irrespective of how payoff-uncertainty and context-uncertainty are specified, i.e., irrespective of which vector of probability functions $(p,f)\in\Phi\times\Xi(\mu)$ is given. In other words, the main purpose is to characterize sequences of learning rules which are efficient in the following sense.

DEFINITION: A sequence of learning rules Γ^{∞} is *efficient* if for every $p\in\Phi$ and every $f\in\Xi(\mu)$,

$$[v(p)>0]\Rightarrow[\Lambda^{\infty}=1] \text{ and } [v(p)<0]\Rightarrow[\Lambda^{\infty}=0].$$

3. Main Theorem

The main theorem of this paper is presented as follows.

THEOREM: *A sequence of learning rules* Γ^{∞} *is efficient, if and only if*

(i) $\lim_{x \to -\infty} \mu(x) = -\infty$ and $\lim_{x \to +\infty} \mu(x) = +\infty$,

(ii) $\gamma = 0$,

(iii) $\overline{w}(c) = 0$ *for all* $c \in C$, *and* $\underline{w} \geq 0$,

(iv) *there exists* $\overline{y}: C \to (0, \infty)$ *such that*

$$\overline{\theta}(v, c) = 1 - \overline{y}(c)|v|,$$

and

(v) *there exists* $\underline{y} \in (0,1)$ *such that*

$$\underline{\theta}(v) \geq 1 - \underline{y}v \text{ for all } v \in (0, \overline{v}], \text{ and}$$

$$\underline{\theta}(v) \leq 1 + \underline{y}v \text{ for all } v \in [\underline{v}, 0).$$

Properties (i) through (v) are very restrictive, but there do exist sequences of learning rules which satisfy these properties, i.e., which are efficient.

Property (i) implies that the probability that the decision maker chooses the uncertain action converges to zero as a state of mind decreases, whereas this probability converges to unity as a state of mind increases.

Property (ii) is the most important necessary condition, which implies that the limit relative maximal strength of confidence is zero. That is, the upper reflecting barrier $\overline{x}_m$, regarded as the maximal strength of confidence that the uncertain action is more profitable than the safe action, greatly surpasses the negative of the lower reflecting barrier $-\underline{x}_m$, regarded as the minimal strength of confidence.

REMARK 3: Property (ii) implies that at the overwhelming majority of states of mind, the decision maker will choose the uncertain action with almost certainty. It is only exceptional states of mind that make the decision maker choose the safe action with almost certainty. The Theorem says that if the uncertain action is less profitable than the safe action, then the decision maker eventually sticks to these exceptional states of mind with almost certainty.

Property (iii) implies that, at almost every state of mind and in almost every context, the state of mind changes upwards only if the realized payoff for the uncertain action is positive, whereas it changes downwards only if the realized payoff is negative.

REMARK 4: Suppose that $w(x,c)$ is regarded as the *subjective* expected payoff for the uncertain action. Then, by choosing the uncertain action, the decision maker changes the state of mind upwards (downwards), if the realized payoff is more than (less than) $w(x,c)$. Property (iii), together with property (ii), says that at the overwhelming majority of states of mind, this subjective expected payoff is almost always equal to zero. Neither the difference of state of mind nor the difference of context is reflected in the difference of subjective expected payoff, i.e., the difference of strength of confidence can not be translated into the difference of subjective expected payoff. This point is in contrast to the Bayesian approach by Savage (1954) in which a decision maker determines which action to be chosen only by maximizing her subjective expected payoff.

REMARK 5: Suppose that the subjective expected payoffs for the uncertain action increase as the state of mind increases. The decision maker chooses the uncertain action if and only if this subjective expected payoff is more than zero. Property (iii) implies that the decision maker fails to update the subjective expected payoff in some coherent way: By choosing the uncertain action, the decision maker increases the subjective expected payoff whenever the realized payoff is more than zero. Since the subjective expected payoff must be strictly more than zero when the decision maker will choose the uncertain action, one concludes that she may increase the state of mind, and therefore, increase the subjective expected payoff, even through the realized payoff is strictly less than the current subjective expected payoff.[7]

Property (iv) implies that, at almost every state of mind and in almost every context, the transition probability of state of mind is *proportional* to the realized payoff. Hence, the probability that the state of mind changes upwards is greater than (smaller than) the

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[7] Keynes (1936) has argued a related point that a vital entrepreneur has a psychological tendency to invest in spite of past disappointments. Keynes thought that investments generally fail to deliver the expected returns —— ". . . it is probable that the actual average results of investments . . . have disappointed the hopes that prompted them . . . If human nature felt no temptation to take a chance, no satisfaction (profit apart) in constructing a factory, a railway, a mine, or a farm, there might not be much investment as a result of cold calculation" (Chapter 12, p.150).

probability that the state of mind changes downwards, only if the objective expected payoff for the uncertain action is more than (less than) zero, i.e., only if $v(p) > 0$ ($v(p) < 0$).

Property (v) implies that when the state of mind is sufficiently small, the probability that the state of mind changes upwards may be smaller than the probability that it changes downwards even if the objective expected payoff for the uncertain action is more than zero, i.e., even if $v(p) > 0$.

EXAMPLE: Consider a situation in which $\bar{v} = -\underline{v} = 1$. Specify a sequence of learning rules Γ^∞ as follows:

$$\mu(x) = x \text{ for all } x \in Z,$$
$$w(x,c) = 0 \text{ for all } (x,c) \in Z \times C,$$
$$\theta(v,x,c) = 1 - |v| \text{ for all } (v,x,c) \in V \times Z \times C,$$
$$\underline{x}_m = -m \text{ and } \bar{x}_m = m^2 \text{ for all } m \geq 1.$$

It is easy to check that this specified Γ^∞ satisfies properties (i) through (v), and therefore, is efficient.

We will prove the theorem in the next section.

4. Proof of the Theorem

Define

$$\overline{\xi}^{+}(\overline{w},p,f)=\overline{\xi}^{+}\equiv\int_{-\infty}^{+\infty}\left\{\int_{\overline{w}(c)}^{\overline{v}}(1-\overline{\theta}(v,c))p(v)dv\right\}f(c)dc,$$

$$\overline{\xi}^{-}(\overline{w},p,f)=\overline{\xi}^{-}\equiv\int_{-\infty}^{+\infty}\left\{\int_{\underline{v}}^{\overline{w}(c)}(1-\overline{\theta}(v,c))p(v)dv\right\}f(c)dc,$$

$$\underline{\xi}^{+}(\underline{w},p)=\underline{\xi}^{+}\equiv\int_{\underline{w}}^{\overline{v}}(1-\underline{\theta}(v))p(v)dv,$$

and

$$\underline{\xi}^{-}(\underline{w},p)=\underline{\xi}^{-}\equiv\int_{\underline{v}}^{\underline{w}}(1-\underline{\theta}(v))p(v)dv.$$

We must note that

$$\lim_{x\to+\infty}\frac{\xi^{+}(x-1)}{\xi^{-}(x)}=\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}},$$

and

$$\lim_{x\to-\infty}\frac{\xi^{+}(x-1)}{\xi^{-}(x)}=\lim_{x\to-\infty}\left\{\frac{\left(\frac{\xi^{+}(x-1)}{F(x-1)}\right)}{\left(\frac{\xi^{+}(x)}{F(x)}\right)}\cdot\frac{F(\mu(x-1))}{F(\mu(x))}\right\}=\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}}\lim_{x\to-\infty}\frac{F(\mu(x-1))}{F(\mu(x))}.$$

These equations and the definition of γ imply

$$\lim_{m\to+\infty}\left(\prod_{x=\underline{x}_m+1}^{\overline{x}_m}\frac{\xi^{+}(x-1)}{\xi^{-}(x)}\right)^{\frac{1}{\overline{x}_m-\underline{x}_m}}=\lim_{m\to+\infty}\left(\prod_{x=0}^{\overline{x}_m}\frac{\xi^{+}(x-1)}{\xi^{-}(x)}\right)^{\frac{1}{\overline{x}_m-\underline{x}_m}}\left(\prod_{x=\underline{x}_m+1}^{0}\frac{\xi^{+}(x-1)}{\xi^{-}(x)}\right)^{\frac{1}{\overline{x}_m-\underline{x}_m}}$$

$$=\left(\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}}\right)^{\frac{1}{\gamma+1}}\left(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}}\lim_{x\to-\infty}\frac{F(\mu(x-1))}{F(\mu(x))}\right)^{\frac{\gamma}{\gamma+1}}. \tag{2}$$

Consider a sequence of learning rules $\Gamma^{\infty}=(\Gamma_m)_{m=1}^{\infty}$. We denote

$g_m(x)\equiv g(x,D,\Gamma_m)$.

From equalities (1),

$$g_m(x)=\frac{\xi^{+}(x-1)}{\xi^{-}(x)}g_m(x-1)\text{ for all }m\geq 1\text{ and all }x\in\{\underline{x}_m+1,\ldots,\overline{x}_m\},$$

that is,

$$\frac{g_m(\overline{x}_m)}{g_m(\underline{x}_m)}=\prod_{x=\underline{x}+1}^{\overline{x}}\frac{\xi^{+}(x-1)}{\xi^{-}(x)}.$$

From equality (2), one gets

$$\lim_{m\to+\infty}\left(\frac{g_m(\overline{x}_m)}{g_m(\underline{x}_m)}\right)^{\frac{1}{\overline{x}_m-\underline{x}_m}}=\left(\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}}\right)^{\frac{1}{\gamma+1}}\left(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}}\lim_{x\to-\infty}\frac{F(\mu(x-1))}{F(\mu(x))}\right)^{\frac{\gamma}{\gamma+1}}.$$

We present the following useful propositions.

PROPOSITION 1: *For every* $(p, f) \in \Phi \times \Xi(\mu)$,

(1-1) $$[\Lambda^{\infty} = 1] \Rightarrow [\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}} \geq 1 \; and \; (\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} \geq 1],$$

and

(1-2) $$[\Lambda^{\infty} = 0] \Rightarrow [\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} \leq 1 \; and \; (\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} \leq 1].$$

PROOF: See Appendix A.

PROPOSITION 2: *For every* $(p, f) \in \Phi \times \Xi(\mu)$,

(2-1) $$[\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}} > 1 \; and \; (\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} > 1] \Rightarrow [\Lambda^{\infty} = 1],$$

and

(2-2) $$[\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} < 1 \; and \; (\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} < 1] \Rightarrow [\Lambda^{\infty} = 0].$$

PROOF: See Appendix B.

First of all, we show that properties (i) through (v) are necessary.

Property (i): Suppose that $\lim_{x \to -\infty} \mu(x) \neq -\infty$. Then, for every $(p, f) \in \Phi \times \Xi(\mu)$, there exists $\varepsilon > 0$ such that $F(\mu(x)) \geq \varepsilon$, and therefore, it must hold that $\Lambda^{\infty} \geq \varepsilon > 0$. This is a contradiction. Similarly we can prove $\lim_{x \to +\infty} \mu(x) = +\infty$.

Property (ii): Suppose that $\gamma = 0$, i.e., $\gamma > 0$. Let $p \in \Phi$ be chosen such that $v(p) > 0$. From Remark 1 in Section 2.3, one gets that for every $b \in (0,1)$, there exists $f \in \Xi(\mu)$ such that

$$\lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} = b.$$

Hence, we can choose $f \in \Xi(\mu)$ such that $\lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))}$ is so close to zero that

$$(\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} < 1.$$

This is a contradiction of property (1-1) in Proposition 1.

Property (iii): Suppose that there exists $c \in C$ such that $\overline{w}(c) > 0$. Then, there exist $f \in \Phi$ and $p \in \Xi(\mu)$ such that

$$v(p) > 0, \quad \frac{\overline{\xi}^{+}}{\overline{\xi}^{-}} < 1,$$

the probability of occurrence of contexts in the neighborhood of c is very large,

and

the probability of occurrence of payoffs in the interval $(0, \overline{w}(c))$ is very large.

Since property (ii) is necessary, i.e., since $\gamma = 0$, one gets

$$\left(\frac{\overline{\xi}^{+}}{\overline{\xi}^{-}}\right)\left(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))}\right)^{\gamma} = \frac{\overline{\xi}^{+}}{\overline{\xi}^{-}} < 1.$$

This is a contradiction of property (1-1) in Proposition 1. Similarly we can check that there exists no $c \in C$ such that $\overline{w}(c) < 0$.

Next, suppose that $\underline{w} < 0$. Then, there exist $f \in \Phi$ and $p \in \Xi(\mu)$ such that

$$\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} > 1,$$

and

the probability of occurrence of payoffs in the interval $(\underline{w}, 0)$ is large enough to satisfy $v(p) < 0$.

Clearly, this is a contradiction of property (1-2) in Proposition 1.

Property (iv): Suppose that property (iv) does not hold. Then, there exist $c \in C$, $y > 0$, $v \in (0, \overline{v}]$, and $v' \in [\underline{v}, 0)$ such that either

$$\overline{\theta}(v, c) > 1 - yv \text{ and } \overline{\theta}(v', c) < 1 + yv',$$

or

$$\overline{\theta}(v, c) < 1 - yv \text{ and } \overline{\theta}(v', c) > 1 + yv'.$$

Consider the former case. From property (iii), we must note that there exist $f \in \Phi$ and $p \in \Xi(\mu)$ such that

$$v(p) > 0 \text{ and } \frac{\overline{\xi}^{+}}{\overline{\xi}^{-}} < 1,$$

where

the probability of occurrence of contexts in the neighborhood of c is very large,

the probability of occurrence of payoffs in the neighborhood of v is more than $\frac{-v'}{v-v'}$,

and

the probability of occurrence of payoffs in the neighborhood of v' is less than $\frac{v}{v-v'}$.

Since property (ii) is necessary, i.e., since $\gamma = 0$, one gets

$$(\frac{\overline{\xi}^+}{\overline{\xi}^-})(\frac{\underline{\xi}^+}{\underline{\xi}^-}\lim_{x\to-\infty}\frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} = \frac{\overline{\xi}^+}{\overline{\xi}^-} < 1.$$

This is a contradiction of property (1-1) in Proposition 1. Similarly we can check the latter case also.

Property (v): Suppose that property (v) does not hold. Then, there exist $y > 0$, $v \in (0, \overline{v}]$, and $v' \in [\underline{v}, 0)$ such that

$$\underline{\theta}(v) < 1 - yv \text{ and } \underline{\theta}(v') > 1 + yv'.$$

From property (iii), we must note that there exist $f \in \Phi$ and $p \in \Xi(\mu)$ such that

$$v(p) < 0 \text{ and } \frac{\underline{\xi}^+}{\underline{\xi}^-}\lim_{x\to-\infty}\frac{F(\mu(x-1))}{F(\mu(x))} > 1,$$

where

the probability of occurrence of payoffs in the neighborhood of v is less than $\frac{-v'}{v-v'}$,

and

the probability of occurrence of payoffs in the neighborhood of v' is more than $\frac{v}{v-v'}$.

This is a contradiction of property (1-2) in Proposition 1.

Hence, we have proven that properties (i) through (v) are necessary.

Next, we show that properties (i) through (v) are sufficient. Suppose that Γ^∞ satisfies properties (i) through (v). Properties (i), (iii) and (iv) say that

$$[v(p) > 0] \Rightarrow [\frac{\overline{\xi}^+}{\overline{\xi}^-} > 1] \text{ and } [v(p) < 0] \Rightarrow [\frac{\overline{\xi}^+}{\overline{\xi}^-} < 1],$$

which together with property (ii), implies

$$[v(p)>0] \Rightarrow [(\frac{\bar{\xi}^{+}}{\bar{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} = \frac{\bar{\xi}^{+}}{\bar{\xi}^{-}} > 1],$$

and

$$[v(p)<0] \Rightarrow [(\frac{\bar{\xi}^{+}}{\bar{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} = \frac{\bar{\xi}^{+}}{\bar{\xi}^{-}} < 1].$$

Properties (i), (iii) and (v) say that

$$[v(p)<0] \Rightarrow [\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} < 1].$$

Hence, one gets that for every $(f,p) \in \Phi \times \Xi(\mu)$,

$$[v(p)>0] \Rightarrow [\frac{\bar{\xi}^{+}}{\bar{\xi}^{-}} > 1 \; and \; (\frac{\bar{\xi}^{+}}{\bar{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} > 1],$$

and

$$[v(p)<0] \Rightarrow [\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))} < 1 \; and \; (\frac{\bar{\xi}^{+}}{\bar{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} < 1],$$

which , together with Proposition 2, imply that Γ^{∞} is efficient.

From these observations, we have completed the proof of the theorem.

5. Concluding Remarks

This paper characterized efficient sequences of learning rules under payoff-uncertainty and context-uncertainty. An ordinal decision maker, who is motivated by looking after her own interest and never experiments with undesirable actions, may fail to gather unbiased information through daily decision making, and therefore, fail to choose the efficient action in the long run. This paper showed that we can construct a sequence of learning rules according to which even such a self-interested decision maker can succeed to gather unbiased information at any time, and eventually choose the efficient action.

The future research may be required to generalize our analysis to several cases such as a case in which a decision maker chooses among multiple uncertain actions and a case in which the other individuals' experiences are available.[8] We would like to emphasize that, in any case, it is important for the understanding of choice under uncertainty that how several psychological, motivational factors are adjusted is explicitly modeled in an appropriate way.

[8] There is a growing literature of *cognitive dissonance* in economic psychology such as Rabin (1995) and Carrillo and Mariotti (1997) in which an individual is apt to exclude information which contradicts her subjective belief. We may strengthen the result of the present paper by taking cognitive dissonance into account. Especially, cognitive dissonance may interrupt a decision maker from gathering information about the other decision makers' experiences, because these may be inconsistent with her belief.

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Appendix A: Proof of Proposition 1

We will prove only property (1-1), because property (1-2) can be proved in the same way.

Suppose $\frac{\bar{\xi}^{+}}{\bar{\xi}^{-}} < 1$. Then, there exists a positive integer x and a positive real number less than unity, $\varepsilon \in (0,1)$, such that

$$\frac{\xi^{+}(x'-1)}{\xi^{-}(x')} \leq \varepsilon < 1 \text{ for all } x' \geq x.$$

Since

$$g_m(x') = \{\prod_{x''=x+1}^{x'} \frac{\xi^{+}(x''-1)}{\xi^{-}(x'')}\} g_m(x) \text{ for all } x' > x,$$

one gets

$$\sum_{x'=x+1}^{\bar{x}_m} g_m(x') = \sum_{x'=x+1}^{\bar{x}_m} \{\prod_{x''=x+1}^{x'} \frac{\xi^{+}(x''-1)}{\xi^{-}(x'')}\} g_m(x)$$
$$\leq \sum_{x'=x+1}^{\bar{x}_m} \varepsilon^{x'-x} g_m(x) = \varepsilon(\frac{1-\varepsilon^{\bar{x}_m - x}}{1-\varepsilon}) g_m(x) \leq (\frac{\varepsilon}{1-\varepsilon}) g_m(x).$$

This implies that the probability that the state of mind is higher than x is at most $(\frac{\varepsilon}{1-\varepsilon})$ times as large as the probability that the state of mind is equal to x. Hence, the probability that the state of mind is higher than x never converges to unity as $m \to +\infty$, and therefore, $\Lambda^{\infty} < 1$. This is a contradiction.

Suppose $(\frac{\bar{\xi}^{+}}{\bar{\xi}^{-}})(\frac{\underline{\xi}^{+}}{\underline{\xi}^{-}} \lim_{x \to -\infty} \frac{F(\mu(x-1))}{F(\mu(x))})^{\gamma} < 1$. Then, there exist a positive integer x, a negative integer $\tilde{x}$, and positive real numbers $\varepsilon \neq 1$ and $\tilde{\varepsilon} \neq 1$ such that

$$\frac{\xi^{+}(x'-1)}{\xi^{-}(x')} \leq \varepsilon \text{ for all } x' \geq x,$$

$$\frac{\xi^{+}(x'-1)}{\xi^{-}(x')} \leq \tilde{\varepsilon} \text{ for all } x' \leq x,$$

and

$$\varepsilon(\tilde{\varepsilon})^{\gamma} < 1.$$

Hence,

$$\sum_{x'=x+1}^{\bar{x}_m} g_m(x') = \sum_{x'=x+1}^{\bar{x}_m} \{\prod_{x''=x+1}^{x'} \frac{\xi^{+}(x''-1)}{\xi^{-}(x'')}\} g_m(x) \leq \varepsilon(\frac{1-\varepsilon^{\bar{x}_m - x}}{1-\varepsilon}) g_m(x),$$

and

$$\sum_{x'=\underline{x}_m}^{\tilde{x}-1} g_m(x') = \sum_{x'=\underline{x}_m}^{\tilde{x}-1} \{\prod_{x''=x'}^{\tilde{x}-1} \frac{\xi^-(x''+1)}{\xi^+(x'')}\} g_m(\tilde{x})$$

$$\geq \sum_{x'=x+1}^{\bar{x}^m} (\tilde{\varepsilon})^{-(\tilde{x}-\underline{x}_m)} g_m(\tilde{x}) = (\frac{1-(\tilde{\varepsilon})^{\underline{x}_m-\tilde{x}}}{\tilde{\varepsilon}-1}) g_m(\tilde{x}),$$

and therefore,

$$\frac{\sum_{x'=x+1}^{\bar{x}_m} g_m(x')}{\sum_{x'=\underline{x}_m}^{\tilde{x}-1} g_m(x')} \leq (\frac{\varepsilon(\tilde{\varepsilon}-1)}{1-\varepsilon})(\frac{1-\varepsilon^{\bar{x}_m-x}}{1-(\tilde{\varepsilon})^{\underline{x}_m-\tilde{x}}})(\prod_{x''=x'+1}^{x} \frac{\xi^+(x''-1)}{\xi^-(x'')}).$$

We can check that the right hand side of this inequality never converges to zero as $m \to +\infty$: Suppose $\varepsilon > 1$. We must note

$$(\frac{\varepsilon(\tilde{\varepsilon}-1)}{1-\varepsilon})(\frac{1-\varepsilon^{\bar{x}_m-x}}{1-(\tilde{\varepsilon})^{\underline{x}_m-\tilde{x}}})(\prod_{x''=x'+1}^{x} \frac{\xi^+(x''-1)}{\xi^-(x'')})$$

$$= (\frac{\varepsilon(\tilde{\varepsilon}-1)}{1-\varepsilon})(\frac{1-\varepsilon^{\bar{x}_m-x}}{1-(\tilde{\varepsilon})^{-(\frac{-\underline{x}_m+\tilde{x}}{\bar{x}_m-x})(\bar{x}_m-x)}})(\prod_{x''=x'+1}^{x} \frac{\xi^+(x''-1)}{\xi^-(x'')})$$

$$= (\frac{\varepsilon(\tilde{\varepsilon}-1)}{1-\varepsilon})(\frac{1-\varepsilon^{-(\bar{x}_m-x)}}{(\varepsilon(\tilde{\varepsilon})^{(\frac{-\underline{x}_m+\tilde{x}}{\bar{x}_m-x})})^{-(\bar{x}_m-x)} - \varepsilon^{-(\bar{x}_m-x)}})(\prod_{x''=x'+1}^{x} \frac{\xi^+(x''-1)}{\xi^-(x'')}),$$

which converges to zero as $m \to +\infty$, because $\lim_{m\to+\infty} \varepsilon^{-(\bar{x}_m-x)} = 0$, $\lim_{m\to+\infty} \frac{-\underline{x}_m+\tilde{x}}{\bar{x}_m-x} = \gamma$, and $\lim_{m\to+\infty} (\varepsilon(\tilde{\varepsilon})^{\gamma})^{-(\bar{x}_m-x)} = +\infty$. This is a contradiction of $\Lambda^{\infty} = 1$, because the stationary probability that the state of mind is more than x is zero. Similarly we can check the case of $\varepsilon < 1$. Hence, we have proven property (1-1).

Appendix B: Proof of Proposition 2

We will prove only property (2-1), because property (2-2) can be proved in the same way.

We must note that there exist a positive integer x, a negative integer $\tilde{x}$, and positive real numbers $\varepsilon > 1$ and $\tilde{\varepsilon} \neq 1$ such that

$$\frac{\xi^{+}(x'-1)}{\xi^{-}(x')} \geq \varepsilon \text{ for all } x' \geq x,$$

$$\frac{\xi^{+}(x'-1)}{\xi^{-}(x')} \geq \tilde{\varepsilon} \text{ for all } x' \leq x,$$

and

$$\varepsilon(\tilde{\varepsilon})^{\gamma} \geq 1.$$

Hence, one gets

$$\sum_{x'=x+1}^{\bar{x}_m} g_m(x') = \sum_{x'=x+1}^{\bar{x}_m} \{\prod_{x''=x+1}^{x'} \frac{\xi^{+}(x''-1)}{\xi^{-}(x'')}\} g_m(x) \geq \varepsilon(\frac{1-\varepsilon^{\bar{x}_m - x}}{1-\varepsilon}) g_m(x),$$

and

$$\begin{aligned}
\sum_{x'=\underline{x}_m}^{\tilde{x}-1} g_m(x') &= \sum_{x'=\underline{x}_m}^{\tilde{x}-1} \{\prod_{x''=x'}^{\tilde{x}-1} \frac{\xi^{-}(x''+1)}{\xi^{+}(x'')}\} g_m(\tilde{x}) \\
&\leq \sum_{x'=\underline{x}_m}^{\tilde{x}-1} (\tilde{\varepsilon})^{-(x'-\underline{x}_m)} g_m(\tilde{x}) = (\frac{1-(\tilde{\varepsilon})^{\underline{x}_m-\tilde{x}}}{\tilde{\varepsilon}-1}) g_m(\tilde{x}) \\
&= (\frac{1-(\tilde{\varepsilon})^{\underline{x}_m-\tilde{x}}}{\tilde{\varepsilon}-1})(\prod_{x'=\tilde{x}+1}^{x} \frac{\xi^{-}(x')}{\xi^{+}(x'-1)}) g_m(x),
\end{aligned}$$

and therefore,

$$\begin{aligned}
\frac{\sum_{x'=x+1}^{\bar{x}_m} g_m(x')}{\sum_{x'=\underline{x}_m}^{\tilde{x}-1} g_m(x')} &\geq (\frac{\varepsilon(\tilde{\varepsilon}-1)}{1-\varepsilon})(\frac{1-\varepsilon^{\bar{x}_m-x}}{1-(\tilde{\varepsilon})^{\underline{x}_m-\tilde{x}}})(\prod_{x'=\tilde{x}+1}^{x} \frac{\xi^{+}(x'-1)}{\xi^{-}(x')}) \\
&= (\frac{\varepsilon(\tilde{\varepsilon}-1)}{1-\varepsilon})(\frac{1-\varepsilon^{-(\bar{x}_m-x)}}{(\varepsilon(\tilde{\varepsilon})^{(\frac{-\underline{x}_m+\tilde{x}}{\bar{x}_m-x})})^{-(\bar{x}_m-x)} - \varepsilon^{-(\bar{x}_m-x)}})(\prod_{x''=x'+1}^{x} \frac{\xi^{+}(x''-1)}{\xi^{-}(x'')}).
\end{aligned}$$

The right hand side of this inequality diverges to $+\infty$ as $m \to +\infty$, because $\lim_{m\to+\infty} \varepsilon^{-(\bar{x}_m-x)} = 0$, $\lim_{m\to+\infty} \frac{-\underline{x}_m+\tilde{x}}{\bar{x}_m-x} = \gamma$, and $\lim_{m\to+\infty} (\varepsilon(\tilde{\varepsilon})^{\gamma})^{-(\bar{x}_m-x)} = 0$. Moreover, we must note

$$\sum_{x'=\tilde{x}}^{x} g_m(x') = \sum_{x'=\tilde{x}}^{x} (\prod_{x''=\tilde{x}+1}^{x'} \frac{\xi^{-}(x'')}{\xi^{+}(x''-1)}) g_m(x),$$

and therefore,

$$\frac{\sum_{x'=x+1}^{\bar{x}_m} g_m(x')}{\sum_{x'=\tilde{x}}^{x} g_m(x')} \geq \varepsilon\left(\frac{1-\varepsilon^{\bar{x}_m - x}}{1-\varepsilon}\right)\left\{\sum_{x'=\tilde{x}}^{x}\left(\prod_{x''=\tilde{x}+1}^{x'} \frac{\xi^-(x'')}{\xi^+(x''-1)}\right)\right\}^{-1}.$$

The right hand side of this inequality diverges to infinity as $m \to +\infty$ because $\lim_{m\to+\infty} \varepsilon^{\bar{x}_m - x} = +\infty$.

The above observations imply that the stationary probability that the state of mind is more than x is unity. Since we can choose x as large as possible, we have proved $\Lambda^\infty = 1$, i.e., property (2-1).