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Multiple Equilibria in the Endogenous Economic Growth Model with the Cash-in-Advance Constraint

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by

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Abstract

In previous studies, it is well known that equilibria can be indeterminate when the cash-in-advance constraint is binding. This paper extends these previous studies to the case where the economy grows endogenously. The extension is particularly noteworthy because temporarily different growth rates can make the levels of capital stocks persistently different in the endogenous growth model. We show that when the cash-in-advance constraint applies only to consumption, multiple equilibrium paths are more likely outcome in the endogenous growth economy. We also show that when the equilibrium path is a sunspot equilibrium, a deviation from the balanced growth path can accelerate the expected growth rates of consumption both in the short-run and in the long-run However, any deviation from the balanced growth path never improves social welfare and requires the role of discretionary monetary policy.

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1. Introduction

Recently, various attempts have been made to clarify indeterminacy of the equilibrium dynamic path in endogenous growth models. In particular, in two-sector growth models with external effects in production, several studies have shown that equilibria are indeterminate (see, for example, Boldrin and Rustichini (1994), Benhabib and Perli (1994), and Xies (1994)). Even in the one-sector model, Benhabib and Farmer (1994) have demonstrated that equilibria are indeterminate when there exists a strong external effect of labor in production[1]. However, in the one sector model, Boldrin and Rustichini (1994) have proved that indeterminacy can be ruled out under fairly weak assumptions that are consistent with those often adopted in the applied literature.[2]

The purpose of this paper is to investigate indeterminacy of equilibrium path in the one-sector endogenous growth model when the cash-in-advance constraint is binding. We show that when liquidity constraints are applied only to consumption purchases, there exist multiple equilibrium paths for reasonable parameters in a simple endogenous growth model. In particular, when the equilibrium path is a sunspot equilibrium, a deviation from the balanced growth path can accelerate the expected growth rates of consumption both in the short-run and in the long-run. However, we also show that any deviation from the balanced growth path never improves social welfare.

In previous studies, there exist a number of studies which investigated indeterminacy of equilibrium dynamics in monetary economies (for example, Brock (1974), Grandmont (1985), Matsuyama (1990), and Fukuda (1993)). In particular, Woodford (1994) has shown that there exist sunspot equilibria in cash-in-advance models.[3] However, few studies have analyzed indeterminacy in cash-in-advance economies with capital accumulation. In addition, none of these studies investigated endogenous growth

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1 Benhabib and Gali (1995) also pointed out the possibility of multiple equilibrium paths in one-sector models of "poverty trap", where at least one of those paths involves sustained endogenous growth.

2 For one-sector models without endogenous growth, there are some exceptional studies which showed indeterminacy of the equilibrium path in one sector models with externalities. These studies include Kehoe-Levine-Romer (1991) which assumed the external effect of aggregate consumption, Kehoe (1991) which assumed the negative externality of aggregate capital, and Spear (1991) which assumed the externality of tomorrow's aggregate capital stock.

3 Michener and Ravikumar (1994) showed the existence of deterministic cycles and chaos in a similar cash-in-advance model.

models. This paper shows that extending the analysis to the endogenous growth case, indeterminacy of equilibrium dynamics becomes more likely outcome in the cash-in-advance model.

Except that we allow no decreasing returns to scale in production, the model in this paper is the same as those of Stockman (1981), Abel (1985), and Cooley and Hansen (1989). In the endogenous growth framework, similar models have been investigated by Marquis and Reffett (1991, 1995), Gomme (1993), Mino (1994), and Jones and Manuelli (1995). In the analysis, we consider the case where the cash-in-advance constraint applies only to consumption. We also suppose that there exists no intrinsic shock in the economy. In this framework, the rate of consumption growth is constant in the balanced growth path. However, when the cash-in-advance constraint is strictly binding, the equilibrium path can be multiple and the rate of consumption growth can fluctuate around the balanced growth path. In particular, self-fulfilling expectations play a crucial role in determining the long-run equilibrium growth.

Our extension to the endogenous growth model is particularly noteworthy because when there exist multiple convergence equilibrium paths, two countries which start from the same initial income level can follow different growth path from then on. In particular, even if they may display a common growth rate in the long run, temporarily different growth rates can make the levels of capital stocks persistently different (see Figure 1). Thus, our result can explain why certain countries never catch up with the leader even if they started out from almost similar conditions.

In the new growth literature, there exist a number of studies that have pointed out the potential role of self-fulfilling expectations in determining the long-run economic growth. For example, assuming prefect-foresight, Krugman (1991) and Matsuyama (1991) have presented growth models of increasing returns where the choice of the long-run equilibrium depends on self-fulfilling expectations as well as history. In addition, Shigoka (1995) and Drugeon and Wagniolle (1996) have shown the existence of sunspot equilibria in endogenous growth models. Because self-fulfilling expectations affect resource allocation in sunspot equilibria, these two studies also imply the potential role of self-fulfilling expectations for the long-run growth.

The analysis of this paper is similar to these previous studies in that we investigate the role of self-fulfilling expectations in determining the long-run economic growth rate. However, Krugman and Matsuyama focused on multiple steady state equilibria rather than multiple transition paths around a unique balanced growth path. In addition, although all of these previous studies used multi-sector models with no outside money, our analysis focuses on the role of self-fulfilling expectations in the one-sector growth model with outside money. In particular, we show that when there exists a sunspot equilibrium, extraneous uncertainty can not only make equilibrium paths highly volatile but also may accelerate the expected growth rates of consumption. We also show that volatile equilibrium paths are never superior to the balanced growth path in terms of social welfare and call for the role of discretionary monetary policy.

Our result that extraneous uncertainty can accelerate the expected growth rates of consumption may be surprising. However, it is comparable to the Levhari and Srinivasan's (1969) result that intrinsic uncertainty may increase the expected amount of consumption when the degree of risk aversion is large. In fact, in our model of the cash-in-advance constraint, a sunspot equilibrium can accelerate the expected growth rates of consumption if and only if the degree of risk aversion is large.

The paper proceeds as follows. Section 2 first analyzes a cash-in-advance model without endogenous growth and shows that indeterminacy can arise around a steady state. Section 3 then extends the analysis to the one-sector model of endogenous growth and demonstrates that indeterminacy can arise around the balanced growth path. Section 4 investigates how the existence of a stationary sunspot equilibrium affects the expected growth rate in our cash-in-advance model. Section 5 discusses welfare implications and the role of monetary policy. Section 6 summarizes our main results and refers to their implications.

2. The Economy without Endogenous Growth

Before investigating indeterminacy in the endogenous growth model, this section first introduces a formal cash-in-advance model without endogenous growth and shows that indeterminacy can arise around the steady state. The model follows the pure currency cash-in-advance framework described in Stockman (1981), Abel (1985), and Cooley and Hansen (1989). However, throughout the text, we assume that liquidity constraints are applied only to consumption purchases.

In the following model, a representative individual has the following expected utility function

$$(1)\quad E_t \sum_{i=0}^{\infty} \beta^i u(c_{t+i}), \qquad 0 < \beta < 1,\ u' > 0,\ u'' < 0,$$

where ß is a discount factor and c_{t+i} is consumption. E_t is the conditional expectation operator based on the information set at period t.

The individual can hold two assets, money and capital. Letting k_t and M_t denote the individual capital stock and the nominal money balances, respectively, held at the beginning of period t, the budget constraint can be written as

(2) $c_t + k_{t+1} + M_{t+1}/P_t = f(k_t) + k_t + (M_t+T_t)/P_t$,

where P_t is the money price of the homogeneous good and T_t is the nominal money transfer received at the beginning of period t. For analytical simplicity, we assume that the capital does not depreciate.

The term $f(k_t)$ is the production function. In this section, we assume that $f(k_t)$ is increasing and concave in k_t (that is, $f'(k_t) > 0$ and $f''(k_t) < 0$) and satisfies the Inada conditions (that is, $\lim_{k\to 0} f(k_t) = \infty$ and $\lim_{k\to\infty} f(k_t) = 0$). Under this assumption, the model shows no unbounded growth because there exists no exogenous technological shock in the economy.

As for the cash-in-advance constraint, we consider the case where the constraint applies only to consumption purchases. Under this constraint, the nominal value of consumption during period t is less than or equal to the money on hand at the beginning of period t,

(3) $c_t \leq (M_t+T_t)/P_t$.

A representative individual's optimization problem is to maximize the expected utility function (1) subject to the budget constraint (2) and the cash-in-advance constraint (3). The constraint optimization problem can be solved using the following Lagrangean:

$$(4)\quad L = E_t\sum_{i=0}^{\infty} \beta^i [\, u(c_{t+i}) + \lambda_{t+i}\{f(k_{t+i}) + (M_{t+i} + T_{t+i})/P_{t+i} - c_{t+i} - k_{t+1+i} + k_{t+i} - M_{t+1+i}/P_{t+i}\} + \gamma_{t+i}\{(M_{t+i} + T_{t+i})/P_{t+i} - c_{t+i}\}],$$

Differentiating (4) with respect to c_t, k_{t+1}, and M_{t+1}, we obtain

(5a) $u'(c_t) = \lambda_t + \gamma_t$,

(5b) $\lambda_t = \beta E_t \lambda_{t+1} (1+f'(k_{t+1}))$,

(5c) $\lambda_t / P_t = \beta E_t [(\lambda_{t+1} + \gamma_{t+1}) / P_{t+1}]$.

Equations (5a) - (5c) lead to

(6) $E_t\,[(P_t/P_{t+1})\,u'(c_t)] = \beta\, E_t\,[(P_{t+1}/P_{t+2})\,u'(c_{t+1})\,\{1 + f'(k_{t+1})\}]$.

Given equation (6), the model is closed by specifying government behavior. In the following analysis, we assume that the (per capita) nominal money supply M_t is exogenously determined as follows:

(7) $M_t = z\,M_{t-1}$ for all t.

We also assume that government adjusts its transfer payments T_t endogenously to balance its budget constraint :

(8) $T_t = M_{t+1} - M_t$.

Under this government behavior, the equilibrium condition of good market at period t is described as follows :

(9) $c_t = f(k_t) + k_t - k_{t+1}$.

In our model, the cash-in-advance constraint is binding if and only if $P_t/P_{t+1} < 1 + f'(k_{t+1})$. Assuming that the cash-in-advance constraint is always binding, it holds that $P_t = M_{t+1}/c_t$. Hence, substituting this into (6) and assuming perfect foresight yield

(10) $\beta\, c_t\, c_{t+2}\, u'(c_{t+2})\,\{1 + f'(k_{t+1})\} = c_{t+1}^{\,2}\, u'(c_{t+1})$.

Equations (9) and (10) determine an equilibrium sequence of $\{c_t, k_t\}$for our economy if and only if it satisfies the transversality condition such that

(11) $\lim_{t\to\infty} \beta^t\, u'(c_t)\, k_{t+1} = 0$.

The steady state is characterized by a constant capital stock, $k_t = k$, and a constant level of consumption, $c_t = c$. Thus, substituting $k_t = k$ and $c_t = c$ into (10), we can show that the steady state capital stock k is characterized by

(12) $\beta\,\{1 + f'(k)\} = 1$.

Because the Inada condition is satisfied, k which satisfies (12) exists uniquely. In addition, the steady state capital stock k is equal to the steady state capital stock without cash-in-advance constraint.

The analysis of dynamic behavior of capital stock requires tedious algebra in the cash-in-advance model. Substituting (9) into (10), we can obtain a third-order difference equation in k_t as follows.

$$(13)\quad \beta\,[f(k_t) + k_t - k_{t+1}]\,[f(k_{t+2}) + k_{t+2} - k_{t+3}]\,u'(f(k_{t+2}) + k_{t+2} - k_{t+3})\,\{1 + f'(k_{t+1})\}$$
$$= [f(k_{t+2}) + k_{t+2} - k_{t+3}]^2\,u'(f(k_{t+2}) + k_{t+2} - k_{t+3}).$$

The strategy of the analysis to derive the local stability condition of (13) is to linearize (13) around the steady state $k_t = k$ and then to analyze the characteristic roots of the linearized system. Let $\bar{K}_t = k_t - k$ denote the deviation of k_t from its steady state value. Then, linearizing (13) around $k_t = k$ yields

$$(14)\quad (1\text{-R})\,\bar{K}_{t+3} - [(3\text{-2R}) + f'\,(1\text{-R})]\,\bar{K}_{t+2} + [(1+f')(2\text{-R}) + 1 - \beta f f'']\,\bar{K}_{t+1} - (1+f')\,\bar{K}_t$$
$$= 0,$$

where $\text{R} \equiv - c\,u''(c) / u'(c)$.

The associated characteristic equation is

$$(15)\quad h(X) \equiv (1\text{-R})\,X^3 - [(3\text{-2R}) + f'(1\text{-R})]\,X^2 + [(1+f')(2\text{-R}) + 1 - \beta f f'']\,X - (1+f')$$
$$= 0.$$

Thus, the linearized equation (14) satisfies the saddle point property if and only if one characteristic root is less than unity and the other two are greater than unity in their absolute values. It is easy to show that

$$(16a)\quad h(0) = - (1 + f') < 0,$$
$$(16b)\quad h(1) = - \beta f f'' > 0,$$
$$(16c)\quad h(-1) = - (2 + f')(3 - 2\text{R}) - (2 + f' - \beta f f'').$$

Because $h'(X) > 0$ when $0 < \text{R} < 1$ and $X < 0$, (16a) and (16b) imply that the (local) saddle point stability is always satisfied when $0 < \text{R} < 1$. Even when $\text{R} > 1$, (16b) and (16c) imply that the dynamic equation (13) has a saddle point path to the steady state if and only if $h(-1) < 0$, or equivalently $R < 2 - [\beta f f''/\{2(2+f')\}]$.

However, when two characteristic roots are less than unity in their absolute values, the linearized equation (14) has multiple convergence equilibrium paths to the steady state. Thus, when

$$(17) \qquad R > 2 - [\beta f f''/\{2(2+f')\}],$$

the third-order difference equation (13) has multiple convergence equilibrium paths to the steady state. That is, when the degree of risk aversion is large,[4] the dynamic capital accumulation path in our cash-in-advance model is indeterminate even with the common initial allocation of the capital stock.

The source of indeterminate dynamic paths arises from the conflict between intertemporal substitution and income effects. That is, a rise of future prices always has a negative substitution effect on the current money demand (or a positive substitution effect on the current demand for consumption), while it always has a positive income effect on the current money demand (or a negative income effect on the current demand for consumption). Hence, the total effect of raised future prices on the current money demand is always ambiguous. This source of indeterminacy is analogous to that in the overlapping generations models of Grandmont (1985) and the money-in-the-utility function model of Fukuda (1993).

3. The Economy of Endogenous Growth

This section considers the one-sector model of endogenous growth with the cash-in-advance constraint. As in the last section, a representative individual maximizes the expected utility function (1) subject to the constraints (2) and (3). However, the production function $f(k_t)$ no longer shows decreasing returns to scale. Instead, we assume that the production function exhibits constant returns to scale in k_t, that is,

$$(18) \quad f(k_t) = A\, k_t.$$

We also assume that the utility function takes the form of the constant relative risk

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4 Based on consumption data, previous empirical studies have shown that the degree of relative risk aversion needs to be small to explain too smooth consumption. However, based on asset prices, other empirical studies have shown that the degree of risk aversion needs to be incredibly large to explain too volatile asset prices.

aversion (CRRA) such that

(19) $u(c_t) = c_t^{1-R} / (1-R), \qquad R > 0.$

The first-order condition of this modified optimization problem can be obtained by substituting (18) and (19) into (5a) - (5c).

(20a) $1 / c_t^{R} = \lambda_t + \gamma_t,$

(20b) $\lambda_t = \beta E_t \lambda_{t+1} (1+A),$

(20c) $\lambda_t / P_t = \beta E_t [(\lambda_{t+1} + \gamma_{t+1}) / P_{t+1}].$

Equations (20a) - (20c) lead to

(21) $E_t [(P_t / P_{t+1}) (1/c_{t+1}^{R})] = \beta (1+A) E_t [(P_{t+1}/P_{t+2}) (1/c_{t+2}^{R})].$

As in the last section, the cash-in-advance constraint is binding if and only if $P_t / P_{t+1} < f'(k_{t+1}) = 1+A$. Assuming that the cash-in-advance constraint is binding, it holds that $P_t = M_{t+1} / c_t$, so that (21) is rewritten as

(22) $E_t [(c_{t+1}^{1-R} / c_t)] = \beta(1+A) E_t [(c_{t+2}^{1-R} / c_{t+1})].$

Then, the balanced growth path, if it exists, is characterized by a constant growth rate of capital stock, $k^g \equiv k_{t+1}/k_t$, and a constant growth rate of consumption, $c^g \equiv c_{t+1}/c_t$. Substituting $c_{t+1}/c_t = c^g$ into (10) and noting that $c_t = (1+A)k_t - k_{t+1}$ in the good market equilibrium, we can show that the balanced growth path is characterized by

(23) $c^g = k^g = \{\beta (1+A)\}^{1/R}.$

The balanced growth path, however, needs to satisfy the transversality condition (11). Under our assumptions, the condition (11) is satisfied if and only if $c_{t+1}/c_t = k_{t+1}/k_t < 1+A$. Thus, to assure the transversality condition, the following analysis assumes that

(24) $\{\beta (1+A)\}^{1/R} < (1+A).$

In order to investigate dynamic behavior of capital stock, substitute the good market equilibrium condition (9) into (22). Then, assuming perfect foresight and defining $c^g_t \equiv c_{t+1}/c_t$ and $k^g_t \equiv k_{t+1}/k_t$, we can obtain a first-order difference equation of (c^g_t, k^g_t) as follows.

(25a) $c^g_{t+1} = [c^g_t / \{\beta(1+A)\}]^{1/(1-R)}$,

(25b) $k^g_{t+1} = (1+A) - c^g_t\,[(1+A)(1/k^g_t) - 1]$.

In order to derive the local stability condition of (25a) and (25b), let $\bar{C}^g_t \equiv c^g_t - c^g$ and $\bar{K}^g_t \equiv k^g_t - k^g$ to denote the deviation of c^g_t and k^g_t from their balanced growth path. Then, linearizing (25a) and (25b) around c^g and k^g yields

$$(26) \quad \begin{pmatrix} \bar{C}^g_{t+1} \\ \bar{K}^g_{t+1} \end{pmatrix} = \begin{pmatrix} 1/(1-R) & 0 \\ 1-B & B \end{pmatrix} \begin{pmatrix} \bar{C}^g_t \\ \bar{K}^g_t \end{pmatrix}$$

where $B \equiv (1+A) / [\beta(1+A)]^{1/R}$.

The linearized dynamics (26) has a unique non-diverging path if and only if both of two characteristic roots in the above matrix are greater than unity in their absolute values. It is easy to see that two characteristic roots are B and 1/(1-R) in the above matrix. Since $B > 1$ under the assumption (24), this implies that the balanced growth path is the only equilibrium path if and only if $|1/(1-R)| > 1$, or equivalently $R < 2$.

On the other hand, when either of characteristic roots are less than unity in their absolute values in the above matrix, the linearized dynamics has multiple convergence equilibrium paths. Thus, if

(27) $R > 2$,

the dynamics (26) has multiple convergence equilibrium paths to the balanced growth path. The condition (27) is analogous to (17) because we can obtain (27) by substituting $f(k_t) = A\,k_t$ into (17). In addition, the source of indeterminate dynamic paths is still the conflict between intertemporal substitution and income effects. However, for the existence of multiple convergence paths, the condition (27) allows relatively smaller degree of risk aversion than the condition (17).

When there exist multiple convergence equilibrium paths, the rate of economic growth can vary depending on how people expect the future growth path. Thus, even if two countries start from the same initial capital stock, they can follow different equilibrium path from then on and display a common growth rate only in the long run.

In addition, because temporarily different growth rates can make the levels of capital stocks persistently different, certain countries never catch-up with the leader even if they started out from almost similar conditions.

4. The Effects of Sunspots on Economic Growth

Until previous sections, we have focused on multiple equilibria under perfect foresight. However, previous studies have shown that whenever there exist multiple convergence equilibrium paths, there exist k-state stationary sunspot equilibria in any neighborhood of the steady state (see, among others, Woodford (1986) and Chiappori, Geoffard, and Guesnerie (1992)). Their results imply that under the condition (27) (that is, $R > 2$), there exist stationary sunspot equilibria around the balanced growth path in our model. Assuming that $R > 2$, this section investigates how some stationary sunspots affect the expected growth rate in our cash-in-advance model.

Suppose that at period t, there is an extraneous shock z_t drawn from a k-element set ; $z_t \in \{1, \ldots, k\}$. Suppose also that this extraneous shock z_t evolves according to a stationary k-state Markov chain. Denote the growth rate of consumption $c^g_t = c_{t+1}/c_t$ at state z_t by $c^g(z_t)$ and assume no uncertainty in $c^g(z_t)$ at period t. Then, (22) implies that

$$(28) \qquad c^g(z_t) = \beta(1+A)\, E_t\, [c^g(z_{t+1})]^{1-R}.$$

When $R > 2$, it is easy to show that there exist $0 < \pi_{ij} < 1$ (i = 1, 2, and j = 1, 2) and $c^g(1) > c^g > c^g(2)$ such that

$$(29) \qquad c^g(i) = \beta(1+A)\, [\pi_{i1}\, c^g(1)^{1-R} + \pi_{i2}\, c^g(2)^{1-R}] \qquad \text{for i = 1 and 2,}$$

where $\pi_{ij} \equiv \text{prob}\,(z_{t+1} = j \mid z_t = i)$. In addition, because $c^g(1) > c^g > c^g(2)$, there exist $0 < q_{31} < 1$, $0 < q_{32} < 1$, and $0 < q_{31} + q_{32} < 1$ such that $c^g = [q_{31}\, c^g(1) + q_{32}\, c^g(2)]/(q_{31} + q_{32})$. Thus, defining $c^g(3) \equiv c^g$, $q_{11} \equiv \pi_{11} - q_{31}$, $q_{22} \equiv \pi_{22} - q_{32}$, and $q_{13} \equiv q_{23} \equiv q_{31} + q_{32}$, (29) implies the existence of three-state stationary sunspot equilibria which satisfy

$$(30) \qquad c^g(i) = \beta(1+A)\, [q_{i1}\, c^g(1)^{1-R} + q_{i2}\, c^g(2)^{1-R} + q_{i3}\, c^g(3)^{1-R}] \qquad \text{for i = 1, 2, and 3,}$$

where $q_{ij} \equiv \text{prob}\,(z_{t+1} = j \mid z_t = i)$ for all i = 1, 2, 3 and j = 1, 2, 3.

These results imply that when $R > 2$, there exist three-state stationary sunspot equilibria where the growth rate of consumption is greater than the balanced growth path at state 1, is less than the balanced growth path at state 2, and is equal to the balanced growth path at state 3 (see Figure 2). Assuming $R > 2$, we consider such there exist three-state stationary sunspot equilibria which include the balanced growth path for one state variable, that is, $c(1) > c^g$, $c(2) < c^g$, and $c(3) = c^g$ where $c^g \equiv \{\beta(1+A)\}^{1/R}$. Then, when the initial period is state 3, the following proposition holds.

Proposition 1: Assume that $R > 2$ and suppose that the initial growth rate of consumption is equal to that in the balanced growth path. Then, the expected growth rate in the three-state stationary sunspot equilibria is greater than that in the balanced growth path at the following periods.

Proof: First, Jensen's inequality implies that

$$(31)\quad E_t[c^g(z_{t+1})]^{1-R} > [E_t\, c^g(z_{t+1})]^{1-R}$$

when $R > 2$. Thus, because of the law of iterated projections, the iterative application of Jensen's inequality leads to $E_t\,[E_{t+1}[\ \dots\ E_{t+n-1}[c^g(z_{t+n})]^{1-R}]^{1-R}]^{1-R} > [E_t\, c^g(z_{t+n})]^{n(1-R)}$, for all positive integer n. Since $c^g(z_t) = [\beta(1+A)]^{n(1-R)} E_t\,[E_{t+1}[\dots E_{t+n-1}[c^g(z_{t+n})]^{1-R}]^{1-R}]^{1-R}$ and $c^g(3) = [\beta(1+A)]^{n(1-R)} c^g(3)^{n(1-R)}$ when $c^g(3) = \{\beta(1+A)\}^{1/R}$, this indicates that when $c^g(z_t) = c^g(3) = \{\beta(1+A)\}^{1/R}$, $c^g(3)^{n(1-R)} > [E_t\, c^g(z_{t+n})]^{n(1-R)}$, or equivalently

$$(32)\quad c^g(3) < E_t\, c^g(z_{t+n})$$

because $R > 2$.

Note that $c^g(3)$ is the growth rate of consumption in the balanced growth path. Note also that the right-hand side of (32) is the conditional expectation of c^g_{t+n} formed at period t. Thus, the inequality (32) implies that when the economy is in the balanced growth path at the initial period, three-state stationary sunspot equilibria always increase the average growth rate of consumption at the following periods. [Q.E.D.]

This proposition implies that when $R > 2$, some stationary sunspot equilibria can accelerate the expected growth rate in our cash-in-advance economy. In particular, because growth rates keep fluctuating forever in stationary sunspot equilibria, people's variable expectations in the sunspot equilibria can cause significant difference in the expected rate of economic growth both in the short-run and in the long-run.

A crucial point in deriving the above result is that the degree of relative risk aversion is always greater than unity when stationary sunspot equilibria exist. In fact, unless R > 1, the inequality in equation (31) is reversed so that the expected growth rate of consumption in the balanced growth path becomes larger. As we mentioned in the introduction, this result is comparable to the Levhari and Srinivasan's (1969) result that intrinsic uncertainty may increase the expected amount of consumption when the degree of risk aversion is large. Intuitively, the result is derived because when R > 1, $E_t u'(c_{t+n})/u'(c_t) = E_t (c_t/c_{t+1})^R$ is convex in c_{t+1}. Given $E_t u'(c_{t+n})/u'(c_t)$, this convexity implies that uncertainty in c_{t+1} increases the expected value of c_{t+1}.

5. Welfare Implication and the Role of Monetary Policy

In the last section, we showed that extraneously stochastic fluctuations in some three-state stationary sunspot equilibria can accelerate the average growth rate of consumption in our model. However, the larger expected rate of growth in sunspot equilibria does not necessarily imply that the sunspots can improve economic welfare. In fact, we can show the following proposition.

Proposition 2: In our model without a positive externality in production, any deviation from the balanced growth path never improves the expected utility of a representative agent in equilibrium.

Proof: We first consider the case where the cash-in-advance constraint is not binding. In this case, $\gamma_t = 0$, so that (20a) and (20b) yield

(33) $1 / c_t^{\,R} = \beta(1+A)\, E_t (1 / c_{t+1}^{\,R}).$

Because $c_t = (1+A)k_t - k_{t+1}$ in the good market equilibrium, (21) leads to

(34) $1 / [(1+A)k_t - k_{t+1}]^R = \beta(1+A)\, E_t (1 / [(1+A)k_{t+1} - k_{t+2}]^R),$

or equivalently,

(35) $k^g_{t+1} = \{(1+A)+[\beta(1+A)]^{1/R}\} - (1+A)\,[\beta(1+A)]^{1/R}\, k^g_t,$

where $k^g_t \equiv k_{t+1}/k_t$. Since $k^g_t < 1+A$ under the transversality condition, this dynamic equation has the balanced growth path such that $k^g_t = \{\beta\,(1+A)\}^{1/R}$, which is equivalent to the balanced growth path in the cash-in-advance economy. In addition, under the assumption (24), this balanced growth path is the only equilibrium growth path which satisfies the transversality condition.

Since the expected utility of a representative individual is larger without the cash-in-advance constraint than with the cash-in-advance constraint, the above result implies that a representative individual with the cash-in-advance constraint can always enjoy the higher expected utility in the balanced growth path than in any sunspot equilibrium or multiple convergent paths. [Q.E.D.]

The above proposition on economic welfare implies a potential role of monetary policy[5]. Until now, we have assumed that the (per capita) money supply follows the rule such that $M_t = z\, M_{t-1}$ for all t. In the balanced growth, this money supply rule is both neutral and superneutral because the equilibrium growth rate is independently determined. However, when there exist multiple convergent paths or sunspot equilibria, another money supply rule may improve economic welfare.

To see this, let assume a new money supply rule such that M_t is determined endogenously to keep the inflation rate constant, i.e., $P_{t+1} / P_t = \Pi$. Then, equation (21) is written as

$$(36) \qquad E_t\,(1/c_{t+1}{}^{R}) = \beta(1+A)\, E_t\,(1/c_{t+2}{}^{R}),$$

which is equivalent to the first-order condition of the economy without cash-in-advance constraint. Therefore, to the extent that the money supply is determined endogenously to keep the inflation rate constant, the only equilibrium is the balanced growth path which is optimal in terms of social welfare. That is, a discretionary money supply rule which keeps the inflation rate constant always leads to the optimal resource allocation which may not be achieved under the rule which keeps the growth rate of money supply constant.

6. Concluding Remarks

This paper has investigated indeterminacy of equilibrium dynamic path in the one-sector endogenous growth model with the cash-in-advance constraint. A main result of this paper is that when the cash-in-advance constraint is binding, there exist multiple equilibrium paths for reasonable parameters in a standard AK model. In particular, we

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[5] In the different context, Fukuda (1997) discussed the role of monetary policy in eliminating non-convergent dynamic paths.

showed that when the equilibrium path is a sunspot equilibrium, a deviation from the balanced growth path can accelerate the expected growth rates of consumption both in the short-run and in the long-run, although it may deteriorate social welfare.

Throughout the analysis in the text, we assumed that the cash-in-advance constraint applies only to consumption. This assumption may, however, be crucial in deriving our main results. In fact, as we show in the Appendix, the balanced growth path is locally the only equilibrium dynamics when the cash-in-advance constraint applies investment purchases as well as consumption purchases. Therefore, in order for self-fulfilling expectations to play a crucial role in determining the long-run equilibrium growth, the form of liquidity constraint may be important in the one-sector growth model.

Appendix: The Model with Another Cash-in-Advance Constraint

In the text, we have focused on the case where liquidity constraints are applied only to consumption purchases. However, previous studies frequently adopted another formulation of the cash-in-advance constraint which requires that liquidity constraints are applied to investment purchases as well as consumption purchases. The purpose of this Appendix is to introduce this alternative cash-in-advance model and to show that our main results are somewhat sensitive to the form of liquidity constraints.

Except for the cash-in-advance constraint, the following model is exactly the same as the endogenous growth model in section 3.[6] Under a new cash-in-advance constraint, the nominal value of consumption plus investment during period t is less than or equal to the money on hand at the beginning of period t,

$$\text{(A1)} \qquad c_t + k_{t+1} - k_t \leq (M_t+T_t)/P_t.$$

Thus, a representative individual's optimization problem is to maximize the expected utility function (19) subject to a budget constraint (2) and a cash-in-advance constraint (A1). The constraint optimization problem can be solved using the following Lagrangean:

[6] In the model without endogenous growth, Abel (1985) showed that the equilibrium dynamic path is locally unique under the same cash-in-advance constraint.

$$(A2)\quad L = E_t \sum_{i=0}^{\infty} \beta^i [\, c_{t+i}^{1-R}/(1-R) + \lambda_{t+i}\{(1+A)k_{t+i} - k_{t+1+i} - c_{t+i} + (M_{t+i}+T_{t+i})/P_{t+i} - M_{t+1+i}/P_{t+i}\} + \gamma_{t+i}\{(M_{t+i}+T_{t+i})/P_{t+i} - c_{t+i} + k_{t+i} - k_{t+1+i}\}],$$

Differentiating (A2) with respect to c_t, k_{t+1}, and M_{t+1}, we obtain

$$(A3a)\quad c_t^{-R} = \lambda_t + \theta_t,$$

$$(A3b)\quad \lambda_t + \theta_t = ß\, E_t\, [\lambda_{t+1}\,(1+A) + \theta_{t+1}],$$

$$(A3c)\quad \lambda_t / P_t = ß\, E_t\, [(\lambda_{t+1} + \theta_{t+1}) / P_{t+1}].$$

As in previous sections, it holds that $c_t = (1+A)\, k_t - k_{t+1}$ in equilibrium. Thus, when the cash-in-advance constraint (A1) is binding, it holds that

$$(A4)\quad P_t = M_{t+1}/(Ak_t).$$

Hence, assuming the perfect foresight, equations (A3a) - (A3c) lead to

$$(A5)\quad (c_{t+1}/c_t)^R = ß\,[1 + A(ß/z)(k_{t+2}/k_{t+1})\,(c_{t+1}/c_{t+2})^R].$$

Substituting $c_t = (1+A)\, k_t - k_{t+1}$ into (A5) and letting $k^g_t = k_{t+1}/k_t$, we obtain

$$(A6)\quad k^{g\,R}_t\, [(1+A) - k^g_{t+1}]^R\, [(1+A) - k^g_{t+2}]^R = ß[(1+A) - k^g_t]^R\, [(1+A) - k^g_{t+2}]^R + A(ß^2/z)\, k^{g\;1-R}_{t+1}\, [(1+A) - k^g_t]^R\, [(1+A) - k^g_{t+1}]^R$$

If we define the balanced growth path of k^g_t by k^g, (A6) leads to

$$(A7)\quad (k^g)^R = ß\,[1 + A(ß/z)\,(k^g)^{1-R}].$$

There exists a positive value of k^g if and only if

$$(A8)\quad R\,(k^g)^{R-1} > A(ß^2/z)\,(1-R)\,(k^g)^{-R}.$$

Thus, to assure the existence of the balanced growth path, we assume (A8) in the following analysis.

Now, let $\bar{K}^g_t = k^g_t - k^g$ to denote the deviation of k^g_t from their balanced growth path. Then linearizing (A7) around k^g yields

(A9) $R\,[(k^g)^R - \beta]\bar{K}^g_{t+2} + [R\,\beta - A(\beta^2/z)\,(1-R)(k^g)^{-R}(1+A - k^g)]\;\bar{K}^g_{t+1}$
$- R\,(k^g)^{R-1}\,(1+A)\bar{K}^g_t = 0.$

The associated characteristic equation is

(A10) $g(X) \equiv R\,[(k^g)^R - \beta]\,X^2 + [R\,\beta - A(\beta^2/z)(1-R)(k^g)^{-R}(1+A - k^g)]\,X - R(k^g)^{R-1}\,(1+A)$
$= 0.$

Noting that $k^g < 1+A$ under the transversality condition (11), it can be shown that

(A11a) $g(1) = -\,[R\,\beta\,k^{g\,-1} + A(\beta^2/z)\,k^{g\,-R}]\,(1+A - k^g) < 0,$
(A11b) $g(-1) = -2R\,\beta - [R(k^g)^{R-1} - (1-R)A(\beta^2/z)\,(k^g)^{-R}]\,(1+A - k^g) < 0.$

The above conditions imply that the balanced growth path is locally the only equilibrium path which satisfies the transversality condition.

Thus, when the cash-in-advance constraint applies investment purchases as well as consumption purchases, the balanced growth path is locally the only equilibrium dynamics. Therefore, in order for self-fulfilling expectations to play a crucial role in determining the long-run equilibrium growth, the form of liquidity constraint may be important in the one-sector growth model.

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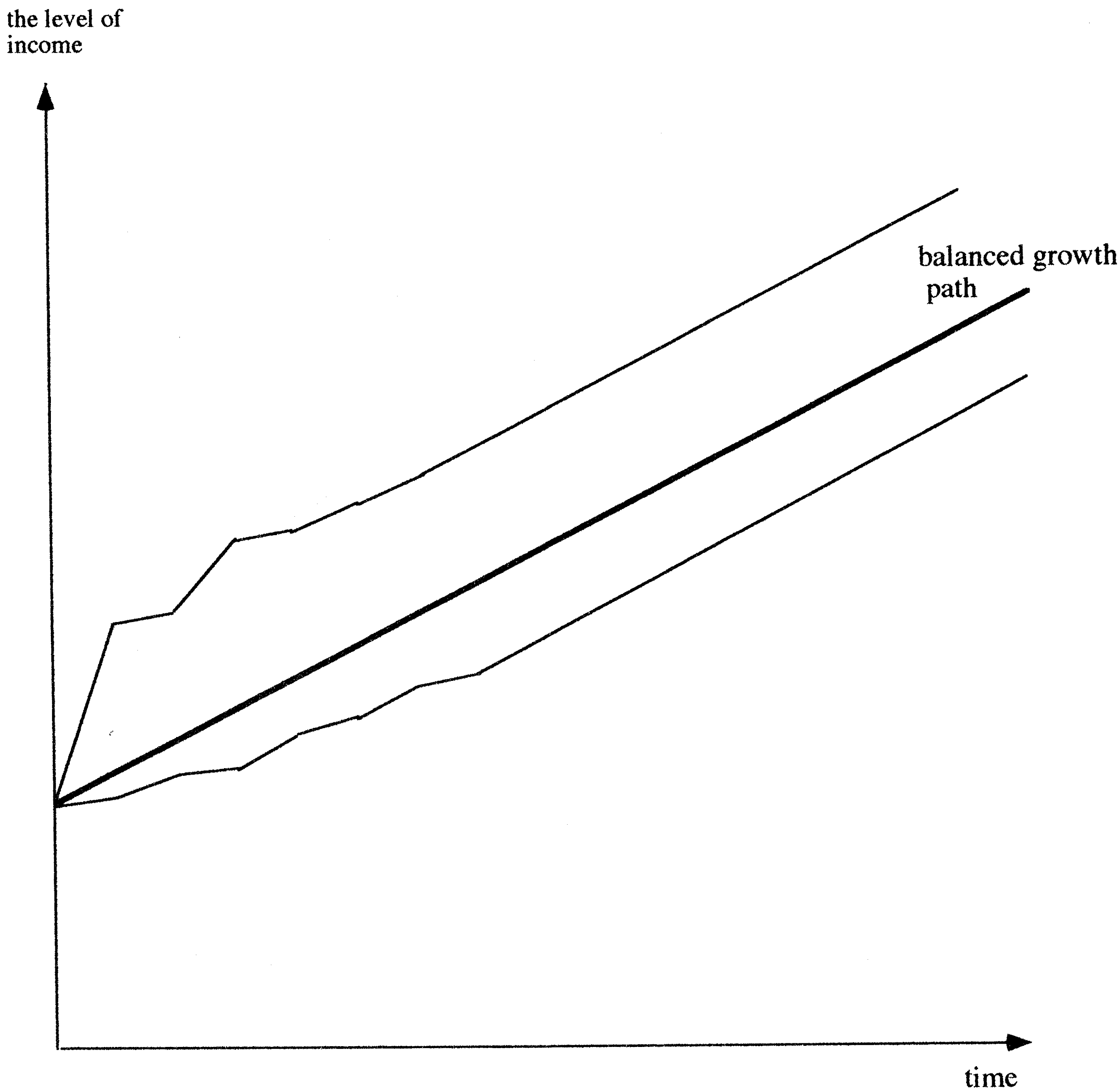
Figure 1 Multiple Convergent Paths in the Endogenous Growth Model



Note) When there exist multiple convergence equilibrium paths in the endogenous growth model, temporarily different growth rates can make the levels of income persistently different even if they display a common growth rate in the long run.

Figure 2   Three state stationary sunspot equilibria