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The Optimal Provision of Public Goods by Local and Central Governments

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Abstract

In this paper, the optimal subsidizing scheme of the central government on the locally provided public good in the decentralized economy is examined when the central government also provides another public good. First we examine the welfare effects of changes in the effective price of the locally provided public good and in the income tax rate, and derive some paradoxical results. Next, we investigate the optimal subsidy rates to the locally provided public good. It is derived that the optimal subsidy scheme is dependent on the relative efficiency of two types of public goods.

Keywords: locally provided public good, central public good
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1. Introduction

In the decentralized economy, each local government provides the locally provided public good without recognizing the spill-over effects over jurisdictions, which creates efficiency losses. Boadway et al. (1989) have examined the effects of tax transfer policy when the local government provides public goods voluntarily and derived the general neutrality theorem. Buchholz and Konrad (1995) and Ihori (1996) have examined the effects which the productivity differentials create in the similar framework of the voluntary provision of public goods. [1] In these existing papers, all public goods have been provided by the local government. However public goods can be also provided by the central government. Generally public goods are provided by both the central government and the local government. It is necessary to extend the model of these papers to the one where both governments provide public goods. By doing so, we will be able to derive more general results and the optimal role of the central government and explore the possible role of the central government to restore the efficient allocation.

On the other hand, in the framework of the private charity model where each agent supplies charitable contribution with externality, Feldstein (1980, 1987), Roberts (1987), Driessen (1987) and Ihori (1995) have investigated the optimal role of the government. These papers compared the relative efficiency between the subsidy to charitable contribution and the direct government provision. Especially Ihori (1995) has derived explicitly the optimal level of tax deductibility. These models can been reformulated to a more general model where each local government supplies public goods and the central government subsidizes to the provision of the locally provided public good and also supplies a public good.

There are two main differences between the previous private charity models and this paper. One is a subsidizing scheme and the other is the generality of the utility function. Most of the private charity papers, except for Roberts (1987), have discussed the flat subsidizing scheme with the same subsidy rate among consumers.[2] However, as Boadway et al. (1989) examined, the central government often sets the heterogeneous subsidizing

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[1] These papers assume pure public goods provided by the local government. On the other hand, Ihori (1994), by considering the locally provided public good with various interregional externalities, has explored the possibility of immiserizing growth in the decentralized economy.

[2] Though Roberts (1987) has discussed about the different subsidizing scheme, he has not derived the optimal scheme explicitly.

schemes among each local government.[3] Therefore we consider subsidizing schemes in the heterogeneous subsidizing economy. Next, we adopt the general utility function where the locally provided public good and the central public good are included separately. In the previous papers it is assumed that the locally provided public good and the central public good are perfectly substitutable.

This paper is organized as follows. Section 2 presents the model. Section 3 examines the welfare effects of changes in several policy variables of the central government. Section 4 explores the optimal heterogeneous subsidizing scheme of the central government which maximizes social welfare, in the decentralized economy where both governments provide public goods simultaneously. Section 5 concludes the paper.

2. Model

Consider an economy with 2 heterogeneous jurisdictions, each of which supplies a public good, G. Each region is constructed by the representative agent and the agent can not move to the other region. A representative agent in region i gets utility from consumption, c^i, a locally provided public good, G, and a central public good, H. Then the utility in region i, is represented by the following utility function:

$$U^i = u(c^i, G, H) \tag{1}$$

where both c, G and H are assumed to be normal goods. Each local government, by setting the local lump sum tax, divides fixed income in each region into the locally provided public good and consumption goods. After all, each local government faces the following budget constraint:

$$c^i + g^i = y^i - Tax^i \tag{2}$$

where g^i is the level of the locally provided public good supplied by the local government in region i, y^i is exogenously given income in region i, and Tax^i is an effective tax burden in region i. The cost of the locally provided public good is the same among all regions and is assumed to be unity in terms of consumption goods. Since the locally provided public good has perfect spillover effects, G becomes equal to the sum of the quantity of the locally provided public good in each region, that is,

$$G \equiv g^1 + g^2 \tag{3}$$

Next, consider the tax and subsidy scheme by the central government. The central government is assumed to control the income tax level and the subsidy rate on locally

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[3] Actually, in Japan, the subsidizing scheme for private charity is homogeneous, but the subsidizing scheme for the local government is heterogeneous.

provided public good provision (Grants matched to the locally provided public good). Then the effective tax burden in region i is represented as follows:

$$Tax^i = ty^i - \beta^i g^i \quad (4)$$

where t and β^i represent a income tax and a rate of subsidy to the locally provided public good in region i, respectively. $\beta = 1$ indicates the 100 percent subsidy, that is, the local government can supply its public good freely. However then there would not exist an inner solution. $\beta = 0$ indicates no subsidy from the central government to the locally provided public good. Therefore we assume $0 \leq \beta < 1$ to have the inner solutions associated with the locally provided public good and consumption. Then the budget constraint of the central government becomes

$$H = ty^1 - \beta^1 g^1 + ty^2 - \beta^2 g^2 \quad (5)$$

Substituting equation (4) into (2), we get

$$c^i + g^i = y^i - (ty^i - \beta^i g^i)$$

Rewriting this, we have

$$c^i + p^i g^i = (1-t)y^i.$$

where p^i [4], defined by $1-\beta^i$, is the effective price of the locally provided public good. This equation means that the effective price of the locally provided public good is lowered by the subsidy from the central government. Using equation (3), the above equation can be rewritten as

$$c^i + p^i G = (1-t)y^i + (1-\beta^i)g^{-i}. \quad (6)$$

Each local government determines its public good provision and the level of consumption in each region, treating the effective price of the locally provided public good, p^i, the income tax rate, t, the public good provision in the other region, g^{-i}, and the public good provision by the central government, H, as given [5]

Here to describe the Nash equilibrium, define the following expenditure function:

$$\text{Minimize } E^i \equiv c^i + p^i G$$

$$\text{Sub to } u^i(c^i, G, H) \geq U^i$$

Then, the expenditure level is represented as a function of the effective price, utility, and the level of the central public good as follows:

$$E^i = E^i(U^i, p^i, H)$$

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[4] By the definition of p and the assumption of β, we get $0 < p \leq 1$. Here $p = 1$ indicates no subsidy from the central government to the locally provided public good.

[5] In order to get the results clearly, we assume that each local government does not bind the non-negativity constraint on providing public goods and consumption goods. In other words, we assume the inner solutions.

Using the definition of E^i and subtracting consumption level, equation (6) becomes

$$(1-t)y^i = E^i - p^i G + p^i g^i$$

Now the expenditure function has the following sign and characteristics under the framework of this paper;

As for p, $E^i_p = G^i(U^i, p^i, H)$、 $G^i_p < 0$, $c^i_p > 0$, $c^i_p = -p^i G^i_p$[6]

As for U, $E^i_U = c^i_U + p^i G^i_U > 0$、 $G^i_U > 0$、 $c^i_U > 0$

As for H, $G^i_H \leq 0$、 $c^i_H < 0$、 $E^i_H = c^i_H + p^i G^i_H < 0$,

where $G^i = G^i(U^i, p^i, H)$ and $c^i = c^i(U^i, p^i, H)$ represent the compensated functions of the locally provided public good and consumption good, respectively. From derivatives on these functions, we can derive the efficiency criterion of the central public good, compared with the locally provided public good. Now we have the following lemma.

Lemma 1

If $1 + c^1_H + c^2_H + G^i_H > 0$, *the central public good is* ***less efficient*** *than the locally provided public good. Also if* $1 + c^1_H + c^2_H + G^i_H = 0$, *the central public good is* ***as efficient as*** *the locally provided public good.*

Proof

Since functions c and G are compensated functions, the derivative on each function with respect to the central public good represents evaluations of H on G and c. In this paper, the marginal cost to supply the central public good is the same as the cost of c and G and is equal to one. In other words, the one unit increase of the central public good creates the one unit decrease of each consumption or each contribution to the locally provided public good. $c^1_H + c^2_H + G^i_H$ represents the compensated change on the sum of each consumption and each contribution by the marginal increase of the central public good. If $c^1_H + c^2_H + G^i_H > -1$, the one unit increase of the central public good requires more than one unit resources in units of consumption goods. Therefore the central public good is less efficient than the locally provided public good. If $1 + c^1_H + c^2_H + G^i_H = 0$, the costs of the marginal increase of both the central public good and the local public good are the same. QED

Then the budget constraints are formulated as follows:

$$(1-t)y^1 = E^1 - p^1 G^1(U^1, p^1, H) + p^1 g^1 \tag{7-1}$$

$$(1-t)y^2 = E^2 - p^2 G^2(U^2, p^2, H) + p^2 g^2 \tag{7-2}$$

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[6] This equation is derived from equation $E^i_p = G^i(U^i, p^i, H)$ and $E^i_p = c^i_p + G^i + p^i G^i_p$.

Also since $G = G^i(U^i, p^i, H) = g^1 + g^2$, we get

$$(1-t)y^2 = E^2 - p^2 g^1$$

Substituting this into equation (7-1) and subtracting g^1, we have

$$(1-t)y^1 = E^1 - p^1 G^1 + \frac{p^1}{p^2}(E^2 - (1-t)y^2),$$

which becomes

$$p^2(1-t)y^1 + p^1(1-t)y^2 = p^2 E^1 + p^1 E^2 - p^1 p^2 G^1$$

Finally the model will reduce to the following three equations;

$$p^2 E^1 + p^1 E^2 - p^1 p^2 G^1(U^1, p^1, H) = p^2(1-t)y^1 + p^1(1-t)y^2 \quad (8\text{-}1)$$

$$G^1(U^1, p^1, H) = G^2(U^2, p^2, H) \quad (8\text{-}2)$$

$$H = y^1 + y^2 - c^1(U^1, p^1, H) - c^2(U^2, p^2, H) - G^1(U^1, p^1, H). \quad (8\text{-}3)$$

Equation (8-1) represents the total budget constraint in local governments considering externality effects. Equation (8-2) implies that the locally provided public good is the same in each region. Equation (8-3) represents the budget constraint of the central government. From these three equations, the utility level in each region and the public good provided by the central government are determined if an income tax, t, and a subsidy rate, β, are exogenous.

3 Comparative Statics

In this section, the welfare effect of changes in some policy variables of the central government is examined. The variables which the central government can control are the income tax rate, the level of the central public good and the subsidy rate to the locally provided public good. However, all policy schemes are not independent due to the budget constraint of the central government. Therefore, at first, we set the assumption that the income tax rate is exogenously given and the level of the central public good is endogenous. Next, we investigate the other case where the income tax rate is endogenous and the level of the central public good is exogenous.

3.1 Case 1: Endogenous Supply of the Central Public Good

Totally differentiating (8-1), (8-2) and (8-3), we have

$$\begin{bmatrix} p^2(E_U^1 - p^1G_U^1) & p^1E_U^2 & p^2(E_H^1 - p^1G_H^1) + p^1E_H^2 \\ G_U^1 & -G_U^2 & G_H^1 - G_H^2 \\ c_U^1 + G_U^1 & c_U^2 & 1 + c_H^1 + c_H^2 + G_H^1 \end{bmatrix} \begin{bmatrix} dU^1 \\ dU^2 \\ dH \end{bmatrix} = \begin{bmatrix} -E^2 + (1-t)y^2 + p^1p^2G_p^1 \\ -G_p^1 \\ -c_p^1 - G_p^1 \end{bmatrix} dp^1 + \begin{bmatrix} -E^1 + (1-t)y^1 \\ G_p^2 \\ -c_p^2 \end{bmatrix} dp^2 + \begin{bmatrix} -p^2y^1 - p^1y^2 \\ 0 \\ 0 \end{bmatrix} dt \tag{9}$$

Using the characteristics of the expenditure equation, we can rewrite (9) as follows.

$$\begin{bmatrix} p^2c_U^1 & p^1E_U^2 & p^2c_H^1 + p^1c_H^2 + p^1p^2G_H^2 \\ G_U^1 & -G_U^2 & G_H^1 - G_H^2 \\ c_U^1 + G_U^1 & c_U^2 & 1 + c_H^1 + c_H^2 + G_H^1 \end{bmatrix} \begin{bmatrix} dU^1 \\ dU^2 \\ dH \end{bmatrix} = \begin{bmatrix} -p^2g^1 + p^1p^2G_p^1 \\ -G_p^1 \\ -c_p^1 - G_p^1 \end{bmatrix} dp^1 + \begin{bmatrix} -p^1g^2 \\ G_p^2 \\ -c_p^2 \end{bmatrix} dp^2 + \begin{bmatrix} -p^2y^1 - p^1y^2 \\ 0 \\ 0 \end{bmatrix} dt \tag{10}$$

where the determinant of $\begin{bmatrix} p^2c_U^1 & p^1E_U^2 & p^2c_H^1 + p^1c_H^2 + p^1p^2G_H^2 \\ G_U^1 & -G_U^2 & G_H^1 - G_H^2 \\ c_U^1 + G_U^1 & c_U^2 & 1 + c_H^1 + c_H^2 + G_H^1 \end{bmatrix}$, defined as Δ^1, has to be negative for the stability condition of Nash equilibrium. For example, if the central public good is less efficient than the locally provided public good ($c_H^1 + c_H^2 + G_H^1 \geq -1$) and $G_H^1 = G_H^2$, we can prove the stability condition of this equilibrium by using the assumption associated with the utility function of this paper.

Welfare Effects of the Change in the Subsidy Rate

By calculating of matrix (10), we have the welfare effect on each region in an increase of the subsidy rate in region 1 as follows;

$$\begin{aligned} \frac{dU^1}{dp^1} = \frac{1}{\Delta^1}[&(p^2\gamma^1 + p^1\gamma^2 + p^1p^2\mu^2)(G_p^1)\{c_U^2 + G_U^2(1-p^1)\} \\ &+ (\mu^1 - \mu^2)\{(-p^2g^1 + p^1p^2G_p^1)c_U^2 - (p^1-1)G_p^1p^1(c_U^2 + p^2G_U^2)\} \\ &+ \{(1-\gamma^1-\gamma^2-\mu^1)(-p^2g^1G_U^2 + p^1c_U^2G_p^1)\}] \end{aligned} \tag{11-1}$$

and

$$\begin{aligned} \frac{dU^2}{dp^1} = \frac{1}{\Delta^1}[&(p^2\gamma^1 + p^1\gamma^2 + p^1p^2\mu^2)(G_U^1c_p^1 - G_p^1c_U^1) \\ &+ (\mu^1 - \mu^2)\{(p^1-1)G_p^1p^2c_U^1 - (-p^2g^1 + p^1p^2G_p^1)(c_U^1 + G_U^1)\} \\ &+ (1-\gamma^1-\gamma^2-\mu^1)(-p^2G_p^1E_U^1 + p^2g^1G_U^1)\}] \end{aligned} \tag{11-2}$$,

where $\gamma^i \equiv -c_H^i > 0$ and $\mu^i \equiv -G_H^i > 0$, the compensated values on consumption good and the locally provided public good of the central public good, respectively.

Equation (11-1) represents the welfare effect in the region where the effective price of subsidy increases, that is, the own subsidy rate decreases. The price change affects the level of the central public good through the behavior of each local government. This effect has been constructed by three terms, which are, first, the effect on the total budget constraint, second, the difference of the evaluation on G of the central public good, third, the effect on the budget constraint of the central government. On the other hand, equation (11-2) represents the welfare effect in the other region. Each term in equation (11-2) is also interpreted as similar as equation (11-1).

Now we have got some results from equations (11-1) and (11-2).

- If the central public good is less efficient than or as efficient as the locally provided public good ($1-\gamma^1-\gamma^2-\mu^1 \geq 0$) and $\mu^1 \geq \mu^2$, then $\frac{dU^2}{dp^1} < 0$
- If the central public good is as efficient as the locally provided public good ($1-\gamma^1-\gamma^2-\mu^1 = 0$) and $\mu^1 \geq \mu^2$, then $\frac{dU^1}{dp^1} > 0$

These represent that the compensated values on consumption good and the locally provided public good, which are associated with the efficiency criterion of *H,* determine the sign of effects of the effective price change. If $1-\gamma^1-\gamma^2-\mu^1 = 0$ and $\mu^1 \geq \mu^2$, we have the strong paradox that utility in region 1 where the subsidy rate increases decreases and utility in region 2 increases. In order to get the intuition of this result, we consider the case with $1-\gamma^1-\gamma^2-\mu^1 = 0$ and $\mu^1 = \mu^2$. This means that H is as efficient as G in each region, that is, $1-\gamma^1-\gamma^2-\mu^i = 0$. Then the change of H does not affect utility directly. An increase of the subsidy rate stimulates the provision of G in region 1, which has the positive externality effect for region 2. Receiving this positive effect, region 2 decreases its contribution of G. This creates the strong paradox.

If $1-\gamma^1-\gamma^2-\mu^1 \geq 0$ and $\mu^1 \geq \mu^2$, then in all regions, H is less efficient or as efficient as G. An increase of the subsidy rate decreases the level of *H,* which is better for all regions. Therefore the effect on utility in region 1 is ambiguous.

Also when we consider the special cases as follows, conditions to determine the sign of effects would be easily understood.

(a) The special case with the utility function of $U^i = u(c^i, G + A^i(H))$

Then $\gamma^i = 0$. Now we get follows.

- If $1 \geq \mu^1 \geq \mu^2$, $\dfrac{dU^2}{dp^1} < 0$
- If $1 = \mu^1 \geq \mu^2$, $\dfrac{dU^1}{dp^1} > 0$.

In this case, a change of H does not affect the consumption level directly. Therefore the efficiency criterion depends on whether μ is smaller than 1 or not. Especially if the utility function is $U^i = u(c^i, G + m^i H)$, then $m^i = \mu^i$ and the efficiency parameter, *m*, becomes constant. Furthermore if the utility function is $U^i = u(c^i, G + \mu H)$, then we have $\mu^i = \mu$ and following simple results.

- If the central public good is less efficient than or as efficient as the locally provided public good ($\mu \leq 1$), then $\dfrac{dU^2}{dp^1} < 0$.
- If the central public good is as efficient as the locally provided public good ($\mu = 1$), then $\dfrac{dU^1}{dp^1} > 0$.

(b) The special case with the utility function of $U^i = u(c^i + A^i(H), G)$

Then $\mu^i = 0$. Now we get follows.

- If the central public good is less efficient than or as efficient as the locally provided public good ($\gamma^1 + \gamma^2 \leq 1$), then $\dfrac{dU^2}{dp^1} < 0$.
- If the central public good is as efficient as the locally provided public good ($\gamma^1 + \gamma^2 = 1$)[7], then $\dfrac{dU^1}{dp^1} > 0$.

Now we have the following proposition:

Proposition 1

If the central public good is as efficient as the locally provided public good in region 1 and $\mu^1 \geq \mu^2$, *an increase of the subsidy rate of region 1 decreases utility in region 1 and increases utility in region 2.*

This extends the results of Boadway et al. (1989 Theorem 2), Roberts (1987) and Ihori (1996) with respect to the existence of the central public good. Even if the supply of the

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[7] Especially, if the utility function is $U^i = u(c^i + \dfrac{H}{2}, G), \gamma^i = 1/2$.

central public good is endogenous, paradoxical results hold as long as the central public good is as efficient as the locally provided public good in region 1 and $\mu^1 \geq \mu^2$.

This proposition has the following policy implication. Let us consider the economy where there exists the utility differential, that is $U^A > U^B$[8]. Then the central government may want to decrease a subsidy rate in region A or increase a subsidy rate in region B if it does not consider the spill-over effects of the locally provided public good. However as long as $\gamma^1 + \gamma^2 + \mu^1 = 1$ and $\mu^1 \geq \mu^2$, this scheme creates the paradoxical result that the degree of the utility differential between regions increases.

Welfare Effects of the Change in the Income Tax Rate

Changes in the income tax rate on utility are given by

$$\frac{dU^1}{dt} = \frac{1}{\Delta^1}(p^2y^1 + p^1y^2)\left\{(1-\gamma^1-\gamma^2-\mu^1)G_U^2 - c_U^2(\mu^1-\mu^2)\right\} \tag{12-1}$$

and

$$\frac{dU^2}{dt} = \frac{1}{\Delta^1}(p^2y^1 + p^1y^2)\left\{(1-\gamma^1-\gamma^2-\mu^1)G_U^1 + (c_U^1 + G_U^1)(\mu^1-\mu^2)\right\} \tag{12-2}$$

If $\gamma^1 + \gamma^2 + \mu^i = 1$ in all regions, the level of utility after the change of the income tax rate does not vary at all. This neutrality result is easily understood as follows. As shown in Lemma 1, if $\gamma^1 + \gamma^2 + \mu^i = 1$, then the central public good is as efficient as the locally provided public good. By the increase of the income tax rate, the central public good increases[9], while the locally provided public good and consumption decrease. However now the efficiency parameter is the same between regions, and therefore the neutrality result holds. As known well, there exists the neutrality theorem associated with the transfer among contributors (see Shibata (1971) and Warr (1983)). Our result is another neutrality result associated with the transfer between the public sector and the private sector. This type of the neutrality result has been derived in Roberts (1984) under the special case with the utility function: $U^i = u(c^i, G + H)$. Then we have $G_H = -1$. The increase of the central public good is perfectly offset by the decrease of the locally provided public good and consumption level is unchanged.

Given the income tax rate, t, if efficiency parameters are in a range of $\gamma^1 + \gamma^2 + \mu < 1$ and $\mu^1 = \mu^2$, the increase of the income tax rate lowers utility in all regions, that is,

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[8] This utility differential occurs from some reasons, for example the differential between the exante subsidy rates, the income differential, the difference of exogenous parameter such as an environment.

[9] Actually, the effect on the level of the central public goods is calculated as $\frac{dH}{dt} = \frac{1}{\Delta^1}(-p^2y^1 - p^1y^2)(G_U^1c_U^2 + G_U^2(c_U^1 + G_U^1)) > 0$, which is positive.

$\frac{dU^i}{dt} < 0$. Intuition is as follows. An increase of the income tax rate means an increase of the supply of the central public good. However the evaluation of the central public good is lower than the one of the locally provided public good. Therefore utility decreases.

Proposition 2

If the central public good is as efficient as the locally provided public good in all regions, then the transfer between the public side and the private side is neutral on the allocation.

3.2 Case 2: Endogenous Income Tax Rate

In order to compare results in the existing case where H is predetermined with above results in the general case, next, consider the case where the income tax rate is endogenous and the central public good is assumed to be constant. Then totally differentiating (8-1), (8-2) and (8-3), we have

$$\begin{bmatrix} p^2 c_U^1 & p^1 E_U^2 & p^2 y^1 + p^1 y^2 \\ G_U^1 & -G_U^2 & 0 \\ c_U^1 + G_U^1 & c_U^2 & 0 \end{bmatrix} \begin{bmatrix} dU^1 \\ dU^2 \\ dt \end{bmatrix} = \begin{bmatrix} -p^2 g^1 + p^1 p^2 G_p^1 \\ -G_p^1 \\ -c_p^1 - G_p^1 \end{bmatrix} dp^1 + \begin{bmatrix} -p^1 g^2 \\ G_p^2 \\ -c_p^2 \end{bmatrix} dp^2 \tag{13}$$
$$\begin{bmatrix} p^2\gamma^1 + p^1\gamma^2 + p^1 p^2 \mu^2 \\ \mu^1 - \mu^2, \\ -(1-\gamma^1-\gamma^2-\mu^1) \end{bmatrix} dH$$

where the determinant of the matrix in the left hand side becomes negative, that is, $\Delta^2 > 0$.

Welfare Effects of the Change in the Subsidy Rate

Welfare effects of a change in the subsidy rate of region 1 become

$$\frac{dU^1}{dp^1} = \frac{-1}{\Delta^2}\{(p^2 y^1 + p^1 y^2) G_p^1 [c_U^2 + G_U^2 (1-p^1)]\} > 0$$

$$\frac{dU^2}{dp^1} = \frac{-1}{\Delta^2}\{(p^2 y^1 + p^1 y^2)(G_U^1 c_p^1 - G_p^1 c_U^1) < 0$$

Therefore we have the following strong paradoxical proposition:

Proposition 3

In the case of the endogenous income tax, an increase of the subsidy rate of region 1 decreases utility in region 1 and increases utility in other region 2, which is independent *of efficiency parameters* γ^i *and* μ^i.

This proposition reduces to Boadway et al. (1989, Theorem 2) when the level of the central public good is zero and the revenue change created by the increase of the subsidy rate is financed by the income tax , instead of the lump sum transfer.[10]

Similar to Proposition 1, this intuition can be understood. In this case, the supply level of the central public good is constant, therefore the efficiency parameters γ^i and μ^i do not affect the utility. Region 2 which receives the positive externality raises utility and region 1 lowers utility through the negative externality due to the decrease of the contribution of region 2.

Welfare Effects of the Change in the Central Public Good

Now the income tax rate is an endogenous variable, and therefore the central government controls the level of the central public good. Then the welfare effect of a change in the central public good is given as follows;

$$\frac{dU^1}{dH} = \frac{1}{\Delta^2}\{(p^2y^1 + p^1y^2)\{(\mu^1 - \mu^2)c_U^2 - G_U^2(1-\gamma^1-\gamma^2-\mu^1)\}$$

$$\frac{dU^2}{dH} = \frac{1}{\Delta^2}\{(p^2y^1 + p^1y^2)\{-G_U^1(1-\gamma^1-\gamma^2-\mu^1) - (\mu^1-\mu^2)(c_U^1 + G_U^1)\}$$

The implication of these welfare effects of a change in H is similar to Proposition 2. If $\gamma^1 + \gamma^2 + \mu^i = 1$ in all regions, a change in H is independent of utility of each region. This derives the neutrality result between the private sector and the public sector. Condition $\gamma^1 + \gamma^2 + \mu^i = 1$ in all regions implies that the central public good is as efficient as the locally provided public good, and hence the increase of H can be perfectly adjusted by the change in c and G in each region. In the range of $\gamma^1 + \gamma^2 + \mu^1 < 1$ and $\mu^1 = \mu^2$, the central public good is less efficient, and hence the decrease of H raises utility, that is, $\frac{dU^i}{dH} < 0$.

4 Optimal Subsidizing Scheme

In order to examine the social optimal subsidizing scheme by the central government, we introduce the social welfare function:

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[10] This strong paradox also depends on the assumption of the same productivity of the public good provision. If the productivity is different, this paradox does not necessarily occur. (See Ihori (1996).)

$$W = \alpha^1 U^1 + \alpha^2 U^2$$

If there exist inner solutions, the optimal subsidy rates in each region and the income tax rate (or the central public good) must be determined such as to satisfy the following three first order conditions;

$$\alpha^1 U^1_{p^1} + \alpha^2 U^2_{p^1} = 0, \quad (14\text{-}1)$$

$$\alpha^1 U^1_{p^2} + \alpha^2 U^2_{p^2} = 0, \quad (14\text{-}2)$$

in case 1, $\alpha^1 U^1_t + \alpha^2 U^2_t = 0$ (14-3)

and in case 2, $\alpha^1 U^1_H + \alpha^2 U^2_H = 0$. (14-4)

4.1 Case 1: Endogenous Supply of the Central Public Good

In this case, considering equations (11-1) and (11-2), condition (14-1) becomes

$$\begin{aligned}
&\alpha^1[(p^2\gamma^1 + p^1\gamma^2 + p^1p^2\mu^1)(G^1_p)\{c^2_U + G^2_U(1-p^1)\} \\
&\quad + (\mu^1 - \mu^2)\{(-p^2g^1 + p^1p^2G^1_p)c^2_U - (p^1-1)G^1_p p^1(c^2_U + p^2G^2_U)\} \\
&\quad + \{(1-\gamma^1-\gamma^2-\mu)(-p^2g^1G^2_U + p^1c^2_UG^1_p)\}] \\
&+\alpha^2[(p^2\gamma^1 + p^1\gamma^2 + p^1p^2\mu^1)(G^1_Uc^1_p - G^1_pc^1_U) \\
&\quad + (\mu^1 - \mu^2)\{(p^1-1)G^1_pp^2c^1_U - (-p^2g^1 + p^1p^2G^1_p)(c^1_U + G^1_U)\} \\
&\quad + (1-\gamma^1-\gamma^2-\mu)(-p^2G^1_pE^1_U + p^2g^1G^1_U)] = 0
\end{aligned}$$

Rewriting this, we get

$$\begin{aligned}
&(p^2\gamma^1 + p^1\gamma^2 + p^1p^2\mu^1)G^1_p\{\alpha^1[c^2_U + G^2_U(1-p^1)]) + \alpha^2(G^1_U\frac{c^1_p}{G^1_p} + c^1_U)\} \\
&+(\mu^1 - \mu^2)[\alpha^1\{(-p^2g^1 + p^1p^2G^1_p)c^2_U - (p^1-1)G^1_pp^1(c^2_U + p^2G^2_U)\} \\
&\qquad + \alpha^2\{(p^1-1)G^1_pp^2c^1_U - (-p^2g^1 + p^1p^2G^1_p)(c^1_U + G^1_U)\}] \\
&+(1-\gamma^1-\gamma^2-\mu^1)\{\alpha^1(p^2g^1G^2_U + p^1c^2_UG^1_p) + \alpha^2(-p^2G^1_pE^1_U + p^2g^1G^1_U)\} = 0.
\end{aligned}$$

By $E_U = c_U + pG_U$ and $c_p = -pG_p$, we have

$$\begin{aligned}
&(p^2\gamma^1 + p^1\gamma^2 + p^1p^2\mu^1)G^1_p\{\alpha^1E^2_U - \alpha^2E^1_U + \alpha^1G^2_U(1-p^1-p^2)\} + \\
&+(\mu^1 - \mu^2)[\alpha^1\{(-p^2g^1 + p^1p^2G^1_p)c^2_U - (p^1-1)G^1_pp^1(c^2_U + p^2G^2_U)\} \\
&\qquad + \alpha^2\{(p^1-1)G^1_pp^2c^1_U - (-p^2g^1 + p^1p^2G^1_p)(c^1_U + G^1_U)\}] \\
&(1-\gamma^1-\gamma^2-\mu^1)\{p^2g^1(\alpha^1G^2_U + \alpha^2G^1_U) + \alpha^1p^1c^2_UG^1_p - \alpha^2p^2G^1_pE^1_U\} = 0
\end{aligned}$$

4.1.1 A case of identical regions

First, consider the optimal subsidy rate in identical regions. Then the first order condition becomes

$$(2p\gamma + pp\mu)G_p(1-2p) + (1-2\gamma-\mu)(2pg - G_ppp) = 0. \quad (15)$$

Defining the compensated price elasticity on G as ε $(\equiv -\frac{pG_p}{G} > 0)$ and noting $2g = G$, the above equation can be rewritten as

$$(2\gamma + p\mu)(2p-1) + p(1-2\gamma - \mu)(\frac{1}{\varepsilon} + 1) = 0$$

From this equation, we can have the following result about the optimal effective price. If $1-2\gamma - \mu \geq 0$ in the equilibrium, then $p \leq \frac{1}{2}$, that is, $\beta \geq \frac{1}{2}$. Intuition is as follows. Note that, under the assumption of identical regions, whether the central public good is less efficient than the locally provided public good or not corresponds to $1-2\gamma - \mu > 0$ or not. If $1-2\gamma - \mu > 0$, then the central public good is inefficient. Therefore the supply of the locally provided public good should be stimulated by the subsidizing scheme, compared with the case where the central public good is as efficient as the locally provided public good. Now we have the following proposition;

Proposition 4

If each region is identical, the optimal subsidy rate becomes higher than $\frac{1}{2}$ as long as the central public good is less efficient. *Especially if the central public good is as efficient as the locally provided public good, then it becomes* $\frac{1}{2}$.

If $1-2\gamma - \mu \geq 0$ in the equilibrium where the income tax rate has been set optimally, then the optimal income tax rate becomes the minimum one such that *H=0*, because $\frac{dU^i}{dt} < 0$ from equations (12-1) and (12-2). Therefore, in the case where the optimal subsidy rate becomes higher than 1/2, the supply of the central public good is zero, that is, the locally provided public good has been only supplied.

We can derive the optimal effective price from equation (15) exactly, but it is complicated and hard to understand. Therefore we consider the special case.

(a) The case with the utility function of $U^i = u(c, G + A(H))$

Then we get $\gamma^i = 0$. Therefore the optimal effective price can be exactly solved as

$$p = \frac{1}{2} - \frac{(1-\mu)(1+\varepsilon)}{2\varepsilon\mu}.$$

This equation represents that if $\mu = 1$, the optimal effective price becomes $\frac{1}{2}$, which means that the optimal subsidy rate is also $\frac{1}{2}$. Also, under the assumption that $\mu<1$, the optimal subsidy rate becomes higher than $\frac{1}{2}$. This is because, under this type of the utility function, whether $\mu<1$ or not represents whether the central public good is less efficient than the locally provided public good or not since $\gamma^i = 0$ is by the assumption associated with the relationship between G and H in the utility function.

Next, we consider a comparative static under the case of $\mu<1$ and the assumption that μ, γ and ε are almost unchanged after changes of other variables. It gives effects on the optimal effective price and the optimal subsidy rate of changes in μ and ε. By calculations, followings are derived. First, the larger the compensated value on G of H, that is, the efficiency parameter of the central public good, μ, is, the larger the optimal effective price is and the smaller the optimal subsidy rate is. Second, the larger the compensated price elasticity on G, ε, is, the larger the optimal effective price is and the smaller the optimal subsidy rate is. These intuitions are as follows. First, we consider a change in μ. An increase of μ means that the central public good becomes more efficient, compared with the locally provided public good. Then it is not as necessary to stimulate the provision by the local government, as the one before μ increases. Therefore the optimal subsidy rate decreases and approaches to 1/2. Next, we consider a change in ε. An increase of ε means that the subsidy becomes more effective for the provision of G. Then by the smaller subsidy rate, the optimal level of the provision of G is achieved. Therefore the optimal subsidy rate becomes small.

(b) The case with the utility function of $U^i = u(c^i + A^i(H), G)$

Then we get $\mu^i = 0$. Now the optimal effective price can be solved as

$$p = \frac{1}{2} - \frac{(\frac{1}{2} - \gamma)(1+\varepsilon)}{1 - 2\gamma + \varepsilon(1+2\gamma)}$$

From this optimal effective price, we get the similar result. If $\gamma < \frac{1}{2}$, the effective price is smaller than $\frac{1}{2}$, that is the subsidy rate is larger than $\frac{1}{2}$. Similar to the above discussions, under this type of the utility function, whether the central public good is inefficient or not corresponds to whether $\gamma < \frac{1}{2}$ or not. Also in the case of $\gamma < \frac{1}{2}$, we have the same comparative static results in the efficiency of the central public good, γ, and the compensated price elasticity, ε, as the former special case. The intuition is also similar to the one of the former special case. Summarizing comparative static results on the optimal subsidy rate in two special cases, we have the following.

Proposition 5

Under the utility function of $U^i = u(c, G + A(H))$ or $U^i = u(c + A(H), G)$, the optimal subsidy rate has following characteristics.

1. *The larger the compensated value on G or c of H, that is, the efficiency of the central public good, μ or γ, is, the smaller the optimal subsidy rate is.*
2. *The larger the compensated price elasticity on G, ε, is, the smaller the optimal subsidy rate is.*

4.1.2 A case of the same efficient central public good as the locally provided public good

Suppose the efficiency parameter of the central public good is the same as the one of the locally provided public good in all regions. Then we get $\gamma^1 + \gamma^2 + \mu^i = 1$. This derives $\mu^1 = \mu^2$. By using these conditions, the optimality condition (14-1) reduces to

$$p^1 + p^2 - 1 = \frac{\alpha^1 E_U^2 - \alpha^2 E_U^1}{\alpha^1 G_U^2}.$$

Similarly, from equation (14-2), we have

$$p^1 + p^2 - 1 = \frac{\alpha^2 E_U^1 - \alpha^1 E_U^2}{\alpha^2 G_U^1}.$$

If there exist solutions, the optimal subsidy rate has to satisfy the following two equations:

$$p^1 + p^2 = 1, \text{ (which represents } \beta^1 + \beta^2 = 1.) \tag{16-1}$$

and

$$\frac{E_U^1}{\alpha^1} = \frac{E_U^2}{\alpha^2}. \tag{16-2}$$

Equation (16-1) means that the sum of the subsidy rates of all regions must be equal to one, which corresponds to the Lindahl equilibrium. In addition, the optimal subsidy rates have to satisfy another condition, (16-2), which means that the monetary value of the marginal utility, adjusted by the social weight of each region, must be equal in all regions. Also since $0 \le \beta^i < 1$, the optimal subsidy rate in each region cannot become zero. If the subsidy rate in one region is zero, the subsidy rate in other region must be one. However the subsidy rate cannot be one by the assumption. Therefore, if the subsidy rates are optimally set, the central government subsidizes to all regions to provide public goods in the economy.

Now we have got the following proposition.

Proposition 6

If $\gamma^1 + \gamma^2 + \mu^i = 1$ *and there exist the optimal subsidy rates, they must be set such as to satisfy the following conditions.*

1: The sum of the optimal subsidy rates of all regions is unity.

2: The monetary value of the marginal utility, adjusted by the social weight of each region, is equal in all regions.

4.1.3 A general case where the central public good is less efficient

In the case where $\gamma^1 + \gamma^2 + \mu^i < 1$ and each region is different, it is necessary to consider how the other scheme has been set. In other words, when $\gamma^1 + \gamma^2 + \mu^i < 1$ or $\mu^1 \neq \mu^2$, the income tax rate affects the utility in each region. The optimal subsidy rate will depend on whether each scheme is set optimally or not. Suppose that other schemes have been optimally set. The optimal scheme associated with the income tax rate is to minimize it since $\frac{dW}{dt} < 0$ as long as $\gamma^1 + \gamma^2 + \mu^i < 1$ and $\mu^1 = \mu^2$ from equations (12-1) and (12-2). Since γ^i and μ^i are endogenous variables, these variables change, depending on the level of the income tax rate. However as long as $\gamma^1 + \gamma^2 + \mu^i < 1$ and $\mu^1 = \mu^2$, it is optimal that the income tax rate is set such that *H=0* ,that is, the revenue from the income tax rate is used only for the subsidy. This means that if the optimal income tax rate is set optimally, then the level of the central public good is always zero and hence is fixed. Then the optimal subsidy rate becomes equal to the same condition as in the next section. On the other hand, if γ^i and μ^i increase seriously as the increase of the income tax rate, there may exist inner solutions. Then it is optimal to select the income tax rate such that $\gamma^1 + \gamma^2 + \mu^i = 1$. The optimal subsidy rate under $\gamma^1 + \gamma^2 + \mu^i = 1$ has been already derived in this subsection (see Proposition 6). From equation (15), in a case of identical regions, the optimal subsidy rate becomes $\frac{1}{2}$ in the economy where the central public good has been supplied, that is, *H>0.*

4.2 Case 2: Endogenous Income Tax Rate

In this case, the first order condition (16-1) simply becomes

$$\alpha^1\{G_p^1[c_U^2 + G_U^2(1-p^1)]\} + \alpha^2\{(G_U^1 c_p^1 + G_p^1 c_U^1)\} = 0$$

Rewriting this, we get

$$G_p^1\{\alpha^1 E_U^2 - \alpha^2 E_U^1 + \alpha^1 G_U^2(1-p^1-p^2)\} = 0,$$

which means

$$p^1 + p^2 - 1 = \frac{\alpha^1 E_U^2 - \alpha^2 E_U^1}{\alpha^1 G_U^2}.$$

This equation is equal to the optimality condition in the case 1.

Similarly, from equation (16-2), we have

$$p^1 + p^2 - 1 = \frac{\alpha^2 E_U^1 - \alpha^1 E_U^2}{\alpha^2 G_U^1}.$$

Therefore we get the same optimality conditions, (16-1) and (16-2), that is, the sum of the subsidy rates in all regions must be one, $\beta^1 + \beta^2 = 1$.

Finally let us consider the first order condition of *H*. (Equation (14-4)). This implication is quite similar to the one of the income tax rate examined in 4.1.3 If $\gamma^1 + \gamma^2 + \mu^i < 1$ and $\mu^1 = \mu^2$ for any *H*, then there does not exist an inner solution of *H*. Then the optimal provision of the central public good becomes zero. On the other hand, if for some *H*, $\gamma^1 + \gamma^2 + \mu^i = 1$, then there exists an inner solution, that is, the positive provision of *H* is optimal.

5. Conclusions

This paper has considered optimal tax expenditures by the central government when the local government contributes to a public good which has a perfect spillover effect over each jurisdiction and the central government can supply another public good. We have investigated the optimal discriminated subsidy rates on the locally provided public good. The optimality conditions depend on the efficiency parameter between the central public good and the locally provided public good. In the case with the same efficiency, a change of the subsidy rate produces the strong paradoxical result. Also it is proved that as long as the other scheme is optimally set, the sum of the optimal subsidy rates must be equal to one, which corresponds to the Lindahl equilibrium. Then the central government should subsidize to the provision of the locally provided public good of all regions. This result also holds when the central public good is predetermined. The efficiency criterion of the central public good is an important factor to determine the optimal scheme.

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