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# Nonparametric Identification and Estimation of the Number of Components in Multivariate Mixtures

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**Abstract**

This article analyzes the identifiability of the number of components in $k$-variate, $M$-component finite mixture models in which each component distribution has independent marginals, including models in latent class analysis. Without making parametric assumptions on the component distributions, we investigate how one can identify the number of components from the distribution function of the observed data. When $k \geq 2$, a lower bound on the number of components ($M$) is nonparametrically identifiable from the rank of a matrix constructed from the distribution function of the observed variables. Building on this identification condition, we develop a procedure to consistently estimate a lower bound on the number of components.

Keywords: finite mixture; latent class analysis; nonnegative rank; rank estimation

## 1 Introduction

Finite mixture models provide flexible ways to model unobserved population heterogeneity. Because of their flexibility, finite mixtures have been used in numerous applications in diverse fields such as biological, physical, and social sciences. Comprehensive theoretical accounts and examples of applications can be found in Everitt and Hand (1981), Titterington et al. (1985), McLachlan and Basford (1988), Lindsay (1995), and McLachlan and Peel (2000).

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A finite mixture model is characterized by three main determinants: the number of components, the component distributions, and the mixing proportions. As emphasized in Hettmansperger and Thomas (2000), there is often little theoretical guidance for selecting the number of components and/or the form of the component distributions despite their key role in the specification of mixtures. Furthermore, it has been known that the estimates of the number of components are sensitive to the choice of the component distributions (see, for example, Schork et al. (1990) and Roeder (1994)), and that imposing incorrect parametric restrictions on the component distributions may lead to erroneous inference on the number of components (Cruz-Medina et al. 2004).

This article analyzes the nonparametric identifiability of the number of components in $k$-variate, $\tilde{M}$-component finite mixture models of $W = (W_1, \ldots, W_k)'$ under the assumption that the $W_j$'s are independently (but not necessarily identically) distributed within each component:

$$F(w) = F(w_1, \ldots, w_k) = \sum_{m=1}^{\tilde{M}} \pi^m F_1^m(w_1) F_2^m(w_2) \cdots F_k^m(w_k), \quad \pi^m > 0, \quad \sum_{m=1}^{\tilde{M}} \pi^m = 1. \quad (1)$$

Here, $F(w)$ is the distribution function of $W$, $\pi^m$ is the mixture proportion of the $m$-th subpopulation, and $F_j^m(w_j)$ is the distribution function of $W_j$ conditional on being from the $m$-th subpopulation, respectively. The number of components in $F(w)$, $M$, is defined as the smallest positive integer $\tilde{M}$ for which a finite mixture representation (1) can be found.

We analyze how one can recover the number of components $M$ from the exact knowledge of the distribution function of observed variables $F(w_1, \ldots, w_k)$ when no parametric assumptions are imposed on the component distributions. Nonparametric identifiability and estimation of finite mixtures has recently attracted increasing attention. Hall and Zhou (2003), Hall et al. (2005), and Allman et al. (2009) analyze nonparametric identifiability of component distributions and mixing proportions in model (1) under known $M$.[1] In particular, Allman et al. (2009) show that if $k \geq 3$, model (1) is nonparametrically identifiable for any $M$ if the $F_j^m(w_j)$'s have sufficient variation. Hettmansperger and Thomas (2000) and Cruz-Medina et al. (2004) analyze the nonparametric identification and estimation of model (1) with iid marginals by partitioning the support of $W_j$ into bins and transforming the data to multinomial vectors. Benaglia et al. (2009) and Levine et al. (2011) develop algorithms for estimating model (1) nonparametrically using kernels. However, no theoretical results on the identification of the number of components in model (1) are provided in the existing literature.

We show that a lower bound on the number of components $M$ is identified without imposing any parametric assumptions if $k \geq 2$. Interestingly, this result holds despite the

[1] Kasahara and Shimotsu (2009) study nonparametric identification of finite mixture dynamic discrete choice models widely used in econometrics.

fact that the component distributions are not identifiable when $k = 2$ (see Clogg 1981; Hall and Zhou 2003). The lower bound is stated in terms of the rank of a matrix constructed from the (multinomial) distribution function of the observed data, where for continuous variables, we transform each element of $W$ to a discrete random variable by partitioning its support as in Elmore et al. (2004). We also illustrate the cases in which the bound is tight except possibly for a set of mixture models with zero Lebesgue measure, and therefore, the bound is tight generically in the sense of Allman et al. (2009, p. 3106). By estimating the rank of its empirical analogue, we develop a procedure to consistently estimate a lower bound on the number of components. Simulations illustrate that our procedure performs well.

The mixture model (1) assumes that the marginal distributions are independent conditional on belonging to a subpopulation. The conditional independence assumption may be viewed as a version of a standard repeated measures random effects model, in which multivariate observations on an individual are often assumed to be independent conditional on the identity of the individual. The model (1) has important applications as demonstrated in some recent works on nonparametric mixture models as well as those on multinomial mixtures (e.g., Zhou et al. 2005; Dunson and Xing 2011; Bhattacharya and Dunson 2011), and encompasses models in latent class analysis that has been widely used in many fields including sociology, psychology, and biostatistics (Lazarsfeld and Henry 1968; Clogg 1995; Hagenaars and McCutcheon 2002; Magidson and Vermunt 2004; Skrondal and Rabe-Hesketh 2004). Once an estimate of a lower bound of $M$ is obtained, one can use algorithms such as Benaglia et al. (2009) and Levine et al. (2011) to nonparametrically estimate the mixture model (1), provided that the mixing proportions and the component distributions are identifiable.

Numerous methods to select the number of components have been proposed in a parametric setting (see, for example, Henna 1985; Leroux 1992; Lindsay and Roeder 1992; Windham and Cutler 1992; Roeder 1994; Chen and Kalbfleisch 1996; Dacunha-Castelle and Gassiat 1999; Keribin 2000; James et al. 2001; Woo and Sriram 2006). Our proposed procedure requires the conditional independence assumption but makes no distributional assumptions on the components. Furthermore, our selection procedure is based on a statistic whose asymptotic distribution is chi-squared or can be easily simulated, and it does not require the estimation of a mixture model with a different number of components.

The remainder of the article is organized as follows. Section 2 discusses the nonparametric identifiability of a lower bound on the number of components under $k \geq 2$. Section 3 introduces a procedure to test a lower bound on the number of mixture components. Section 4 reports simulation results, and empirical examples are provided in section 5. The supplementary appendix contains the proofs, mathematical details, and detailed results from simulations and empirical examples.

## 2 Nonparametric identification of a lower bound on the number of components

### 2.1 Two-variable case

We first analyze the nonparametric identification of a *lower bound* on the number of components for the mixture model (1) with $k = 2$. For notational clarity, we use $X$ and $Y$ in place of $W_1$ and $W_2$. Specifically, consider the following finite mixture models of variable $(X, Y)$:

$$F(x, y) = \sum_{m=1}^{\tilde{M}} \pi^m F_x^m(x) F_y^m(y), \quad \pi^m > 0, \quad \sum_{m=1}^{\tilde{M}} \pi^m = 1, \tag{2}$$

where $F_x^m(x)$ and $F_y^m(y)$ are the distribution functions of $X$ and $Y$ conditional on being from the $m$-th subpopulation. No assumptions are imposed on $F_x^m(x)$'s and $F_y^m(y)$'s except that they are distribution functions. Define the number of components in $F(x, y)$, $M$, as the smallest positive integer $\tilde{M}$ for which a finite mixture representation (2) can be found.

We proceed to construct a partition, $\Delta$, of the support of $(X, Y)$, and form a matrix that represents the distribution of $(X, Y)$ over $\Delta$. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the support of $X$ and $Y$. Partition $\mathcal{X}$ and $\mathcal{Y}$ into $|\Delta_x|$ and $|\Delta_y|$ mutually exclusive and exhaustive subsets, respectively, as $\Delta_x = \{\delta_1^x, \ldots, \delta_{|\Delta_x|}^x\}$ and $\Delta_y = \{\delta_1^y, \ldots, \delta_{|\Delta_y|}^y\}$, where $|\mathcal{S}|$ denotes the number of elements in a set $\mathcal{S}$. Define $\Delta = \Delta_x \times \Delta_y$. Given a partition $\Delta$, collect the distributions of $X$ and $Y$ conditional on being from the $m$-th subpopulation into a vector as

$$p_x^m = (\Pr(x \in \delta_1^x | m), \ldots, \Pr(x \in \delta_{|\Delta_x|}^x | m))' \quad \text{and} \quad p_y^m = (\Pr(y \in \delta_1^y | m), \ldots, \Pr(y \in \delta_{|\Delta_y|}^y | m))', \tag{3}$$

respectively. The vectors $p_x^m$ and $p_y^m$ implicitly depend on $\Delta_x$ and $\Delta_y$.

Arrange $\Pr(X \in \delta_a^x, Y \in \delta_b^y)$ for partition level $(a, b) = (1, 1), \ldots, (|\Delta_x|, |\Delta_y|)$ into a $|\Delta_x| \times |\Delta_y|$ bivariate probability matrix as

$$P_\Delta = \begin{bmatrix} \Pr(X \in \delta_1^x, Y \in \delta_1^y) & \cdots & \Pr(X \in \delta_1^x, Y \in \delta_{|\Delta_y|}^y) \\ \vdots & \ddots & \vdots \\ \Pr(X \in \delta_{|\Delta_x|}^x, Y \in \delta_1^y) & \cdots & \Pr(X \in \delta_{|\Delta_x|}^x, Y \in \delta_{|\Delta_y|}^y) \end{bmatrix}. \tag{4}$$

Then, $P_\Delta$ represents the distribution of $(X, Y)$ on the partition $\Delta$ and can be expressed in terms of $\pi^m$'s, $p_x^m$'s, and $p_y^m$'s as

$$P_\Delta = \sum_{m=1}^{\tilde{M}} \pi^m p_x^m (p_y^m)', \quad \pi^m > 0, \quad \sum_{m=1}^{\tilde{M}} \pi^m = 1. \tag{5}$$

Equation (5) is a finite mixture model (2) that is restricted to the partition $\Delta$.

For a partition $\Delta$, define *the number of components in* $P_\Delta$ as the smallest integer $\tilde{M}$ such that the finite mixture representation (5) is possible. The number of components in $P_\Delta$ is closely related to the concept of *nonnegative rank* developed by Cohen and Rothblum (1993). For a nonnegative matrix $A$, its nonnegative rank is denoted by $\text{rank}_+(A)$ and defined as the smallest number of nonnegative rank-one matrices such that $A$ equals their sum. Because $P_\Delta$ is a nonnegative matrix and the right-hand side of equation (5) is the sum of nonnegative rank-one matrices, by definition, the number of components in $P_\Delta$ is the nonnegative rank of $P_\Delta$.

The following proposition, originally from Cohen and Rothblum (1993), states the properties of the nonnegative rank of $P_\Delta$ and its relation to the rank of $P_\Delta$.

**Proposition 1 (Cohen and Rothblum, 1993)** *(a)* $\textit{rank}(P_\Delta) \leq \textit{rank}_+(P_\Delta) \leq \min\{|\Delta_x|, |\Delta_y|\}$. *(b) If* $\textit{rank}(P_\Delta) \leq 2$, *then* $\textit{rank}(P_\Delta) = \textit{rank}_+(P_\Delta)$. *(c) If* $|\Delta_x| \leq 3$ *or* $|\Delta_y| \leq 3$, *then* $\textit{rank}_+(P_\Delta) = \textit{rank}(P_\Delta)$.

From Proposition 1(a), $\text{rank}(P_\Delta)$ gives a lower bound on the number of components in $P_\Delta$ whereas the number of support points of $X$ and $Y$ gives an upper bound on the number of identifiable components since $|\Delta_x| \leq |\mathcal{X}|$ and $|\Delta_y| \leq |\mathcal{Y}|$. It follows from Proposition 1 that $\text{rank}_+(P_\Delta) = \text{rank}(P_\Delta)$ if $\text{rank}_+(P_\Delta) \leq 3$.

The number of components in $P_\Delta$ is identified with the nonnegative rank of $P_\Delta$. Determining the nonnegative rank of a matrix is computationally difficult[2], however, and is still a subject of ongoing research (see, for example, Dong, Lin, and Chu 2009). Therefore, it is useful to characterize a *lower bound* on the number of components in $P_\Delta$ in terms of the rank of $P_\Delta$.

An obvious limitation of the lower bound based on the rank of $P_\Delta$ is a possible discrepancy between the lower bound and the actual number of components. This is because the latter requires that the components $\pi^m$'s, $p_x^m$'s, and $p_y^m$'s in (5) to be nonnegative while the former does not.[3] We investigate the size of a set of mixture models wherein $\text{rank}_+(P_\Delta) > \text{rank}(P_\Delta)$. Given a positive integer $M_0$, define the space of $M_0$-component mixture models $\theta = \{p_x^m, p_y^m, \pi^m\}_{m=1}^{M_0}$ by $\Theta \subset (\mathcal{S}_{|\Delta_x|-1})^{M_0} \times (\mathcal{S}_{|\Delta_y|-1})^{M_0} \times \mathcal{S}_{M_0-1}$ as in Allman et al. (2009, p. 3107), where $\mathcal{S}_k$ denotes the standard $k$-simplex. The following proposition shows that if we randomly draw a mixture model $\theta$ from $\Theta$ and construct a matrix $P(\theta) = \sum_{m=1}^{M_0} \pi^m p_x^m (p_y^m)'$, then we have $\text{rank}(P(\theta)) = M_0$ with probability one. This result holds because, when $\text{rank}(P(\theta)) < M_0$, either the vectors $\{p_x^m\}_{m=1}^{M_0}$ or $\{p_y^m\}_{m=1}^{M_0}$ are

[2]Vavasis (2009) shows that determining the nonnegative rank of a matrix is NP-hard.

[3]For example, suppose that $|\Delta_x| = |\Delta_y| = 4$ and $P_\Delta = \sum_{m=1}^{4} \pi^m p_x^m (p_y^m)'$, where $\pi^m > 0$ and $p_x^m$s are linearly independent but $p_y^1 + p_y^2 - p_y^3 - p_y^4 = 0$, so that $\text{rank}(P_\Delta)$ is at most 3. Writing one $p_y^m$ in terms of the other $p_y^m$s and substituting into $P_\Delta$ will give a three-term mixture representation of $P_\Delta$. However, if $-\pi^1 p_x^1 + \pi^2 p_x^2$ and $-\pi^3 p_x^3 + \pi^4 p_x^4$ have both positive and negative elements, then the resulting three-term mixture representation necessarily contains negative components, and $\text{rank}_+(P_\Delta)$ is strictly larger than 3.

linearly dependent, but the set of linearly dependent $\{p_x^m\}_{m=1}^{M_0}$'s (or $\{p_y^m\}_{m=1}^{M_0}$'s) has Lebesgue measure zero in $\mathbb{R}^{\Delta_x \times M_0}$ (or $\mathbb{R}^{\Delta_y \times M_0}$).

**Proposition 2** *If $M_0 \leq \min\{|\Delta_x|, |\Delta_y|\}$, then $M_0 = rank_+(P(\theta)) = rank(P(\theta))$ holds for all the points in $\Theta$ except possibly for a set of Lebesgue measure zero.*

When $X$ and $Y$ are discrete, taking the support of $(X, Y)$ as $\Delta$ and applying Proposition 2 gives that $\text{rank}(P_\Delta) = \text{rank}_+(P_\Delta) = M$ with probability one if we draw an $M$-component bivariate mixture model with conditionally independent marginals. Hence, the bound is tight with probability one. When $X$ and $Y$ are continuous, there is no obvious choice of a single partition $\Delta$. The nonnegative rank of $P_\Delta$ could be strictly smaller than $M$ when a single partition $\Delta$ does not fully reveal the information for identifying the number of components in $F(x, y)$. A tighter lower bound of $M$ may be obtained by taking the maximum value of the rank of $P_\Delta$s across different partitions.

**Proposition 3** *Suppose that in model (2), the distribution of $(X, Y)$ is continuous. (a) If $\{F_x^m(x)\}_{m=1}^M$ are linearly independent and $\{F_y^m(y)\}_{m=1}^M$ are linearly independent, then there exists a partition $\Delta$ with $|\Delta_x| = |\Delta_y| = M$ such that rank$(P_\Delta) = M$. (b) If we draw $\Delta$ randomly, then the probability that rank$_+(P_\Delta) =$ rank$(P_\Delta)$ is one.*

Proposition 3(a) gives a sufficient condition under which the rank of $P_\Delta$ is equal to $M$ for some choice of $\Delta$; in this case, $M$ is identified with the maximum value of $\text{rank}(P_\Delta)$'s over all possible partitions of $\mathcal{X} \times \mathcal{Y}$ into $M \times M$ subsets. Proposition 3(b) implies that whether $\text{rank}(P_\Delta) = M$ holds depends on whether $\text{rank}_+(P_\Delta) = M$ holds.

### 2.2 General $k$-variable case

We now illustrate how our approach in Section 2.1 can be applied to the mixture model (1) with $k \geq 3$ to obtain a lower bound on $M$. Consider a hyperrectangle partition $\Delta = \Delta_1 \times \cdots \times \Delta_k$ of $\mathcal{W}$. Let $P_\Delta$ denote $F(w)$ on $\Delta$. Note that $P_\Delta$ is a $k$-dimensional array. $P_\Delta$ is written as a weighted average of $k$-dimensional tensors as follows:

$$P_\Delta = \sum_{m=1}^{\text{rank}_+(P_\Delta)} \pi^m P^m, \quad P^m = \otimes_{j=1}^k p_j^m, \quad \pi^m > 0, \quad \sum_{m=1}^{\text{rank}_+(P_\Delta)} \pi^m = 1, \tag{6}$$

where $\otimes$ denotes a tensor product and $p_j^m$ is a $|\Delta_j| \times 1$ vector. Here, $\text{rank}_+(P_\Delta)$ is defined as the smallest positive integer for which a representation (6) holds and called the nonnegative (tensor) rank of $P_\Delta$ (see, for example, Lim and Common 2009). As in the two-variable case, $\text{rank}_+(P_\Delta)$ provides a lower bound on $M$.

We construct a matrix from $P_\Delta$ by grouping the variables in $W = (W_1, \ldots, W_k)'$ into two groups. We index the groupings by $\alpha$. For the grouping $\alpha$, let $X^\alpha$ and $Y^\alpha$ be the grouped

variables. Let $\Delta_{x^\alpha} = \{\delta_1^{x^\alpha}, \ldots, \delta_{|\Delta_{x^\alpha}|}^{x^\alpha}\} = \prod_{j \in S_x(\alpha)} \Delta_j$ be the partition of the support of $X^\alpha$, where $S_x(\alpha)$ is the set of indices such that $W_j \in X^\alpha$, and define $\Delta_{y^\alpha}$ similarly. Then, we construct a $|\Delta_{x^\alpha}| \times |\Delta_{y^\alpha}|$ bivariate probability matrix $P_\Delta^\alpha$ by arranging $\Pr(X^\alpha \in \delta_a^{x^\alpha}, Y^\alpha \in \delta_b^{y^\alpha})$ for partition level $(a, b) = (1, 1), \ldots, (|\Delta_{x^\alpha}|, |\Delta_{y^\alpha}|)$ as in (4).

A lower bound on $M$ can be obtained in terms of $\text{rank}_+(P_\Delta)$, $\text{rank}_+(P_\Delta^\alpha)$, and $\text{rank}(P_\Delta^\alpha)$ as $M \geq \text{rank}_+(P_\Delta) \geq \text{rank}_+(P_\Delta^\alpha) \geq \text{rank}(P_\Delta^\alpha)$. Taking the maximum value of $\text{rank}_+(P_\Delta)$, $\text{rank}_+(P_\Delta^\alpha)$, and $\text{rank}(P_\Delta^\alpha)$ across different partitions, $\Delta$'s, and different groupings, $\alpha$'s, gives tighter lower bounds. Such bounds may still be, however, strictly smaller than $M$.

We investigate when $\text{rank}_+(P_\Delta) = \text{rank}(P_\Delta^\alpha)$ holds. With a slight abuse of notation, given a positive integer $M_0$, define the space of $M_0$-component mixture models $\theta = \{p_1^m, \ldots, p_k^m, \pi^m\}_{m=1}^{M_0}$ as $\Theta \subset (\mathcal{S}_{|\Delta_1|-1})^{M_0} \times \cdots \times (\mathcal{S}_{|\Delta_k|-1})^{M_0} \times \mathcal{S}_{M_0-1}$. Given an element $\theta$ of $\Theta$, group $W$ into $X^\alpha$ and $Y^\alpha$ and construct a bivariate probability matrix $P^\alpha(\theta) = \sum_{m=1}^{M_0} \pi^m p_{x^\alpha}^m (p_{y^\alpha}^m)'$, where $p_{x^\alpha}^m$ is $|\Delta_{x^\alpha}| \times 1$ and $p_{y^\alpha}^m$ is $|\Delta_{y^\alpha}| \times 1$. The following proposition shows that if the grouped variables have sufficiently large state spaces relative to $M_0$, an analogous result to Proposition 2 holds for a $k$-variable model.

**Proposition 4** *If $M_0 \leq \min\{|\Delta_{x^\alpha}|, |\Delta_{y^\alpha}|\}$, then $M_0 = \text{rank}(P^\alpha(\theta))$ holds for all the points in $\Theta$ except possibly for a set of Lebesgue measure zero.*

In the continuous variable case, we have a simple corollary of Proposition 3(a).

**Corollary 1** *Suppose that in model (1), the distribution of $W$ is continuous. If there are two variables $W_j$ and $W_\ell$ such that $\{F_j^m(w_j)\}_{m=1}^M$ are linearly independent and $\{F_\ell^m(w_\ell)\}_{m=1}^M$ are linearly independent, then there exists a grouping $\alpha$ and partition $\Delta$ such that $\text{rank}(P_\Delta^\alpha) = M$.*

### 2.3 Relation to latent class analysis

Consider a special case in which an observation vector $W = (W_1, \ldots, W_k)'$ consists of $k$ dichotomous or polytomous responses, which are typically answers to questions or results of diagnoses. In this case, our model (1) becomes identical to the model used in *latent class analysis*. For recent surveys and applications of latent class analysis, see the references in the introduction.

The latent class analysis with $k = 2$ (two-way contingency table) is also known as *latent budget analysis* (Goodman 1974; Clogg 1981; de Leeuw and van der Heijden 1988). Testing the number of components in a latent budget model is particularly difficult because the parameters of the model are not identified unless some restrictions are imposed. Using our result, it is possible to identify a lower bound on $M$ without imposing restrictions on the parameters, even though identifying a lower bound of $M$ does not solve the problem of parameter non-identification.

# 3 Estimating a lower bound on the number of components

Proposition 1 in Section 2 shows that the rank of a matrix $P_\Delta$ in (5) gives a lower bound on the number of mixture components. In this section, we develop procedures to estimate the rank of $P_\Delta$ for a given partition $\Delta$ and extend these procedures to the case when there are more than two variables.

## 3.1 Statistic by Kleibergen and Paap (2006)

Kleibergen and Paap (2006) develop a procedure to test the null hypothesis that the rank of $P_\Delta$ is equal to $r$ as described below. For notational brevity, let $s = |\Delta_x|$ and $t = |\Delta_y|$, and write the singular value decomposition of an $s \times t$ matrix $P_\Delta$ as

$$P_\Delta = USV' = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} V_{11}' & V_{12}' \\ V_{21}' & V_{22}' \end{pmatrix},$$

where $U$ is an $s \times s$ orthogonal matrix, $V$ is a $t \times t$ orthogonal matrix, and $S$ is an $s \times t$ matrix that contains the singular values of $P_\Delta$ in decreasing order on its main diagonal and is equal to zero elsewhere. In the partition of $U$, $S$, and $V$ on the right-hand side, $U_{11}$, $S_1$, and $V_{11}$ are $r \times r$, and the dimensions of the other submatrices are defined conformably. Then, the null hypothesis $H_0 : \text{rank}(P_\Delta) = r$ is equivalent to $H_0 : S_2 = 0$ because the rank of a matrix is equal to the number of non-zero singular values.

The statistic by Kleibergen and Paap is based on an orthogonal transformation of $S_2$ given by $\Lambda_r = A_{r,\perp}' P_\Delta B_{r,\perp}'$, where $A_{r,\perp}' = (U_{22}U_{22}')^{1/2}(U_{22}')^{-1}[U_{12}' \vdots U_{22}']$ and $B_{r,\perp} = (V_{22}V_{22}')^{1/2}(V_{22}')^{-1}[V_{12}' \vdots V_{22}']$. Unlike $S_2$, $\Lambda_r$ is not restricted to be non-negative.[4] Then, the null hypothesis $H_0 : rank(P_\Delta) = r$ is equivalent to $H_0 : \Lambda_r = 0$. Let $\hat{P}_\Delta$ be an estimator of the matrix $P_\Delta$ with sample size $N$. We assume that $\text{vec}(\hat{P}_\Delta)$ is asymptotically normally distributed.

**Assumption 1** *$\sqrt{N}vec(\hat{P}_\Delta - P_\Delta) \to_d N(0, \Sigma)$ as $N \to \infty$, where $\Sigma$ is an $st \times st$ covariance matrix.*

When the distribution of $W$ is discrete or $\Delta$ is predetermined, $\text{vec}(\hat{P}_\Delta)$ follows a multinomial distribution, and a formula for $\Sigma$ is easily available. If $W$ has a continuous distribution and the empirical quantiles of the $W_j$'s are used to construct $\Delta$, then $\text{vec}(\hat{P}_\Delta)$ follows the empirical multivariate quantile-partitioned (EMQP) distribution (Borkowf, 2000) described in the supplemental appendix, and one can use bootstrap to estimate $\Sigma$.

We estimate $\Lambda_r$ by $\hat{\Lambda}_r = \hat{A}_{r,\perp}' \hat{P}_\Delta \hat{B}_{r,\perp}'$ and test $H_0 : \Lambda_r = 0$, where $\hat{A}_{r,\perp}$ and $\hat{B}_{r,\perp}$ are the estimators of $A_{r,\perp}$ and $B_{r,\perp}$ obtained from the singular value decomposition of $\hat{P}_\Delta$. Kleibergen and Paap (2006) derive the asymptotic distribution of $\hat{\lambda}_r = \text{vec}(\hat{\Lambda}_r)$, as summarized below.

[4] Robin and Smith (2000) propose a rank statistic based on a consistent estimator of $S_2$. However, because $S_2$ is nonnegative, the asymptotic distribution of their estimator of $S_2$ is not Gaussian when $S_2 = 0$.

**Proposition 5** *(Kleibergen and Paap, 2006, Theorem 1) Suppose that Assumption 1 holds and that* $\Omega_r = (B_{r,\perp} \otimes A'_{r,\perp})\Sigma(B_{r,\perp} \otimes A'_{r,\perp})'$ *is nonsingular. If rank*$(P_\Delta) \leq r$*, then* $\sqrt{N}\hat{\lambda}_r \to_d N(0, \Omega_r)$ *as* $N \to \infty$*.*

Kleibergen and Paap (2006, Corollary 1) propose the statistic called the *rk statistic*:

$$\mathrm{rk}(r) = N\hat{\lambda}_r'\hat{\Omega}_r^{-1}\hat{\lambda}_r, \tag{7}$$

where $\hat{\Omega}_r$ is a consistent estimator for $\Omega_r$. If the assumptions of Proposition 5 hold, then $\mathrm{rk}(r)$ converges in distribution to a $\chi^2((s-r)(t-r))$ random variable under $H_0 : \mathrm{rank}(P_\Delta) = r$. The nonsingularity assumption on $\Omega_r$ can be relaxed by using the Moore-Penrose (M-P) pseudoinverse as discussed in Section 3.4.

The choice of $\Delta$ is left to the researcher. As for the number of partitions, it is desirable to use a partition that is as fine as possible from the perspective of pure identification, but using a finer partition increases the variance of $\hat{P}_\Delta$. In practice, we suggest setting the number of partitions equal to one plus the maximum number of components we want to allow for in modeling the data. As for the choice of partitions, a natural choice would be to use equiprobable intervals as in Pearson's chi-squared test, but there may be cases where using a non-equiprobable partition gives a stronger power because mixture models often have fat tails. The optimal choice of partitions remains an open question.

### 3.2 Sequential hypothesis testing

Denote the population rank of $P_\Delta$ by $r_0$. To estimate $r_0$, we sequentially test $H_0 : \mathrm{rank}(P_\Delta) = r$ against $H_1 : \mathrm{rank}(P_\Delta) > r$ starting from $r = 0$, and then $r = 1, \ldots, t^*$, where $t^* = \min\{s, t\}$. The first value for $r$ that leads to a nonrejection of $H_0$ gives our estimate for $r_0$.

For $r = 0, \ldots, t^*$, let $c^r_{1-\alpha_N}$ denote the $100(1-\alpha_N)$ percentile of the cumulative distribution function of a $\chi^2\left((s-r)(t-r)\right)$ random variable. Then, our estimator based on sequential hypothesis testing (SHT, hereafter) is defined as

$$\hat{r} = \min_{r \in \{0,\ldots,t^*\}} \{r : \mathrm{rk}(i) \geq c^i_{1-\alpha_N}, i = 0, \ldots, r-1, \mathrm{rk}(r) < c^r_{1-\alpha_N}\}. \tag{8}$$

The estimator $\hat{r}$ depends on the choice of the significance level $\alpha_N$. As shown by Robin and Smith (2000, Theorem 5.2), $\hat{r}$ converges to $r_0$ in probability as $N \to \infty$ if we choose $\alpha_N$ such that $\alpha_N = o(1)$ and $-N^{-1}\ln\alpha_N = o(1)$.

### 3.3 Information criteria

We also consider a selection procedure by information criteria to estimate $r_0$ consistently. Consider the criterion function $Q(r) = \mathrm{rk}(r) - f(N)g(r)$, where $g(r)$ is a (possibly stochastic)

penalty function. Define $\tilde{r} = \arg\min_{1\le r\le t^*} Q(r)$. Under a standard condition on $f(N)$ and $g(r)$, this gives a consistent estimate of $r_0$:

**Proposition 6** *Suppose that the conditions of Proposition 5 hold, and $\hat{\Omega}_r$ converges to a nonsingular matrix for any $r \ge r_0$. Suppose that $f(N) \to \infty$, $f(N)/N \to 0$, and $\Pr(g(r) - g(r_0) < 0) \to 1$ for all $r > r_0$ as $N \to \infty$. Then, $\tilde{r} \to_p r_0$.*

For the choice of $f(N)$ and $g(r)$, we consider the penalty terms in the Akaike (AIC), Bayesian (BIC), and Hannan-Quinn (HQ) information criteria. We choose $g(r) = (s-r)(t-r)$ with $f(N) = 2$ for AIC, $f(N) = \log(N)$ for BIC, and $f(N) = 2\log(\log(N))$ for HQ. The BIC and HQ model selection procedures provide a consistent estimate of $r_0$ since their choice of $f(N)$ and $g(r)$ satisfies the conditions in Proposition 6. In contrast, AIC is not necessarily consistent and tends to overestimate $r_0$ with a large sample size.

### 3.4 Case of multiple variables

Suppose that $W = (W_1, \ldots, W_k)'$ with $k \ge 3$ follows the distribution function (1). As in Section 2.2, we group the variables in $W$ into two groups $X^\alpha$ and $Y^\alpha$, with the grouping index $\alpha$, and let $P^\alpha_\Delta$ denote a $|\Delta_{x^\alpha}| \times |\Delta_{y^\alpha}|$ bivariate probability matrix derived from the joint distribution of $X^\alpha$ and $Y^\alpha$ on a partition $\Delta$. We test the null hypothesis that $\mathrm{rank}(P^\alpha_\Delta) \le r$ for all $\alpha \in \mathcal{A}_0$, where $\mathcal{A}_0$ is a set of the $\alpha$s over which we construct test statistics.

We assume that all the variables in $W$ are included in the first grouping $\{X^1, Y^1\}$. Then, for every $\alpha \in \mathcal{A}_0$, the elements of the probability matrix $P^\alpha_\Delta$ can be expressed as a linear combination of the elements of $P^1_\Delta$, and therefore, there exists a matrix $\Pi^\alpha$ such that $\mathrm{vec}(P^\alpha_\Delta) = \Pi^\alpha \mathrm{vec}(P^1_\Delta)$.

Define $A^\alpha_{r,\perp}$, $B^\alpha_{r,\perp}$, and $\lambda^\alpha_r$ analogously to $A_{r,\perp}$, $B_{r,\perp}$, and $\lambda_r$ in Section 4.1 using $P^\alpha_\Delta$ in place of $P_\Delta$. Define $\hat{\lambda}^\alpha_r = \mathrm{vec}((\hat{A}^\alpha_{r,\perp})'\hat{P}^\alpha_\Delta(\hat{B}^\alpha_{r,\perp})') = (\hat{B}^\alpha_{r,\perp} \otimes (\hat{A}^\alpha_{r,\perp})')\Pi^\alpha \mathrm{vec}(\hat{P}^1_\Delta)$ using the estimators of $P^1_\Delta$, $A^\alpha_{r,\perp}$, and $B^\alpha_{r,\perp}$. To test the null hypothesis that $\mathrm{rank}(P^\alpha_\Delta) \le r$ for all $\alpha \in \mathcal{A}_0$, we stack $\hat{\lambda}^\alpha_r$'s into a vector as $\hat{\lambda}_r(\mathcal{A}_0) = ((\hat{\lambda}^1_r)', \ldots, (\hat{\lambda}^{|\mathcal{A}_0|}_r)')'$ and test the null hypothesis $\lambda_r(\mathcal{A}_0) = 0$. Extending Proposition 5, the following corollary establishes the asymptotic normality of $\hat{\lambda}_r(\mathcal{A}_0)$. We omit its proof to save space, because it is a straightforward consequence of Slutsky's theorem.

**Corollary 2** *Suppose that $\sqrt{N}vec(\hat{P}^1_\Delta - P^1_\Delta) \to_d N(0, \Sigma^1_\Delta)$ and that $\Omega_r(\mathcal{A}_0)$ defined in (9) below is nonsingular. If $rank(P^\alpha_\Delta) \le r$ for all $\alpha \in \mathcal{A}_0$, we have $\sqrt{N}\hat{\lambda}_r(\mathcal{A}_0) \to_d N(0, \Omega_r(\mathcal{A}_0))$ as $N \to \infty$, where*

$$\Omega_r(\mathcal{A}_0) = \begin{bmatrix} \Psi^1\Sigma^1_\Delta(\Psi^1)' & \cdots & \Psi^1\Sigma^1_\Delta(\Psi^{|\mathcal{A}_0|})' \\ \vdots & \ddots & \vdots \\ \Psi^{|\mathcal{A}_0|}\Sigma^1_\Delta(\Psi^1)' & \cdots & \Psi^{|\mathcal{A}_0|}\Sigma^1_\Delta(\Psi^{|\mathcal{A}_0|})' \end{bmatrix} \tag{9}$$

*and* $\Psi^\alpha = (B^\alpha_{r,\perp} \otimes (A^\alpha_{r,\perp})')\Pi^\alpha$.

We can test the null hypothesis $H_0 : \text{rank}(P^\alpha_\Delta) \leq r$ for all $\alpha \in \mathcal{A}_0$ by the *average rk statistic* defined as

$$\text{ave-rk}(r, \mathcal{A}_0) = N(\hat{\lambda}_r(\mathcal{A}_0))'(\hat{\Omega}_r(\mathcal{A}_0))^{-1}\hat{\lambda}_r(\mathcal{A}_0), \tag{10}$$

where $\hat{\Omega}_r(\mathcal{A}_0)$ is a consistent estimator of $\Omega_r(\mathcal{A}_0)$. Thus, ave-rk$(r, \mathcal{A}_0)$ combines information from $\hat{\lambda}^\alpha_r$'s across different $\alpha$'s using the inverse of their covariance matrix as the weight. Under the assumptions in Corollary 2, ave-rk$(r, \mathcal{A}_0)$ converges in distribution to a $\chi^2(\nu(\mathcal{A}_0))$ random variable, where $\nu(\mathcal{A}_0) \equiv \sum_{\alpha\in\mathcal{A}_0}(|\Delta_{x^\alpha}|-r)(|\Delta_{y^\alpha}|-r)$ is the number of elements in $\hat{\lambda}_r(\mathcal{A}_0)$. We note, however, that the average rk statistic may give a slack lower bound when enumerating sufficiently many of the groupings and partitions of the data is not computationally feasible.

When $\nu(\mathcal{A}_0)$ is larger than the rank of $\Sigma^1_\Delta$, the covariance matrix $\Omega_r(\mathcal{A}_0)$ becomes singular and the assumption of Corollary 2 is violated. In such a case, if $\Pr(\text{rank}(\hat{\Omega}_r(\mathcal{A}_0)) = \text{rank}(\Omega_r(\mathcal{A}_0))) \to 1$, using the M-P pseudoinverse of $\hat{\Omega}_r(\mathcal{A}_0)$ in the ave-rk statistic (10) gives a test statistic whose asymptotic distribution is $\chi^2(\text{rank}(\Omega_r(\mathcal{A}_0)))$ (Andrews, 1987). However, in finite samples, if $\hat{\Omega}_r(\mathcal{A}_0)$ has a very small but nonzero eigenvalue, its pseudoinverse may take a very large value and behave erratically. To deal with the singularity of $\Omega_r(\mathcal{A}_0)$, we follow Lütkepohl and Burda (1997) to use a suitable reduced rank estimator in place of $\hat{\Omega}_r(\mathcal{A}_0)$. Given a small constant $c$, we apply a singular decomposition to $\hat{\Omega}_r(\mathcal{A}_0)$ and replace the eigenvalues smaller than $c$ with zero. Let $\hat{\Omega}_{r,c}(\mathcal{A}_0)$ denote this low-rank approximation of $\hat{\Omega}_r(\mathcal{A}_0)$, and define the *modified* average rk statistic as

$$\text{ave-rk}^+(r, \mathcal{A}_0) = N(\hat{\lambda}_r(\mathcal{A}_0))'(\hat{\Omega}_{r,c}(\mathcal{A}_0))^+\hat{\lambda}_r(\mathcal{A}_0). \tag{11}$$

The asymptotic distribution of ave-rk$^+(r, \mathcal{A}_0)$ is $\chi^2(J_c)$, where $J_c$ is the number of eigenvalues of $\Omega_r(\mathcal{A}_0)$ that are no smaller than $c$. The behavior of ave-rk$^+(r, \mathcal{A}_0)$ could be sensitive to the choice of $c$. In the simulations in Section 5, we set $c$ equal to 0.01 times the largest eigenvalue of $\Omega_r(\mathcal{A}_0)$.[5]

We also consider an alternative statistic that is applicable even when $\nu(\mathcal{A}_0)$ is large. In the alternate statistic, we first choose $K$ subsets of $\mathcal{A}_0$ as $\{\mathcal{A}_1, \ldots, \mathcal{A}_K\}$ so that $\mathcal{A}_0 = \bigcup_{j=1}^K A_j$, and construct the ave-rk$^+(r, \mathcal{A}_j)$ as in (11) but using $\mathcal{A}_j$ in place of $\mathcal{A}_0$. We then combine the information in ave-rk$^+(r, \mathcal{A}_j)$ for $j = 1, \ldots, K$ into the modified max-rk statistic defined as max-rk$^+(r) = \max_{j=1,\ldots,K}$ ave-rk$^+(r, \mathcal{A}_j)$. By choosing $\mathcal{A}_j$'s so that the degree of freedom $\nu(\mathcal{A}_j)$ is sufficiently small, max-rk$^+(r)$ would be less sensitive to the choice of $c$ than ave-rk$^+(r, \mathcal{A}_0)$. We can apply the sequential hypothesis testing procedure to max-rk$^+(r)$. Its asymptotic null distribution is not chi-squared but it can be easily simulated using the relation $\sqrt{N}\hat{\lambda}^\alpha_r = \hat{\Psi}^\alpha\sqrt{N}(\text{vec}(\hat{P}^1_\Delta) - \text{vec}(P^1_\Delta))$.

[5] See Lütkepohl and Burda (1997) for other choices of $c$.

# 4 Simulation study

We conduct simulation experiments to assess the finite sample performance of our proposed procedures for selecting the number of components. The reported results are based on 1000 simulated samples from mixtures with $M = 3$ components with three different sample sizes: $N = 500$, 2000, and 8000. To construct the rk statistic (7) and the ave-rk$^+$ statistic (11) for each sample, we estimate $\Omega_r$ and $\Omega_r(\mathcal{A}_0)$ consistently by nonparametric bootstrap using 1000 random samples with replacement from empirical distributions.

In the first experiment, we generate samples of $(X, Y)$ from a 3-component normal mixture $\sum_{m=1}^{3} \pi^m N_2(\mu^m, I_2)$ with $\mu^1 = (0, 0)'$, $\mu^2 = (1.0, 2.0)'$, $\mu^3 = (2.0, 1.0)'$, and $\pi^1 = \pi^2 = \pi^3 = 1/3$. We denote the $100 \times q$ percentile of empirical distributions of $X$ and $Y$ by $z_q^x$ and $z_q^y$, respectively. We consider three different partitions of the form $\Delta^j = \{(-\infty, z_{q_1}^j], (z_{q_1}^j, z_{q_2}^j], \ldots, (z_{q_{t-2}}^j, z_{q_{t-2}}^j], (z_{q_{t-1}}^j, \infty)\}$ for $j = x, y$, where $t$ is the number of partitions. We choose $t = 4$ with $(q_1, q_2, q_3) = (0.25, 0.5, 0.75)$ for Partition 1 and $(q_1, q_2, q_3) = (0.1, 0.5, 0.9)$ for Partition 2. Thus, the support of $X$ or $Y$ is partitioned into 4 (asymptotically) equiprobable subsets in Partition 1 while Partition 2 provides the finer partitions in the tail part of distributions than Partition 1. Partition 3 combines Partitions 1 and 2 as $(q_1, q_2, q_3, q_4, q_5) = (0.1, 0.25, 0.5, 0.75, 0.9)$.

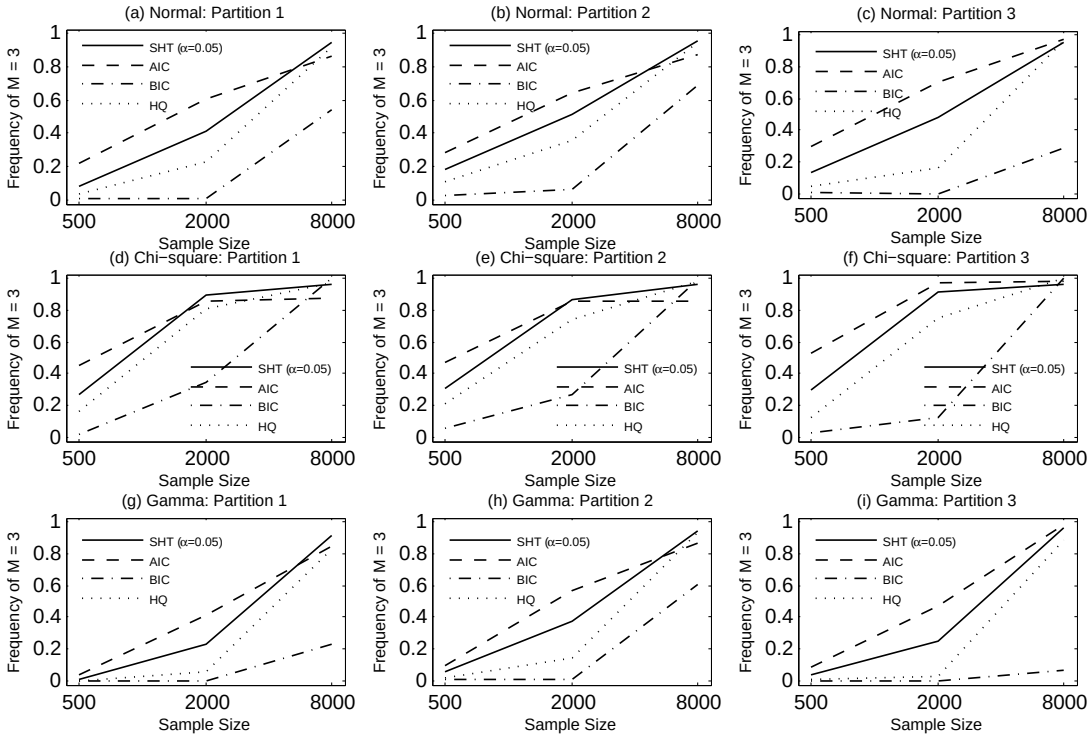


Figure 1: Selection Frequencies of the Number of Components: Two Variables

Figures 1(a), (b), and (c) show the frequency that the SHT with $\alpha = 0.05$, AIC, BIC, and

HQ based on the rk statistic correctly select $M = 3$ for Partitions 1, 2, and 3, respectively. Across different partitions, the performance of all the procedures improves as the sample size increases. With sample sizes 500 and 2000, the AIC outperforms other statistics, but overestimates the number of components at $N = 8000$ in Partitions 1 and 2. BIC exhibits the worst performance among all of the methods in this setup. HQ is a better choice than BIC but is outperformed by SHT in most cases. SHT in Partition 2 performs better than in Partition 1, even though the improvement is not substantial. This is probably because the tail part of distributions provides important information for separately identifying different components in this experiment. SHT in Partition 2 also performs better than SHT in Partition 3; using a larger number of partitions does not necessarily improve performance because $\text{var}(\hat{P}_\Delta)$ increases with the number of partitions.

Figures 1(d), (e), and (f) show the selection frequency across different partitions when we generate samples from 3-component chi-squared mixtures, where $\pi^m = 1/3$, $X^m \sim \chi^2(k_x^m)$, and $Y^m \sim \chi^2(k_y^m)$ with $(k_x^1, k_y^1) = (1, 1)$, $(k_x^2, k_y^2) = (3, 6)$, and $(k_x^3, k_y^3) = (6, 3)$. Figures 1(g), (h), and (i) report the selection frequency for 3-component gamma mixtures with component-specific shape parameters, where $\pi^m = 1/3$, $X^m \sim \text{Gamma}(k_1^m, 1)$, and $Y^m \sim \text{Gamma}(k_2^m, 1)$ with $(k_1^1, k_2^1) = (1, 1)$, $(k_1^2, k_2^2) = (1.5, 3)$, and $(k_1^3, k_2^3) = (3, 1.5)$. In both the chi-squared and gamma mixtures, the relative performance across SHT, AIC, BIC, and HQ and the relative performance of SHT across different partitions are qualitatively similar to the case of the normal mixture discussed above.

Next, we consider a 4-variable, 3-component normal mixture, where $W = (W_1, \ldots, W_4)'$ follows $\sum_{m=1}^{M} \pi^m N_4(\mu^m, I_4)$ with $\mu^1 = (0, 0, 0, 0)'$, $\mu^2 = (1.0, 2.0, 0.5, 1.0)'$, $\mu^3 = (2.0, 1.0, 1.0, 0.5)'$, and $\pi^1 = \pi^2 = \pi^3 = 1/3$. Following the approach in Section 3.4, we consider three groupings: $\{X^1, Y^1\} = \{(W_1, W_2), (W_3, W_4)\}$, $\{X^2, Y^2\} = \{(W_1, W_3), (W_2, W_4)\}$, and $\{X^3, Y^3\} = \{(W_1, W_4), (W_2, W_3)\}$. We then estimate the probability matrix $P_\Delta^\alpha$ for each $\alpha \in \{1, 2, 3\}$, and construct the ave-rk$^+$ statistic (11) by setting $c$ equal to 0.01 times the largest eigenvalue of $\Omega_r(\mathcal{A}_0)$. The support of $W_i$ is partitioned into 2 equiprobable subsets based on its empirical median, so that the dimension of $P_\Delta^\alpha$ is $4 \times 4$. As an alternative method, we also consider the maximum likelihood estimator (MLE)-based parametric model selection procedure with AIC, BIC, and HQ, where each component distribution is correctly specified as a 4-dimensional normal distribution with unknown means and an unknown diagonal covariance matrix. Figure 2(a) reports the result. The MLE-based AIC substantially overestimates the number of components. While the MLE-based HQ outperforms the ave-rk$^+$-based SHT, their performances are comparable when $N \geq 2000$. This is encouraging given that our ave-rk$^+$-based methods do not use parametric restrictions of the normal mixture model.

We also consider an 8-variable, 3-component normal mixture, where $W = (W_1, \ldots, W_8)'$ follows $\sum_{m=1}^{3} \pi^m N_8(\mu^m, I_8)$ with $\mu^1 = (0, 0, 0, 0, 0, 0, 0, 0)'$, $\mu^2 = (1.0, 2.0, 0.5, 1.0, 0.75, 1.25, 0.25, 0.5)'$, $\mu^3 = (2.0, 1.0, 1.0, 0.5, 1.25, 0.75, 0.5, 0.25)'$, and $\pi^1 = \pi^2 = \pi^3 = 1/3$. We first choose 4 vari-

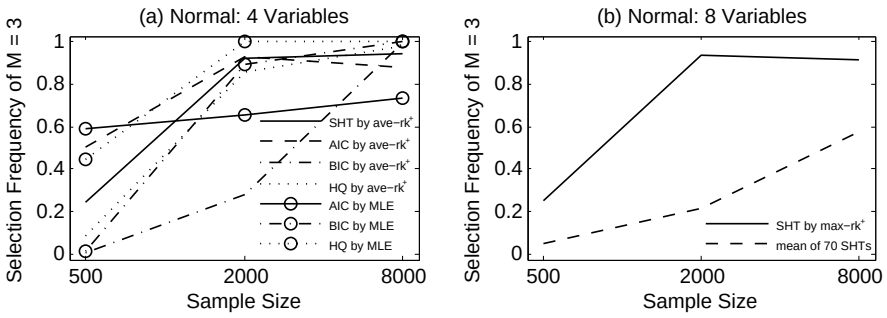


Figure 2: Selection Frequencies of the Number of Components: Four and Eight Variables

ables out of 8, and then construct the ave-rk$^+$ statistic using the procedure for the 4-variable model discussed in the previous paragraph. Because ${}_8C_4 = 70$, we construct 70 ave-rk$^+$ statistics. Finally, we compute the max-rk$^+$ statistic from these 70 ave-rk$^+$ statistics. Figure 2(b) reports the selection frequency of SHT using the max-rk$^+$ statistic and the mean selection frequencies by SHT across 70 different ave-rk$^+$ statistics. The max-rk$^+$ statistic performs substantially better than individual ave-rk$^+$ statistics, suggesting that combining information from different ave-rk$^+$ statistics improves the performance of our procedures.

# 5 Examples

## 5.1 Intergenerational occupational mobility in Great Britain

We estimate the number of latent classes in the table of intergenerational mobility from father's occupation to subject's occupation in Great Britain, originally studied by Clogg (1981) using latent class models. Clogg estimates the 2-class and 3-class models using these data by imposing *a priori* restrictions on a set of parameters. Panel (1) of Table 1 presents the result of the SHT procedure applied to the $5 \times 5$ table of social mobility in Great Britain taken from Table 1.B of Clogg (1981); the null hypothesis that the number of latent classes is no more than 4 is rejected at any significance level. The AIC, BIC, and HQ procedures also indicate that the number of latent classes is at least 5 (not reported in the table). As reported in Panel (2) of Table 1, we further examine the number of latent classes in the $8 \times 8$ table using Table 1.C of Clogg (1981) starting from the null hypothesis of no more than 5 classes. SHT suggests that this intergenerational occupational mobility data could be generated from 7 latent classes while BIC, AIC, and HQ suggest 5, 8, and 6 latent classes, respectively. Overall, the results of our procedures suggest that there are more than 5 latent classes, rejecting the 2- and 3-class models studied by Clogg.

Table 1: Intergenerational Social Mobility in Great Britain

| | (1) 5×5 Table | | | | (2) 8×8 Table | | |
|---|---|---|---|---|---|---|---|
| Null hypothesis ($H_0$) | $M=1$ | $M=2$ | $M=3$ | $M=4$ | $M=5$ | $M=6$ | $M=7$ |
| rk statistic | 557.08 | 144.64 | 48.18 | 15.71 | 35.59 | 12.33 | 2.27 |
| d.f. | 16 | 9 | 4 | 1 | 9 | 4 | 1 |
| $p$-value | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.015 | 0.132 |

Notes: The data are from Tables 1.B and 1.C of Clogg (1981).

Table 2: Type of Trade and Ethnic Group Data, Amsterdam and Rotterdam

| Values of rk statistics and the degrees of freedom | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Amsterdam | | | | Rotterdam | | | |
| Null hypothesis ($H_0$) | $M=1$ | $M=2$ | $M=3$ | $M=4$ | $M=1$ | $M=2$ | $M=3$ | $M=4$ |
| rk statistic | 318.09 | 57.87 | 13.48 | 0.23 | 190.23 | 60.82 | 9.20 | 1.88 |
| d.f. | 20 | 12 | 6 | 2 | 20 | 12 | 6 | 2 |
| $p$-value | 0.000 | 0.000 | 0.036 | 0.891 | 0.000 | 0.000 | 0.163 | 0.391 |

Notes: The data are from Table 2a of van der Heijden et al. (2002).

### 5.2 Types of trades started by different ethnic groups

The second example analyzes the difference across ethnic groups in the types of trades they start in two large cities in the Netherlands, Amsterdam and Rotterdam, studied by van der Heijden, van der Ark, and Mooijaart (2002). There are 6 types of trades and 5 ethnic groups for each of the two cities. The members of some ethnic groups are more likely to start certain types of trades because of such factors as the number of clients in the same ethnic group or their level of human capital, including knowledge of the Dutch language. From this viewpoint, each latent class could be reflecting a specific type of network and human capital. Based on likelihood ratio statistics, van der Heijden et al. (2002) conclude that the number of latent classes $M = 3$ "seems adequate" for both Amsterdam and Rotterdam. We apply our procedures to examine if the number of latent classes is at least 3 or not. Table 2 shows the rk statistics and the corresponding $p$-values from the SHT procedure. For Amsterdam, SHT suggests 3 or 4 latent classes, whereas AIC, BIC, and HQ suggest 4, 2, and 3 latent classes, respectively. For Rotterdam, all of our procedures suggest 3 latent classes.

### 5.3 Response patterns in five-item subsets of LSAT

In our third example, we analyze the response patterns in two different five-item subsets of LSAT, denoted by LSAT-6 and LSAT-7, originally studied by Mislevy (1984). We employ the max-rk$^+$ statistic to these data. The response to five items is represented by $\{W_1, W_2, W_3, W_4, W_5\}$ where $W_i \in \{0, 1\}$. We first choose 4 items out of 5 and then construct the ave-rk$^+$ statistic from the estimates of $P_\Delta^\alpha$s for three different groupings $\alpha = 1, 2, 3$, where we estimate the covariance matrix $\Omega_{r,c}(\mathcal{A}_0)$ using the asymptotic formula. Because there are

${}_4C_5 = 5$ different ways of choosing 4 items out of 5, we construct the max-rk$^+$ statistic from the 5 ave-rk$^+$ statistics. SHT based on the max-rk$^+$ statistic suggests that $M \geq 2$ in LSAT-6 at $\alpha = 0.1$, 0.05, and 0.01, and that $M \geq 3$ at $\alpha = 0.1$ and 0.05 and $M \geq 2$ at $\alpha = 0.01$ in LSAT-7.

### 5.4 Example 3 of Hettmansperger and Thomas (2000)

We also apply our procedure to the data that consist of 83 college-age women each with eight replications of Witkin's rod-and-frame task. The response variable, measured as the rod's error deviation in degrees from the vertical, is continuously distributed. Hettmansperger and Thomas apply various tests of the number of components to the data transformed to a binomial mixture: Lindsay's (1995) gradient function method suggests $M = 4$, the Hellinger and the Pearson penalized distances suggest $M = 2$, and the bootstrapped likelihood ratio test suggests $M = 3$. Following Hettmansperger and Thomas, we use the known cut-off point of 5 degree to define the 8 response variables $\{W_1, \ldots, W_8\}$ with $W_j = 1(|X_j| \leq 5^\circ)$, where $X_j$ is the $j$-th error. We then calculate the max-rk$^+$ statistic from the 70 ave-rk$^+$ statistics with $c = 0.01$, each of which is constructed from the 4 chosen variables out of the 8 response variables using the covariance matrix $\Omega_r(\mathcal{A}_0)$ estimated by the asymptotic formula. The SHT based on the max-rk$^+$ statistic suggests that $M \geq 3$ at $\alpha = 0.1$, 0.05, and 0.01, consistent with the Hettmansperger and Thomas's result from the bootstrapped likelihood ratio test.

# Supplementary Appendix

This supplementary appendix contains the following details: (A) proof of the results in the paper, (B) the asymptotic variance of a multivariate contingency table when its categories are defined by the empirical quantiles of the marginal data (from Borkowf (2000)), (C) tables of simulation results, and (D) additional results from empirical examples.

## A Proofs of the results in the main text

### A.1 Proof of Proposition 1

The proofs are given in Cohen and Rothblum (1993). Proposition 1(a),(b), and (c) correspond to Lemma 2.3, Theorem 4.1, and Corollary 4.2 of Cohen and Rothblum (1993), respectively.

### A.2 Proof of Proposition 2

First, note that $M_0 = \text{rank}_+(P(\theta))$ holds from the definition of $\Theta$. Define $P_x$, $P_y$, and $V$ as $P_x = [p_x^1, \ldots, p_x^{M_0}]'$, $P_y = [p_y^1, \ldots, p_y^{M_0}]'$, and $V = \text{diag}(\pi^1, \ldots, \pi^{M_0})$, respectively, where $P_x$ is $M_0 \times |\Delta_x|$ and $P_y$ is $M_0 \times |\Delta_y|$. Then, $P(\theta)$ is written as $P(\theta) = P_x' V P_y$. Applying the Frobenius inequality to the right-hand side of $\text{rank}(P(\theta)) = \text{rank}(P_x' V P_y)$ and noting that $\text{rank}(P_x' V) = \text{rank}(P_x')$, $\text{rank}(V P_y) = \text{rank}(P_y)$, and $\text{rank}(V) = \text{rank}_+(P(\theta))$, we obtain $\text{rank}(P(\theta)) \geq \text{rank}(P_x) + \text{rank}(P_y) - \text{rank}_+(P(\theta))$. Suppose that $\text{rank}_+(P(\theta)) > \text{rank}(P(\theta))$. Then, we have $2\text{rank}_+(P(\theta)) > \text{rank}(P_x) + \text{rank}(P_y)$, and thus, either $\text{rank}(P_x)$ or $\text{rank}(P_y)$ must be strictly smaller than $\text{rank}_+(P(\theta))$. Because $\text{rank}_+(P(\theta)) \leq \min\{|\Delta_x|, |\Delta_y|\}$, either $P_x$ or $P_y$ does not have full rank. Note that the elements of a rank-deficient matrix must satisfy a set of polynomial restrictions, and hence, must lie in a zero set of a finite collection of polynomials. Therefore, in the space of $M_0 \times |\Delta_x|$ matrices that represents the space of $\{p_x^m\}_{m=1}^{M_0}$, the set of $P_x$'s that do not have full rank has zero Lebesgue measure (see Allman et al. 2009, p. 3105), and a similar argument holds for $P_y$. This proves the stated result. □

### A.3 Proof of Proposition 3

From Lemma 17 of Allman et al. (2009), there exists a positive integer $\kappa$ and real numbers $x_1 < x_2 < \cdots < x_{\kappa-1}$ such that the vectors $\{(F_x^m(x_1), \ldots, F_x^m(x_{\kappa-1}), 1)\}_{1 \leq m \leq M}$ are linearly independent. Therefore, it is possible to construct $\Delta$ such that $P_x = [p_x^1, \ldots, p_x^M]'$ has rank $M$. Similarly, it is possible to construct $\Delta$ such that $P_y = [p_y^1, \ldots, p_y^M]'$ has rank $M$. Write $P_\Delta$ as $P_\Delta = P_x' V P_y$, where $V = \text{diag}(\pi^1, \ldots, \pi^M)$. Applying the Frobenius inequality to the right-hand side of $\text{rank}(P_\Delta) = \text{rank}(P_x' V P_y)$ and noting that $\text{rank}(P_x' V) = \text{rank}(P_x')$, $\text{rank}(V P_y) = \text{rank}(P_y)$, and $\text{rank}(V) = M$, we obtain $\text{rank}(P_\Delta) \geq M$. The stated result of part (a) then follows because $\text{rank}(P_\Delta) \leq M$. For part (b), Theorem 2.2 of Dong, Lin and

Chu (2009) shows that when we draw a matrix $P$ from a set of matrices whose nonnegative rank is $M_0$, the probability of $\text{rank}(P) = M_0$ is one. It follows that $\Pr(\text{rank}_+(P_\Delta) = \text{rank}(P_\Delta)) = \sum_{m=1}^{M} \Pr[\text{rank}_+(P_\Delta) = \text{rank}(P_\Delta) | \text{rank}_+(P_\Delta) = m] \Pr(\text{rank}_+(P_\Delta) = m) = \sum_{m=1}^{M} \Pr(\text{rank}_+(P_\Delta) = m) = 1$, and part (b) consequently follows. □

### A.4 Proof of Proposition 4

As in the proof of Theorem 4 of Allman et al. (2009; p. 3119), given an $n \times a_1$ matrix $A_1$ and an $n \times a_2$ matrix $A_2$, define an $n \times a_1 a_2$ matrix $A = A_1 \otimes^{\text{row}} A_2$ as the row-wise tensor product, so that $A(i, a_2(j-1)+k) = A_1(i,j)A_2(i,k)$. Let $P_j = [p_j^1, \ldots, p_j^{M_0}]'$ be an $(M_0 \times |\Delta_j|)$ matrix collecting the distribution of $W_j$ on $\Delta_j$ across all the components. Define $P_{x^\alpha} = \otimes^{\text{row}}_{j \in S_x(\alpha)} P_j$, and then, from Lemma 12 of Allman et al. (2009), we have that $P_{x^\alpha}$ collects the distribution of $X^\alpha$ on $\Delta_{x^\alpha}$ across all the components. Define $P_{y^\alpha}$ similarly, and then, $P^\alpha(\theta)$ may be written as $P^\alpha(\theta) = P'_{x^\alpha} V P_{y^\alpha}$, where $V = \text{diag}(\pi^1, \ldots, \pi^{M_0})$. From Lemma 13 of Allman et al. (2009), we have $\text{rank}(P_{x^\alpha}) = \min\{M_0, |\Delta_{x^\alpha}|\}$ and $\text{rank}(P_{y^\alpha}) = \min\{M_0, |\Delta_{y^\alpha}|\}$ for generic $P_j$'s; that is, all the $P_j$'s except for a set of Lebesgue measure zero. Because $|\Delta_{x^\alpha}|, |\Delta_{y^\alpha}| \geq M_0$ by assumption, we have $\text{rank}(P_{x^\alpha}) = \text{rank}(P_{y^\alpha}) = M_0$ for generic $P_j$'s. Therefore, $\text{rank}(P^\alpha(\theta)) = M_0$ holds for generic $P_j$'s, from the Frobenius inequality and proceeding as in the proof of Proposition 2. □

### A.5 Proof of Proposition 5

The proof is given by the proof of Theorem 1 in Kleibergen and Paap (2006). □

### A.6 Proof of Proposition 6

First, we show that $\Pr(\tilde{r} < r_0) \to 0$. If $\tilde{r} < r_0$, $Q(r) < Q(r_0)$ for some $r < r_0$. Thus, $\Pr(\tilde{r} < r_0) \leq \sum_{r=1}^{r_0-1} \Pr(Q(r) < Q(r_0))$. Observe that $\Pr(Q(r) < Q(r_0)) = \Pr(\text{rk}(r) - \text{rk}(r_0) - f(N)g(r) + f(N)g(r_0) < 0) = \Pr(N\hat{\lambda}'_r \hat{\Omega}_r^{-1} \hat{\lambda}_r - N\hat{\lambda}'_{r_0} \hat{\Omega}_{r_0}^{-1} \hat{\lambda}_{r_0} + f(N)(g(r_0) - g(r)) < 0)$. For any $r < r_0$, this probability tends to 0 as $N \to \infty$ because $f(N)/N \to 0$, $\hat{\lambda}'_r \hat{\Omega}_r^{-1} \hat{\lambda}_r \to_p \lambda'_r \Omega_r^{-1} \lambda_r > 0$, and $\hat{\lambda}'_{r_0} \hat{\Omega}_{r_0}^{-1} \hat{\lambda}_{r_0} \to_p \lambda'_{r_0} \Omega_{r_0}^{-1} \lambda_{r_0} = 0$.

Second, we show that $\Pr(\tilde{r} > r_0) \to 0$. As above, we have $\Pr(\tilde{r} > r_0) \leq \sum_{r=r_0+1}^{t^*} \Pr(Q(r) < Q(r_0))$ and $\Pr(Q(r) < Q(r_0)) = \Pr(N\hat{\lambda}'_r \hat{\Omega}_r^{-1} \hat{\lambda}_r - N\hat{\lambda}'_{r_0} \hat{\Omega}_{r_0}^{-1} \hat{\lambda}_{r_0} + f(N)(g(r_0) - g(r)) < 0)$. For any $r > r_0$, this probability tends to 0 as $N \to \infty$ because both $N\hat{\lambda}'_r \hat{\Omega}_r^{-1} \hat{\lambda}_r$ and $N\hat{\lambda}'_{r_0} \hat{\Omega}_{r_0}^{-1} \hat{\lambda}_{r_0}$ converge to a chi-square random variable, $f(N) \to \infty$, and $\Pr(g(r_0) - g(r) > 0) \to 1$ as $N \to \infty$. □

# B Asymptotic variance of vec($\hat{P}_\Delta$) when the empirical quantiles are used for partitioning

In simulations in Section 4, we use the empirical quantiles of the marginal data to construct the partitions in forming $\hat{P}_\Delta$. When the categories of a multivariate contingency table are defined by the empirical quantiles of the marginal data, the resulting contingency table does not follow a multinomial distribution, but a distribution called the empirical multivariate quantile-partitioned (EMQP) distribution. Borkowf (2000) derives the asymptotic variance of the EMQP distribution, which we summarize in the following. See Borkowf (2000) for further details.

Let $W = (W_1, \ldots, W_k)'$ be a $k$-dimensional random variable with the distribution function $F(w)$. Let $d_j$ be the number of categories in the $j$-th dimension, and define an ordered set of population cumulative marginal proportions in the $j$-th dimension as $\{\gamma_j(i)\}_{i=0}^{d_j}$ with $0 = \gamma_j(0) < \gamma_j(1) < \cdots < \gamma_j(d_j) = 1$. Let $F_j(w_j)$ denote the marginal distribution in the $j$-th dimension, and let $\xi_j(i) = F_j^{-1}(\gamma_j(i))$ denote the population quantile for the $i$-th category of the $j$-th dimension. For a vector of indices $a = (a_1, \ldots, a_k)'$ of categories with $1 \le a_j \le d_j$, let $\xi(a) = (\xi_1(a_1), \ldots, \xi_k(a_k))'$ be the $k \times 1$ vector of population quantiles, and let $\phi(a) = F(\xi(a))$ denote a cumulative joint proportion.

Given the iid observations of $W_i = (W_{i1}, \ldots, W_{ik})'$ for $i = 1, \ldots, n$ from $F(w)$, define the joint and marginal empirical distribution functions as $\hat{F}(w) = n^{-1} \sum_{i=1}^n \prod_{j=1}^k I\{W_{ij} \le w_j\}$ and $\hat{F}_j(w_j) = n^{-1} \sum_{i=1}^n I\{W_{ij} \le w_j\}$. Let $u_j(i) = \inf\{u : \gamma_j(i) \le \hat{F}_j(u)\}$ denote the empirical quantile for the $i$-th category of the $j$-th dimension, and let $u(a) = (u_1(a_1), \ldots, u_k(a_k))'$ be a vector of empirical quantiles. In our simulations, we estimate $P_\Delta$ by the proportion of observations in each cell of the partition $\Delta$, which is constructed using the empirical quantiles. Let $\hat{p}(a)$ be the proportion of observations in the $a$-th cell, then $\hat{p}(a)$ can be derived from $\hat{F}(a)$ by the relation (Borkowf (2000), equation (2.1))

$$\hat{p}(a) = \sum_{b_1=a_1-1}^{a_1} \cdots \sum_{b_k=a_k-1}^{a_k} \left[ \prod_{j=1}^k (-1)^{a_j - b_j} \right] \hat{F}(u(b)). \tag{12}$$

Borkowf (2000, equation (3.10)) shows that the asymptotic covariance between $n^{1/2}\hat{F}(u(a))$ and $n^{1/2}\hat{F}(u(b))$ is given by

$$\lambda(a)'\Omega(a,b)\lambda(b), \quad \lambda(a) = [-\eta(a)'\ 1]', \tag{13}$$

where $\eta(a) = (\eta_1(a), \ldots, \eta_k(a))'$ is a $k \times 1$ vector of conditional proportions with $\eta_j(a) = \Pr[\bigcap_{\ell=1,\ell\neq j}(W_\ell \le \xi_\ell(a_\ell))|W_j = \xi_j(a_j)]$ denoting the probability that $W_\ell \le \xi_\ell(a_\ell)$ for all $\ell \neq j$ given that $W_j = \xi_j(a_j)$. For $\Omega(a,b) = \{\{\omega_{j\ell}\}_{\ell=1}^{k+1}\}_{j=1}^{k+1}$, its first $k$ diagonal elements are $\omega_{jj} = \gamma_j(\min\{a_j, b_j\}) - \gamma_j(a_j)\gamma_j(b_j)$. The $(j,\ell)$-th off-diagonal element in the upper left $k \times k$

submatrix is $\omega_{j\ell} = \phi(m^a) - \gamma_j(a_j)\gamma_\ell(b_\ell)$, where $m^a$ is a $k \times 1$ vector with $m^a_j = a_j$, $m^a_\ell = b_\ell$, and $m^a_i = d_i$ for $i \neq \{j, \ell\}$. The first $k$ elements in the last row are $\omega_{(k+1)j} = \phi(m^b) - \phi(a)\gamma_j(b_j)$, where $m^b_j = \min\{a_j, b_j\}$ and $m^b_i = a_i$ for $i \neq j$, and the elements in the last column are similar, but with the indices transposed. Finally, the lower right cell is $\omega_{(k+1)(k+1)} = \phi(m^c) - \phi(a)\phi(b)$, where $m^c_j = \min\{a_j, b_j\}$.

The asymptotic covariance of $\hat{p}(a)$ and $\hat{p}(b)$ is derived from the asymptotic covariance between $n^{1/2}\hat{F}(u(a))$ and $n^{1/2}\hat{F}(u(b))$, and relation (12). To estimate $\lambda(a)'\Omega(a,b)\lambda(b)$ in (13), one needs to estimate the conditional distribution function $\eta_j(a) = \Pr[\bigcap_{\ell=1,\ell\neq j}(W_\ell \leq \xi_\ell(a_\ell))|W_j = \xi_j(a_j)]$ for $j = 1, \ldots, k$ nonparametrically by a kernel estimator. Alternatively, one may estimate the asymptotic variance of the EMQP distribution by bootstrap.

# C Additional simulation results

Tables 3, 4, and 5 report the simulation results from 2-variable, 3-component normal, chi-squared, and gamma mixtures, respectively. Table 6 reports the simulation result from 4-variable, 3-component normal mixture while Tables 7-8 report the simulation results from 8-variable, 3-component normal mixtures. Figure 1 is based on Tables 3-5 while Figure 2 is based on Tables 6-7.

We also consider an 8-variable, 3-component normal mixture in which the distribution of $W_1, \ldots, W_5$ is the same as in the previous experiment, but $W_6$, $W_7$, and $W_8$ are set to have identical distributions across sub-populations with $\mu^1 = (0,0,0,0,0,0,0,0)'$, $\mu^2 = (1.0, 2.0, 0.5, 1.0, 0.75, 0, 0, 0)'$, and $\mu^3 = (2.0, 1.0, 1.0, 0.5, 1.25, 0, 0, 0)'$. This is a challenging setup because only five out of eight variables can be used to identify the number of components. As shown in Table 8, the performance of our procedures in this experiment is generally worse than the performance in the experiment in Figure 2(b) but the max-rk$^+$ statistic successfully chooses the correct $M$ at $N = 8000$.

Table 3: Selection Frequencies of the Number of Components for Normal Mixtures: Two Variables

| Selection frequencies by the rk statistic using $(X, Y)$ from normal mixtures | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Partition 1: $t = 4$ with $(q_1, q_2, q_3) = (0.25, 0.5, 0.75)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.021 | 0.891 | 0.082 | 0.006 | 0.000 | 0.566 | 0.414 | 0.020 | 0.000 | 0.008 | 0.950 | 0.042 |
| AIC | 0.004 | 0.757 | 0.215 | 0.024 | 0.000 | 0.317 | 0.609 | 0.074 | 0.000 | 0.001 | 0.866 | 0.133 |
| BIC | 0.464 | 0.533 | 0.003 | 0.000 | 0.000 | 0.989 | 0.011 | 0.000 | 0.000 | 0.456 | 0.543 | 0.001 |
| HQ | 0.092 | 0.876 | 0.031 | 0.001 | 0.000 | 0.766 | 0.226 | 0.008 | 0.000 | 0.043 | 0.921 | 0.036 |
| d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| | Partition 2: $t = 4$ with $(q_1, q_2, q_3) = (0.1, 0.5, 0.9)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.024 | 0.778 | 0.183 | 0.015 | 0.000 | 0.454 | 0.512 | 0.034 | 0.000 | 0.003 | 0.955 | 0.042 |
| AIC | 0.008 | 0.654 | 0.287 | 0.051 | 0.000 | 0.246 | 0.639 | 0.115 | 0.000 | 0.000 | 0.874 | 0.126 |
| BIC | 0.385 | 0.593 | 0.022 | 0.000 | 0.000 | 0.939 | 0.061 | 0.000 | 0.000 | 0.305 | 0.692 | 0.003 |
| HQ | 0.088 | 0.795 | 0.113 | 0.004 | 0.000 | 0.629 | 0.353 | 0.018 | 0.000 | 0.020 | 0.949 | 0.031 |
| d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| | Partition 3: $t = 6$ with $(q_1, q_2, q_3, q_4, q_5) = (0.1, 0.25, 0.5, 0.75, 0.9)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.027 | 0.830 | 0.135 | 0.008 | 0.000 | 0.493 | 0.483 | 0.024 | 0.000 | 0.000 | 0.957 | 0.043 |
| AIC | 0.004 | 0.695 | 0.297 | 0.004 | 0.000 | 0.281 | 0.701 | 0.018 | 0.000 | 0.000 | 0.971 | 0.029 |
| BIC | 0.521 | 0.472 | 0.007 | 0.000 | 0.000 | 0.997 | 0.003 | 0.000 | 0.000 | 0.708 | 0.292 | 0.000 |
| HQ | 0.133 | 0.821 | 0.045 | 0.001 | 0.000 | 0.836 | 0.164 | 0.000 | 0.000 | 0.034 | 0.966 | 0.000 |
| d.f. | 25.000 | 16.000 | 9.000 | 0.000 | 25.000 | 16.000 | 9.000 | 0.000 | 25.000 | 16.000 | 9.000 | 0.000 |

Notes: The true number of components is $M = 3$. $(X, Y)'$ follows a 3-component normal mixture distribution, where each component distribution is $N_2(\mu^m, I_2)$ with $\mu^1 = (0, 0)'$, $\mu^2 = (1.0, 2.0)'$, and $\mu^3 = (2.0, 1.0)'$. The mixing proportion is $\pi^1 = \pi^2 = \pi^3 = 1/3$. We set $\Delta = \{(-\infty, z_{q_1}], (z_{q_1}, z_{q_2}), \ldots, (z_{q_{t-1}}, \infty)\}$, where $z_q$ is the $100 \times q$ percentile of the empirical distribution.

Table 4: Selection Frequencies of the Number of Components for Chi-squared Mixtures: Two Variables

| Selection frequencies by the rk statistic using $(X, Y)$ from chi-squared mixtures | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Partition 1: $t = 4$ with $(q_1, q_2, q_3) = (0.25, 0.5, 0.75)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.000 | 0.714 | 0.273 | 0.013 | 0.000 | 0.069 | 0.898 | 0.033 | 0.000 | 0.000 | 0.958 | 0.042 |
| AIC | 0.000 | 0.489 | 0.456 | 0.055 | 0.000 | 0.015 | 0.860 | 0.125 | 0.000 | 0.000 | 0.873 | 0.127 |
| BIC | 0.012 | 0.970 | 0.018 | 0.000 | 0.000 | 0.655 | 0.345 | 0.000 | 0.000 | 0.000 | 0.998 | 0.002 |
| HQ | 0.000 | 0.826 | 0.165 | 0.009 | 0.000 | 0.169 | 0.811 | 0.020 | 0.000 | 0.000 | 0.969 | 0.031 |
| d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| | Partition 2: $t = 4$ with $(q_1, q_2, q_3) = (0.1, 0.5, 0.9)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.000 | 0.663 | 0.303 | 0.034 | 0.000 | 0.094 | 0.863 | 0.043 | 0.000 | 0.000 | 0.965 | 0.035 |
| AIC | 0.000 | 0.449 | 0.469 | 0.082 | 0.000 | 0.019 | 0.857 | 0.124 | 0.000 | 0.000 | 0.852 | 0.148 |
| BIC | 0.003 | 0.938 | 0.058 | 0.001 | 0.000 | 0.726 | 0.273 | 0.001 | 0.000 | 0.001 | 0.996 | 0.003 |
| HQ | 0.000 | 0.771 | 0.209 | 0.020 | 0.000 | 0.230 | 0.738 | 0.032 | 0.000 | 0.000 | 0.977 | 0.023 |
| d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| | Partition 3: $t = 6$ with $(q_1, q_2, q_3, q_4, q_5) = (0.1, 0.25, 0.5, 0.75, 0.9)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.000 | 0.691 | 0.293 | 0.016 | 0.000 | 0.055 | 0.913 | 0.032 | 0.000 | 0.000 | 0.965 | 0.035 |
| AIC | 0.000 | 0.465 | 0.525 | 0.010 | 0.000 | 0.012 | 0.969 | 0.019 | 0.000 | 0.000 | 0.977 | 0.023 |
| BIC | 0.005 | 0.966 | 0.029 | 0.000 | 0.000 | 0.873 | 0.127 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| HQ | 0.000 | 0.877 | 0.122 | 0.001 | 0.000 | 0.248 | 0.752 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| d.f. | 25.000 | 16.000 | 9.000 | 0.000 | 25.000 | 16.000 | 9.000 | 0.000 | 25.000 | 16.000 | 9.000 | 0.000 |

Notes: The true number of components is $M = 3$. $(X, Y)'$ follows a 3-component chi-squared mixture distribution, where $X^m$ and $Y^m$ are independently drawn from chi-squared distributions with $k_x^m$ and $k_y^m$ degrees of freedom, respectively, with $(k_x^1, k_y^1) = (1, 1)$, $(k_x^2, k_y^2) = (3, 6)$, and $(k_x^3, k_y^3) = (6, 3)$. The mixing proportion is $\pi^1 = \pi^2 = \pi^3 = 1/3$. We set $\Delta = \{(-\infty, z_{q_1}], (z_{q_1}, z_{q_2}), \ldots, (z_{q_{t-1}}, \infty)\}$, where $z_q$ is the $100 \times q$ percentile of the empirical distribution.

Table 5: Selection Frequencies of the Number of Components for Gamma Mixtures: Two Variables

| Selection frequencies by the rk statistic using $(X, Y)$ from gamma mixtures | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Partition 1: $t = 4$ with $(q_1, q_2, q_3) = (0.25, 0.5, 0.75)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.828 | 0.161 | 0.011 | 0.000 | 0.203 | 0.560 | 0.227 | 0.010 | 0.000 | 0.038 | 0.919 | 0.043 |
| AIC | 0.653 | 0.306 | 0.036 | 0.005 | 0.107 | 0.424 | 0.415 | 0.054 | 0.000 | 0.006 | 0.843 | 0.151 |
| BIC | 0.998 | 0.002 | 0.000 | 0.000 | 0.993 | 0.007 | 0.000 | 0.000 | 0.339 | 0.434 | 0.227 | 0.000 |
| HQ | 0.964 | 0.036 | 0.000 | 0.000 | 0.689 | 0.253 | 0.057 | 0.001 | 0.002 | 0.145 | 0.826 | 0.027 |
| d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| | Partition 2: $t = 4$ with $(q_1, q_2, q_3) = (0.1, 0.5, 0.9)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.699 | 0.240 | 0.055 | 0.006 | 0.108 | 0.488 | 0.378 | 0.026 | 0.000 | 0.008 | 0.946 | 0.046 |
| AIC | 0.540 | 0.341 | 0.097 | 0.022 | 0.058 | 0.296 | 0.564 | 0.082 | 0.000 | 0.002 | 0.864 | 0.134 |
| BIC | 0.959 | 0.035 | 0.006 | 0.000 | 0.954 | 0.039 | 0.007 | 0.000 | 0.090 | 0.299 | 0.610 | 0.001 |
| HQ | 0.862 | 0.117 | 0.019 | 0.002 | 0.510 | 0.337 | 0.144 | 0.009 | 0.000 | 0.032 | 0.930 | 0.038 |
| d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| | Partition 3: $t = 6$ with $(q_1, q_2, q_3, q_4, q_5) = (0.1, 0.25, 0.5, 0.75, 0.9)$ | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.742 | 0.216 | 0.039 | 0.003 | 0.142 | 0.591 | 0.248 | 0.019 | 0.000 | 0.008 | 0.961 | 0.031 |
| AIC | 0.556 | 0.356 | 0.086 | 0.002 | 0.092 | 0.426 | 0.470 | 0.012 | 0.000 | 0.000 | 0.977 | 0.023 |
| BIC | 0.978 | 0.022 | 0.000 | 0.000 | 0.997 | 0.003 | 0.000 | 0.000 | 0.535 | 0.395 | 0.070 | 0.000 |
| HQ | 0.922 | 0.073 | 0.005 | 0.000 | 0.770 | 0.205 | 0.025 | 0.000 | 0.001 | 0.114 | 0.885 | 0.000 |
| d.f. | 25.000 | 16.000 | 9.000 | 0.000 | 25.000 | 16.000 | 9.000 | 0.000 | 25.000 | 16.000 | 9.000 | 0.000 |

Notes: The true number of components is $M = 3$. $(X, Y)'$ follows a 3-component chi-squared mixture distribution, where each component distribution is $(X^m, Y^m) \sim (\text{Gamma}(k_1^m, 1), \text{Gamma}(k_2^m, 1))$ with $(k_1^1, k_2^1) = (1, 1)$, $(k_1^2, k_2^2) == (1.5, 3)$, and $(k_1^3, k_2^3) == (3, 1.5)$. The mixing proportion is $\pi^1 = \pi^2 = \pi^3 = 1/3$. We set $\Delta = \{(-\infty, z_{q_1}], (z_{q_1}, z_{q_2}), \ldots, (z_{q_{t-1}}, \infty)\}$, where $z_q$ is the $100 \times q$ percentile of the empirical distribution.

Table 6: Selection Frequencies of the Number of Components: Four Variables

| Selection frequencies by the average rk statistic constructed from simultaneously using 3 different groupings | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| ave-rk ($\alpha = 0.05$) | 0.002 | 0.749 | 0.244 | 0.005 | 0.000 | 0.041 | 0.924 | 0.035 | 0.000 | 0.000 | 0.941 | 0.059 |
| AIC by ave-rk | 0.000 | 0.483 | 0.504 | 0.013 | 0.000 | 0.006 | 0.927 | 0.067 | 0.000 | 0.000 | 0.876 | 0.124 |
| BIC by ave-rk | 0.052 | 0.944 | 0.004 | 0.000 | 0.000 | 0.722 | 0.278 | 0.000 | 0.000 | 0.000 | 0.994 | 0.006 |
| HQ by ave-rk | 0.001 | 0.916 | 0.083 | 0.000 | 0.000 | 0.132 | 0.856 | 0.012 | 0.000 | 0.000 | 0.977 | 0.023 |
| mean d.f. | 11.000 | 6.992 | 2.513 | 0.000 | 11.000 | 6.181 | 2.213 | 0.000 | 11.000 | 6.004 | 2.015 | 0.000 |
| Selection frequencies by the MLE-based model selection under parametric multi-dimensional normal distribution | | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| AIC by MLE | 0.000 | 0.034 | 0.592 | 0.374 | 0.000 | 0.000 | 0.658 | 0.342 | 0.000 | 0.000 | 0.733 | 0.267 |
| BIC by MLE | 0.000 | 0.985 | 0.015 | 0.000 | 0.000 | 0.105 | 0.895 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| HQ by MLE | 0.000 | 0.553 | 0.444 | 0.003 | 0.000 | 0.000 | 0.998 | 0.002 | 0.000 | 0.000 | 1.000 | 0.000 |
| d.f. | 4 | 9 | 14 | 19 | 4 | 9 | 14 | 19 | 4 | 9 | 14 | 19 |
| Selection frequencies by the rk statistic using a single grouping $(X^\alpha, Y^\alpha)$ | | | | | | | | | | | | |
| $X^1 = (W_1, W_2)$ | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| $Y^1 = (W_3, W_4)$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.004 | 0.704 | 0.274 | 0.018 | 0.000 | 0.071 | 0.893 | 0.036 | 0.000 | 0.000 | 0.963 | 0.037 |
| AIC | 0.002 | 0.490 | 0.445 | 0.063 | 0.000 | 0.024 | 0.855 | 0.121 | 0.000 | 0.000 | 0.849 | 0.151 |
| BIC | 0.297 | 0.689 | 0.013 | 0.001 | 0.000 | 0.644 | 0.354 | 0.002 | 0.000 | 0.000 | 0.997 | 0.003 |
| HQ | 0.038 | 0.792 | 0.158 | 0.012 | 0.000 | 0.154 | 0.818 | 0.028 | 0.000 | 0.000 | 0.971 | 0.029 |
| mean d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| $X^2 = (W_1, W_3)$ | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| $Y^2 = (W_2, W_4)$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.135 | 0.828 | 0.033 | 0.004 | 0.000 | 0.888 | 0.104 | 0.008 | 0.000 | 0.549 | 0.431 | 0.020 |
| AIC | 0.042 | 0.825 | 0.111 | 0.022 | 0.000 | 0.735 | 0.223 | 0.042 | 0.000 | 0.321 | 0.590 | 0.089 |
| BIC | 0.775 | 0.225 | 0.000 | 0.000 | 0.001 | 0.999 | 0.000 | 0.000 | 0.000 | 0.991 | 0.009 | 0.000 |
| HQ | 0.297 | 0.696 | 0.007 | 0.000 | 0.000 | 0.969 | 0.029 | 0.002 | 0.000 | 0.796 | 0.198 | 0.006 |
| mean d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |
| $X^3 = (W_1, W_4)$ | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| $Y^3 = (W_2, W_3)$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ | $M = 1$ | $M = 2$ | $M = 3$ | $M \geq 4$ |
| SHT ($\alpha = 0.05$) | 0.001 | 0.735 | 0.247 | 0.017 | 0.000 | 0.123 | 0.839 | 0.038 | 0.000 | 0.000 | 0.959 | 0.041 |
| AIC | 0.000 | 0.535 | 0.403 | 0.062 | 0.000 | 0.041 | 0.837 | 0.122 | 0.000 | 0.000 | 0.861 | 0.139 |
| BIC | 0.166 | 0.819 | 0.015 | 0.000 | 0.000 | 0.754 | 0.244 | 0.002 | 0.000 | 0.006 | 0.994 | 0.000 |
| HQ | 0.006 | 0.848 | 0.139 | 0.007 | 0.000 | 0.261 | 0.710 | 0.029 | 0.000 | 0.000 | 0.968 | 0.032 |
| mean d.f. | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 | 9.000 | 4.000 | 1.000 | 0.000 |

Notes: The true number of components is $M = 3$. $W = (W_1, W_2, W_3, W_4)'$ follows a 3-component normal mixture distribution, where each component distribution is $N_4(\mu^m, I_4)$ for $m = 1, 2, 3$. The parameter values are $\pi^1 = \pi^2 = \pi^3 = 1/3$, $\mu^1 = (0, 0, 0, 0)'$, $\mu^2 = (1.0, 2.0, 0.5, 1.0)$, and $\mu^3 = (2.0, 1.0, 1.0, 0.5)$.

Table 7: Selection Frequencies of the Number of Components: Eight Variables with $\mu^1 = (0,0,0,0,0,0,0,0)'$, $\mu^2 = (1.0,2.0,0.5,1.0,0.75,1.25,0.25,0.5)'$, and $\mu^3 = (2.0,1.0,1.0,0.5,1.25,0.75,0.5,0.25)'$

| Selection frequencies based on the maximum of 70 modified ave-rk statistics | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ |
| max ($\alpha=0.10$) | 0.000 | 0.655 | 0.344 | 0.001 | 0.000 | 0.022 | 0.956 | 0.022 | 0.000 | 0.000 | 0.875 | 0.125 |
| $\text{rk}^+$ ($\alpha=0.05$) | 0.000 | 0.746 | 0.253 | 0.001 | 0.000 | 0.057 | 0.933 | 0.010 | 0.000 | 0.000 | 0.912 | 0.088 |
| ($\alpha=0.01$) | 0.000 | 0.858 | 0.142 | 0.000 | 0.000 | 0.165 | 0.835 | 0.000 | 0.000 | 0.000 | 0.956 | 0.044 |
| Mean of the selection frequencies across 70 modified ave-rk statistics | | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ |
| mean ($\alpha=0.10$) | 0.101 | 0.817 | 0.078 | 0.003 | 0.003 | 0.704 | 0.281 | 0.012 | 0.000 | 0.329 | 0.634 | 0.037 |
| of 70 ($\alpha=0.05$) | 0.142 | 0.810 | 0.047 | 0.001 | 0.005 | 0.776 | 0.214 | 0.005 | 0.000 | 0.408 | 0.574 | 0.019 |
| SHT ($\alpha=0.01$) | 0.245 | 0.740 | 0.015 | 0.000 | 0.012 | 0.866 | 0.121 | 0.001 | 0.000 | 0.549 | 0.446 | 0.005 |
| mean of AICs | 0.012 | 0.867 | 0.119 | 0.003 | 0.000 | 0.587 | 0.399 | 0.013 | 0.000 | 0.222 | 0.734 | 0.044 |
| mean of BICs | 0.284 | 0.715 | 0.001 | 0.000 | 0.035 | 0.942 | 0.023 | 0.000 | 0.000 | 0.832 | 0.167 | 0.000 |
| mean of HQs | 0.078 | 0.909 | 0.013 | 0.000 | 0.004 | 0.878 | 0.117 | 0.001 | 0.000 | 0.573 | 0.424 | 0.004 |

Notes: The true number of components is $M = 3$. $W$ follows a 3-component normal mixture distribution $\sum_{m=1}^{3} \pi^m N_8(\mu^m, I_8)$. The parameter values are $\pi^1 = \pi^2 = \pi^3 = 1/3$, $\mu^1 = (0,0,0,0,0,0,0,0)'$, $\mu^2 = (1.0,2.0,0.5,1.0,0.75,1.25,0.25,0.5)'$, and $\mu^3 = (2.0,1.0,1.0,0.5,1.25,0.75,0.5,0.25)'$. The modified rk statistic with $c = 0.01 \times \hat{s}_1$ is used, where $\hat{s}_1$ is the estimated largest singular value of $\Omega_r$.

Table 8: Selection Frequencies of the Number of Components: Eight Variables, where Three Variables do not have a mixture distribution with $\mu^1 = (0,0,0,0,0,0,0,0)'$, $\mu^2 = (1.0,2.0,0.5,1.0,0.75,0,0,0)'$, and $\mu^3 = (2.0,1.0,1.0,0.5,1.25,0,0,0)'$

| Selection frequencies based on the maximum of 70 modified ave-rk statistics | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ |
| max ($\alpha=0.10$) | 0.000 | 0.848 | 0.152 | 0.000 | 0.000 | 0.200 | 0.792 | 0.008 | 0.000 | 0.000 | 0.974 | 0.026 |
| $\text{rk}^+$ ($\alpha=0.05$) | 0.000 | 0.880 | 0.120 | 0.000 | 0.000 | 0.279 | 0.718 | 0.003 | 0.000 | 0.000 | 0.979 | 0.021 |
| ($\alpha=0.01$) | 0.003 | 0.934 | 0.063 | 0.000 | 0.000 | 0.450 | 0.550 | 0.000 | 0.000 | 0.000 | 0.985 | 0.015 |
| Mean of selection frequencies across the 70 modified ave-rk statistics | | | | | | | | | | | | |
| | $N = 500$ | | | | $N = 2000$ | | | | $N = 8000$ | | | |
| | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ | $M=1$ | $M=2$ | $M=3$ | $M\geq 4$ |
| mean ($\alpha=0.10$) | 0.310 | 0.655 | 0.034 | 0.001 | 0.127 | 0.789 | 0.081 | 0.004 | 0.070 | 0.811 | 0.112 | 0.008 |
| of 70 ($\alpha=0.05$) | 0.372 | 0.608 | 0.019 | 0.000 | 0.149 | 0.789 | 0.060 | 0.002 | 0.075 | 0.831 | 0.089 | 0.004 |
| SHT ($\alpha=0.01$) | 0.493 | 0.501 | 0.006 | 0.000 | 0.192 | 0.769 | 0.038 | 0.000 | 0.086 | 0.843 | 0.070 | 0.001 |
| mean of AICs | 0.077 | 0.874 | 0.048 | 0.001 | 0.034 | 0.846 | 0.117 | 0.004 | 0.025 | 0.806 | 0.160 | 0.009 |
| mean of BICs | 0.466 | 0.534 | 0.000 | 0.000 | 0.210 | 0.780 | 0.010 | 0.000 | 0.101 | 0.850 | 0.049 | 0.000 |
| mean of HQs | 0.229 | 0.766 | 0.004 | 0.000 | 0.108 | 0.854 | 0.037 | 0.000 | 0.066 | 0.865 | 0.068 | 0.001 |

Notes: The true number of components is $M = 3$. $W$ follows a 3-component normal mixture distribution $\sum_{m=1}^{3} \pi^m N_8(\mu^m, I_8)$. The parameter values are $\pi^1 = \pi^2 = \pi^3 = 1/3$, $\mu^1 = (0,0,0,0,0,0,0,0)'$, $\mu^2 = (0.5,1.0,0.25,0.5,0.75,0,0,0)'$, and $\mu^3 = (1.0,0.5,0.5,0.25,0.25,0,0,0)'$. The modified rk statistic with $c = 0.01 \times \hat{s}_1$ is used, where $\hat{s}_1$ is the estimated largest singular value of $\Omega_r$.

# D Additional results from empirical examples

Tables 9-11 report additional tables for the empirical examples in Section 5.

Table 9: Intergenerational Social Mobility in Great Britain

| (1) British Social Mobility Data ($8 \times 8$ Table) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Father's | Subject's Status | | | | | | | |
| Status | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 50 | 19 | 26 | 8 | 7 | 11 | 6 | 2 |
| 2 | 16 | 40 | 34 | 18 | 11 | 20 | 8 | 3 |
| 3 | 12 | 35 | 65 | 66 | 35 | 88 | 23 | 21 |
| 4 | 11 | 20 | 58 | 110 | 40 | 183 | 64 | 32 |
| 5 | 2 | 8 | 12 | 23 | 25 | 46 | 28 | 12 |
| 6 | 12 | 28 | 102 | 162 | 90 | 553 | 230 | 177 |
| 7 | 0 | 6 | 19 | 40 | 21 | 158 | 143 | 71 |
| 8 | 0 | 3 | 14 | 32 | 15 | 126 | 91 | 106 |

| (2) Selected Value of a Lower Bound on $M$ ($8\times8$ Table) | | | |
|---|---|---|---|
| | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
| SHT | $M = 7$ | $M = 7$ | $M = 6$ |
| | AIC | BIC | HQ |
| Information Criteria | $M = 8$ | $M = 5$ | $M = 6$ |
| No. of Observations | 3497 | | |

Notes: The data are from Table 1.C of Clogg (1981). Occupational categories in Panel (1) are as follows. 1: professional and high administrative; 2: managerial and executive; 3, inspectional, supervisory, and other non-manual (high grade); 4: inspectional, supervisory, and other non-manual (low grade); 5: routine grades of nonmanual; 6: skilled manual; 7: semi-skilled manual; and 8: unskilled manual. Panel (2) reports the result from the $5 \times 5$ table in which categories 2 and 3, categories 5 and 6, and categories 7 and 8 were combined.

Table 10: Type of Trade and Ethnic Group Data, Amsterdam and Rotterdam

| (1) Cross-Classification by Ethnic Group and Type of Trade | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Amsterdam | | | | | | Rotterdam | | | | | |
| | Types of Trade | | | | | | Types of Trade | | | | | |
| Ethnic Group | 1 | 2 | 3 | 4 | 5 | Total | 1 | 2 | 3 | 4 | 5 | Total |
| Dutch | 382 | 367 | 788 | 113 | 28 | 1678 | 323 | 209 | 459 | 91 | 153 | 1235 |
| Turks | 14 | 21 | 3 | 8 | 10 | 56 | 29 | 30 | 2 | 15 | 14 | 90 |
| Moroccans | 12 | 36 | 2 | 5 | 7 | 62 | 8 | 17 | 2 | 13 | 5 | 45 |
| Antilleans | 8 | 6 | 2 | 1 | 2 | 19 | 5 | 4 | 3 | 4 | 3 | 19 |
| Surinamese | 44 | 33 | 33 | 17 | 24 | 151 | 35 | 31 | 28 | 19 | 33 | 146 |
| Others | 208 | 97 | 86 | 26 | 39 | 456 | 82 | 18 | 19 | 16 | 12 | 147 |
| Total | 668 | 560 | 914 | 170 | 110 | 2422 | 482 | 309 | 513 | 158 | 220 | 1682 |

Notes: The data are from Table 2a of van der Heijden et al. (2002). In the original table, there are 8 ethnic groups but we have merged the "Cape Verdeans" and the "Ghanaians" into the "Other" ethnic group because they are relatively small ethnic minorities. Types of trade in Panel (1) are as follows. 1: wholesale trade; 2: retail trade; 3: producer services; 4: catering and restaurants; 5: personal services.

| (2) Selected Value of a Lower Bound on $M$ | | | | | | |
|---|---|---|---|---|---|---|
| | Amsterdam | | | Rotterdam | | |
| | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
| SHT | M=4 | M=4 | M=3 | M=3 | M=3 | M=3 |
| | AIC | BIC | HQ | AIC | BIC | HQ |
| Information Criteria | M=4 | M=2 | M=3 | M=3 | M=3 | M=3 |
| No. of Observations | 2422 | | | 1682 | | |

Table 11: Response Patterns in Five-item Subsets of LSAT and the Estimated Number of Latent Ability Distributions

| | Number of Components Selected based on 5 Items | | | | | |
|---|---|---|---|---|---|---|
| | LSAT 6 | | | LSAT 7 | | |
| | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
| max-rk$^+$ statistic | M = 2 | M = 2 | M = 2 | M = 3 | M = 3 | M = 2 |
| | Number of Components Selected based on 4 items | | | | | |
| | LSAT 6 | | | LSAT 7 | | |
| SHT | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
| $\{W_1, W_2, W_3, W_4\}$ | M = 2 | M = 2 | M = 2 | M = 3 | M = 2 | M = 2 |
| $\{W_2, W_3, W_4, W_5\}$ | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 |
| $\{W_1, W_3, W_4, W_5\}$ | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 |
| $\{W_1, W_2, W_4, W_5\}$ | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 |
| $\{W_1, W_2, W_3, W_5\}$ | M = 3 | M = 2 | M = 2 | M = 3 | M = 3 | M = 2 |
| Model Selection | *AIC* | *BIC* | *HQ* | *AIC* | *BIC* | *HQ* |
| $\{W_1, W_2, W_3, W_4\}$ | M = 2 | M = 2 | M = 2 | M = 3 | M = 2 | M = 2 |
| $\{W_2, W_3, W_4, W_5\}$ | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 | M = 2 |
| $\{W_1, W_3, W_4, W_5\}$ | M = 2 | M = 2 | M = 2 | M = 3 | M = 2 | M = 2 |
| $\{W_1, W_2, W_4, W_5\}$ | M = 2 | M = 1 | M = 2 | M = 2 | M = 2 | M = 2 |
| $\{W_1, W_2, W_3, W_5\}$ | M = 3 | M = 1 | M = 2 | M = 3 | M = 2 | M = 2 |
| No. of observations | 1000 | | | 1000 | | |

Notes: The data are from Table 1 of Mislevy (1984).

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