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The SIML Estimation of Integrated Covariance and Hedging Coefficient under Round-off Errors, Micro-market Price Adjustments and Random Sampling

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Abstract

For estimating the integrated volatility and covariance by using high frequency data, Kunitomo and Sato (2011, 2013) have proposed the Separating Information Maximum Likelihood (SIML) method when there are micro-market noises. The SIML estimator has reasonable finite sample properties and asymptotic properties when the sample size is large when the hidden efficient price process follow a Brownian semi-martingale. We shall show that the SIML estimation is useful for estimating the integrated covariance and hedging coefficient when we have round-off errors, micro-market price adjustments, noises and high-frequency data are randomly sampled. The SIML estimation is consistent, asymptotically normal in the stable convergence sense under a set of reasonable assumptions and it has reasonable finite sample properties with these effects.

Key Words

Integrated Covariance, Hedging Coefficient, High-Frequency Financial Data, Round-off Errors, Micro-Market Price Adjustments and Noises, Random Sampling, Separating Information Maximum Likelihood (SIML).

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1. Introduction

Recently a considerable interest has been paid on the estimation problem of the integrated volatility and covariance by using high-frequency financial data. Although the earlier studies often had ignored the presence of micro-market noises in financial markets, there have been arguments that the micro-market noises have important roles in high-frequency financial data, and then several new statistical estimation procedures have been proposed. Among many studies in recent literature on the related topics three methods have attracted some attention : the linear combination of realised volatilities constructed with subsampling by Zhang, Mykland and Ait-Sahalia (2005), the realised kernel method with autocovariances and kernels by Bandorff-Nielsen, Hansen, Lund and Shephard (2008, 2011), and the pre-averaging method by Jacod, Li, Mykland, Podolskij and Vetter (2009), Christensen, K., Podolskij, M. and M. Vetter (2013). See Jacod and Protter (2012) and Ait-Sahalia and Jacod (2014) for the recent developments on the related issues and their references.

As an alternative estimation method Kunitomo and Sato (2008, 2011, 2013) have proposed a simple statistical method called the Separating Information Maximum Likelihood (SIML) method for estimating the integrated volatility and the integrated covariance by using high frequency data under the presence of micro-market noises. The SIML estimator has reasonable asymptotic properties as well as finite sample properties ; it is consistent, asymptotically normal in the the stable convergence sense under a set of regularity conditions and it has reasonable finite sample properties. The SIML estimator has the asymptotic robustness properties, that is, it is consistent and asymptotically normal even when there are round-off errors and (non-linear) price adjustments with the hidden efficient market price process.

In this paper we shall further investigate some properties of the SIML estimation of integrated volatility, covariance and the hedging coefficient when we have the round-off errors, the micro-market noises and randomly sampled data. For actual high frequency financial data, they are recorded at random times and the effects of randomness could be significant when we have the round-off errors and micro-market noises. The statistical problem on the round-off error models has been previously investigated by Delattre and Jacod (1997), Rosenbaum (2009), and Li and Myckland (2012). in the univariate case and our formulation is more general than the previous works. Empirically we have the tick-size effects (the minimum price change size and the minimum order size) and we observe bid-ask spreads in financial markets. Also the non-synchronous sampling on the covariance estimation in high frequency financial data is common, which was investigated by Hayashi and Yoshida (2005, 2008) and they proposed the *so-called* Hayashi-Yoshida (H-Y) method under the situation that there is no micro-market noise. There can be several schemes of random sampling for the covariance estimation and we shall adopt the refreshing scheme developed by Bandorff-Nielsen, Hansen, Lund and Shephard (2011) although other methods could be applied. By synchronizing financial data, we shall consider the estimation problem of hedging coefficient and correlation coefficient when we have micro-market noises, which have important roles in hedging and risk managements. Since we need the volatility estimation as well as the covariance estimation for this purpose, it may be difficult to use the H-Y method directly when we have micro-market noises. We show that the SIML estimator is asymptotically robust; that is, it is consistent and asymptotically normal in the stable convergence sense as the sample size increases under a reasonable set of assumptions. The asymptotic property of the SIML method on the integrated volatility and covariance has desirable properties over other estimation methods for the underlying continuous stochastic process with micro-market noise. Because the SIML estimation is a simple method, it can be practically used for analyzing the multivariate (high frequency) financial time series. In this paper we shall focus on the estimation of covariance with two dimension and the hedging coefficient while as companion papers Sato and Kunitomo (2015) and Misaki and Kunitomo (2013) have investigated the estimation problem of the integrated volatility.

In Section 2 we introduce the round-off error model and micro-market price adjustment models with micro-market noise, and then discuss the Separating Information Maximum Likelihood (SIML) method. Then in Section 3 we give some numerical results of the SIML estimation in the basic simulation framework with randomly sample data. In Section 4 we shall give some asymptotic properties of the SIML estimator under a set of assumptions. Then in Section 5 we shall report further simulation results in a variety of round-off models and micro-market price adjustment models. Finally, in Section 6 some brief remarks will be given. Some mathematical details shall be given in Appendix.

2. Micro-market noise models and the estimation of covariance

2.1 A General Formulation

Let $y_s(t_i^s)$ be the $i-$th observation of the (log-) price of the first asset at t_i^s for $0 = t_0^s < t_1^s < \cdots < t_{n_s^*}^s \leq 1$ and $y_f(t_j^f)$ be the $j-$th observation of the (log-) price of the second asset at t_j^f for $0 = t_0^f < t_1^f < \cdots < t_{n_f^*}^f \leq 1$, where $t_{n_s^*}^s = \max_{t_i^s \leq 1}\{t_i^s\}$, $t_{n_f^*}^f = \max_{t_i^f \leq 1}\{t_i^f\}$ and we denote n_a $(a = s, f)$ as constant indexes and n_a^* $(a = s, f)$ as (bounded) stochastic indexes.

We consider the situation when the two observed (log-)prices are different from the corresponding underlying continuous process $X_s(t)$ and $X_f(t)$ $(0 \leq t \leq 1)$, respectively. The two dimension continuos stochastic process $X(t) = (X_s(t), X_f(t))'$ is a Brownian semi-martingale with

$$X(t) = X(0) + \int_0^t \mu_x(s)ds + \int_0^t \sigma_x(s)dB(s) \;\; (0 \leq t \leq 1), \tag{2.1}$$

where $\mu(s)$ and $\sigma_x(s)$ are the 2×1 drift terms and the 2×2 volatility matrix, which are progressively measurable with respect to the $\sigma-$field $\mathcal{F}_t$ and $B(s)$ is the two-dimensional Brownian motion. The first statistical objective is to estimate the quadratic variation or the integrated volatility matrix

$$\Sigma_x = \int_0^1 \Sigma_x(s)ds = \begin{pmatrix} \sigma_{ss}^{(x)} & \sigma_{sf}^{(x)} \\ \sigma_{sf}^{(x)} & \sigma_{ff}^{(x)} \end{pmatrix} \tag{2.2}$$

$(\Sigma_x(s) = \sigma_x(s)\sigma_x(s)^{'})$ of the underlying continuous process $X(t)$ $(0 \leq t \leq 1)$ from the set of discrete observations on $(y_s(t_i^s), y_f(t_j^f))$ with $i = 1, \cdots, n_s^*$ and $j = 1, \cdots, n_f^*$. In this paper we make a set of assumptions on the underlying process given by (2.1).

Assumption I : The Brownian semi-martingale (2.1) satisfies the condition on drift and volatility terms such that $\mu(s)$ and $\sigma_x(s)$ are continuous and bounded in $s \in [0, 1]$.

The basic high-frequency financial market model with micro-market noises can be represented by

$$(2.3) \qquad y_s(t_i^s) = X_s(t_i^s) + v_s(t_i^s) \ , \ y_f(t_j^f) = X_f(t_j^f) + v_f(t_j^f) \ ,$$

where the underlying process $X(t) = (X_s(t), X_f(t))'$ is the Brownian semi-martingale given by (2.1). In the basic model we assume that $v_s(t_i^s)$ and $v_f(t_j^f)$ are a sequence of independently and identically distributed random variables with $\mathbf{E}(v_s(t_i^s)) = 0, \mathbf{E}(v_f(t_j^f)) = 0$, $\mathbf{E}(v_s(t_i^s)^2) = \sigma_{ss}^{(v)}, \mathbf{E}(v_f(t_j^f)^2) = \sigma_{ff}^{(v)}, \mathbf{E}(v_s(t_i^s)v_f(t_j^f)) = \delta(t_i^s, t_j^f)\sigma_{sf}^{(v)}$, where $\delta(\cdot, \cdot)$ is the indicator function with $\delta(a, a) = 1$ and $\delta(a, b) = 0$ $(a \neq b)$.
It is possible to extend the following discussions with some notational complications to the cases when $v_s(t_i^s)$ and $v_f(t_j^f)$ are the discrete stationary time series process satisfying

$$(2.4) \qquad v_s(t_i^s) = \sum_{j=0}^{\infty} \theta_j^s w_s(t_{i-j}^s) \ , \ v_f(t_i^f) = \sum_{j=0}^{\infty} \theta_j^f w_f(t_{i-j}^f) \ ,$$

where there exist ρ_a $(0 < \rho_a < 1;\ a = s, f)$ such that $\theta_j^a = O(\rho_a^j)$ and $w_s(t_i^a)$ $(a = s, f)$ are a sequence of independent random variables with $\mathbf{E}(w_s(t_i^s)) = 0, \mathbf{E}(w_f(t_j^f)) = 0$, $\mathbf{E}(w_s(t_i^s)^2) < \infty$, and $\mathbf{E}(w_f(t_j^f)^2) < \infty$. We define the sequence of random variables $w_a(t_i^a) = 0$ and $\theta_j^a = 0$ for $t_i^a < 0$ and $t_j^a < 0$ $(a = s, f)$ and we maintain the (weak) stationarity conditions of $v_a(t_i^a)$ in the MA representation (2.4) for the resulting simplicity of our arguments.
The important statistical aspect of (2.3) is the fact that it is an additive (signal-plus-noise) measurement error model. However, there are some reasons why the basic

model as (2.3) is not enough for applications. For instance, the high frequency financial models with the round-off-errors models and micro-market price adjustments for financial prices are not in the form of (2.3). Then we shall further consider several examples of the more general situation when the observed (log-)prices $y_a(t_i^a)$ are the sequence of discrete stochastic processes generated by

$$y_a(t_i^a) = h_a\left(\mathbf{X}(t_i^a), y_a(t_{i-1}^a), u_a(t_i^a)\right) \ (a = s, f), \tag{2.5}$$

where $h_a(\ \cdot\ )$ are measurable functions, the (unobservable) continuous martingale process $\mathbf{X}(t)$ $(0 \leq t \leq 1)$ is defined by (2.3) and the micro-market noises $u_a(t_i^a)$ are the discrete stochastic processes.

There are special cases of our interest in the form of (2.1) and (2.5), which reflect the important aspects on modeling financial markets and the high frequency financial data.

As the first example of non-linear models, we consider the round-off-error model with the micro-market noise. One motivation for this model has been the fact that in actual financial markets transactions occur with the minimum tick size and the observed price data do not have continuous paths over time. The traded price and quantity usually have the minimum size and the Nikkei-225-futures, which have been the most important traded derivatives in Japan (as explained in Kunitomo and Sato (2011)) for instance, has the minimum 10 yen (about 0.1 U.S. dollar) size while the Nikkei-225-stocks are traded with the minimum 1 yen. It is important to see the effects of round-off-errors in different markets. We assume that $y_a(t_i^n) = P_a(t_i^n)$ and

$$P_a(t_i^a) = g_{a,\eta}\left[X_a(t_i^a) + u_a(t_i^a)\right] \ , \tag{2.6}$$

where the micro-market noise term $u_a(t_i^a)$ are sequences of i.i.d. random variables with $\mathbf{E}[u_a(t_i^a)] = 0$, $\mathbf{E}[u_a(t_i^a)^2] = \sigma_{a,u}^2$ and the nonlinear function

$$g_{a,\eta}(x) = \eta_a \left[\frac{x}{\eta_a}\right] \tag{2.7}$$

is the round-off part of x and $[x]$ is the largest integer being equal or less than x, and the threshold parameter η_a are (small) positive constants.
This model corresponds to the micro-market model with the restriction of the minimum price change with micro-market noises and η_a represents the parameter of the level for the minimum price change. As an illustration we set the same value for two threshold parameters (although they can be different in practice) and give typical simulation paths in the basic (two-dimensional) round-off error model as Figure 2-1 in Appendix.

Second, the (linear) micro-market price adjustment model is written as $y_a(t_i^a) = P_a(t_i^a)$ $(i = 1, \cdots, n_a^*; a = s, f)$ and

$$P_a(t_i^a) - P_a(t_{i-1}^a) = g_a \left[X_a(t_i^a) - P_a(t_{i-1}^a)\right] + u_a(t_a^a) \ , \tag{2.8}$$

where $X_a(t)$ (the intrinsic value of a security at t) and $P_a(t_i^a)$ are the observed log-price at t_i^a, the adjustment (constant) coefficients g_a $(0 < g_a < 2)$, and $u_a(t_i^a)$ is an i.i.d. sequence of noises with $\mathbf{E}[u(t_i^a)] = 0$ and $\mathbf{E}[u(t_i^a)^2] = \sigma_{aa}^2$. Sato and Kunitomo (2015) have considered nonlinear micro-market price adjustments models, which are the extensions of (2.8).

Third, if we set $y_a(t_i^a) = P_a(t_i^a)$ and

$$P_a(t_i^a) - P_a(t_{i-1}^a) = g_{\eta,a} \left[X_a(t_i^a) - P_a(t_{i-1}^a) + u_a(t_i^a)\right] \ , \tag{2.9}$$

where the round-ff error function $g_{a,\eta}(x)$ is defined by (2.7), and $u_a(t_i^a)$ is a sequence of i.i.d. noises with $\mathbf{E}[u_a(t_i^a)] = 0$ and $\mathbf{E}[u_a(t_i^a)^2] = \sigma_{aa}^2$.
This model corresponds to the micro-market model with the restriction of the minimum price change and η_a are the parameters of minimum price changes. It has been common that the market maker usually set the minimum price change such as a dollar, yen or other currencies and the minimum quantity traded in actual financial markets.

The basic (high-frequency) financial model with micro market noises is the special case when the underlying process $X(t) = (X_s(t), X_f(t))'$ is given by (2.1). The synchronous sampling means $t_i^s = t_i^f$ and the fixed grid observation means

$t_i^a - t_{i-1}^a = n^{-1}$. There can be special cases of (2.5) such as (2.6),(2.8) and (2.9) when we have micro-market adjustment models and the non-synchronous observations as well as the random sampling. Sato and Kunitomo (2015) have discussed several examples of the micro-market models with round-off errors, pice adjustments and noises, but all of them one dimensional cases. There have been micro-market models in economics, which may be related the micro-market price adjustment models. (See Hansbrouck (2007) for instance.)

In this paper we also consider the situation that the high-frequency data are observed at random times t_i^a ($a = s$ or f) under some conditions on random sampling. For non-synchronously observed data for covariance estimation, there have been several ways to handle the problem and we adopt *the refreshing time method* developed by Harris, Mcinish, Shoesmith and Wood (1995), and used by Bandorff-Nielsen, Hansen, Lund and Shepard (2011). Define $t_0 = 0, t_1 = \max\{t_1^s, t_1^f\}$ and $t_{j+1}^n = \max\{t^s_{N_j^s+1}, t_{N_j^f+1}\}$, where $t^a_{N_j^a+1}$ are random times for $y_a(t_i^a)$, and N_j^a is the corresponding counting process. We denote the resulting random times as $0 = t_0 < t_1 < \cdots < t_{n^*}$ and the random number of observations as n^*. Then the resulting counting process N_j^n and n^* are finite-valued random variables in $[0,1]$ for any finite n and we denote $t^n_{n^*}$ (≤ 1) is the last transaction time before the market closing time in a day. We sometimes abuse the notations n and n^* for n_a and n_a^* ($a = s, f$) for the resulting notational simplicity in the following analysis and statements whenever we do not have any confusion.

Assumption II : The sampling process $\{t_j^n\}$ is independent of the underlying process $\{X(t)\}$ and n^* is a finite-valued random variable in $[0,1]$ for any n. There exist positive constants c_s, c_f, c and an increasing sequence of fixed n such that as $n \to \infty$

$$(2.10) \qquad t_{n^*} \xrightarrow{p} 1 \ , \ \frac{n_a^*}{n} \xrightarrow{p} c_a \ (a = s \text{ or } f) \ , \ \frac{n^*}{n} \xrightarrow{p} c \ .$$

Let $\Delta_j^n t_j^n = t_j^n - t_{j-1}^n = (1/n) D_j^n$ be a sequence of $\mathcal{F}_{t^n_{j-1}}$−adapted random variables. The (bounded) increasing continuous process $\tau(t)$ with $\tau(0) = 0, \tau(1) = 1$ and the continuous process $d(t)$ $(0 \leq t \leq 1)$ are well-defined such that $t_j^n = \tau(t) +$

$O_p(n^{-\gamma_1})$ $(\gamma_1 > 0)$ and $D_j^n = d(t) + O_p(n^{-\gamma_2})$ $(\gamma_2 > 0)$ uniformly in t $(t \in (0,1])$ as $j(n)/n \to t$ and $n \to \infty$.

These conditions imply that $t_j^n \xrightarrow{p} \tau(t)$, $D_j^n \xrightarrow{p} d(t)$ and $[D_j^n(t)]^2 \xrightarrow{p} d(t)^2$ uniformly (as $j(n)/n \to t$ and $n \to \infty$), where $\mathbf{E}\left[|t_i^n - t_{i-1}^n|\right] = O(n^{-1})$ is proportional to the average duration on intervals in $[0,1]$ for any fixed n. For the standard normalization we often take $c_s = c_f = 1$ or $c = 1$ in the following analysis. A typical example of random sampling is the Poisson Process Sampling on t_i^a with the intensity functions $\lambda_n^{(a)} = nc_a$ $(a = s, f)$. In this case the (mutually) independent random durations D_i^a $(a = s, f)$ are exponentially distributed with $\mathbf{E}(D_i^a) = 1$, $nD_i^a = t_i^a - t_{i-1}^a$ and $\tau(t) = t$ $(0 < t \leq 1)$ if we take the normalization $c = 1$.

Integrated Hedging Ratio and Correlation

For the financial risk management problems, the use of hedging coefficients and correlation coefficients has been often discussed in the literatures on financial futures. (See Duffie (1989) for instance.) Then it is important to estimate the hedging ratio and the correlation coefficient from a set of dicrete sampled price data. The (integrated) hedging ratio based on high-frequency financial data can be defined by

$$H = \frac{\sigma_{sf}^{(x)}}{\sigma_{ff}^{(x)}} \ . \tag{2.11}$$

The (integrated) correlation coefficient between two prices can be defined by

$$\rho_{sf} = \frac{\sigma_{sf}^{(x)}}{\sqrt{\sigma_{ss}^{(x)}\sigma_{ff}^{(x)}}} \ . \tag{2.12}$$

2.2 The SIML Method

When the two dimensional financial data are synchronously observed and equally spaced, the derivation of the separating information maximum likelihood (SIML) estimation has been given by Kunitomo and Sato (2008, 2011). For non-synchronous observed data, there have been several ways to handle the problem and we shall adopt *the refreshing time method* as explained in the previous sub-section in the following discussions.

We consider the standard situation when $\Sigma_s(s) = \Sigma_x$ and $X(t)$ $(0 \le t \le 1)$ are independent of $u_s(t_i^s)$ and $u_f(t_j^f)$. Let $\mathbf{y}_{n^*}^a = (y_a(t_i^n))$ $(a = s, f)$ be $n^* \times 1$ vectors of the re-arranged (synchronous data) observations, where n^* is a finite valued random variable. We transform $\mathbf{y}_{n^*}^s$ and $\mathbf{y}_{n^*}^f$ to $\mathbf{z}^s$ $(= (\mathbf{z}_k^s))$ and $\mathbf{z}^f$ $(= (\mathbf{z}_k^f))$ by

$$\mathbf{z}^s = \sqrt{n^*}\mathbf{P}'\mathbf{C}^{-1}\left(\mathbf{y}_{n^*}^s - \bar{\mathbf{y}}_{s0}\right)\ ,\ \mathbf{z}^f = \sqrt{n^*}\mathbf{P}'\mathbf{C}^{-1}\left(\mathbf{y}_{n^*}^f - \bar{\mathbf{y}}_{f0}\right)\ , \tag{2.13}$$

where $\mathbf{C}$ be an $n^* \times n^*$ matrix such that

$$\mathbf{C}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix}^{-1}\ , \tag{2.14}$$

$$\mathbf{P} = (p_{jk})\ ,\ p_{jk} = \sqrt{\frac{2}{n^* + \frac{1}{2}}}\cos\left[\pi(\frac{2k-1}{2n^*+1})(j - \frac{1}{2})\right]\ , \tag{2.15}$$

and $\mathbf{1}_{n^*} = (1, \cdots, 1)'$ ($n^* \times 1$ vector),

$$\bar{\mathbf{y}}_{a0} = \mathbf{1}_{n^*} \cdot (y_s(0), y_f(0))\ . \tag{2.16}$$

Under the assumption of the Gaussian distributions both on the signal terms and noise terms with equally spaced observation case, Kunitomo and Sato (2008) have introduced the SIML estimator. By using the similar argument, we define the SIML estimator of $\sigma_{ss}^{(x)}$ and $\sigma_{ss}^{(v)}$ by

$$\hat{\sigma}_{ss}^{(x)} = \frac{1}{m_{n^*}}\sum_{k=1}^{m_{n^*}}[z_k^s]^2\ , \tag{2.17}$$

and

$$\hat{\sigma}_{ss}^{(v)} = \frac{1}{l_{n^*}}\sum_{k=n^*+1-l_{n_s^*}}^{n^*} a_{k,n^*}^{-1}[z_k^s]^2\ , \tag{2.18}$$

where

$$a_{k,n^*} = 4n_s^*\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n^*+1}\right)\right]\ . \tag{2.19}$$

Similarly, we define the SIML estimator of $\sigma_{ff}^{(x)}$ and $\sigma_{ff}^{(v)}$ as $\hat{\sigma}_{ff}^{(x)}$ and $\hat{\sigma}_{ff}^{(v)}$. Then the SIML covariance estimator with *the refreshing time scheme* can be defined by

$$\hat{\sigma}_{sf}^{(x)} = \frac{1}{m_{n^*}} \sum_{k=1}^{m_{n^*}} [z_k^s z_k^f] \ . \tag{2.20}$$

For both $\hat{\sigma}_{ss}^{(x)}, \hat{\sigma}_{ff}^{(x)}$, and $\hat{\sigma}_{ss}^{(v)}, \hat{\sigma}_{ff}^{(v)}$, the number of terms m and l are dependent on n^* and we require the order requirements that $m_{n^*} = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $l_{n^*} = O(n^{\beta})$ $(0 < \beta < 1)$. If we set

$$\hat{\Sigma}_x = \begin{pmatrix} \hat{\sigma}_{ss}^{(x)} & \hat{\sigma}_{sf}^{(x)} \\ \hat{\sigma}_{sf}^{(x)} & \hat{\sigma}_{ff}^{(x)} \end{pmatrix} , \tag{2.21}$$

$$\hat{\Sigma}_v = \begin{pmatrix} \hat{\sigma}_{ss}^{(v)} & \hat{\sigma}_{sf}^{(v)} \\ \hat{\sigma}_{sf}^{(v)} & \hat{\sigma}_{ff}^{(v)} \end{pmatrix} , \tag{2.22}$$

then we have one important aspect of the SIML estimator such that it is positive definite by construction.

There are several remarks on the SIML estimation. First, we can modify the SIML method such that the asymptotic bias can be removed. The modified SIML (MSIML) estimator of $\Sigma^{(x)}$ is given by

$$\hat{\Sigma}_{x,m} = \frac{1}{m_{n^*}} \sum_{k=1}^{m_{n^*}} \mathbf{z}_k \mathbf{z}_k^{'} - \left[\frac{1}{m_{n^*}} \sum_{k=1}^{m_{n^*}} a_{k,n^*} \right] \hat{\Sigma}_v \ , \tag{2.23}$$

where $\mathbf{z}_k = (z_k^s, z_k^f)^{'}$ are 2×1 vectors. Sato and Kunitomo (2015) have discussed this modification (the MSIML estimation) in the integrated volatility case. (See their Corollary 3.3.)

Secondly, it has been a common ptactice to use the hedging ratio in the statistical risk management. (See Duffie (1989) for instance.) Then by using the estimators for the integrated volatility and the integrated covariance, the SIML estimator of the hedging ratio $H = \sigma_{sf}^{(x)} / \sigma_{ff}^{(x)}$ can be defined by

$$\hat{H} = \frac{\hat{\sigma}_{sf}^{(x)}}{\hat{\sigma}_{ff}^{(x)}} \ . \tag{2.24}$$

Thirdly, when the two dimensional financial data (spot price and future price) are non-synchronously observed, there are several different ways to use the SIML estimation. (i) The refreshing time method we have adopted. (ii) We set the transaction time of the second asset is to synchronize the transaction time of the first asset. The first method can be geberalized to the multidimensional cases with loosing some information. The second method is to use the Hayashi-Yoshida method of covariance estimation and then we can apply the SIML method. Since there can be several estimation methods and different combinations of using the integrated volatility and covariance, Misaki (2013) have compared the performance of alternative methods. In our simulations we are comparing several alternative estimation methods.

3. Basic Simulation

We have investigated the robust properties of the SIML estimator for the integrated volatility based on a set of simulations with the number of replications being 1,000. We have taken the (fixed) sample size $n = 1,800$ and $n = 18,000$ (the realized n^* depends on each simulation) and we have chosen $\alpha = 0.4$ and $\beta = 0.8$. The details of the simulation procedure are similar to the corresponding ones reported by Sato and Kunitomo (2014), and Misaki and Kunitomo (2013).

In our basic simulations we consider two cases when the observations are the sum of signal and micro-market noise. We use two Poisson Random Sampling as the basic stochastic sampling with the parameter $\lambda_n^{(a)} = n$ $(a = s, f)$. In the first example the signal is the Brownian motion with no drift coefficients and the volatility function

$$\Sigma_x(s) = \Sigma_x(0)\left[a_0 + a_1 s + a_2 s^2\right] \tag{3.1}$$

where a_i $(i = 0, 1, 2)$ are constants and we have some restrictions such that $\Sigma_x(s)$ is positive definite for $s \in [0, 1]$. In this case the integrated volatility is given by

$$\Sigma_x = \int_0^1 \Sigma_x(s)ds = \Sigma_x(0)\left[a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right] . \tag{3.2}$$

In this example we have taken several intra-day instantaneous volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.

In the second example let $n_a^* = \max_{1 \geq i, t_i^a \leq 1}\{t_i^a\}$ $(a = s, f)$ and the instantaneous volatility function follows the stochastic volatility model that

$$\sigma_{ss}^{(x)} = \frac{1}{n_s^*}\sum_{i=1}^{n_s^*} \sigma_{ss}^{(x)}(t_i^s) \ , \tag{3.3}$$

where $\sigma_{ss}^{(x)}(t_i^s) = \sigma_{ss}^{(x)}(0)e^{H(t_i^s)}$ $(0 < t_1^s < \cdots < t_{n_s^*}^s)$, $\sigma_{ff}^{(x)}(t_j^f) = \sigma_{ff}^{(x)}(0)e^{H(t_j^f)}$ $(0 < t_1^n < \cdots < t_{n_f^*}^n)$ and

$$h_{aa}(t_i^a) = \gamma \ h_{aa}(t_{i-1}^a) + \delta \ u_a^{\epsilon}(t_i^a) \tag{3.4}$$

for $a = s$ or f. In our experiments we have set $\gamma = 0.9, \delta = 0.2$ and each of $(u_s^{\epsilon}(t_i^s)$ and $u_f^{\epsilon}(t_j^f))$ are the white noise process followed by $N(0, \sigma_{ss}^{(\epsilon)})$ and $N(0, \sigma_{ff}^{(\epsilon)})$ as the typical case.

We summarize our estimation results of the first example in Tables 3.1-3.3 and the second example in Table 3.4, respectively. In each table we have also calculated the historical volatility (the realized volatility, RV), the historical covariance (RCV) and the HY (Hayashi-Yoshida) estimator, and the SIML estimator, in order to make comparisons. In Tables Raw means the estimates based on all simulated data and 10 sec, for instance, means the estimates based on the simulated data at each grids of 10 second. We have used the refreshing scheme in order to calculate the SIML covariance estimator because of the non-synchronization of random times observations. There can be several possibilities to estimate the hedging coefficient since there are several ways to combine the variance and covariance, that is, RCV-RV, HY-RV, HY-SIML and SIML-SIML. In tables $\sigma_{x_1}^2, \sigma_{x_2}^2, \sigma_{x12}$ denote for the volatility of the spot data and future data, and covariance, respectively while $\sigma_{v_1}^2, \sigma_{v_2}^2, \sigma_{v12}$ denote for the variances of the noises and their covariance. We have computed several combinations of the true integrated volatilities, covariance, and noise variances, and then compared their performances.

In Table 3.1 we take the case when $n = 1,800$ while in Table 3.2 we take the case when $n = 18,000$. We find that the general features of the performance of the SIML including its bias in Table 3-1 are very similar to those in Table 3-2 except the variances of the estimators. (It is obvious that they are quite different.) Then we have omitted this comparison in other tables. Table 3.3 gives the simulation results for the case when the volatility function has a U-shape while Table 3.4 gives the ones for the case of the volatility is generated by (3.3) and (3.4). They correspond to the typical situations in the basic case.

When there are micro-market noise components with the martingale signal part, the value of RCV often differs substantially from the true integrated covariance of the signal part. However, we have found that it is possible to estimate the integrated variances, the integrated covariance and the noise variances when we have the signal-noise ratio as $10^{-2} \sim 10^{-6}$ by the SIML estimation method. Although we have omitted the details of the second example, the estimation results are similar in the stochastic volatility model.

For the estimation of the hedging coefficients, We have confirmed that by using the H-Y method we can improve the historical covariance method as was pointed out by Hayashi and Yoshida (2005). However, when we have micro-market noise terms, the SIML-SIML combination of the integrated variance and the integrated covariance dominate the H-Y method and others. This point is vivid and important on the hedging coefficient estimation as Tables 3.1-3.4 have shown.

By our basic simulations we can conclude that we can estimate both the integrated covariance of the hidden martingale part and the (integrated) hedging coefficients reasonably in all cases we have examined by the SIML estimation.

Table 3.1 : Estimation of covariance and hedging coefficient :

Case 1 ($a_0 = 1, a_1 = a_2 = 0; \lambda = 1800$)

1800	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.01E-04	2.21E-04	2.03E-04	2.00E-04	2.05E-04
		6.60E-05	4.58E-05	6.51E-05	9.53E-05	1.30E-04
σ^2_{v1}	2.00E-06	2.03E-06	1.05E-07	9.88E-07	2.17E-06	3.01E-06
		1.49E-07	6.07E-09	8.64E-08	3.19E-07	8.34E-07
RV1	2.00E-04	7.40E-03	7.06E-03	4.75E-03	1.39E-03	4.39E-04
		3.53E-04	3.40E-04	2.54E-04	1.38E-04	8.70E-05
σ^2_{x2}	2.00E-04	2.07E-04	2.22E-04	2.09E-04	2.07E-04	2.07E-04
		6.74E-05	4.49E-05	6.63E-05	1.01E-04	1.37E-04
σ^2_{v2}	2.00E-06	2.03E-06	1.05E-07	9.84E-07	2.17E-06	3.00E-06
		1.43E-07	6.02E-09	8.51E-08	3.06E-07	8.14E-07
RV2	2.00E-04	7.41E-03	7.06E-03	4.75E-03	1.39E-03	4.41E-04
		3.42E-04	3.28E-04	2.54E-04	1.33E-04	8.43E-05
σ^2_{x12}	1.00E-04	9.93E-05	9.98E-05	9.99E-05	1.01E-04	1.02E-04
		5.59E-05	3.43E-05	5.04E-05	7.66E-05	1.04E-04
σ^2_{v12}	0.00E+00	5.46E-09	-4.10E-11	2.09E-09	5.86E-08	4.52E-07
		1.21E-07	2.11E-09	4.97E-08	2.20E-07	5.82E-07
RCV	1.00E-04	6.75E-05	7.33E-06	4.30E-05	8.03E-05	9.53E-05
		1.78E-04	5.34E-05	1.24E-04	9.33E-05	6.15E-05
HY	1.00E-04	1.05E-04				
		1.25E-04				
RCV-RV	5.00E-01	9.08E-03	1.04E-03	8.98E-03	5.77E-02	2.20E-01
		2.41E-02	7.57E-03	2.60E-02	6.68E-02	1.39E-01
HY-RV	5.00E-01	1.42E-02	1.49E-02	2.21E-02	7.59E-02	2.49E-01
		1.70E-02	1.78E-02	2.64E-02	9.11E-02	3.04E-01
HY-SIML	5.00E-01	5.81E-01	4.87E-01	5.71E-01	6.86E-01	8.72E-01
		7.54E-01	5.98E-01	7.42E-01	1.06E+00	1.75E+00
SIML-SIML	5.00E-01	4.97E-01	4.51E-01	4.91E-01	5.17E-01	5.05E-01
		2.42E-01	1.29E-01	2.05E-01	3.30E-01	5.11E-01
1800	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	1.96E-04	2.00E-04	1.97E-04	1.95E-04	1.98E-04
		6.46E-05	4.14E-05	6.36E-05	9.26E-05	1.27E-04
σ^2_{v1}	2.00E-08	4.92E-08	3.84E-09	3.85E-08	2.01E-07	1.02E-06
		3.73E-09	1.83E-10	2.81E-09	3.00E-08	2.84E-07
RV1	2.00E-04	2.72E-04	2.69E-04	2.45E-04	2.11E-04	2.02E-04
		1.19E-05	1.18E-05	1.22E-05	1.85E-05	3.86E-05
σ^2_{x2}	2.00E-04	2.02E-04	2.00E-04	2.02E-04	2.02E-04	2.02E-04
		6.47E-05	4.03E-05	6.38E-05	9.75E-05	1.33E-04
σ^2_{v2}	2.00E-08	4.90E-08	3.85E-09	3.85E-08	2.02E-07	1.03E-06
		3.65E-09	1.86E-10	2.84E-09	2.98E-08	2.77E-07
RV2	2.00E-04	2.72E-04	2.69E-04	2.46E-04	2.12E-04	2.04E-04
		1.18E-05	1.18E-05	1.21E-05	1.82E-05	3.63E-05
σ^2_{x12}	1.00E-04	9.95E-05	9.98E-05	1.00E-04	1.02E-04	1.03E-04
		5.45E-05	3.14E-05	4.87E-05	7.51E-05	1.02E-04
σ^2_{v12}	0.00E+00	7.94E-09	-2.45E-13	1.31E-09	6.34E-08	4.70E-07
		3.97E-09	7.85E-11	2.00E-09	2.22E-08	2.30E-07
RCV	1.00E-04	6.67E-05	4.92E-06	3.73E-05	8.35E-05	9.76E-05
		8.61E-06	2.17E-06	6.10E-06	1.37E-05	3.01E-05
HY	1.00E-04	1.00E-04				
		1.11E-05				
RCV-RV	5.00E-01	2.45E-01	1.83E-02	1.52E-01	3.96E-01	4.82E-01
		2.96E-02	8.08E-03	2.40E-02	5.52E-02	1.16E-01
HY-RV	5.00E-01	3.68E-01	3.73E-01	4.09E-01	4.77E-01	5.12E-01
		3.76E-02	3.81E-02	4.17E-02	5.22E-02	1.06E-01
HY-SIML	5.00E-01	5.69E-01	5.22E-01	5.65E-01	6.65E-01	9.02E-01
		2.10E-01	1.18E-01	2.01E-01	4.31E-01	1.21E+00
SIML-SIML	5.00E-01	5.10E-01	5.01E-01	5.11E-01	5.29E-01	5.24E-01
		2.39E-01	1.23E-01	2.03E-01	3.36E-01	5.02E-01

Table 3.2 : Estimation of covariance and hedging coefficient :

Case 1 ($a_0 = 1, a_1 = a_2 = 0; \lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.04E-04	2.05E-04	2.05E-04	2.08E-04	2.11E-04
		4.18E-05	4.17E-05	6.74E-05	1.00E-04	1.33E-04
σ^2_{v1}	2.00E-06	2.00E-06	9.37E-07	2.02E-06	2.19E-06	3.01E-06
		5.68E-08	3.03E-08	1.39E-07	3.10E-07	8.43E-07
RV1	2.00E-04	7.22E-02	4.57E-02	7.39E-03	1.41E-03	4.42E-04
		1.08E-03	8.07E-04	2.78E-04	1.29E-04	8.66E-05
σ^2_{x2}	2.00E-04	2.05E-04	2.06E-04	2.06E-04	2.09E-04	2.10E-04
		4.25E-05	4.26E-05	6.68E-05	1.01E-04	1.31E-04
σ^2_{v2}	2.00E-06	2.00E-06	9.40E-07	2.04E-06	2.20E-06	3.00E-06
		5.53E-08	3.06E-08	1.46E-07	3.13E-07	8.41E-07
RV2	2.00E-04	7.22E-02	4.57E-02	7.41E-03	1.41E-03	4.41E-04
		1.08E-03	7.74E-04	2.89E-04	1.33E-04	8.64E-05
σ^2_{x12}	1.00E-04	1.00E-04	9.99E-05	1.02E-04	1.07E-04	1.07E-04
		3.59E-05	3.35E-05	5.10E-05	7.90E-05	1.05E-04
σ^2_{v12}	0.00E+00	-2.30E-10	-9.84E-10	1.40E-08	9.76E-08	5.14E-07
		4.64E-08	1.89E-08	1.02E-07	2.22E-07	5.85E-07
RCV	1.00E-04	8.27E-05	4.15E-05	1.02E-04	1.03E-04	1.02E-04
		5.44E-04	3.83E-04	2.10E-04	9.32E-05	6.32E-05
HY	1.00E-04	9.94E-05				
		3.86E-04				
RCV-RV	5.00E-01	1.14E-03	9.13E-04	1.38E-02	7.36E-02	2.31E-01
		7.53E-03	8.39E-03	2.83E-02	6.60E-02	1.37E-01
HY-RV	5.00E-01	1.38E-03	2.18E-03	1.35E-02	7.09E-02	2.32E-01
		5.35E-03	8.45E-03	5.23E-02	2.78E-01	9.25E-01
HY-SIML	5.00E-01	5.11E-01	5.07E-01	5.20E-01	5.70E-01	9.00E-01
		2.00E+00	1.99E+00	2.24E+00	2.89E+00	5.28E+00
SIML-SIML	5.00E-01	4.91E-01	4.86E-01	4.99E-01	5.18E-01	5.19E-01
		1.44E-01	1.30E-01	2.02E-01	3.17E-01	5.16E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	1.99E-04	1.99E-04	2.02E-04	2.02E-04	2.04E-04
		4.08E-05	4.06E-05	6.58E-05	9.89E-05	1.30E-04
σ^2_{v1}	2.00E-08	2.28E-08	1.22E-08	4.88E-08	2.02E-07	1.01E-06
		6.51E-10	3.76E-10	3.40E-09	2.96E-08	2.92E-07
RV1	2.00E-04	9.20E-04	6.55E-04	2.72E-04	2.12E-04	2.02E-04
		1.24E-05	1.03E-05	8.98E-06	1.75E-05	3.76E-05
σ^2_{x2}	2.00E-04	2.00E-04	2.00E-04	2.01E-04	2.03E-04	2.03E-04
		4.14E-05	4.13E-05	6.55E-05	9.74E-05	1.26E-04
σ^2_{v2}	2.00E-08	2.28E-08	1.22E-08	4.93E-08	2.01E-07	1.02E-06
		6.31E-10	3.68E-10	3.46E-09	2.92E-08	2.92E-07
RV2	2.00E-04	9.20E-04	6.55E-04	2.73E-04	2.11E-04	2.03E-04
		1.27E-05	1.01E-05	9.09E-06	1.76E-05	3.83E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.00E-04	1.03E-04	1.07E-04	1.08E-04
		3.49E-05	3.21E-05	5.00E-05	7.73E-05	1.03E-04
σ^2_{v12}	0.00E+00	7.25E-10	9.42E-11	1.18E-08	8.79E-08	4.92E-07
		5.54E-10	2.41E-10	2.47E-09	2.28E-08	2.24E-07
RCV	1.00E-04	6.67E-05	3.68E-05	9.00E-05	9.82E-05	1.00E-04
		6.77E-06	5.28E-06	6.86E-06	1.37E-05	2.98E-05
HY	1.00E-04	9.99E-05				
		6.36E-06				
RCV-RV	5.00E-01	7.25E-02	5.62E-02	3.31E-01	4.63E-01	4.95E-01
		7.34E-03	8.04E-03	2.27E-02	5.18E-02	1.18E-01
HY-RV	5.00E-01	1.09E-01	1.53E-01	3.68E-01	4.74E-01	5.11E-01
		6.90E-03	9.59E-03	2.45E-02	4.67E-02	1.00E-01
HY-SIML	5.00E-01	5.22E-01	5.23E-01	5.57E-01	6.39E-01	8.14E-01
		1.13E-01	1.13E-01	1.98E-01	4.04E-01	9.23E-01
SIML-SIML	5.00E-01	5.05E-01	5.02E-01	5.15E-01	5.33E-01	5.40E-01
		1.40E-01	1.25E-01	1.99E-01	3.12E-01	4.85E-01

Table 3.3 : Estimation of covariance and hedging coefficient :
Case 2 ($a_0 = 1, a_1 = -1, a_2 = 1; \lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	1.66E-04	1.71E-04	1.72E-04	1.72E-04	1.74E-04	1.76E-04
		3.54E-05	3.53E-05	5.69E-05	8.41E-05	1.12E-04
σ^2_{v1}	2.00E-06	2.00E-06	9.37E-07	2.02E-06	2.16E-06	2.84E-06
		5.68E-08	3.02E-08	1.39E-07	3.06E-07	7.94E-07
RV1	1.66E-04	7.21E-02	4.56E-02	7.36E-03	1.37E-03	4.08E-04
		1.08E-03	8.07E-04	2.78E-04	1.26E-04	8.08E-05
σ^2_{x2}	1.66E-04	1.72E-04	1.73E-04	1.73E-04	1.75E-04	1.76E-04
		3.60E-05	3.61E-05	5.61E-05	8.49E-05	1.10E-04
σ^2_{v2}	2.00E-06	2.00E-06	9.39E-07	2.03E-06	2.17E-06	2.83E-06
		5.52E-08	3.06E-08	1.46E-07	3.08E-07	7.94E-07
RV2	1.66E-04	7.22E-02	4.57E-02	7.38E-03	1.37E-03	4.08E-04
		1.08E-03	7.73E-04	2.89E-04	1.31E-04	8.06E-05
σ^2_{x12}		8.36E-05	8.31E-05	8.50E-05	8.85E-05	8.91E-05
		3.04E-05	2.84E-05	4.30E-05	6.65E-05	8.77E-05
σ^2_{v12}	0.00E+00	-3.52E-10	-9.98E-10	1.20E-08	8.31E-08	4.31E-07
		4.64E-08	1.89E-08	1.02E-07	2.19E-07	5.51E-07
RCV	8.33E-04	7.16E-05	3.55E-05	8.70E-05	8.71E-05	8.51E-05
		5.44E-04	3.83E-04	2.09E-04	9.14E-05	5.88E-05
HY	8.33E-04	8.28E-05				
		3.86E-04				
RCV-RV	5.00E-01	9.90E-04	7.81E-04	1.18E-02	6.35E-02	2.09E-01
		7.53E-03	8.39E-03	2.84E-02	6.64E-02	1.39E-01
HY-RV	5.00E-01	1.15E-03	1.81E-03	1.13E-02	6.04E-02	2.09E-01
		5.35E-03	8.45E-03	5.26E-02	2.85E-01	1.00E+00
HY-SIML	5.00E-01	5.09E-01	5.04E-01	5.13E-01	5.60E-01	9.17E-01
		2.39E+00	2.37E+00	2.69E+00	3.47E+00	6.06E+00
SIML-SIML	5.00E-01	4.89E-01	4.83E-01	4.96E-01	5.15E-01	5.18E-01
		1.47E-01	1.32E-01	2.05E-01	3.21E-01	5.20E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	1.66E-04	1.66E-04	1.66E-04	1.66E-04	1.69E-04	1.70E-04
		3.42E-05	3.39E-05	5.51E-05	8.26E-05	1.09E-04
σ^2_{v1}	2.00E-08	2.24E-08	1.17E-08	4.40E-08	1.72E-07	8.47E-07
		6.38E-10	3.63E-10	3.05E-09	2.54E-08	2.46E-07
RV1	1.66E-04	8.86E-04	6.21E-04	2.38E-04	1.79E-04	1.69E-04
		1.21E-05	9.87E-06	7.91E-06	1.49E-05	3.15E-05
σ^2_{x2}	1.66E-04	1.67E-04	1.67E-04	1.68E-04	1.69E-04	1.69E-04
		3.47E-05	3.46E-05	5.45E-05	8.10E-05	1.04E-04
σ^2_{v2}	2.00E-08	2.24E-08	1.17E-08	4.44E-08	1.71E-07	8.52E-07
		6.16E-10	3.57E-10	3.13E-09	2.48E-08	2.44E-07
RV2	1.66E-04	8.86E-04	6.22E-04	2.39E-04	1.78E-04	1.70E-04
		1.23E-05	9.71E-06	8.00E-06	1.49E-05	3.22E-05
σ^2_{x12}		8.40E-05	8.34E-05	8.54E-05	8.87E-05	8.94E-05
		2.93E-05	2.69E-05	4.18E-05	6.46E-05	8.52E-05
σ^2_{v12}	0.00E+00	6.03E-10	7.63E-11	9.82E-09	7.33E-08	4.09E-07
		5.38E-10	2.32E-10	2.23E-09	1.94E-08	1.88E-07
RCV	8.33E-04	5.56E-05	3.06E-05	7.50E-05	8.19E-05	8.34E-05
		6.50E-06	5.03E-06	6.02E-06	1.16E-05	2.51E-05
HY	8.33E-04	8.32E-05				
		5.90E-06				
RCV-RV	5.00E-01	6.27E-02	4.93E-02	3.15E-01	4.58E-01	4.94E-01
		7.32E-03	8.08E-03	2.29E-02	5.20E-02	1.19E-01
HY-RV	5.00E-01	9.39E-02	1.34E-01	3.50E-01	4.68E-01	5.10E-01
		6.65E-03	9.40E-03	2.56E-02	4.84E-02	1.02E-01
HY-SIML	5.00E-01	5.23E-01	5.23E-01	5.58E-01	6.40E-01	8.17E-01
		1.14E-01	1.14E-01	2.00E-01	4.04E-01	9.37E-01
SIML-SIML	5.00E-01	5.05E-01	5.02E-01	5.15E-01	5.32E-01	5.40E-01
		1.42E-01	1.26E-01	2.01E-01	3.15E-01	4.87E-01

Table 3.4 : Estimation of covariance and hedging coefficient :
Case 3 (Stochastic volatility; $\lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}		2.05E-04	2.06E-04	2.07E-04	2.07E-04	2.08E-04
		4.11E-05	4.11E-05	6.70E-05	9.43E-05	1.29E-04
σ^2_{v1}	2.00E-06	2.00E-06	9.40E-07	2.03E-06	2.18E-06	3.03E-06
		5.48E-08	3.02E-08	1.45E-07	3.31E-07	8.33E-07
RV1		7.22E-02	4.57E-02	7.41E-03	1.40E-03	4.42E-04
		1.05E-03	7.70E-04	3.01E-04	1.38E-04	8.47E-05
σ^2_{x2}		2.07E-04	2.07E-04	2.09E-04	2.07E-04	2.07E-04
		4.24E-05	4.22E-05	6.71E-05	9.72E-05	1.35E-04
σ^2_{v2}	2.00E-06	2.00E-06	9.40E-07	2.03E-06	2.19E-06	3.01E-06
		5.65E-08	3.03E-08	1.45E-07	3.26E-07	8.68E-07
RV2		7.22E-02	4.57E-02	7.40E-03	1.40E-03	4.41E-04
		1.10E-03	8.16E-04	3.01E-04	1.38E-04	8.86E-05
σ^2_{x12}		1.01E-04	1.00E-04	1.02E-04	1.02E-04	1.03E-04
		3.43E-05	3.14E-05	4.93E-05	7.31E-05	1.05E-04
σ^2_{v12}	0.00E+00	1.54E-09	-1.12E-09	6.37E-09	8.62E-08	4.70E-07
		4.61E-08	1.86E-08	1.02E-07	2.25E-07	5.98E-07
RCV		8.23E-05	4.50E-05	7.87E-05	9.35E-05	9.86E-05
		5.36E-04	3.80E-04	2.10E-04	9.46E-05	6.15E-05
HY		1.31E-04				
		3.86E-04				
RCV-RV	5.00E-01	1.14E-03	9.88E-04	1.07E-02	6.71E-02	2.25E-01
		7.42E-03	8.31E-03	2.84E-02	6.79E-02	1.37E-01
HY-RV	5.00E-01	1.81E-03	2.86E-03	1.76E-02	9.24E-02	3.11E-01
		5.34E-03	8.43E-03	5.20E-02	2.78E-01	9.22E-01
HY-SIML	5.00E-01	6.79E-01	6.76E-01	7.17E-01	8.27E-01	1.04E+00
		2.01E+00	2.01E+00	2.18E+00	2.83E+00	4.02E+00
SIML-SIML	5.00E-01	4.93E-01	4.89E-01	4.98E-01	4.95E-01	4.78E-01
		1.40E-01	1.26E-01	1.97E-01	3.21E-01	4.85E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}		2.00E-04	2.00E-04	2.02E-04	2.01E-04	2.01E-04
		3.95E-05	3.94E-05	6.48E-05	9.17E-05	1.27E-04
σ^2_{v1}	2.00E-08	2.28E-08	1.22E-08	4.92E-08	2.02E-07	1.03E-06
		6.36E-10	3.80E-10	3.46E-09	2.87E-08	2.90E-07
RV1		9.20E-04	6.55E-04	2.72E-04	2.12E-04	2.04E-04
		1.26E-05	1.04E-05	9.28E-06	1.76E-05	3.71E-05
σ^2_{x2}		2.01E-04	2.01E-04	2.03E-04	2.02E-04	2.02E-04
		4.07E-05	4.04E-05	6.51E-05	9.46E-05	1.31E-04
σ^2_{v2}	2.00E-08	2.28E-08	1.22E-08	4.89E-08	2.03E-07	1.01E-06
		6.50E-10	3.83E-10	3.53E-09	2.91E-08	2.88E-07
RV2		9.20E-04	6.55E-04	2.72E-04	2.13E-04	2.03E-04
		1.26E-05	1.06E-05	8.92E-06	1.75E-05	3.74E-05
σ^2_{x12}		1.01E-04	1.01E-04	1.03E-04	1.02E-04	1.03E-04
		3.34E-05	3.03E-05	4.79E-05	7.17E-05	1.02E-04
σ^2_{v12}	0.00E+00	7.50E-10	9.85E-11	1.17E-08	8.82E-08	4.95E-07
		5.77E-10	2.38E-10	2.48E-09	2.24E-08	2.29E-07
RCV		6.68E-05	3.70E-05	8.99E-05	9.84E-05	1.01E-04
		7.08E-06	5.24E-06	6.89E-06	1.34E-05	2.86E-05
HY		1.00E-04				
		6.32E-06				
RCV-RV	5.00E-01	7.26E-02	5.65E-02	3.31E-01	4.65E-01	4.94E-01
		7.74E-03	8.00E-03	2.29E-02	5.14E-02	1.11E-01
HY-RV	5.00E-01	1.09E-01	1.53E-01	3.70E-01	4.77E-01	5.09E-01
		6.89E-03	9.63E-03	2.45E-02	4.75E-02	9.81E-02
HY-SIML	5.00E-01	5.21E-01	5.21E-01	5.51E-01	6.28E-01	8.22E-01
		1.11E-01	1.10E-01	1.90E-01	3.66E-01	7.92E-01
SIML-SIML	5.00E-01	5.05E-01	5.03E-01	5.12E-01	5.11E-01	5.04E-01
		1.38E-01	1.22E-01	1.92E-01	3.19E-01	4.99E-01

4. Asymptotic Properties of the SIML Estimation

It is important to investigate the asymptotic properties of the SIML estimator when the volatility function $\Sigma_x^2(s)$ is not constant over time. When the integrated variance-covariance matrix is a positive (deterministic) constant (i.e. Σ_x is not stochastic) while the instantaneous covariance function is time varying, we have the consistency and the asymptotic normality of the SIML estimator as $n \to \infty$. For the case of non-stochastic sampling, the asymptotic properties of the SIML estimator have been given by Kunitomo and Sato (2015). We give a slightly general result as the next proposition when the observations are ramdomly sampled. For the stochastic volatility and covariamce case in the continuous time, we assume that $\Sigma_x(t) = (\sigma_{ab}^{(x)})$ follows

$$\sigma_{ab}^{(x)}(t) = \sigma_{ab}^{(x)}(0) + \int_0^t \mu_{ab}^{\sigma}(s)ds + \int_0^t \gamma_{ab}^{\sigma}(s)dB(s) + \int_0^t \gamma_{ab}^{\sigma,*}(s)dB^*(s) , \tag{4.1}$$

where the coefficient μ_{ab}^{σ}, and the 2×1 vectors $\gamma_{ab}^{\sigma}(s), \gamma_{ab}^{\sigma,*}(s)$ are extensively measurable, continuous and bounded, and $B^*(s)$ is a 2×1 Brownian motion vector which is orthogonal to $B(s)$.

We extend the probability space $(\Omega, \mathcal{F}, P)$ and denote the extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ as explained by Chapter VIII of Jacod and Shiryaev (2003) or Jacod and Protter (2012). Then we say that a sequence of random variables Z_n with an index n stably converges in law if $\mathbf{E}[Yf(Z_n)] \longrightarrow \tilde{\mathbf{E}}[Yf(Z)]$ for all bounded continuous functions $f(\cdot)$ and all bounded random variables Y, and $\tilde{\mathbf{E}}[\cdot]$ is the expectation operator with respect to the extended probability space. We denote this convergence as $Z_n \stackrel{\mathcal{L}-s}{\longrightarrow} Z$ and write

$$Z_n \stackrel{\mathcal{L}-s}{\longrightarrow} N(0, 2\int_0^t \sigma_z^4(s)ds) \tag{4.2}$$

if Z is a continuous process defined on a very good filtered extension of $(\Omega, \mathcal{F}, P)$, which conditionally on the $\sigma-$field $\mathcal{F}$ is a Gaussian proces with independent increments satisfying

$$V_t = \tilde{\mathbf{E}}[(Z_t^*)^2|\mathcal{F}] = 2\int_0^t \sigma_z^4(s)ds \tag{4.3}$$

for $0 < t \leq 1$. The asymptotic properties of the SIML estimator in the stochastic volatility and covariance case can be summarized as Theorem 4.1 and the proof will be given in Appendix-A. We often use the notations n and n^* for n_a and n_a^* $(a = s, f)$ whenever we do not make any confusion.

Theorem 4.1 : We assume that $X(t)$ and $v_a(t_i^a)$ $(a = s, f,\ i, j \geq 1)$ in (2.1) and (2.3) are independent with (2.4), $\sigma_x = \int_0^1 \sigma_x(s)ds$ is positive definite, Assumptions I and II with $c = 1$ as the normalization. We further assume that $\mathbf{E}[w_s(t_i^s)^4] < \infty$, $\mathbf{E}[w_f(t_j^f)^4] < \infty$.

(i) For m_{n^*} and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$, $m_{n^*}/m_n \stackrel{p}{\to} 1$,

$$\hat{\sigma}_{ss}^{(x)} - \sigma_{ss}^{(x)} \stackrel{p}{\longrightarrow} 0\ ,\ \hat{\sigma}_{ff}^{(x)} - \sigma_{ff}^{(x)} \stackrel{p}{\longrightarrow} 0 \tag{4.4}$$

and

$$\hat{\sigma}_{sf}^{(x)} - \sigma_{sf}^{(x)} \stackrel{p}{\longrightarrow} 0\ . \tag{4.5}$$

(ii) For m_{n^*} and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$, $m_{n^*}/m_n \stackrel{p}{\to} 1$,

$$\sqrt{m_n}\left[\hat{\sigma}_{ss}^{(x)} - \sigma_{ss}^{(x)}\right] \stackrel{\mathcal{L}-s}{\longrightarrow} N\left[0, V_{ss}\right]\ ,\ \sqrt{m_n}\left[\hat{\sigma}_{ff}^{(x)} - \sigma_{ff}^{(x)}\right] \stackrel{\mathcal{L}-s}{\longrightarrow} N\left[0, V_{ff}\right]\ , \tag{4.6}$$

and

$$\sqrt{m_n}\left[\hat{\sigma}_{sf}^{(x)} - \sigma_{sf}^{(x)}\right] \stackrel{\mathcal{L}-s}{\longrightarrow} N\left[0, V_{sf}\right]\ , \tag{4.7}$$

where $Z_n \stackrel{\mathcal{L}-s}{\longrightarrow} Z$ is the stable convergence in law with

$$V_{ss} = 2\int_0^1 \left[\sigma_{ss}^{(x)}(\tau(s))\right]^2 d(s)^2 ds\ ,\ V_{ff} = 2\int_0^1 \left[\sigma_{ff}^{(x)}(\tau(s))\right]^2 d(t)^2 ds \tag{4.8}$$

and

$$V_{sf} = \int_0^1 \left[\sigma_{ss}^{(x)}(\tau(s))\sigma_{ff}^{(x)}(\tau(s)) + (\sigma_{sf}^{(x)}(\tau(s)))^2\right] d(s)^2 ds\ . \tag{4.9}$$

(iii) Assume $v_a(t_i^a)$ $(= w_a(t_i^a);\ a = s, f)$ in (2.4) are (mutually) independent randomv ariables. For the modified SIML estimator, we take $0 < \alpha < 0.5$. Then as $n \longrightarrow \infty$, $m_{n^*}/m_n \stackrel{p}{\to} 1$,

$$\sqrt{m_n}\left[\hat{\sigma}_{m.ss}^{(x)} - \sigma_{ss}^{(x)}\right] \stackrel{\mathcal{L}-s}{\longrightarrow} N\left[0, V_{ss}\right]\ ,\ \sqrt{m_{n_f}}\left[\hat{\sigma}_{m.ff} - \sigma_{ff}\right] \stackrel{\mathcal{L}-s}{\longrightarrow} N\left[0, V_{ff}\right] \tag{4.10}$$

and

$$\sqrt{m_n}\left[\hat{\sigma}^{(x)}_{m.sf}-\sigma^{(x)}_{sf}\right]\xrightarrow{\mathcal{L}-s}N\left[0,V_{sf}\right] \tag{4.11}$$

where V_{ss}, V_{ff} and V_{sf} are the same when $0<\alpha<0.5$.

There are some remarks on the conditions we have used in Theorem 4.1 and some possibilities of their extensions.

First, we have the condition $0<\alpha<0.4$ for the asymptotic normality of the SIML estimator. When we use the modified SIML (MSIML) estimation we can improve the convergence rate as $0<\alpha<0.5$. Due to the asymptotic bias in the SIML estimation, it is slightly smaller than the optimal rate but the SIML estimotor can be esasily implemented. There have been several discussions on the convergence rates since Gloter and Jacod (2001). It seems that there is a trade-off between the asymptotic efficiency and robustness of alternative estimators.

Second, when $v_s(t_i^s), v_f(t_j^f)$ $(i,j\geq 1)$ are autocorrelated and have the representation (2.4) with the fourth-order moment conditions that $\mathbf{E}[w_s(t_i^s)^4]<\infty$, $\mathbf{E}[w_f(t_j^f)^4]<\infty$, we can strengthen the same result in Theorem 4.1 with $0<\alpha<0.4$.

Third, when the variance-covariance matrix is constant, the number of observations are constant $n^*=n$, then $t_i^s-t_{i-1}^s=1/n, t_j^f-t_{j-1}^f=1/n$ $(i,j=1,\cdots,n)$ and $\tau(t)=t, d(t)=1, \tau(t)-\tau(s)=t-s$ for $0<s<t<1$, $\sigma^{(x)}_{ss}(s)=\sigma^{(x)}_{ss}, \sigma^{(x)}_{ff}(s)=\sigma^{(x)}_{ff}, \sigma^{(x)}_{sf}(s)=\sigma^{(x)}_{sf}$, $\tau(t)=t, d(t)=1, \tau(t)-\tau(s)=t-s$ for $0\leq s<t\leq 1$. The asymptotic variance and covariance are given by

$$V_{ss}=2[\sigma^{(x)}_{ss}]^2\ ,\ V_{sf}=\sigma^{(x)}_{ss}\sigma^{(x)}_{ff}+[\sigma^{(x)}_{sf}]^2\ . \tag{4.12}$$

Fourth, there are several cases when we have the representation of (2.5) and (2.6) instead of (2.3). Sato and Kunitomo (2015) have given several conditions when the similar asymptotic results for the SIML estimation of the integrated volatility as Theorem 4.1 holds in the univariate case.

By using the estimators of the integrated volatility and the integrated covariance, the SIML estimator of the hedging ratio $H=\sigma^{(x)}_{sf}/\sigma^{(x)}_{ff}$ has been defined by (2.8).

From Theorem 4.1, we use the standard delta method and the resulting asymptotic variance can be expressed as

$$(4.13) V[\hat{H}] = [\frac{1}{\sigma_{ff}^{(x)}}]^2 V_{sf} + [\frac{\sigma_{sf}^{(x)}}{\sigma_{ff}^{(x)2}}]^2 V_{ff} - 4\frac{\sigma_{sf}^{(x)}}{\sigma_{ff}^{(x)3}} \int_0^1 \sigma_{sf}^{(x)}(\tau(s))\sigma_{ff}^{(x)}(\tau(s))d(s)^2 ds \ .$$

Although the above formula looks complicated, when the volatility and covariance functions are constant we have the next result.

Corollary 4.2 : Assume that the instantaneous volatility matrix $\mathbf{\Sigma}_x$ is constant and $\Sigma_x(s)$ is a deterministic function of time. Then the asymptotic variance of the limiting distribution of $\sqrt{m_n}\left[\hat{H} - H\right]$ is given by

$$(4.14) \qquad \omega_H = \frac{\sigma_{ss}^{(x)}}{\sigma_{ff}^{(x)}}\left[1 - \frac{\sigma_{sf}^{(x)2}}{\sigma_{ss}^{(x)}\sigma_{ff}^{(x)}}\right] .$$

As the final remark on the asymptotic properties of the SIML estimator and the MSIML estimator, it would be possible to derive the corresponding results on the asymptotic distribution of the SIML and MSIML estimators when we have the non-linear transformations of (2.5) and (2.6). Some of the results have been reported in Sato and Kunitomo (2015) for the case of fixed observation intervals with one dimension. Since it is natural to consider the case when the integrated volatility is random, the extensions of our arguments have been currently under investigation.

5. Further Simulations

By extending the basic additive noise model (Model 1) reported in Section 3, we have conducted a large number of simulations. In each table we have calculated the historical volatility (the realized volatility, RV), the historical covariance (RCV), the HY (Hayashi-Yoshida) estimator and the SIML estimator. In Tables Raw means the estimates based on all simulated data and 10 sec means the estimates based on the simulated data at each grids of 10 second.

We have adopted the round-off error models and the micro-market adjustment models, which have been investigated by Sato and Kunitomo (2015) for the corresponding one dimensional cases. Among many Monte-Carlo simulations, we summarize our main results as Tables 5.1-5.7.

In Table 5.3 we use the EACD(1,1) model, which was originally proposed by Engle and Russell (1998). Financial econometricians have been interested in the (exponential) autoregressive conditional duration (EACD) models because the assumption of Poisson random sampling leads to the sequence of i.i.d. random variables for durations while we may have the dependent structure on the observed durations. Although there can be many types of duration dependence models, we shall use the EACD(1,1) model as a representative one. Let $\tau_i^a = t_i^a - t_{i-1}^a$ and $\tau_i^a = \psi_i^a \epsilon_i^a$, where

$$\psi_i^a = \omega + \delta \epsilon_i^a + \gamma \psi_{i-1}^a \tag{5.1}$$

and ϵ_i^a are a sequence of i.i.d. exponential random variables with $\delta > 0, \gamma > 0, \omega > 0$ $\mathbf{E}[\epsilon_i^a] = 0$ and $V[\epsilon_i^a] = 1$ for $a = s$ or f. We have set $\delta = 0.06$, $\gamma = 0.9$ and $\omega = 0.04$ because $\omega =$ (average duration)$\times[1-(\delta+\gamma)]$ when $n = 18,000$ with the standardization of 1 second.

In our simulations we use several non-linear transformation models in the form of (2.5). (Sato and Kunitomo (2015) have discussed the economic meaning of some of these models in details.) We take $X_s(t_i^s)$ and $X_f(t_j^f)$ individually and each model in our simulation corresponds to

Model 1 $\quad h_{a,1}(x,y,u) = x + u$,

Model 2 $\quad h_{a,2}(x,y,u) = g_{\eta_a,a}(x+u)$ (where $g_{\eta,a}(z) = \eta_a[z/\eta_a]$ (2.7)) ,

Model 3 $\quad h_{a,3}(x,y,u) = y + g_a(x-y) + u$ (g_a : constants) ,

Model 4 $\quad h_{a,4}(x,y,u) = y + g_{\eta_a,a}(x-y+u)$ (where $g_{a,\eta}(z) = \eta_a[z/\eta_a]$ (2.7)) ,

Model 5 $\quad h_{a,5}(x,y,u) = y + g_{\eta,a}(x-y) + u$ (where $g_{\eta,a}(z) = \eta_a[z/\eta_a]$ (2.7)) ,

Model 6 $\quad h_{a,6}(x,y,u) = y + u + \begin{cases} g_{1,a}(x-y) & \text{if } y \geq 0 \text{ (where } g_{1,a} \text{ : a constant)} \\ g_{2.a}(x-y) & \text{if } y < 0 \text{ (where } g_{2,a} \text{ : a constant)} \end{cases}$,

respectively.

Model 1 is the basic (signal-plus-noise) additive model and Model 2 is the basic round-off error model. Model 3 corresponds to the linear price adjustment with the micro-market noise and when the adjustment coefficient $0 < g_a < 2$, which is a sufficient condition that the corresponding adjustment mechanism has an ergodic property. Model 4 and Model 5 are the micro-market models with the round-off errors and price adjustments, Model 4 is the basic round-off model with price adjustment and Model 5 has a more complicated structure. Model 6 is the micro-market noise model with asymmetric non-linear price adjustment mechanism.

When there are micro-market noise components with the martingale signal part, the value of RCV often differs substantially from the true integrated covariance of the signal part. However, we have found that it is possible to estimate the integrated volatilities, the integrated covariance and the noise variances when we have the signal-noise ratio as $10^{-2} \sim 10^{-6}$ by the SIML estimation method. Although we have omitted the details of the second example, the estimation results are similar in the stochastic volatility model.

For the estimation of the covariance, we have confirmed that by using the H-Y method we can improve the historical covariance method as was pointed out by Hayashi and Yoshida (2005) when there are no micro-market noises. However, when we have micro-market noise terms, the bias of the H-Y method can be large in the volatility estimation while the SIML estimation gives reasonable estimates of the hedging coefficient. There can be alternative ways to use the integrated volatility and the integrated covariance for estimating the hedging coefficient, and the SIML-SIML combination gives reasonable estimates in all cases we have investigated. This point is vivid and important on the practical risk management purposes as Tables 5.1-5.7 have shown.

Table 5.1 and Table 5.2 give the simulation results in the basic roud-off error modes when the sampling scheme is a Poisson process with a constant intensity λ. As we have shown in Sato and Kunitomo (2015), the SIML estimator may dominate the existing methods including the realized kernel method and the pre-average method in some round-off-error cases. The selected simulation results in other linear and non-linear price adjustments models have been shown in Tables 5.4-Table 5.7. By examining these results of our simulations in addition to the basic cases in Section 3 we conclude that we can estimate the integrated volatility and integrated covariance of the hidden continuous part reasonably well by the SIML estimation method. It may be surprising to find that the SIML method gives reasonable estimates even when we have nonlinear transformations of the original unobservable security (intrinsic) values. We have conducted a number of further simulations, but the results are quite similar as we have reported in this section.

Table 5.1 : Estimation of covariance and hedging coefficient :

Model 2 ($a_0 = 1, a_1 = a_2 = 0; \lambda = 18000; \eta = 0.001$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ_{x1}^2	2.00E-04	2.04E-04	2.05E-04	2.06E-04	2.08E-04	2.10E-04
		4.20E-05	4.20E-05	6.76E-05	1.00E-04	1.33E-04
σ_{v1}^2	2.00E-06	2.09E-06	9.77E-07	2.10E-06	2.27E-06	3.09E-06
		5.94E-08	3.14E-08	1.43E-07	3.25E-07	8.60E-07
RV1	2.00E-04	7.52E-02	4.76E-02	7.69E-03	1.46E-03	4.52E-04
		1.14E-03	8.33E-04	2.90E-04	1.34E-04	8.81E-05
σ_{x2}^2	2.00E-04	2.05E-04	2.06E-04	2.07E-04	2.09E-04	2.10E-04
		4.26E-05	4.27E-05	6.70E-05	1.01E-04	1.31E-04
σ_{v2}^2	2.00E-06	2.09E-06	9.78E-07	2.12E-06	2.28E-06	3.09E-06
		5.75E-08	3.20E-08	1.51E-07	3.25E-07	8.75E-07
RV2	2.00E-04	7.52E-02	4.76E-02	7.71E-03	1.46E-03	4.52E-04
		1.13E-03	8.11E-04	2.99E-04	1.40E-04	8.90E-05
σ_{x12}^2	1.00E-04	1.00E-04	9.99E-05	1.02E-04	1.07E-04	1.07E-04
		3.60E-05	3.36E-05	5.11E-05	7.90E-05	1.05E-04
σ_{v12}^2		5.68E-10	-8.17E-10	1.48E-08	9.77E-08	5.24E-07
		4.81E-08	1.98E-08	1.05E-07	2.31E-07	6.01E-07
RCV	1.00E-04	8.59E-05	4.57E-05	1.02E-04	1.04E-04	1.03E-04
		5.66E-04	3.99E-04	2.18E-04	9.60E-05	6.46E-05
HY	1.00E-04	1.00E-04				
		4.03E-04				
RCV-RV	5.00E-01	1.14E-03	9.64E-04	1.32E-02	7.12E-02	2.28E-01
		7.52E-03	8.40E-03	2.83E-02	6.57E-02	1.37E-01
HY-RV	5.00E-01	1.34E-03	2.11E-03	1.31E-02	6.97E-02	2.29E-01
		5.36E-03	8.46E-03	5.25E-02	2.81E-01	9.48E-01
HY-SIML	5.00E-01	5.20E-01	5.16E-01	5.21E-01	5.71E-01	9.34E-01
		2.08E+00	2.07E+00	2.35E+00	3.08E+00	6.42E+00
SIML-SIML	5.00E-01	4.91E-01	4.85E-01	4.99E-01	5.18E-01	5.17E-01
		1.45E-01	1.30E-01	2.02E-01	3.17E-01	5.21E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ_{x1}^2	2.00E-04	2.00E-04	1.99E-04	2.00E-04	2.03E-04	2.05E-04
		4.08E-05	4.05E-05	6.58E-05	9.93E-05	1.31E-04
σ_{v1}^2	2.00E-08	8.52E-08	4.24E-08	1.27E-07	2.85E-07	1.10E-06
		3.70E-09	2.00E-09	9.21E-09	4.15E-08	3.13E-07
RV1	2.00E-04	3.23E-03	2.17E-03	5.62E-04	2.62E-04	2.12E-04
		9.54E-05	6.66E-05	2.11E-05	2.13E-05	4.01E-05
σ_{x2}^2	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.04E-04	2.03E-04
		4.15E-05	4.14E-05	6.55E-05	9.77E-05	1.26E-04
σ_{v2}^2	2.00E-08	8.49E-08	4.23E-08	1.28E-07	2.85E-07	1.10E-06
		3.74E-09	1.94E-09	9.87E-09	4.03E-08	3.14E-07
RV2	2.00E-04	3.23E-03	2.17E-03	5.64E-04	2.62E-04	2.13E-04
		9.65E-05	6.66E-05	2.30E-05	2.22E-05	4.04E-05
σ_{x12}^2	1.00E-04	1.01E-04	1.00E-04	1.03E-04	1.07E-04	1.08E-04
		3.49E-05	3.21E-05	5.01E-05	7.74E-05	1.03E-04
σ_{v12}^2		7.22E-10	9.18E-11	1.22E-08	8.85E-08	4.93E-07
		2.09E-09	8.81E-10	6.21E-09	3.08E-08	2.40E-07
RCV	1.00E-04	6.70E-05	3.68E-05	9.05E-05	9.85E-05	1.00E-04
		2.52E-05	1.77E-05	1.46E-05	1.65E-05	3.09E-05
HY	1.00E-04	1.01E-04				
		1.97E-05				
RCV-RV	5.00E-01	2.08E-02	1.70E-02	1.61E-01	3.76E-01	4.74E-01
		7.80E-03	8.17E-03	2.55E-02	5.54E-02	1.19E-01
HY-RV	5.00E-01	3.12E-02	4.64E-02	1.79E-01	3.86E-01	4.92E-01
		6.09E-03	9.08E-03	3.52E-02	7.93E-02	1.36E-01
HY-SIML	5.00E-01	5.26E-01	5.27E-01	5.60E-01	6.41E-01	8.15E-01
		1.53E-01	1.53E-01	2.22E-01	4.00E-01	9.01E-01
SIML-SIML	5.00E-01	5.05E-01	5.01E-01	5.15E-01	5.33E-01	5.40E-01
		1.40E-01	1.25E-01	1.99E-01	3.12E-01	4.86E-01

Table 5.2 : Estimation of covariance and hedging coefficient :

Model 2 ($a_0 = 1, a_1 = a_2 = 0; \lambda = 18000; \eta = 0.002$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.05E-04	2.06E-04	2.06E-04	2.08E-04	2.11E-04
		4.23E-05	4.24E-05	6.80E-05	1.01E-04	1.34E-04
σ^2_{v1}	2.00E-06	2.34E-06	1.09E-06	2.35E-06	2.51E-06	3.36E-06
		6.47E-08	3.49E-08	1.59E-07	3.62E-07	9.36E-07
RV1	2.00E-04	8.42E-02	5.33E-02	8.59E-03	1.60E-03	4.84E-04
		1.27E-03	9.40E-04	3.24E-04	1.48E-04	9.52E-05
σ^2_{x2}	2.00E-04	2.06E-04	2.07E-04	2.07E-04	2.10E-04	2.11E-04
		4.27E-05	4.28E-05	6.73E-05	1.01E-04	1.32E-04
σ^2_{v2}	2.00E-06	2.33E-06	1.09E-06	2.37E-06	2.53E-06	3.34E-06
		6.57E-08	3.61E-08	1.64E-07	3.62E-07	9.42E-07
RV2	2.00E-04	8.41E-02	5.33E-02	8.61E-03	1.61E-03	4.82E-04
		1.27E-03	9.07E-04	3.28E-04	1.55E-04	9.63E-05
σ^2_{x12}	1.00E-04	1.00E-04	9.98E-05	1.02E-04	1.07E-04	1.07E-04
		3.61E-05	3.37E-05	5.12E-05	7.91E-05	1.05E-04
σ^2_{v12}		5.01E-10	-2.92E-10	1.60E-08	8.71E-08	5.22E-07
		5.40E-08	2.23E-08	1.21E-07	2.60E-07	6.58E-07
RCV	1.00E-04	9.36E-05	5.91E-05	1.05E-04	1.00E-04	1.03E-04
		6.25E-04	4.50E-04	2.48E-04	1.08E-04	6.94E-05
HY	1.00E-04	1.04E-04				
		4.47E-04				
RCV-RV	5.00E-01	1.11E-03	1.11E-03	1.21E-02	6.26E-02	2.12E-01
		7.42E-03	8.46E-03	2.88E-02	6.75E-02	1.38E-01
HY-RV	5.00E-01	1.24E-03	1.96E-03	1.22E-02	6.47E-02	2.19E-01
		5.31E-03	8.39E-03	5.21E-02	2.83E-01	9.85E-01
HY-SIML	5.00E-01	5.40E-01	5.36E-01	5.34E-01	5.77E-01	9.24E-01
		2.32E+00	2.30E+00	2.61E+00	3.43E+00	6.27E+00
SIML-SIML	5.00E-01	4.88E-01	4.83E-01	4.97E-01	5.17E-01	5.17E-01
		1.45E-01	1.30E-01	2.02E-01	3.21E-01	4.92E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.03E-04	2.03E-04	2.02E-04	2.04E-04	2.06E-04
		4.14E-05	4.11E-05	6.63E-05	1.00E-04	1.33E-04
σ^2_{v1}	2.00E-08	1.70E-07	8.49E-08	2.56E-07	5.26E-07	1.34E-06
		1.39E-08	7.42E-09	2.87E-08	7.85E-08	3.72E-07
RV1	2.00E-04	6.46E-03	4.34E-03	1.11E-03	4.08E-04	2.42E-04
		4.48E-04	3.07E-04	8.28E-05	3.67E-05	4.47E-05
σ^2_{x2}	2.00E-04	2.05E-04	2.05E-04	2.03E-04	2.05E-04	2.05E-04
		4.21E-05	4.20E-05	6.62E-05	9.86E-05	1.27E-04
σ^2_{v2}	2.00E-08	1.70E-07	8.48E-08	2.58E-07	5.28E-07	1.36E-06
		1.39E-08	7.15E-09	2.96E-08	7.67E-08	3.83E-07
RV2	2.00E-04	6.47E-03	4.34E-03	1.12E-03	4.09E-04	2.43E-04
		4.44E-04	2.95E-04	8.35E-05	3.74E-05	4.54E-05
σ^2_{x12}	1.00E-04	1.02E-04	1.01E-04	1.04E-04	1.08E-04	1.08E-04
		3.54E-05	3.27E-05	5.08E-05	7.82E-05	1.03E-04
σ^2_{v12}		8.00E-10	1.42E-10	1.13E-08	8.61E-08	4.97E-07
		4.23E-09	1.70E-09	1.25E-08	5.51E-08	2.87E-07
RCV	1.00E-04	6.77E-05	3.77E-05	8.98E-05	9.74E-05	1.01E-04
		4.92E-05	3.49E-05	2.96E-05	2.52E-05	3.46E-05
HY	1.00E-04	9.97E-05				
		3.92E-05				
RCV-RV	5.00E-01	1.05E-02	8.68E-03	8.08E-02	2.39E-01	4.17E-01
		7.67E-03	7.99E-03	2.63E-02	5.88E-02	1.24E-01
HY-RV	5.00E-01	1.55E-02	2.30E-02	8.97E-02	2.45E-01	4.25E-01
		6.04E-03	9.02E-03	3.51E-02	9.63E-02	1.83E-01
HY-SIML	5.00E-01	5.10E-01	5.10E-01	5.51E-01	6.38E-01	8.19E-01
		2.24E-01	2.24E-01	2.98E-01	4.87E-01	1.01E+00
SIML-SIML	5.00E-01	5.01E-01	4.97E-01	5.15E-01	5.32E-01	5.44E-01
		1.40E-01	1.26E-01	2.01E-01	3.12E-01	4.91E-01

Table 5.3 : Estimation of covariance and hedging coefficient :
Model 1 (ACD ; $\lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.05E-04	2.06E-04	2.05E-04	2.02E-04	2.03E-04
		4.30E-05	4.27E-05	6.64E-05	9.33E-05	1.27E-04
σ^2_{v1}	2.00E-06	2.00E-06	9.32E-07	2.03E-06	2.18E-06	2.98E-06
		5.55E-08	3.36E-08	1.41E-07	3.16E-07	8.14E-07
RV1	2.00E-04	7.22E-02	4.51E-02	7.40E-03	1.40E-03	4.38E-04
		1.62E-03	9.34E-04	3.01E-04	1.37E-04	8.63E-05
σ^2_{x2}	2.00E-04	2.05E-04	2.06E-04	2.07E-04	2.12E-04	2.12E-04
		4.25E-05	4.27E-05	6.85E-05	1.03E-04	1.42E-04
σ^2_{v2}	2.00E-06	2.01E-06	9.31E-07	2.03E-06	2.20E-06	3.04E-06
		5.58E-08	3.34E-08	1.46E-07	3.07E-07	8.41E-07
RV2	2.00E-04	7.22E-02	4.52E-02	7.41E-03	1.41E-03	4.45E-04
		1.78E-03	9.57E-04	3.02E-04	1.32E-04	8.89E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.00E-04	1.02E-04	1.04E-04	1.03E-04
		3.70E-05	3.37E-05	5.13E-05	7.66E-05	1.03E-04
σ^2_{v12}	0.00E+00	-9.58E-10	-3.15E-10	6.07E-09	1.05E-07	4.83E-07
		4.56E-08	1.83E-08	1.03E-07	2.17E-07	5.84E-07
RCV	1.00E-04	7.51E-05	3.09E-05	8.39E-05	1.04E-04	1.00E-04
		5.01E-04	3.78E-04	2.16E-04	9.30E-05	6.06E-05
HY	1.00E-04	1.31E-04				
		3.80E-04				
RCV-RV	5.00E-01	1.04E-03	6.88E-04	1.13E-02	7.38E-02	2.32E-01
		6.94E-03	8.36E-03	2.91E-02	6.60E-02	1.39E-01
HY-RV	5.00E-01	1.81E-03	2.90E-03	1.78E-02	9.60E-02	3.10E-01
		5.26E-03	8.41E-03	5.14E-02	2.72E-01	9.22E-01
HY-SIML	5.00E-01	6.77E-01	6.76E-01	7.16E-01	8.07E-01	1.03E+00
		2.00E+00	1.98E+00	2.14E+00	2.68E+00	4.58E+00
SIML-SIML	5.00E-01	4.90E-01	4.85E-01	4.98E-01	5.21E-01	5.24E-01
		1.46E-01	1.28E-01	2.06E-01	3.46E-01	5.14E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.00E-04	2.00E-04	1.99E-04	1.96E-04	1.96E-04
		4.15E-05	4.13E-05	6.40E-05	9.08E-05	1.24E-04
σ^2_{v1}	2.00E-08	2.28E-08	1.21E-08	4.89E-08	2.04E-07	1.02E-06
		6.36E-10	4.01E-10	3.47E-09	3.01E-08	2.83E-07
RV1	2.00E-04	9.20E-04	6.49E-04	2.72E-04	2.13E-04	2.01E-04
		1.73E-05	1.09E-05	9.35E-06	1.79E-05	3.72E-05
σ^2_{x2}	2.00E-04	2.00E-04	2.00E-04	2.02E-04	2.06E-04	2.06E-04
		4.10E-05	4.11E-05	6.63E-05	1.02E-04	1.38E-04
σ^2_{v2}	2.00E-08	2.29E-08	1.21E-08	4.91E-08	2.03E-07	1.02E-06
		6.27E-10	4.11E-10	3.38E-09	3.09E-08	2.87E-07
RV2	2.00E-04	9.20E-04	6.50E-04	2.72E-04	2.12E-04	2.04E-04
		1.90E-05	1.17E-05	9.09E-06	1.73E-05	3.78E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.00E-04	1.02E-04	1.03E-04	1.03E-04
		3.54E-05	3.22E-05	4.98E-05	7.47E-05	9.97E-05
σ^2_{v12}	0.00E+00	8.14E-10	1.11E-10	1.14E-08	9.04E-08	5.00E-07
		5.59E-10	2.40E-10	2.53E-09	2.37E-08	2.23E-07
RCV	1.00E-04	6.84E-05	3.59E-05	8.92E-05	9.93E-05	1.01E-04
		6.91E-06	5.18E-06	6.92E-06	1.34E-05	2.88E-05
HY	1.00E-04	1.00E-04				
		6.34E-06				
RCV-RV	5.00E-01	7.44E-02	5.53E-02	3.28E-01	4.67E-01	5.01E-01
		7.53E-03	7.97E-03	2.32E-02	5.14E-02	1.14E-01
HY-RV	5.00E-01	1.09E-01	1.55E-01	3.70E-01	4.76E-01	5.16E-01
		7.11E-03	9.88E-03	2.53E-02	4.97E-02	1.02E-01
HY-SIML	5.00E-01	5.25E-01	5.26E-01	5.60E-01	6.58E-01	8.42E-01
		1.16E-01	1.16E-01	1.93E-01	4.24E-01	8.77E-01
SIML-SIML	5.00E-01	5.03E-01	5.02E-01	5.13E-01	5.30E-01	5.35E-01
		1.42E-01	1.25E-01	1.99E-01	3.29E-01	4.94E-01

Table 5.4 : Estimation of covariance and hedging coefficient :
Model 3 ($g = 0.2; \lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.25E-04	2.28E-04	2.09E-04	2.10E-04	2.15E-04
		4.53E-05	4.53E-05	6.83E-05	1.02E-04	1.38E-04
σ^2_{v1}	2.00E-06	6.27E-07	5.63E-07	4.31E-06	5.73E-06	6.53E-06
		1.78E-08	1.67E-08	3.12E-07	8.19E-07	1.80E-06
RV1	2.00E-04	4.00E-02	3.63E-02	1.74E-02	3.52E-03	8.59E-04
		5.26E-04	5.97E-04	7.05E-04	3.38E-04	1.81E-04
σ^2_{x2}	2.00E-04	2.27E-04	2.29E-04	2.10E-04	2.12E-04	2.13E-04
		4.74E-05	4.78E-05	6.83E-05	1.02E-04	1.34E-04
σ^2_{v2}	2.00E-06	6.27E-07	5.63E-07	4.30E-06	5.72E-06	6.45E-06
		1.73E-08	1.71E-08	2.99E-07	8.15E-07	1.81E-06
RV2	2.00E-04	4.00E-02	3.63E-02	1.74E-02	3.51E-03	8.55E-04
		5.37E-04	5.98E-04	6.80E-04	3.36E-04	1.74E-04
σ^2_{x12}	1.00E-04	9.98E-05	9.96E-05	1.02E-04	1.06E-04	1.06E-04
		3.80E-05	3.66E-05	5.16E-05	7.95E-05	1.07E-04
σ^2_{v12}	0.00E+00	5.23E-11	-3.47E-10	5.64E-09	8.30E-08	5.67E-07
		2.06E-08	1.14E-08	2.12E-07	5.76E-07	1.30E-06
RCV	1.00E-04	2.40E-05	1.16E-05	6.91E-05	9.35E-05	1.05E-04
		3.42E-04	2.87E-04	4.83E-04	2.31E-04	1.30E-04
HY	1.00E-04	2.74E-05				
		3.75E-04				
RCV-RV	5.00E-01	5.94E-04	3.18E-04	3.96E-03	2.70E-02	1.23E-01
		8.55E-03	7.91E-03	2.77E-02	6.58E-02	1.50E-01
HY-RV	5.00E-01	6.83E-04	7.46E-04	1.57E-03	7.87E-03	2.98E-02
		9.39E-03	1.04E-02	2.16E-02	1.08E-01	4.63E-01
HY-SIML	5.00E-01	1.23E-01	1.23E-01	8.24E-02	6.97E-02	1.51E-01
		1.77E+00	1.75E+00	2.13E+00	2.80E+00	4.26E+00
SIML-SIML	5.00E-01	4.43E-01	4.36E-01	4.89E-01	5.11E-01	5.08E-01
		1.45E-01	1.36E-01	2.05E-01	3.20E-01	4.93E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	1.99E-04	1.99E-04	2.00E-04	2.02E-04	2.04E-04
		4.07E-05	4.05E-05	6.57E-05	9.89E-05	1.30E-04
σ^2_{v1}	2.00E-08	6.31E-09	5.95E-09	5.29E-08	2.14E-07	1.02E-06
		1.79E-10	1.77E-10	3.87E-09	3.08E-08	2.95E-07
RV1	2.00E-04	4.22E-04	4.01E-04	2.96E-04	2.19E-04	2.03E-04
		5.35E-06	6.44E-06	1.05E-05	1.78E-05	3.82E-05
σ^2_{x2}	2.00E-04	2.00E-04	2.00E-04	2.01E-04	2.03E-04	2.03E-04
		4.14E-05	4.13E-05	6.55E-05	9.74E-05	1.25E-04
σ^2_{v2}	2.00E-08	6.31E-09	5.95E-09	5.29E-08	2.13E-07	1.03E-06
		1.74E-10	1.78E-10	3.69E-09	3.10E-08	2.94E-07
RV2	2.00E-04	4.22E-04	4.01E-04	2.96E-04	2.18E-04	2.04E-04
		5.50E-06	6.37E-06	1.01E-05	1.78E-05	3.82E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.00E-04	1.03E-04	1.07E-04	1.07E-04
		3.49E-05	3.21E-05	5.00E-05	7.72E-05	1.03E-04
σ^2_{v12}	0.00E+00	-6.54E-13	1.29E-12	3.78E-09	7.83E-08	4.82E-07
		2.13E-10	1.20E-10	2.65E-09	2.34E-08	2.24E-07
RCV	1.00E-04	1.34E-05	9.35E-06	5.67E-05	9.20E-05	9.88E-05
		3.58E-06	3.14E-06	7.35E-06	1.37E-05	2.97E-05
HY	1.00E-04	2.00E-05				
		4.21E-06				
RCV-RV	5.00E-01	3.17E-02	2.33E-02	1.92E-01	4.20E-01	4.87E-01
		8.45E-03	7.83E-03	2.40E-02	5.27E-02	1.17E-01
HY-RV	5.00E-01	4.74E-02	4.98E-02	6.76E-02	9.18E-02	1.02E-01
		9.96E-03	1.05E-02	1.42E-02	2.03E-02	2.81E-02
HY-SIML	5.00E-01	1.04E-01	1.04E-01	1.11E-01	1.27E-01	1.61E-01
		3.01E-02	3.01E-02	4.34E-02	8.28E-02	1.80E-01
SIML-SIML	5.00E-01	5.04E-01	5.01E-01	5.15E-01	5.33E-01	5.40E-01
		1.40E-01	1.25E-01	2.00E-01	3.12E-01	4.84E-01

Table 5.5 : Estimation of covariance and hedging coefficient :
Model 4 ($\eta = 0.001; \lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.04E-04	2.05E-04	2.06E-04	2.08E-04	2.11E-04
		4.20E-05	4.19E-05	6.76E-05	1.01E-04	1.34E-04
σ^2_{v1}	2.00E-06	2.09E-06	9.76E-07	2.11E-06	2.27E-06	3.09E-06
		5.82E-08	3.12E-08	1.45E-07	3.25E-07	8.68E-07
RV1	2.00E-04	7.52E-02	4.76E-02	7.69E-03	1.46E-03	4.51E-04
		1.12E-03	8.45E-04	2.90E-04	1.35E-04	8.90E-05
σ^2_{x2}	2.00E-04	2.05E-04	2.06E-04	2.06E-04	2.09E-04	2.10E-04
		4.26E-05	4.26E-05	6.68E-05	1.01E-04	1.31E-04
σ^2_{v2}	2.00E-06	2.09E-06	9.79E-07	2.12E-06	2.28E-06	3.07E-06
		5.75E-08	3.22E-08	1.56E-07	3.25E-07	8.75E-07
RV2	2.00E-04	7.52E-02	4.76E-02	7.71E-03	1.46E-03	4.51E-04
		1.13E-03	8.21E-04	3.05E-04	1.39E-04	8.98E-05
σ^2_{x12}	1.00E-04	1.00E-04	9.99E-05	1.02E-04	1.07E-04	1.07E-04
		3.59E-05	3.35E-05	5.11E-05	7.90E-05	1.05E-04
σ^2_{v12}	0.00E+00	-3.00E-10	-1.33E-09	1.39E-08	9.67E-08	5.15E-07
		4.74E-08	1.97E-08	1.04E-07	2.31E-07	6.03E-07
RCV	1.00E-04	7.50E-05	4.12E-05	1.02E-04	1.04E-04	1.02E-04
		5.60E-04	3.98E-04	2.17E-04	9.66E-05	6.47E-05
HY	1.00E-04	9.48E-05				
		3.99E-04				
RCV-RV	5.00E-01	9.95E-04	8.70E-04	1.32E-02	7.16E-02	2.27E-01
		7.45E-03	8.37E-03	2.82E-02	6.62E-02	1.38E-01
HY-RV	5.00E-01	1.26E-03	2.00E-03	1.23E-02	6.51E-02	2.12E-01
		5.30E-03	8.38E-03	5.19E-02	2.76E-01	9.31E-01
HY-SIML	5.00E-01	4.84E-01	4.81E-01	4.97E-01	5.49E-01	8.87E-01
		2.08E+00	2.06E+00	2.33E+00	3.10E+00	5.53E+00
SIML-SIML	5.00E-01	4.91E-01	4.86E-01	4.99E-01	5.18E-01	5.19E-01
		1.44E-01	1.30E-01	2.02E-01	3.19E-01	5.17E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.00E-04	2.00E-04	2.00E-04	2.03E-04	2.05E-04
		4.10E-05	4.08E-05	6.61E-05	9.92E-05	1.30E-04
σ^2_{v1}	2.00E-08	8.51E-08	4.24E-08	1.27E-07	2.84E-07	1.10E-06
		3.63E-09	1.94E-09	9.69E-09	4.06E-08	3.08E-07
RV1	2.00E-04	3.23E-03	2.17E-03	5.61E-04	2.62E-04	2.13E-04
		9.51E-05	6.61E-05	2.26E-05	2.18E-05	3.93E-05
σ^2_{x2}	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.03E-04	2.03E-04
		4.15E-05	4.14E-05	6.56E-05	9.75E-05	1.26E-04
σ^2_{v2}	2.00E-08	8.50E-08	4.25E-08	1.28E-07	2.84E-07	1.11E-06
		3.78E-09	2.00E-09	9.50E-09	4.06E-08	3.07E-07
RV2	2.00E-04	3.23E-03	2.17E-03	5.63E-04	2.62E-04	2.14E-04
		9.94E-05	6.91E-05	2.27E-05	2.14E-05	3.97E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.00E-04	1.02E-04	1.06E-04	1.08E-04
		3.50E-05	3.22E-05	5.02E-05	7.73E-05	1.03E-04
σ^2_{v12}	0.00E+00	6.72E-10	1.12E-10	1.17E-08	8.75E-08	4.96E-07
		2.03E-09	8.42E-10	6.40E-09	3.07E-08	2.34E-07
RCV	1.00E-04	6.67E-05	3.68E-05	8.95E-05	9.85E-05	1.01E-04
		2.34E-05	1.75E-05	1.48E-05	1.64E-05	3.10E-05
HY	1.00E-04	9.94E-05				
		1.97E-05				
RCV-RV	5.00E-01	2.07E-02	1.70E-02	1.60E-01	3.75E-01	4.73E-01
		7.24E-03	8.02E-03	2.56E-02	5.47E-02	1.20E-01
HY-RV	5.00E-01	3.08E-02	4.59E-02	1.77E-01	3.81E-01	4.83E-01
		6.11E-03	9.13E-03	3.57E-02	8.03E-02	1.35E-01
HY-SIML	5.00E-01	5.18E-01	5.18E-01	5.52E-01	6.32E-01	8.05E-01
		1.50E-01	1.50E-01	2.24E-01	4.11E-01	9.52E-01
SIML-SIML	5.00E-01	5.03E-01	5.00E-01	5.13E-01	5.30E-01	5.39E-01
		1.40E-01	1.25E-01	2.00E-01	3.13E-01	4.83E-01

Table 5.5 : Estimation of covariance and hedging coefficient :

Model 5 ($g = 0.001; \lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.04E-04	2.06E-04	2.06E-04	2.08E-04	2.11E-04
		4.20E-05	4.20E-05	6.77E-05	1.00E-04	1.34E-04
σ^2_{v1}	2.00E-06	2.09E-06	9.76E-07	2.11E-06	2.27E-06	3.09E-06
		5.89E-08	3.18E-08	1.45E-07	3.25E-07	8.57E-07
RV1	2.00E-04	7.52E-02	4.76E-02	7.70E-03	1.46E-03	4.51E-04
		1.12E-03	8.46E-04	2.91E-04	1.34E-04	8.82E-05
σ^2_{x2}	2.00E-04	2.06E-04	2.07E-04	2.07E-04	2.10E-04	2.11E-04
		4.27E-05	4.27E-05	6.71E-05	1.01E-04	1.32E-04
σ^2_{v2}	2.00E-06	2.09E-06	9.79E-07	2.12E-06	2.28E-06	3.08E-06
		5.74E-08	3.16E-08	1.50E-07	3.31E-07	8.68E-07
RV2	2.00E-04	7.52E-02	4.76E-02	7.72E-03	1.46E-03	4.51E-04
		1.13E-03	8.02E-04	3.02E-04	1.40E-04	8.91E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.00E-04	1.03E-04	1.07E-04	1.08E-04
		3.60E-05	3.36E-05	5.13E-05	7.89E-05	1.06E-04
σ^2_{v12}	0.00E+00	-2.79E-10	-1.04E-09	1.41E-08	9.49E-08	5.12E-07
		4.82E-08	1.97E-08	1.05E-07	2.30E-07	6.01E-07
RCV	1.00E-04	8.49E-05	3.92E-05	1.02E-04	1.03E-04	1.02E-04
		5.70E-04	3.98E-04	2.16E-04	9.63E-05	6.48E-05
HY	1.00E-04	1.03E-04				
		4.01E-04				
RCV-RV	5.00E-01	1.13E-03	8.25E-04	1.32E-02	7.08E-02	2.26E-01
		7.58E-03	8.37E-03	2.80E-02	6.60E-02	1.37E-01
HY-RV	5.00E-01	1.37E-03	2.17E-03	1.34E-02	7.11E-02	2.41E-01
		5.34E-03	8.43E-03	5.22E-02	2.79E-01	9.39E-01
HY-SIML	5.00E-01	5.26E-01	5.23E-01	5.36E-01	5.74E-01	8.80E-01
		2.09E+00	2.07E+00	2.36E+00	2.97E+00	4.96E+00
SIML-SIML	5.00E-01	4.93E-01	4.87E-01	5.01E-01	5.20E-01	5.18E-01
		1.44E-01	1.30E-01	2.02E-01	3.17E-01	5.14E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.00E-04	2.00E-04	2.01E-04	2.03E-04	2.05E-04
		4.09E-05	4.06E-05	6.60E-05	9.91E-05	1.31E-04
σ^2_{v1}	2.00E-08	6.77E-08	3.85E-08	1.32E-07	2.85E-07	1.10E-06
		2.74E-09	1.62E-09	9.24E-09	4.18E-08	3.16E-07
RV1	2.00E-04	2.86E-03	2.07E-03	5.71E-04	2.62E-04	2.13E-04
		6.90E-05	5.09E-05	1.96E-05	2.18E-05	3.94E-05
σ^2_{x2}	2.00E-04	2.01E-04	2.01E-04	2.02E-04	2.04E-04	2.04E-04
		4.15E-05	4.13E-05	6.58E-05	9.76E-05	1.26E-04
σ^2_{v2}	2.00E-08	6.79E-08	3.86E-08	1.32E-07	2.84E-07	1.11E-06
		2.78E-09	1.60E-09	9.02E-09	4.02E-08	3.09E-07
RV2	2.00E-04	2.86E-03	2.07E-03	5.71E-04	2.62E-04	2.14E-04
		6.81E-05	4.99E-05	2.03E-05	2.15E-05	3.95E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.01E-04	1.03E-04	1.07E-04	1.08E-04
		3.50E-05	3.21E-05	5.02E-05	7.74E-05	1.03E-04
σ^2_{v12}	0.00E+00	7.58E-10	1.56E-10	1.19E-08	8.67E-08	4.93E-07
		1.74E-09	7.47E-10	6.63E-09	3.21E-08	2.35E-07
RCV	1.00E-04	6.76E-05	3.77E-05	8.97E-05	9.78E-05	1.01E-04
		2.26E-05	1.63E-05	1.46E-05	1.66E-05	3.05E-05
HY	1.00E-04	1.00E-04				
		1.91E-05				
RCV-RV	5.00E-01	2.36E-02	1.83E-02	1.57E-01	3.73E-01	4.74E-01
		7.92E-03	7.86E-03	2.51E-02	5.52E-02	1.18E-01
HY-RV	5.00E-01	3.50E-02	4.85E-02	1.76E-01	3.85E-01	4.88E-01
		6.64E-03	9.20E-03	3.37E-02	8.09E-02	1.33E-01
HY-SIML	5.00E-01	5.23E-01	5.24E-01	5.59E-01	6.41E-01	8.17E-01
		1.48E-01	1.48E-01	2.31E-01	4.28E-01	1.01E+00
SIML-SIML	5.00E-01	5.06E-01	5.03E-01	5.16E-01	5.34E-01	5.39E-01
		1.40E-01	1.24E-01	2.00E-01	3.13E-01	4.87E-01

Table 5.7 : Estimation of covariance and hedging coefficient :
Model 6 ($g_1 = 0.2, g_2 = 5; \lambda = 18000$)

18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	2.15E-04	2.17E-04	2.08E-04	2.10E-04	2.15E-04
		4.36E-05	4.35E-05	6.83E-05	1.01E-04	1.36E-04
σ^2_{v1}	2.00E-06	7.24E-07	6.00E-07	3.59E-06	4.17E-06	4.98E-06
		2.07E-08	1.81E-08	2.59E-07	5.99E-07	1.37E-06
RV1	2.00E-04	4.29E-02	3.69E-02	1.37E-02	2.59E-03	6.75E-04
		5.75E-04	6.15E-04	5.55E-04	2.44E-04	1.40E-04
σ^2_{x2}	2.00E-04	2.16E-04	2.18E-04	2.09E-04	2.11E-04	2.13E-04
		4.51E-05	4.53E-05	6.79E-05	1.02E-04	1.33E-04
σ^2_{v2}	2.00E-06	7.23E-07	6.01E-07	3.59E-06	4.17E-06	4.94E-06
		2.00E-08	1.86E-08	2.47E-07	5.98E-07	1.39E-06
RV2	2.00E-04	4.29E-02	3.69E-02	1.37E-02	2.59E-03	6.74E-04
		5.91E-04	6.10E-04	5.44E-04	2.50E-04	1.37E-04
σ^2_{x12}	1.00E-04	1.01E-04	1.01E-04	1.03E-04	1.07E-04	1.08E-04
		3.71E-05	3.51E-05	5.17E-05	7.92E-05	1.07E-04
σ^2_{v12}	0.00E+00	6.82E-11	-4.05E-10	9.73E-09	9.09E-08	5.53E-07
		2.26E-08	1.22E-08	1.81E-07	4.22E-07	9.86E-07
RCV	1.00E-04	3.27E-05	1.98E-05	8.14E-05	9.79E-05	1.05E-04
		3.54E-04	2.95E-04	3.88E-04	1.72E-04	1.01E-04
HY	1.00E-04	3.81E-05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
		3.70E-04	0.00E+00	0.00E+00	0.00E+00	0.00E+00
RCV-RV	5.00E-01	7.58E-04	5.39E-04	5.92E-03	3.83E-02	1.56E-01
		8.26E-03	8.00E-03	2.84E-02	6.65E-02	1.48E-01
HY-RV	5.00E-01	8.89E-04	1.03E-03	2.80E-03	1.48E-02	5.56E-02
		8.63E-03	1.01E-02	2.71E-02	1.44E-01	5.80E-01
HY-SIML	5.00E-01	1.79E-01	1.79E-01	1.52E-01	1.50E-01	2.65E-01
		1.83E+00	1.81E+00	2.11E+00	2.74E+00	4.27E+00
SIML-SIML	5.00E-01	4.69E-01	4.63E-01	4.96E-01	5.18E-01	5.14E-01
		1.44E-01	1.33E-01	2.04E-01	3.18E-01	4.95E-01
18000	True	Raw	1 sec.	10 sec.	60 sec.	300 sec.
σ^2_{x1}	2.00E-04	1.99E-04	1.99E-04	2.00E-04	2.02E-04	2.04E-04
		4.07E-05	4.05E-05	6.58E-05	9.89E-05	1.30E-04
σ^2_{v1}	2.00E-08	7.37E-09	6.56E-09	5.10E-08	2.07E-07	1.02E-06
		2.12E-10	1.95E-10	3.69E-09	2.99E-08	2.93E-07
RV1	2.00E-04	4.67E-04	4.29E-04	2.83E-04	2.15E-04	2.03E-04
		6.05E-06	6.90E-06	9.72E-06	1.75E-05	3.80E-05
σ^2_{x2}	2.00E-04	2.00E-04	2.00E-04	2.01E-04	2.03E-04	2.03E-04
		4.14E-05	4.13E-05	6.55E-05	9.74E-05	1.25E-04
σ^2_{v2}	2.00E-08	7.37E-09	6.56E-09	5.10E-08	2.07E-07	1.02E-06
		2.04E-10	1.98E-10	3.56E-09	3.02E-08	2.91E-07
RV2	2.00E-04	4.68E-04	4.29E-04	2.83E-04	2.14E-04	2.03E-04
		6.20E-06	6.79E-06	9.66E-06	1.77E-05	3.80E-05
σ^2_{x12}	1.00E-04	1.01E-04	1.00E-04	1.06E-04	1.07E-04	1.08E-04
		3.49E-05	3.21E-05	5.00E-05	7.72E-05	1.03E-04
σ^2_{v12}	0.00E+00	3.63E-11	7.22E-12	6.01E-09	8.22E-08	4.87E-07
		2.39E-10	1.33E-10	2.65E-09	2.30E-08	2.23E-07
RCV	1.00E-04	2.03E-05	1.37E-05	6.76E-05	9.44E-05	9.94E-05
		3.84E-06	3.38E-06	7.13E-06	1.36E-05	2.97E-05
HY	1.00E-04	3.06E-05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
		4.43E-06	0.00E+00	0.00E+00	0.00E+00	0.00E+00
RCV-RV	5.00E-01	4.35E-02	3.19E-02	2.39E-01	4.39E-01	4.91E-01
		8.18E-03	7.90E-03	2.40E-02	5.23E-02	1.17E-01
HY-RV	5.00E-01	6.54E-02	7.14E-02	1.08E-01	1.43E-01	1.56E-01
		9.46E-03	1.03E-02	1.56E-02	2.30E-02	3.62E-02
HY-SIML	5.00E-01	1.60E-01	1.60E-01	1.70E-01	1.95E-01	2.48E-01
		3.90E-02	3.90E-02	6.22E-02	1.26E-01	2.84E-01
SIML-SIML	5.00E-01	5.05E-01	5.01E-01	5.15E-01	5.33E-01	5.39E-01
		1.40E-01	1.25E-01	2.00E-01	3.12E-01	4.85E-01

6. Conclusions

In this paper, we have shown that the Separating Information Maximum Likelihood (SIML) estimator is useful and it has the asymptotic robustness in the sense that it is consistent and it has the asymptotic normality in the stable convergence sense under a fairly general conditions when there are non-linear micro-market adjustments and errors, and the high frequency financial data are randomly sampled. They include not only the cases when we have the micro-market noises but also the cases when the micro-market structure has the round-off errors and the nonlinear price adjustments under a set of reasonable assumptions. This paper has focused on the estimation of the integrated covariance and the hedging coefficients and we have shown that the SIML estimator has reasonable asymptotic as well as finite sample properties. Micro-market factors in financial markets are common in the sense that we have the minimum price change and the minimum order size rules and also we often observe the bid-ask differences in stock markets. Sato and Kunitomo (2015) have shown that the SIML estimator dominates the existing methods including the realized kernel method and the pre-average method in some situations. Therefore the robustness of the estimation methods of the integrated covariance and the integrated hedging coefficient is quite important.

Also we should stress on the fact that the SIML estimator is very simple and it can be practically used not only for the integrated volatility but also the integrated covariance and the hedging coefficients from the multivariate high frequency financial series. An application on the analysis of stock-index futures market has been reported in Misaki (2013) for instance.

As a concluding remark there are several theoretical and practical problems arisen in our discussions mentioned in this paper. It is an obvious direction to weaken the underlying assumptions including Assumptions I and II. Also it may be interesting to see what to the extent our results would be valid when we have different threshold parameters in different assets in the round-off error models, which may be reasonable in actual financial markets such as the spot and futures Nikkei-

225 in Japan. Also it should be important to generalize our method to the cases with many dimensions. They are currently under investigation.

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APPENDIX : On Mathematical Derivations

In this Appendix, we give some details of the proof of Theorem 4.1. The method of our derivations is based on the results reported by Kunitomo and Sato (2013) and Sato and Kunitomo (2015), but we need some extra arguments. We shall use the notations K_i $(i \geq 1)$ as positive constants. Our proof consists of several steps.

(**Step-1**) The first step is to argue that the effects of the randomness of sampling and the effects of drift terms in the underlying stochastic processes are stochastically negligible under our assumptions.

Although t_i^n and n^* are random variables, which are finite-valued and bounded in $[0,1]$ for any n. We write $y_i^a = x_i^a + v_i^a$ for $a = s$ or f, where $y_i^a = y_a(t_i^n)$ and $x_i^a = X_a(t_i^n)$ in the basic case. We set $y_i^n = (y^s(t_i^n), y^f(t_i^n))'$, $x_i^n = (x^s(t_i^n), x^f(t_i^n))'$ and also we write the underlying (unobservable) returns in the period $(t_{i-1}^n, t_i^n]$ as

$$\text{(A.1)}\quad r_i^* = x_i^n - x_{i-1}^n = \int_{t_{i-1}^n}^{t_i^n} \mu_x(s, X(s))ds + \int_{t_{i-1}^n}^{t_i^n} \sigma_x(s, X(s))dB_s \ \ (i = 1, \cdots, n^*)$$

and the martingale part as

$$\text{(A.2)}\qquad r_i^n = \int_{t_{i-1}^n}^{t_i^n} \sigma_x(s, X(s))dB_s \ \ (i = 1, \cdots, n^*)$$

with $0 = t_0 \leq t_1^n < \cdots < t_{n^*}^n \leq 1$. By using Assumptions I and II, we have $n^*/n = 1 + o_p(1)$,

$$\mathbf{E}[\|r_j^*\|^2] = \mathbf{E}[\| \int_{t_{i-1}^n}^{t_i^n} \mu_x(s, X(s))ds|^2] + 2\mathbf{E}[(\int_{t_{i-1}^n}^{t_i^n} \mu_x(s, X(s))ds)' r_i^n] + \mathbf{E}[\|r_i\|^2] \ ,$$

and

$$\mathbf{E}[\| \int_{t_{i-1}^n}^{t_i^n} \sigma_x(s)dB_s - \int_{t_{i-1}^n}^{t_i^n} \sigma_x(t_{i-1}^n)dB_s\|^2] = O((\frac{1}{n})^2) \ .$$

Then we can evaluate as

$$\text{(A.3)}\qquad \mathbf{E}[\|r_i^*\|^2 - \int_{t_{i-1}^n}^{t_i^n} \mathrm{tr}(\Sigma_x(s))ds] = O((\frac{1}{n})^{3/2}) \ .$$

Hence we find that the effects of drift terms are negligible for the estimation of integrated volatility function under Assumptions I and II.

Similarly, we can use that $n^*/n \xrightarrow{p} 1$ and

$$\text{(A.4)} \qquad \sum_{j=1}^{n^*} \int_{t_{j-1}^n}^{t_j^n} \Sigma_x(s)ds - \int_0^1 \Sigma_x(s)ds \xrightarrow{p} O ,$$

and then

$$\text{(A.5)} \qquad \mathbf{E}[\int_{t_{i-1}^n}^{t_i^n} \Sigma_x(s)ds] = O(\frac{1}{n}) .$$

We note that the volatility function $\Sigma_x(s)$ $(0 \leq s \leq 1)$ and the integrated volatility $\Sigma_x = \int_0^1 \Sigma_x(s)ds$ can be stochastic.

By above arguments we can proceed the present proof as if n^* were fixed and was replaced by the corresponding fixed n. For instance we can use the standard method for the random sums (martingales) to the following proof of CLT (central limit theorem). (See Section 35 of Billingsley (1995) for instance.)

Then we shall use the fixed n (and m_{n^*}) as if it were n^* (and m_n) in the following developments of this Appendix for the sake of the resulting simplicities. (Basically we need to replace n by n^* in each step, and then we need to many tedious arguments because n^* is stochastic.)

(**Step-2**) Let $Z_{in}^{(1)}$ and $Z_{in}^{(2)}$ $(i = 1, \cdots, n)$ be the i-th elements of $n \times 2$ matrices

$$\text{(A.6)} \qquad \mathbf{Z}_n^{(1)} = h_n^{-1/2}\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{X}_n - \bar{\mathbf{Y}}_0) \ , \ \mathbf{Z}_n^{(2)} = h_n^{-1/2}\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{V}_n,$$

respectively, where we denote $\mathbf{X}_n = (x_i^{'}) = (x_i^s, x_i^f)$, $\mathbf{V}_n = (v_i^{'}) = (v_i^s, v_i^f)$, $\mathbf{X}_n = (z_{in}^{'})$ are $n \times 2$ matrices with $z_{kn} = z_{kn}^{(1)} + z_{kn}^{(2)}$ and $\mathbf{P}$ is defined by (2.15).

We write z_{kn}^s and z_{kn}^f as the first and second components of z_{kn}, and also we use the notations $z_{kn}^{s,j}$ and $z_{kn}^{f,j}; (j = 1, 2)$ for the j-th components of $z_{kn}^{(j)}$. By following the proof developed by Kunitomo and Sato (2013) for the case of fixed n, we use the arguments for investigating on the asymptotic distribution of $\sqrt{m_n}[\hat{\sigma}_{ss}^{(x)} - \sigma_{ss}^{(x)}]$ and $\sigma_{ss}^{(x)} = \int_0^1 \sigma_{ss}^{(x)}(s)ds$. (We can use the similar arguments for $\sigma_{ff}^{(x)} = \int_0^1 \sigma_{ff}^{(x)}(s)ds$ and $\sigma_{sf}^{(x)} = \int_0^1 \sigma_{sf}^{(x)}(s)ds$, but we omit the details.) We use the decomposition

$$\begin{aligned} \sqrt{m_n}\left[\hat{\sigma}_{ss}^{(x)} - \sigma_{ss}^{(x)}\right] &= \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(z_{kn}^s)^2 - \sigma_{ss}^{(x)}\right] \\ \text{(A.7)} \qquad &= \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(z_{kn}^{s,1})^2 - \sigma_{ss}^{(x)}\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathbf{E}[(z_{kn}^{s,2})^2] \end{aligned}$$

$$+\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[(z_{kn}^{s,2})^2-\mathbf{E}[(z_{kn}^{s,2})^2]\right]+2\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[(z_{kn}^{s,1})(z_{kn}^{s,2})\right] .$$

Because under Assumption II the effects of differences due to n^* and n are small, we notice that we can ignore the effects of $\sum_{m_{n^*}}^{n}(z_{kn}^{s})^2$ and $[m_{n^*}-m_n]/m_n$ in the following evaluations.

Then we shall investigate the conditions that three terms except the first one of (A.7) are $o_p(1)$. If they are satisfied, we could estimate the integrated volatility consistently as if there were no noise terms because other terms can be ignored asymptotically as $n\to\infty$.

Let $\mathbf{b}_k=(b_{kj})=\mathbf{e}_k^{'}\mathbf{P}_n\mathbf{C}_n^{-1}=(b_{kj})$ and $\mathbf{e}_k^{'}=(0,\cdots,1,0,\cdots)$ be an $n\times 1$ vector. We write $z_{kn}^{(2)}=\sum_{j=1}^{n}b_{kj}v_j^{'}$ and use the relation

$$(\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n^{'})_{k,k'}=\delta(k,k^{'})4\sin^2[\frac{\pi}{2n+1}(k-\frac{1}{2})].$$

Then because we have $n\sum_{j=1}^{n}b_{kj}b_{k^{'}j}=\delta(k,k^{'})a_{kn}$, $\mathbf{E}(v_{ss}^{(v)})$ are bounded and (2.4), it is straightforward to find K_1 such that

$$\text{(A.8)}\qquad \mathbf{E}[(z_{kn}^{s,2})]^2=n\mathbf{E}[\sum_{i=1}^{n}b_{ki}v_i^s\sum_{j=1}^{n}b_{kj}v_j^s]\le K_1\times a_{kn} ,$$

where v_j^s is the first element of v_j and we use the notation $\theta_j^a(t_j^a)$ $(a=s,f)$, $b_{kj}=0$ $(j\le 0)$. (We take $\rho=\max\{\rho_s,\rho_f\}$ and apply (32) of Kunitomo and Sato (2011).) Also Kunitomo and Sato (2013) have shown that

$$\text{(A.9)}\qquad \frac{1}{m_n}\sum_{k=1}^{m_n}a_{kn}=\frac{1}{m_n}2n\sum_{k=1}^{m_n}\left[1-\cos(\pi\frac{2k-1}{2n+1})\right]=O(\frac{m_n^2}{n})$$

and the second term of (A.7) becomes

$$\text{(A.10)}\qquad \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathbf{E}[z_{kn}^{s,2}]^2\le K_1\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}a_{kn}=O(\frac{m_n^{5/2}}{n}) .$$

This term is $o(1)$ if $0<\alpha<0.4$. (When we use the MSIML estimation, the effect of this term can be removed from $\sqrt{m_n}[\hat{\sigma}_{m.ss}^{(x)}-\sigma_{ss}^{(x)}]$ and we can improve the condition such that $0<\alpha<0.5$ in Theorem 4.1.)

For the fourth term of (A.7),

$$\mathbf{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n}z_{kn}^{s,1}z_{kn}^{s,2}\right]^2 \quad=\quad \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathbf{E}\left[z_{kn}^{s,1}z_{k',n}^{s,1}z_{kn}^{s,2}z_{k',n}^{s,2}\right]$$

$$= \mathbf{E}\left[2\sum_{j,j'=1}^{n} s_{jk}s_{j'k'}\mathbf{E}(r_j r_{j'}|\mathcal{F}_{\min(j,j')})z_{kn}^{s,2}z_{k'n}^{s,2}\right]$$

$$= O(\frac{m_n^2}{n}) ,$$

where

$$s_{jk} = \cos[\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})]$$

for $j,k = 1,2,\cdots,n$. (See Lemma 1.3 of Kunitomo and Sato (2008a).) In the above evaluation we have used the relation

$$|\sum_{j=1}^{n} s_{jk}s_{j,k'}| \le [\sum_{j=1}^{n} s_{jk}^2] = n/2 + 1/4 \text{ for any } k \ge 1 .$$

For the third term of (A.7), we need to consider the variance of

$$(z_{kn}^{s,2})^2 - \mathbf{E}[(z_{kn}^{s,2})^2] = n\sum_{j,j'=1}^{n} b_{kj}b_{k,j'}\left[v_j^s v_{j'}^s - \mathbf{E}(v_j^s v_{j'}^s)\right]$$

and then itis enough to evaluate the expectation of $\left[(z_{kn}^{s,2})^2 - \mathbf{E}[(z_{kn}^{s,2})^2]\right]\left[(z_{k',n}^{s,2})^2 - \mathbf{E}[(z_{k',n}^{s,2})^2]\right]$. When v_j^s are correlated and satisfy (2.4), becuse of the independence assumption on w_i $(i = 1,\cdots,n)$, it is possible to evaluate that there exists a positive constant K_3 such that

$$\text{(A.11)}\mathbf{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n}((z_{kn}^{s,2})^2 - \mathbf{E}[(z_{kn}^{s,2})^2])\right]^2 \le \frac{K_3}{m_n}(\sum_{k=1}^{m_n} a_{kn})^2 = O(\frac{1}{m_n}\times(\frac{m_n^2}{n})^2) .$$

Thus the third term of (A.7) is negligible if $0 < \alpha < 0.4$. (see Section 5 of Kunitomo and Sato (2011). When $\{v_j^s\}$ have the moving average (MA) representation as (2.4), the term is $O(m^5/n^2)$ because of (A.9) and then it can be negligible if $0 < \alpha < 0.4$.) When $\{v_j^s\}$ are mutually independent, it is possible to evaluate that the left-hand side of (A.11) is less that $(K_3/m_n)\sum_{k=1}^{m_n} a_{kn}^2$, which is $O(m_n^4/n^2)$. Then the third term of (A.7) is negligible if $0 < \alpha < 0.5$. (See (A.13) of Kunitomo and Sato (2013). This point the important step for the proof of Part (iii) because then the only concern is on (A.10).)

(**Step-3**) The third step is to give the asymptotic variance of the first term of (A.7), that is,

$$\sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(z_{kn}^{s,(1)})^2-\sigma_{ss}^{(x)}\right] \tag{A.12}$$

because it is of the order $O_p(1)$. We write

$$z_{kn}^{s,(1)}=\sqrt{\frac{n}{2n+1}}\sum_{j=1}^{n} r_j^s(e^{i\theta_{kj}}+e^{-i\theta_{kj}})\ , \tag{A.13}$$

where r_j^s is the first component of r_j in (A.2) and $\theta_{kj}=[2\pi/(2n+1)](k-1/2)(j-1/2)$. By using the relation that $(e^{i\theta_{kj}}+e^{-i\theta_{kj}})^2=2+e^{2i\theta_{kj}}+e^{-2i\theta_{kj}}$, we represent

$$\begin{aligned}
&[\frac{2n+1}{2n}]\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(z_{kn}^{s,(1)})^2-\int_0^1\sigma_{ss}^{(x)}(s)ds\right] \\
=\ & \frac{1}{m_n}\sum_{k=1}^{m_n}\left\{\frac{1}{2}[\sum_{j=1}^{n}(r_j^s)^2(e^{i\theta_{kj}}+e^{-i\theta_{kj}})^2-2(1+\frac{1}{2n})\int_0^1\sigma_{ss}^{(x)}(s)ds]\right. \\
& \left.+[\sum_{j\neq j'=1}^{n} r_j^s r_{j'}^s(e^{i\theta_{kj}}+e^{-i\theta_{kj}})(e^{i\theta_{kj'}}+e^{-i\theta_{kj'}})]\right\} \\
=\ & 2\sum_{j>j'=1}^{n} r_j^s r_{j'}^s[\frac{1}{m_n}\sum_{k=1}^{m_n}(e^{i\theta_{kj}}+e^{-i\theta_{kj}})(e^{i\theta_{kj'}}+e^{-i\theta_{kj'}})] \\
& +\sum_{j=1}^{n}[(r_j^s)^2-\int_{t_{j-1}^n}^{t_j^n}\sigma_{ss}^{(x)}(s)ds] \\
& +\frac{1}{2}\sum_{j=1}^{n}[(r_j^s)^2-\int_{t_{j-1}}^{t_j^n}\sigma_{ss}^{(x)}(s)ds][\sum_{k=1}^{m_n}(e^{2i\theta_{kj}}+e^{-2i\theta_{kj}})] \\
& -\frac{1}{2n}\sum_{j=1}^{n}[\int_{t_{j-1}}^{t_j^n}\sigma_{ss}^{(x)}(s)ds] \\
=\ & (A)+(B)+(C)+(D)\ ,\text{(say)}.
\end{aligned} \tag{A.14}$$

Then by using the derivations in Kunitomo and Sato (2013) (also Sato and Kunitomo (2015)), it is possible to show that except the first term (A), $\sqrt{m_n}(B)\xrightarrow{p}0$, $\sqrt{m_n}(C)\xrightarrow{p}0$, and $\sqrt{m_n}(D)\xrightarrow{p}0$ as $m_n\to\infty$ $(n\to\infty)$. Also by using the simple relation

$$(e^{i\theta_{kj}}+e^{-i\theta_{kj}})(e^{i\theta_{kj'}}+e^{-i\theta_{kj'}})=(e^{i(\theta_{kj}+\theta_{kj'})}+e^{-i(\theta_{kj}+\theta_{kj'})})+(e^{i(\theta_{kj}-\theta_{kj'})}+e^{-i(\theta_{kj}-\theta_{kj'})})\ ,$$

the random quantity $\sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(z_{kn}^{s,(1)})^2-\sigma_{ss}^{(x)}\right]$ is asymptotically equivalent to

$$(A)^{'}=2\sum_{j>j'=1}^{n} r_j^s r_{j'}^s h_m(j,j^{'})\ , \tag{A.15}$$

where for $j, j^{'} = 1, \cdots, n$

$$h_m(j,j^{'}) = \frac{1}{2\sqrt{m_n}} \frac{\sin 2m(\theta_j + \theta_{j'})/2}{\sin(\theta_j + \theta_{j'})/2} + \frac{1}{2\sqrt{m_n}} \frac{\sin 2m(\theta_j - \theta_{j'})/2}{\sin(\theta_j - \theta_{j'})/2}$$

and $\theta_j = [2\pi/(2n+1)](j - 1/2)$.

By extending the evaluation method on Féje-kernel (see Chapter 8 of Anderson (1971)), we can derive the variance of the asymptotic distribution of $(A)^{'}$, which is asymptotically equivalent to (A.12).

Lemma A-1 : Under Assumption II, as $n \to \infty$ the asymptotic variance of $(A)^{'}$ is given by

(A.16) $$V_{ss} = 2\int_0^1 \left[\sigma_{ss}^{(x)}(\tau(s))\right]^2 d(s)^2 ds \ .$$

Proof of Lemma A-1 : From the representation of (A.15), given the sampling process t_j^n $(j = 1, \cdots, n^*)$ we have the conditional expectation

$$\begin{aligned} & \mathbf{E}[((A)^{'})^2 | \{t_j^s\}] \\ = & \ 2\sum_{j>j^{'}=1}^{n^*} \sigma_{ss.j}^{(x)} \sigma_{ss.j^{'}}^{(x)} [\frac{1}{4m}][\frac{\sin 2m\pi(j+j^{'}-1)/(2^{n^*}+1)}{\sin\pi(j+j^{'})/(2^{n^*}+1)} + \frac{\sin 2m\pi(j-j^{'}-1)/(2^{n^*}+1)}{\sin\pi(j-j^{'})/(2^{n^*}+1)}]^2, \end{aligned}$$

where we use the notation

(A.17) $$\sigma_{ss.j}^{(x)} = \mathbf{E}[\int_{t_{j-1}^n}^{t_j^n} \sigma_{ss}^{(x)}(s) ds | t_{j-1}^n, t_j^n] \ .$$

Under Assumption I by using the standard approximation argument on integrals we find that

(A.18) $$\mathbf{E}\left[\int_{t_{j-1}^n}^{t_j^n} \sigma_{ss}^{(x)}(s) ds - \int_{t_{j-1}^n}^{t_j^n} \sigma_{ss}^{(x)}(t_{j-1}^n) ds\right] = o(\frac{1}{n})$$

and

(A.19) $$\int_{t_{j-1}^n}^{t_j^n} \sigma_{ss}^{(x)}(t_{j-1}) ds = (t_n^n - t_{j-1}^n)\sigma_{ss}^{(x)}(t_{j-1}^n) = O_p(\frac{1}{n}) \ .$$

Also for any continuous functions $a(s)$ and $b(t)$ in $[0,1]$

(A.20) $$2\int_0^1\int_0^1 \frac{1}{4m}[\frac{\sin m\pi(s-t)}{\sin(\pi/2)(s-t)} + \frac{\sin m\pi(s+t)}{\sin(\pi/2)(s+t)}]^2 a(s)b(t) ds dt$$

$$
\begin{aligned}
&= 2\int_0^1\int_0^1 \frac{1}{4m}\left\{[\frac{\sin m\pi(s-t)}{\sin(\pi/2)(s-t)}]^2+[\frac{\sin m\pi(s+t)}{\sin(\pi/2)(s+t)}]^2\right.\\
&\qquad \left.+[\frac{\sin m\pi(s-t)}{\sin(\pi/2)(s-t)}][\frac{\sin m\pi(s+t)}{\sin(\pi/2)(s+t)}]\right\} a(s)b(t)dsdt\\
&= (E)+(F)+(G)\ , \text{(say)}.
\end{aligned}
$$

By changing the order of integrations, as $m\to\infty$ we can evaluate the first term as

$$
\begin{aligned}
\text{(A.21)}\qquad & 2\int_0^1 \frac{1}{2m}[\frac{\sin^2(2m)\frac{\pi}{2}u}{\sin^2(\pi/2)u}]\left[\int_0^{1-u} a(u+t)b(t)dt\right]du\\
\longrightarrow\ & 2\lim_{u\to 0}\int_0^{1-u} a(u+t)b(t)dt\\
=\ & 2\int_0^1 a(t)b(t)dt\ .
\end{aligned}
$$

(See Lemma 8.3.3 of Anderson (1971).) Also by changing the order of integrations, we have found that the second term is negligible as

$$
\text{(A.22)}\qquad (F)=2\int_0^1 \frac{1}{2m}[\frac{\sin^2(2m)\frac{\pi}{2}u}{\sin^2(\pi/2)u}][\int_0^u a(s)b(u-s)ds]du\longrightarrow 0
$$

as $m\to\infty$. By applying the similar argument to the third term and we can find that the third term (G) is also negligible when m is large. Under Assumption I and Assumption II, as $(i(n)-1)/n\to s$ and $(j(n)-1)/n\to t$ for n being large while a fixed m, we have $t_i^n\xrightarrow{p}\tau(s)$, $t_j^n\to\tau(t)$, $n(t_i^n-t_{i-1}^n)\xrightarrow{p} d(s)$, and $n(t_j^n-t_{j-1}^n)\xrightarrow{p} d(t)$. Then the only non-negligible term in (A.17) corresponds to

$$
\text{(A.23)}\quad V^{'}=\int_0^1\int_0^1 \frac{1}{4m}[\frac{\sin m\pi(s-t)}{\sin(\pi/2)(s-t)}]^2\sigma_{ss}^{(x)}(\tau(s))\sigma_{ss}^{(x)}(\tau(t))d(s)d(t)dsdt\ .
$$

Then by letting $m\to\infty$, we have the desired result as (A.14).
(**Q.E.D.**)

(**Step-4**) As the next step we need to show the stable convergence in law of (A.12), but the arguments are quite similar to (**Step 4**) of the proof of Theorem 3.2 in Sato and Kunitomo (2015), which is based on the method explained by Chapter VIII of Jacod and Shiryaev (2003) or Jacod and Protter (2012). Thus we have omitted the details of our arguments in this version.

(**Step-5**) Finally we need to deal with the integrated covariance. By modifying the derivations for the proof of the integrated covariance, we use $\hat{\sigma}_{ss}^{(x)}, \hat{\sigma}_{ff}^{(x)}$ and $\hat{\sigma}_{sf}^{(x)}$. (It is straightforward to develp the similar arguments as for $\hat{\sigma}_{ss}^{(x)}$ but they are tedious. Thenwe have omittted the details.) Then we can evaluate the variance of the asymptotic distributions of the integrated covariance SIML estimator. Then the resulting variance formula becomes

$$V_{sf} = \int_0^1 \left[\sigma_{ss}^{(x)}(\tau(s))\sigma_{ff}^{(x)}(\tau(s)) + (\sigma_{sf}^{(x)}(\tau(s)))^2\right] d(s)^2 ds \ . \tag{A.24}$$

(Q.E.D.)

Fig.2-1:Round-off model