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Hitoshi Matsushima
The University of Tokyo

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Hitoshi Matsushima+

Faculty of Economics, University of Tokyo

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## Abstract

This paper investigates general two-player infinitely repeated games where the discount factor is less than but close to unity. We assume that monitoring is imperfect and private, and players' private signals are correlated through the unobservable macro shock. We show that a sequential equilibrium payoff vector approximates efficiency in a wide class of environments when the size of the set of private signals for each player is sufficiently large in comparison with the size of the set of possible macro shocks as well as the size of the set of actions for the opponent. We require almost no condition on the accuracy of players' monitoring technology. We argue that the use of review strategy works very well in the private monitoring case, although it does not work well in the public monitoring case. We apply our efficiency result to a model of price-setting duopoly a la Stigler (1964), where each firm's price choice is unobservable to its rival firm and the sales level for each firm is regarded as its private signal. Contrarily to Stigler's conjecture, the full cartel collusion can be self-enforcing even if firms cannot communicate and have the option of making secret price cuts.

**Keywords:** Discounted Repeated Games, Imperfect Private Monitoring, Review Strategies, Correlated Signals, Random Macro Shocks, Efficiency, Secret Price Cuts.

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+ Faculty of Economics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan. E-mail: hitoshi@e.u-tokyo.ac.jp

# 1. Introduction

We investigate general two-player infinitely repeated games, where the common discount factor is less than, but close to, unity. We assume that monitoring is imperfect and *private*. At the end of each period, each player observes her private signal, which is randomly drawn according to the probability function conditional on players' action choices. Each player cannot observe the opponent's action choice as well as the opponent's private signal. The present paper provides a sufficient condition under which efficiency is sustainable in the sense that there exists a nearly efficient sequential equilibrium payoff vector whenever the discount factor is close to unity. Based on this sufficiency, we show that efficiency can be sustained in a wide class of environments with private monitoring. Of particular importance, players' private signals can be *correlated* each other, and almost *no* conditions on the accuracy of players' monitoring technology are required.

The study of sequential equilibria in the private monitoring case has the severe difficulty, because each player's anticipation on which strategies the other players will play may depend on her private history in a complicated way.[1] Ely and Valimaki (2002) overcame this difficulty by investigating only a restricted class of equilibria that satisfy the *interchangeability* in the sense that all combinations of players' possible sub-strategies are Nash equilibria.[2] This restriction makes the equilibrium analysis drastically simplified. Hence, Ely and Valimaki could show that the folk theorem might hold in repeated prisoner-dilemma games even if monitoring is private.[3]

Ely and Valimaki, however, assumed that monitoring is almost perfect.[4] Following their paper, Matsushima (2002) investigated the case in which private monitoring is far from perfect. Matsushima constructed *review strategies* that are originated in Radner (1985) and cultivated by Matsushima (2001).[5] A review strategy is constructed as follows. The infinite time horizon is divided into finite period intervals named review phases. In each review phase, each player counts the number of periods in which a particular event occurs. If the resultant number of the event occurring during the review phase is larger than a threshold level, the player will be likely to punish the opponent in the next review phase. Here, the

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[1] Radner (1986) firstly investigated repeated games with private monitoring where no discounting was assumed. The two papers by Matsushima (1990a, 1990b) firstly investigated discounted repeated games with private monitoring. See also Kandori (2001) for the brief survey on this field.

[2] A related idea is found in Piccione (2002). See also Obara (1999) and Kandori and Obara (2000).

[3] Sekiguchi (1997), Bhaskar (1999), and Bhaskar and Obara (2002) showed the efficiency results in a repeated prisoner-dilemma game without using this interchangeability.

[4] Matsushima (2000) showed that the result in Ely and Valimaki could be extended to the case with the zero likelihood ratio condition.

[5] Kandori and Matsushima (1998) used similar review strategies in the study of two-player repeated games with communication.

player chooses the event as being ‘bad’ in that the probability of its occurrence is the smallest when the opponent plays collusive. According to the law of large numbers, a review strategy profile approximately induces the efficient payoff vector.

Matsushima (2002) weakened the interchangeability a la Ely and Valimaki by requiring only that all combinations of players’ sub-strategies starting from the first period of a review phase be Nash equilibria. By combining the idea of review strategy with this weakened interchangeability, Matsushima (2002) could show that the folk theorem might hold in repeated prisoner-dilemma games even if almost no conditions on the accuracy of players’ monitoring technology are required.[6]

Matsushima (2002) assumed that players’ private signals are *conditionally independent*, i.e., never correlated each other. Conditional independence in fact played the crucial role in the use of review strategies. Consider the case that players’ private signals are correlated. The probability of a bad event occurring may depend on the opponent’s private signal. Suppose that a player played collusive, but observed a private signal according to which she updates the probability of the opponent’s counting the bad event very high. Then, she expects to fail the opponent’s review with very high probability, and therefore, she may stop playing collusive from the next period.

Conditional independence is a restrictive condition, and might be even inappropriate to require in many economic situations. For example, consider a *price-setting duopoly* where each firm’s price choice is not observable to the rival firm. Each firm has to monitor the rival firm’s price choice through the observation of its sales level, which is not observable to the rival firm. The market demand conditions are likely to fluctuate between ‘boom’ and ‘recession’ according to an exogenous *random macro shock* that is unobservable to the firms. Hence, firms’ sales levels as their private signals are random variables and are correlated through this shock. This means that it should be appropriate *not* to assume conditional independence.

In the repeated game literature it has long been conjectured that the use of review strategy does not work very well when monitoring is imperfect and players’ private signals are correlated. When monitoring is public in that players’ private signals are perfectly correlated, each player can perfectly know which private signal the opponent has observed. This implies that for each player there exists *no* bad event such that the opponent’s private signal has no information on whether the player has counted this event. This interferes with any review strategy profile being an equilibrium.

In contrast to the negative conjecture from the public monitoring case, the present paper shows that the use of review strategy *does* work even in the correlated private monitoring case. We assume that the size of the set of possible private signals for each player is sufficiently *large* in comparison with the size of the set of possible macro shocks as well as the size of the set of actions for the opponent. This assumption guarantees that there almost always exists a bad event for each player such that the probability of its occurrence does not depend on the macro shock. Hence, the opponent’s private signal has *no* information about

[6] See also Piccione (2002) and Ely and Valimaki (2000) for the discussion on the case that private monitoring is not almost perfect.

whether the player has counted this event. Based on this, it follows that efficiency can be sustained by review strategy equilibria *almost everywhere* in the set of possible signal structures with the assumption above.

The paper will prove this efficiency not only in prisoner-dilemma games but also in more general two-player games. Especially, we allow each players to have three or more possible actions.[7]

The efficiency result of the paper would provide the study of cartel oligopoly with the following substantial impact. The classical work by Stigler (1964) has pointed out that each firm may have the option of making the *secret price cuts* in that it offers to consumers a sales price that is lower than the cartel price in secret from the other firms. Stigler then emphasized that secret price cuts would be the main course of preventing the firms' cartel agreement from being self-enforcing. Since Stigler did not provide a systematic analysis, we should carefully check to what extent his arguments was correct by making an appropriate model, which would be a discounted repeated game with correlated private monitoring such as what the present paper will investigate. In contrast to Stigler's argument, the efficiency result of the paper implies that the full cartel agreement can be self-enforcing even if firms have the option of making the secret price cuts.

Kandori and Matsushima (1998) and Compte (1998) have assumed that players could communicate in repeated games with private monitoring.[8] These works provided their respective folk theorem, and then concluded that communication enhances the possibility of self-enforcing cartel agreement. In real economic situations, however, the Anti-Trust Law prohibits communication among the firms' executives. The present paper does *not* allow firms to communicate. Hence, contrary to these works, we will conclude that firms can make a self-enforcing cartel agreement even if their communication is severely regulated.

The earlier work by Green and Porter (1994) has investigated repeated games of cartel oligopoly in different ways from Stigler and the present paper. In their model, not price but quantity is the only choice variable for each firm, which is not observable to the other firms. Each firm has to monitor the other firms' quantity choices through the market-clearing price. This price is observable to all firms, and fluctuates according to the exogenous macro shock, which the firms cannot observe. Hence, unlike what might be Stigler's primary concern, Green and Porter did consider the public monitoring case. By using the trigger strategy construction, Green and Porter showed that business cycle takes place on the equilibrium path, and that the partial cartel collusion can be self-enforcing. Subsequently, Fudenberg, Levine and Maskin (1994) could prove that in the general public monitoring case the folk theorem holds, and therefore, the full cartel collusion can be self-enforcing.

The basic logic behind the present paper has the following substantial difference from Green and Porter, and Fudenberg, Levine and Maskin. In the public monitoring case, it is inevitable that the macro shock influences players' future behaviors, because each player cannot distinguish the impact of the opponent's deviation from that of the macro shock.

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[7] Ely and Valimaki (2002) investigated games with three or more actions in the almost perfect monitoring case.

[8] See also Aoyagi (2002).

Because of this, Fudenberg, Levine, and Maskin couldn't use the idea of review strategy, and instead used the device of punishment and reward on the hyperplanes originated in Matsushima (1989), combined with the self-generation originated in Abreu, Pearce and Stacchetti (1986). In the private monitoring case studied by the present paper, however, players' future behaviors *never* depend on the macro shock, because we choose a bad event so as not to have the probability of its occurrence depend on this shock.

The above point is also in contrast to Rotemberg and Saloner (1984), which has investigated repeated games with perfect monitoring where the market demand condition fluctuates according to the exogenous macro shock that is observable to the firms. In their paper the market price fluctuates counter-cyclically to the macro shock fluctuation between boom and recession, whereas in the present paper firms' pricing behaviors are never influenced by the macro shock fluctuation.

The organization of the paper is as follows. Section 2 provides the model of repeated games. As a generalization of Matsushima (2002), Section 3 provides a sufficient condition under which the collusive payoff vector is sustainable. Section 4 provides the main theorem that the collusive payoff vector is sustainable almost everywhere. Section 5 considers the symmetric case and provides a condition under which efficiency is sustainable. Section 6 considers price-setting duopoly with product differentiation, and shows that the full cartel agreement can be self-enforcing even if firms have the option of making the secret price cuts and communication is prohibited. Section 7 concludes.

## 2. The Model

A two-player infinitely repeated game with discounting and with imperfect private monitoring is denoted by $\Gamma(\delta) = ((A_i, u_i, \Omega_i)_{i\in\{1,2\}}, \delta, p)$, and defined as follows. In every period $t \geq 1$, players 1 and 2 play the component game defined by $(A_i, u_i)_{i\in\{1,2\}}$, where $A_i$ is the finite set of actions for player $i \in \{1,2\}$, $A \equiv A_1 \times A_2$, and player $i's$ payoff function is given by $u_i : A \to R$. A mixed action for player $i$ is denoted by $\alpha_i : A_i \to [0,1]$, where $\sum_{a_i \in A_i} \alpha_i(a_i) = 1$. We denote by $\Delta_i$ the set of all mixed actions for player $i$. We denote $j = 1$ ($j = 2$) when $i = 2$ ($i = 1$, respectively). The common discount factor is denoted by $\delta \in [0,1)$.

At the end of every period, player $i$ observes her *private* signal $\omega_i$. The opponent $j \neq i$ *cannot* observe player $i's$ private signal $\omega_i$. The finite set of possible private signals for player $i$ is denoted by $\Omega_i$.[9] Let $\Omega \equiv \Omega_1 \times \Omega_2$. A signal profile $\omega \equiv (\omega_1, \omega_2) \in \Omega$ occurs with probability $p(\omega|a)$ when players choose the action profile $a \in A$. We denote a *signal structure* by $(\Omega, p)$, where $p \equiv (p(\cdot\,|\,a))_{a\in A}$. A signal structure associated with the set of private signal profiles $\Omega$ will be simply denoted by $p$ instead of $(\Omega, p)$. We assume that the signal structure $p$ has the *full support* in that

$$p(\omega\,|\,a) > 0 \text{ for all } (a,\omega) \in A \times \Omega.$$[10]

Let $p_i(\omega_i\,|\,a) \equiv \sum_{\omega_j \in \Omega_j} p(\omega\,|\,a)$. We may regard $u_i(a)$ as the expected value defined as

$$u_i(a) = \sum_{\omega_i \in \Omega_i} \pi_i(\omega_i, a_i) p_i(\omega_i\,|\,a),$$

where $\pi_i(\omega_i, a_i)$ is the realized payoff for player $i$ in the component game when player $i$ chooses $a_i$ and observes $\omega_i$. We denote by $P = P(\Omega)$ the set of possible signal structures associated with $\Omega$.

A *private history* for player $i$ up to period $t$ is denoted by $h_i^t \equiv (a_i(\tau), \omega_i(\tau))_{\tau=1}^t$, where $a_i(\tau) \in A_i$ is the action for player $i$ in period $\tau$, and $\omega_i(\tau) \in \Omega_i$ is the private signal for player $i$ in period $\tau$. The null history is denoted by $h_i^0$. The set of all private histories for player $i$ up to period $t$ is denoted by $H_i^t$. A *strategy for player* $i$ is defined as a function

[9] Because of the finiteness, we use strategies that induce players to choose mixed actions. The purification is possible when the sets of private signals are the continuum sets. See Matsushima (2002).

[10] We require the full support assumption for simplicity of arguments. We can derive the same results of the paper without it, but at the expense of complexity.

$s_i : \bigcup_{t=0}^{\infty} H_i^t \to \Delta_i$. Player $i$ chooses the action $a_i$ with probability $s_i(h_i^{t-1})(a_i)$ in period $t$ when the realized private history for player $i$ up to period $t-1$ is $h_i^{t-1}$. The *sub*-strategy of $s_i$, which player $i$ plays after period $t$ when the realized private history is $h_i^{t-1}$, is denoted by $s_i |_{h_i^{t-1}}$. The set of strategies for player $i$ is denoted by $S_i$. Let $S \equiv S_1 \times S_2$. Player $i's$ normalized long-run payoff induced by a strategy profile $s \in S$ after period $t$ when her private history is $h_i^{t-1}$ is given by

$$v_i(\delta, s, h_i^{t-1}) \equiv (1-\delta)E[\sum_{\tau=1}^{\infty} \delta^{\tau-1} u_i(a(\tau+t-1)) \mid s, h_i^{t-1}],$$

where $E[\cdot \mid s, h_i^{t-1}]$ implies the expectation conditional on $(s, h_i^{t-1})$. Player $i's$ normalized long-run payoff induced by $s \in S$ is denoted by

$$v_i(\delta, s) \equiv v_i(\delta, s, h_i^0).$$

Let

$$v(\delta, s) \equiv (v_1(\delta, s), v_2(\delta, s)).$$

A strategy profile $s \in S$ is said to be a *Nash equilibrium* in $\Gamma(\delta)$ if for every $i \in \{1,2\}$, and every $s_i' \in S_i$,

$$v_i(\delta, s) \geq v_i(\delta, s / s_i').$$

A strategy profile $s \in S$ is said to be a *sequential equilibrium* in $\Gamma(\delta)$ if for every $i \in \{1,2\}$, every $s_i' \in S_i$, every $t \geq 1$, and every $h_i^{t-1} \in H_i^{t-1}$,

$$v_i(\delta, s, h_i^{t-1}) \geq v_i(\delta, s_i', s_j, h_i^{t-1}).$$

Since the signal structure has the full support, it follows that the set of Nash equilibrium payoff vectors equals the set of sequential equilibrium payoff vectors.

## 3. Sustainability

A payoff vector $v = (v_1, v_2) \in R^2$ is said to be *sustainable* if for every $\varepsilon > 0$, and every infinite sequence of discount factors $(\delta^m)_{m=1}^{\infty}$ satisfying $\lim_{m \to +\infty} \delta^m = 1$, there exists an infinite sequence of strategy profiles $(s^m)_{m=1}^{\infty}$ such that for every large enough $m$, $s^m$ is a sequential equilibrium in $\Gamma(\delta^m)$ and

$$v_i - \varepsilon \leq v_i(\delta^m, s^m) \leq v_i + \varepsilon \text{ for all } i \in \{1,2\}.$$

Hence, $v$ is sustainable if a sequential equilibrium payoff vector approximates it whenever players are patient enough. Since the set of Nash equilibrium payoff vectors equals the set of sequential equilibrium payoff vectors, it follows that $v$ is sustainable if a Nash equilibrium payoff vector approximates it whenever players are patient enough.

We shall fix two action profiles $c = (c_1, c_2) \in A$ and $d = (d_1, d_2) \in A$ arbitrarily.

**Condition 1:**[11] For every $i \in \{1,2\}$,

$$u_i(d_i, c_j) > u_i(c),\ u_i(d) > u_i(c_i, d_j), \tag{1}$$

and

$$u_i(c) > \max_{a_i \in A_i} u_i(a_i, d_j). \tag{2}$$

We will call $c_i$ and $d_i$ the *collusive* action and the *defective* action respectively. Inequalities (1) imply that each player prefers the defective action $d_i$ to the collusive action $c_i$ in the component game irrespective of which the opponent $j$ will choose between the collusive action $c_j$ and the defective action $d_j$. Inequality (2) implies that each player prefers the collusive action profile $c = (c_1, c_2)$ to any action profile that induces the opponent to choose the defective action.

If a payoff vector $v$ is sustainable, it must be *individually rational* in that

$$v_i > \min_{a_j \in A_j} \max_{a_i \in A_i} u_i(a) \text{ for all } i \in \{1,2\}.$$

We can check that Condition 1 is more restrictive than individual rationality. Suppose that the defective action profile $d$ is given by $d_i \in \arg\min_{a_j \in A_j} [\max_{a_i \in A_i} u_i(a)]$ for all $i \in \{1,2\}$. Individual rationality requires that $u_i(c) > \max_{a_i \in A_i} u_i(a_i, d_j)$ for all $i \in \{1,2\}$. Condition 1 requires the inequalities above, but also requires that $u_i(c) \leq u_i(d_i, c_j)$ for all $i \in \{1,2\}$, which be not necessarily implied by individual rationality.

For every $i \in \{1,2\}$, we define *a random event on* $\Omega_i$ as a function $\psi_i : \Omega_i \to [0,1]$. A random event $\psi_i$ is interpreted as follows. Suppose that player $i$ observes not only the private signal $\omega_i$ but also a real number $x_i$ that is drawn according to the uniform distribution on the interval [0,1].

[11] Ely and Valimaki (2002) provided a related condition in the almost perfect monitoring case.

We will say that the random event $\psi_i$ occurs when player $i's$ observation $(\omega_i, x_i)$ satisfies $0 \le x_i < \psi_i(\omega_i)$. Hence, we can say that the random event $\psi_i$ occurs with probability $\psi_i(\omega_i)$ when player $i$ observes the private signal $\omega_i$. The probability that the random event $\psi_i$ occurs when players choose the action profile $a$ is given by

$$p_i(\psi_i \mid a) \equiv \sum_{\omega_i \in \Omega_i} \psi_i(\omega_i) p_i(\omega_i \mid a).$$

Let $p_i(\omega_i \mid a, \omega_j) \equiv \dfrac{p(\omega \mid a)}{p_j(\omega_j \mid a)}$ denote the probability that player $i$ observes the private signal $\omega_i \in \Omega_i$ when players choose the action profile $a \in A$ and the opponent $j$ observes the private signal $\omega_j \in \Omega_j$. The probability that the random event $\psi_i$ occurs when players choose the action profile $a$ and the opponent $j$ observes $\omega_j$ is given by

$$p_i(\psi_i \mid a, \omega_j) \equiv \sum_{\omega_i \in \Omega_i} \psi_i(\omega_i) p_i(\omega_i \mid a, \omega_j).$$

Let $\Psi_i$ denote the set of all random events on $\Omega_i$.

**Condition 2:** For every $i \in \{1,2\}$, there exist four random events $\psi_i^*$, $\psi_i^+$, $\psi_i^{**}$ and $\psi_i^{++}$ that satisfies the following two properties.

(i) For every $a_j \notin \{c_j, d_j\}$,

$$p_i(\psi_i^* \mid c) < p_i(\psi_i^* \mid c_i, d_j) = p_i(\psi_i^* \mid c_i, a_j),$$
$$p_i(\psi_i^+ \mid c) = p_i(\psi_i^+ \mid c_i, d_j) < p_i(\psi_i^+ \mid c_i, a_j),$$
$$p_i(\psi_i^{**} \mid d_i, c_j) < p_i(\psi_i^{**} \mid d) = p_i(\psi_i^{**} \mid d_i, a_j),$$

and

$$p_i(\psi_i^{++} \mid d_i, c_j) = p_i(\psi_i^{++} \mid d) < p_i(\psi_i^{++} \mid d_i, a_j).$$

(ii) For every $a_j \in A_j$, and every $\omega_j \in \Omega_j$,

$$p_i(\psi_i^* \mid c_i, a_j, \omega_j) = p_i(\psi_i^* \mid c_i, a_j),$$
$$p_i(\psi_i^+ \mid c_i, a_j, \omega_j) = p_i(\psi_i^+ \mid c_i, a_j),$$
$$p_i(\psi_i^{**} \mid d_i, a_j, \omega_j) = p_i(\psi_i^{**} \mid d_i, a_j),$$

and

$$p_i(\psi_i^{++} \mid d_i, a_j, \omega_j) = p_i(\psi_i^{++} \mid d_i, a_j).$$

Property (i) means that that each of these random events is 'bad' in the sense that the probability of its occurrence is the *lowest* when the opponent chooses the collusive action. Note that for every $i \in \{1,2\}$, if $c_j$ and $d_j$ are the only available actions for the opponent $j$, then there exist $\psi_i^*$, $\psi_i^+$, $\psi_i^{**}$ and $\psi_i^{++}$ that satisfy property (i) *almost everywhere* in $P$, i.e., in the set of possible signal structures associated with $\Omega$. In other words, there exist $\psi_i^*$, $\psi_i^+$, $\psi_i^{**}$ and $\psi_i^{++}$

that satisfy property (i) whenever the signal structure $p$ satisfies *minimal information requirement* in that for every $i \in \{1,2\}$

$$p_i(\cdot \mid a) \neq p_i(\cdot \mid a') \text{ for all } a \in A \text{ and all } a' \in A/\{a\}.$$[12]

Here, we can easily check the existences of $\psi_i^{+}$ and $\psi_i^{++}$ by choosing the null events, i.e.,

$$\psi_i^{+}(\omega_i) = \psi_i^{++}(\omega_i) = 0 \text{ for all } \omega_i \in \Omega_i.$$

Property (ii) means that the opponent $j's$ private signal has *no* information about whether the random events $\psi_i^{*}$ and $\psi_i^{+}$ ($\psi_i^{**}$ and $\psi_i^{++}$) on $\Omega_i$ occur or not, provided that player $i$ chooses the collusive action $c_i$ (the defective action $d_i$, respectively). Note that if the signal structure $p$ satisfies *conditional independence* in that

$$p(\omega \mid a) = p_1(\omega_1 \mid a) p_2(\omega_2 \mid a) \text{ for all } a \in A \text{ and all } \omega \in \Omega,$$

then for any random event $\psi_i$ on $\Omega_i$, the opponent $j's$ private signal has no information about whether this random event occurs or not, irrespective of players' pure action choices, i.e.,

$$p_i(\psi_i \mid a, \omega_j) = p_i(\psi_i \mid a) \text{ for all } a \in A \text{ and all } \omega_j \in \Omega_j,$$

which implies that property (ii) trivially holds.

A special case of $2 \times 2$ component games is so-called a *prisoner-dilemma game*, where for every $i \in \{1,2\}$, $A_i = \{c_i, d_i\}$, $u_i(c) > u_i(d)$, and $u_i(d_i, a_j) > u_i(c_i, a_j)$ for all $a_j \in A_j$. The inequalities above imply that Condition 1 automatically holds. Matsushima (2002) showed that the collusive payoff vector $u(c)$ is sustainable when the component game is a prisoner-dilemma game and the signal structure $p$ satisfies minimal information requirement and conditional independence. The following proposition generalizes this result.

**Proposition 1:** *With Conditions 1 and 2, $u(c)$ is sustainable.*

**Proof:** See the Appendix.

Proposition 1 states that we can replace both minimal information requirement and conditional independence with Condition 2. Proposition 1 states also that the sustainability result holds even in general two-player games where each player has two or more actions. The first statement is particularly important because players' private signals can be *correlated*. In order to understand it, we will provide the outline of its proof as follows.

Consider the repeated prisoner-dilemma game. Since $c_i$ and $d_i$ are the only available actions for every agent $i \in \{1,2\}$, it follows that Condition 2 holds if and only if for every $i \in \{1,2\}$, there exist $\psi_i^{*}$ and $\psi_i^{**}$ such that

$$p_i(\psi_i^{*} \mid c) < p_i(\psi_i^{*} \mid c_i, d_j), \; p_i(\psi_i^{**} \mid d_i, c_j) < p_i(\psi_i^{**} \mid d), \tag{3}$$

and for every $a_j \in A_j$, and every $\omega_j \in \Omega_j$,

[12] Minimal information requirement is the terminology in Matsushima (2002).

$$p_i(\psi_i^* \mid c_i, a_j, \omega_j) = p_i(\psi_i^* \mid c_i, a_j) \text{ and } p_i(\psi_i^{**} \mid d_i, a_j, \omega_j) = p_i(\psi_i^{**} \mid d_i, a_j) \,. \quad (4)$$

Here, the inequalities (3) and the equalities (4) correspond to property (i) and property (ii), respectively. Let the discount factor $\delta \in (0,1)$ be close to 1. Fix a sufficiently large integer $T > 0$ arbitrarily. For every $i \in \{1,2\}$, and every $a_i \in A_i$, fix a real number $\xi_i(a_i) \in [0,1]$ arbitrarily. Moreover, for each $i \in \{1,2\}$, fix an integer $M_i \in \{0,...,T\}$ arbitrarily. Since $T$ is sufficiently large, it follows from the inequalities (3) that we can choose $M_i$ satisfying that

$$p_i(\psi_i^* \mid c) < \frac{M_i}{T} < p_i(\psi_i^* \mid c_i, d_j) \,.$$

Similarly to Matsushima (2002), consider the following strategy $\bar{s}_i$, named the *collusive review strategy*, for each player $i$. We will divide the infinite time horizon into infinitely many $T$ period intervals. Each interval is called a *review phase*. According to $\bar{s}_i$, player $i$ continues choosing the collusive action $c_i$, i.e., playing collusive, during the first review phase, i.e., from period 1 to period $T$. When the number of periods in which the random event $\psi_i^*$ occurs is less than or equals $M_i$, the opponent $j$ will *pass* player $i's$ review, and player $i$ will certainly continue choosing $c_i$ during the next review phase. When this number is more than $M_i$, the opponent $j$ will *fail* player $i's$ review, and player $i$ will continue choosing the defective action $d_i$, i.e., playing defective (choosing $c_i$, i.e., playing collusive) during the next review phase with probability $\xi_i(c_i)$ (probability $1-\xi_i(c_i)$, respectively).

When player $i$ continues choosing $d_i$ during the next review phase and there exists no period in which the random event $\psi_i^{**}$ occurs, the opponent $j$ will *pass* player $i's$ review, and player $i$ will continue choosing $c_i$ (choosing $d_i$) during the review phase after the next with probability $1-\xi_i(d_i)$ (probability $\xi_i(d_i)$, respectively). When such a period exists, the opponent $j$ will *fail* player $i's$ review, and player $i$ will certainly continue choosing $d_i$. Similarly, we specify the remainder of $\bar{s}_i$.

We specify another strategy $\underline{s}_i$, named the *defective review strategy*, for player $i$, in the same way as $\bar{s}_i$, *except* for the first review phase. In contrast to $\bar{s}_i$, during the first review phase, player $i$ continues choosing $d_i$ instead of choosing $c_i$. Note that all sub-strategies of $\bar{s}_i$ and $\underline{s}_i$ that start from period $T+1$ are mixtures of $\bar{s}_i$ and $\underline{s}_i$.[13]

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[13] Here, we did not explicitly mention the random events $\psi_i^+$ and $\psi_i^{++}$. In the three or more action case, we need to construct more complicated review strategies where we will carefully check $\psi_i^+$ and $\psi_i^{++}$ as well as $\psi_i^*$ and $\psi_i^{**}$. By using additional devices of reviewing on the basis of $\psi_i^+$ and $\psi_i^{++}$, each player $i$ can make a very severe punishment on the opponent $j$ when she continues choosing other actions than $c_j$ and $d_j$. The reason why we require the inequality (2) in Condition 1 is that by

According to the following four steps, we will show that $v(\delta,\bar{s})$ and $v(\delta,\underline{s})$ approximate $u(c)$ and $u(d)$, respectively, and that $\bar{s}$ and $\underline{s}$ are both Nash equilibria. These steps are alike the proof in Matsushima (2002), *except for the final step.*

**Step 1:** We will show that $v(\delta,\bar{s})$ and $v(\delta,\underline{s})$ approximate $u(c)$ and $u(d)$, respectively. The law of large numbers implies that when players play $\bar{s}=(\bar{s}_1,\bar{s}_2)$, it is almost certain that the number of periods in which $\psi_i^*$ occurs, divided by $T$, is close to $p_i(\psi_i^* \mid c)$. Since $\frac{M_i}{T} > p_i(\psi_i^* \mid c)$, it is almost certain that the opponent $j$ will pass player $i's$ review. Hence, $v(\delta,\bar{s})$ approximates $u(c)$. The law of large numbers implies that when players play $\underline{s}=(\underline{s}_1,\underline{s}_2)$, it is almost certain that the number of periods in which $\psi_i^{**}$ occurs, divided by $T$, is close to $p_i(\psi_i^{**} \mid d)$. Since the threshold for this review equals zero and $p_i(\psi_i^{**} \mid d)$ is positive, it is almost certain that the opponent $j$ will fail the review. Hence, $v(\delta,\underline{s})$ approximates $u(d)$.

**Step 2:** We will show that a weaker version of interchangeability a la Ely and Valimaki holds in that the opponent $j$ is indifferent to the choice between $\bar{s}_j$ and $\underline{s}_j$ whenever player $i$ plays either $\bar{s}_i$ or $\underline{s}_i$, i.e.,

$$v_j(\delta,\bar{s}) = v_j(\delta,\bar{s}_i,\underline{s}_j) \text{ and } v_j(\delta,\underline{s}) = v_j(\delta,\underline{s}_i,\bar{s}_j).[14]$$

Suppose that player $i$ plays $\bar{s}_i$, while the opponent $j$ plays $\underline{s}_j$. Then, it is almost certain that the number of periods in which $\psi_i^*$ occurs, divided by $T$, is close to $p_i(\psi_i^* \mid c_i,d_j)$. Since $\frac{M_i}{T} < p_i(\psi_i^* \mid c_i,d_j)$, it is almost certain that the opponent $j$ will fail player $i's$ review. Hence, the opponent $j$ might not prefer $\underline{s}_j$ to $\bar{s}_j$ even if she can earn the short-term benefit from playing $\underline{s}_j$ instead of $\bar{s}_j$. By choosing $\xi_i(c_i)$ appropriately, we can make the opponent $j$ indifferent to the choice between $\bar{s}_j$ and $\underline{s}_j$, i.e., $v_j(\delta,\bar{s}) = v_j(\delta,\bar{s}_i,\underline{s}_j)$. Next, suppose that player $i$ plays $\underline{s}_i$, while the opponent $j$ plays $\bar{s}_j$. Then, the probability that $\psi_i^{**}$ occurs in all periods is $p_i(\psi_i^{**} \mid d_i,c_j)^T$. On the other hand, when players play $\underline{s}$ instead of $(\underline{s}_i,\bar{s}_j)$, this

continuing choosing the action maximizing her instantaneous payoff, the opponent $j$ is guaranteed to obtain about $\max_{a_j\in A_j} u_j(d_i,a_j)$ as her long-run payoff. See the Appendix.

[14] In the study of the almost perfect monitoring case, Ely and Valimaki (2002) required as the restriction of interchangeability that the opponent is indifferent to the choice among all possible sub-strategies. Kandori and Obara (2000) required a similar restriction in the study of private strategies in the perfect monitoring case. The requirement of interchangeability makes the equilibrium analysis drastically simplified. We then do not need to calculate the updated probability of each player $i's$ playing $\bar{s}_i$.

probability equals $p_i(\psi_i^{**} \mid d)^T$. Since $p_i(\psi_i^{**} \mid d_i, c_j) < p_i(\psi_i^{**} \mid d)$ and $T$ is sufficiently large, it follows that the ratio in probabilities $(\frac{p_i(\psi_i^{**} \mid d_i, c_j)}{p_i(\psi_i^{**} \mid d_i)})^T$ is close to zero. Hence, the opponent $j$ might not prefer $\underline{s}_j$ to $\bar{s}_j$ even if she can earn the short-term benefit from playing $\underline{s}_j$ instead of $\bar{s}_j$. By choosing $\xi_i(d_i)$ appropriately, we can make the opponent $j$ indifferent to the choice between $\bar{s}_j$ and $\underline{s}_j$, i.e., $v_j(\delta, \underline{s}) = v_j(\delta, \underline{s}_i, \bar{s}_j)$.

From Step 2, we can easily check that each player $i$ is indifferent to the choice among all strategies that induce her to make either the repeated choices of the collusive action or the repeated choices of the defective action in every review phase. We denote by $\hat{S}_i$ the set of all strategies $s_i$ for player $i$ satisfying that the sub-strategy after period $T+1$ is either the collusive review strategy or the defective review strategy, i.e.,

$$s_i|_{h_i^T} \in \{\bar{s}_i, \underline{s}_i\} \text{ for all } h_i^T \in H_i^T,$$

and there exists $(a_i(1), ..., a_i(T)) \in A^T$ such that

$$s_i(h_i^{t-1}) = a(t) \text{ for all } t = 1, ..., T \text{ and all } h_i^{t-1} \in H_i^{t-1}.$$

Here, the latter equalities imply that player $i's$ action choice is history-independent in every period during the first review phase.

**Step 3:** We will show that each player $i$ has no strict incentive to choose any strategy that belongs to $\hat{S}_i$ instead of $\bar{s}_i$, i.e., for every $i \in \{1,2\}$, and every $s_i \in \hat{S}_i$,

$$v_i(\delta, \bar{s}) \geq v_i(\delta, s_i, \bar{s}_j) \text{ and } v_i(\delta, \underline{s}) \geq v_i(\delta, s_i, \underline{s}_j).$$

The basic idea of Step 3 is originated in Matsushima (2002). Suppose that player $i$ plays $\bar{s}_i$. Then, by letting $\frac{M_i}{T}$ as close to $p_i(\psi_i^* \mid c)$ as possible, we can make the increase in the probability of the opponent $j$ being punished when she plays any strategy in $\hat{S}_j$ other than $\bar{s}_j$, sufficiently large. Next, suppose that player $i$ plays $\underline{s}_i$. Since the threshold for player $i's$ review equals zero, it follows that the increase in the probability of the opponent $j$ being punished when she plays any strategy in $\hat{S}_j$ other than $\bar{s}_j$, is sufficiently large. These increases in probabilities are the driving forces to prevent the opponent $j$ from playing any strategy in $\hat{S}_j$ other than $\bar{s}_j$.

In the final step below, we will have the substantial difference between Matsushima (2002) and the present paper.

**Step 4:** All we have to check for the rest is that each player $i$ has no strict incentive to play any strategy $s_i \notin \hat{S}_i$ whenever the sub-strategy after period $T+1$ is either the collusive review strategy or the defective review strategy, i.e.,

$$s_i|_{h_i^T} \in \{\bar{s}_i, \underline{s}_i\} \text{ for all } h_i^T \in H_i^T,$$

but her action choice is history-dependent in every period during the first review phase, in that there exists no $(a_i(1),...,a_i(T))$ such that

$$s_i(h_i^{t-1}) = a(t) \text{ for all } t = 1,...,T \text{ and all } h_i^{t-1} \in H_i^{t-1}.$$

Matsushima (2002) proved this step by using conditional independence, whereas the present paper will prove it by using a weaker condition such as property (ii). Property (ii) implies that when each player $i$ plays $\bar{s}_i$ (plays $\underline{s}_i$), the opponent $j's$ private signal $\omega_j$ has no information about whether the random event $\psi_i^*$ ($\psi_i^{**}$, respectively) occurs, and therefore, it has no information about which strategy the opponent will play from the next period. Hence, in every period during the first review phase, the best response for a player can be chosen *independently* of which private signals she has ever observed. This implies that if there exists $s_i \in S_i$ such that

$$\text{either } v_i(\delta, \bar{s}) \geq v_i(\delta, s_i, \bar{s}_j) \text{ or } v_i(\delta, \underline{s}) \geq v_i(\delta, s_i, \underline{s}_j),$$

then we can always find such a $s_i$ in the set $\hat{S}_i$. Since we have proved that no such $s_i$ exists in $\hat{S}_i$, it follows that $\bar{s}$ and $\underline{s}$ are both Nash equilibria.

The role of the signal-independence implied by property (ii) is crucial in the use of review strategies. Suppose that there exists $\hat{\omega}_j \in \Omega_j$ such that

$$p_i(\psi_i^* | c, \hat{\omega}_j) > p_i(\psi_i^* | c_i, d_j).$$

When the opponent $j$ chooses the collusive action $c_j$ and observes $\hat{\omega}_j$, she will update the probability of the event $\psi_i^*$ occurring so that it is higher than that when she chooses the defective action $d_j$. This implies that the opponent $j$ will expect to fail player $i's$ review with very high probability, so that she will have strict incentive to stop choosing $c_j$ from the next period. This contradicts the Nash equilibrium property. The observation above is the essence of the reason why previous researchers could not use review strategies in the 'both-side' imperfect public monitoring cases, with which the signal-independence is always inconsistent.[15]

In the next section, we will provide a sufficient condition for Condition 2, which does not imply conditional independence and includes a wide class of signal structures that allow players' private signals to be correlated.

[15] Radner (1985) used review strategies in the study of repeated principal-agent games with imperfect public monitoring under one-side uncertainty. Radner, Myerson and Maskin (1986) could not use review strategies in the study of repeated partnership games under both-side uncertainty, and derived the negative result of perfect public strategy equilibria. Radner (1986) could use review strategies in repeated partnership games but assumed no discounting.

## 4. Condition 2 and Generic Sustainability

The following condition is sufficient for Condition 2.

**Condition 3:** For every $i \in \{1,2\}$, and every $a_i \in \{c_i, d_i\}$, there exists no function $e: A_j \times \Omega_j \to R$ such that

$$\sum_{a_j \in A_j, \omega_j \in \Omega_j} e(a_j, \omega_j) p_i(\cdot \mid a, \omega_j) = 0,$$

and

$$\text{either } \sum_{\omega_j \in \Omega_j} e(c_j, \omega_j) p_i(\cdot \mid a_i, c_j, \omega_j) \neq 0 \text{ or } \sum_{\omega_j \in \Omega_j} e(d_j, \omega_j) p_i(\cdot \mid a_i, d_j, \omega_j) \neq 0.$$

**Proposition 2:** *Condition 3 implies Condition 2.*

**Proof:** Fix $i \in \{1,2\}$ arbitrarily. From Condition 3 it follows that there exist three positive integers $k_1$, $k_2$, $k_3$, and $k_1 + k_2 + k_3$ linearly independent $|\Omega_i|$ – dimensional vectors $\rho_1, \ldots, \rho_{k_1+k_2+k_3}$ such that for every $\omega_j \in \Omega_j$, $p_i(\cdot \mid a_i, c_j, \omega_j)$ is a linear combination of $\rho_1, \ldots,$ and $\rho_{k_1}$, $p_i(\cdot \mid a_i, d_j, \omega_j)$ is a linear combination of $\rho_{k_1+1}, \ldots,$ and $\rho_{k_1+k_2}$, and for every $a_j \notin \{c_j, d_j\}$, $p_i(\cdot \mid a, \omega_j)$ is a linear combination of $\rho_{k_1+k_2+1}, \ldots,$ and $\rho_{k_1+k_2+k_3}$. Hence, for every $(b, b', b'') \in R^3$, there exists a random event $\psi_i \in \Psi_i$ such that for every $\omega_j \in \Omega_j$,

$$p_i(\psi_i \mid a_i, c_j, \omega_j) = b,$$
$$p_i(\psi_i \mid a_i, d_j, \omega_j) = b',$$

and

$$p_i(\psi_i \mid a, \omega_j) = b'' \text{ for all } a_j \notin \{c_j, d_j\}.$$

This means that there exist $\psi_i^*$ and $\psi_i^+$ that satisfy the properties of Condition 2. Similarly, we can check also that there exist $\psi_i^{**}$ and $\psi_i^{++}$ that satisfy the properties of Condition 2.

**Q.E.D.**

We must note that Condition 3 does not hold in a non-negligible subset of $P = P(\Omega)$, i.e., the set of possible signal structures associated with $\Omega$. Since it always holds that

$$|\Omega_i| < |A_j| \times |\Omega_j| \text{ for some } i \in \{1,2\},$$

the collection $\{p_i(\cdot \mid a, \omega_j) \mid (a_j, \omega_j) \in A_j \times \Omega_j\}$ is *never* linearly independent for such a player $i \in \{1,2\}$. This means that in a non-negligible subset of $P$, there exist $\omega_j \in \Omega_j$ and $a'_j \in A_j / \{c_j\}$ such that $p_i(\cdot \mid c, \omega_j)$ is a convex combination of $\{p_i(\cdot \mid c_i, a_j, \omega_j) \mid (a_j, \omega_j) \in A_j \times \Omega_j\}$ that puts a positive weight on $p_i(\cdot \mid c_i, a'_j, \omega_j)$. This contradicts Condition 3.

However, as we will show below, with some reasonable restriction on the set of possible signal structures, Condition 3 *almost everywhere* holds. We will decompose a signal structure $p$ into two functions denoted by $q$ and $f_0$ as follows. In every period, after players' choosing the action profile, a *macro shock* $\theta_0$ is randomly drawn according to the conditional probability function $f_0(\cdot \mid a):\Xi_0 \to [0,1]$, where $\Xi_0$ denote the finite set of possible macro shocks. Players *cannot* observe the realization of this shock. After player's choosing the action profile $a \in A$ and the macro shock $\theta_0 \in \Xi_0$ occurring, the private signal profile $\omega \in \Omega$ is randomly drawn according to the conditional probability function $q(\cdot \mid a,\theta_0):\Omega \to [0,1]$. Hence, the probability of the private signal profile $\omega \in \Omega$ occurring when players chooses the action profile $a \in A$, i.e., $p(\omega \mid a)$, is described by

$$p(\omega \mid a) = \sum_{\theta_0 \in \Xi_0} q(\omega \mid a,\theta_0) f_0(\theta_0 \mid a).$$

Let $q_i(\omega_i \mid a,\theta_0) \equiv \sum_{\omega_j \in \Omega_j} q(\omega \mid a,\theta_0)$ and $q_i(\psi_i \mid a,\theta_0) \equiv \sum_{\omega_i \in \Omega_i} \psi_i(\omega_i) q_i(\omega_i \mid a,\theta_0)$. We assume that players' private signals are correlated *only* through this unobservable macro shock. This assumption implies that for every $(a,\theta_0) \in A \times \Xi_0$, $q(\cdot \mid a,\theta_0)$ be *conditionally independent* in that

$$q(\omega \mid a,\theta_0) = q_1(\omega_1 \mid a,\theta_0) q_2(\omega_2 \mid a,\theta_0) \text{ for all } \omega \in \Omega.$$

Based on the argument above, a signal structure will be denoted by $(\Xi_0,\Omega,f_0,q)$ instead of $(\Omega,p)$. The following condition is sufficient for Condition 2.

**Condition 4:** For every $i \in \{1,2\}$, and every $a_i \in \{c_i,d_i\}$, the collection of the probability functions $\{q_i(\cdot \mid a,\theta_0) \mid (a_j,\theta_0) \in A_j \times \Xi_0\}$ is *linearly independent* in that there exists no function $e: A_j \times \Xi_0 \to R$ such that

$$(e(a_j,\theta_0))_{(a_j,\theta_0)\in A_j \times \Xi_0} \neq 0 \text{ and } \sum_{(a_j,\theta_0)\in A_j \times \Xi_0} e(a_j,\theta_0) q_i(\cdot \mid a,\theta_0) = 0.$$

**Proposition 3:** *Condition 4 implies Condition 2.*

**Proof:** Fix $i \in \{1,2\}$ arbitrarily. Since $\{q_i(\cdot \mid a,\theta_0) \mid (a_j,\theta_0) \in A_j \times \Xi_0\}$ is linearly independent for all $a_i \in \{c_i,d_i\}$, it follows that there exist $\psi_i^*$, $\psi_i^+$, $\psi_i^{**}$ and $\psi_i^{++}$ such that for every $a_j \notin \{c_j,d_j\}$,

$$p_i(\psi_i^* \mid c) < p_i(\psi_i^* \mid c_i,d_j) \leq p_i(\psi_i^* \mid c_i,a_j),$$
$$p_i(\psi_i^+ \mid c) = p_i(\psi_i^+ \mid c_i,d_j) < p_i(\psi_i^+ \mid c_i,a_j),$$
$$p_i(\psi_i^{**} \mid d_i,c_j) < p_i(\psi_i^{**} \mid d) \leq p_i(\psi_i^{**} \mid d_i,a_j),$$
$$p_i(\psi_i^{++} \mid d_i,c_j) = p_i(\psi_i^{++} \mid d) < p_i(\psi_i^{++} \mid d_i,a_j),$$

for every $(a_j, \theta_0) \in A_j \times \Xi_0$,

$$p_i(\psi_i^* \mid c_i, a_j, \theta_0) = p_i(\psi_i^* \mid c_i, a_j),$$
$$p_i(\psi_i^+ \mid c_i, a_j, \theta_0) = p_i(\psi_i^+ \mid c_i, a_j),$$
$$p_i(\psi_i^{**} \mid d_i, a_j, \theta_0) = p_i(\psi_i^{**} \mid d_i, a_j),$$

and

$$p_i(\psi_i^{++} \mid d_i, a_j, \theta_0) = p_i(\psi_i^{++} \mid d_i, a_j).$$

Since $q(\cdot \mid a, \theta_0)$ is conditionally independent, it follows that

$$p_i(\omega_i \mid a, \omega_j) = \frac{\sum_{\theta_0 \in \Xi_0} q_1(\omega_1 \mid a, \theta_0) q_2(\omega_2 \mid a, \theta_0) f(\theta_0 \mid a)}{\sum_{\theta_0 \in \Xi_0} q_j(\omega_j \mid a, \theta_0) f(\theta_0 \mid a)},$$

and therefore, for every $\psi_i \in \Psi_i$,

$$p_i(\psi_i \mid a, \omega_j)$$
$$= \sum_{\omega_i \in \Omega_i} \frac{\psi_i(\omega_i) \sum_{\theta_0 \in \Xi_0} q_1(\omega_1 \mid a_i, c_j, \theta_0) q_2(\omega_2 \mid a_i, c_j, \theta_0) f(\theta_0 \mid a_i, c_j)}{\sum_{\theta_0 \in \Xi_0} q_j(\omega_j \mid a_i, c_j, \theta_0) f(\theta_0 \mid a_i, c_j)}$$
$$= \frac{\sum_{\theta_0 \in \Xi_0} p_i(\psi_i \mid a, \theta_0) q_j(\omega_j \mid a_i, c_j, \theta_0) f(\theta_0 \mid a_i, c_j)}{\sum_{\theta_0 \in \Xi_0} q_j(\omega_j \mid a_i, c_j, \theta_0) f(\theta_0 \mid a_i, c_j)}.$$

Since $p_i(\psi_i^* \mid c_i, a_j, \theta_0) = p_i(\psi_i^* \mid c_i, a_j)$, it follows that

$$p_i(\psi_i^* \mid c_i, a_j, \omega_j) = p_i(\psi_i^* \mid c_i, a_j).$$

Similarly, we can derive

$$p_i(\psi_i^+ \mid c_i, a_j, \omega_j) = p_i(\psi_i^+ \mid c_i, a_j),$$
$$p_i(\psi_i^{**} \mid d_i, a_j, \omega_j) = p_i(\psi_i^{**} \mid d_i, a_j),$$

and

$$p_i(\psi_i^{++} \mid d_i, a_j, \omega_j) = p_i(\psi_i^{++} \mid d_i, a_j).$$

Hence, Conditions 3 implies Condition 1. **Q.E.D.**

We denote by $P^* = P^*(\Xi_0, \Omega)$ the set of possible signal structures associated with $\Omega$ and $\Xi$. We must note that if

$$|\Omega_i| \geq |A_j| \times |\Xi_0| \text{ for all } i \in \{1,2\}, \tag{5}$$

then Condition 4 holds *almost everywhere* in the set $P^*$. From this, together with Propositions 1 and 3, we have proved the generic sustainability result as follows.

**Theorem 4:** *If Condition 1 and inequalities (5) hold, then* $u(c)$ *is sustainable almost everywhere* *in* $P^*$.

Theorem 4 states that with Condition 1 the collusive payoff vector $u(c)$ is *generically* sustainable whenever the size of the set of possible private signals for each player is sufficiently *large* in comparison with the size of the set of possible macro shocks as well as the size of the set of the opponent's actions.

Of particular importance, we can check below that every signal structure, irrespective of whether it has full support or not, can be approximated by other signal structures that satisfy Condition 4. For every $\varepsilon > 0$, a signal structure $(\Omega, p)$ is said to be $\varepsilon -$*close to* another signal structure $(\hat{\Omega}, \hat{p})$ if

$$\Omega \subset \hat{\Omega},$$

and for every $\omega \in \Omega$, and every $a \in A$,

$$|p(\omega \mid a) - \hat{p}(\omega \mid a)| \leq \varepsilon .$$

Fix a signal structure $(\Omega, p)$ arbitrarily where it may not have the full support. We can rewrite $(\Omega, p)$ as $(\Xi_0, \hat{\Omega}, f_0, q)$, where

$$\Xi_0 = \Omega,$$

$$\Omega \subset \hat{\Omega},$$

$$f_0(\omega \mid a) = p(\omega \mid a) \text{ and } q(\omega \mid a, \omega) = 1 \text{ for all } \omega \in \Omega \text{ and all } a \in A,$$

and $(\Xi_0, \hat{\Omega}, f_0, q)$ satisfies inequalities (5), i.e.,

$$|\hat{\Omega}_i| \geq |A_j| \times |\Xi_0| \text{ for all } i \in \{1,2\} .$$

Note that for every $(a, \theta_0) \in A \times \Xi_0$, $q(\cdot \mid a, \theta_0)$ is conditionally independent. Hence, it follows from the argument above that for every $\varepsilon > 0$, there exists a signal structure in the set $P^*(\Xi_0, \Omega)$ satisfying Condition 4 such that $(\Xi_0, \hat{\Omega}, f_0, q)$ is $\varepsilon -$close to this signal structure. From this observation, we have shown that for every two-player component game, every signal structure, every $(c, d) \in A^2$ satisfying Condition 1, and every $\varepsilon > 0$, there exists a $\varepsilon -$ close signal structure such that by replacing the former signal structure with the latter, the collusive payoff vector $u(c)$ is sustainable.

For example, consider the public monitoring case as follows, where a signal structure $p$ is said to be *public* if for every $i \in \{1,2\}$, there exists a function $\eta_i : \Xi_0 \to \Omega_i$ such that

$$\eta_1(\theta_0) \neq \eta_1(\theta_0') \text{ if and only if } \eta_2(\theta_0) \neq \eta_2(\theta_0'),$$

and

$$q_i(\eta_i(\theta_0) \mid a, \theta_0) = 1 \text{ for all } i \in \{1,2\} \text{ and all } \theta_0 \in \Xi_0 .$$

Whenever the signal structure is public then Condition 2 does not hold as follows. Note that for every $i \in \{1,2\}$, and every $\psi_i \in \Psi_i$,

$$p_i(\psi_i \mid a, \eta_j(\theta_0)) = \psi_i(\eta_i(\theta_0)) \text{ for all } a \in A \text{ and all } \theta_0 \in \Xi_0 .$$

This contradicts Condition 2, because $p_i(\psi_i \mid a, \omega_j)$ is independent of $a$ and depends on $\omega_j$.

Hence, it follows that in the perfect monitoring case, there exists no random event that satisfies Condition 2, and therefore, $u(c)$ is unlikely to be sustainable via review strategy equilibria.

In contrast, we can show that $u(c)$ is sustainable when monitoring is private but almost public.[16] Fix a positive real number $\varepsilon > 0$ arbitrarily, which is close to zero. A signal structure $p \in P$ is said to be $\varepsilon - public$ if for every $i \in \{1,2\}$, there exists $\eta_i : \Xi_0 \to \Omega_i$ such that

$$\eta_1(\theta_0) \neq \eta_1(\theta_0') \text{ if and only if } \eta_2(\theta_0) \neq \eta_2(\theta_0'),$$

and

$$q_i(\eta_i(\theta_0) \,|\, a, \theta_0) \geq 1 - \varepsilon, \text{ and } q_i(\omega_i \,|\, a, \theta_0) \leq \varepsilon \text{ for all } \omega_i \in \Omega_i / \{\eta_i(\theta_0)\}.$$

We denote by $P^*(\varepsilon) = P^*(\varepsilon, \Xi_0, \Omega) \subset P^*(\Xi_0, \Omega)$ the set of possible $\varepsilon -$public signal structures associated with $\Xi_0$ and $\Omega$. In the same way as in $P^*$, we can check that if $\Xi_0$ and $\Omega$ satisfy inequalities (5), then Condition 4 holds almost everywhere in the set $P^*(\varepsilon)$. Hence, it follows from Propositions 1 and 3 that if Condition 1 and inequalities (5) hold, then $u(c)$ is sustainable almost everywhere in $P^*(\varepsilon)$.

[16] Mailath and Morris (2002) studied the almost public monitoring case.

## 5. Condition 1 and Efficient Sustainability

This section investigates a component game that has *ordered* action sets and is *symmetric* in that

$$A_i = \{0,1,...,\bar{a}_i\} \text{ for all } i \in \{1,2\},$$

$$\bar{a}_1 = \bar{a}_2,$$

and for every $a \in A$, and every $a' \in A$,

$$u_1(a) = u_2(a') \text{ if } a_1 = a'_2 \text{ and } a_2 = a'_1,$$

where $\bar{a}_i$ is a positive integer. We assume that for every $i \in \{1,2\}$, *the opponent $j$'s action makes player $i$ soft* in that $u_i(a)$ is increasing with respect to $a_j$. Let $r : \{0,1,...,\bar{a}_2\} \to \{0,1,...,\bar{a}_1\}$ denote the *best response function* defined by

$$u_1(r(a_2), a_2) \geq u_1(a) \text{ for all } a \in A.$$

Since the component game is symmetric, $r$ is the best response function not only for player 1 but also for player 2, i.e.,

$$u_2(a_1, r(a_1)) \geq u_2(a) \text{ for all } a \in A.$$

We assume that players' actions are *strategic complements* in the sense that $r$ is a non-decreasing function. Moreover, we assume that $u_1(a)$ is *single-peaked* in the sense that it is increasing with respect to $a_1 \in A_1$ when $0 \leq a_1 < r(a_2)$, whereas it is decreasing when $r(a_2) < a_1 \leq \bar{a}_1$.

Based on the assumptions above, we choose the collusive action profile $c$ as a symmetric action profile that is *efficient* among all symmetric action profiles in that $c_1 = c_2$, and for every $a \in A$,

$$u(c) \geq u(a) \text{ if } a_1 = a_2.$$

We choose the defective action profile $d$ as a symmetric action profile satisfying that $d_1 = d_2$,

$$u_1(c) > u_1(r(d_2), d_2), \tag{6}$$

and there exists no symmetric action profile $a \in A$ such that $a_1 = a_2$, $a > d$, and $u_1(c) > u_1(r(a_2), a_2)$. Hence, $d$ is specified as the *maximal* symmetric action profile $a$ satisfying the inequality $u_1(c) > u_1(r(a_2), a_2)$. Since $u_1$ is increasing with respect to $a_2$, it follows that

$$c > d.$$

We assume that $u_1(r(d_2), d_2)$ is close to, and therefore, approximates, $u_1(c)$. Since $u_1$ is increasing with respect to $a_2$ and $c > d$, it follows that

$$u_1(r(d_2), c_2) > u_1(r(d_2), d_2).$$

Since $u_1(r(d_2), d_2)$ approximates $u_1(c)$, it would be plausible to assume that

$$u_1(r(d_2), c_2) > u_1(c). \tag{7}$$

**Proposition 5:** *In the case of symmetric component game with the assumptions above, the action profiles* $c$ *and* $d$ *specified above satisfy Condition 1.*

**Proof:** Inequality (6) and the symmetry imply inequalities (2). We can check $d_2 \geq r(d_1)$ in the following way. If $d_2 < r(d_1)$, then it must hold that

$$u_1(r(d_2), r(d_1)) > u_1(r(d_2), d_2),$$

because $u_1$ is increasing with respect to $a_2$. Since $u_1(r(d_2), d_2)$ is approximated by $u_1(c)$, it follows that

$$u_1(r(d_2), r(d_1)) > u_1(c).$$

This contradicts the fact that $c$ is efficient among all symmetric action profiles, because $(r(d_2), r(d_1))$ is symmetric and dominates $c$. Hence, we have proved

$$c_1 > d_1 \geq r(d_2). \tag{8}$$

Inequalities (8), the single-peakedness, and the symmetry imply that

$$u_1(d) > u_1(c_1, d_2) \text{ and } u_2(d) > u_1(d_1, c_2).$$

Inequalities (7) and (8), the single-peakedness, and the symmetry imply that

$$u_1(d_1, c_2) > u_1(c) \text{ and } u_2(c_1, d_2) > u_2(c).$$

These inequalities imply inequalities (1). Hence, we have proved that Condition 1 holds.

**Q.E.D.**

In many price-setting duopoly games it is quite plausible to require the assumptions such as the strategic complements, the single-peakedness, and the increasingness of firms' profit functions with respect to their rivals' prices.[17] Hence, it is natural to assume the existence of the cartel price vector $c$ and the defective price vector $d$ that are consistent with Condition 1. In the next section, we will provide a detailed model of price-setting duopoly that satisfies Condition 2 also.

The same result as Proposition 5 holds even when we replace the strategic complements and the increasingness with the strategic substitutes and the decreasingness, respectively. In many quantity-setting duopoly games it is plausible to require the assumptions such as the strategic substitutes, the single-peakedness, and the decreasingness of firms' profit functions with respect to their rivals' quantities. Hence, even in models of quality-setting duopoly, it is natural to assume the existence of the cartel agreement with Condition 1.

[17] There exist exceptions. See Tirole (1988).

## 6. Secret Price Cuts

This section considers the following price-setting dynamic duopoly with product differentiation. For every $i \in \{1,2\}$, let

$$A_i = \{0,1,2,...,\overline{a}_i\},$$

and

$$\Omega_i = \{0,1,2...,\overline{\omega}_i\},$$

where $\overline{a}_i > 0$ and $\overline{\omega}_i > 0$ are positive integers. In every period $t$, each firm $i \in \{1,2\}$ chooses the unit price level $a_i(t) \in A_i$ for its own commodities. The rival firm $j \neq i$ cannot observe firm $i's$ price choice $a_i(t)$. Here, $\overline{a}_i$ is the price level such that firm $i$ will never set the price bigger than $\overline{a}_i$, because if so, then no consumer will buy from it. At the end of the period, firm $i$ receives its sales level $\omega_i(t) \in \Omega_i$, which is the only available information that firm $i$ can observe about the rival firm $j's$ price choice $a_j(t)$. The rival firm $j$ cannot observe firm $i's$ sales level $\omega_i(t)$. Firm $i's$ production capacity is limited so that $\overline{\omega}_i$ is the maximal amount up to which firm $i$ can produce and supply at one time. Hence, $\overline{\omega}_i$ is also the upper bound of firm $i's$ sales level.

When firm $i$ chooses the price level $a_i \in A_i$ and receives the sales level $\omega_i \in \Omega_i$, its instantaneous profit is given by

$$\pi_i(\omega_i, a_i) = a_i\omega_i - z_i(\omega_i),$$

where $z_i(\omega_i)$ is the cost for firm $i's$ production. We assume that $z : \Omega_i \to R$ is increasing and $z_i(0) = 0$. Hence, the expected instantaneous profit for firm $i$ is given by

$$u_i(a) = \sum_{\omega_i \in \Omega_i} \{a_i\omega_i - z_i(\omega_i)\} p_i(\omega_i \mid a) \text{ for all } a \in A.$$

How each firm $i's$ sales level to be determined is modeled as follows. There exist $n$ consumers. For every $t \geq 1$, and every $h \in \{1,...,n\}$, all the macro shock $\theta_0 = \theta_0(t)$, her *private* shock $\theta_h = \theta_h(t)$, and the price vector $a(t) \in A$ influence consumer $h's$ buying behavior in period $t$. When firms choose the price vector $a$ and the macro shock $\theta_0$ and the private shock $\theta_h$ occur, consumer $h's$ demand level for firm $i$ is given by a non-negative integer $l_i(h, a, \theta_0, \theta_h)$. Hence, the total demand level for firm $i$ is defined by

$$D_i(a,\theta) \equiv \sum_{h=1}^{n} l_i(h, a, \theta_0, \theta_h),$$

where $\theta \equiv (\theta_0,...,\theta_n)$ denote a *shock profile*. Since $\overline{\omega}_i$ is the upper bound of firm $i's$ production, the sales level for firm $i$ is given by

$$\omega_i(a,\theta) \equiv \min[D_i(a,\theta), \overline{\omega}_i] \in \Omega_i.$$

Let $\omega(a,\theta) \equiv (\omega_1(a,\theta), \omega_2(a,\theta)) \in \Omega$.

We assume that each consumer will *never* buy commodities from both firms. We assume that which firm she will buy from does *not* depend on her private shock. Hence, for every $(a,\theta_0)\in A\times\Xi_0$, and every $h\in\{1,...,n\}$, there exists a firm $\iota(h,a,\theta_0)\in\{1,2\}$ such that consumer $h$ buys either nothing or only from the firm $\iota(h,a,\theta_0)$, i.e., for every $\theta_h\in\Xi_h$,

$$l_i(h,a,\theta_0,\theta_h)=0 \text{ if } i\neq\iota(h,a,\theta_0).$$ [18] [19]

For every $h\in\{0,...,n\}$, we denote by $\Xi_h$ the finite set of possible private shocks $\theta_h$. Let $\Xi\equiv\prod_{h=0}^{n}\Xi_h$ denote the set of possible shock profiles. The shock profile $\theta\in\Xi$ is randomly drawn according to the conditional probability function $f(\cdot\mid a):\Xi\to[0,1]$ where $\sum_{\theta\in\Xi}f(\cdot\mid a)=1$ and in every period the realization of the shock profile $\theta(t)\in\Xi$ is not observable to the firms. The expected instantaneous profit for each firm $i\in\{1,2\}$ equals

$$u_i(a)=\sum_{\theta\in\Xi}\{a_i\omega_i(a,\theta)-z_i(\omega_i(a,\theta))\}f(\theta\mid a).$$

Let $\Xi_{-h}\equiv\prod_{h'\neq h}\Xi_{h'}$, $\theta_{-h}\equiv(\theta_{h'})_{h'\neq h}\in\Xi_{-h}$, $\Xi_{-0-h}\equiv\prod_{h'\notin\{0,h\}}\Xi_{h'}$,

$$f_0(\theta_0\mid a)\equiv\sum_{\theta_{-0}\in\Xi_{-0}}f(\theta\mid a),\ f_{-0}(\theta_{-0}\mid a,\theta_0)\equiv\frac{f(\theta\mid a)}{f_0(\theta_0\mid a)},$$

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[18] Note that the private signal $\theta_h$ may or may not influence the amount that consumer $h$ will buy from the firm $\iota(h,a,\theta_0)$.

[19] Consider the following example. When consumer $h$ buys $l_1$ units of firm 1's commodity and $l_2$ units of firm 2's commodity, her utility equals

$$l_1\{v_i(h,\theta_0)-a_1\}+l_2\{v_2(h,\theta_0)-a_2\}+w(h,l_1,l_2,\theta_0,\theta_h),$$

where $w(h,l_1,l_2,\theta_0,\theta_h)$ satisfies the concavity in that for every $(l_1,l_2)$,

$$w(h,l_1,l_2,\theta_0,\theta_h)\leq w(h,l_1+l_2,0,\theta_0,\theta_h)=w(h,0,l_1+l_2,\theta_0,\theta_h).$$

Consumer $h$ maximizes her utility on the assumption that she expects to buy as many as she wants. Consumer $h's$ demand vector $(l_1,l_2)$ must be the solution to the maximization problem given by

$$\max_{l_1,l_2}[l_1\{v_1(h,\theta_0)-a_1\}+l_2\{v_2(h,\theta_0)-a_2\}+w(h,l_1,l_2,\theta_0,\theta_h)].$$

We specify $\iota(h,a,\theta_0)$ by

$$\iota(h,a,\theta_0)=1 \text{ if and only if } v_1(h,\theta_0)-a_1\geq v_2(h,\theta_0)-a_2,$$

and $(l_1(h,a,\theta_0,\theta_h),l_2(h,a,\theta_0,\theta_h))$ by

$$l_i(h,a,\theta_0,\theta_h)\in\arg\max_{l_i}[l_i\{v_i(h,\theta_0)-a_i\}+w(h,0,l_i,\theta_0,\theta_h)] \text{ if } i=\iota(h,a,\theta_0),$$

and

$$l_i(h,a,\theta_0,\theta_h)=0 \text{ if } i\neq\iota(h,a,\theta_0).$$

The demand vector $(l_1(h,a,\theta_0,\theta_h),l_2(h,a,\theta_0,\theta_h))$ is the solution to the maximization problem.

and

$$f_h(\theta_h \mid a,\theta_0) \equiv \sum_{\theta_{-0-h}\in\Xi_{-0-h}} f(\theta_{-0} \mid a,\theta_0) \text{ for all } h \in \{1,...,n\}.$$

We assume that the consumers' demands are correlated only through the macro shock $\theta_0$, so that for every $(a,\theta_0) \in A \times \Xi_0$, $f_{-0}(\cdot \mid a,\theta_0)$ is conditionally independent in that

$$f_{-0}(\theta_{-0} \mid a,\theta_0) = \prod_{h\in\{1,...,n\}} f_h(\theta_h \mid a,\theta_0) \text{ for all } \theta_{-0} \in \Xi_{-0}.$$

We can write the signal structure $p$, or $(f_0, q)$, as

$$p(\omega \mid a) = \sum_{\theta\in\Xi:\omega(a,\theta)=\omega} f(\theta \mid a) \text{ for all } \omega \in \Omega \text{ and all } \theta \in \Xi,$$

or

$$q(\omega \mid a,\theta_0) = \sum_{\theta_{-0}\in\Xi_{-0}:\omega(a,\theta)=\omega} f_{-0}(\theta_{-0} \mid a,\theta_0) \text{ for all } \omega \in \Omega, \text{ all } a \in A \text{ and all } \theta_0 \in \Xi_0.$$

The following proposition states that firms' sales levels are correlated only through the macro shock $\theta_0$.

**Proposition 6:** *In the price-setting dynamic duopoly above, $q(\cdot \mid a,\theta_0)$ is conditionally independent for every $(a,\theta_0) \in A \times \Xi_0$.*

**Proof:** For each $i \in \{1,2\}$, we denote by $N(i,a,\theta_0)$ the set of all consumers $h$ satisfying that $\iota(h,a,\theta_0) = i$. Since $\omega_i(a,\theta)$ does not depend on $\theta_h$ for every $h \notin N(i,a,\theta_0)$, we can write $\omega_i(a,\theta_0,(\theta_h)_{h\in N(i,a,\theta_0)})$ instead of $\omega_i(a,\theta)$. Note

$$\begin{aligned}
q(\omega \mid a,\theta_0) &= \sum_{\theta_{-0}\in\Xi_{-0}:\omega(a,\theta)=\omega} \prod_{h=1}^{n} f_h(\theta_h \mid a,\theta_0) \\
&= \sum_{\theta_{-0}\in\Xi_{-0}:\omega(a,\theta)=\omega} \{ \prod_{h\in N(1,a,\theta_0)} f_h(\theta_h \mid a,\theta_0) \prod_{h\in N(2,a,\theta_0)} f_h(\theta_h \mid a,\theta_0)\} \\
&= [\sum_{\substack{(\theta_h)_{h\in N(1,a,\theta_0)} \in \prod_{h\in N(1,a,\theta_0)} \Xi_h : \\ \omega_1(a,\theta_0,(\theta_h)_{h\in N(1,a,\theta_0)})=\omega_1}} \{ \prod_{h\in N(1,a,\theta_0)} f_h(\theta_h \mid a,\theta_0)\}] [\sum_{\substack{(\theta_h)_{h\in N(2,a,\theta_0)} \in \prod_{h\in N(2,a,\theta_0)} \Xi_h : \\ \omega_2(a,\theta_0,(\theta_h)_{h\in N(2,a,\theta_0)})=\omega_2}} \{ \prod_{h\in N(2,a,\theta_0)} f_h(\theta_h \mid a,\theta_0)\}] \\
&= q_1(\omega_1 \mid a,\theta_0) q_2(\omega_2 \mid a,\theta_0),
\end{aligned}$$

which implies that $q(\cdot \mid a,\theta_0)$ is conditionally independent. **Q.E.D.**

Fix the cartel price vector $c \in A$ and the defective price vector $d \in A$ arbitrarily, which satisfy Condition 1. From Propositions 1 and 6, it follows that the cartel profit vector $u(c)$ is sustainable if Condition 2 holds. Note that the inequalities (5) equals

$$\bar{\omega}_i + 1 \geq (\bar{a}_j + 1)|\Xi_0| \text{ for all } i \in \{1,2\}. \qquad (9)$$

The inequalities (9) imply that each firm *i's* production capacity $\bar{\omega}_i$ is sufficiently large in comparison with the size of the set of possible macro shocks as well as the size of the set of possible price levels for the rival firm's product.[20] From Theorem 4, it follows that the cartel profit vector $u(c)$ is sustainable almost everywhere when each firm $i's$ production capacity is large so as to satisfy the inequalities (9). Hence, we can conclude that the full cartel collusion can be self-enforcing in a wide class of price-setting duopoly even if the firms cannot communicate and have the option of making the secret price cuts.

Note that all of the probabilities of firms' choosing the price vectors $c$, $d$, $(c_1, d_2)$, and $(d_1, c_2)$ in every period $t$ are independent of the history $(\theta_0(1),...,\theta_0(t-1)) \in \Xi_0^{t-1}$ of the macro shocks up to period $t-1$. This implies that the price fluctuation, or the occurrence of price wars, does not depend on exogenous factors on business conditions between boom and recession. This is in contrast to the studies of public monitoring such as Green and Porter (1984) Abreu, Pearce and Stacchetti (1986), and Rotemberg and Saloner (1986), because these papers showed that the occurrence of price wars crucially depends on such exogenous factors.

Note that we may not be able to make the defective price vector $d$ very low because of Condition 1. Suppose that we have $d$ equaled the one-shot Nash equilibrium action profile, and choose $c$ so as to satisfy $u(c) > u(d)$. Such a pair of $c$ and $d$ does not necessarily satisfy the first inequality of (1), which is necessary for weakened interchangeability. Hence, we have to make $d$ higher than the one-shot Nash equilibrium action profile. This implies that although a price war surely occurs on the equilibrium path, the fall in prices during the price war is not so drastic and this war goes on for a long time. This is in contrast to the previous works such as Abreu, Pearce and Stacchetti (1986), which showed that in repeated quantity-setting oligopoly with imperfect public monitoring, a price war on the optimal symmetric equilibrium path goes on only one period, and induces firms to make very severe competitive prices.

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[20] Note that the inequalities (9) hold when the product capacity for each firm is restrictive but its product can be divided into small sales units. Moreover, we assume that each firm can change its price level only in the discrete way.

## 7. Concluding Remarks

This paper investigated general two player infinitely repeated games where the discount factor is less than but close to unity. Monitoring was assumed to be imperfect and private. We allowed players' private signals to be correlated through the unobservable macro shock. We showed that efficiency is sustainable in a wide class of environments when the size of the set of private signals for each player is sufficiently large in comparison with the size of the set of possible macro shocks as well as the size of the set of the opponent's actions. We applied our efficiency result to a price-setting duopoly, and showed that the full cartel collusion can be self-enforcing even if firms cannot communicate and have the option of making the secret price cuts.

In order to derive our results, we constructed review strategy equilibria. A modification of the law of large numbers implied that players could almost perfectly monitor whether the opponent has continued playing collusive or playing defective, and have no incentive to deviate from the repeated collusive choices. Moreover, the linear independency of the conditional probabilities implied that we could find a bad event for each player such that the probability of its occurrence is the lowest when the opponent playing collusive, and does not depend on the opponent's private signal. By combining the idea of review strategy above with weakened interchangeability a la Ely and Valimaki (2002), we could show that efficiency is sustainable.

It would be important to extend our results to the three or more player case. We may expect to be able to prove the efficiency result according to the following two steps.

(1) When private monitoring is far from perfect, can we apply the idea of review strategy above to the three or more player case?

(2) When private monitoring is almost perfect, can we derive the efficiency result in the three or more player case by using the device of interchangeability a la Ely and Valimaki or its variants?

Step (1) is closely related to the main concern of the present paper. My answer to Step (1) is optimistic. Similarly to the two player case, it would follow that whenever the size of private signals for a player is sufficiently large in comparison with the size of the set of possible macro shocks, then we can find a bad event for each opponent such that the probability of its occurrence is the lowest when this opponent playing collusive, and it does not depend on the opponent's private signal as well as the other opponents' action choices. Because of this, we may expect each player to have incentive to play a review strategy during each review phase.

Step (2) is much more problematic than Step (1). Although Ely and Valimaki (2002) and Bhaskar and Obara (2002) have provided a partial answer to this step, we still have many unclear points such as how wide is the class of three or more player games to which we can apply the interchangeability or its variant. However, these points are beyond the purpose of the paper because Step (2) is not much related to our main concern. As a result, the three or more player cases should be intensively studied in future researches.[21]

[21] This paper did not provide the full folk theorem, whereas the previous papers such as Ely and Valimaki (2002) and Matsushima (2002) did. I do not have checked carefully, but it would not be difficult to extend our result to the sustainability of payoff vectors that are not necessarily efficient. We may need to use the idea of Steps 2 and 3 in the proof of the folk theorem in Matsushima (2002).

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## Appendix: The Proof of Proposition 1

We denote $a_i^t \equiv (a_i(1),...,a_i(t))$, $\omega_i^t \equiv (\omega_i(1),...,\omega_i(t))$, and

$$\omega_i^T(h_i^t) \equiv (\omega_i(t-T+1),...,\omega_i(t)).$$

We recursively define the probability $f_i^*(r,t,a_j^t)$ that $\psi_i^*$ occurs $r$ times during the first $t$ periods when she chooses $c_i$ in all the periods and the opponent $j$ chooses $a_j^t$ as follows. For every $a_j^1$, let

$$f_i^*(1,1,a_j^1) = 1 - f_i^*(0,1,a_j^1) = p_i(\psi_i^* \mid c_i, a_j(1)).$$

For every $t \geq 1$, every $r \notin \{0,...,t\}$, and every $a_j^t$, let

$$f_i^*(r,t,a_j^t) = 0.$$

For every $t \geq 2$, every $r \in \{0,...,t\}$, and every $a_j^t$, let

$$f_i^*(r,t,a_j^t) = f_i^*(r-1,t-1,a_j^{t-1})p_i(\psi_i^* \mid c_i, a_j(t))$$
$$+ f_i^*(r,t-1,a_j^{t-1})(1 - p_i(\psi_i^* \mid c_i, a_j(t))).$$

Similarly, we define the probability $f_i^+(r,t,a_j^t)$ that $\psi_i^+$ occurs $r$ times during the first $t$ periods when she chooses $c_i$ in all the periods and the opponent $j$ chooses $a_j^t$. Similarly, we define also the probability $f_i^+(r,t,a_j^t)$ ($f_i^{++}(r,t,a_j^t)$) that $\psi_i^{**}$ ($\psi_i^{++}$, respectively) occurs $r$ times during the first $t$ periods when she chooses $d_i$ in all the periods and the opponent $j$ chooses $a_j^t$.

Fix an infinite sequence of positive integers $(T^m)_{m=1}^\infty$ arbitrarily, where

$$\lim_{m\to\infty} T^m = +\infty.$$

According to Lemma A1 in Matsushima (2002), we can choose an infinite sequence of positive integers $(M_i^{*m})_{m=1}^\infty$ satisfying that $M_i^{*m} \in \{0,...,T^m\}$ for all $m \geq 1$,

$$\lim_{m\to\infty} F_i^*(M_i^{*m}, T^m, c_j^{T^m}) = 1,$$

$$\lim_{m\to\infty} \frac{M_i^{*m}}{T^m} = p_i(\psi_i^* \mid c), \text{ and}$$

$$\lim_{m\to\infty} T^m f_i^*(M_i^{*m}, T^m, c_j^{T^m}) = +\infty,$$

where $F_i^*(r,t,a_j^t) \equiv \sum_{r'=0}^{r} f_i^*(r',t,a_j^t)$. Similarly, we can choose $(M_i^{+m})_{m=1}^\infty$ and $(M_i^{++m})_{m=1}^\infty$ satisfying that $M_i^{+m} \in \{0,...,T^m\}$ and $M_i^{++m} \in \{0,...,T^m\}$ for all $m \geq 1$,

$$\lim_{m\to\infty} F_i^+(M_i^{+m}, T^m, c_j^{T^m}) = 1, \quad \text{(A1)}$$

$$\lim_{m\to\infty}\frac{M_i^{+m}}{T^m}=p_i(\psi_i^+\mid c), \tag{A2}$$

$$\lim_{m\to\infty}T^m f_i^+(M_i^{+m},T^m,c_j^{T^m})=+\infty, \tag{A3}$$

$$\lim_{m\to\infty}F_i^{++}(M_i^{++m},T^m,c_j^{T^m})=1, \tag{A4}$$

$$\lim_{m\to\infty}\frac{M_i^{++m}}{T^m}=p_i(\psi_i^{++}\mid d)\text{, and} \tag{A5}$$

$$\lim_{m\to\infty}T^m f_i^{++}(M_i^{++m},T^m,c_j^{T^m})=+\infty, \tag{A6}$$

where $F_i^+(r,t,a_j^t)\equiv\sum_{r'=0}^{r}f_i^+(r',t,a_j^t)$ and $F_i^{++}(r,t,a_j^t)\equiv\sum_{r'=0}^{r}f_i^{++}(r',t,a_j^t)$.

From Condition 1, we can choose a payoff vector $\underline{v}\in R^2$ satisfying that

$$\max_{a_i\in A_i}u_i(a_i,d_i)<\underline{v}_i<u_i(c) \text{ for all } i\in\{1,2\}.$$

From the specifications of $((M_i^{*m},M_i^{+m},M_i^{++m})_{i\in\{1,2\}})_{m=1}^{\infty}$ and $\underline{v}$ above, we can choose an infinite sequence $(\delta^m,\bar{v}^m,\underline{v}^m,(\bar{\xi}_i^m,\underline{\xi}_i^m,\bar{\varsigma}_i^m,\underline{\varsigma}_i^m)_{i\in\{1,2\}})_{m=1}^{\infty}$ as follows. Let

$$\gamma^m\equiv(\delta^m)^{T^m}.$$

For every $i\in\{1,2\}$,

$$0<\bar{\xi}_i^m<\frac{1}{2},\ 0<\underline{\xi}_i^m<\frac{1}{2},\ 0<\bar{\varsigma}_i^m<\frac{1}{2} \text{ and } 0<\underline{\varsigma}_i^m<\frac{1}{2} \text{ for all } m\geq 1,$$

$$\lim_{m\to\infty}\gamma^m=1,$$

$$\lim_{m\to\infty}\bar{v}_i^m=u_i(c),$$

$$\lim_{m\to\infty}\underline{v}_i^m=\underline{v}_i,$$

$$\lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}\bar{\varsigma}_i^m=+\infty, \tag{A7}$$

$$\lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}\underline{\varsigma}_i^m>0, \tag{A8}$$

$$\lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}\underline{\xi}_i^m f_i^{**}(0,T^m,d_j^{T^m})=0, \tag{A9}$$

and for every large enough $m$,

$$\bar{v}_j^m=u_j(c)-\frac{\gamma^m}{1-\gamma^m}\{\bar{\xi}_i^m(1-F_i^*(M_i^{*m},T^m,c_j^{T^m}))$$

$$+\bar{\zeta}_i^m(1-F_i^+(M_i^{+m},T^m,c_j^{T^m}))\}(\bar{v}_j^m-\underline{v}_j^m)$$

$$=u_j(c_i,d_j)-\frac{\gamma^m}{1-\gamma^m}\{\bar{\xi}_i^m(1-F_i^*(M_i^{*m},T^m,d_j^{T^m}))$$

$+\bar{\zeta}_i^m(1-F_i^+(M_i^{+m},T^m,c_j^{T^m}))\}(\bar{v}_j^m-\underline{v}_j^m)$, and

$$\underline{v}_j^m = u_j(d)+\frac{\gamma^m}{1-\gamma^m}\{\underline{\xi}_i^m f_i^{**}(0,T^m,d_j^{T^m}) + \underline{\zeta}_i^m F_i^{++}(M_i^{++m},T^m,c_j^{T^m})\}(\bar{v}_j^m-\underline{v}_j^m)$$

$$= u_j(d_i,c_j)+\frac{\gamma^m}{1-\gamma^m}\{\underline{\xi}_i^m f_i^{**}(0,T^m,c_j^{T^m})$$

$$+\underline{\zeta}_i^m F_i^{++}(M_i^{++m},T^m,c_j^{T^m})\}(\bar{v}_j^m-\underline{v}_j^m).$$

Note from the equality (A9) that

$$\underline{v}_j = u_j(d)+\lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}\,\underline{\zeta}_i^m F_i^{++}(M_i^{++m},T^m,c_j^{T^m})(\bar{v}_j^m-\underline{v}_j^m). \tag{A10}$$

The equality (A10) implies that $\lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}\,\underline{\zeta}_i^m F_i^{++}(M_i^{++m},T^m,c_j^{T^m}) = \underline{v}_j - u_j(d) < +\infty$. This, together with Condition 2 and the equality (A4), implies that for every $a_j^\infty=(a_j(1),a_j(2),...)$, whenever $\lim_{m\to\infty}F_i^{++}(M_i^{++m},T^m,a_j^{T^m})=0$ then

$$\lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}\underline{\varsigma}_i^m F_i^{++}(M_i^{++m},T^m,a_j^{T^m})=0. \tag{A11}$$

We recursively define the probability $w_i^*(r,t,\omega_i^t)$ that $\psi_i^*$ occurs $r$ times during the first $t$ periods when she observes $\omega_j^t$ at the end of period $t$ as follows. For every $\omega_j^1$, let

$$w_i^*(1,1,\omega_i^1)=1-w_i^*(0,1,\omega_i^1)=\psi_i^*(\omega_i(1)).$$

For every $t\geq 1$, every $r\notin\{0,...,t\}$, and every $\omega_i^t$, let

$$w_i^*(r,t,\omega_i^t)=0.$$

For every $t\geq 2$, every $r\in\{0,...,t\}$, and every $\omega_i^t$, let

$$w_i^*(r,t,\omega_i^t)=w_i^*(r-1,t-1,\omega_i^{t-1})\psi_i^*(\omega_i(t)) + w_i^*(r,t-1,\omega_i^{t-1})(1-\psi_i^*(\omega_i)).$$

Similarly, we define also the probability $w_i^{**}(r,t,\omega_i^t)$ ($w_i^+(r,t,\omega_i^t)$, and $w_i^{++}(r,t,\omega_i^t)$) that $\psi_i^{**}$ ($\psi_i^+$, and $\psi_i^{++}$, respectively) occurs exactly $r$ times during the first $t$ periods when she observes $\omega_j^t$ at the end of period $t$. We define

$$\bar{\rho}_i^m(\omega_i^{T^m})\equiv 1-\bar{\xi}_i^m(1-W_i^*(M_i^{*m},T^m,\omega_i^{T^m}))-\bar{\varsigma}_i^m(1-W_i^+(M_i^{+m},T^m,\omega_i^{T^m})),$$

and

$$\underline{\rho}_i^m(\omega_i^{T^m})\equiv\underline{\xi}_i^m w_i^{**}(0,T^m,\omega_i^{T^m})+\underline{\varsigma}_i^m W_i^{++}(M_i^{++m},T^m,\omega_i^{T^m}),$$

where

$$W_i^*(r,t,\omega_i^t)\equiv\sum_{r'=0}^{r}w_i^*(r',t,\omega_i^t),\quad W_i^+(r,t,\omega_i^t)\equiv\sum_{r'=0}^{r}w_i^+(r',t,\omega_i^t), \text{ and}$$

$$W_i^{++}(r,t,\omega_i^t) \equiv \sum_{r'=0}^{r} w_i^{++}(r',t,\omega_i^t) .$$

Based on the specifications above, we specify an infinite sequence of two strategy profiles $(\bar{s}^m, \underline{s}^m)_{m=1}^{\infty}$ as follows. For every $t \geq 1$, and every $h_i^{t-1} \in H_i$,

$$\bar{s}_i^m(h_i^{t-1})(a_i) = 0 \text{ and } \underline{s}_i^m(h_i^{t-1})(a_i) = 0 \text{ for all } a_i \in A_i / \{c_i, d_i\} .$$

For every $t \in \{1,...,T^m\}$, and every $h_i^{t-1} \in H_i$,

$$\bar{s}_i^m(h_i^{t-1})(c_i) = 1 \text{ and } \underline{s}_i^m(h_i^{t-1})(d_i) = 1 .$$

For every $k \in \{1,2,...\}$, every $t \in \{kT^m + 2,...,(k+1)T^m\}$, and every $h_i^{t-1} \in H_i$,

$$\bar{s}_i^m(h_i^{t-1})(c_i) = 1 \text{ if } a_i(t-1) = c_i ,$$

$$\bar{s}_i^m(h_i^{t-1})(d_i) = 1 \text{ if } a_i(t-1) \neq c_i ,$$

$$\underline{s}_i^m(h_i^{t-1})(c_i) = 1 \text{ if } a_i(t-1) = c_i ,$$

and

$$\underline{s}_i^m(h_i^{t-1})(d_i) = 1 \text{ if } a_i(t-1) \neq c_i .$$

For every $k \in \{1,2,...\}$, and every $h_i^{kT^m} \in H_i$,

$$\bar{s}_i^m(h_i^{kT^m})(c_i) = \bar{\rho}_i^m(\omega_i^{T^m}(h_i^{kT^m})) \text{ if } a_i(kT^m) = c_i ,$$

$$\bar{s}_i^m(h_i^{kT^m})(c_i) = \underline{\rho}_i^m(\omega_i^{T^m}(h_i^{kT^m})) \text{ if } a_i(kT^m) = d_i ,$$

$$\underline{s}_i^m(h_i^{kT^m+1})(c_i) = \bar{\rho}_i^m(\omega_i^{T^m}(h_i^{kT^m})) \text{ if } a_i(kT^m) = c_i ,$$

and

$$\underline{s}_i^m(h_i^{kT^m})(c_i) = \underline{\rho}_i^m(\omega_i^{T^m}(h_i^{kT^m})) \text{ if } a_i(kT^m) = d_i .$$

According to $\bar{s}_i^m$, player $i$ continues choosing $c_i$ from period 1 to period $T^m$. When the number of periods in which $\psi_i^*$ occurs is more than the threshold $M_i^{*m}$, she will play $\underline{s}_i^m$ from period $T^m + 1$ with probability $\bar{\xi}_i^m$. When the number of periods in which $\psi_i^+$ occurs is more than the threshold $M_i^{+m}$, she will play $\underline{s}_i^m$ with probability $\bar{\varsigma}_i^m$. When both numbers are more than their respective thresholds, she will play $\underline{s}_i^m$ with probability $\bar{\xi}_i^m + \bar{\varsigma}_i^m$. Otherwise she will play $\bar{s}_i^m$. Hence, $\bar{\rho}_i^m(\omega_i^{T^m})$ is regarded as the probability that she will play $\bar{s}_i^m$ from period $T^m + 1$ when she observes $\omega_i^{T^m}$.

According to $\underline{s}_i^m$, player $i$ continues choosing $d_i$ from period 1 to period $T^m$. When the number of periods in which $\psi_i^{**}$ occurs equals the threshold zero, she will play $\bar{s}_i^m$ from period $T^m + 1$ with probability $\bar{\xi}_i^m$. When the number of periods in which $\psi_i^{++}$ occurs is less than or equals the threshold $M_i^{++m}$, she will play $\bar{s}_i^m$ with probability $\underline{\varsigma}_i^m$. When both numbers are less than or equal their respective thresholds, she will play $\bar{s}_i^m$

with probability $\bar{\xi}_i^m + \underline{\varsigma}_i^m$. Otherwise she will play $\underline{s}_i^m$. Hence, $\underline{\rho}_i^m(\omega_i^{T^m})$ is regarded as the probability that she will play $\bar{s}_i^m$ from period $T^m + 1$ when she observes $\omega_i^{T^m}$.

Since

$$f_i^*(r,t,a_j^t) = E[w_i^*(r,t,\omega_i^{t-1}) \mid c_i^t, a_j^t],$$
$$f_i^{**}(r,t,a_j^t) = E[w_i^{**}(r,t,\omega_i^{t-1}) \mid d_i^t, a_j^t],$$
$$f_i^+(r,t,a_j^t) = E[w_i^+(r,t,\omega_i^{t-1}) \mid c_i^t, a_j^t], \text{ and}$$
$$f_i^{++}(r,t,a_j^t) = E[w_i^{++}(r,t,\omega_i^{t-1}) \mid d_i^t, a_j^t],$$

it follows that for every large enough $m$,

$$v_j(\delta^m, \bar{s}^m) = v_j(\delta^m, \bar{s}^m / \underline{s}_j^m) = \bar{v}_j^m, \text{ and}$$
$$v_j(\delta^m, \underline{s}^m) = v_j(\delta^m, \underline{s}^m / \bar{s}_j^m) = \underline{v}_j^m.$$

The device of interchangeability above was originated in Ely and Valimaki (2002). Note that

$$\lim_{m\to\infty} v_j(\delta^m, \bar{s}^m) = \lim_{m\to\infty} v_j(\delta^m, \bar{s}^m / \underline{s}_j^m) = u_j(c), \text{ and}$$
$$\lim_{m\to\infty} v_j(\delta^m, \underline{s}^m) = \lim_{m\to\infty} v_j(\delta^m, \underline{s}^m / \bar{s}_j^m) = \underline{v}_j^m.$$

By using the same logic as Step 1 in the proof of the folk theorem in Matsushima (2002), we can show that $\bar{s}^m$ and $\underline{s}^m$ are Nash equilibria for every large enough $m$, in the following way. Because of the interchangeability above, we can show as its byproduct that $\bar{s}^m / \underline{s}_j^m$ and $\underline{s}^m / \bar{s}_j^m$ are Nash equilibria for every large enough $m$.

Matsushima (2002) assumed conditional independence, but its proof depended only on the property that the probabilities of the events occurring are independent of the opponent's private signal. Condition 2 of the present paper implies this property, i.e., implies that the probabilities of $\psi_i^*$, $\psi_i^{**}$, $\psi_i^+$ and $\psi_i^{++}$ occurring are independent of the opponent's private signal $\omega_j$. From this property, all we have to do is to prove that each player has no incentive to deviate during the first $T^m$ periods by making history-independent action choices, i.e., to prove that for every $s_i \in S_i$, whenever there exists $\hat{a}_i^{T^m} = (\hat{a}_i(1),...,\hat{a}_i(T^m))$ such that $s_i |_{h_i^{T^m}} = \bar{s}_i^m |_{h_i^{T^m}}$ for all $h_i^{T^m} \in H_i$, and

$$s_i(h_i^{t-1})(\hat{a}_i(t)) = 1 \text{ for all } t \in \{1,...,T^m\},$$

then

$$\bar{v}_i^m \geq v_i(\delta^m, \bar{s}^m / s_i) \text{ and } \underline{v}_i^m \geq v_j(\delta^m, \underline{s}^m / s_i).$$

Suppose that $\hat{a}_i^{T^m}$ satisfies that

$$\hat{a}_i(t) \in \{c_i, d_i\} \text{ for all } t \in \{1,...,T^m\}.$$

From Condition 2, it follows that

$$F_j^{+}(M_i^{+m},T^m,\hat{a}_j^{T^m}) = F_j^{+}(M_i^{+m},T^m,c_j^{T^m}),$$
$$F_j^{++}(M_i^{++m},T^m,\hat{a}_j^{T^m}) = F_j^{++}(M_i^{++m},T^m,c_j^{T^m}),$$

and therefore, for every large enough $m$,

$$\bar{v}_i^m - v_i(\delta^m,\bar{s}^m/s_i) = (1-\delta^m)\sum_{t=0}^{T^m}(\delta^m)^{t-1}\{u_i(c)-u_i(\hat{a}_i(t),c_j)\}$$
$$+\gamma^m\bar{\xi}_j^m\{F_j^*(M_j^{*m},T^m,c_i^{T^m}) - F_j^*(M_j^{*m},T^m,\hat{a}_i^{T^m})\}(\bar{v}_i^m - \underline{v}_i^m) \text{, and}$$
$$\underline{v}_i^m - v_j(\delta^m,\underline{s}^m/s_i) = (1-\delta^m)\sum_{t=0}^{T^m}(\delta^m)^{t-1}\{u_i(d)-u_i(\hat{a}_i(t),d_j)\}$$
$$+\gamma^m\underline{\xi}_i^m\{f_j^{**}(0,T^m,d_i^{T^m}) - f_j^{**}(0,T^m,\hat{a}_i^{T^m})\}(\bar{v}_i^m - \underline{v}_i^m).$$

Since the payoff differences above do not depend on $\psi_j^{+}$ and $\psi_j^{++}$, $(M_i^{*m})_{m=1}^{\infty}$ is specified according to Lemma A1 in Matsushima (2002), and the threshold for $\psi_j^{**}$ is set equal zero, we can directly apply Matsushima (2002) to our problem, and therefore, it follows that $\bar{v}_i^m \geq v_i(\delta^m,\bar{s}^m/s_i)$ and $\underline{v}_i^m \geq v_j(\delta^m,\underline{s}^m/s_i)$.

The following arguments are basically the same as above.

Fix $\hat{a}_i^{\infty} = (\hat{a}_i(1),\hat{a}_i(2),...,)$ arbitrarily, where

$$\hat{a}_i(t) \notin \{c_i,d_i\} \text{ for some } t \geq 1.$$

For every positive integer $T \geq 1$, we denote by $r(\hat{a}_i^T)$ the number of periods during the first $T$ periods in which $\hat{a}_i(t) \notin \{c_i,d_i\}$. We denote by $\hat{s}_i^m \in S_i$ the strategy for player $i$ satisfying that $s_i|_{h_i^{T^m}} = \bar{s}_i^m|_{h_i^{T^m}}$ for all $h_i^{T^m} \in H_i$, and

$$s_i(h_i^{t-1})(\tilde{a}_i(t)) = 1 \text{ for all } t \in \{1,...,T^m\}.$$

Choose another strategy $\tilde{s}_i^m \in S_i$ satisfying that there exists $\tilde{t} \in \{1,...,T^m\}$ such that $\tilde{s}_i^m(h_i^{t-1}) = \hat{s}_i^m(h_i^{t-1})$ for all $t \neq t'$ and all $h_i^{t-1} \in H_i^{t-1}$,

$$\tilde{a}_i(\tilde{t}) \notin \{c_i,d_i\} \text{, and}$$
$$s_i(h_i^{\tilde{t}-1})(d_i) = 1 \text{ for all } h_i^{\tilde{t}-1} \in H_i^{\tilde{t}-1}.$$

First, suppose that

$$\lim_{m\to\infty} T^m f_j^{+}(M_j^{+m},T^m,\hat{a}_i^{T^m}) = +\infty. \tag{A12}$$

From Condition 2 and the equality (A3),

$$\lim_{m\to+\infty}\frac{r(\hat{a}_i^{T^m})}{T^m} = 0.$$

Hence, the payoff difference $\dfrac{v_i(\delta^m,\bar{s}^m/\tilde{s}_i^m) - v_i(\delta^m,\bar{s}^m/\hat{s}_i^m)}{1-\delta^m}$ is approximated by, or even larger than, the value given by

$$u_i(d_i, c_j) - \max_{a_i \in A_i} u_i(a_i, c_j) + \frac{\gamma^m}{1-\gamma^m} \overline{\varsigma}_j^m T^m f_j^+(M_j^{+m}, T^m, \hat{a}_i^{T^m})\{p_j(\psi_j^+ \mid d_i, c_j)$$
$$- p_j(\psi_j^+ \mid \hat{a}_i(\widetilde{t}), c_j)\}(\overline{v}_i^m - \underline{v}_i^m).$$

Here $f_j^+(M_j^{+m}, T^m, \hat{a}_i^{T^m})\{p_j(\psi_j^+ \mid d_i, c_j) - p_j(\psi_i^+ \mid \hat{a}_j(\widetilde{t}), c_j)\}$ approximates the difference in the probability that the number of periods in which $\psi_j^+$ occurs exceeds $M_j^{+m}$. The equality (A12) implies that this difference is very large. Hence, it follows from the equality (A7) that the loss of future payoff, which is represented by

$$\frac{\gamma^m}{1-\gamma^m} \overline{\varsigma}_j^m T^m f_j^+(M_j^{+m}, T^m, \hat{a}_i^{T^m})\{p_j(\psi_j^+ \mid d_i, c_j)$$
$$- p_j(\psi_j^+ \mid \hat{a}_i(\widetilde{t}), c_j)\}(\overline{v}_i^m - \underline{v}_i^m),$$

diverges to infinity as $m$ increases, and therefore, exceeds the maximal short-run gain, which is represented by

$$u_i(d_i, c_j) - \max_{a_i \in A_i} u_i(a_i, c_j).$$

Hence, it follows that $v_i(\delta^m, \overline{s}^m / \widetilde{s}_i^m) > v_i(\delta^m, \overline{s}^m / \hat{s}_i^m)$.

Next, suppose that

$$\lim_{m \to \infty} T^m f_j^{++}(M_j^{++m}, T^m, \hat{a}_i^{T^m}) = +\infty. \tag{A13}$$

From Condition 2 and the equality (A6),

$$\lim_{m \to +\infty} \frac{r(\hat{a}_i^{T^m})}{T^m} = 0.$$

Hence, the payoff difference $\dfrac{v_i(\delta^m, \underline{s}^m / \widetilde{s}_i^m) - v_i(\delta^m, \underline{s}^m / \hat{s}_i^m)}{1-\delta^m}$ is approximated by, or even larger than, the value given by

$$u_i(d) - \max_{a_i \in A_i} u_i(a_i, d_j) + \frac{\gamma^m}{1-\gamma^m} \underline{\varsigma}_j^m T^m f_j^{++}(M_j^{++m}, T^m, \hat{a}_i^{T^m})\{p_j(\psi_j^{++} \mid d)$$
$$- p_j(\psi_j^{++} \mid \hat{a}_i(\widetilde{t}), d_j)\}(\overline{v}_i^m - \underline{v}_i^m).$$

Here $f_j^{++}(M_j^{++m}, T^m, \hat{a}_i^{T^m})\{p_j(\psi_j^{++} \mid d) - p_j(\psi_j^+ \mid \hat{a}_i(\widetilde{t}), d_j)\}$ approximates the difference in the probability that the number of periods in which $\psi_j^{++}$ occurs exceeds $M_j^{++m}$. The equality (A13) implies that this difference is very large. Hence, it follows from the inequality (A8) that the loss of future payoff, which is represented by

$$\frac{\gamma^m}{1-\gamma^m} \underline{\varsigma}_j^m T^m f_j^{++}(M_j^{++m}, T^m, \hat{a}_i^{T^m})\{p_j(\psi_i^{++} \mid d)$$
$$- p_j(\psi_i^{++} \mid \hat{a}_i(\widetilde{t}), d_j)\}(\overline{v}_i^m - \underline{v}_i^m),$$

diverges to infinity as $m$ increases, and therefore, exceeds the maximal short-run gain, which is represented by

$$u_i(d) - \max_{a_i \in A_i} u_i(a_i, c_j).$$

Hence, it follows that $v_i(\delta^m, \underline{s}^m / \tilde{s}_i^m) > v_i(\delta^m, \underline{s}^m / \hat{s}_i^m)$.

Suppose that

$$\lim_{m \to \infty} T^m f_i^+(M_i^{+m}, T^m, \hat{a}_j^{T^m}) < +\infty.$$

From Condition 2 and the equality (A3),

$$\lim_{m \to +\infty} \frac{r(\hat{a}_i^{T^m})}{T^m} > 0,$$

and therefore, from Condition 2 and the equality (A2), it must hold that

$$\lim_{m \to \infty} F_i^+(M_i^{+m}, T^m, \hat{a}_j^{T^m}) = 0. \tag{A14}$$

The payoff difference $\dfrac{v_i(\delta^m, \bar{s}^m / \underline{s}_i^m) - v_i(\delta^m, \bar{s}^m / \hat{s}_i^m)}{r(\hat{a}_i^{T^m})(1-\delta^m)}$ is approximated by, or even larger than, the value given by

$$u_i(d_i, c_j) - \max_{a_i \in A_i} u_i(a_i, c_j)$$
$$+ \frac{T^m}{r(\hat{a}_j^{T^m})} \frac{\gamma^m}{1-\gamma^m} \bar{\varsigma}_j^m \{F_j^+(M_j^{+m}, T^m, d_i^{T^m}) - F_j^+(M_j^{+m}, T^m, \hat{a}_i^{T^m})\}(\bar{v}_i^m - \underline{v}_i^m).$$

From the equalities (A1) and (A14), it follows that the difference in probability $F_j^+(M_j^{+m}, T^m, d_i^{T^m}) - F_j^+(M_j^{+m}, T^m, \hat{a}_i^{T^m})$ is near unity. Hence, from the equality (A7), the loss of future payoff, which is represented by

$$\frac{T^m}{r(\hat{a}_j^{T^m})} \frac{\gamma^m}{1-\gamma^m} \bar{\varsigma}_j^m \{F_j^+(M_j^{+m}, T^m, d_i^{T^m}) - F_j^+(M_j^{+m}, T^m, \hat{a}_i^{T^m})\}(\bar{v}_i^m - \underline{v}_i^m),$$

diverges to infinity as $m$ increases, and therefore, exceeds the maximal short-run gain, which is represented by

$$u_i(d_i, c_j) - \max_{a_i \in A_i} u_i(a_i, c_j).$$

Hence, it follows that $v_i(\delta^m, \bar{s}^m) > v_i(\delta^m, \bar{s}^m / \hat{s}_i^m)$.

Moreover, suppose that

$$\lim_{m \to \infty} T^m f_i^{++}(M_i^{++m}, T^m, \hat{a}_j^{T^m}) < +\infty.$$

From Condition 2 and the equality (A6),

$$\lim_{m \to +\infty} \frac{r(\hat{a}_i^{T^m})}{T^m} > 0,$$

and therefore, from Condition 2 and the equality (A5), it must hold that

$$\lim_{m \to \infty} F_i^{++}(M_i^{++m}, T^m, \hat{a}_j^{T^m}) = 0. \tag{A15}$$

The payoff difference $\dfrac{v_i(\delta^m, \underline{s}^m) - v_i(\delta^m, \underline{s}^m / \hat{s}_i^m)}{r(\hat{a}_i^{T^m})(1-\delta^m)}$ is approximated by, or even larger than, the value given by

$$u_i(d) - \max_{a_i \in A_i} u_i(a_i, d_j)$$
$$+ \frac{T^m}{r(\hat{a}_j^{T^m})} \frac{\gamma^m}{1-\gamma^m} \bar{\varsigma}_j^m \{F_j^{++}(M_j^{+m}, T^m, d_i^{T^m}) - F_j^{++}(M_j^{+m}, T^m, \hat{a}_i^{T^m})\}(\bar{v}_i^m - \underline{v}_i^m).$$

From the equalities (A10) and (A15), it follows that this value is approximated by, or even larger than $\underline{v}_i^m - \max_{a_i} u_i(a_i, d_j) > 0$. Hence, it follows that $v_i(\delta^m, \bar{s}^m) > v_i(\delta^m, \bar{s}^m / \hat{s}_i^m)$.

The above arguments imply that for every $s_i \in S_i$, whenever there exists $\hat{a}_i^{T^m} = (\hat{a}_i(1),...,\hat{a}_i(T^m))$ such that $s_i|_{h_i^{T^m}} = \bar{s}_i^m|_{h_i^{T^m}}$ for all $h_i^{T^m} \in H_i$, and $s_i(h_i^{t-1})(\hat{a}_i(t)) = 1$ for all $t \in \{1,...,T^m\}$, then $\bar{v}_i^m \geq v_i(\delta^m, \bar{s}^m / s_i)$ and $\underline{v}_i^m \geq v_j(\delta^m, \underline{s}^m / s_i)$. Hence, it follows that $\bar{s}^m$, $\underline{s}^m$, $\bar{s}^m / \underline{s}_j^m$ and $\underline{s}^m / \bar{s}_j^m$ are all Nash equilibria for every large enough $m$. Since $\lim_{m\to\infty} v_j(\delta^m, \bar{s}^m) = u_j(c)$, we have completed the proof of Proposition 1.

**Q.E.D.**