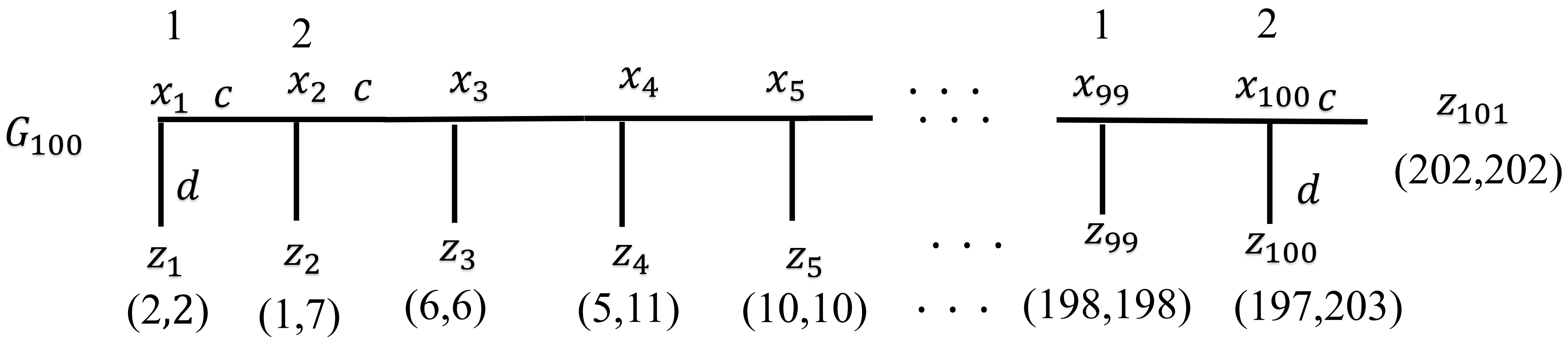
Conceptual Analysis of the Centipede Paradox and its Practical Resolution

by M. Kaneko & R. Ishikawa, 09 April 2024

1) The *centipede game with* $n = 100$.



2) The paradox is an antagonism between the (seemingly) well founded theory (BI theory) and its implication strange from practical point of view.

3) We conduct a conceptual study of the centipede paradox. Does the BI theory include drawbacks?

4) Our study leads to a practical recommendation such as $c^{\ell} \cdot d^{100-\ell}$ $(100 - \ell \leq 2)$. In what sense?

1) Identification of the *centipede paradox*

- the centipede games - Individual Motive (IM) & Cooperative Motive (CM)
- the paradox – based on a paragraph from Selten ('78, TD).

2) The BI (backward induction) theory is modified to the CIB (conscious/inertial behavior) theory based on Kaneko's ('20) EU theory with probability grids.

- In BI theory, IM makes CM *ineffective*.
- In CIB theory, IM may become inactive, and then CM would be effective with inertia.

3) Resulting outcome from the CIB theory:

$$\sigma = \begin{cases} c^{\ell} \cdot d^{n-\ell} & \text{if } \textit{cognitive ability} \text{ is low} \\ d^{n} & \text{if cognitive ability is high} \end{cases}$$

ℓ is close to n, e.g., $n - \ell = 0, 1 \; or \; 2$.

5) We argue that these form a resolution of the centipede game.

Selten ('78), 132-133: Chain-Store Paradox

(*) ... If I had to play the game in the role of the chain-store,
I would follow deterrence theory, …

- I get the impression that most people share this inclination.
- My experiences suggest that mathematically trained persons recognize the logical validity of the induction argument, but they refuse to accept it as a guide to practical behavior.

i. He explains the BI argument to his people
ii. They are mathematical trained people
iii. How does he explain his theory to them?
   The full theory? Or an algorithm?
iv. But they are not "completely rational people", yet ordinary people

Questions:

- Q-o: What are the classes of centipede games and variants?

  Some hints may be hidden? Germ and germination of cooperation.
- Q-i: What is the conceptual bases of the BI theory? What are wrong?

  Two bases are wrong.
- Q-ii: What is our modification, the CIB theory, of the BI theory?

  The above two are weakened.
- Q-iii: Is Selten's question about refusal of the BI theory meaningful?

  How about the CIB theory?

  A drawback suggested in Q-i nullifies his question.
- Q-iv: How do we evaluate the resulting outcomes?

Q-*o*: Centipede Game
- P1 (Indi. Mo.) — only P1 is effective
- P2 (Coop. Mo.) ⟵ nullified by P1

Q-*i*:
- P1 (Com. Compara.) - - - → weakened
- P2 (For. Bygones)

CIB
- Incom.
- Inertia

⇒ P1 is partial; P2 revives **Q-*ii***

CIB sol. $c^{\ell} \cdot d^{n-\ell}$ with $\ell = \ell(n)$

⇩

Q-*iii* Explanation of $c^{\ell} \cdot d^{n-\ell}$ by the behavioral algorithm with the oracle

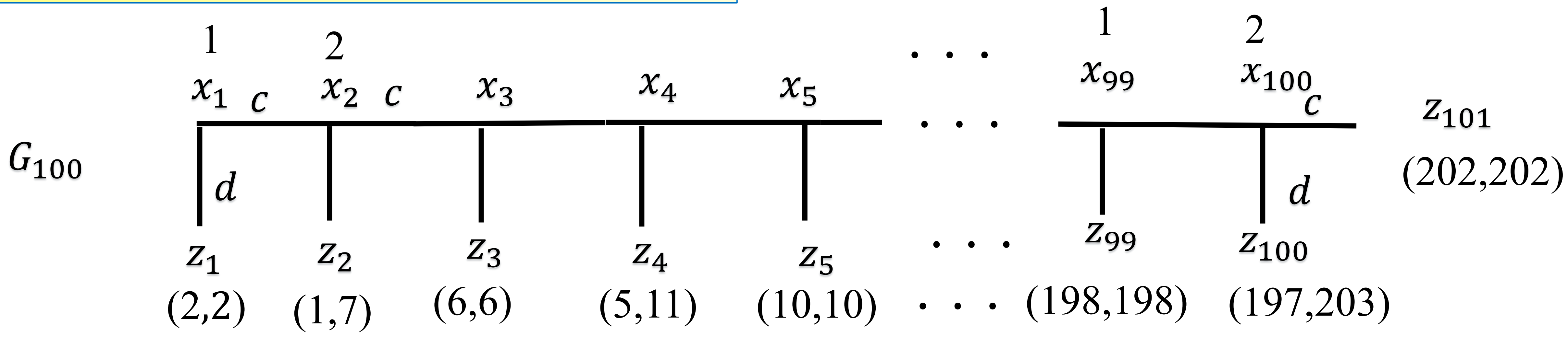
Incidentally, P1 makes Selten's question about practical behavior almost empty

Q-*iv* Is behavioral divide $n - \ell(n)$ small such as 0,1,2? Yes, it is. O2 is satisfied

⇩

This tendency is confirmed in CIB

Identification of the centipede games



A *centipede game* G_n: the *decision nodes* $x_1, \ldots, x_n$, *end nodes* $z_1, \ldots, z_{n+1}$, and the payoff functions (g_1, g_2) satisfy

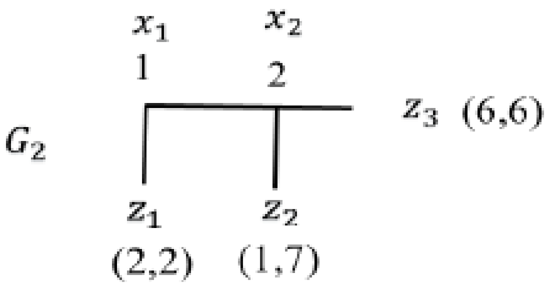
(i) Individual Motive (IM): $g_{\pi(t)}(z_t) > g_{\pi(t)}(z_{t+1})$ for all $t = 1, \ldots, n$;

(ii) Cooperative Motive (CM): $g_{\pi(t)}(z_t) < g_{\pi(t)}(z_{t+2})$ for all $t = 1, \ldots, n-1$.

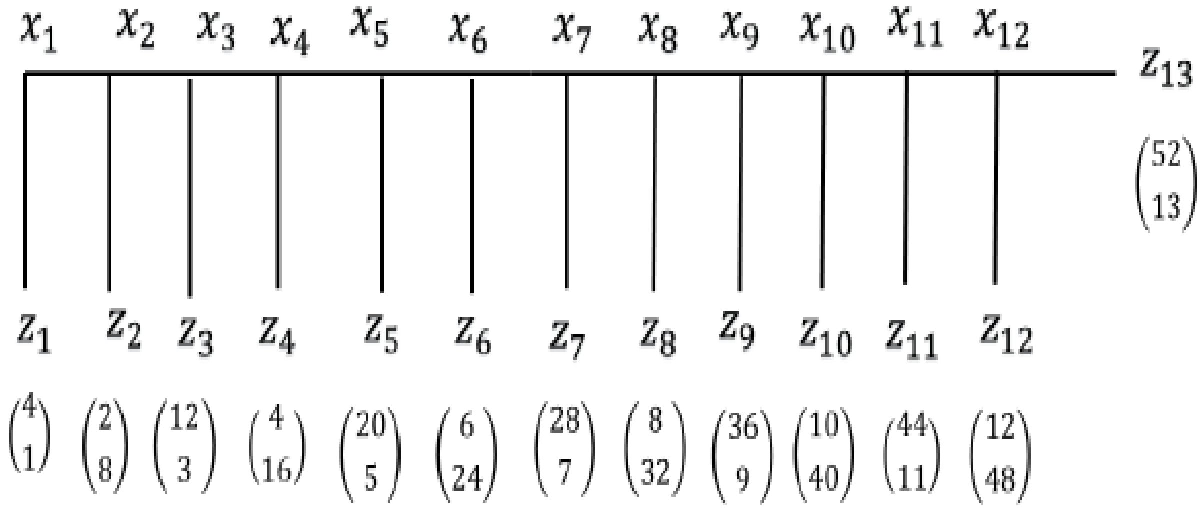
We say that G_n is a *pre-centipede* game if it satisfies IM.

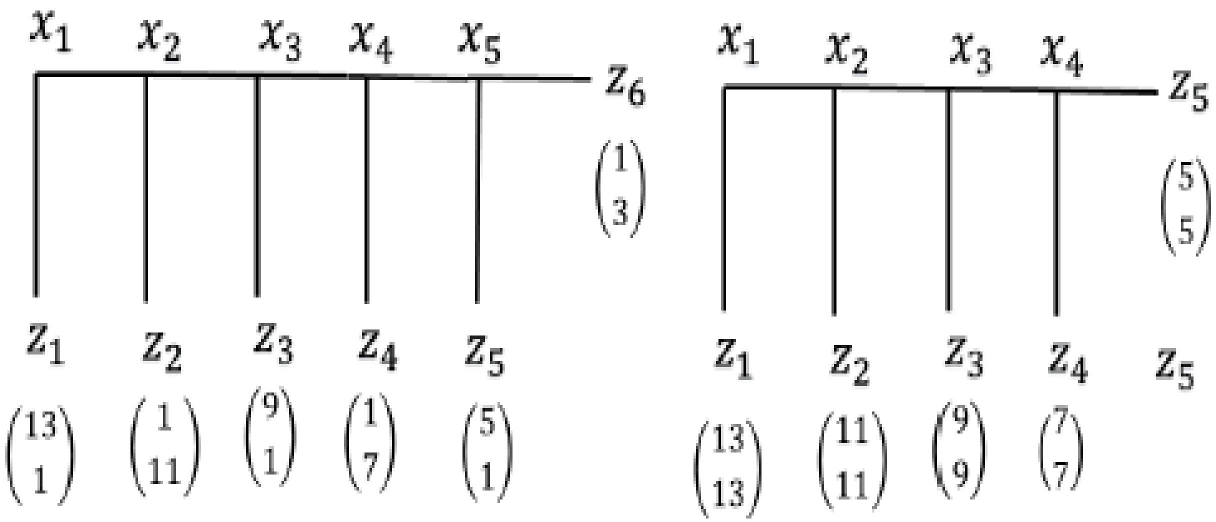
- ✓ We always assume that monetary payoffs and all distinct.
- ✓ The above game is a centipede game, which has the BI solution $\sigma = d^{100}$.

Centipede and pre-centipede games



The initial segment, of length 2, of G_{100}





Pre-centipede games without CM

The above is an *sk-convex* centipede game

- IM is quickly increasing;
- CM is effective only in the beginning of the game, but IM is dominant later.

➢ We focus on the class of sk-linear and sk-concave centipede games.

BI solutions

- A pair of plans $\sigma = (\sigma_1, \sigma_2): \{x_1, \dots, x_n\} \to \{c, d\}$.
- $r(\sigma_{x_t\|})$ is the *realization* of σ, conditional upon that x_t is reached.

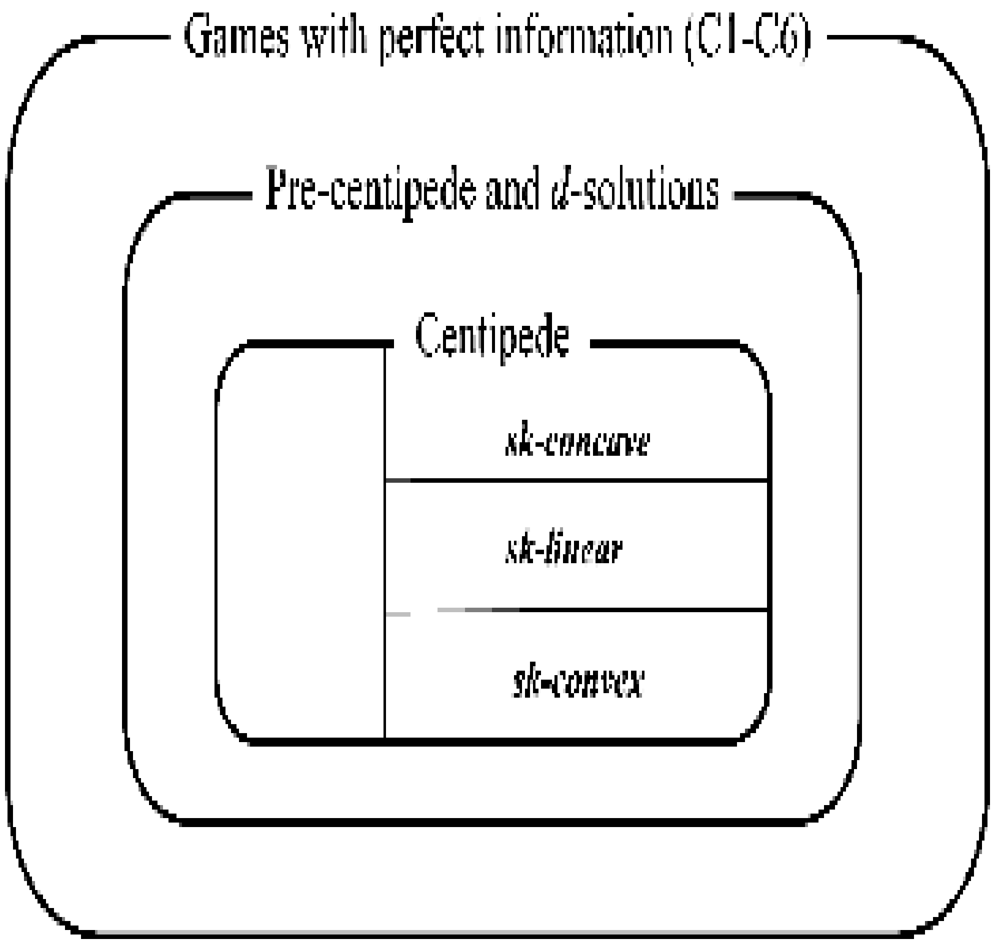
➢ σ is a *BI solution* of G_n iff σ satisfies

$$\sigma(x_t) = \begin{cases} c & if\ g_{\pi(t)}\left(r\left(\sigma_{x_{t+1}\|}\right)\right) > g_{\pi(t)}(z_t) \\ d & if\, g_{\pi(t)}(z_t) > g_{\pi(t)}\left(r\left(\sigma_{x_{t+1}\|}\right)\right). \end{cases}$$

Lemma 2.2. There is a unique BI solution in G_n, without assuming IM & CM.

Theorem 2.1. G_n is a pre-centipede game, i.e., IM holds, if and only if the d-solution d^n is a BI solution.

C1-C5



2): Conceptual bases of the BI theory

P0E (evaluation of outcomes): decision making requires evaluation of future outcomes.

P0M (mathematical induction): the principle of mathematical induction.

- ◆ We do not modify these postulates.
- ➢ P0E motivates to define "reversed causality degree".
- ➢ P0M is used because a game involves some generality.
  In this paper, we use P0M as a method but not study it as an object.

P1: (Perfect comparability): Payoffs are perfectly comparable.

P2: (Forget the bygones): The past is ignored and the future is only taken into account.

- ➢ P1 is modified based on Kaneko's ('20, ET) EU theory with probability grids.
- ➢ P2 is modified by introducing "inertia"; at x_t with some distance from x_1, he would choose c again unless his preference asserts to choose d.

Wishful thinking in a centipede game

- **C**ooperative **M**otive implies the following

Lemma 2.1 (Germ for Cooperation). Let G_n be a centipede game. Let $\ell \leq n$.

- $g_{\pi(\ell)}(z_t) < g_{\pi(\ell)}(z_\ell)$ if $t < \ell$ or $t = \ell + 1$
- $g_{\pi(\ell-1)}(z_t) < g_{\pi(\ell-1)}(z_{\ell+1})$ if $t < \ell + 1$.

  - Here, CM revives.

EU Theory with probability grids

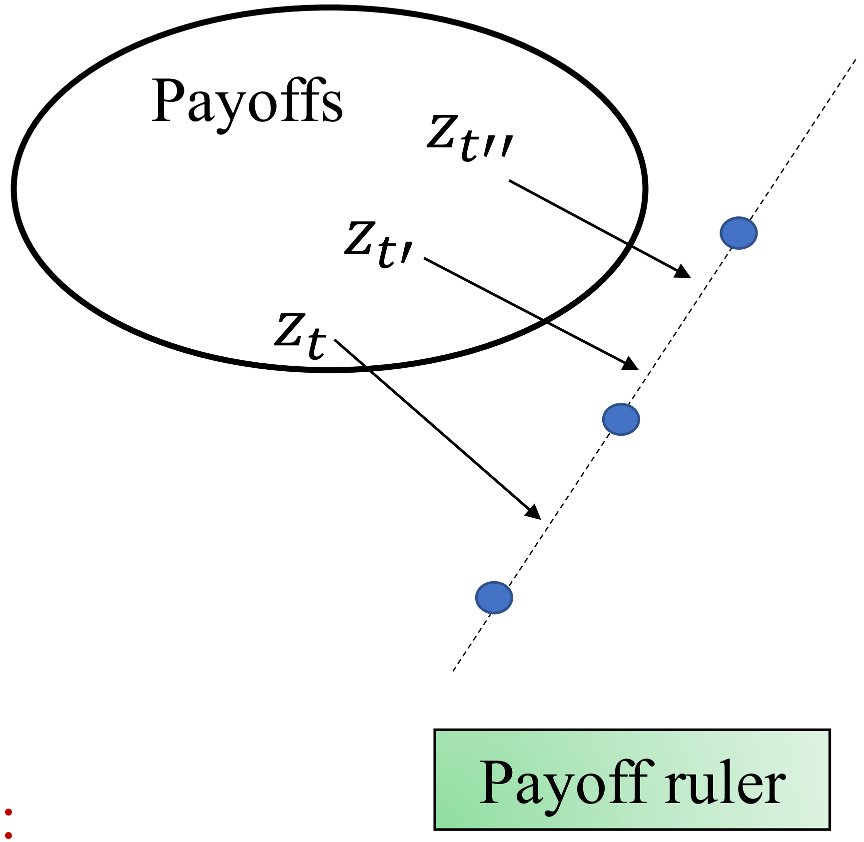
- The payoff ruler consists of scale grids (simple lotteries)

➢ $\underline{\alpha}_i < min_{z_t}\, g_i(z_t) < max_{z_t} g_i(z_t) < \overline{\alpha}_i$.

➢ ρ_i is the cognitive ability of payoffs, where $\rho_i = 0, 1, \ldots$

➢ A scale grid is expressed as a simple lottery

$$\frac{\nu}{2^{\rho_i}} \cdot \overline{\alpha}_i + \left(1 - \frac{\nu}{2^{\rho_i}}\right) \cdot \underline{\alpha}_i = \overline{\alpha}_i + \nu \cdot \frac{\overline{\alpha}_i - \underline{\alpha}_i}{2^{\rho_i}}.$$



Using the payoff ruler, we define the bounded preferences:

◆ Given ρ_i, this is a purely finite construct.

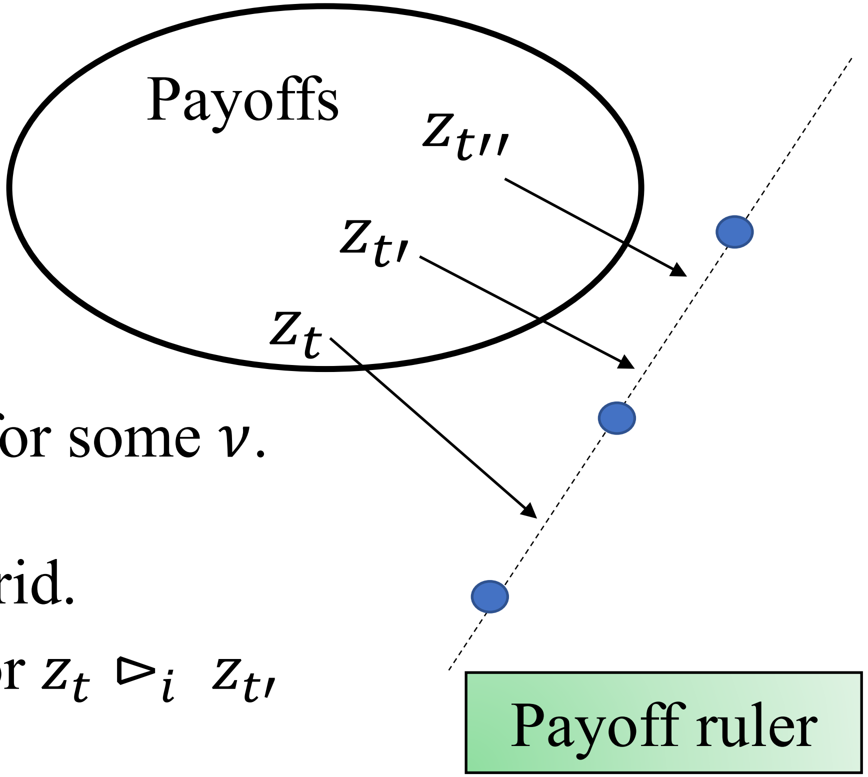
When ρ_i goes $+\infty$, the theory tends to the classical EU theory.

◆ In this sense, this differs from "similarity" (e.g., Rubinstein ('88)).

◆ But our concern is finite ρ_i.

3): From the BI theory to the CIB (*conscious/inertial behavior*) theory

- Payoff ruler consists of scale grids.



Using the payoff ruler, we define the bounded preferences:

◆ $z_{t'} \rhd_i \ z_t \Leftrightarrow_{def} \ g_{\pi(k)}(z_{t'}) \ \geq \underline{\alpha}_i + \nu \cdot \frac{\overline{\alpha}_i - \underline{\alpha}_i}{2^{\rho_i}} \geq g_{\pi(k)}(z_t)$ for some ν.

- **(Strict preference):** $z_{t'}$ and z_t are separated by a scale grid.
- **(Incomparability):** $z_{t'} \bowtie_i \ z_t \Leftrightarrow_{def}$ neither $z_{t'} \rhd_i \ z_t$ nor $z_t \rhd_i \ z_{t'}$

◆ $z_{t'} \unrhd_i \ z_t \Leftrightarrow_{def} z_{t'} \rhd_i \ z_t$ or $z_t \bowtie_i \ z_{t'}$.

Lemma 4.1. $\unrhd_i$ is a complete preordering with incomparability.

- Cooperative Motive implies the following

Lemma 2.1 (Germ for Cooperation). Let G_n be a centipede game. Let $\ell \leq n$.

- $g_{\pi(\ell)}(z_t) < g_{\pi(\ell)}(z_\ell)$ if $t < \ell$ or $t = \ell + 1$
- $g_{\pi(\ell-1)}(z_t) < g_{\pi(\ell-1)}(z_{\ell+1})$ if $t < \ell + 1$.

◆ But this is nullified by IM.

Theorem 3.1 (Germination of cooperation) Let $G_n(\Sigma, b)$ be a centipede game, and let x_ℓ be a decision node. Then,

(1) if $x_{\ell+1} \bowtie_{\pi(\ell)} x_\ell$, then $x_{\ell+1} \trianglerighteq_{\pi(t)} x_t$ for all $t \leq \ell$;

(2) If $x_{\ell+1} \triangleright_{\pi(t)} x_t$, then $x_{\ell+1} \triangleright_{\pi(t')} x_{t'}$ for all $t' \leq t$ with $\pi(t) = \pi(t')$.

① Germination is ready in $G_n(\Sigma, b)$.

② *Inertia* helps start germination.

The CIB theory

The consciousness boundary b_i $(1 \leq b_i \leq n,\ \pi(b_i) = i)$:

- Within this boundary, PLi makes a decision consciously.
  - Beyond b_i, he follows the inertia c, unless he has a strict preference z_t over the realized node $r\left(\sigma_{x_{t+1}\|}\right)$ conditional upon x_{t+1}.
  - he takes d, if he does.

➢ σ is a *CBI solution* of $G_n(\Sigma, b)$ iff σ is defined by

$$\sigma(x_t) = \begin{cases} d & if\ z_t \rhd_{\pi(t)} r\left(\sigma_{x_{t+1}\|}\right) & (\textit{conscious choice}) \\ c \text{ or } d & if\ r\left(\sigma_{x_{t+1}\|}\right) \bowtie_{\pi(t)} z_t \ \&\ t \leq b_{\pi(t)} & \\ c & if\ r\left(\sigma_{x_{t+1}\|}\right) \rhd_{\pi(t)} z_t \ \&\ t \leq b_{\pi(t)} & (\textit{conscious choice}) \\ c & if\ r\left(\sigma_{x_{t+1}\|}\right) \trianglerighteq_{\pi(t)} z_t \ \&\ t > b_{\pi(t)} & (\textit{inerial behavor}) \end{cases}$$

Canonical CIB solution

- Compare the adjacent endnodes, $z_t \rhd_{\pi(t)} z_{t+1}$ or $z_t \bowtie_{\pi(t)} z_{t+1}$.
- Let ℓ be the number satisfying

  (a): $z_\ell \bowtie_{\pi(\ell)} z_{\ell+1}$ and (b) it is the maximum among such ℓ's.

Theorem 4.1: Let $G_n(\Sigma, b)$ be a centipede game, and let $\ell = \ell(n)$ be given above. Then,

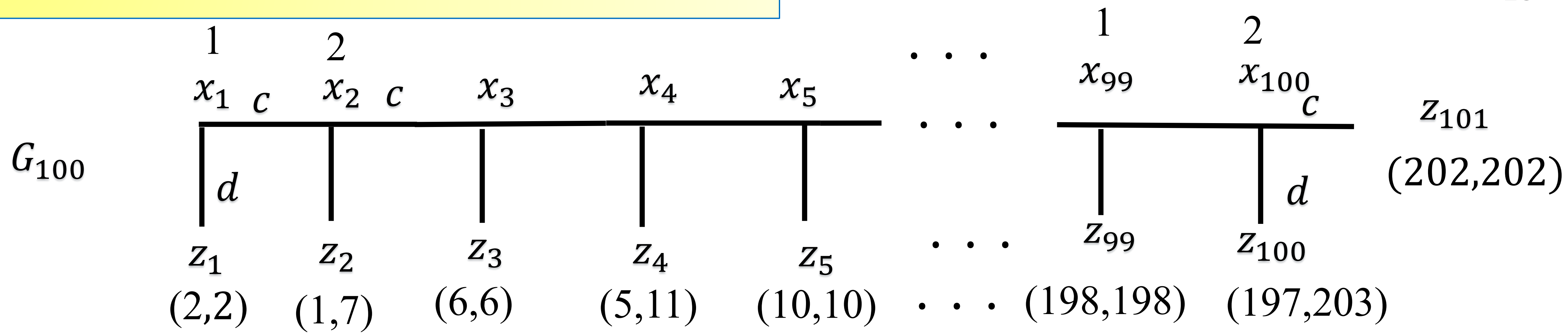
(i): $\sigma = c^\ell d^{n-\ell}$ is a CIB solution.

(ii): $\sigma = c^\ell d^{n-\ell}$ is a unique CIB-solution if and only if

$$\ell > \max(b_1, b_2) \text{ and } z_{\ell+1} \rhd_i z_{b_i} \text{ for } i = 1,2.$$

- ➢ We call $\sigma = c^\ell d^{n-\ell}$ is the *canonical* CIB solution.
- ➢ It is used as the representative of CIB solutions.

What are the canonical CIB solution?



Let $\underline{\alpha}_i = 0$ and $\overline{\alpha}_i = 300$ for $i = 1,2.$,

In $G_{100}(\Sigma, b)$,

$$\sigma = \begin{cases} c^{100} & \text{if } \rho_1 = \rho_2 \leq 7 \\ c^{99}d & \text{if } \rho_1 = \rho_2 = 8 \\ d^{100} & \text{if } \rho_1 = \rho_2 \geq 9 \end{cases}$$

In $G_{68}(\Sigma, b)$,

$$\sigma = \begin{cases} c^{68} & \text{if } \rho_1 = \rho_2 \leq 6 \\ c^{66}d^2 & \text{if } \rho_1 = \rho_2 = 7 \\ c^{65}d^3 & \text{if } \rho_1 = \rho_2 = 8 \\ d^{68} & \text{if } \rho_1 = \rho_2 \geq 9 \end{cases}$$

- Strong tendencies for cooperation for low cognitive abilities;
- d-solution for high cognitive abilities.

➢ Are the above examples typical? - - more examples and results in the paper.

Selten people's response and the reversed causality degree

1)) The payoffs in the last area of G_{100} are much larger than those in the beginning.

2) The *cause-and-effect* for decision making:

- the cause around z_{101} ➡ the realization of σ.

Define the *reversed causality degree* $RC_n(\sigma)$ in G_n by

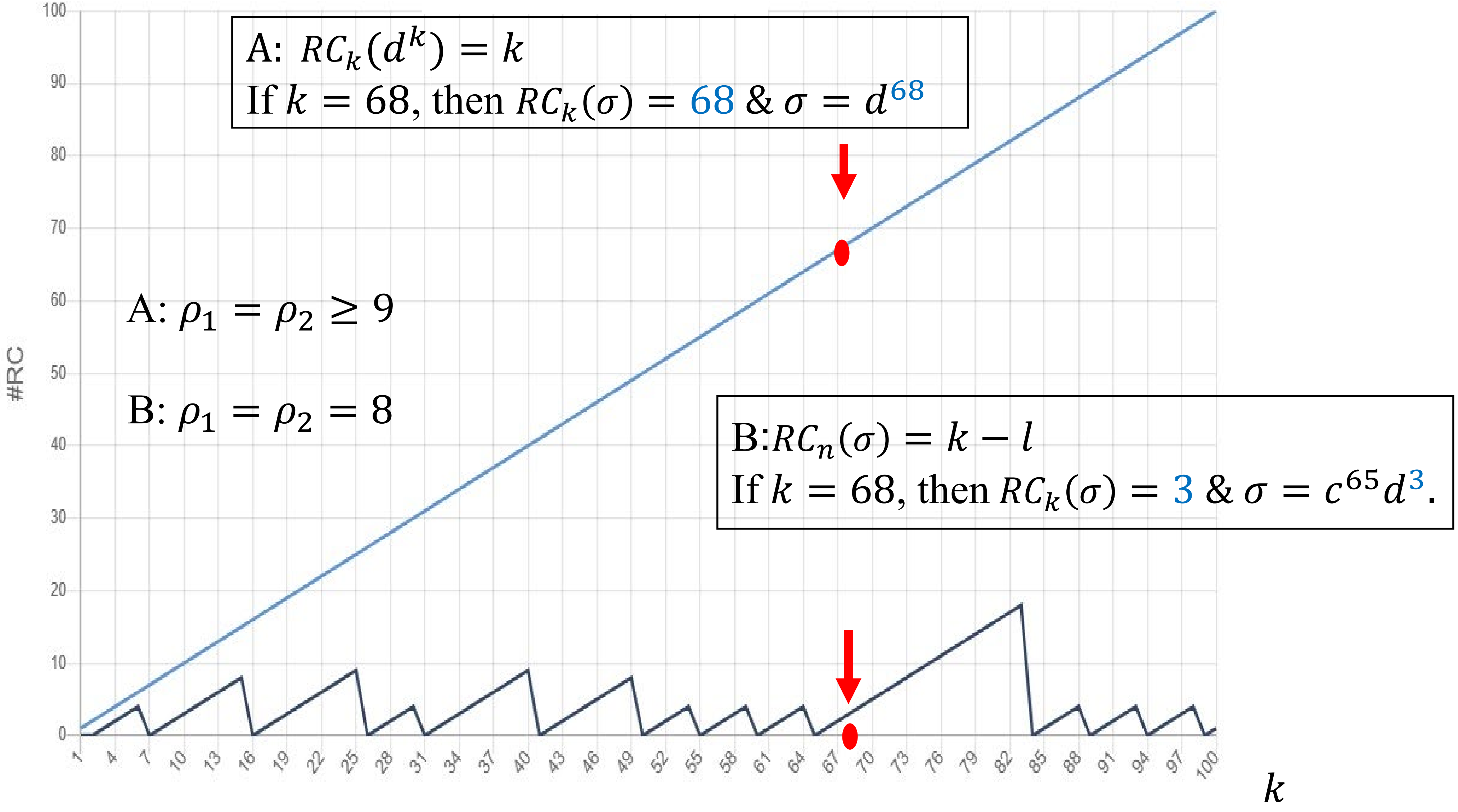
$$RC_n(\sigma) = (n+1) - r_T(\sigma).$$

- For the canonical CIB solution $\sigma = c^{\ell} d^{n-\ell}$,

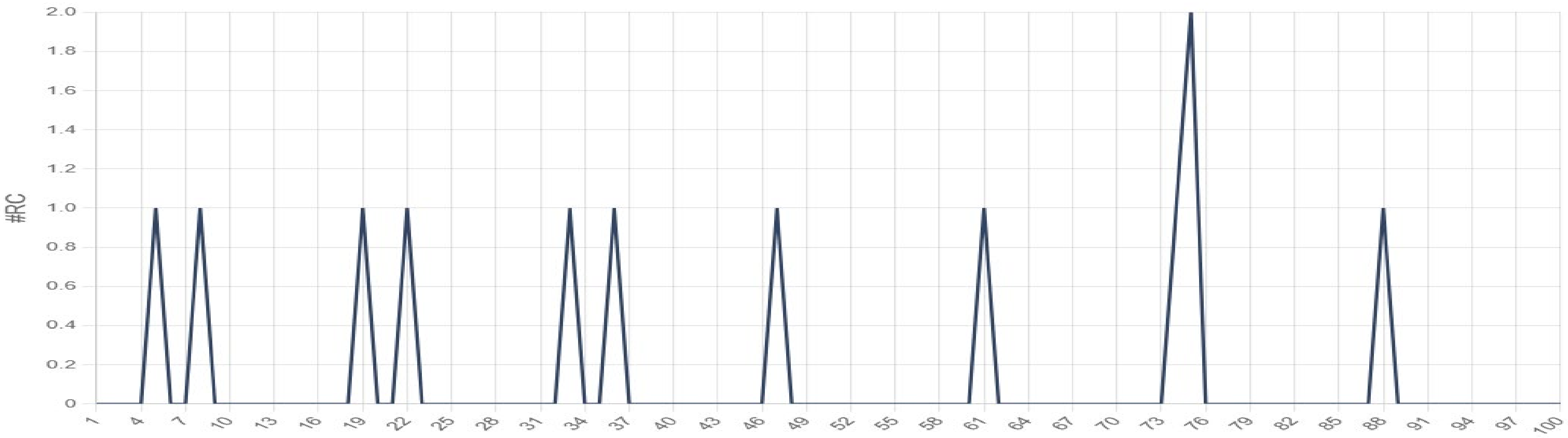
$$RC_n(\sigma) = n - \ell(n).$$

- if $\sigma = d^n$, then $RC_n(\sigma) = n$;
- if $\sigma = c^n$, then $RC_n(\sigma) = 0$;
- if $\sigma = c^{n-1} d^1$, then $RC_n(\sigma) = 1$.

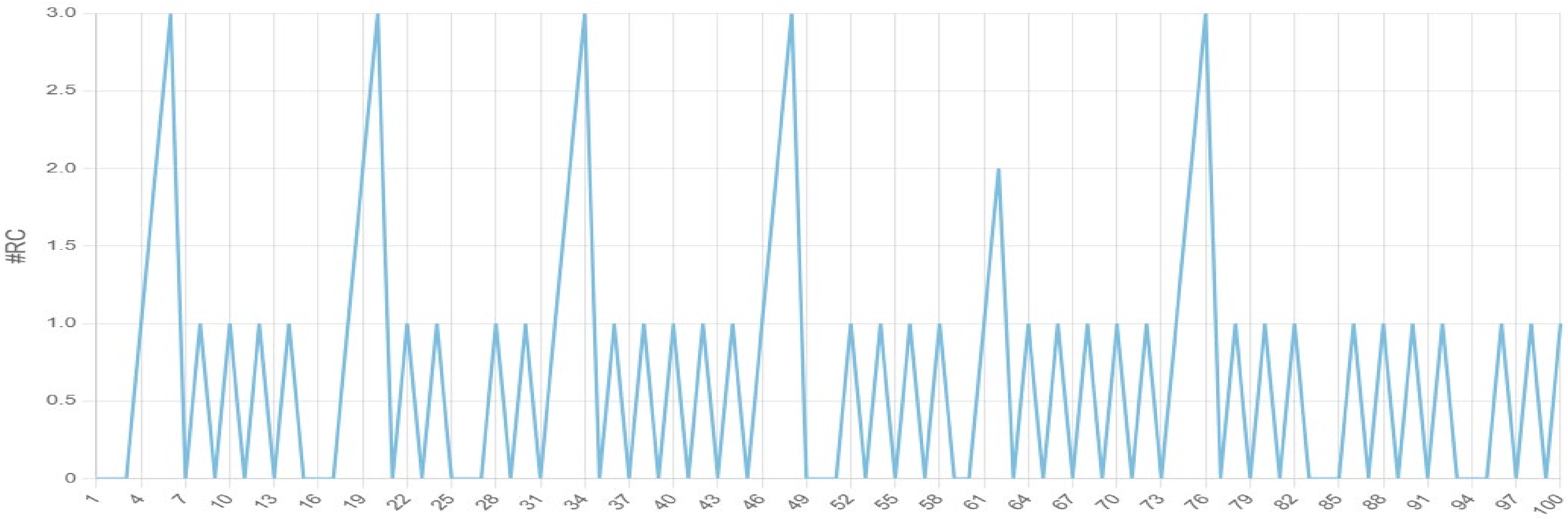
$$RC_k(\sigma) = k - \ell \text{ in } G_k(\Sigma, b) \text{ for } \sigma = c^{\ell} d^{k-\ell}, k = 1, \ldots, n$$



$RC_k(\sigma) = k - \ell$ in $G_k(\Sigma, b)$ for $\sigma = c^{\ell} d^{k-\ell}, k = 1, \ldots, 100$



$\rho_1 = \rho_2 = 5$



$\rho_1 = 5 \ \ \rho_2 = 8$

Summary of the calculation results

Three cases of cognitive abilities ρ_1 and ρ_2 in a centipede game $G_n(\Sigma, b)$

(a): Both ρ_1 and ρ_2 are high; the resulting outcome is d^n;

- the reversed causality degree is the highest $RC_n(d^n) = n$;
- this is compatible with the Selten people's complaints, yet the cognitive abilities are high.

(b): Both are low; the resulting outcome is $\sigma = c^\ell d^{n-\ell}$ for ℓ colse to n;

- the reversed causality degree is $RC_n(\sigma) = n - \ell$, small, e.g., $RC_n(c^n) = 0$;
- this is compatible with what the Selten people want.

(c): ρ_1 is high and ρ_2 is low;

- This case is similar to (b).

An algorithm with an oracle - - guide for practical behavior

Al suggests that at a decision node x_t from the last decision node x_n, you make a comparison between z_t *and* z_{t+1}, until you find $z_\ell \bowtie_{\pi(\ell)} z_{\ell+1}$.

These comparisons are based on your own inner feeling.

A) If you do not find such a pair, you are recommended to take the strategy taking *d* always.

B) If you find $z_\ell \bowtie_{\pi(\ell)} z_{\ell+1}$, then you should jump to the first decision node x_1 to take the strategy taking *c* and then *d* up to the end of the game.

B) is based on your knowledge on the CIB theory - - Oracle.

In Selten's (*), his colleague were taught his theory including PC (perfect comparability), requiring no his own comparisons.

A Resolution of the Centipede Paradox.

(1) The antagonism faced by the Selten people

(2) Identification of the BI argument.

(3) Modification of the BI theory to the CIB (consciousness-inertial behavior) theory

- Full cognitive separability - - the BI theory (*d*-solution)
- Partial cognitive inseparability - - the CIB theory (*canonical* CIB solution).

(4) Thought experiments on the Selten people's responses in terms of $RC_n(\sigma)$.

- When the PL's have high cognitive abilities of payoffs, $RC_n(\sigma)$ is large.
  - ✓ This expresses the Selten people's responses in (*).
- When at least one of them has a small low ability, $RC_n(\sigma)$ is small.
  - ✓ The Selten people's have no complaints.

➢ After all, in what sense is it a resolution of the centipede paradox? In what sense, not?

- It is for the Selten people to whom the CIB theory is explained.
- Not to fresh people without such knowledge, i.e., not in the sense of standard experiments.