Conceptual/Practical Resolution of the Centipede Paradox[*] [†]

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Abstract

We start to identify the centipede paradox as the antagonism between the BI (backward induction) outcome in a centipede game and people's responses that Selten indicated for the chain-store paradox. We explore the underlying postulates for the centipede paradox as well as a centipede game. Based on them, we weaken the BI theory so that payoffs are possibly incomparable for a player depending upon his bounded cognitive ability. In this case, he is assumed to follow inertial behavior when decision nodes have some distance from the start. In the CIB (conscious choice/inertial behavior) theory which we develop, when both players have high cognitive abilities, the CIB theory exhibits the same outcome as the BI theory, but when at least one has a low ability, it induces quite opposite outcomes to go to the ending area of the game. We argue that these results are compatible with the postulate for a centipede game leading to cooperation, which is the reason for with people's responses indicated by Selten. These considerations form a resolution of the centipede paradox.

Key Words: Centipede Games, BI theory, Cognitive Bounds for Payoffs, Inertia, CIB theory, Degree of Reversed Causality

1 Introduction

We have met some paradoxical results from the BI (*backward induction*) theory in game theory. Two results directly relevant to this paper are the *chain-store paradox* and the *centipede paradox* due to Selten [24] and Rosenthal [22]; the latter shows the difficulty in a clearer manner than the former. We focus on the centipede paradox and attempt to give its resolution. Selten states what the paradox is for chain-store games; we apply his statement to centipede games. We modify the BI theory to the CIB *(conscious-choice/ inertial behavior*) theory by introducing players' bounded cognitive abilities for payoffs and a postulate based on inertial behavior. The new theory keeps the same prediction as the BI theory when both players have high cognitive abilities, but it dictates quite opposite results when they are low. We measure the degree of reversed causality on these results. In this introduction, we discuss the background of the paradox and how we approach it.

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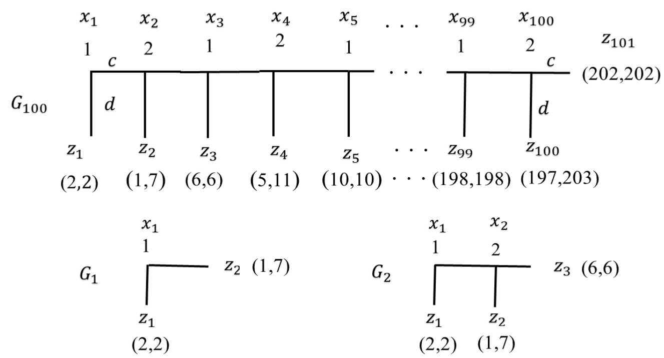

Figure 1: Centipede games

Look at the centipede game of length 100 in Fig.1: Players 1 and 2 alternatively choose action c (continue) or d (down, finish), except for the decision node x_{100} where either choice goes to an end node. The BI theory starts with the hypothetical consideration of player 2's decision making at x_{100} and suggests to choose d since payoff 203 from d is larger than payoff 202 from c. Assuming this choice, player 1 chooses d at x_{99}, the theory continues suggesting the player at each node to choose d, and finally, 1 chooses d at the first node x_1. We call these choices the *BI outcome*, denoted by d^n. There is a large discrepancy between the hypothesis of player 2's decision making at x_{100} and the resulting outcome to finish the game at x_1.

A paradox is an antagonism between an authoritative view and some other view.[1] Selten [24], pp.132-133, expresses what he means by the chain-store paradox.

> (*) ... If I had to play the game in the role of the chain-store, I would follow deterrence theory (*which is opposite to the BI theory*). ... I get the impression that most people share this inclination. My experiences suggest that mathematically trained persons recognize the logical validity of the induction argument, but they refuse to accept it as a guide to practical behavior.

This is an antagonism between the BI outcome and the responses of "mathematically trained persons". The BI theory is authoritative in that it is rigorously formulated in game theory. Selten asked his colleagues about the first sentence of (*), perhaps, with an simple explanation of the BI outcome. They, including Selten himself, refused the theoretical conclusion because it is far from a guide to practical behavior. This can be applied to the centipede game; we regard (*) as stated for the centipede game.[2]

The paradox has still remained but experimental game theory has shown large deviations from the BI outcome (see Garciao-Pola, *et.al* [7] for an exhaustive survey[3]). In experiments,

[1] A logical paradox means that a theory includes a sentence entailing a contradiction such as Russel's paradox (see Mendelson [20], p.1 to p.9). The centipede paradox is not a logical paradox in that the BI theory contradicts some other reference.

[2] In the chain-store game, the deterrent strategy can be a threat to stop retailers to enter the market. In a centipede game, a player's choice d is not a threat because it terminates the game immediately.

[3] Three groups of works are found in the literature. One is represented by Aumann [1] about logical validity on the BI theory from the viewpoint of epistemic game theory and he showed its validity but not discussed a resolution. The other two groups have taken moves in the same direction as ours; the CIB theory shares similar objects with both literatures of Bayesian game theory (cf., Binmore [2], Kreps [15]) and of behavioral/experimental game theory (cf., McKelvey-Palfrey [19], Kawagoe-Takizawa [13], Ho-Su [8], and Garcia-Pola, *et al.* [7]). These show some possible deviations from the d-behavior; they did not explicitly discuss what the centipede paradox is.

subjects are typically taken from university students, and well "mathematically trained persons" are often avoided. In the present paper, we take "mathematically trained persons", who are yet ordinary people and called the *Selten people.* Also, the experimental literature focuses on small centipede games, e.g., the centipede games studied in [7] are of length 6. On the other hand, our central concern is a relatively long centipede games.

In order to study the centipede paradox as a whole, we ask the following questions:

Q-o: What are the classes of centipede games and variants?

Q-i: What is the conceptual bases of the BI theory? What are wrong?

Q-ii: What is our modification, the CIB theory, of the BI theory?

Q-iii: Is Selten's question about refusal of the BI theory meaningful?
How about the CIB theory?

Q-iv: How do we evaluate the resulting outcomes?

We develop our conceptual/practical arguments along these questions. The entire development consists of conceptual/practical studies of concepts in the centipede paradox, which will be summarized in Fig.2.

Questions Q-o to Q-iv provide full consideration of a reconciliation of the antagonism. Here, we discuss these questions and describe corresponding developments in the paper.

Q-o**: Centipede and pre-centipede games**: In Section 2, we characterize the class of *centipede games* by *Individual Motive* (IM) and *Cooperation Motive* (CM). Condition IM is the main condition which leads to the BI outcome d^{100}, but Condition CM could induce the people to refuse the BI outcome. The argument leading to d^{100} includes the following doubts*:*

O*1*: its cause-effect has a large discrepancy from the cause of decision making at x_{100}
to the effect of leading to the realization outcome z_1;

O*2*: the payoffs in the last area of G_{100} are much larger than those in the beginning.

It is shown that the *pre-centipede games* requiring only condition IM are necessary and sufficient for the BI theory to have d^n. That is, though people have a hope, by CM, to avoid the BI outcome d^n, their hopeful thought become ineffective by IM. The class of pre-centipede games include many games irrelevant to the centipede paradox.

Q-i**: Conceptual bases of the BI theory**: It consists of the following postulates;

P0E *(Evaluations of outcomes)*: decision making requires evaluations of future outcomes;

P0M *(Mathematical induction)*: the principle of mathematical induction is used;

P1 *(Complete comparability)*: payoffs are perfectly comparable;

P2 *(Forget the bygones)*: the past is ignored and the future is only taken into account.

Postulate P0E is basic for decision making in that it values the available options. In the game G_2 of Fig.1, player 1 compares between choices d and c, but for choice c, he values the consequence from choice c, which requires him to think hypothetically about player 2's choice at x_2. In general, he values each future outcome and then comes back to the present decision making, which allows the possibility to value only near futures.[4] But if P0E is unlimited, it would lead to the difficulty mentioned in O*1*; this difficulty is included in the BI outcome d^n.

[4] P0E has some similarity to "counterfactual" (cf., Lewis [17] and Menzies-Beebee [21]). It retraces the process from a possible event not happened at present to a past cause. P0E first evaluates future outcomes, and then, it gives decision making. Thus, P0E conceptually differs from "counterfactual".

Postulate P0M is fundamental for mathematical studies involving some abstract structures, even about finite targets.[5] A study about G_{100} involves such a structue, but not G_2. In this paper, we use P0M as a research method but do not touch it as a research target.

Although P1 is standard in the literature of game theory/economics; utility theory without P1 has recently been studied (cf., Fishburn [6], Dubra, *et al.* [4], Kaneko [9]). The point of P2 is well expressed by the phrase "*let bygones be bygones.*" The BI theory depends only upon the future payoffs. Note that P0E does not exclude looking on a history. In fact, P1 and IM prevent CM from being effective. We relax P1 and P2 so as to find a way out from the paradox.

Q-*ii*: Our modification of the BI theory: The CIB theory to be presented in this paper is a modification of the BI theory by weakening P1 and P2. It allows various new behavioral outcomes having choice c. For P1, we adopt Kaneko's [9] expected utility theory with probability grids; utility values are measured by a payoff ruler with payoff scales based on the bounded cognitive ability of a player. When he has a low cognitive ability, his ruler has rough grids and his preferences may show incompatibilities between two payoffs.

To weaken P2, we introduce the concept of *inertial choice*. The general idea is:

(#): *unless a new stimulus comes, the object continues to move in the same manner as before, and if some comes, it may change the move responding to it.*

In a centipede game, the player at a decision node x_t in some distance from the start x_1 follows the same choice c as before, unless he has a new stimulus. In a centipede game, "when x_t has far enough from x_1 and the choices c and d are incomparable or c is preferred to d, he follows the inertial choice c." Here, a distance and incomparability play a crucial role to express the concept of inertia.[6,7]

The CIB theory suggests the following behavior for a centipede game G_n;

$$c^{\ell} d^{n-\ell} \ (0 \leq \ell \leq n). \tag{1}$$

That is, action c is taken up to x_ℓ and then d is chosen to the end of the game. This $c^{\ell} d^{n-\ell}$ will be called the *canonical* CIB solution, which is the representative of the CIB solutions in G_n with multiple CIB solution.

Q-*iii*: Selten's question about refusal for the CIB solution: The last phrase "they refuse to accept it as a guide to practical behavior" in (*) is the key for this question. The Selten people are taught the BI theory including already the detailed preferences by P1; the people are asked only to answer "yes" or "no" in the sense that there are no rooms remaining to consider their own choice behavior.

In the CIB theory, payoffs are monetary but their preferences over them depend upon the cognitive bounds; if cogntive bounds are high, the resulting recommendation is d^n, but otherwise, the recommended outcome could be $c^{\ell} d^{n-\ell}$. Cognitive bounds are private and their actual values are not mentioned in the CIB theory. Instead of the details of the CIB theory, a summary form, a *behavior algorithm with an oracle;* private information is checked by choices but some theoretical

[5] cf., Mendelson [20], Kleene [14], Section 7.

[6] A modification of P1 was given in Kaneko [10] in the context of the St.Petersburg (1-person) game. There, a participant makes a decision to buy or not a ticket before the game. This game has no room for inertia.

[7] This idea was used in Marschak-Selten [18] to introduce an inertial behavior in a repeated game in that unless a player finds an enough motivation to change behavior, he would keep the same action. In a broader sense, the concept of convention in Lewis [16] uses a similar idea.

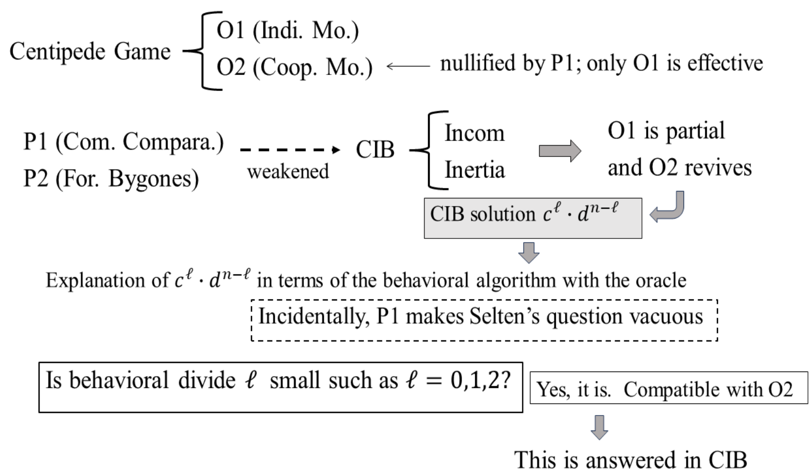

Figure 2: Conceptual/practical diagram

structure is mentioned by the oracle, which will be discussed in Section 5. The Selten people need to think about their private preferences. The question is whether they choose the behavior based on their private preferences together with the help given by the algorithm.

Q-*iv*: Evaluations of behavior $c^{\ell}d^{n-\ell}$ $(0 \leq \ell \leq n)$ by the reversed causality degree: Possible reasons for the refusal of the BI outcome d^n are O*1*, particularly, O*2*. We represent the underlying idea of O*1* and O*2* by the *degree of reversed causality* RC_n between the hypothesis and the resulting outcome. In the case of the BI outcome d^{100} in G_{100}, the degree RC_{100} is 100 representing the discrepancy between the hypothesis z_{101} and the resulting outcome z_1, which is the maximum in the centipede game G_{100} of Fig.1. In the case of $c^{\ell}d^{100-\ell}$ in (1) with $n = 100$, the value is calculated as $RC_{100} = 100 - \ell$. Thus, it is a crucial question for the resolution of the centipede paradox how ℓ is close to n. We argue that ℓ is typically $n, n-1$ or $n-2$ when the cogntive ability of at least one person is relatively low.

The relations between the above mentioed steps are dipicted in Fig.2. These form a resolution of the centipede paradox.

The present paper constitutes of six sections in addition to this introduction. Section 2 gives definitions of a few classes of centipede games, and discusses the foundations of the BI theory. Sections 3 and 4 introduce the CIB theory, and we presents various theorems in the CIB theory. Section 5 gives central results, and give an algorithm with which a Selten person is taught how to play a centipede game with the CIB theory. Section 6 introduces the concept of reversed causality degree and discusses its behavior over canonical CIB solutions for various lengths of centipede games. Section 7 concludes the paper and gives some comments on possible extensions of the CIB theory. Proofs of the results in each section will be given in its last subsection.

2 Centipede Games and the BI solution

We define centipede games and a BI solution. In order to have the clear scopes of the BI theory and of the CIB theory, we give the definitions in a quite formal manner with a few examples. A centipede game is defined by two conditions on the payoff functions, and a pre-centipede game is defined by one of them. We prove that the BI outcome d^n is a BI solution if and only if the game is a pre-centipede game.

2.1 Basic definitions for a centipede game

We consider a 2-person game with perfect information given as $G_n = (X_n, Z_n, \pi, \{c, d\}, (g_1, g_2))$, where

C1 $X_n = \{x_1, ..., x_n\}$ is the set of *decision nodes*;

C2 $Z_n = \{z_1, ..., z_n, z_{n+1}\}$ is the set of *end nodes*;

C3 π is the *player assignment* over X_n with $\pi(t) = 1$ if t is odd and $\pi(t) = 2$ if t is even;

C4 for $t \leq n$, player $\pi(t)$ at node x_t chooses either *action* c or d, where c connects x_t to x_{t+1} ($x_{t+1} = z_{n+1}$ if $t = n$) and d connects x_t to z_t;

C5 $g_i : Z_n \to \mathbb{R}_+$ is a one-to-one monetary *payoff function* for $i = 1, 2$,

where $\mathbb{R}_+$ is the set of nonnegative real numbers. Players 1 and 2 alternatively move, and the player assigned at node x_t is denoted by $\pi(t)$. C1 to C4 require the game structure of G_n to be the same as that of Fig.1. Here, we take $\mathbb{R}_+$ to be the region of either payoff function; here, payoffs are monetary. C5 also requires all payoffs to be distinct for each player. Theorem 5.2 assumes that the payoff functions takes integral values.

We specify either when we need to be more specific.

Definition 2.1 (Centipede game): We say that G_n is a *centipede game* iff

$$\textit{Individualistic Motive (IM)}\ g_{\pi(t)}(z_t) > g_{\pi(t)}(z_{t+1}) \text{ for all } t \leq n; \tag{2}$$

$$\textit{Cooperative Motive (CM)}\ g_{\pi(t)}(z_t) < g_{\pi(t)}(z_{t+2}) \text{ for all } t \leq n-1. \tag{3}$$

IM represents the individual motive for player $\pi(t)$ at a decision node x_t to choose d, as long as the next player $\pi(t+1)$ behaves in the same manner. On the other hand, CM is the cooperative motive for player $\pi(t)$ at any decision node x_t to continue the game. CM is given by Garcia-Pola, *et at.* [7], p.396, (1), and IM corresponds to [7], p.396, (2).[8]

Combining (2) and (3), we have:

$$g_{\pi(t)}(z_{t+1}) < g_{\pi(t)}(z_{t+2}) \text{ for all } t \leq n-1. \tag{4}$$

His payoff values $g_{\pi(t)}(z_t)$, $g_{\pi(t)}(z_{t+1})$, and $g_{\pi(t)}(z_{t+2})$ form a *dent*; the value gets once smaller and then larger. The other player $\pi(t+1)$'s IM prevents the achievement of CM for $\pi(t)$. This is a cause of the centipede paradox. If we require only CM, it may be the case that the payoff functions may have no dents, i.e., they are monotone increasing; in this case, the players simply choose c at every decision node z_t, and we have no paradox.

Numerical examples are informative to understand the above conditions and centipede games.

[8] Their individual motive condition is formulated using the term “the payoff value of the non-deciding player in a position” Although the formulation is ambiguous, it is observed from the other part of the paper that their intention is the same as our IM.

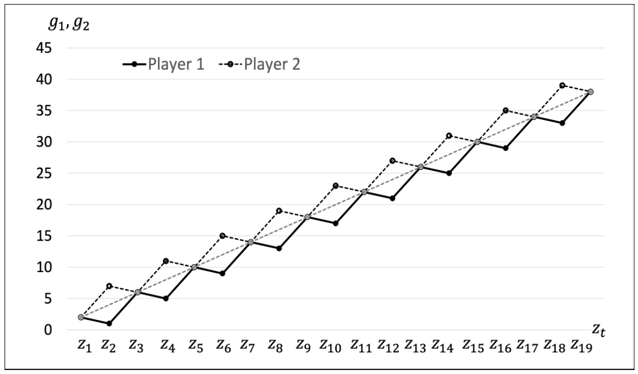

Figure 3: Classical centipede game

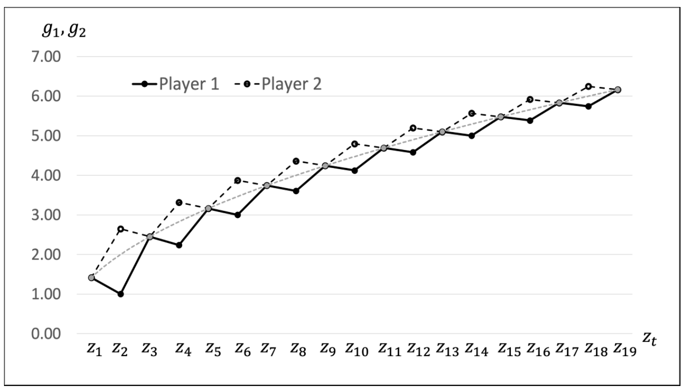

Figure 4: Shapes of an sk-concave centipede

Example 2.1 (Classical centipede 1): The game of Fig.1 is given by the payoff functions $(g_1(z_t), g_2(z_t))$: for $z_t \in Z_{100}$,

$$(g_1(z_t), g_2(z_t)) = \begin{cases} (2t, 2t) & \text{if } t \text{ is odd} \\ (2t-3, 2t+3) & \text{if } t \text{ is even.} \end{cases} \tag{5}$$

These satisfy both (2) and (3). Fig.3 depicts these payoff functions $g_1(z_t)$ and $g_2(z_t)$. Player $\pi(t)$'s payoff values $g_{\pi(t)}(z_t)$, $g_{\pi(t)}(z_{t+1})$, and $g_{\pi(t)}(z_{t+2})$ form a dent, which overlaps with the dent from $g_{\pi(t+1)}(z_{t+1})$, $g_{\pi(t+1)}(z_{t+2})$, and $g_{\pi(t+1)}(z_{t+2})$ for player $\pi(t+1)$. These functions are linear in the sense that the dents are linearly lined up.

For better understanding of the centipede paradox, we will consider some subclass of centipede games. We say that G_n is *sk-concave* iff for $t \leq n-1$,

$$g_{\pi(t)}(z_t) - g_{\pi(t)}(z_{t+1}) \text{ and } g_{\pi(t)}(z_{t+2}) - g_{\pi(t)}(z_t) \text{ are weakly decreasing.} \tag{6}$$

Thus, the differences corresponding to both IM and CM are weakly decreasing. Hence, the difference $g_{\pi(t)}(z_{t+2}) - g_{\pi(t)}(z_{t+1})$ of (4) is also weakly decreasing, since it is expressed as $[g_{\pi(t)}(z_{t+2}) - g_{\pi(t)}(z_t)] + [g_{\pi(t)}(z_t) - g_{\pi(t)}(z_{t+1})]$. When both are constant, the centipede game is *sk-linear*.

If we replace "weak decreasing" in (6) by "weak increasing", a centipede game G_n is said to be *sk-convex*. In the literature on experimental studies since McKelvey-Palfrey [19], sk-convex

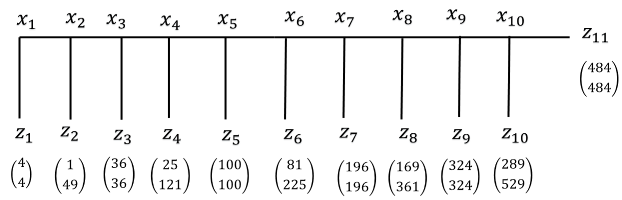

Figure 5: Sk-convex centipede game

centipede games have often been studied. In this class, both IM and CM are getting larger as the game goes further. The centipede paradox targets centipede games with relatively large n, but sk-convexity is not well compatible with large n in that monetary payoffs in experiments may be too large. For the main development of this paper, we will focus on sk-concave centipede games. Nevertheless, it would be informative to look at what happens in the outside of the class of sk-concave centipede games.

The game in Fig.5 is an sk-convex centipede game with length 10, where the payoff function of player i is given as the square value $(g_i(z_t))^2$ of $g_i(z_t)$ given by (5) for $t = 1, ..., 11$. As mentioned, the payoff values could be absurd for large n such as 100.[9]

Nevertheless, we consider the broader class of games than that of centipede games. To consider the role of CM. When G_n satisfies IM, we call G_n a *pre-centipede game*. The class of pre-centipede games is very large, but it is a useful benchmark for our development from the BI theory to the CIB theory. It will be proved in Section 2.2 that the class of pre-centipede games is equivalent to that of games such that the BI theory entails the BI outcomes. Also, some basic theorems in the CIB theory rely only upon (2).

Fig.6 gives three examples, where the first two satisfy only IM but violates CM and the third violates IM but not CM.[10] Neither game requires interactive decision making in that in the first two, player 1 can simply choose action d at x_1 because no future possibilities give higher payoffs[11] and in the third, each player chooses c. These games are free from the centipede paradox. The centipede paradox is created by the conflict between IM and CM.

In contrast to pre-centipede games, centipede games contain germ of cooperation, which is expressed in Lemma 2.1.$\langle 1\rangle$. $\langle 2\rangle$ is the dual of $\langle 1\rangle$. A proof is given in Section 2.3.

Lemma 2.1(Wishfulness). Let G_n be a centipede game and let ℓ be a given number with $\ell \leq n$. Then,

$\langle 1\rangle$(Germ of cooperation) $g_{\pi(\ell)}(z_t) < g_{\pi(\ell)}(z_\ell)$ if $t < \ell$ or $t = \ell + 1$, and $g_{\pi(\ell-1)}(z_t) < g_{\pi(\ell-1)}(z_{\ell+1})$ if $t < \ell + 1$;

[9] Crosetto-Mantovani [3] studied the example given as $(g_1(z_t), g_2(z_t)) = (4t, t)$ if t is odd and $(2t, 4t)$ if t is even. This formula itself is linear, but the game is an sk-convex centipede game. In this example, the case $n = 100$ is less absurd.

[10] The first example violates C5 but is modified, with small perturbations, keeping C5 as well as the BI result.

[11] e.g., the dominant action choice is enough.

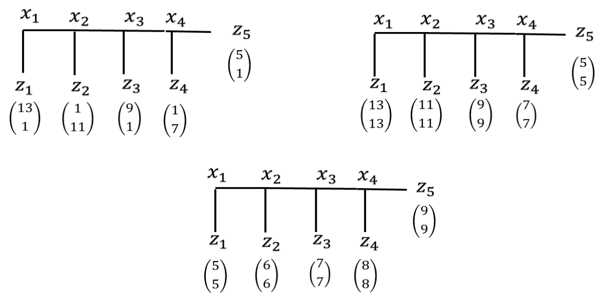

Figure 6: Games violating IM or CM

$\langle \mathbf{2} \rangle$**(Dual of** $\langle 1 \rangle$**)** $g_{\pi(\ell)}(z_\ell) < g_{\pi(\ell)}(z_t)$ if $t > \ell + 1$, and $g_{\pi(\ell-1)}(z_{\ell+1}) < g_{\pi(\ell-1)}(z_t)$ if $t > \ell + 1$.

Let $\ell = n$. The first states that player $\pi(n)$ prefers end node z_n to the others and player $\pi(n-1)$ prefers end node z_{n+1} to the others. A difficulty is that the most preferred end nodes do not coincide; here, IM prevents cooperation: Thus, it is still a germ of cooperation. This fact holds for any $\ell \leq n$, which is the exact assertion of in $\langle 1 \rangle$. IM does not stop ℓ to go to 1. $\langle 2 \rangle$ is the dual of $\langle 1 \rangle$.

2.2 The BI theory

We call a function $\sigma_i : \{x_t \in X_n : \pi(t) = i\} \to \{c, d\}$ a *behavioral plan* of player i. Let $\sigma = (\sigma_1, \sigma_2)$ be a *pair of behavioral plans*. For simplicity, we write $\sigma(x_t)$ for $\sigma_i(x_t)$ with $i = \pi(x_t)$. We may regard σ as an n-vector in $A^n = A \times \cdots \times A$, where $A = \{c, d\}$; we write a vector $(a_1, \cdots, a_n)$ as $a_1 \cdots a_n$, and, the d-behavior, c-behavior are $d^n = d \cdots d$, $c^n = c \cdots c$.

We say that $\sigma_{x_t|i}$ is the behavioral plan *conditional upon* x_t ($t \leq n$) iff the domain of σ_i is restricted to the set $\{x_s : s \geq t \text{ and } \pi(s) = i\}$. The pair $(\sigma_{x_t|1}, \sigma_{x_t|2})$ of behavioral plans conditional upon x_t is denoted as $\sigma_{x_t|}$, and we write $\sigma_{x_t|}(x_{t'})$ for $\sigma_{x_t|\pi(t')}(x_{t'})$. The *realization* $r(\sigma_{x_t|}) = z_s$ *of a given pair* $\sigma_{x_t|} = (\sigma_{x_t|1}, \sigma_{x_t|2})$ means that under the hypothesis that decision node x_t is reached, each player i follows σ_i until d is chosen at x_s, or the game goes to endnode $z_s = z_{n+1}$. We write $r(\sigma_{x_1|})$ simply as $r(\sigma)$.

At x_t, if he chooses action d, the game goes immediately to endnode z_t, but if he chooses c, then he expects the outcome $r(\sigma_{x_{t+1}|})$ under the hypothesis that the players follow their behavioral plan $\sigma_{x_{t+1}|}$. The choice is based on P1 (complete comparability), by making a comparison between $g_{\pi(t)}(z_t)$ and $g_{\pi(t)}(r(\sigma_{x_{t+1}|}))$. When $t = n$, the predicted outcome from $\sigma_{x_{n+1}|}$ is z_{n+1}.

We say that a pair of behavioral plans $\sigma = (\sigma_1, \sigma_2)$ is a *BI solution* iff for any x_t ($t = 1, ..., n$),

$$\sigma(x_t) = \begin{cases} c & \text{if } g_{\pi(t)}(r(\sigma_{x_{t+1}|})) > g_{\pi(t)}(z_t) \\ d & \text{if } g_{\pi(t)}(r(\sigma_{x_{t+1}|})) < g_{\pi(t)}(z_t). \end{cases} \tag{7}$$

The postulates P1 and P2 are used in (7); without P1, the two cases in (7) might not be exhaustive and the definition is incomplete. In these cases, the choice is determined solely by the future possible payoffs, which is P2. On the other hand, (7) does not rely upon P0M in the sense that for a given σ, (7) checks if it satisfies the right-hand side. We prove, using mathematical induction, the unique existence of a BI solution, though it is not necessarily the

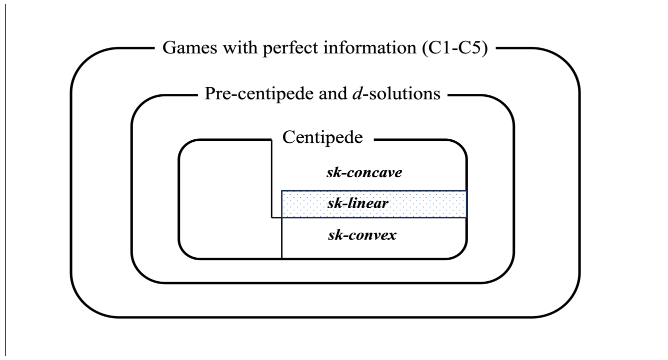

Figure 7: Classification of pre-centipede games

d-behavior d^n. This needs neither IM nor CM.

Lemma 2.2. (Unique existence): Let G_n be a derived game given by C1 to C5. Then, G_n has a unique BI solution σ.

Now, IM is necessary and sufficient for the BI solution σ to be the d-behavior.

Theorem 2.1 (Pre-centipede and the d-behavior): Let G_n be a derived game given by C1 to C5. Then, G_n is a pre-centipede game if and only if the d-behavior d^n is a BI solution.

Lemma 2.2 says that the unique existence of a BI-solution is obtained for any game with C1 to C5. Theorem 2.1 states that the BI solution serving the d-behavior is equivalent solely to IM. Thus, the class of games having the d-behavior is that of pre-centipede games including the games in Fig.6. Fig.7 depicts the subclasses of the pre-centipede games. To have a resolution of the centipede paradox, we need CM as well as IM. that is, the class of centipede games.

2.3 The Selten people's refusal of the BI theory

Let us interpret the quotation (*) in our context. Three key terms are relevant:

(i) mathematically trained persons;

(ii) the logical validity of the induction argument;

(iii) a guide to practical behavior.

The third key term (iii) is indicative that the Selten people are boundedly rational ordinary people, though they are quite rational in the sense of (i) and (ii). They understand the induction argument but not very details of the argument.

As stated in (iii), the Selten people expect that the theory could help how they should behave in such a game. Looking at the centipede game in Fig.1, they may find that it would be better to wait to go far from the start x_1. This is the implication of Lemma 2.1. Nevertheless, it does not yet suggest to deviate from the BI theory. If they ask what causes only wishfulness, they may understand P1 and P2 as the cause.

Section 3 discusses how the BI theory is modified by weakening P1 and P2. These modifications look related to O*1* (and O*2*), but rigorously speaking, the bases for the modifications are independent of O*1*.

2.4 Proofs

Proof of Lemma 2.1. $\langle 2\rangle$ is the dual of $\langle 1\rangle$. We prove only the left assertion of $\langle 1\rangle$; the right assertion is proved in a simpler manner. $\langle 2\rangle$ is proved in the dual manner.

Let x_t be any decision node with $t < \ell$ and $\pi(t) = \pi(\ell)$. By (2), we have $g_{\pi(\ell)}(z_{t+1}) < g_{\pi(\ell)}(z_t)$, and by (4), $g_{\pi(\ell)}(z_{t-1}) < g_{\pi(\ell)}(z_t)$. Also, by (3), $g_{\pi(\ell)}(z_{t-2}) < g_{\pi(\ell)}(z_t)$. These are summarized; for all t with $\pi(t) = \pi(\ell)$,

$$g_{\pi(\ell)}(z_{t+1}) < g_{\pi(\ell)}(z_t), g_{\pi(\ell)}(z_{t-1}) < g_{\pi(\ell)}(z_t), \text{ and } g_{\pi(\ell)}(z_{t-2}) < g_{\pi(\ell)}(z_t). \tag{8}$$

In this expression, all end nodes occur for $t < \ell$ or $t = \ell + 1$. Since $g_{\pi(\ell)}(z_t) < g_{\pi(\ell)}(z_\ell)$ by the repeated use of (3), $g_{\pi(\ell)}(z_t) < g_{\pi(\ell)}(z_\ell)$ if $t < \ell$ or $t = \ell + 1$.■

Proof of Lemma 2.2. Let σ be a BI solution. We prove by mathematical induction that for any $t \leq n$, $\sigma_{x_t|}(x_s)$ is uniquely determined. The induction base is the case $x_t = x_n$. By (7), we have the unique $\sigma(x_n)$. Let x_t be any node in X_n with $t > 1$. Suppose the induction hypothesis that each of $\sigma(x_n), ..., \sigma(x_{t+1})$ is uniquely determined. This hypothesis determines $r(\sigma_{x_{t+1}|}) = z_{t'}$ uniquely. By C5, either $g_{\pi(t)}(z_t) > g_{\pi(t)}(z_{t'}) = g_{\pi(t)}(r(\sigma_{x_{t+1}|}))$ or $g_{\pi(t)}(z_t) < g_{\pi(t)}(z_{t'}) = g_{\pi(t)}(r(\sigma_{x_{t+1}|}))$. By (7), we have $\sigma(x_t) = d$ in the first case, and we have $\sigma(x_t) = c$ in the second case. This choice is unique. By the principle of mathematical induction, σ is uniquely determined and satisfies (7) for all $x_t \in X_n$.■

Proof of Theorem 2.1. The only-if part is straightforward. We show the if part. Consider the contrapositive of this. Suppose that (2) does not hold. Then, $g_{\pi(t)}(z_t) < g_{\pi(t)}(z_{t+1})$ for some $t \leq n$, since $g_{\pi(t)}$ is a 1-to-1 function by C5. Let t be the maximum in such t's. When $t = n$, we have $g_{\pi(n)}(z_n) < g_{\pi(n)}(z_{n+1})$; thus $\sigma(x_n) = c$. Let $t < n$. Then, $\sigma(x_s) = d$ for all $s > t$. Hence, choice c at x_t leads to x_{t+1} and then to z_{t+1}. But this gives a larger payoff than $g_{\pi(t)}(z_t)$; by (7), we have $\sigma(x_s) = c$. Thus, we have the contrapositive of the if assertion.■

3 Measurements of Payoffs and Incomparabilities

Here, we introduce cognitive bounds to a game G_n, based on Kaneko's [9] expected utility theory with probability grids. In this theory, each player uses a finite ruler to measure payoffs; its precision depends upon his cognitive ability. When the ruler is precise enough, he can separate each payoff from the others and his preferences are complete; in this case, the results given in Section 2 hold. In the imprecise case, he faces Incomparabilities over such payoffs. In Section 4, we will modify the BI solution into the CIB solution.

3.1 Cognitive bounds and Incomparabilities

Let $G_n = (X_n, Z_n, \pi, \{c, d\}, (g_1, g_2))$ be a game with C1 to C5. We introduce the cognitive abilities of payoffs for the players, adding two components $\Sigma = (\Sigma_1, \Sigma_2)$ and $b = (b_1, b_2)$ to G_n:

C6 $\Sigma_i = (\rho_i, [\underline{\gamma}_i, \overline{\gamma}_i])$ consists of a cognitive degree $\rho_i \geq 0$ and a pair of *lower and upper bounds* $\underline{\gamma}_i, \overline{\gamma}_i$ of payoffs with $\underline{\gamma}_i < g_i(z_t) < \overline{\gamma}_i$ for all $z_t \in Z_n$ and $i = 1, 2$;

C7 b_i $(1 \leq b_i \leq n)$ is the *consciousness boundary* with $\pi(b_i) = i$ for $i = 1, 2$.

The pair $\Sigma_i = (\rho_i, [\underline{\gamma}_i, \overline{\gamma}_i])$ defines a bounded ruler for payoffs. The boundary b_i will be used in the definition of a CIB solution to separate inertial behavior from conscious behavior. We denote G_n associated with (Σ, b) by $G_n(\Sigma, b)$. When G_n is a centipede (pre-centipede) game, we simply call $G_n(\Sigma, b)$ a centipede (pre-centipede) game.

The BI theory is modified based on $\Sigma_i, i = 1, 2$. The modification is a restriction of expected utility theory in that the set of available probabilities is a finite and gets more accurate as the cognitive ability of player i becomes high. Kaneko [9] developed a general theory in an axiomatic manner. The full theory is costly for consideration the centipede paradox. We adopt the risk neutrality case of Kaneko's theory; in doing so, we can skip the axiomatic development directly to the payoff evaluations of payoffs while keeping incomparabilities of some payoffs. This sacrifices some important cases such as risk neutrality, but it is not in the central part of our resolution of the paradox.

When a cognitive degree $\rho_i \geq 0$ of player i is given, the set of available *probability grids* is given as $\{\frac{\nu}{2^{\rho_i}} : 0 \leq \nu \leq 2^{\rho_i}\}$. The probability grids are used to define the *payoff ruler* for i as the set of *simple lotteries* that are probability distributions over $\overline{\gamma}_i$ with probability $\frac{\nu}{2^{\rho_i}}$ and $\underline{\gamma}_i$ with the remaining probability $1 - \frac{\nu}{2^{\rho_i}}$ $(0 \leq \nu \leq 2^{\rho_i})$. Here, we assume risk-neutrality on evaluations of simple lotteries; hence, each simple lottery is evaluated as the expected payoff $\frac{\nu}{2^{\rho_i}} \cdot \overline{\gamma}_i + (1 - \frac{\nu}{2^{\rho_i}}) \cdot \underline{\gamma}_i$. The payoff ruler for i is given as the set of expected values:

$$\Lambda_{\rho_i} = \{\frac{\nu}{2^{\rho_i}} \cdot \overline{\gamma}_i + (1 - \frac{\nu}{2^{\rho_i}}) \cdot \underline{\gamma}_i : 0 \leq \nu \leq 2^{\rho_i}\}. \tag{9}$$

Each value of Λ_{ρ_i} is called a *payoff scale* and is denoted by $\lambda_{\rho_i}(\nu)$.[12] Preferences over the payoff ruler Λ_{ρ_i} are *completely* defined by the expect value $\lambda_i(\nu)$; that is, for all ν with $0 \leq \nu \leq 2^{\rho_i}$,

$$\lambda_i(\nu) > \lambda_i(\nu') \text{ if and only if } \nu > \nu'. \tag{10}$$

That is, the payoff scales are linearly ordered by the order on the natural numbers ν.

The payoff ruler plays a crucial role for considerations of the centipede paradox. However, the central part is the evaluation of the payoffs $\{g_i(z_t) : z_t \in Z_n\} \cup \{\underline{\gamma}_i, \overline{\gamma}_i\}$. In order to avoid complications, we add artificial end nodes z_0, z_{n+2}, so that

$$\underline{\gamma}_i = g_i(z_0) \text{ and } \overline{\gamma}_i = g_i(z_{n+2}) \text{ for } i = 1, 2. \tag{11}$$

Then, let $Z_n^* = Z_n \cup \{z_0, z_{n+2}\}$. The game structure of $G_n(\Sigma, b)$ is not changed; it has only additional symbols z_0, z_{n+2} so as to have uniform expressions. That is, the set of possible payoffs is denoted as

$$\Gamma_i = \{g_i(z_t) : z_t \in Z_n\} \cup \{\underline{\gamma}_i, \overline{\gamma}_i\} = \{g_i(z_t) : z_t \in Z_n^*\}. \tag{12}$$

By this, we can focus only on preferences on the set Z_n^*, instead of mixing the endnodes Z_n with lower and upper bounds of payoffs $\underline{\gamma}_i, \overline{\gamma}_i$.

Payoff comparisons over Z_n^* are central for our development, but these are not directly made. Instead, they are based on the payoff ruler Λ_{ρ_i}. The first step is to evaluate each of two payoff values with the payoff ruler. Then, the second step is to make comparisons between the payoffs from two end nodes in Z_n^*, based on the evaluations of the payoffs given in the first step. This is formally described in the following binary relations of $\rhd_i$ and $\bowtie_i$ over Z_n^*.

[12] Exactly speaking, the set Λ_{ρ_i} should be expressed as Λ_{i,ρ_i} since each element depends upon $\underline{\gamma}_i$ and $\overline{\gamma}_i$. However, this makes no confusions.

Formally, we say that for endnodes z_t, $z_{t'}$ in Z_n^*, player i *strictly prefers* z_t to $z_{t'}$, denoted by $z_t \rhd_i z_{t'}$, iff for some $\lambda_i(\nu)$ in Λ_{ρ_i},

$$g_i(z_t) \geq \lambda_i(\nu) \geq g_i(z_{t'}). \tag{13}$$

Since at least one inequality is strict by C5 and C6, $\lambda_i(\nu)$ separates strictly between $g_i(z_t)$ and $g_i(z_{t'})$. Thus, player i finds the payoff scale $\lambda_i(\nu)$ separating between $g_i(z_t)$ and $g_i(z_{t'})$, and thinks that $g_i(z_t)$ is better than $g_i(z_{t'})$. When he does no succeed in finding such a payoff scale, $g_i(z_t)$ and $g_i(z_{t'})$ are incomparable for him. For any distinct $z_t, z_{t'} \in Z_n^*$, we define the *incomparability relation* $z_t \bowtie_{\rho_i} z_{t'}$ by

$$\lambda_i(\nu+1) > g_i(z_t), g_i(z_{t'}) > \lambda_i(\nu) \tag{14}$$

for some ν $(0 \leq \nu < 2^{\rho_i} - 1)$. That is, the two payoffs are strictly between the adjacent payoff scales. When $z_t \rhd_i z_{t'}$, by (13), z_t and $z_{t'}$ are distinct. In the following, when we talk about two end nodes in Z_n^*, we assume that they are distinct.

We have the *trichotomy* result: for any endnodes z_t and $z_{t'}$ in Z_n^*,

$$\text{exactly one of } z_t \rhd_i z_{t'},\ z_{t'} \rhd_i z_t \text{ and } z_t \bowtie_i z_{t'} \text{ holds.} \tag{15}$$

This is because the negation of (13) is (14) as long as z_t and $z_{t'}$ are distinct. Now, we define the relation $z_t \trianglerighteq_i z_{t'}$ iff $z_t \rhd_i z_{t'}$ or $z_t \bowtie_i z_{t'}$. We have the following fact, which is familiar in the standard case with strict preferences and indifferences. This is verified by observing (13), (14), and is summarized in Lemma 3.1.

Lemma 3.1.$\langle\mathbf{1}\rangle$: Relations $\rhd_i$, $\bowtie_i$, and $\trianglerighteq_i$ are transitive;

$\langle\mathbf{2}\rangle$**(Completeness with incomparabilities)**: the relation $\trianglerighteq_i$ is a complete transitive relation over the pairs of distinct end nodes in Z_n^*.[13]

It follows from Lemma 3.1 that for distinct $z_t, z_{t'}, z_{t''}$,

$$z_t \bowtie_{\pi(t)} z_{t'} \text{ and } z_{t'} \trianglerighteq_{\pi(t)} z_{t''} \text{ imply } z_t \trianglerighteq_{\pi(t)} z_{t''}; \tag{16}$$

$$z_{t''} \trianglerighteq_{\pi(t)} z_{t'} \text{ and } z_{t'} \bowtie_{\pi(t)} z_t \text{ imply } z_{t''} \trianglerighteq_{\pi(t)} z_t.$$

In these, we can substitute $\rhd_{\pi(t)}$ for all occurrences of $\trianglerighteq_{\pi(t)}$. These will be often used.

The two statements of Lemma 3.2 are simple observations. For the second, we denote the unit interval of the payoff ruler by $\Delta_{\rho_i} = \frac{\overline{\gamma}_i - \underline{\gamma}_i}{2^{\rho_i}}$. This is the unit interval of the payoff ruler $\lambda_i(\cdot)$, i.e., $\lambda_i(\nu+1) - \lambda_i(\nu) = \Delta_{\rho_i}$.

Lemma 3.2.$\langle\mathbf{1}\rangle$: If $g_i(z_t) > g_i(z_{t'})$, then $z_t \trianglerighteq_{\rho_i} z_{t'}$;

$\langle\mathbf{2}\rangle$: If $z_t \bowtie_{\rho_i} z_{t'}$, then $g_i(z_t) - g_i(z_{t'}) < \Delta_{\rho_i}$; equivalently, if $g_i(z_t) - g_i(z_{t'}) \geq \Delta_{\rho_i}$, then $z_t \rhd_{\rho_i} z_{t'}$.

The converse of $\langle 2 \rangle$ does not necessarily hold. We need to be careful when we go from the case $g_i(z_t) - g_i(z_{t'}) \geq \Delta_{\rho_i}$ to the case $g_i(z_t) - g_i(z_{t'}) < \Delta_{\rho_i}$.

Fig.8 describes how the payoff ruler is used. Two (payoffs of) end nodes are compared through the payoff ruler but not directly. The first step is to compare a given end node with the payoff ruler, and the second is to make comparisons between two end nodes through the compared payoff scales.

[13] Lemma 3.1 does not hold in a more general framework allowing probability distributions other than the simple lotteries in Kaneko [9], specifically, Theorem 6.2, p.751. $\langle\mathbf{2}\rangle$ implies that some real-valued function v_i over Z_n represents the relation $\trianglerighteq_i$. We, however, should not forget that $\bowtie_i$ means incomparability but not indifference.

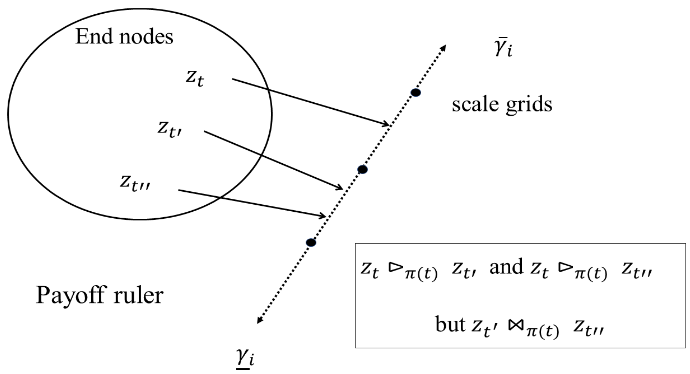

Figure 8: Payoff ruler measures end nodes

Here, we give one theorem showing that for a centipede game, our theory gives a new light on the centipede paradox. Recall Lemma 2.1 that a germ of cooperation is hidden in a centipede game but it is suppressed by Individual Motive (2). Theorem 3.1 states that our theory removes this suppression, but this is not enough to go further. For this, Section 4 will add one more concept "inertia".

Theorem 3.1 (Germination of Cooperation). Let $G_n(\Sigma, b)$ be a centipede game and let x_ℓ be a decision node. Then,

$\langle \mathbf{1} \rangle$ if $z_{\ell+1} \bowtie_{\rho_{\pi(\ell)}} z_\ell$, then $z_{\ell+1} \unrhd_{\rho_{\pi(t)}} z_t$ for all t with $t \le \ell$;

$\langle \mathbf{2} \rangle$ if $z_{\ell+1} \rhd_{\rho_{\pi(t)}} z_t$, then $z_{\ell+1} \rhd_{\rho_{\pi(t)}} z_{t'}$ for all $t' < t$ with $\pi(t') = \pi(t)$.

Let $\ell = n$. Let $z_{n+1} \bowtie_{\rho_{\pi(n)}} z_n$. Then, player $\pi(t)$ at x_t with $t \le n$ with $\pi(t) = \pi(n)$ prefers z_{n+1} mostly in the sense of $\unrhd_{\rho_i}$. The point is that both players may agree to go to z_{n+1}, while in Lemma 2.1, player $\pi(n)$ prefers z_n mostly but $\pi(n-1)$ prefers z_{n+1} mostly. This incoincidence makes IM effective in the BI theory. In our theory, Theorem 3.1 removes this incoincidence. For this argument to induce the behavior to take c, however, we need one more concept, inertial behavior, which will be discussed in Section 4. In the situation where $\ell < n$, the backward induction inducing the d-behavior after decision node x_ℓ.

Remark 3.1. Some authors have introduced a similarity relation in the standard EU theory (cf., Rubinstein [23]). The EU theory with probability grids with $\Gamma_{n,i}$ and Λ_{ρ_i} differs ontologically from a typical similarity theory in that our theory is a purely finite construct. On the other hand, a typical similarity theory is constructed as a part of the EU theory with a continuum of probabilities and, possibly, with a continuum of monetary payoffs. In our theory, Kaneko [9], Section 7, showed that when ρ_i goes to infinity, it converges to a part of the standard EU theory.

3.2 Proofs

Proof of Lemma 3.2 $\langle \mathbf{1} \rangle$. Let $\gamma_i = g_i(z_t) > g_i(z_{t'}) = \gamma_i'$. We have not $(z_{t'} \rhd_{\rho_i} z_t)$ by (13). By (15), $z_t \rhd_{\rho_i} z_{t'}$ or $z_t \bowtie_{\rho_i} z_{t'}$. Hence, $z_t \unrhd_{\rho_i} z_{t'}$.

$\langle \mathbf{2} \rangle$. Let $z_t \bowtie_{\rho_i} z_{t'}$. By (22), $\pi_i(\nu + 1) > g_i(z_t), g_i(z_{t'}) > \pi_i(\nu)$ for some ν. Since $\Delta_{\rho_i} = \pi_i(\nu + 1) - \pi_i(\nu)$. Hence, $\Delta_{\rho_i} > g_i(z_t) - g_i(z_{t'})$.■

Proof of Theorem 3.1.$\langle \mathbf{1} \rangle$. Let $z_{\ell+1} \bowtie_{\rho_{\pi(\ell)}} z_\ell$. First, consider the case when ℓ is even. For the assertion that $z_{\ell+1} \trianglerighteq_{\rho_{\pi(t)}} z_t$ for all t with $t \leq \ell$, we show the equivalent assertion that for any even $k < \ell$ including $k = 0$,

$$z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-k)}} z_{\ell-k} \text{ and } z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-(k+1))}} z_{\ell-(k+1)}. \tag{17}$$

The equivalence is verified as follows: Let $k = 0$, (17) is written as $z_{\ell+1} \trianglerighteq_{\pi(\ell)} z_\ell$ and $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-1)}} z_{\ell-1}$, and for arbitrary even $k < \ell$, (17) is written as $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-k)}} z_{\ell-k}$ and $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-(k+1))}} z_{\ell-(k+1)}$. Finally, when $k = \ell - 2$, this is written as $z_{\ell+1} \trianglerighteq_{\rho_{\pi(2)}} z_2$ and $z_{\ell+1} \trianglerighteq_{\rho_{\pi(1)}} z_1$. These three cases are written as $z_{\ell+1} \trianglerighteq_{\rho_{\pi(t)}} z_t$ for all t with $t \leq \ell$.

Now, let prove (17) by induction on even k up to $\ell - 2$. The induction base is (17) for $k = 0$; the left comparison is the assumption $z_{\ell+1} \bowtie_{\rho\pi(\ell)} z_\ell$. Consider the right comparison. Since $g_{\pi(\ell-1)}(z_{\ell+1}) > g_{\pi(\ell-1)}(z_{\ell-1})$ by (3), we have $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-1)}} z_{\ell-1}$ by Lemma 3.2.$\langle 1 \rangle$; we have (17) for $k = 0$. The remaining is the inductive step. Suppose the induciton hypothesis that (17) holds for a given even $k < \ell - 1$. Consider $k + 2$. Since $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-k)}} z_{\ell-k}$ by the induction hypothesis and $z_{\ell-k} \trianglerighteq_{\rho_{\pi(\ell-(k+2))}} z_{\ell-(k+2)}$ by (3) with $\pi(\ell - k) = \pi(\ell - (k + 2))$, we have $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-(k+2))}} z_{\ell-(k+2)}$ by transitivity for $\trianglerighteq_{\rho\pi(\ell-(k+2))}$, which is the left formula for $k+2$. Since $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-(k+1))}} z_{\ell-(k+1)}$ by the induction hypothesis and $z_{\ell-(k+1)} \trianglerighteq_{\rho_{\pi(\ell-(k+3))}} z_{\ell-(k+3)}$ by (3), we have $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-(k+3))}} z_{\ell-(k+3)}$ by transitivity, which is the right formula of (17) for $k+2$. By the induction principle, we have the assertion.

Consider the case where ℓ is odd. we show that for any odd $k < \ell$,

$$z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-k)}} z_{\ell-k} \text{ and } z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell-(k+1))}} z_{\ell-(k+1)}; \tag{18}$$

This does not include $z_{\ell+1} \trianglerighteq_{\rho_{\pi(\ell)}} z_\ell$, but it is the assumption. Hence, it suffices to show (18) for any odd $k < \ell$. This is done in the same manner as in the previous paragraph.

$\langle \mathbf{2} \rangle$. Let $z_{\ell+1} \triangleright_{\rho_{\pi(t)}} z_t$. Let $t' < t$ with $\pi(t') = \pi(t)$. Since $g_{\pi(t)}(z_t) > g_{\pi(t)}(z_{t'})$, we have $z_t \trianglerighteq_{\rho_{\pi(t)}} z_{t'}$ by Lemma 3.2.$\langle 1 \rangle$. Then, we have $z_{\ell+1} \triangleright_{\rho_{\pi(t)}} z_{t'}$ by (16).■

4 The CIB Theory

We modify the BI solution into the CIB solution based on the EU theory with probability grids and the other concept "inertia". General results such as existence and uniqueness properties on a CIB solution are given for pre-centipede games. Directly relevant results to the centipede paradox are given in Section 5.

4.1 Consciousness vs. inertial; a CIB solution

The distinction between conscious choices and inertial behavior plays a crucial role in the CIB theory. Within the boundary $b_{\pi(t)}$, player $\pi(t)$ consciously thinks about his choice; when he meets incomparability, his choice is arbitrary between c and d. After $b_{\pi(t)}$, he follows the inertia, i.e., (#) in Section 1. The idea is described in the present context; because reaching x_t requires the repetition of the same action c up to x_{t-1}, player $\pi(t)$ follows the inertial action c unless d is strictly preferred to c.

The latter part needs formal definitions. Consider a decision node x_t with $t > b_{\pi(t)}$. We say that player $\pi(t)$ takes a *conscious choice at* x_t in σ iff

$$z_t \triangleright_{\pi(t)} r(\sigma_{x_{t+1}|}) \text{ and } \sigma(x_t) = d, \tag{19}$$

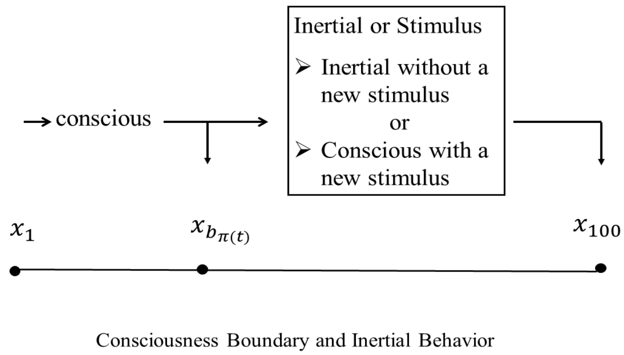

Figure 9: Consciousness boundary and inertial behavior

and he follows *inertial behavior at* x_t in σ iff

$$r(\sigma_{x_{t+1}|}) \trianglerighteq_{\pi(t)} z_t \text{ and } \sigma(x_t) = c. \tag{20}$$

These are a formulation of the interpretation of (#) in Section 1. In Fig.8, (19) is applied to all the region from x_1 to x_n, and (20) is to the the later segment for inertial behavior. Based on these ideas, we modify the definition (7) of a BI solution. Let $G_n(\Sigma, b)$ be a game satisfying C1 to C7.

Definition 4.1 (CIB solution). A pair of behavioral plan $\sigma = (\sigma_1, \sigma_2)$ is a *CIB solution* in $G_n(\Sigma, b)$ iff for any decision node x_t $(t \leq n)$,

$$\sigma(x_t) = \begin{cases} d & \text{if } z_t \rhd_{\pi(t)} r(\sigma_{x_{t+1}|}) \\ c \text{ or } d & \text{if } r(\sigma_{x_{t+1}|}) \bowtie_{\pi(t)} z_t \;\;\&\;\; t \leq b_{\pi(t)} \\ c & \text{if } r(\sigma_{x_{t+1}|}) \rhd_{\pi(t)} z_t \;\;\&\;\; t \leq b_{\pi(t)} \\ c & \text{if } r(\sigma_{x_{t+1}|}) \trianglerighteq_{\pi(t)} z_t \;\;\&\;\; t > b_{\pi(t)}. \end{cases} \tag{21}$$

The first line was suggested by (19), the second and third suggests different behavior. The fourth is (20). These cases are exclusive and exhaustive; thus, (21) is well defined. Postulate P1 of Section 1 is violated in the $2nd$ and $4th$ lines, and P2 is violated in the $4th$ line in that the behavior depends upon the previous choices. We note that if $z_t \rhd_{\pi(t)} r(\sigma_{x_{t+1}|})$ or $r(\sigma_{x_{t+1}|}) \rhd_{\pi(t)} z_t$ holds for all $t \leq n$, (21) is equivalent to the definition (7) of a BI solution; thus, incomparability $r(\sigma_{x_{t+1}|}) \bowtie_{\pi(t)} z_t$ occurs for some t if and only if the CIB solution differs from the BI solution.

The CIB theory allows a variety of possible solutions including the c-behavior and the d-behavior, depending upon the payoff functions and the parameters (Σ, b). First, we analyze the general behavior of a CIB solution. Theorem 4.1 states the existence of a CIB solution for a game $G_n(\Sigma, b)$ with C1 to C7, and gives two uniqueness statements on a CIB solution in $G_n(\Sigma, b)$.

Theorem 4.1. Let $G_n(\Sigma, b)$ be a game with C1 to C7.

⟨1⟩(Existence) $G_n(\Sigma, b)$ has at least one CIB solution.

⟨2⟩(Unique trajectory after $\boldsymbol{b = (b_1, b_2)}$) If σ and σ' are CIB solutions, then, $\sigma(x_t) = \sigma'(x_t)$

for all t with $t > b_{\pi(t)}$.

$\langle \mathbf{3} \rangle$**(Uniqueness for the entire domain)** σ is a unique CIB solution in $G_n(\Sigma, b)$ if and only if any CIB solution σ' satisfies

$$\text{for any } x_t \text{ with } t \leq b_{\pi(t)},\ \sigma(x_t) = \begin{cases} c & \text{if } r(\sigma'_{x_{t+1}|}) \rhd_{\pi(t)} z_t \\ d & \text{if } z_t \rhd_{\pi(t)} r(\sigma'_{x_{t+1}|}). \end{cases} \tag{22}$$

The structure of the set of CIB solutions varies with payoff functions, parameters $\Sigma = (\Sigma_1, \Sigma_2)$ and $b = (b_1, b_2)$. The two uniqueness claims are useful for the subsequent developments of the paper. In order to study a CIB solution for a centipede game $G_n(\Sigma, b)$, we focus on the *adjacent preferences*: $z_t \rhd_t z_{t+1}$ or $z_t \bowtie_t z_{t+1}$ for $t = 1, ..., n$. We consider the following two cases:

FC $G_n(\Sigma, b)$ is *fully comparable* iff $z_t \rhd_t z_{t+1}$ for all $t = 1, ..., n$;

PIC $G_n(\Sigma, b)$ is *partially incomparable* iff $z_t \bowtie_t z_{t+1}$ for some $t = 1, ..., n$.

Case PIC is the key for a resolution of the centipede paradox. In this case, Theorem 3.1 (Germination of cooperation) is indicative that the choice the maximum of such t's leads to a CIB solution, which will be discussed in Section 5. Nevertheless, either of Cases PIC and FC occurs with different values of cognitive abilities $\rho_i, i = 1, 2$. A resolution of the centipede paradox requires to consider these two cases.

First, consider the condition for the d-solution. Recall $\Delta_{\rho_{\pi(t)}} := \frac{\overline{\gamma}_i - \underline{\gamma}_i}{2^{\rho_i}} = \pi_i(\nu + 1) - \pi_i(\nu)$ $(0 \leq \nu < 2^{\rho_i})$.

Theorem 4.2 (Conditions for the d-behavior) Let $G_n(\Sigma, b)$ be a pre-centipede game with bounds (Σ, b).

$\langle \mathbf{1} \rangle$ The d-behavior $\sigma = d^n$ is a unique CIB solution if and only if $G_n(\Sigma, b)$ is fully comparable.

$\langle \mathbf{2} \rangle$(**Sufficient condition**) $G_n(\Sigma, b)$ is fully comparable if

$$g_{\pi(t)}(z_t) - g_{\pi(t)}(z_{t+1}) \geq \Delta_{\rho_{\pi(t)}} \quad \text{for all } t \leq n. \tag{23}$$

For a pre-centipede game G_n without bounds, (2) is enough to guarantee the d-solution d^n, as stated in Theorem 2.1.$\langle 1 \rangle$. When it has bounds (Σ, b), FC is necessary and sufficient to have the d-behavior. Then $\langle 2 \rangle$ gives a sufficient condition for FC. When the game is sk-concave, the inequality (23) only for $t = n$ implies the entire (23).

4.2 Proofs

Proof of Theorem 4.1.$\langle \mathbf{1} \rangle$ We construct $\sigma = (\sigma_1, \sigma_2)$ by induction from the last decision node x_n. As the induction base, we define

$$\sigma(x_n) = \begin{cases} c & \text{if } z_{n+1} \trianglerighteq_{\pi(n)} z_n \\ d & \text{if } z_n \rhd_{\pi(n)} z_{n+1}. \end{cases} \tag{24}$$

This is well defined by trichotomy (15). Suppose the induction hypothesis that we have constructed $\sigma(x_n), ..., \sigma(x_{t+1})$ so that each takes the value c or d. These are regarded as the conditional behavioral play $\sigma_{x_{t+1}|}$ upon the hypothesis that x_{t+1} is reached. Now, we define $\sigma(x_t)$

by

$$\sigma(x_t) = \begin{cases} c & \text{if } r(\sigma_{x_{t+1}|}) \trianglerighteq_{\pi(t)} z_t \\ d & \text{if } z_t \vartriangleright_{\pi(t)} r(\sigma_{x_{t+1}|}). \end{cases} \tag{25}$$

By the mathematical induction principle, we have $\sigma(x_n), ..., \sigma(x_1)$. Thus, we have a pair of behavioral plans $\sigma = (\sigma_1, \sigma_2)$.

It remains to show that σ satisfies (21) for $t \leq n$. When $t = n$, (24) implies that $\sigma_{\pi(n)}(x_n)$ satisfies (21) for $t = n$. Now, let t be arbitrary with $1 \leq t < n$. If $z_t \vartriangleright_{\pi(t)} r(\sigma_{x_{t+1}|})$, then, the first case of (21) holds for t. Now, if $r(\sigma_{x_{t+1}|}) \vartriangleright_{\pi(t)} z_t$ or $[r(\sigma_{x_{t+1}|}) \bowtie_{\pi(t)} z_t \ \& \ t > b_{\pi(t)}]$, the first or third of (21) holds for t. The last case is $[r(\sigma_{x_{t+1}|}) \bowtie_{\pi(n)} z_t$ and $t \leq b_{\pi(t)}]$. Then, the second of the definition (21) of a CIB solution holds for t.

$\langle \mathbf{2} \rangle$ In the proof $\langle 1 \rangle$, the construction of σ is uniquely determined, and also, it corresponds uniquely to (21) when $t > b_{\pi(t)}$. Any CIB-solution σ' coincides with σ when $t > b_{\pi(t)}$.

$\langle \mathbf{3} \rangle$ $\langle 2 \rangle$ asserts uniqueness for the behavior after the boundaries, i.e., $t > b_{\pi(t)}$. The entire uniqueness of a CIB solution holds if and only if its behavior for each player i is unique at x_t within $t \leq b_i = b_{\pi(t)}$. The latter follows from (22); indeed, when (22) holds, we can use the induction from the boundary $b_{\pi(t)}$. Then, using the 1st and 3rd cases of (21), the behavior is uniquely determined. This is the *if* part of $\langle 3 \rangle$.

Consider the *only-if* part. We prove the contrapositive. Suppose that (22) does not hold for some $t \leq b_{\pi(t)}$. Then, we construct a CIB solution $\sigma' \neq \sigma$. Let t^o be the maximum among such t's. Then, $r(\sigma_{x_{t^o+1}|}) \bowtie_{\pi(t^o)} z_{t^o}$. Suppose that $\sigma_{x_{t^o+1}|}$ is already defined in $\langle 2 \rangle$. Now, we define $\sigma'(x_{t^o}) \neq \sigma(x_{t^o})$. Let $\sigma'_{x_{t^o}|} = (\sigma'(x_{t^o}), \sigma_{x_{t^o+1}|})$. Then, $\sigma'_{x_{t^o}|}$ satisfies (21) for $x_{t^o}, ..., x_n$. This is the induction base. Let s be a number with $1 < s \leq t^o$. Suppose the induction hypothesis that $\sigma'_{x_s|} = (\sigma'(x_s), \sigma'_{s+1|})$ satisfies(21) for $x_s, ..., x_n$. Then, we choose $\sigma'(x_{s-1})$ so that

$$\sigma'(x_{s-1}) = \begin{cases} c & \text{if } r(\sigma'_{x_s|}) \trianglerighteq_{\pi(t)} z_{s-1} \\ d & \text{if } z_{s-1} \vartriangleright_{\pi(t)} r(\sigma'_{x_s|}). \end{cases}$$

Then, this together with the induciton hypothesis implies that (21) for $x_{s-1}, ..., x_n$. Hence, by the induction principle, we have $\sigma' = \sigma'_{x_1|} = (\sigma'(x_1), \sigma'_{2|})$ satisfies (21). That is, σ' is a CIB solution but it differs from σ.■

Proof of Theorem 4.2 $\langle \mathbf{1} \rangle$ (Only-If) Let $\sigma = d^n$ be a unique CIB solution. Then, $r(\sigma_{x_{t+1}|}) = z_{t+1}$ for all $t = 1, ..., n-1$. By (21), $\sigma(x_t) = d$ implies $z_t \vartriangleright_{\pi(t)} z_{t+1}$ for all $t = 1, ..., n$.

(If) This is is proved by the induction from x_n.

$\langle \mathbf{2} \rangle$ Consider z_t, z_{t+1} satisfying (23). We can choose ν so that $\lambda_{\rho_i}(\nu+1) > g_i(z_{t+1}) \geq \lambda_{\rho_i}(\nu)$. Since $g_i(z_t) \geq g_i(z_{t+1}) + \Delta_{\rho_i}$ by (23), we have, $g_i(z_t) \geq g_i(z_{t+1}) + \Delta_{\rho_i} > \lambda_{\rho_i}(\nu) + \Delta_{\rho_i} = \lambda_{\rho_i}(\nu+1)$. Thus, $\lambda_{\rho_i}(\nu+1)$ separates between z_t and z_{t+1}; so, $z_t \vartriangleright_{\pi(t)} z_{t+1}$. Since t is arbitrary, we have $z_t \vartriangleright_{\pi(t)} z_{t+1}$ for all $t \leq n$. It follows from Theorem 3.2.$\langle 1 \rangle$ that a CIB solution is unique and $\sigma(x_t) = d$ for all $t \leq n$.■

5 Behavior of Canonical CIB Solutions for Centipede Games

The CIB solution is not uniquely determined for a centipede game $G_n(\Sigma, b)$. We adopt one type of a CIB solution, called *canonical*, as a representative of multiple CIB solutions, which is expressed as $c^\ell d^{n-\ell}$ for some ℓ $(0 \leq \ell \leq n)$. This brings about the study of the behavior of a CIB solution. In this section, we argue that for large cognitive abilities $\rho_i, i = 1, 2$, the canonical $c^\ell d^{n-\ell}$ is simply d^n but for small cognitive abilities, typically ℓ is around n including n. This supports the final step to our resolution of the centipede paradox in Section 6.

5.1 A canonical CIB solution for a centipede game

Let $G_n(\Sigma, b)$ be a centipede game with bounds (Σ, b). We say that ℓ is the *behaviral divide* iff

$$z_t \rhd_{\pi(t)} z_{t+1} \text{ for all } t \geq \ell + 1 \text{ and } z_\ell \bowtie_{\pi(\ell)} z_{\ell+1}. \tag{26}$$

We stipulate that $\ell = 0$ when there is no t with $z_t \bowtie_{\pi(t)} z_{t+1}$. Looking at adjacent endnodes from the end of the game, ℓ is the first place showing incomparability. We say that $\sigma = c^\ell d^{n-\ell}$ is the *canonical pair of behavior plans.* When $\ell = 0$, the canonical $\sigma = c^\ell d^{n-\ell}$ is d^n and is the unique CIB solution by Theorem 4.2.$\langle 1 \rangle$. Although there may be multiple CIB solutions in $G_n(\Sigma, b)$, the canonical pair $\sigma = c^\ell d^{n-\ell}$ is necessarily a CIB solution and it is regarded as a representative among the CIB solutions for a centipede games $G_n(\Sigma, b)$.

Let $\ell \geq 1$ be the behavioral divide. Since incomparability occurs at x_ℓ, Theorem 3.1.$\langle 1 \rangle$(germination of cooperation) states that the weak preference $z_{\ell+1} \unrhd_{\pi(t)} z_t$ holds for all $t \leq \ell$, and by (26), $z_t \rhd_{\pi(t)} z_{t+1}$ for all $t \geq \ell + 1$. This means that the canonical $c^\ell d^{n-\ell}$ is a CIB solution, which is the if part of Theorem 5.1.$\langle 1 \rangle$. The assertion $\langle 2 \rangle$ of Theorem 5.1 gives a condition for the uniqueness of a CIB solution, and $\langle 3 \rangle$ is its implication that after the boundaries b_i, any CIB solution has the same trajectory as that of the canonical CIB solution.

Theorem 5.1 (Canonical CIB solution) Let $G_n(\Sigma, b)$ be a centipede game with bounds (Σ, b). Let ℓ be any number with $0 \leq \ell \leq n$.

$\langle \mathbf{1} \rangle$ $c^\ell d^{n-\ell}$ is a CIB solution if and only if ℓ is the behavioral divide.

$\langle \mathbf{2} \rangle$ Let ℓ be the behavior divide. Then, $c^\ell d^{n-\ell}$ is a unique CIB solution if and only if $\ell > b^* = \max(b_1, b_2)$ and $z_{\ell+1} \rhd_i z_{b_i}$ for $i = 1, 2$.

$\langle \mathbf{3} \rangle$ Let σ be the canonical CIB solution and let σ' be any CIB solution. Then, $\sigma(x_t) = \sigma'(x_t)$ for all t with $t > b_{\pi(t)}$.

We adopt the canonical CIB solution $c^\ell d^{n-\ell}$ as the representative of the CIB solutions. When the condition in $\langle 2 \rangle$ holds, the canonical CIB is the unique CIB solution; when b_i is small for $i = 1, 2$, we can expect $\langle 2 \rangle$ to hold. Even if $\langle 2 \rangle$ is not the case, $\langle 3 \rangle$ states that the difference from the canonical solution occur only within the consciousness boundaries $b_i, i = 1, 2$.[14]

Remark 5.1 Theorem 5.1.$\langle 1 \rangle$ states that the consciousness boundaries $b = (b_1, b_2)$ have no effects on the canonical CIB solution. We may use this fact in the subsequent studies.

Let us look Example 2.1. We assume that $\underline{\gamma}_1 = \underline{\gamma}_2 = 0$, $\overline{\gamma}_1 = \overline{\gamma}_2 = 300$, $b_1 = 7$, $b_2 = 6$, and $\rho_1 = \rho_2 \geq 0$. Here, only the cognitive abilities are variables.

Example 5.1 (Classical centipedes) Consider the three centipede games given by (5) with $n = 100$, $n = 68$, and $n = 10$. $G_{68}(\Sigma, b)$ is the restriction of $G_{100}(\Sigma, b)$ to the nodes $\{x_1, ..., x_{68}\}$ and $\{z_1, ..., z_{69}\}$ only with the modifications that the last endnode z_{69} are connected directly to x_{68} with choice c at x_{68}. $G_{10}(\Sigma, b)$ is defined similarly. In either of $G_{100}(\Sigma, b)$ and $G_{68}(\Sigma, b)$, the

[14]When $1 \leq \ell \leq \min(b_1, b_2)$, $c^{n-\ell} d^\ell$ is still a canonical one.

CIB solution is uniquely determined for each $\rho_1 = \rho_2 \geq 0$. In $G_{100}(\Sigma, b)$, a CIB solution σ is the c-solution or d-solution for $\rho_1 = \rho_2 \leq 7$ or $\rho_1 = \rho_2 \geq 9$, and in the middle case $\rho_1 = \rho_2 = 8$, it is $c^{99}d$. In $G_{68}(\Sigma, b)$, we find the same tendency to have d in the last few decision nodes for $\rho_1 = \rho_2 = 7, 8$.

$$\begin{array}{ccc} \text{In } G_{100}(\Sigma, b) & & \text{In } G_{68}(\Sigma, b) \\ \sigma = \begin{cases} c^{100} & \text{if } \rho_1 = \rho_2 \leq 7 \\ c^{99}d & \text{if } \rho_1 = \rho_2 = 8 \\ d^{100} & \text{if } \rho_1 = \rho_2 \geq 9 \end{cases} & \text{and } \sigma = & \begin{cases} c^{68} & \text{if } \rho_1 = \rho_2 \leq 6 \\ c^{66}d^2 & \text{if } \rho_1 = \rho_2 = 7 \\ c^{65}d^3 & \text{if } \rho_1 = \rho_2 = 8 \\ d^{68} & \text{if } \rho_1 = \rho_2 \geq 9. \end{cases} \end{array} \tag{27}$$

Consider $G_{10}(\Sigma, b)$ with the same parameter values. The CIB solution is given as

$$\sigma \in \begin{cases} \{c^{10}\} & \text{if } \rho_1 = \rho_2 \leq 7 \\ \{c^7d^3, d^{10}\} & \text{if } \rho_1 = \rho_2 = 8 \\ \{d^{10}\} & \text{if } \rho_1 = \rho_2 \geq 9. \end{cases}$$

Multiplicity of CIB solutions happens when $\rho_1 = \rho_2 = 8$, where the canonical one is c^7d^3. Theorem 5.1.$\langle 3 \rangle$ claims that $\sigma(x_8) = \sigma(x_9) = \sigma(x_{10}) = d$. The calculation of the results in (27) for $G_{100}(\Sigma, b)$ is given in Section 5.4.

From Theorem 5.1 and Example 5.1, we predict the following:

Behavioral Dichotomy:

(BD1) If the cognitive bounds ρ_1 and ρ_2 are low, the behavior divide ℓ is close to n, i.e., the canonical CIB solution is $c^\ell d^{n-\ell}$.

(BD2) If they are high, the behavioral divide ℓ is 0, i.e., the canonical CIB solution is d^n.

This includes still ambiguity. In $G_{100}(\Sigma, b)$ of (27), BD1 has two cases c^{100} and $c^{99}d^1$, and BD2 has the third line. In $G_{68}(\Sigma, b)$, BD1 has the first three lines, and BD2 has the fourth line. In either examples, we have a tendency of the behavioral dichotomy.

5.2 Locations of the behavior divide ℓ

Theorem 5.1 means that the canonical CIB solution $c^\ell d^{n-\ell}$ can be a good representative of the set of CIB solutions. The behavior divide ℓ is may be regarded as close to n if it exists, since the divide ℓ is the maximum of ℓ''s with $z_{\ell'} \bowtie_{\ell'} z_{\ell'+1}$. Therefore, behavioral dichotomy holds quite likely. Indeed, the three cases of Example 5.1 show this conjecture. Yet, this is observed in these examples. We should check the locations of ℓ in more cases. Here, we see relatively simple cases, and postpone more complex cases to Section 6. That is, we look at sufficient conditions for a canonical CIB solution to be c^n or $c^{n-1}d$.

Let $G_n(\Sigma, b)$ be a centipede game with bounds Σ and b. Let $\nu_i^* = \min\{\nu : \lambda_i(\nu) > \max_{z_t \in Z_n} g_i(z_t)\}$ for $i = 1, 2$. Then, we consider the following conditions for players $\pi(k)$, $k = n, n-1$:

$$\lambda_{\pi(k)}(\nu^*_{\pi(k)} - 1) < g_{\pi(k)}(z_{k+1}) < g_{\pi(k)}(z_k) < \lambda_{\pi(k)}(\nu^*_{\pi(k)}). \tag{28}$$

If $k = n$, this is $z_n \bowtie_{\pi(n)} z_{n+1}$ and if $k = n-1$, it is $z_{n-1} \bowtie_{\pi(n-1)} z_n$. In the former, the behavioral divide is $\ell = n$, and Theorem 5.1 implies the canonical CIB solution c^n. When the former does not hold but the latter does, the behavior divide is $\ell = n-1$, and Theorem 5.1

implies the canonical CIB solution $c^{n-1}d$. These cases are summarized as the following lemma.

Lemma 5.1 (Conditions for c^n or $c^{n-1}d$) Let $G_n(\Sigma_n, b)$ be a centipede game.

$\langle \mathbf{1} \rangle$ Let (28) hold for n. Then, the canonical CIB solution is c^n.

$\langle \mathbf{2} \rangle$ Let (28) hold not for n but for $n-1$. Then, the canonical CIB solution is $c^{n-1}d$.

This holds for any parameter values in $G_n(\Sigma, b)$, but (28) itself may not hold either for $k = n$ or $k = n-1$. One extreme example is; when $\rho_{\pi(n)} = 0$, (28) holds for n because $2^{\rho_{\pi(n)}} = 2$ and $\lambda_{\pi(k)}(0) < g_{\pi(n)}(z_t) < \lambda_{\pi(k)}(1)$ for all $z_t \in Z_n$. When $\rho_{\pi(n)}$ is large but $\rho_{\pi(n-1)} = 0$, we could have the condition of $\langle 2 \rangle$. These will be discussed in Section 6.

Here, we give one type of a sufficient condition to have c^n, $c^{n-1}d$, or d^n. First, we assume that $g_i(\cdot)$ is integral-valued for $i = 1, 2$, i.e., their range is the set of positive integers $\mathbb{I}_+$, and that the ruler is *tight* in the sense that

$$\overline{\gamma}_i = \max_{z_t \in Z_n} g_i(z_t) + 1 \text{ for } i = 1, 2. \tag{29}$$

Since the payoffs are monetary and integral-valued, $\overline{\gamma}_i$ is simply the smallest payoff satisfying $\overline{\gamma}_i > g_i(z_t)$ for all $z_t \in Z_n$. We show that the canonical CIB solution is given as $c^\ell d^{n-\ell}$ with $\ell = n, n-1$ or 0 depending upon the cognitive bounds.

Theorem 5.2 (Tight payoff rulers and behavioral dichotomy) Let $G_n(\Sigma_n, b)$ be an sk-concave centipede game where $g_i(\cdot)$ is integral-valued with (29) for $i = 1, 2$. Then, there are ρ_1^* and ρ_2^* such that for any cognitive bounds $\rho_1 \geq 0$ and $\rho_2 \geq 0$, $G_n(\Sigma_n, b)$ has the canonical CIB solution σ given as

$$\sigma = \begin{cases} c^n & \text{if } \rho_{\pi(n)} \leq \rho^*_{\pi(n)} \\ c^{n-1}d & \text{if } \rho_{\pi(n)} > \rho^*_{\pi(n)} \ \& \ \rho_{\pi(n-1)} \leq \rho^*_{\pi(n-1)} \\ d^n & \text{if } \rho_{\pi(n)} > \rho^*_{\pi(n)} \ \& \ \rho_{\pi(n-1)} > \rho^*_{\pi(n-1)}. \end{cases} \tag{30}$$

Her, the sk-concavity assumption is used only to derive the third line. The mixture with c and d occurs only in the second case. In this theory, the behavioral dichotomy holds in a clear-cut manner; the first and second lines belong to BD1 and the third is in BD2.

Finally, we give a sufficient condition for the canonical CIB solution to be the unique CIB solution. Let ℓ be the behavior divide in $G_n(\Sigma, b)$. Consider the following condition: for $i = 1, 2$,

$$g_i(z_{b_i}) < \frac{\overline{\gamma}_i - \underline{\gamma}_i}{2} \leq g_i(z_t) < g_i(z_{\ell+1}) \text{ for some } t, \tag{31}$$

This means that z_t separates between z_{b_i} and $z_{\ell+1}$; thus $z_{\ell+1} \rhd_i z_{b_i}$ for $i = 1, 2$. This is the condition of Theorem 5.1.$\langle 2 \rangle$.

Lemma 5.2 When (31) holds, then the canonical CIB solution is the unique CIB solution.

Since (31) is a relatively weak condition, the lemma supports that the canonical CIB solution is the representative of the CIB solutions.

5.3 A behavioral algorithm for the Selten people

In (*) of Section 1, the Selten people were asked about their opinions on the BI theory. Before it, they were explained about the BI theory, and then, each person makes a thought experiment.

We suppose that the explanation takes a form of an algorithm, because the full understanding is costly for non-professional people, even trained mathematically, as discussed in Section 2.3. Here, we explain, to each of the Selten people, the basic language and some concepts for the CIB theory. It differs from the BI theory that the algorithm here has an oracle, in addition to detailed choices coming from reflections on one's mind. The oracle is a theoretical result, specifically Theorem 5.1, of the CIB theory, and suggests to skip some detailed comparisons.

First, we teach the Selten people the two concepts with their intended meanings:

K1 Strict preferences $\rhd_{\pi(t)}$ over the adjacent pairs (z_t, z_{t+1}).

K2 Incomparabilities $\bowtie_{\pi(t)}$ over the adjacent pairs (z_t, z_{t+1}).

These are explained with the payoff ruler consisting payoff scales. Then, each of the Selten people reflects upon his mind and estimates $\rhd_{\pi(t)}$ and $\bowtie_{\pi(t)}$. This is done at the intuition level.

Then, we teach the general (abstract) knowledge, K3, for $G_n(\Sigma, b)$:

K3 The behavioral divide ℓ and Theorem 5.1 for the canonical CIB solution.

Each person uses his strict preference $\rhd_{\pi(t)}$ and incomparability $\bowtie_{\pi(t)}$ to find the behavioral divide ℓ. Thus, he follows the suggestion of the voice from the outside, which is a guide from the CIB theory, mentioned in Section 2.3, (*iii*), and is called an *oracle*.

Let us formulate the CIB algorithm with a performer, abbreviated as SP. Now, the SP faces a centipede game G_n. He takes each player's position described by the game G_n. At decision node x_t, he thinks about what to do as if he is player $\pi(t)$. The algorithm tells him at each step to go to the next step, to change the behavior, or to terminate the algorithm.

The CIB Algorithm with an oracle: It starts with the last decision node x_n. Because the steps are all uniform, we describe the step at an arbitrary decision node x_t.

Step Nb$_t$ The SP checks whether $z_t \rhd_{\pi(t)} z_{t+1}$ or $z_t \bowtie_{\pi(t)} z_{t+1}$ holds.

Nb$_t$1 Let $z_t \rhd_{\pi(t)} z_{t+1}$. He puts $\sigma(x_t) = d$, and the algorithm goes to Step Nb$_{t-1}$ when $t > 1$, but it terminates when $t = 1$.

Nb$_t$2 Let $z_t \bowtie_{\pi(t)} z_{t+1}$. He puts $\sigma(x_{t'}) = c$ for all $t' \leq t$, and the algorithm terminates.

The algorithm has at most n steps, and in each termination case, the algorithm assigns a pair of behavior plans σ. Indeed, when the SP finds $z_t \bowtie_{\pi(t)} z_{t+1}$ in Nb$_t$2, the algorithm terminates with the resulting outcome $c^t d^{n-t}$. When the SP does not find it at any t, the algorithm reaches Nb$_1$1 and the resulting outcome is d^n. In either case, the outcome is the canonical CIB solution.

In the case Nb$_t$2, the oracle suggests that the algorithm assigns choice c for node x_t for all $t \leq \ell$ based on K3. It connects his choices suggested from the theoretical statement in his mind.

The number of steps to Nb$_\ell$ to the termination of the CIB algorithm is $n - \ell$. It is a crucial problem to think about how large is the difference $n - \ell$ is. In the examples in Section 5.1, ℓ is quite close to n when the cognitive abilities are not large. This will be considered in Section 6.

A variant is an algorithm without the oracle, going down to the very first decision node x_1, but the oracle is natural in the context of explaining the CIB theory to the Selten people.

5.4 Proofs

Calculation of the CIB solution σ in $G_{100}(\Sigma, b)$ Let us verify the CIB solution σ in $G_{100}(\Sigma, b)$. When $\rho_1 = \rho_2 \geq 9$, Theorem 4.2.$\langle 2\rangle$ is a sufficient condition $\sigma = d^n$. Indeed, $g_{\pi(t)}(z_t) - g_{\pi(t)}(z_{t+1}) = 1 > \Delta_{\rho_{\pi(t)}} = (300-0)/2^9 \doteqdot 0.586$ for all t, which together with Theorem 4.2.$\langle 2\rangle$ implies $z_{t+1} \rhd_{\pi(t)} z_{t+1}$ for all t. Thus, $\sigma = d^n$ by $\langle 1\rangle$. Let $\rho_1 = \rho_2 = 8$. Since $\Delta_{\rho_{\pi(t)}} = (300-0)/2^8 = 1.171875$, we have $203 > \underline{\gamma}_{\pi(100)} + \Delta_{\rho_{\pi(100)}} \times 173 \doteqdot 202.27 > 202$; thus, $z_{100} \rhd_{\pi(100)} z_{101}$. But since $g_1(z_{99}) = 198$ and $g_1(z_{100}) = 197$, we have $198.05 \doteqdot \underline{\gamma}_1 + \Delta_{\rho_1} \times 169 > 198 > 197 > \underline{\gamma}_1 + \Delta_{\rho_1} \times 168 \doteqdot 196.9$. So, $z_{99} \bowtie_1 z_{100}$ by (14). By Theorem 5.1.$\langle 2\rangle$, $\sigma = c^{99}d$ is the unique CIB solution . Finally, let $\rho_1 = \rho_2 \leq 7$. Then, $\Delta_{\rho_i} \geq (300-0)/2^7 = 2.34375$. Let $\rho_i = 7$. Then, since $\pi(100) = 2$, $g_2(z_{101}) = 203$, and $g_2(z_{100}) = 202$, we have $203.9 \doteqdot \underline{\gamma}_2 + \Delta_{\rho_2} \times 87 > 203 > 202 > \underline{\gamma}_2 + \Delta_{\rho_2} \times 86 \doteqdot 201.6$. Hence, $z_{100} \bowtie_{\pi(100)} z_{101}$ by (14); $\sigma = c^{100}$ is the unique CIB solution $\sigma = c^{100}$ by Theorem 5.1.$\langle 2\rangle$.$\blacksquare$

Proof of Theorem 5.1 $\langle \mathbf{1}\rangle$ The *if* part was explained before the theorem. We prove the *only-if* part. We prove the contrapositive of the assertion. Suppose that $\sigma = c^\ell d^{n-\ell}$ is not canonical. Then, ℓ is not the maximum of ℓ' with $z_{\ell'+1} \bowtie_{\pi(\ell')} z_{\ell'}$. Then, there is an ℓ' with $\ell' > \ell$ and $z_{\ell'+1} \bowtie_{\pi(\ell')} z_{\ell'}$. By the condition, we can assume that ℓ' is the maximum and $\ell' > b^* = \max(b_1, b_2)$. However, $\sigma(x_{\ell'}) = d$. By (21), σ is not a CIB solution.

$\langle \mathbf{2}\rangle$ The additional claim $\sigma = c^\ell d^{n-\ell}$ follows from the only-if part of the main assertion.

(Only-if) Suppose that $G_n(\Sigma, b)$ has a unique CIB solution σ. Then, by (26) and Theorem 3.1.$\langle 1\rangle$ the latter part of (26) implies that $\sigma(x_t) = d$ for $t > \ell$. Then, since $z_\ell \bowtie_{\pi(\ell)} z_{+1}$, we have $\sigma(x_\ell) = c$ by (21). Thus, $r(\sigma_{x_{\ell+1}|}) = z_{\ell+1}$. Since $z_{\ell+1} \unrhd_t x_t$ for all $t > \max(b_1, b_2)$ by Lemma 5.1.$\langle 1\rangle$, we have $\sigma(x_t) = c$ for all $t > \max(b_1, b_2)$ by (21). Now, let $b_i = \max(b_1, b_2)$. Then, because of the uniqueness of the CIB solution σ, we have $z_{\ell+1} \rhd_i z_{b_i}$ by (21); so, $\sigma(x_{b_i}) = c$ again by (21). Consider $t \leq b_i - 1$. We assume the induction hypothesis that $\sigma(x_{t'}) = c$ for all $t' \geq t$. Hence, $r(\sigma_{x_{t+1}|}) = z_{\ell+1} \unrhd z_{t+1}$. Let $t \geq \min(b_1, b_2) = b_j$. Then $z_{\ell+1} \unrhd_{\pi(t)} z_t$. If $\pi(t) = i$, then $r(\sigma_{x_{t+1}|}) = z_{\ell+1} \rhd_i z_{b_i} \unrhd_i z_{t+1}$, which implies $\sigma(x_t) = c$ by (21). If $\pi(t) = j$, then $r(\sigma_{x_{t+1}|}) = z_{\ell+1} \unrhd_i z_t$ by Lemma 5.1.$\langle 1\rangle$. Then, if $t > b_j$, then $\sigma(x_t) = c$ by (21), and if $t = b_j$, then $z_{\ell+1} \rhd_j z_t$ by (21). As a whole, we have shown $\sigma(x_t) = c$ for all $t \geq \min(b_1, b_2)$. This means that $z_{\ell+1} \rhd_i z_{b_i}$ for $i = 1, 2$. This and Lemma 3.2 imply $z_{\ell+1} \rhd_i z_t$ for all $t \leq \min(b_1, b_2)$. By Theorem 3.1.$\langle 3\rangle$, σ is uniquely determined.

(If) By Theorem 3.1, it suffices to show condition (22). Let $z_{\ell+1} \rhd_i z_{b_i}$. For any z_t with $t < b_i$ and $\pi(t) = i$, we have $g_i(z_{b_i}) > g_i(z_t)$ by (3). Hence, by Lemma 3.2.$\langle 1\rangle$, we have $z_{b_i} \unrhd_i z_t$. Hence, $z_{\ell+1} \rhd_i z_t$ by (16). We have shown,

$$\text{for } i = 1, 2,\ z_{\ell+1} \rhd_i z_t \text{ for all } z_t \text{ with } \pi(t) = i \text{ and } t \leq b_i.$$

Now, by condition (26) and Theorem 3.1, a CIB solution σ is determined uniquely in the domain $\{x_t : t \geq b_{\pi(t)}\}$. Then, it holds that $\sigma(x_t) = c$ for x_t with $b_{\pi(t)} \leq t \leq \ell$. Finally, we let $\sigma(x_t) = c$ for x_t with $t \leq b_{\pi(t)}$. Then, it holds that

$$r(\sigma_{x_{t+1}|}) = z_{\ell+1} \rhd_{\pi(t)} z_t \text{ for } t \leq b_{\pi(t)}.$$

Thus, by Theorem 3.1.$\langle 3\rangle$, σ is the unique CIB solution.$\blacksquare$

Proof of Theorem 5.2 Let $k = n, n-1$. Recall $\Delta_{\rho_{\pi(k)}} = \frac{\overline{\gamma}_{\pi(k)} - \underline{\gamma}_{\pi(k)}}{2^{\rho_{\pi(k)}}}$. Let $\rho^*_{\pi(k)}$ be the maximum of $\rho_{\pi(k)}$'s satisfying

$$\Delta_{\rho_{\pi(k)}} > g_{\pi(k)}(z_k) - g_{\pi(k)}(z_{k+1}) \tag{32}$$

By (32), for $\rho_{\pi(k)} > \rho^*_{\pi(k)}$, it holds that $\Delta_{\rho^*_{\pi(k)}} > g_{\pi(k)}(z_k) - g_{\pi(k)}(z_{k+1}) \geq \Delta_{\rho_{\pi(k)}}$. Thus,

$$\Delta_{\rho_{\pi(k)}} \begin{cases} > g_{\pi(k)}(z_k) - g_{\pi(k)}(z_{k+1}) & \text{if } \rho_{\pi(k)} \leq \rho^*_{\pi(k)} \\ \leq g_{\pi(k)}(z_k) - g_{\pi(k)}(z_{k+1}) & \text{if } \rho_{\pi(k)} > \rho^*_{\pi(k)}. \end{cases} \tag{33}$$

Now, let $k = n$. Consider the first case $\rho_{\pi(n)} \leq \rho^*_{\pi(n)}$ of (30). Then, $\lambda_{\rho_{\pi(n)}}(2^{\rho_{\pi(n)}}) = \overline{\gamma}_{\pi(n)} > g_{\pi(n)}(z_n)$. Now, we have

$$\begin{aligned} \overline{\gamma}_{\rho_{\pi(n)}} - \Delta_{\rho_{\pi(n)}} &< \overline{\gamma}_{\rho_{\pi(n)}} - [g_{\pi(n)}(z_n) - g_{\pi(n)}(z_{n+1})] \\ &< (\overline{\gamma}_{\rho_{\pi(n)}} - g_{\pi(n)}(z_n)) + g_{\pi(n)}(z_{n+1}) = g_{\pi(n)}(z_{n+1}). \end{aligned} \tag{34}$$

Thus, $g_{\pi(n)}(z_{n+1}) > \lambda_{\rho_{\pi(n)}}(2^{\rho_{\pi(i)}} - 1)$. Hence, $\lambda_{\rho_{\pi(n)}}(2^{\rho_{\pi(n)}}) = \overline{\gamma}_{\pi(n)} > g_{\pi(n)}(z_n)$ and $g_{\pi(n)}(z_{n+1}) > \lambda_{\rho_{\pi(n)}}(2^{\rho_{\pi(n)}} - 1)$. Thus, $z_n \bowtie_{\pi(n)} z_{n+1}$ by Lemma 4.1.$\langle 2 \rangle$. By Theorem 5.1.$\langle 1 \rangle$, we have $\sigma = c^n$.

Consider the second case $\rho_{\pi(n)} > \rho^*_{\pi(n)}$ and $\rho_{\pi(n-1)} \leq \rho^*_{\pi(n-1)}$ of (30). It follows from (33) that

$$\overline{\gamma}_{\rho_{\pi(n)}} - \Delta_{\rho_{\pi(n)}} \geq \overline{\gamma}_{\rho_{\pi(n)}} - [g_{\pi(n)}(z_n) - g_{\pi(n)}(z_{n+1})] > g_{\pi(n)}(z_{n+1}).$$

Hence, $g_{\pi(n)}(z_{n+1}) < \overline{\gamma}_{\rho_{\pi(n)}} - \Delta_{\rho_{\pi(n)}}$. On the other hand, since $g_{\pi(n)}(z_n) - g_{\pi(n)}(z_{n+1}) \geq \Delta_{\rho_{\pi(n)}}$ by (33) and the payoff functions are integer-valued and bijective, we have

$$\begin{aligned} \overline{\gamma}_{\rho_{\pi(n)}} - \Delta_{\rho_{\pi(n)}} &= \overline{\gamma}_{\rho_{\pi(n)}} - [g_{\pi(n)}(z_n) - g_{\pi(n)}(z_{n+1})] \\ &= (g_{\pi(n)}(z_n) + 1) - [g_{\pi(n)}(z_n) - g_{\pi(n)}(z_{n+1})] \\ &= 1 + g_{\pi(n)}(z_{n+1}) \leq g_{\pi(n)}(z_n). \end{aligned}$$

This together with the previous result implies that $g_{\pi(n)}(z_n) \geq \overline{\gamma}_{\rho_{\pi(n)}} - \Delta_{\rho_{\pi(n)}} = \lambda_{\pi(n)}(2^{\rho_{\pi(n)}-1} > g_{\pi(n)}(z_{n+1})$. By (13), we have $z_n \rhd_{\pi(n)} z_{n+1}$. Since $\rho_{\pi(n-1)} \leq \rho^*_{\pi(n-1)}$, it is proved in the same manner as (34) that $z_{n-1} \bowtie_{\pi(n-1)} z_n$. Hence, by Theorem 5.1.$\langle 1 \rangle$, we have $\sigma = c^{n-1}d$.

Consider the third case $\rho_{\pi(n)} > \rho^*_{\pi(n)}$ and $\rho_{\pi(n-1)} > \rho^*_{\pi(n-1)}$of (30). In this case, we have $\Delta_{\rho_{\pi(n)}} \leq g_{\pi(n)}(z_n) - g_{\pi(n)}(z_{n+1})$. Also, by sk-concavity, we have $\Delta_{\rho_{\pi(n)}} \leq g_{\pi(n)}(z_n) - g_{\pi(n)}(z_{n+1}) \leq ... \leq g_{\pi(t)}(z_t) - g_{\pi(t)}(z_{t+1})$ for all $t \leq n$ with $\pi(t) = n$. In the same manner, we have $\Delta_{\rho_{\pi(n-1)}} \leq g_{\pi(n-1)}(z_{n-1}) - g_{\pi(n-1)}(z_n) \leq ... \leq g_{\pi(t)}(z_t) - g_{\pi(t)}(z_{t+1})$ for all $t \leq n$ with $\pi(t) = n - 1$. By Theorem 3.2.$\langle 1 \rangle$, we have $\sigma = d^n$.$\blacksquare$

6 Reversed Causality Degrees and Centipede Games with Different Lengths

This section has two-fold developments; the concept of the *reversed causality degree* RC_n is introduced to $G_n(\Sigma, b)$. It is applied to the *class,* $\mathbb{G}_n(\Sigma, b)$, *of initial segments* $G_k(\Sigma, b)$ of a centipede game $G_n(\Sigma, b)$. The degree expresses a discrepancy between the depth of the hypothesis for decision making and that of the resulting outcome. By looking at values RC_k for $G_k(\Sigma, b) \in \mathbb{G}_n(\Sigma, b)$, we observe a quite strong tendency of behavioral dichotomy. Finally, it is argued that these are compatible with the refusals/acceptances of the solutions by the Selten people.

6.1 Reversed causality degrees in $G_n(\Sigma, b)$

Let $G_n(\Sigma, b)$ be a centipede game and σ a pair of behavior plans. We define the *reversed causality degree* RC_n *in* σ in $G_n(\Sigma, b)$ by:

$$RC_n = (n+1) - r_T(\sigma), \tag{35}$$

where $r_T(\sigma) = k+1$ is the depth of the realization $r(\sigma) = z_{k+1}$. Since $1 \leq r_T(\sigma) \leq n+1$, we have the range of RC_k as $0 \leq RC_k \leq n$. This concept expresses the degree of cause-effect included in σ; the cause-effect is reversed as discussed for P0E in Section 1. It is calculated along the CIB algorithm described in Section 5.3; to get a solution, a player hypothetically starts thinking at the final decision node x_n with the very last end node z_{n+1}, and when it stops at $r(\sigma) = z_{k+1}$, the causality degree is given by (35).

We focus on the canonical CIB solution σ. Theorem 5.1 stated that the canonical CIB solution is expressed as $\sigma = c^{\ell} d^{n-\ell}$ $(0 \leq \ell \leq n)$. Since $r_T(c^{\ell} d^{n-\ell}) = \ell + 1$, we have $RC_n = (n+1) - r_T(c^{\ell} d^{n-\ell}) = n - \ell$, which is stated as the next lemma.

Lemma 6.1 (Reversed causality and conscious choice): $RC_n = n - \ell$ for any centipede game $G_n(\Sigma, b)$.

The degree $RC_n = n - \ell$ is interpreted as the number of conscious decision making at $x_{\ell+1}, ..., x_n$. If we count the decision nodes within consciousness boundaries $b^* = \max(b_1, b_2)$, $RC_n(\sigma)$ needs the additional constant b^*, but for simplicity, we ignore the constant b^*.

For a centipede game $G_n(\Sigma, b)$, we have the 1-to-1 correspondence:

$$\text{the canonical CIB solution } \sigma \text{ is } c^{\ell} d^{n-\ell} \text{ if and only if } RC_n = n - \ell. \tag{36}$$

The degree RC_n is regarded as a function over a class of centipede games. It suffices to focus on the behavior divide ℓ for each centipede game in the class. In the cases of Theorem 5.2, there are only three types of CIB solutions, $c^n, c^{n-1}d$, or d^n. By (36), we have $RC_n = 0$, 1, or n. The value $RC_n = n$ happens when the players have high cognitive abilities, and the other two cases $RC_n = 0$ and $RC_n = 1$ are when they have low cognitive abilities. The separation between the former and latter cases is the behavioral dichotomy. Now, we study this separation by looking at the behavior of the reversed causality degree RC_n.

6.2 The class of initial segments of a centipede game $G_n(\Sigma, b)$

Let $G_n(\Sigma, b)$ be a centipede game. Let $k \leq n$. The *initial segment* $G_k(\Sigma, b)$ of length k is the restriction of $G_n(\Sigma, b)$ to the decision nodes $x_1, ..., x_k$ and the endnodes $z_1, ..., z_k, z_{k+1}$ so that the payoff functions are simply restricted to these end nodes but we change the connection from x_k directly to z_{k+1} with choice c at x_k. Then, we define the set

$$\mathbb{G}_n(\Sigma, b) = \{G_k(\Sigma, b) : \text{it is an initial segment of } G_n(\Sigma, b) \text{ and } 1 \leq k \leq n\}.$$

Each $G_k(\Sigma, b) \in \mathbb{G}_n(\Sigma, b)$ is a centipede game, and the canonical CIB solution of $G_k(\Sigma, b)$ is denoted by $c^{\ell(k)} d^{k-\ell(k)}$, where $\ell(k)$ is the behavioral divide of $G_k(\Sigma, b)$. Then, $RC_k = k - \ell(k)$ is a function of k $(1 \leq k \leq n)$. Instead of focussing on the value RC_k for a single game $G_k(\Sigma, b) \in \mathbb{G}_n(\Sigma, b)$, we focus on the function $RC_k = k - \ell(k)$ $(1 \leq k \leq n)$.

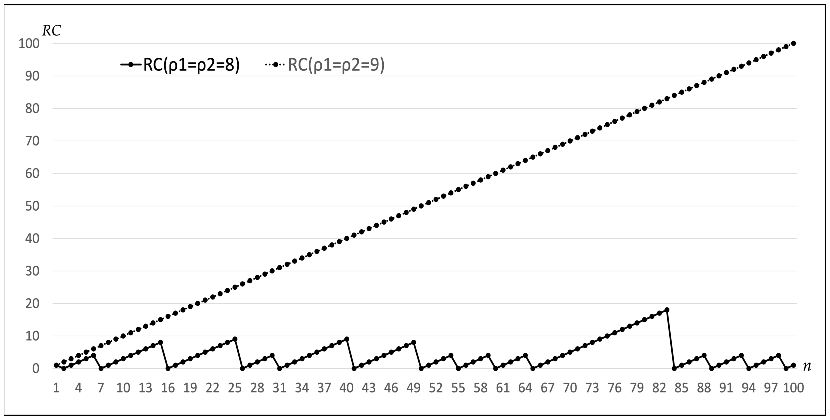

Figure 10: RC_k of $G_k(\Sigma) \in \mathbb{G}_n(\Sigma)$

To study the function $RC_k = k - \ell(k)$, we consider the following sets: for $i = 1, 2$,

$$L_n(\Sigma_i) = \{\ell : 1 \leq \ell \leq n,\ z_\ell \bowtie_{\pi(\ell)} z_{\ell+1} \text{ and } \pi(\ell) = i \text{ in } G_n(\Sigma, b)\}. \tag{37}$$

It is the set of indices where the adjacent endnodes are incomparable for i. Let $L_n(\Sigma) = L_n(\Sigma_1, \Sigma_2) = L_n(\Sigma_1) \cup L_n(\Sigma_2)$.

For $k \in L_n(\Sigma)$, by Lemma 5.1.⟨1⟩, $z_k \bowtie_{\pi(k)} z_{k+1}$ implies that the canonical CIB solution of $G_k(\Sigma, b)$ is c^k. Consider k with $\ell \leq k < \ell'$ for $\ell, \ell' \in L_n(\Sigma)$ where ℓ' is adjacent to ℓ in $L_n(\Sigma)$. We ignore ℓ' when ℓ is the maximum in $L_n(\Sigma)$. It follows from Theorem 5.1.⟨1⟩ that for any k with $\ell \leq k < \ell'$, the canonical CIB solution σ^k of $G_k(\Sigma, b)$ and RC_k are given:

$$\sigma^k = c^\ell d^{k-\ell(k)} \text{ and } RC_k = k - \ell(k). \tag{38}$$

Thus, the function $RC_k = k - \ell(k)$ is obtained from $L_n(\Sigma) = L_n(\Sigma_1) \cup L_n(\Sigma_2)$.

Let $G_n(\Sigma, b)$ be the centipede game $G_{100}(\Sigma, b)$ of Example 5.1. In Fig.10, the function $RC_k = k - \ell(k)$ is depicted in the two cases for $\rho_1 = \rho_2 = 9$ and $\rho_1 = \rho_2 = 8$. The straight line is $RC_k = k$ for the case $\rho_1 = \rho_2 = 9$, and the saw-shaped line is $RC_k(\rho) = k - \ell(k)$ for $\rho_1 = \rho_2 = 8$. These lines have full information about the canonical CIB solution for $G_k(\Sigma, b) \in \mathbb{G}_{100}(\Sigma, b)$. For example, When $\rho_1 = \rho_2 = 9$ and $k = 83$, $RC_{83}(\rho) = 83$ implies $\sigma^{83} = d^{83}$. When $\rho_1 = \rho_2 = 8$, $RC_{83}(\rho) = 18$ implies $\sigma^{83} = c^{65} d^{18}$. Here, the result differs from Theorem 5.2. Incidentally, in this case, when $k = 65$, we have $RC_{65} = 0$ and the CIB solution c^{65} for $G_{65}(\Sigma, b)$.

The following theorem describes the behavior of the function $RC_k(\cdot)$. ⟨1⟩ states that $RC_k(\cdot)$ is weakly increasing, and ⟨2⟩ states that $RC_k(\cdot, \cdot)$ has the lowest boundary at $\rho = (0, 0)$, i.e., $RC_k(0, 0) = 0$ for all $k \leq n$ and the upper bound at some $\rho^* = (\rho_1^*, \rho_2^*)$. ⟨2⟩ implies that the canonical CIB solution is c^k for all $k \leq n$ at $\rho = (0, 0)$, and it is d^k for all $k \leq n$ after ρ^*. ⟨1⟩ implies that the saw-shaped line of the function $RC_k(\cdot)$ is monotonically moving from $(0, 0)$ to ρ^*. In Fig.9, the lines for $\rho_1, \rho_2 \leq 8$ are also saw-shaped and are located underneath the line of $\rho_1 = \rho_2 = 8$.

Theorem 6.1 ⟨1⟩(Monotonicity) $RC_k(\cdot)$ is weakly increasing w.r.t. $\rho = (\rho_1, \rho_2)$, i.e., if $\rho = (\rho_1, \rho_2) \leq \rho' = (\rho_1', \rho_2')$, then $RC_k(\rho) \leq RC_k(\rho')$ for all $k \leq n$.

⟨2⟩(Boundaries) $RC_k(0, 0) = 0$ for all $k \leq n$, and there is some ρ^* such that for all $\rho \geq \rho^*$, $RC_k(\rho) = k$ for all $k \leq n$.

This is indicative of the behavioral dichotomy. Nevertheless, $\langle 1 \rangle$ is yet qualitative and $\langle 2 \rangle$ is about boundaries. To study the behavior of the function $RC_k(\rho)$ quantatively, we prepare the average of the function.

Average reversed causality degrees $\overline{RC}(\rho)$: RC_k is a function of k and is determined by cognitive abilities $\rho = (\rho_1, \rho_2)$; so, we denote this by $RC_k(\rho)$. Now, we define the average of $RC_k(\rho) = k - \ell(k)$ over $\mathbb{G}_n(\Sigma, b)$ by

$$\overline{RC}(\rho) = \frac{1}{n} \sum_{k=1}^{n} RC_k(\rho). \tag{39}$$

Consider the class $\mathbb{G}_{100}(\Sigma, b)$ derived from $G_{100}(\Sigma, b)$ of Example 5.1. In the case $\rho_1 = \rho_2 = 9$, it holds that $RC_k(9,9) = k$ for $k \leq n$, it is calculated using the well-known formula that $\overline{RC}(9,9) = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n+1}{2} = 50.5$. When $\rho_1 = \rho_2 = 8$, the average is calculated by an computer algorithm based on (39) that $\overline{RC}(8,8) = 4.14$.

Since $RC_{83}(\rho) = 18$, it follows from $\overline{RC}(\rho) = 4.14$ that $RC_k(\rho)$ is much smaller than $RC_{83}(\rho) = 18$ for $k \neq 83$. We may predict that the decrease from $\overline{RC}(9,9) = 50.5$ to $\overline{RC}(8,8) = 4.14$ continues more to smaller $\rho_1 = \rho_2$. Table 6.1 shows the behavior of $\overline{RC}(\rho)$.

Table 6.1 the average $\overline{RC}(\rho)$

$\rho_1 = \rho_2$	0	1	2	3	4	5	6	7	8	≥ 9
$\overline{RC}(\rho_1, \rho_2)$	0	0.03	0.04	0.05	0.07	0.12	0.34	0.62	4.14	50.5

For example, when $\rho_1 = \rho_2 = 5$, $\overline{RC}(\rho)$ becomes 0.12, which is much smaller than $\overline{RC}(8,8) = 4.14$. It is interpreted as meaning that at least 88 games $G_k(\Sigma, b) \in \mathbb{G}_{100}(\Sigma, b)$ have the canonical CIB solutions c^k, equivalently at most 12 games have the canonical CIB solutions $c^{k-1}d^1$. By scrutinizing the calculation data of the function RC_k, we find only one exception that $c^{73}d^2$ is the CIB solution in $G_{75}(\mathbf{\Sigma})$.

The average $\overline{RC}(\rho)$ is even smaller when $\rho_1 = \rho_2$ is smaller than 5. The statement that the behavior divide $\ell(k)$ is close to 0 is interpreted as meaning the behavior of $\overline{RC}(\rho)$ shown in Table 6.1.

Initial segments $G_k(\Sigma, b)$ of $\mathbb{G}_n(\Sigma, b)$ with $\rho_1 \neq \rho_2$ In the previous examples, we assumed $\rho_1 = \rho_2$. This assumption can be eliminated without difficulty. This leads to an important implication for considerations of the Selten people's responses. In (37), the set $L_n(\Sigma_i)$ is defined independent of $L_n(\mathbf{\Sigma}_j)$ $(j \neq i)$. Hence, the above arguments hold except the examples. Consider the class $\mathbb{G}_{100}(\Sigma, b)$ derived from $G_{100}(\Sigma, b)$ of Example 5.1. Now, it is assumed that ρ_1 is variable but $\rho_2 = 9$ is variable. Then, $\overline{RC}(\rho_1, 9)$ is given by Table 6.2.

Table 6.2 $\overline{RC}(\rho_1, \rho_2)$ for ρ_1 with constant $\rho_2 = 9$

ρ_1	0	1	2	3	4	5	6	7	8	≥ 9
$\overline{RC}(\rho)$	0.5	0.54	0.54	0.58	0.62	0.74	1.02	1.62	7.94	50.5

Table 6.2 is similar to Table 6.1 in that $\overline{RC}(\rho)$ decreases quickly as ρ_1 moves downward from 8. The difference is that the lowest $\overline{RC}(\rho)$ is 0 in Table 6.1, but that of Table 6.2 is 0.5. In Table 6.2, player $\pi(k) = 2$ at x_k in $G_k(\Sigma, b)$ chooses always d, but when $\rho_1 = 0$, player 1 chooses always c at x_{k-1}. Hence, each value in Table 6.2 is 0.5, while each in Table 6.1 is 0 when $\rho_1 = \rho_2 = 0$.

Let us see how Table 6.2 changes when $\rho_2 = 5$. The new table is given as Table 6.3.

Table 6.3 $\overline{RC}(\rho)$ for ρ_1 with constant $\rho_2 = 5$

ρ_1	0	1	2	3	4	5	6	7	8	≥ 9
$\overline{RC}(\rho)$	0.05	0.07	0.07	0.08	0.09	0.12	0.19	0.3	0.58	0.69

This is quite different from Table 6.2. Here, the d-behavior d^n disappears entirely; the case of $\rho_1 \geq 9$ and $\rho_2 = 5$ has the highest value $\overline{RC}(\rho) = 0.69$. From the calculation data, we have the following distribution of the canonical CIB solutions σ^k over $G_k(\Sigma, b) \in \mathbb{G}_{100}(\Sigma, b)$ $(1 \leq k \leq 100)$:

$$\begin{array}{rcll} c^k \text{ and } RC_k(\rho) & = & 0 & 46 \text{ times;} \\ c^{k-1}d \text{ and } RC_k(\rho) & = & 1 & 44 \text{ times;} \\ c^{k-2}d^2 \text{ and } RC_k(\rho) & = & 2 & 5 \text{ times;} \\ c^{k-3}d^3 \text{ and } RC_k(\rho) & = & 3 & 5 \text{ times.} \end{array}$$

This distribution gives $\overline{RC}(\rho) = 0.69$. This case is already quite close the case where the cognitive abilities are low. If ρ_1 is smaller than 9, say $\rho_1 = 7$, then $\overline{RC}(\rho) = 0.3$. In this case, the distribtion of $RC_k(\rho)$ is

$$\begin{array}{rcll} c^k \text{ and } RC_k(\rho) & = & 0 & 74 \text{ times;} \\ c^{k-1}d \text{ and } RC_k(\rho) & = & 1 & 23 \text{ times;} \\ c^{k-2}d^2 \text{ and } RC_k(\rho) & = & 2 & 2 \text{ times;} \\ c^{k-3}d^3 \text{ and } RC_k(\rho) & = & 3 & 1 \text{ times.} \end{array}$$

When ρ_1 approaches to 0 but $\rho_2 = 5$, $\overline{RC}(\rho)$ is very close to 0. Here, almost all are c^k or $c^{k-1}d$.

Sk-concave payoff functions We have considered various classes of parameter values on cognitive abilities $\Sigma = (\Sigma_1, \Sigma_2)$ and consciousness boundaries $b = (b_1, b_2)$. There are still other cases, but here, we consider an example of sk-concave payoff functions. Let $G_n(\Sigma, b)$ be a centipede game where the payoff functions are given by (5) in Example 2.1. These $g_1(z_t)$ and $g_2(z_t)$ are sk-linear. Then, we transform them into

$$g_i^*(z_t) = \sqrt{g_i(z_t)} \text{ for } z_t \in Z_n^* \text{ and } i = 1, 2.$$

In this case, the payoff difference $g_{\pi(t)}^*(z_t) - g_{\pi(t)}^*(z_{t+1})$ is decreasing with t. It is an sk-concave function; the graph is depicted in Fig.4. This implies that $z_t \bowtie_{\pi(t)} z_{t+1}$ holds more likely as t increases. When $\rho_1 = \rho_2 = 7$, the function RC_k of k is depicted as the lower saw-shap line in Fig.11; it takes some high values in the smaller domain of k and it takes a smaller value for larger k. Thus, the saw-shape line is less uniform than the case of sk-linear payoff functions.

To see the averages of this non-uniform RC_k, we see the three cases: $\overline{RC}(\rho)_{1\text{\textasciitilde}k} = \frac{1}{k}\sum_{k=1}^{k} RC_t(\rho)$ for $k = 30, 60$, and 100. In all cases, $\overline{RC}(\rho)_{1\text{\textasciitilde}k}$ takes the same tendency: only when $\rho_1 = \rho_2 \geq 9$, it takes a quite large value, but when $\rho_1 = \rho_2 \leq 8$, the average becomes suddenly small, and decreases to very small magnitude.

Table 6.4 $\overline{RC}(\rho)$

$\rho_1 = \rho_2$	0	1	2	3	4	5	6	7	8	≥ 9
$\overline{RC}(\rho)_{1\text{\textasciitilde}30}$	0	0	0.03	0.07	0.17	0.23	0.50	3.97	15.5	15.5
$\overline{RC}(\rho)_{1\text{\textasciitilde}60}$	0	0.02	0.03	0.07	0.13	0.20	0.42	2.27	8.95	30.5
$\overline{RC}(\rho)_{1\text{\textasciitilde}100}$	0	0.01	0.02	0.04	0.10	0.16	0.31	1.49	5.83	50.5

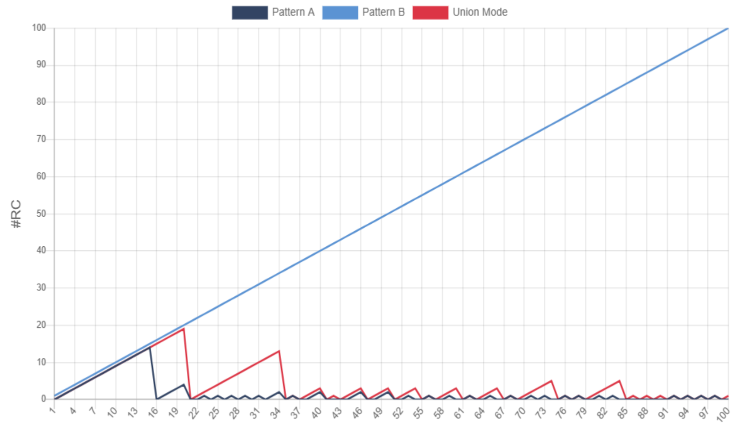

Figure 11: $RC_k(\rho)$ with $\rho = (9,9)$, $(9,7)$, and $(7,7)$

6.3 Thought experiments by the Selten people

Here, we conduct a hypothetical thought experiment, in order for the Selten people to consider how they would respond to the behavioral dichotomy shown in Sections 5 and 6:

(a) An experimental subject (i) is mathematically trained, (ii) believes the logical validity of the backward induction, and (iii) wants a guide to practical behavior;

(b) the CIB theory in Sections 3, 4, and the results in Sections 5 6 are shown and explained to him.

We argue that this hypothetical person does not find a conflict in having (a) and (b) together, including the behavioral recommendation from (b) including (iii) of (a).

Although the person is mathematically trained as well as understands the logical validity of the BI theory, he is still boundedly rational because (iii) means that he needs a concise guide to avoid complicated thinking. In the CIB theory, bounded rationality is described by a degree ρ_i of bounded cognitive ability. Behavioral dichotomy means that BD1: if ρ_1 and ρ_2 are low, the outcome is $c^\ell d^{n-\ell}$ and the behavioral divide ℓ is close to n, and BD2: if they are high, the resulting outcome is d^n. This is yet a relativistic statement having two exclusive cases, but since the hypothetical person characterized by (a) and (b) is boundedly rational, BD1 is only the case. Thus, he is facing BD1 of behavioral dichotomy where the outcome is $c^\ell d^{n-\ell}$ with the behavioral divide ℓ close to n.

The CIB solution $c^{65}d^{18}$ mentioned in Section 6.2 has a large value $k - \ell = 83 - 65 = 18$ for $\rho_1 = \rho_2 = 8$. Nevertheless, when $\rho_1 = \rho_2 \leq 7$, Table 6.1 states that the average $\overline{RC}(\rho)$ becomes much smaller, e.g., 0.62 for $\rho_1 = \rho_2 = 7$, from 4.18 for $\rho_1 = \rho_2 = 8$. The case $\rho_1 = \rho_2 = 8$ is the boundary of the two cases of behavioral dichotomy; perhaps, the case where ρ_1 and ρ_2 are low should be interpreted as $\rho_1 = \rho_2 \leq 7$. The other tables in Section 6.2 show quite similar tendencies.

Table 6.2 and Table 6.3 show the behavior the CIB solution when player 2 is the subject with $\rho_2 = 9$ and $\rho_2 = 5$. By the argument of the previous paragraph, we should consider only the case $\rho_2 = 5$. The case $\rho_1 \geq 9$ or $\rho_2 = 5$ may not be excluded since player 1 is imagined by the subject and his imagination may not exclude the case where the other person is quite rational. In either case, we find a strong tendency to have $c^\ell d^{n-\ell}$ with behavioral divide ℓ close to n. The point of Table 6.4 is that the length of the centipede game $G_k(\Sigma, b)$ is important to have behavioral dichotomy, that is, when k is larger, BD1 holds more likely.

Finally, we return to O*1* and O*2* in Section 1; the subject characterized by (a) and (b) faces a centipede game G_n and finds that the payoffs in the very last part of the game are much higher than those in the beginning of the game. As stated in Lemma 2.1, a centipede game gives a wishful thinking to the subject, but P1, P2 together with Individual motive lead to the d-solution d^n. This outcome is quite contrary to O*1* and O*2*. Then, he learns the modification of P1 and P2. Then, it leads to $c^\ell d^{n-\ell}$ with ℓ close to n. This is consistent with O*1* and O*2*. Thus, the subject person does not find a conflict in having (a) and (b).

6.4 Proof

Proof of Theorem 6.1 $\langle \mathbf{1} \rangle$ Let $\rho' \geq \rho$. Since the set $L_n(\Sigma_i)$ depends upon ρ_i together with the other parameters, we denote it by $L_n(\Sigma_i)_{\rho_i}$ and $L_n(\Sigma_i)_{\rho_i'}$ only with the change of ρ_i to ρ_i'. Then, we show that $\rho_i' \geq \rho_i$ implies $L_n(\Sigma_i')_{\rho_i'} \subseteq L_n(\Sigma_i)_{\rho_i}$. Since $\{\frac{\nu}{2^{\rho_i}} : 0 \leq \nu \leq 2^{\rho_i}\} \subseteq \{\frac{\nu}{2^{\rho_i'}} : 0 \leq \nu \leq 2^{\rho_i'}\}$, the payoff ruler Λ_{ρ_i} is a subset of the ruler $\Lambda_{\rho_i'}$, that is, the payoff scales in $\Lambda_{\rho_i'}$ are finner than those in Λ_{ρ_i}. By (37), $L_n(\Sigma_i)_{\rho_i'}$ is a subset of $L_n(\Sigma_i)_{\rho_i}$. Thus $L_n(\Sigma)_{\rho'} = L_n(\Sigma_1)_{\rho_1'} \cup L_n(\Sigma_2)_{\rho_2'}$ $\subseteq L_n(\Sigma)_\rho$. Equivalently, $L_n(\Sigma)_{\rho'}^C = \{1, ..., n\} - L_n(\Sigma)_{\rho'} \supseteq L_n(\Sigma)_\rho^C$.

We show by induction on $k = 1, ..., n$ that $RC_k(\rho) \leq RC_k(\rho')$. Let $k = 1$. If $1 \in L_n(\Sigma)_\rho^C$, then $z_1 \rhd_{\pi(1)} z_2$ in $G_1(\Sigma, b)_\rho$ and so is in $G_1(\Sigma, b)_{\rho'}$. Thus, d^1 is a CIB solution in $G_1(\Sigma, b)_\rho$ and $G_1(\Sigma, b)_{\rho'}$. Hence, $RC_1(\rho) = RC_1(\rho') = 1$.

Now, the induction hypothesis is $RC_k(\rho) := k - \ell \leq RC_k(\rho') := k - \ell'$ for $k < 100$. If $k+1 \in L_n(\Sigma)_\rho^C$, then $z_{k+1} \rhd_{\pi(k+1)} z_{k+2}$ in $G_{k+1}(\Sigma, b)_\rho$ and so is in $G_{k+1}(\Sigma, b)_{\rho'}$. Thus, a CIB solution in $G_{k+1}(\Sigma, b)_\rho$ and $G_{k+1}(\Sigma, b)_{\rho'}$. Hence, their CIB solutions are $c^\ell d^{k-\ell+1}$ and $c^{\ell'} d^{k-\ell'+1}$, so, $RC_{k+1}(\rho) = k - \ell + 1 \leq k - \ell' + 1 = RC_{k+1}(\rho')$. If $k+1 \in L_n(\Sigma)_\rho$, then $z_{k+1} \bowtie_{\pi(k+1)} z_{k+2}$ in $G_{k+1}(\Sigma, b)_\rho$. Hence, the CIB solution is c^{k+1}, so $RC_{k+1}(\rho) = 0$. We have the induction step.

$\langle \mathbf{2} \rangle$ Since $\Lambda_{\rho_i} = \{\underline{\gamma}_i, \overline{\gamma}_i\}$, it holds that $L_n(\Sigma)_{(0,0)} = \{1, ..., n\}$. Hence, all canonical CIB solution is c^k, i.e., $RC_k(0,0) = 0$ for all $k = 1, ..., n$. Conversely, we can find a sufficiently large ρ_1^*, ρ_2^* so that $L_n(\Sigma)_{(0,0)} = \emptyset$. So, for all $\rho \geq \rho^*$, $RC_k(\rho) = k$ for all $k \leq n$.■

7 Summary and Conclusions

We have taken four steps toward a resolution of the centipede paradox. First, we identified the centipede paradox by the quotation (*) from Selten [24] on a chain-store game and the BI theory. The quotation (*) was suggestive in the two ways to identify what the centipede paradox is and to motivate the concept of the reversed causality degree, which was discussed in Section 6.

A centipede game is much simpler than a chain-store game in that the choice of action d leads immediately to the end of the game in the centipede game, while the game continues (unless it is not at the last stage) if the chain-store takes the deterrence action. This difference is crucial for the concept of inertia in the CIB theory. We focussed on centipede games, though we have taken a slightly larger domain of games given by C1 to C6. Lemma 2.1 states that a centipede game includes a germ of coordination, which is Cooperative Motive (3) but Individual Motive (2) disables CM (3) to germinate and lead to the d-solution d^n.

We identified the difficulties hidden in the BI theory to be P1 (complete comparability) and P2 (forget the bygones). We weakened P1, by adopting the expected utility theory with probability grids due to Kaneko [9]. This theory defines a finite payoff ruler to measure payoffs, and the payoff scales of the ruler correspond to the cognitive ability of a player. When the cognitive ability of a player is low, the payoff ruler allows incomparability. Postulate P2 was modified so that when incomparability holds, the player follows inertial behavior when the decision node has some distance from the start x_1. With these two modifications, the CIB solution is defined.

Then, we studied the behavior of the CIB solution for centipede games. In general, a centipede game may have multiple CIB solutions, but the one type, called *canonical*, is uniquely determined as $c^\ell d^{n-\ell}$ and was regarded as the representative of all CIB solutions. This canonical form was indicative for a resolution of the paradox in that the behavior divide ℓ is close to n when the cognitive ability of at least one of the players is small. This was extensively discussed in Sections 5.2 and 6.

The last step was the introduction of the reversed causality degree $RC_k = k - \ell(k)$ in $G_k(\Sigma, b) \in \mathbb{G}_n(\Sigma, b)$. It was argued that the degree RC_k represents the Selten people's responses to the resulting outcomes of the CIB theory. The behavior divide $\ell(k)$ in $G_k(\Sigma, b)$ is quite close to k when at least one of the players has a low cognitive ability. Only when both have high cognitive abilities, the outcome is d^n; it was argued that the Selten people might refuse the condition for both players to have high cognitive abilities, but not the outcome d^n itself. Thus, our evaluation results are regarded as representing the Selten people's reactions. Section 6.3 synthesized the developments as a resolution of the centipede paradox.

Various questions are expected. First, what is the scope of the CIB theory together with the reversed causality degree? The structure of a centipede game is crucial for the concept of inertia in our developments. It would be difficult to define "inertia" in a chain-store game, since the "retailers" are changing in the game. Since this difficulty is avoided in the finitely repeated prisoner's dilemma, this game has a smaller distance than the chain-store games; Marschak-Selten [18] and Selten-Stoecker [25] could be considered from this point of view. The concept of "inertia" is better suited to inductive game theory due to Kaneko-Matsui [11] and Kaneko-Kline [12]. Attempts of studies in these directions may entail important future researches.

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