Axiomatization of Random Utility Model with Unobservable Alternatives

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Abstract

The random utility model is one of the most fundamental models in economics. Falmagne (1978) provides an axiomatization but his axioms can be applied only when choice frequencies of *all* alternatives from all subsets are observable. In reality, however, it is often the case that we do not observe choice frequencies of some alternatives. For such a dataset, we obtain a finite system of linear inequalities that is necessary and sufficient for the dataset to be rationalized by a random utility model. Moreover, the necessary and sufficient condition is tight in the sense that none of the inequalities is implied by the other inequalities, and dropping any one of the inequalities makes the condition not sufficient.

Keywords: Random utility, axiom, network flow, polytope

1 Introduction

Consider a population of individuals choosing an alternative across choice sets. It is often the case that we do not observe choice frequencies of some alternatives. For ex-

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ample, consider the set of transportation methods that consists of bus, train, walking, and driving. We may be able to estimate the market share of public transportation methods (bus or train) based on the revenues of bus or train companies. However, it may be difficult for us to know whether a person drives or walks (unless we conduct a survey); thus, choice frequencies of walking and driving may not be available. There are many other economically important examples in which we do observe choice frequencies of some but not all alternatives.[1] In this paper we investigate a necessary and sufficient condition on the observable choice frequencies under which they are rationalized by a random utility model.

Falmagne (1978) provided an axiomatization of the random utility model, but datasets we consider are outside his framework because he requires that frequencies of choices of *all* alternatives from all subsets of the set of alternatives must be observed. Not so much is known about the characterization when a dataset fails to meet this requirement. (See our discussion on related literature for details.) One exception is McFadden and Richter (1990), who have a characterization for such cases, but their condition involves infinitely many inequalities and contains redundant inequalities. In fact, it is known that obtaining a *tight* characterization of the random utility model is very difficult in many cases. In fact, when choice frequencies are observed only on binary choice sets, it has been unknown how to obtain a tight characterization of a random utility since the 1980s.[2] See Martí and Reinelt (2011) for a survey. A more recent paper by Sprumont (2022) also mentions the difficulty of the problem.

We focus on a setup in which choice frequencies of some alternatives are always missing but those for the others are observable. Given such a dataset, we obtain a finite system of linear inequalities that is necessary and sufficient for the dataset to be rationalized by a random utility model. Moreover, the necessary and sufficient condition is tight in the sense that none of the inequalities is implied by any other, and dropping one of them makes the condition insufficient.[3] Thus, our characterization extends Falmagne (1978) and is useful in practice compared with that of McFadden

[1] For another example, consider school choice among private schools. The government can often obtain students' choice data from public schools but not from private schools. In this situation, choice frequencies over public schools are observable but those over private schools are not. See Section 2.1 for details.

[2] Tight characterization has been obtained only for the case where the number of alternatives is less than eight. See Reinelt (1993).

[3] Falmagne (1978)'s characterization is also almost tight. Suck (2002) and Fiorini (2004) prove that omitting just a few inequalities from Falmagne (1978) gives a tight characterization.

and Richter (1990).

Our necessary and sufficient conditions consist of two types of conditions. The first condition is the classical nonnegativity of the Block-Marschak polynomials, which appear in Falmagne (1978)'s characterization. The second condition is novel: it consists of inequalities obtained by the summation and subtraction of the Block-Marschak polynomials. This novel condition means that the nonnegativity of the Block-Marschak polynomials is not enough for the dataset to be rationalized by a random utility model; balances across the values of Block-Marschak polynomials are essential for the case of incomplete datasets.

To prove our result, we translate our problem into a network flow problem. This approach is originally developed by Fiorini (2004), who gives a shorter proof of the result of Falmagne (1978). Our methodological innovation is to use a feasibility theorem in a network, which provides a necessary and sufficient condition for the existence of a desirable network flow. This novel tool gives us a clear insight even in the presence of some missing alternatives.

Our results have two practical implications. First, in the empirical literature, researchers often put all unobservable alternatives together and treat them as an aggregated alternative, called an *outside option*, even when they know which elements are unobservable. Our result implies that this approach may ignore some features of the random utility restriction; more precisely, it does not consider the second condition (i.e., the balances across the values of the Block-Marschak polynomials). Second, we demonstrate that our approach to the problem using of network flows not only is useful for axiomatization but also provides efficiency when we test our conditions with given datasets.

Once the dataset has turned out to be consistent with a random utility model, another important question is "What can we say about missing parts in the dataset?" To answer this question, we provide an efficient algorithm to obtain a tight bound for the missing choice frequencies. This algorithm makes the most of the network structure of our problem. Without the network formulation, the efficiency of the algorithm is not guaranteed. See section 4 for details.

We now briefly discuss the related literature. As mentioned above, little is known about the characterization of the random utility model in the case of incomplete data. Our result differs from that of McFadden and Richter (1990) in that their characterization involves infinitely many inequalities and entails redundancy, while ours consists of finite inequalities and contains no redundant ones. In Section 5, we will explain that the theorem obtained by McFadden and Richter (1990) is based on the nonnegativity of some polynomials; and the polynomials contain redundancy in an essential way, unlike BM polynomials. Thus, removing some redundancy from the theorem of McFadden and Richter (1990) is possible but removing all redundancy would be difficult.

Other than the papers mentioned so far, only a few papers have studied the random utility characterization with incomplete data. McFadden (2006) considers a nested structure of choice sets: if choice frequencies are observable in a set D of alternatives, then choice frequencies are observable in any larger set E (i.e., $E \supseteq D$). In this paper, we discuss the random utility characterization under this restriction of available choice sets. Suck (2016) addresses the truncated complete choice environment, in which only choice sets with at least $k \geq 2$ alternatives are observable. Nevertheless, to the best of our knowledge, our setup, in which choice frequencies of some alternatives are missing, is novel in the literature. Moreover, these results are special cases of our theorem–cases in which there are no unobservable alternatives.

As mentioned, we use the network-flow theory to prove our results. Since the publication of Fiorini (2004), some more recent papers have used the network-flow theory to investigate different topics on random utility models. Turansick (2022) characterizes the condition for the identification of random utility models. Chambers, Masatlioglu, and Turansick (2021) provide a new model of random utility with more than one agent. Doignon and Saito (2022) characterize the adjacency of vertices and facets of a mulitiple-choice polytope, which corresponds to the set of random utility models. None of these papers studies incomplete datasets.

2 Model

Let X be a finite set of alternatives. Let X^* be a subset of X. We assume that the choice frequencies of the elements of X^* are not observable (even if a choice set includes the alternatives).

Let $\mathcal{D} \subseteq 2^X \setminus \emptyset$ be the set of choice sets. Unlike Falmagne (1978), we do not assume that $\mathcal{D} = 2^X \setminus \emptyset$. Note that $(\mathcal{D}, \subseteq)$ is a partially ordered set, where $\subseteq$ is the set inclusion. Like McFadden (2006), we assume that $\mathcal{D}$ is an *upper set* (i.e., $\mathcal{D}$ satisfies the following: $D \in \mathcal{D},\ E \supseteq D \implies E \in \mathcal{D}$). To make our notation simple, let $\mathcal{M} := \{(D,x) \in \mathcal{D} \times X \mid x \in D\}$ and $\mathcal{M}^* := \{(D,x) \in 2^X \times X \mid x \in D$ and $[x \in X^*$ or $D \notin \mathcal{D}]\}$. Note that for any $(D,x) \in \mathcal{M}$, the choice frequency over (D,x) is unobservable if and only if $(D,x) \in \mathcal{M}^*$.

Definition 2.1. *A nonnegative vector $\rho \in \mathbf{R}_+^{\mathcal{M} \setminus \mathcal{M}^*}$ is called an incomplete dataset if it satisfies the following conditions: for any $D \in \mathcal{D}$,*

(i) if $D \cap X^* = \emptyset$, then $\sum_{x \in D} \rho(D,x) = 1$; and

(ii) if $D \cap X^* \neq \emptyset$, then $\sum_{x \in D \cap (X^*)^c} \rho(D,x) \leq 1$.

When the context is clear, we will simply call ρ a *dataset* instead of an *incomplete dataset*. If ρ is an incomplete dataset, ρ is not defined on $(D,x) \in \mathcal{M}^*$, i.e., $\rho(D,x)$ is not observeable for $(D,x) \in \mathcal{M}^*$. This does not mean that we cannot know anything about the choice frequencies of elements in X^*. When $x^* \in X^*$ is the only one unobservable alternative in the choice set D (i.e., when $D \cap X^* = \{x^*\}$), we can calculate $\rho(D,x^*)$ as

$$\rho(D,x^*) = 1 - \sum_{y \in D \setminus x^*} \rho(D,y).^{4}$$

Definition 2.2. *A nonnegative vector $\hat{\rho} \in \mathbf{R}_+^{\{(D,x) \mid x \in D \in 2^X\}}$ is called a complete dataset if, for any $D \in 2^X$, $\sum_{x \in D} \hat{\rho}(D,x) = 1$.*

2.1 Examples

Example 1 (Transportation): An analyst is often able to estimate the market share of public transportation methods (i.e., bus or train) based on the revenues of bus or train companies. However, it is sometimes difficult for the analyst to know separately the percentages of people who drive or walk. In this case, $X = \{\text{walk, drive, bus, train}\}$ and $X^* = \{\text{walk, drive}\}$. An example of the set of choice sets is $\mathcal{D} = \big\{\{w,b,t\}, \{w,d,b\}, \{w,d,t\}, \{w,d,b,t\}\big\}$, where w, d, b, and t stand for *walk, drive, bus*, and *train*, respectively. This set $\mathcal{D}$ can be obtained from the assumption that depending on the location of homes, some transportation methods are

[4] We often omit braces for singletons. Here, $D \setminus x^*$ means $D \setminus \{x^*\}$.

not available.[5]

Example 2 (Market Shares of Private Companies): One definition of market share is the percentage of a company's total sales divided by the market's total sales. The market's total sales can be estimated by consumer surveys. However, private companies occasionally do not disclose their financial information, including their total sales; thus the market shares of private companies are sometimes unobservable. For example, suppose that there are four companies (i.e., $X = \{a, b, c, d\}$). If companies c and d are private companies, then we do not know their sales (i.e., $X^* = \{c, d\}$). Other companies $\{a, b\}$ are public and the information from these companies is disclosed. In addition, the availability of products may vary across stores, which would give a variation of choice sets (i.e., $\mathcal{D}$).

Example 3 (School Choice for Private Schools): Applicants submit their choices among public schools so the government knows the percentage of students choosing each public school. However, it might not have access to information on how many students choose each private school. For example, suppose that there are four schools (i.e., $X = \{a, b, c, d\}$). Among them, c and d are private schools for which we do not know the choice frequencies (i.e., $X^* = \{c, d\}$). The availability of schools may depend on the location of homes, which would give a variation of choice sets (i.e., $\mathcal{D}$).

2.2 Random-Utility Rationalization

Let $\mathcal{L}$ be the set of rankings or strict preference relations on X, i.e., binary relations that are irreflexive, asymmetric, transitive, and weakly complete.[6]

Definition 2.3. *An incomplete dataset ρ is random-utility (RU) rationalizable if there exists $\mu \in \Delta(\mathcal{L})$ such that, for any $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$,*

$$\rho(D, x) = \mu(\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x).$$

We then say that μ rationalizes ρ.

[5] Assuming that the analyst believes that the distribution of preferences is independent of the location of homes, it would make sense to find a single distribution over ranking that describes the choice frequencies across $\mathcal{D}$.

[6] A binary relation is weakly complete if, for any distinct elements $x, y \in X$, either $x \succ y$ or $y \succ x$.

Definition 2.4. *Let $p \in \mathbf{R}^{\{(D,x)|x \in D \in 2^X\}}$. For any (D, x) such that $x \in D$, define*

$$K(p, D, x) = \sum_{E:E \supseteq D} (-1)^{|E \setminus D|} p(E, x).$$

$K(p, D, x)$ is called a Block-Marschak (BM) polynomial.[7]

Note that, given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M} \setminus \mathcal{M}^*}$, the BM polynomial $K(\rho, D, x)$ can be calculated if and only if $(D, x) \notin \mathcal{M}^*$ (i.e., $x \notin X^*$ and $D \in \mathcal{D}$).

3 Theorem

Recall that $(2^{X^*}, \subseteq)$ is a partially ordered set with the set inclusion $\subseteq$. Consider a collection $\mathcal{E}$ of subsets of X^*; we assume that $\mathcal{E}$ is an upper set.[8] The complement $\mathcal{E}^c$ is a *lower set* (i.e., $\mathcal{E}^c$ satisfies the following: $E \in \mathcal{E}^c, D \subseteq E \implies D \in \mathcal{E}^c$).[9]

To characterize the RU-rationalizability of incomplete data, the class of choice set defined in the following is fundamental.

Definition 3.1. *A nonempty collection $\mathcal{C}$ of subsets of X is called a test collection if there exist $A \subseteq X \setminus X^*$ and a nonempty upper set $\mathcal{E} \subseteq 2^{X^*}$ such that $\mathcal{C} = \{A \cup E \mid E \in \mathcal{E}\}$. Moreover, the test collection is said to be essential if $\emptyset \neq A \neq X \setminus X^*$ and $\mathcal{E} \neq 2^{X^*}$.*

The following is our main theorem.

Theorem 3.2. *(a) An incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M} \setminus \mathcal{M}^*}$ is RU-rationalizable if and only if the following two conditions hold:*

- *(i) for any $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$ such that $1 < |D| < |X|$, the polynomial $K(\rho, D, x)$ is nonnegative; and*
- *(ii) for any essential test collection $\mathcal{C} \subseteq \mathcal{D}$,*

[7] As we will explain below, BM polynomial is crucial concept to characterize random utility models. BM polynomial appears in other contexts. For example, Brady and Rehbeck (2016) observe that one of their axiom is equivalent to a multiplicative version of the BM polynomial.

[8] Recall the property of an upper set: if $D \in \mathcal{E}, D \subseteq E \Longrightarrow E \in \mathcal{E}$. In the example in which $X^* = \{d, e\}$, all upper sets in 2^{X^*} are $\emptyset$, $\{\{d, e\}\}$, $\{\{d, e\}, \{d\}\}$, $\{\{d, e\}, \{e\}\}$, $\{\{d, e\}, \{d\}, \{e\}\}$, and $\{\{d, e\}, \{d\}, \{e\}, \emptyset\}$.

[9] We use the concept of lower set in the proof.

$$\left(\sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C}} K(\rho, D\cup x, x) - \sum_{(F,y):F\notin\mathcal{C},F\cup y\in\mathcal{C},y\notin X^*} K(\rho, F\cup y, y) \right) \geq 0 \quad (1)$$

(b) Moreover, the inequality conditions in (i) and (ii) are independent: for any inequality condition in (i) or (ii), there exists an incomplete dataset $\hat{\rho} \in \mathbf{R}_+^{\mathcal{M}\setminus\mathcal{M}^*}$ that violates the inequality but satisfies all the other conditions in (i) and (ii).

We first make comments on statement (a) of the theorem. Recall that for any $(D, x) \in \mathcal{M}$, the BM polynomial $K(\rho, D, x)$ is computable based on the observable data if and only if $(D, x) \notin \mathcal{M}^*$. Thus, condition (i) is testable. Also, when $\mathcal{C}$ is a test collection, $D \in \mathcal{C}$ and $D \cup x \notin \mathcal{C}$ imply that $x \notin X^*$.[10] Thus, the first term as well as the second term in condition (ii) can also be tested based on the available data.

The necessity of condition (i) in Theorem 3.2 follows from Falmagne (1978) and McFadden (2006), who show that a *complete* dataset is RU-rationalizable if and only if all BM polynomials are nonnegative. In fact, when $X^* = \emptyset$, our theorem reduces to the statement of Falmagne (1978) and McFadden (2006), although our proof does not rely on their proofs. Novel conditions appear in (ii), which mean that the nonnegativity of the Block-Marschak polynomials is in sufficient for the dataset to be RU-rationalizasble because balances across the values of Block-Marschak polynomials is essential for RU-rationalizability when the dataset is incomplete. For example, one Block-Marschak polynomial being too large may not be a good sign for RU-rationalizability. In Remark 3.10, we provide a further explanation of condition (ii) in terms of network flows.

Statement (b) is an essential part of Theorem 3.2; not only does the theorem give a necessary and sufficient condition, but it is also minimal in the sense of (b).[11] This is in contrast to the approach taken by McFadden and Richter (1990). As we will explain in Section 5, the conditions in McFadden and Richter (1990) are redundant in an essential way. Statement (b) in Theorem 3.2 may be surprising given the known difficulty of obtaining a tight characterization of the random utility model when datasets are incomplete. For instance, when choice frequencies are observed only on binary choice sets, obtaining a tight characterization of a random utility has

[10] Since $\mathcal{C}$ is a test collection, $\mathcal{C} = \{A \cup E | E \in \mathcal{E}\}$ for some $A \subseteq X \setminus X^*$ and an upper set $\mathcal{E} \subseteq 2^{X^*}$. If $x \in X^*$, then $D \in \mathcal{C}$ implies $D \cup x \in \mathcal{C}$ by the definition of test collections (especially by the fact that $\mathcal{E}$ is an upper set).

[11] Geometrically speaking, our theorem identifies the set of all facet-defining inequalities of the *random utility polytope*. (See B.1 for the definition of the polytope.)

been an open question since the 1980s, despite continuous effort across mathematics, psychology, and economics. See Chapter 6 of Martí and Reinelt (2011) for a survey.

Finally, we mention two important implications of our results for applied work as follows. In the existing empirical literature on industrial organization (IO), it is common practice for researchers to aggregate all unobservable alternatives into a single category, commonly referred to as the *outside option*. This aggregation is undertaken even when there is clarity on the constituents of X^*.

Our theorem posits that this *outside option approach* might overlook certain facets of the random utility restriction. Specifically, it neglects the stipulations in (ii) of Theorem 3.2. One central implication of our work is elucidating this discrepancy clearly. We achieve this by providing a minimal set of testable conditions for observed choice probabilities to be consistent with a random utility model.

Upon establishing that a dataset adheres to a random utility model, a pertinent inquiry emerges: "What insights can be gleaned about unobserved choice frequencies?" In Section 4, we identify sets of possible values of unobservable choice frequencies given Theorem 3.2. In Remark 4.4, we observe that the outside option approach yields rather basic identified sets, while our method results in more insightful and refined identified sets.

3.1 Sketch of the proof

In this subsection, we outline the proof of Theorem 3.2. All formal proofs are in the appendix. Given an incomplete dataset ρ, RU-rationalizability is rewritten as follows:[12]

$$(P1) \quad \exists\, \mu \in \Delta(\mathcal{L})$$

[12] An alternative way of formulating the problem would be as follows: there exists a complete dataset $\hat{\rho}$ that conincides on ρ on $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$ and all BM polynomials are nonnegative for any $(D, x) \in \mathcal{M}$. Given such $\hat{\rho}$, Falmagne (1978) and McFadden and Richter (1990) guarantee the existence of a desirable random utility model. An earlier verion of our paper adopt the formulation. Aguiar, Kashaev, Gauthier, and Plavala (2023) also adopt such a formulation to obtain an axiomatization of random utility model in a dynamic setup. Unlike this approach, our approach directly show the existence of a desirable random utility model without using the results by Falmagne (1978) and McFadden and Richter (1990).

such that for any $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$,

$$\rho(D, x) = \mu(\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x). \tag{2}$$

Based on Fiorini (2004), we can show the rationalization is possible if and only if there exists a nonnegative solution r to the following problem:

$$(P2) \quad \exists\, r \in \mathbf{R}_+^{\{(D \setminus x, D) | x \in D \in 2^X\}} \tag{3}$$

such that

$$\sum_{x \in X} r(X \setminus x, X) = 1, \tag{4}$$

$$\sum_{x \in D} r(D \setminus x, D) = \sum_{y \notin D} r(D, D \cup y) \text{ for all } D \in 2^X \text{ such that } 1 \leq |D| \leq |X| - 1, \tag{5}$$

$$r(D \setminus x, x) = K(\rho, D, x) \text{ for all } (D, x) \in \mathcal{M} \setminus \mathcal{M}^*. \tag{6}$$

Remember that for all $(D, x) \in \mathcal{M}$, $K(\rho, D, x)$ can be calculated based on the incomplete dataset if and only if $(D, x) \notin \mathcal{M}^*$.

We first review Fiorini (2004)'s result:

Lemma 3.3 (Fiorini (2004))**.** *Given a complete dataset* $\hat{\rho} \in \mathbf{R}_+^{\{(D,x) | x \in D \in 2^X\}}$*, there exists* $\mu \in \Delta(\mathcal{L})$ *satisfying (2) for any* (D, x) *such that* $x \in D \in 2^X$ *if and only if there exists* $r \in \mathbf{R}_+^{\{(D \setminus x, D) | x \in D \in 2^X\}}$ *satisfying (4), (5) and* $r(D \setminus x, x) = K(\hat{\rho}, D, x)$ *for all* (D, x) *such that* $x \in D \in 2^X$*.*

Based on the result by Fiorini (2004), we prove the following equivalence for incomplete datasets:

Lemma 3.4. *Given an incomplete dataset* $\rho \in \mathbf{R}_+^{\mathcal{M} \setminus \mathcal{M}^*}$*, a solution* μ *exists to* $(P1)$ *if and only if a solution* r *exists to* $(P2)$*.*

Now, we can focus on the new problem (P2) instead of (P1). The advantage of this translation is that (P2) can be seen as a feasibility problem on a network flow. To see this, we review basic notions of the network-flow theory. Recall that a *network* is a pair of a node set $\mathcal{N}$ and a set of edges $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$. [13] Two nodes s (source) and t (terminal) play special roles as explained below.

Consider a function $f : \mathcal{A} \to \mathbf{R}$. For any node $D \in \mathcal{N}$, let $f(D, \mathcal{N}) \equiv \sum_{E \in \mathcal{N}} f(D, E)$; $f(\mathcal{N}, D) \equiv \sum_{E \in \mathcal{N}} f(E, D)$[14]. $f(D, \mathcal{N})$ is the sum of outflows from D; $f(\mathcal{N}, D)$ is the sum of inflows to D.

A function $f : \mathcal{A} \to \mathbf{R}$ is called a *flow* on a network $(\mathcal{N}, \mathcal{A})$ if it satisfies the following conditions:

$$f(s, \mathcal{N}) - f(\mathcal{N}, s) = 1, \tag{7}$$

$$f(D, \mathcal{N}) - f(\mathcal{N}, D) = 0 \quad \forall D \in \mathcal{N} \setminus \{s, t\}, \tag{8}$$

$$f(\mathcal{N}, t) - f(t, \mathcal{N}) = 1. \tag{9}$$

(7) means the net outflows from s is one; (8) means the inflows equal to outflows at each node $D \notin \{s, t\}$; (9) means the net inflows to t is one.

The following lemma provides a necessary and sufficient condition for the existence of a nonnegative flow satisfying some capacity constraints. For each arc (D, E), let $l(D, E)$ and $u(D, E)$ be exogenously given lower and upper bounds of the flow $f(D, E)$ of the arc. We prove the result using maximum-flow theorem from Ford Jr and Fulkerson (2015).[15]

Lemma 3.5. *Let $l, u : \mathcal{A} \to \mathbf{R}_+$ be such that $l(D, E) \leq u(D, E)$ for $(D, E) \in \mathcal{A}$. There exists a flow $f : \mathcal{A} \to \mathbf{R}_+$ such that*

$$l(D, E) \leq f(D, E) \leq u(D, E) \quad \forall (D, E) \in \mathcal{A} \tag{10}$$

if and only if the following condition holds for all $\mathcal{C} \subseteq \mathcal{N}$

$$\sum_{(D,E) \in \mathcal{C} \times \mathcal{C}^c} u(D, E) - \sum_{(D,E) \in \mathcal{C}^c \times \mathcal{C}} l(D, E) \geq \begin{cases} 1 & \text{if } t \notin \mathcal{C}, s \in \mathcal{C}, \\ -1 & \text{if } t \in \mathcal{C}, s \notin \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

We now interpret equation (11): for any collection $\mathcal{C} \subseteq \mathcal{N}$, $f(D, E)$ is called

[13] We define a specific network flow in the next page.

[14] We define $f(D, \mathcal{N}) = 0$ if $(D, E) \notin \mathcal{A}$ for any $E \in \mathcal{N}$; Similarly, $f(\mathcal{N}, D) = 0$ if $(E, D) \notin \mathcal{A}$ for any $E \in \mathcal{N}$.

[15] We appreciate prof. Ui who pointed out a similar result appears in Rockafellar (1998).

an *outflow* from $\mathcal{C}$ if $D \in C$ and $E \notin \mathcal{C}$; $f(D, E)$ is called an *inflow* to $\mathcal{C}$ if $D \notin C$ and $E \in \mathcal{C}$. Thus the left-hand side is the sum of the upper bounds of outflows from $\mathcal{C}$ minus the sum of the lower bounds of inflows to $\mathcal{C}$.[16] On the other hand, the right-hand side is the net outflow from $\mathcal{C}$.

By applying Lemma 3.5 to the network flow defined by (P2), we can obtain a necessary and sufficient condition that is testable for the existence of a solution r to (P2). Note that (4) corresponds to (7) ; (5) corresponds to (8); and (4) and (5) implies $\sum_{x \in X} r(\emptyset, x) = 1$, which corresponds to (9). To incorporate our problem into the framework of Lemma 3.5, let

$$\begin{aligned}
&\mathcal{N} = 2^X, \\
&\mathcal{A} = \{(D, D \cup x) \mid D \subseteq X, x \notin D\}, \\
&s = \emptyset, \\
&t = X, \\
&l(D, D \cup x) = u(D, D \cup x) = K(\rho, D \cup x, x) \text{ if } (D \cup x, x) \notin \mathcal{M}^*, \\
&l(D, D \cup x) = 0 \text{ if } (D \cup x, x) \in \mathcal{M}^*, \\
&u(D, D \cup x) = +\infty \text{ if } (D \cup x, x) \in \mathcal{M}^*.
\end{aligned} \tag{12}$$

In the following, we say an arc $(D, D \cup x)$ is *observable* if $(D \cup x, x) \notin \mathcal{M}^*$[17]; $(D, D \cup x)$ is *unobservable* if $(D \cup x, x) \in \mathcal{M}^*$.[18] In the above formulation, we set the values of flows on observable arcs by BM polynomials (i.e., $l(D, D \cup x) = K(\rho, D \cup x, x) = u(D, D \cup x)$), which correponds to constraint (6). For unobservable arcs, we just require the nonnegativity (i.e., $u(D, D \cup x) = +\infty; l(D, D \cup x) = 0$). See Figure 1 for the case $|X| = 4$.

Remark 3.6. *Given the setup (12), we can provide an intuition behind Lemma 3.3. See Fiorini (2004) for details.*

- *In the setup, each $\emptyset - X$ directed path corresponds to a unique ranking $\succ$. For example, in the figure, the directed path $\emptyset - \{a\} - \{a, b\} - \{a, b, c\} - X$ corresponds to the ranking: $d \succ c \succ b \succ a$. For each ranking $\succ$, let $\Pi^{\succ}$ be the corresponding $\emptyset - X$ directed path.*

[16] In network-flow theory, this value is called residual capacity of a cut $(\mathcal{C}, \mathcal{C}^c)$.
[17] Equivalently, $D \cup x \in \mathcal{D}$ and $x \notin X^*$.
[18] Equivalently, $D \cup x \notin \mathcal{D}$ or $x \in X^*$.

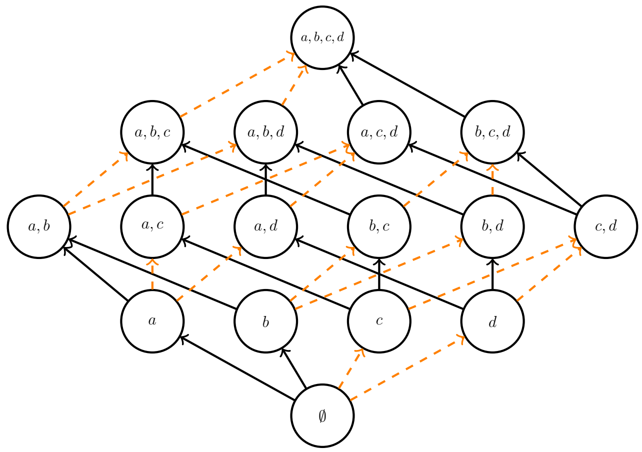

Figure 1: Network Flow. The figure shows the Boolean lattice of degree four, which corresponds to the network defined by (12) for the case in which $X = \{a, b, c, d\}$ and $X^* = \{c, d\}$ and $\mathcal{D} = 2^X \setminus \emptyset$. The solid arrows correspond to observable arcs: the dotted arrows correspond to unobservable arcs.

- *For each random utility model $\mu \in \Delta(\mathcal{L})$, we can construct a flow $r \in \mathbf{R}_+^{\{(D, D\cup x) | x \in D \in 2^X\}}$ on the network as follows: for each raking $\succ$, assign the value $\mu(\succ)$ on the directed path $\Pi^{\succ}$.*

- *Given the construction of the network flow, the values of an arc $(D, D \cup x)$ becomes $\mu(\{\succ | \ D^c \succ x \succ D \setminus x\}) = K(\rho, D, x)$, where the equality holds by the Möbius inversion. Note the constructed flow r satisfies all constraints in (4) and (5) as well as (6) for all $x \in D \in 2^X$.*

Definition 3.7. *A collection $\mathcal{C} \subseteq 2^X$ is said to be complete if*

$$D \in \mathcal{C} \implies \forall x \in X^*, D \cup x \in \mathcal{C}.$$

To apply Lemma 3.5, for $\mathcal{C} \subseteq 2^X$, define

$$\delta_\rho(\mathcal{C}) = \left(\sum_{\substack{(D,x): D\in\mathcal{C}, D\cup x \notin \mathcal{C}, \\ (D\cup x, x)\notin \mathcal{M}^*}} K(\rho, D \cup x, x) - \sum_{\substack{(E,y): E\notin\mathcal{C}, E\cup y \in \mathcal{C}, \\ (E\cup y, y)\notin \mathcal{M}^*}} K(\rho, E \cup y, y) \right) + 1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}. \quad (13)$$

Remark 3.8.

- *For any* $\mathcal{C} \subseteq 2^X$, $\delta_\rho(\mathcal{C})$ *is the net observable outflows from* $\mathcal{C}$. *To see this notice that the first term is the values of the observable outflows from* $\mathcal{C}$ *and the second term is the values of the observable inflows to* $\mathcal{C}$.

- δ *can also be defined with* $\mathcal{D} = 2^X \setminus \emptyset$ *and* $X^* \neq \emptyset$. *That is, for any complete dataset* $\hat{\rho}$, *define*

$$\delta_{\hat{\rho}}(\mathcal{C}) = \left(\sum_{\substack{(D,x) \\ D \in \mathcal{C}, D \cup x \notin \mathcal{C}}} K(\hat{\rho}, D \cup x, x) - \sum_{\substack{(E,y) \\ E \notin \mathcal{C}, E \cup y \in \mathcal{C}}} K(\hat{\rho}, E \cup y, y) \right) \\ +1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}$$

We will use this definition later.

Now, we can state our first feasibility result using Lemma 3.5.

Proposition 3.9. *Given an incomplete dataset* ρ, *a solution to (P2) exists if and only if* $\delta_\rho(\mathcal{C}) \geq 0$ *for any complete collection* $\mathcal{C}$ *such that* $\emptyset \notin \mathcal{C}$.

The condition given in Proposition 3.9 uses only the observable choice data to characterize a solution to (P2) since the above value (13) of δ_ρ depend only on the values of ρ on $\mathcal{M} \setminus \mathcal{M}^*$.

Remark 3.10. *We now can provide an intuition behind Proposition 3.9 by using Figure 2 in which* $X = \{a, b, c, d\}$, $X^* = \{c, d\}$, *and* $\mathcal{D} = 2^X \setminus \emptyset$.

- *Let* $\mathcal{C} = \{\{a, c\}, \{a, d\}, \{a, c, d\}\}$. *Then* $\mathcal{C}$ *is a complete collection.*

- *Red flows are observable outflows from* $\mathcal{C}$*; yellow flows are unobservable inflows to* $\mathcal{C}$*; blue flows are observable inflows to* $\mathcal{C}$. *Note that there are no unobservable outflows. This is because* $\mathcal{C}$ *is complete.*

- *By the equality between inflows and outflows, we have (Red outflows)=(Yellow inflows) + (Blue inflows).*

- *Note also that (Red outflows)=* $\sum_{(D,x):D \in \mathcal{C}, D \cup x \notin \mathcal{C}, (D \cup x, x) \notin \mathcal{M}^*} K(\rho, D \cup x, x)$*; (Blue inflows)=* $\sum_{(E,y):E \notin \mathcal{C}, E \cup y \in \mathcal{C}, (E \cup y, y) \notin \mathcal{M}^*} K(\rho, E \cup y, y)$. *Thus,* $\delta_\rho(\mathcal{C}) \equiv$ *(Red outflows)* $-$ *(Blue inflows).*

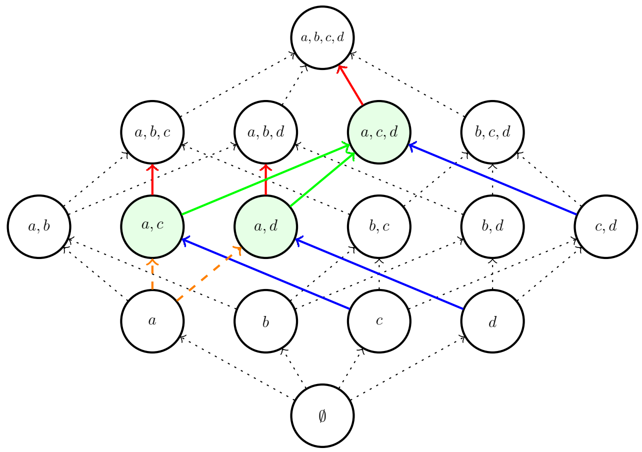

Figure 2: Outflows from $\mathcal{C}$ and inflows to $\mathcal{C}$. In the figure $X = \{a, b, c, d\}$ and $X^* = \{c, d\}$ and $\mathcal{D} = 2^X \backslash \emptyset$ and $\mathcal{C} = \{\{a, c\}, \{a, d\}, \{a, c, d\}\}$. Red flows are observable outflows from $\mathcal{C}$; yellow flows are unobservable inflows to $\mathcal{C}$; blue flows are observable inflows to $\mathcal{C}$. Note that green flows are flows contained in $\mathcal{C}$ and are not relevant to the value of $\delta_\rho(\mathcal{C})$, which is net observable outflows from $\mathcal{C}$.

- *Although yellow flows are unobserbables we know they are nonnegative Thus we have $\delta_\rho(\mathcal{C})$ = (Red outflows) − (Blue inflows) ≥ 0. This explains the necessity of our conditions (ii).*

Although Proposition 3.9 together with Lemma 3.5 successfully characterizes RU-rationalizability based only on the available data, the condition has some redundancy. In the following, we will obtain a tight characterization.

First, we show checking all essential test collections belong to $\mathcal{D}$, rather than all complete collections, is enough (i.e., statement (a-ii) of Theorem 3.2);

Lemma 3.11. *If $\delta_\rho(\mathcal{C}) \geq 0$ is for any test collection $\mathcal{C} \subseteq \mathcal{D}$, then $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$.*

This lemma reduces the number of conditions to be checked because any test collection is complete and it allows us to focus only on $\mathcal{D}$. The next lemma shows that we do not have to check the nonnegativity of δ_ρ for the nonessential test collections.

Lemma 3.12. *Let $\mathcal{C} \equiv \{A \cup E \mid E \in \mathcal{E}\}$ be a test collection with $A \subseteq X \setminus X^*$ and $\mathcal{E} \subseteq 2^{X^*}$. Assume that $\mathcal{C} \subseteq \mathcal{D}$. (i) If $\mathcal{E} = 2^{X^*}$, then $\delta_\rho(\mathcal{C}) = 0$; (ii) Suppose $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$. If $A = X \setminus X^*$ or $A = \emptyset$, then $\delta_\rho(\mathcal{C}) \geq 0$.*

Combining Proposition 3.9, Lemmas 3.11 and 3.12 immediately imply that checking the essential collections belonging to $\mathcal{D}$, rather than all test collections, is enough. This is stated formally in the following corollary, which proves statement (a) of Theorem 3.2.

Corollary 3.13. *A solution to (P2) exists if and only if $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$ and $\delta_\rho(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C} \subseteq \mathcal{D}$*

Finally, we explain the outline of the proof of statement (b) of Theorem 3.2, which claims the nonredundancy of conditions appear in (i) and (ii). This proof is the most intricate part of our proofs. In the proof, we first obtain the the nonredundancy results assuming $\mathcal{D} = 2^X \setminus \emptyset$; then translate the results into the given incomplete datasets. To explain the outline of the proof of the nonredundancy of inequality condition (ii), fix an essential test collection $\mathcal{C}^*$. It suffices to show that there exists an incomplete dataset ρ that satisfies all inequalities in (i) and all inequalities in (ii) except the one for $\mathcal{C}^*$.

We first provide a preliminary lemma that allows us to convert a flow from $\emptyset$ to X into a complete dataset:

Lemma 3.14. *Let $\mathcal{D} = 2^X \setminus \emptyset$. If there exists $r \in \mathbf{R}^{\{(D\setminus x, D) | x \in D \in 2^X\}}$ satisfying the following three conditions: (i) $\sum_{x\in X} r(X \setminus x, X) = 1$; (ii) for any $D \in \mathcal{D}$ such that $1 \leq |D| \leq |X| - 1$, $\sum_{x\in D} r(D \setminus x, D) = \sum_{y\notin D} r(D, D \cup y)$; (iii) for any $x \in D \in \mathcal{D}$, $\sum_{E:E\supseteq D} r(E \setminus x, E) \geq 0$, then there exists an complete dataset $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ such that $\sum_{x\in D} \hat{\rho}(D, x) = 1$ for all $D \in D$ and $K(\hat{\rho}, D, x) = r(D \setminus x, D)$ for any (D, x) such that $x \in D \in 2^X$.*

For any $\mathcal{C} \subseteq \mathcal{N}$, define

$$\delta_r(\mathcal{C}) \equiv \sum_{\substack{(D,x): \\ D\in\mathcal{C}, D\cup x\notin\mathcal{C}, x\notin X^*}} r(D, D \cup x) - \sum_{\substack{(E,y): \\ E\notin\mathcal{C}, E\cup y\in\mathcal{C}, y\notin X^*}} r(E, E \cup y) + 1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}. \tag{14}$$

Given Lemma 3.14, we will construct a flow $r \in \mathbf{R}^{\{(D\setminus x, D)|(D,x)\in\mathcal{M}\}}$ that satisfies conditions (i)–(iii) in the lemma and the following two conditions: (a) $r(D \setminus x, D) \geq 0$ for all $x \notin X^*$; (b) $\delta_r(\mathcal{C}^*) < 0$ and $\delta_r(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$ except $\mathcal{C}^*$. (See Lemma A.3 in the appendix for the complete statements.) By using Lemma

3.14, we translate the flow r into a complete data set $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x \in D \in 2^X\}}$; then convert $\hat{\rho}$ to a incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M} \setminus \mathcal{M}^*}$. (See Corollary A.4 in the appendix.) The most difficult part is the construction of the flow r. The difficulty comes from the fact that we need to change the value of δ_r only on one particular essential test collection $\mathcal{C}^*$ but not the others; the values of $\delta_r(\mathcal{C})$ across test collections $\mathcal{C}$ are interdependent through the conservation law of the network flow; and essential test collections exist across the network. We overcome this difficulty by constructing several flows and combine them into one desirable flow r in an intricate way.

4 Bounds for Unobservable Choice Probabilities

In the previous section, we established a necessary and sufficient condition for an incomplete dataset to be RU-rationalizable. Given this result, in this section we obtain bounds for unobservable choice probabilities.

In practice, predicting unobservable choice probabilities is important. Recall, for instance, the transportation example (Example 1) in Section 2.1. In this example, how people commute is not observable unless they use public transportation (i.e., $X = \{\text{bus, train, walk, drive}\}$ and $X^* = \{\text{walk, drive}\}$). Suppose that the government is considering introducing a new tax on gasoline to encourage people to commute by public transportation. To assess the potential impact of the new policy, it is crucial for the government to know the percentage of people who commute by private car.

The most naive approach is merely to bound the fraction below by zero and above by the percentage of people who did *not* use public transportation. We will observe that this naive approach corresponds to the *outside option approach*, in which the analyst aggregates all unobservable alternatives into a singular category, termed the outside option.

A more careful way, which is the main object of interest here, is to characterize the upper and lower bounds of missing choice frequencies, assuming the choices are consistent with the random utility model. While the naive approach, which corresponds to the outside option approach, is almost uninformative, the more careful approach that we are proposing in this section is likely to produce a finer prediction and to be helpful in policy making.

In this section, we will propose a method to compute the upper and lower bounds of missing choice frequencies assuming that the dataset is RU-rationalizable. We apply the method to a rich dataset obtained by McCausland, Davis-Stober, Marley, Park, and Brown (2020).[19] We then compare the bounds we obtain with the bounds the naive approach implies. By doing so, we clarify the difference between the naive approach and our approach and demonstrate the practical importance of our approach.

4.1 Computation of Bounds

Let $\rho \in \mathbf{R}_+^{\mathcal{M}\setminus\mathcal{M}^*}$ be a given incomplete dataset.

Definition 4.1. *Assume that ρ is RU-rationalizable. A complete dataset $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ is RU-consistent with ρ if (i) $\rho = \hat{\rho}$ on $\mathcal{M}\setminus\mathcal{M}^*$, and (ii) there exists $\mu \in \Delta(\mathcal{L})$ such that for any (D,x) with $x \in D \in 2^X$, $\hat{\rho}(D,x) = \mu(\succ\in\mathcal{L} \mid x \succ y \text{ for all } y \in D\setminus x)$.*

Let Γ be the set of $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ that is RU-consistent with the given incomplete dataset ρ. Remember that $\rho(D,x)$ is undefined (i.e., unobservable) if and only if $(D,x) \in \mathcal{M}^*$. With $(D,x) \in \mathcal{M}^*$ fixed, the goal in this section is to obtain bounds of $\hat{\rho}(D,x)$ for some $\rho \in \Gamma$. Note that by using (P1) in section 3.1, Γ can be written as follows:

$$\left\{ \hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}} \;\middle|\; \begin{array}{l} \text{There exists a solution } \mu \in \Delta(\mathcal{L}) \text{ to (P1) that satisfies} \\ \text{the condition (i) and (ii) in Definition 4.1} \end{array} \right\} \tag{15}$$

By Lemma 3.4, (P1) and (P2) are equivalent. Moreover, by the Möbius inversion, the condition (ii) in Definition 4.1 can be written as follows: for all (D,x) such that $x \in D \in 2^X$, $\mu(\{\succ\in\mathcal{L}|D^c \succ x \succ D\setminus x\}) = K(\hat{\rho},D,x)$. Since $\mu(\{\succ\in\mathcal{L}|D^c \succ x \succ D\setminus x\}) = r(D\setminus x, D)$, where μ is a solution to (P1); and r is a solution to (P2), the condition (ii) is equivalent to

$$r(D\setminus x, x) = K(\hat{\rho},D,x) \text{ for all } (D,x) \text{ such that } x \in D \in 2^X. \tag{16}$$

[19] The dataset is rich and unique in the sense that the authors collected all choice frequencies from all subsets of alternatives to test the random utility model. The dataset is available through the journal webpage.

Thus we can rewrite the set Γ (i.e., (15)) as follows:

$$\left\{ \hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}} \;\middle|\; \begin{array}{l} \text{There exists a solution } r \in \mathbf{R}_+^{\{(D\setminus x,D)|x\in D\in 2^X\}} \text{ to (P2) that} \\ \text{satisfies (16) and } \hat{\rho} = \rho \text{ on } \mathcal{M}\setminus\mathcal{M}^*. \end{array} \right\} \tag{17}$$

By eliminating observable flows r (i.e., $r(D\setminus x, D) = K(\rho, D, x)$ for all $(D,x) \in \mathcal{M}\setminus\mathcal{M}^*$) in (P2), it can be verified that the conditions (4) and (5) of (P2) are equivalent to[20]

$$\sum_{(D,y)\in\mathcal{M}^*:y\in D} r(D\setminus y, D) - \sum_{(D\cup y,y)\in\mathcal{M}^*:y\notin D} r(D, D\cup y) = \delta_\rho(D) \text{ for all } D \subseteq X, \tag{18}$$

where $\delta_\rho(\cdot)$ is defined by (13). Note that $\mathcal{M}^*$ is the set where BM polynomials are not computable based on the observable dataset. The left-hand side is based on unobservable r, while the right-hand side $\delta_\rho(D)$ is based on the observed incomplete choice data ρ.

Using the Möbius inversion formula, we also can rewrite (16) into the following: for all (D,x) such that $x \in D \in 2^X$, we have

$$\hat{\rho}(D,x) = \sum_{E:E\supseteq D} r(E\setminus x, E), \tag{19}$$

where $r(E\setminus x, E) = K(\rho, E, x)$ for all $(E,x) \notin \mathcal{M}\setminus\mathcal{M}^*$. These observations imply that we can rewrite the set (17) into the following set:

Proposition 4.2. Γ *is equal to*

$$\left\{ \hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}} \;\middle|\; \begin{array}{l} \hat{\rho} = \rho \textit{ on } \mathcal{M}\setminus\mathcal{M}^* \textit{ and there exists } r \in \mathbf{R}_+^{\{(D\setminus x,D)|x\in D\in 2^X\}} \\ \textit{that satisfies (18) and (19).} \end{array} \right\}. \tag{20}$$

For each $(D,x) \in \mathcal{M}^*$, we are interested in the set of possible values $\hat{\rho}(D,x)$ for some $\hat{\rho} \in \Gamma$. As is pointed out in Manski (2007), since Γ is convex and all conditions in (20) are linear, the identified set is an interval, and its upper and lower bounds are given by $\overline{\rho}(D,x) \equiv \max_{\hat{\rho}\in\Gamma} \hat{\rho}(D,x)$ and $\underline{\rho}(D,x) \equiv \min_{\hat{\rho}\in\Gamma} \hat{\rho}(D,x)$, respectively.

[20] Condition (6) in (P2) is implied by (16) and $\hat{\rho} = \rho$ on $\mathcal{M}\setminus\mathcal{M}^*$.

Corollary 4.3. *For any $(D, x) \in \mathcal{M}^*$, the upper bound is obtained by*

$$\overline{\rho}(D, x) = \max_{r \in \mathbf{R}_+^{\{(D,x)|x \in D \in 2^X\}}} \sum_{E: E \supseteq D} r(E \setminus x, E) \tag{21}$$

subject to (18), where $r(E \setminus x, E) = K(\rho, E, x)$ for all $(E, x) \notin \mathcal{M} \setminus \mathcal{M}^*$. The lower bound $\underline{\rho}(D, x)$ solves a similar problem with a min replacing the max.

Compared with the original formulation (15) based on (P1), the derivation of bounds in Corollary 4.3 is computationally more efficient. This is because this problem can be seen as a minimum-cost transshipment problem, which is well known in the network-flow theory literature. (See, for example, Ahuja, Magnanti, and Orlin (1988) and Ford Jr and Fulkerson (2015) for details.) One of the key properties of this problem is that it is a linear program with a constraint that has an incidence matrix as its coefficient. An incidence matrix is a matrix-form representation of network structure, which is defined as a matrix consisting only of $0, 1$ and -1 with each column having exactly one element of 1 and -1. For this specific problem, a practical polynomial time algorithm, called the *network simplex algorithm*, can be applied. Since this algorithm relies heavily on the fact that the coefficient of the constraint is an incidence matrix, the original form (P1) does not have its benefit in terms of computational efficiency. We refer readers to Orlin, Plotkin, and Tardos (1993) and Orlin (1997) for further computational aspects of the algorithm. When $\mathcal{D} = 2^X \setminus \emptyset$, the bound (21) can be further simplified as shown in online appendix B.2.

In the following remark, we formalize the outside option approach. We observe that in the outside option approach, the constraint of the random utility model does not have any implication other than the naive constraint that the probabilities must sum up to one.

Remark 4.4. *Consider the naive approach in which we assume one outside option x_0 that represents all unobservable alternatives. Let $\hat{X} = (X \setminus X^*) \cup x_0$. For simplicity, we consider the choice sets which do not contain any elements in X^* or which contain all elements in X^*. (In other words, we ignore the data on choice sets that contain only some (but not all) element(s) of X^*.) For any $D \in \mathcal{D}$ such that $D \cap X^*$ or*

$X^* \subseteq D$, define $\hat{D}$ as follows:

$$\hat{D} = \begin{cases} D & \text{if } D \cap X^* = \emptyset; \\ (D \setminus X^*) \cup x_0 & \text{if } X^* \subseteq D. \end{cases}$$

Define $\hat{\mathcal{D}} = \{\hat{D} | D \in \mathcal{D}$ and $[D \cap X^* = \emptyset$ or $X^* \subseteq D]\}$. Note that for any $\hat{D} \in \hat{\mathcal{D}}$, (i) if $x_0 \notin \hat{D}$, then $D \in \mathcal{D}$; and (ii) if $x_0 \in \hat{D}$, then $(\hat{D} \setminus x_0) \cup X^* \in \mathcal{D}$.

For any $\hat{D} \in \hat{\mathcal{D}}$, (i) if $x_0 \notin \hat{D}$, then define $\hat{\rho}(\hat{D}, x) = \rho(D, x)$ for any $x \in \hat{D}$; and (ii) if $x_0 \in \hat{D}$, then define

$$\hat{\rho}(\hat{D}, x) = \begin{cases} 1 - \sum_{y \in \hat{D} \setminus x_0} \rho((\hat{D} \setminus x_0) \cup X^*, y) & \text{if } x = x_0; \\ \rho((\hat{D} \setminus x_0) \cup X^*, x) & \text{if } x \neq x_0. \end{cases}$$

Then $\hat{\rho}$ is a complete dataset with the reduced choice sets $\hat{\mathcal{D}} \subseteq 2^{\hat{X}}$.

The question is how much the requirement of $\hat{\rho}$ being RU-rationalizable restricts the identification bound for $\rho(D, x)$ for some $x \in X^*$.[21] With the outside option approach, it turns out the RU-rationalizability does not have any implications beyond the fact that the probabilities must sum up to one; that is, the identified set is

$$\left[0, 1 - \sum_{y \in \hat{D} \setminus x_0} \rho((\hat{D} \setminus x_0) \cup X^*, y)\right]. \tag{22}$$

To see this, notice that if the complete data $\hat{\rho}$ is RU-rationalizable, then the identified set will become (22). If $\hat{\rho}$ is not RU-rationalizable, then the identified set will become the empty set.

4.2 Application to Lottery Data

We now apply this method to a stochastic-choice dataset from the experiment conducted by McCausland et al. (2020). In the experiment, the authors fixed a set $X = \{0, 1, 2, 3, 4\}$ of five lotteries and asked 141 participants to choose one from each subset of X. Each participant made decision six times for each choice set. See McCausland et al. (2020) for further details. We aggregate these choice frequencies to

[21] It is easy to see that if the incomplete data ρ is RU-rationalizable, then the complete data $\hat{\rho}$ is RU-rationalizable.

construct a complete dataset denoted by ρ^{obs}.

In this exercise, we mask the choice probabilities of lotteries 0 and 1 and pretend not to observe them; in other words, we set $X^* = \{0,1\}$ and $\mathcal{D} = 2^X \setminus \emptyset$. Let $\mathcal{D}^*$ be the set of choice sets that contain at least two unobservables. Then, it follows that

$$\mathcal{D}^* = \{\{0,1\},\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,1,2,3\},\{0,1,2,4\},\{0,1,3,4\},\{0,1,2,3,4\}\}.$$

Under this setup, we will compute two types of bounds of the probability of lottery 0 being chosen in a given choice set D that contains both lotteries 0 and 1. One of them is the trivial bound (22) that is calculated by

$$\left[0, 1 - \sum_{x \in D \cap \{2,3,4\}} \rho(D,x)\right].$$

As explained in Remark 4.4, this bound corresponds to the outside option approach that treats all unobservable alternatives as one aggregated alternative.

The other bound is the one that takes RU-rationalizabity into account and is computed by the linear program (21).[22] The goal here is to examine how much the random utility assumption shrinks the identified set and improves the prediction of unobservable choice probabilities.

The lottery dataset is nearly but not exactly RU-rationalizable. As our method can be applied only to RU-rationalizable datasets, we will first fit a multinomial logit model to the dataset to get a calibrated dataset that is close to the original one but is RU-rationalizable.[23]

We will then solve the linear programs described above for the calibrated dataset and obtain the bounds of the probability of lottery 0 being chosen. Specifically, for a given choice set that contains both lotteries 0 and 1, we compute the identified set of the probability of lottery 0 being chosen by applying our method to the calibrated

[22] In the dataset, we have $\mathcal{D} = 2^X \setminus \emptyset$. Under this setup, we can further simplify the bound (21) into (27), as shown in the online appendix B.2. We use (27) to calculate the values.

[23] The detail is as follows: To find a RU-rationalizable dataset that approximates the original one, we consider a multinomial logit model with parameter $(a_1,\dots,a_4) \in \mathbf{R}^4$ $\rho(D,x) = \frac{e^{a_x}}{\sum_{y\in D} e^{a_y}}$,where $a_0 = 0$. We search for a parameter $(a_1,\dots,a_4)$ that minimizes the least square loss; that is, $(a_1^*,\dots,a_4^*) \in \operatorname{argmin}_{(a_1,\dots,a_4)\in\mathbf{R}^4} \sum_{D\in\mathcal{D}\setminus\mathcal{D}^*} \sum_{x\in D} \left(\frac{e^{a_x}}{\sum_{y\in D} e^{a_y}} - \rho^{obs}(D,x)\right)^2$. We define the calibrated (incomplete) dataset as the choice data induced by the multinomial logit model with the minimizer $(a_1^*,\dots,a_4^*)$.

RU-rationalizable dataset. The result is presented in Figure 3, which reports the two types of bounds and the actual choice probabilities in the data. Overall, the identified sets of the random utility model, shown in red, are much smaller than the naive bounds, shown in blue, especially when the choice set is large. Notice that some data points are outside the identified sets because the original dataset is not RU-rationalizable. However, even in such cases, they are sufficiently close to the identified sets, and we believe that the drastic shrinkage of the intervals due to the random utility assumption outweighs the risk of misspecification.

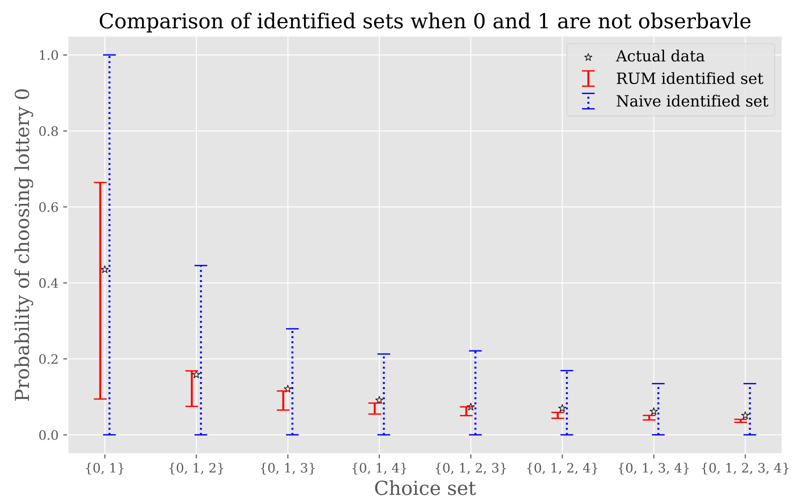

Figure 3: Comparison of identified sets of probabilities of choosing lottery 0 when lotteries 0 and 1 are unobservable. The identified sets for the random utility model are shown in red while those calculated in the naive way are shown in blue. The stars represent the actual probabilities in the dataset.

5 Relationship with McFadden and Richter (1990)

McFadden and Richter (1990) provide a characterization of the random utility model. Unlike the characterization of Falmagne (1978), the characterization of McFadden and Richter (1990) holds even for the case in which the dataset is incomplete. On the other hand, the conditions of McFadden and Richter (1990) involve an infinite number of sequences and some of the conditions are redundant. These features result from the fact that their characterization is obtained as a dual of the existence of rationalizing random utility model through the Farkas's lemma; and the dual condition, in general, involves an infinite number of inequalities. In this section, we further clarify the relationship between our approach and the approach taken by McFadden and Richter (1990). The main message is that the approach by McFadden and Richter (1990) is based on the nonnegativity of polynomials, which we call *McFadden and Richter polynomials*; and the polynomials contain redundancy in an essential way, unlike BM polynomials. Thus, one can improve the results obtained by McFadden and Richter (1990) by removing some redundancy; however it would be difficult to remove all redundancy from the approach.[24]

To clarify whether or not the dataset is incomplete, let $\hat{\mathcal{M}}$ be the set of pairs (D, x) such that $x \in D \in 2^X$ and $\rho(D, x)$ is well defined (i.e., $\rho(D, x)$ is observable to the analysts). We write the set of datasets as $\mathcal{P}(\hat{\mathcal{M}})$.[25] In the previous section, we assumed $\hat{\mathcal{M}} \equiv \mathcal{M} \setminus \mathcal{M}^*$.[26]

Definition 5.1. *Let $\hat{\mathcal{M}} \subseteq \{(D, x) \mid x \in D \in 2^X\}$ and $\rho \in \mathcal{P}(\hat{\mathcal{M}})$. For any sequence $(D_i, x_i)_{i=1}^n$ in $\hat{\mathcal{M}}$ define*

$$R((D_i, x_i)_{i=1}^n, \rho) = \max_{\succ \in \mathcal{L}} \sum_{i=1}^n 1\{x_i \succ D_i \setminus x_i\} - \sum_{i=1}^n \rho(D_i, x_i).$$

We call $R((D_i, x_i)_{i=1}^n, \rho)$ a McFadden and Richter (MR) polynomial.

Theorem 5.2. *(McFadden and Richter (1990)) Let $\hat{\mathcal{M}} \subseteq \{(D, x) \mid x \in D \in 2^X\}$ and $\rho \in \mathcal{P}(\hat{\mathcal{M}})$. There exists $\mu \in \Delta(\mathcal{L})$ such that for any $(D, x) \in \hat{\mathcal{M}}$*

$$\rho(D, x) = \mu\big(\{\succ \in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x\}\big) \tag{23}$$

if and only if $R((D_i, x_i)_{i=1}^n, \rho) \geq 0$ for any sequence $(D_i, x_i)_{i=1}^n$ in $\hat{\mathcal{M}}$.

Notice that same $(D, x) \in \hat{\mathcal{M}}$ appears arbitrary many times in the sequence $(D_i, x_i)_{i=1}^n$. Thus, the number of sequences to be tested is infinite, although there are finitely many pairs $(D, x) \in \hat{\mathcal{M}}$,

[24]In other words, if one removes all redundancy from the results by McFadden and Richter (1990), such results should reduce to Falmagne (1978) for the case of complete datasets and our results for the case of incomplete datasets (in our sense).

[25]Formally, $\mathcal{P}(\hat{\mathcal{M}}) = \{\rho \in \mathbf{R}_+^{\hat{\mathcal{M}}} \mid$ (i) if $(D, x) \in \hat{\mathcal{M}}$ for any $x \in D$, then $\sum_{x \in D} \rho(D, x) = 1$; (ii) if there exists $x \in D$ such that $(D, x) \notin \hat{\mathcal{M}}$, then $\sum_{x \in D} \rho(D, x) \leq 1\}$.

[26]In Falmagne (1978), $\hat{\mathcal{M}}$ is assumed to be $\{(D, x) \in \mathcal{D} \times X \mid x \in D \in 2^X\}$.

To clarify the relationship between the result in McFadden and Richter (1990) and ours, we will prove their result for the case when $\hat{\mathcal{M}} = \{(D,x) \mid x \in D \in 2^X\}$ by using the result in Falmagne (1978). Moreover, as shown in Proposition 5.7, we will show that (i) checking the nonnegativity of MR polynomials for sequences (D_i, x_i) in which each (D,x) appears at most twice is enough for RU-rationalilzability; and (ii) however, even such an improved result would contain redundant inequalities. Then, for incomplete datasets, in Remark 5.8 below, we observe that it is necessary to check the nonnegativity of MR polynomials for sequences in which each (D,x) appears *more than twice*, but some of such sequences would be redundant.

Definition 5.3. *A sequence $(D_i, x_i)_{i=1}^n$ in $\mathcal{M}$ is called redundant if there exists $D \in \{D_i\}_{i=1}^n$ such that*

$$\forall y \in D, \exists i \text{ such that } (D,y) = (D_i, x_i).$$

Remark 5.4.

- *Consider a sequence $\{(D_i, x_i)\}_{i=1}^n$. If there exists $D = \{y_1, \ldots, y_n\}$ with $y_i \neq y_j$ such that $x_i = y_i$ for each $i \in \{1, \ldots, n\}$, then the sequence is redundant and $R((D, y_i)_{i=1}^n, \rho) = 0$ for any ρ.*
- *If an original sequence contains a redundant subsequence $(D, y_i)_{i=1}^n$ like the one above, removing the subsequence does not affect the value of the MR polynomial. In this sense, it is without loss to focus on nonredundant sequences.*

Definition 5.5. *Let $\hat{\mathcal{M}} \subseteq \{(D,x) \mid x \in D \in 2^X\}$. Define*

$$\mathcal{P}_{MR}(m|\hat{\mathcal{M}}) = \left\{ \rho \in \mathcal{P}(\hat{\mathcal{M}}) \middle| \begin{array}{l} R((D_i,x_i)_{i=1}^n, \rho) \geq 0 \text{ for any non-redundant sequence } (D_i,x_i)_{i=1}^n \\ \text{in which no } (D,x) \in \hat{\mathcal{M}} \text{ appears more than } m \text{ times} \end{array} \right\}.$$

$\mathcal{P}_{MR}(m+1|\hat{\mathcal{M}}) \subseteq \mathcal{P}_{MR}(m|\hat{\mathcal{M}})$ for any m. As m increases, the number of sequences to be tested increases and, the number of stochastic choice functions that satisfy the condition therefore becomes smaller. Let $\mathcal{P}_r(\hat{M})$ be the set of random utility models. That is $\mathcal{P}_r(\hat{M}) \equiv \{\rho \in \mathcal{P}(\hat{\mathcal{M}}) \mid$ there exist $\mu \in \Delta(\mathcal{L})$ such that (23) holds for any $(D,x) \in \hat{\mathcal{M}}\}$.

Remark 5.6. *Let $\hat{\mathcal{M}} \subseteq \{(D,x) \mid x \in D \in 2^X\}$. The result in McFadden and Richter (1990) can be written as follows: For any $\rho \in \mathcal{P}(\hat{\mathcal{M}})$, $\rho \in \mathcal{P}_{MR}(m|\hat{\mathcal{M}})$ for any positive* *integer m if and only if $\rho \in \mathcal{P}_r(\hat{\mathcal{M}})$.*[27] *That is,* $\cap_{m=1}^{+\infty}\mathcal{P}_{MR}(m|\hat{\mathcal{M}}) = \mathcal{P}_r(\hat{\mathcal{M}})$.[28]

McFadden and Richter (1990) discuss the difficulty of providing an upper bound on m. They prove that repetition is necessary in general. In other words, they have proved that $\mathcal{P}_{MR}(1|\hat{\mathcal{M}})$ is too big (i.e., $\mathcal{P}_{MR}(1|\hat{\mathcal{M}}) \supsetneq \mathcal{P}_r(\hat{\mathcal{M}})$). Our next result shows that considering $m = 2$ is enough for the case in which there are no missing data.

Proposition 5.7. *Suppose that $\hat{\mathcal{M}} = \{(D, x) \mid x \in D \in 2^X\}$. Then (i) $\mathcal{P}_{MR}(2|\hat{\mathcal{M}}) = \mathcal{P}_r(\hat{\mathcal{M}})$; but (ii) some of sequences considered in $\mathcal{P}_{MR}(2|\hat{\mathcal{M}})$ are redundant.*

In the appendix, we prove this result by using the result of Falmagne (1978). This result not only clarifies the mathematical relationship between McFadden and Richter (1990) and Falmagne (1978), but also refines the result of McFadden and Richter (1990) for the case of complete datasets. The proof is based on the observation that (a) a BM polynomial can be written as a MR polynomial; (b) but not all MR polynomials can be written as BM polynomials. Observation (a) together with the result by Falmagne (1978) proves statement (i) in Proposition 5.7; Observation (b) implies statement (ii) in Proposition 5.7.

The remaining question is how large should m be for the case of general incomplete datasets. Our theorem implies the following:

Remark 5.8. *If $\hat{\mathcal{M}} = \mathcal{M} \setminus \mathcal{M}^*$, then $\mathcal{P}_{MR}(2|\hat{\mathcal{M}}) \supsetneq \mathcal{P}_r(\hat{\mathcal{M}})$.*

To see how Remark 5.8 is implied by Theorem 3.2, notice that for each test collection, each inequality condition (ii) in Theorem 3.2 involves multiple BM polynomials and each BM polynomial corresponds to a sequence $(D_i, x_i)_{i=1}^n$ in which some $(D, x) \in \hat{\mathcal{M}}$ appears twice. Thus, each inequality condition (ii) would correspond to a sequence in which some $(D, x) \in \hat{\mathcal{M}}$ appears more than twice. It would be possible to obtain a nontrivial upper bound m for which we have $\mathcal{P}_r(\hat{\mathcal{M}}) \subseteq \mathcal{P}_{MR}(m|\hat{\mathcal{M}})$ given Theorem 3.2. However, unlike our theorem, even such a result would contain redundant conditions, as some of MR polynomials are redundant even for the complete datasets, unlike BM polytnomials. In this way, our approach based on BS polynomials provides a more tight characterization than the approach based on MR polynomials.

[27] McFadden and Richter (1990) allow redundant sequences as explained above.

[28] $\mathcal{P}_{MR}(m|\hat{\mathcal{M}})$ becomes smaller and "converges" to $\mathcal{P}_r(\hat{\mathcal{M}})$ as m increases.

A Proofs

A.1 Proof of Lemma 3.4

We first prove that (P1) implies (P2). Fix a solution μ to (P1). Define a complete dataset $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$ by $\hat{\rho}(D,x) = \mu(\{\succ\in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x\})$ for all $x \in D \in 2^X$. Then $\hat{\rho} = \rho$ on $\mathcal{M} \setminus \mathcal{M}^*$. Moreover, by Lemma 3.3, we have $r \in \mathbf{R}_+^{\{(D\setminus x,D)|x\in D\in 2^X\}}$ that satisfies (4), (5) and $r(D\setminus x, x) = K(\hat{\rho}, D, x)$ for all (D,x) such that $x \in D \in 2^X$. Since $\hat{\rho} = \rho$ on $\mathcal{M}\setminus\mathcal{M}^*$, thus we have $r(D\setminus x, x) = K(\rho, D, x)$ for all $(D,x) \in \mathcal{M}$ such that $x \notin X^*$. Thus, r is a solution to (P2).

Next we prove that (P2) implies (P1). Fix a solution r to (P2). For any $(D,x) \in \mathcal{M}^*$, $\hat{\rho}(D,x) \equiv \sum_{E:E\supseteq D} r(E\setminus x, E)$. Define $\hat{\rho} = \rho$ on $\{(D,x) \mid x \in D \in 2^X\} \setminus \mathcal{M}^*$, where ρ is the given incomplete dataset. Thus, we obtain a complete dataset $\hat{\rho} \in \mathbf{R}_+^{\{(D,x)|x\in D\in 2^X\}}$. Then by the Möbius inversion, we have $r(D\setminus x, x) = K(\hat{\rho}, D, x)$ for all (D,x) such that $x \in D \in 2^X$. Then r satisfies (4), (5), and (6). Then by Lemma 3.3, there exists $\mu \in \Delta(\mathcal{L})$ such that $\hat{\rho}(D,x) = \mu(\{\succ\in \mathcal{L} \mid x \succ y \text{ for all } y \in D \setminus x\})$ for all (D,x) such that $x \in D \in 2^X$. Since $\rho = \hat{\rho}$ on $\mathcal{M}\setminus\mathcal{M}^*$, (2) holds for any $(D,x) \in \mathcal{M}\setminus\mathcal{M}^*$. Thus, μ is a solution to (P1).

A.2 Proof of Lemma 3.5

To prove the lemma we prove the following general result:

Theorem A.1. *Let $T, S \subseteq \mathcal{N}$ such that $S \cap T = \emptyset$, $a : S \to \mathbf{R}_+$, $b : T \to \mathbf{R}_+$ such that $\sum_{s\in S} a(s) = 1 = \sum_{t\in T} b(t)$. There exists $f : \mathcal{A} \to \mathbf{R}_+$ such that*

$$\begin{aligned}
&f(s,\mathcal{N}) - f(\mathcal{N},s) = a(s) \quad \forall s \in S,\\
&f(D,\mathcal{N}) - f(\mathcal{N},D) = 0 \quad \forall D \in \mathcal{N}\setminus(S\cup T),\\
&f(\mathcal{N},t) - f(t,\mathcal{N}) = b(t) \quad \forall t \in T,\\
&l(D,E) \le f(D,E) \le u(D,E) \quad \forall (D,E) \in \mathcal{A},
\end{aligned}$$

if and only if the following conditions hold for any $\mathcal{C} \subseteq \mathcal{N}$:

$$\sum_{(D,E)\in\mathcal{C}\times\mathcal{C}^c} u(D,E) - \sum_{(D,E)\in\mathcal{C}^c\times\mathcal{C}} l(D,E) \ge \sum_{t\in\mathcal{C}^c\cap T} b(t) - \sum_{s\in\mathcal{C}^c\cap S} a(s). \tag{24}$$

Proof.

Necessity: Suppose a feasible flow f exists.

$$\sum_{t\in\mathcal{C}^c\cap T} b(t) - \sum_{s\in\mathcal{C}^c\cap S} a(s) \leq f(\mathcal{C},\mathcal{C}^c) - f(\mathcal{C}^c,\mathcal{C}) \leq \sum_{(D,E)\in\mathcal{C}\times\mathcal{C}^c} u(D,E) - \sum_{(D,E)\in\mathcal{C}^c\times\mathcal{C}} l(D,E).$$

Sufficiency: Define an extended network with lower bound by $\mathcal{N}^* = \mathcal{N}\cup\{s^*,t^*\}$ and $\mathcal{A}^* = \mathcal{A}\cup\{(s^*,s) \mid s\in S\}\cup\{(t,t^*) \mid t\in T\}$ and

$$\begin{aligned}
&u^*(s^*,s) = a(s), l^*(s^*,s) = 0 \text{ for all } s\in S,\\
&u^*(t,t^*) = b(t), l^*(t,t^*) = 0 \text{ for all } t\in T,\\
&u^*(D,E) = u(D,E), l^*(D,E) = l(D,E) \text{ for all other arcs.}
\end{aligned}$$

We first define the residual capacity function e in the augmented network as follows: for any $\mathcal{C}^*\subseteq\mathcal{N}^*$

$$e(\mathcal{C}^*,\mathcal{N}^*\setminus\mathcal{C}^*) = \sum_{(D,E)\in\mathcal{C}^*\times(\mathcal{N}^*\setminus\mathcal{C}^*)} u^*(D,E) - \sum_{(D,E)\in(\mathcal{N}^*\setminus\mathcal{C}^*)\times\mathcal{C}^*} l^*(D,E).^{29}$$

(Similarly, we define the residual capacity function in the original network as follows: for any $\mathcal{C}\subseteq\mathcal{N}$ $e(\mathcal{C},\mathcal{N}\setminus\mathcal{C}) = \sum_{(D,E)\in\mathcal{C}\times(\mathcal{N}\setminus\mathcal{C})} u^*(D,E) - \sum_{(D,E)\in(\mathcal{N}\setminus\mathcal{C})\times\mathcal{C}} l^*(D,E)$.)

Then we will prove that $(\mathcal{N}^*\setminus\{t^*\},t^*)$ is a minimum s^*-t^* cut. Let be any $\mathcal{C}^*\subseteq\mathcal{N}^*$ such that $s^*\in\mathcal{C}^*$ and $t^*\notin\mathcal{C}^*$. That is, $(\mathcal{C}^*,\mathcal{N}^*\setminus\mathcal{C}^*)$ be an arbitrary cut separating s^* and t^*. Let $\mathcal{C} = \mathcal{C}^*\cap\mathcal{N}$. Then by the structure of network,

$$\begin{aligned}
&e(\mathcal{C}^*,\mathcal{N}^*\setminus\mathcal{C}^*) - e(\mathcal{N}^*\setminus\{t^*\},t^*)\\
&= e(s^*,\mathcal{N}^*\setminus\mathcal{C}^*) + e(\mathcal{C},\mathcal{N}^*\setminus\mathcal{C}^*) - e(\mathcal{N}^*\setminus\{t^*\},t^*) \quad (\because s^*\in\mathcal{C}^*)\\
&= e(s^*,\mathcal{N}^*\setminus\mathcal{C}^*) + e(\mathcal{C},\mathcal{N}\setminus\mathcal{C}) + e(\mathcal{C},t^*) - e(\mathcal{N}^*\setminus\{t^*\},t^*) \quad (\because t^*\in\mathcal{N}^*\setminus\mathcal{C}^*)\\
&= e(s^*,(\mathcal{N}\setminus\mathcal{C})\cap S) + e(\mathcal{C},\mathcal{N}\setminus\mathcal{C}) + e(\mathcal{C}\cap T,t^*) - e(T,t^*)\\
&= e(s^*,(\mathcal{N}\setminus\mathcal{C})\cap S) + e(\mathcal{C},\mathcal{N}\setminus\mathcal{C}) - e((\mathcal{N}\setminus\mathcal{C})\cap T,t^*)\\
&= \sum_{s\in(\mathcal{N}\setminus\mathcal{C})\cap S} a(s) + e(\mathcal{C},\mathcal{N}\setminus\mathcal{C}) - \sum_{t\in(\mathcal{N}\setminus\mathcal{C})\cap T} b(t)
\end{aligned}$$

[29] We write $\mathcal{N}^*\setminus\mathcal{C}^*$ instead of $(\mathcal{C}^*)^c$ to clarify the underling space is $\mathcal{N}^*$ not $\mathcal{N}$.

$$= \sum_{s\in(\mathcal{N}\setminus\mathcal{C})\cap S} a(s) + \left(\sum_{(D,E)\in\mathcal{C}\times(\mathcal{N}\setminus\mathcal{C})} u(D,E) - \sum_{(D,E)\in(\mathcal{N}\setminus\mathcal{C})\times\mathcal{C}} l(D,E) \right) - \sum_{t\in(\mathcal{N}\setminus\mathcal{C})\cap T} b(t),$$

which is nonnegative by (24). Thus $e(\mathcal{C}^*, \mathcal{N}^* \setminus \mathcal{C}^*) \geq e(\mathcal{N}^* \setminus \{t^*\}, t^*)$ for any cut $(\mathcal{C}^*, \mathcal{N}^* \setminus \mathcal{C}^*)$ separating s^* and t^* if and only (24) holds for any $\mathcal{C} \subseteq \mathcal{N}$.

It follows from the maximum-flow theorem with lower bounds (Theorem 6.1 Ahuja, Magnanti, and Orlin (1988)) that (24) implies the existence of a flow f^* from s^* to t^* that saturates all arcs of (T, t^*), that is, $f^*(t, t^*) = b(t)$ for all $t \in T$. Since $\sum_{s\in S} a(s) = 1 = \sum_{t\in T} b(t)$, we must have $f^*(s^*, s) = a(s)$ for all $s \in S$. These equalities imply that $f^*(S, \mathcal{N}) = 1$ and $f^*(\mathcal{N}, T) = 1$ and $f^*(D, \mathcal{N}) = f^*(\mathcal{N}, D)$ for all $D \in \mathcal{N} \setminus (T \cup S)$. Now define f as a restriction of f^* on $(\mathcal{N}, \mathcal{A})$. Then f satisfies all desired conditions. □

By the theorem, we obtain the lemma by letting both T and S singletons.

A.3 Proof of Proposition 3.9

Under the setup (12), there exists a solution to (P2) ⇔ there exists a flow f that satisfies the conditions in Lemma 3.5 ⇔ the condition (11) holds for any $\mathcal{C} \subseteq \mathcal{N}$, where the first equivalence holds by the setup and the second equivalence holds by Lemma 3.5. Thus, to show the proposition, it suffices to prove that the condition (11) holds for any $\mathcal{C} \subseteq \mathcal{N}$ if and only if $\delta_\rho(\mathcal{C}) \geq 0$ for any complete collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$.

Step 1: Suppose that (11) holds for any $\hat{\mathcal{C}} \subseteq \mathcal{N}$. For any $\mathcal{C} \subseteq \mathcal{N}$ such that all outflow from $\mathcal{C}$ is observable, then $\delta_\rho(\mathcal{C}) \geq 0$.

Proof. Fix any $\mathcal{C} \subseteq \mathcal{N}$. Note that $\sum_{(D,E)\in\mathcal{C}^c\times\mathcal{C}} l(D,E) = \sum_{(E,y):E\notin\mathcal{C},E\cup y\in\mathcal{C},(E\cup y,y)\notin\mathcal{M}^*} K(\rho, E\cup y, y)$. Assume that any outflow from $\mathcal{C}$ is observable. Then u does not take the value of $+\infty$. Thus we have $\sum_{(D,E)\in\mathcal{C}\times\mathcal{C}^c} u(D,E) = \sum_{(D,x):D\in\mathcal{C},D\cup x\notin\mathcal{C},(D\cup x,x)\notin\mathcal{M}^*} K(\rho, D\cup x, x)$. Thus the left-hand side minus the right-hand side of (11) equals to the value of $\delta_\rho(\mathcal{C})$. By the suppostion of the statement that (11) holds for any $\hat{C} \subset \mathcal{N}$, we have $\delta_\rho(\mathcal{C}) \geq 0$. ■

We use Step 1 to prove the following:

Step 2: If the condition (11) holds for any $\hat{\mathcal{C}} \subseteq \mathcal{N}$, then $\delta_\rho(\mathcal{C}) \geq 0$ for any complete collection $\mathcal{C} \subseteq \mathcal{N}$ such that $\emptyset \notin \mathcal{C}$.

Proof. Fix any complete collection $\mathcal{C} \subseteq \mathcal{N}$ such that $\emptyset \notin \mathcal{C}$. By the disjoint additivity, we have $\delta_\rho(\mathcal{C}) = \delta_\rho(\mathcal{C} \cap \mathcal{D}) + \delta_\rho(\mathcal{C} \setminus \mathcal{D})$.

Since $\mathcal{D}$ is an upper set and $\mathcal{C}$ is complete, there are no unobservable outflows from $\mathcal{C} \cap \mathcal{D}$. Thus by the supposition of this step, it follows from Step 1 that $\delta_\rho(\mathcal{C} \cap \mathcal{D}) \geq 0$.

Since $\emptyset \notin \mathcal{C} \setminus \mathcal{D}$ and $\mathcal{C} \setminus \mathcal{D}$ has no observable inflows, by the definition of δ_ρ, there are no negative terms in $\delta_\rho(\mathcal{C} \setminus \mathcal{D})$. Thus, we have $\delta_\rho(\mathcal{C} \setminus \mathcal{D}) \geq 0$. It follows that $\delta_\rho(\mathcal{C}) \geq 0$. ■

Step 2 proves one way of the proposition. In the following, we show the other way.

Step 3: If $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}} \subseteq \mathcal{N}$ such that $\emptyset \notin \hat{\mathcal{C}}$, then the condition (11) holds for any $\mathcal{C} \subseteq \mathcal{N}$.

Proof. Fix any $\mathcal{C} \subseteq \mathcal{N}$. If $\mathcal{C}$ has an unobservable outflow, then the left-hand side of (11) becomes infinite and (11) holds as desired. In the following consider the case where $\mathcal{C}$ has no unobservable outflows (i.e., all outflows are observable), which implies $\mathcal{C}$ is complete. As in the proof of Step 1, this implies that the left-hand side minus the right-hand side of (11) equals to $\delta_\rho(\mathcal{C})$.

If $\emptyset \notin \mathcal{C}$, by the supposition of this step, $\delta_\rho(\mathcal{C}) \geq 0$ because $\mathcal{C}$ is complete. Thus, (11) holds.

Assume $\emptyset \in \mathcal{C}$ in the following. Since $\mathcal{C}$ is complete, $2^{X^*} \subseteq \mathcal{C}$. Remember, by the assumption of the case, there exist no unobservable arcs $(D, D \cup x)$ coming out from $\mathcal{C}$. This means that any arc $(D, D \cup x)$ coming out from $\mathcal{C}$ is observable. Thus, we have $D \cup x \in \mathcal{D}$. Since $D \cup x \notin \mathcal{C}$, by the completeness of $\mathcal{C}$, we have $x \notin X^*$.

Now we build a certain subset of $\mathcal{C}$ inductively: initialize $\mathcal{G}_0 = 2^{X^*}$. At the first step, take all of the unobservable flows coming out of $\mathcal{G}_0$. The collection $\mathcal{G}_0$ is complete so all of these flows are going to a node of $\mathcal{D}^c$ (because we are considering unobservable flows). Call the set of all these nodes $\mathcal{F}_0 \subseteq \mathcal{D}^c$. Since each of these nodes is in $\mathcal{D}^c$, all of their subsets are also in $\mathcal{D}^c$ because $\mathcal{D}^c$ is a lower set.[30] In particular, for each $F \in \mathcal{F}_0$ we have $F \setminus X^* \in \mathcal{D}^c$.

Notice also that there is an arc from $\mathcal{G}_0$ to $F \setminus X^*$.[31] So we conclude that $F \setminus X^* \in \mathcal{C}$ (otherwise it contradicts with the fact that all outflows from $\mathcal{C}$ are

[30]Remember $E \in \mathcal{D}^c \implies D \in \mathcal{D}^c$ for all $E \supseteq D$.

[31]From $\emptyset$ to $F \setminus X^*$ which is a singleton in the first step. In step n, such an arc exists because we may write $F = G \cup x$ for $G \in \mathcal{G}_n$ and $x \in X \setminus X^*$. Notice $F \setminus X^* = (G \cup x) \setminus X^*$. Then by the construction of $\mathcal{G}_n$ we have $G \setminus X^* \in \mathcal{G}_n$ and therefore we have the arc $(G \setminus X^*, F \setminus X^*)$ coming out of $\mathcal{G}_n$

observable). By the completeness of $\mathcal{C}$, we have $\{(F \setminus X^*) \cup E : E \in 2^{X^*}\} \subseteq \mathcal{C}$.

Define $\mathcal{H}_0 = \bigcup_{F \in \mathcal{F}_0} \{(F \setminus X^*) \cup E : E \in 2^{X^*}\}$. Now define $\mathcal{G}_1 = \mathcal{G}_0 \cup \mathcal{H}_0$. At the nth step, define $\mathcal{H}_{n-1}$ in the same way as $\mathcal{H}_0$ and let $\mathcal{G}_n = \mathcal{G}_{n-1} \cup \mathcal{H}_{n-1}$. Since 2^X is finite, at some step n the set $\mathcal{G}_n$ will have no unobservable outflows. Call this terminal collection $\mathcal{G}$.

The collection $\mathcal{G}$ has no unobservable outflows and contains $\emptyset$ by its construction. It is straightforward to show that $\mathcal{G}$ has no inflows, which implies $\mathcal{G}$ is a lower set. To see why, suppose that $(D, D \cup x) \in \mathcal{G}^c \times \mathcal{G}$. Since $D \notin \mathcal{G}$, by the construction of $\mathcal{G}$, we have $D \setminus X^* \notin \mathcal{G}$. Since $D \cup x \in \mathcal{G}$ we have $(D \cup x) \setminus X^* \in \mathcal{D}^c$ by the construction of $\mathcal{G}$ again. It follows from that $D \setminus X^*$ and all subsets of $D \setminus X^*$ belong to $\mathcal{D}^c$.[32] Since $\emptyset \in \mathcal{G}$ there must be an arc from $\mathcal{G}$ to a subset E of $D \setminus X^*$, where $E \notin \mathcal{G}$. Since $E \notin \mathcal{D}$, this means $\mathcal{G}$ has an unobservable outflow, which is impossible by its construction.

Since (i) all the outflows from $\mathcal{G}$ are observable; (ii) there are no observable inflows to $\mathcal{G}$; (iii) and $\emptyset \in \mathcal{G}$, therefore we conclude that $\delta_\rho(\mathcal{G}) = 0$.

Note that $\mathcal{C} \setminus \mathcal{G} = \mathcal{C} \cap \mathcal{G}^c$ is complete because $\mathcal{G}$ is a lower set (in particular $\mathcal{G}^c$ is an upper set and thus is complete) and the intersection of complete sets is complete. Since $\emptyset \in \mathcal{G}$, $\mathcal{C} \setminus \mathcal{G}$ does not contain $\emptyset$. It follows from the supposition of the step that $\delta_\rho(\mathcal{C} \setminus \mathcal{G}) \geq 0$.

Since $\delta_\rho(\mathcal{C}) = \delta_\rho(\mathcal{C} \setminus \mathcal{G}) + \delta_\rho(\mathcal{G})$, we have $\delta_\rho(\mathcal{C}) \geq 0$, which means the inequality (11) for $\mathcal{C}$. ■

A.4 Disjoint Additivity of δ

Lemma A.2. *For any $\mathcal{C} \subseteq 2^X$, $\delta_\rho(\mathcal{C}) = \sum_{D \in \mathcal{C}} \delta_\rho(D)$.*

Proof. It suffices to show $\delta_\rho(\mathcal{C} \cup \mathcal{E}) = \delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$ for any disjoint sets $\mathcal{C}, \mathcal{E} \subseteq 2^X$. If there are no arcs connecting $\mathcal{C}$ and $\mathcal{E}$, then the result is trivial. Suppose otherwise. The values of a flow on the connecting arcs will be canceled out in $\delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$. Without loss of generality, suppose that there is a connecting arc $(D, D \cup x)$ and $D \in \mathcal{C}$ and $D \cup x \in \mathcal{E}$. Then the value $K(\rho, D \cup x, x)$ of the flow on the arc is added to $\delta_\rho(\mathcal{C})$ and subtracted from $\delta_\rho(\mathcal{E})$. Thus the value is canceled out in $\delta_\rho(\mathcal{C}) + \delta_\rho(\mathcal{E})$. In the same way, the value $K(\rho, D \cup x, x)$ of the flow on the connecting connecting arc $(D, D \cup x)$ and $D \in \mathcal{E}$ and $D \cup x \in \mathcal{C}$ will be canceled out. ■

[32] This is because $\mathcal{D}^c$ is a lower set.

A.5 Proof of Lemma 3.11

We will prove the statement by the following two steps.

Step 1: If $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any test collection $\hat{\mathcal{C}} \subseteq \mathcal{D}$, then $\delta_\rho(\mathcal{C}) \geq 0$ for any test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$.

Proof. Fix a test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$. Assume that $\mathcal{C} \nsubseteq \mathcal{D}$.

Case 1: Suppose that $\mathcal{C} \cap \mathcal{D} = \emptyset$. By the property of $\mathcal{D}$ (i.e., $D \in \mathcal{D}$ & $E \supseteq D \implies E \in \mathcal{D}$), there are no observable inflow into $\mathcal{C}$. That is, if there exists (E, y) such taht $E \notin \mathcal{C}$ and $E \cup y \in \mathcal{C}$, then $E \cup y \notin \mathcal{D}$, which shows that $\delta_\rho(\mathcal{C})$ does not contain any $-\sum_{(E,y):E\notin\mathcal{C},E\cup y\in\mathcal{C},E\cup y\in\mathcal{D},y\notin X^*} K(\rho, E \cup y, y)$. Moreover, since $\emptyset \notin \mathcal{C}$, $\delta_\rho(\mathcal{C})$ does not contain $-1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}$ either. This means that $\delta_\rho(\mathcal{C})$ does not contain any negative terms. Thus $\delta_\rho(\mathcal{C}) \geq 0$.

Case 2: Suppose that $\mathcal{C} \cap \mathcal{D} \neq \emptyset$. Let $\mathcal{C}^* = \mathcal{C} \cap \mathcal{D}$. By the property of $\mathcal{D}$ (i.e., $D \in \mathcal{D}$ & $E \supseteq D \implies E \in \mathcal{D}$), $\mathcal{D}$ is complete. Since $\mathcal{C}$ is complete, its union $\mathcal{C}^*$ is also complete. Since $\mathcal{C}^* \subseteq \mathcal{D}$, it follows from our supposition that $\delta_\rho(\mathcal{C}^*) \geq 0$. By the disjoint additivity of δ_ρ (Lemma A.2), $\delta_\rho(\mathcal{C}) = \delta_\rho(\mathcal{C}^*) + \sum_{D\in\mathcal{C}\setminus\mathcal{C}^*} \delta_\rho(\{D\})$. Since for any $D \in \mathcal{C} \setminus \mathcal{C}^* \equiv \mathcal{C} \cap \mathcal{D}^c$, we have $\{D\} \cap \mathcal{D} = \emptyset$. Since $D \neq \emptyset$, by Case 1, we have $\delta_\rho(\{D\}) \geq 0$. Thus, we have $\delta_\rho(\mathcal{C}) \geq 0$. ■

Step 2: If $\delta_\rho(\mathcal{C}) \geq 0$ is for any test collection $\mathcal{C}$ such that $\emptyset \notin \mathcal{C}$, then $\delta_\rho(\hat{\mathcal{C}}) \geq 0$ for any complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$.

Proof. Fix a complete collection $\hat{\mathcal{C}}$ such that $\emptyset \notin \hat{\mathcal{C}}$. Decompose $\hat{\mathcal{C}}$ as follows: for each $A \subseteq X \setminus X^*$ write $\mathcal{C}_A = \{D \in \hat{\mathcal{C}} : D \setminus X^* = A\}$. Clearly $\hat{\mathcal{C}} = \bigcup_{A\subseteq X\setminus X^*} \mathcal{C}_A$. It is easy to see that each $\mathcal{C}_A$ is a test collection and $\emptyset \notin \mathcal{C}_A$. Thus by the assumption of the step, $\delta_\rho(\mathcal{C}_A) \geq 0$. Notice that for $A \neq B$, $\mathcal{C}_A$ and $\mathcal{C}_B$ are disjoint. By Lemma A.2, $\delta_\rho(\mathcal{C})$ can be written as $\delta_\rho(\hat{\mathcal{C}}) = \sum_{A\subseteq X\setminus X^*} \delta_\rho(\mathcal{C}_A) \geq 0$. ■

A.6 Proof of Lemma 3.12

Let $\mathcal{C} \equiv \{A \cup E \mid E \in \mathcal{E}\}$ be a test collection with $A \subseteq X \setminus X^*$ and $\mathcal{E} \subseteq 2^{X^*}$. Assume that $\mathcal{C} \subseteq \mathcal{D}$.

Step 1: If $\mathcal{E} = 2^{X^*}$, then $\delta_\rho(\mathcal{C}) = 0$.

Proof. By the fact that $\mathcal{E} = 2^{X^*}$ and $\mathcal{C} \subseteq \mathcal{D}$, all flows into and out of $\mathcal{C}$ are observable.[33] By the equality of inflows and outflows (not necessarily non-negative),

[33]That is, (i) if there exists (D, x) such taht $D \in \mathcal{C}$ and $(D, x) \notin \mathcal{C}$, then $D \in \mathcal{D}$; and (ii) if there

it follows that $\delta_\rho(\mathcal{C})$ is zero. ■

Step 2: Suppose $K(\rho, D, x) \geq 0$ for all $(D, x) \in \mathcal{M}$ such that $x \notin X^*$. If $A = X \setminus X^*$ or $A = \emptyset$, then $\delta_\rho(\mathcal{C}) \geq 0$.

Proof. Assume $A = X \setminus X^*$. Fix any $D \in \mathcal{C}$ such that $D \neq X$. By the supposition, there is no observable flows coming out from D. Since $K(\rho, D, x) \geq 0$ for all $x \in X \setminus X^*$, it follows from the definition of δ_ρ, $\delta_\rho(D) \leq 0$ for all $D \neq X$. Remember $\delta_\rho(\mathcal{C}) = \sum_{D \in \mathcal{C}} \delta_\rho(D)$. Since $\delta_\rho(D) \leq 0$ for all $D \in \mathcal{C} \setminus x$, it suffices to prove $\delta_\rho(\mathcal{C}) = 0$ where $\mathcal{C}$ is the largest, or $\mathcal{C} = \{(X \setminus X^*) \cup E \mid E \in 2^{X^*}\}$. By Step 1, we have $\delta_\rho(\{(X \setminus X^*) \cup E \mid E \in 2^{X^*}\}) = 0$.

Assume $A = \emptyset$. If $\emptyset \in \mathcal{C}$, then $\mathcal{C} = 2^{X^*}$ by the fact that $\mathcal{C}$ is complete. Thus, all inflows into $\mathcal{C}$ are not observable and all outflows from $\mathcal{C}$ are observable, $\delta_\rho(\mathcal{C}) = \sum_{(D,x): D \setminus x \in \mathcal{C}, D \notin \mathcal{C}} K(\rho, D, x) \geq 0$. ■

A.7 Proof of Lemma 3.14

For any (D, x) such that $x \in D \in 2^X$, define $\rho(D, x) = \sum_{E \supseteq D} r(E \setminus x, x)$. By (iii), we have $\rho(D, x) \geq 0$ for all (D, x) such that $x \in D \in 2^X$. Fix any D to show $\sum_{x \in D} \rho(D, x) = 1$. Then we have $\sum_{x \in D} \rho(D, x) = \sum_{x \in D} \sum_{E \supseteq D} r(E \setminus x, E) = \sum_{y \in D} r(D \setminus y, D) + \sum_{x \in D} \sum_{\substack{E \supseteq D \\ |E| \geq |D|+1}} r(E \setminus x, E) = \sum_{\substack{E \supseteq D \\ |E| = |D|+1}} r(D, E) + \sum_{x \in D} \sum_{\substack{E \supseteq D \\ |E| \geq |D|+1}} r(E \setminus x, E) = \sum_{\substack{E \supseteq D \\ |E| = |D|+1}} \sum_{y \in E \cap D^c} r(E \setminus y, E) + \sum_{x \in D} \sum_{\substack{E \supseteq D \\ |E| = |D|+1}} r(E \setminus x, E) + \sum_{x \in D} \sum_{\substack{E \supseteq D \\ |E| \geq |D|+2}} r(E \setminus x, E) = \sum_{y \in E} \sum_{\substack{E \supseteq D \\ |E| = |D|+1}} r(E \setminus y, E) + \sum_{x \in D} \sum_{\substack{E \supseteq D \\ |E| \geq |D|+2}} r(E \setminus x, E) = \sum_{x \in E} \sum_{\substack{E \supseteq D \\ |E| \geq |D|+1}} r(E \setminus x, E)$, where the third equality holds by appling (ii) for the first term; the fourth equality is obtained by rewriting the first term and dividing the second term into the two terms; and the second to the last equality is obtained by combining the first two terms into one. Note that the last term has the same form as the term in the first equation but in the last term the summation over $E = D$ is deleted. By repeating this, we get $\sum_{x \in D} \rho(D, x) = \sum_{x \in E} \sum_{\substack{E \supseteq D \\ |E| \geq |D|+2}} r(E \setminus x, E)$. Finally we get $\sum_{x \in D} \rho(D, x) = \sum_{y \in X} r(X \setminus y, X)$, which is equal to 1 by (i).

A.8 Proof of Statement (b)

To prove statement (b) of Theorem 3.2, we prove the following lemma:

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exists (E, y) such taht $E \notin \mathcal{C}$ and $E \cup y \in \mathcal{C}$, then $E \cup y \in \mathcal{D}$.

Lemma A.3. *Let $\mathcal{D} = 2^X \setminus \emptyset$.*
(i) For each essential test collection $\mathcal{C}^*$, there exists $r \in \mathbf{R}^{\{(D\setminus x,D)|(D,x)\in\mathcal{M}\}}$ that satisfies conditions (i)–(iii) in Lemma 3.14 and the following two conditions: (a)$r(D \setminus x, D) \geq 0$ for all $x \notin X^*$; (b) $\delta_r(\mathcal{C}^*) < 0$ and $\delta_r(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$ except $\mathcal{C}^*$.
(ii) For each $(D, x) \in \mathcal{M}$ such that $1 < |D| < |X|$ and $x \notin X^*$, there exists $r \in \mathbf{R}^{\{(D\setminus x,D)|(D,x)\in\mathcal{M}\}}$ that satisfies conditions (i)–(iii) in Lemma 3.14 and the following two conditions: (a) $r(D \setminus x, D) < 0$; $r(E \setminus y, E) \geq 0$ for all $(E, y) \in \mathcal{M}$ s.t. $y \notin X^*$ and $(E, y) \neq (D, x)$; (b)$\delta_r(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.

Lemma A.3 implies Corollary A.4, which in turn implies statement (b) of Theorem 3.2.

Corollary A.4.
(i) For each $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$ such that $1 < |D| < |X|$, there exists an incomplete dataset $\rho^* \in \mathbf{R}^{\mathcal{M}\setminus\mathcal{M}^*}$ such that (a) $K(\rho^*, D, x) < 0$; and $K(\rho^*, E, y) \geq 0$ for all $(E, y) \in \mathcal{M} \setminus (\mathcal{M}^* \cup \{(D, x)\})$; (b) $\delta_{\rho^*}(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$.
(ii) For each essential test collection $\mathcal{C}^* \subseteq \mathcal{D}$, there exists an incomplete dataset $\rho^* \in \mathbf{R}^{\mathcal{M}\setminus\mathcal{M}^*}$ such that (a) $K(\rho^*, D, x) \geq 0$ for all $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$;(b) $\delta_{\rho^*}(\mathcal{C}^*) < 0$ and $\delta_{\rho^*}(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$.

The proofs of the lemma and the corollary are in the following subsections.

A.8.1 Proof of statement (i) in Lemma A.3

Fix an essential test collection $\mathcal{C}^*$. In the following, we will construct a flow r from $\emptyset$ to X such that $\delta_r(\mathcal{C}^*) < 0$ and $\delta_r(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$.

Let A^* be such that $D \setminus X^* = A^*$ for all $D \in \mathcal{C}^*$. (Such A^* exists because $\mathcal{C}^*$ is a test collection.) Since $\mathcal{C}^*$ is essential, $A^* \neq \emptyset$ and $A^* \neq X \setminus X^*$. Let $\hat{\mathcal{D}} \equiv \{D \mid$ there exists an essential test collection $\mathcal{C}$ such that $D \in \mathcal{C}\}$.

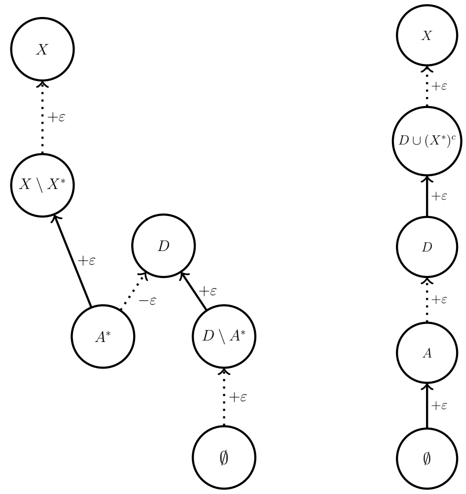
In the following we prove five claims to prove this lemma.

Claim A.5. *There exists a $\emptyset - X$ directed path Π_1 avoiding any nodes in $\hat{\mathcal{D}}$.*

Proof. We first construct a directed path $\emptyset - X \setminus X^*$ that avoids any node in $\hat{\mathcal{D}}$ by adding each element of $X \setminus X^*$ one by one. Note that each node A does not appear in any essential test collection since the only test collection containing A is the nonessential test collection $\{A \cup E \mid E \in 2^{X^*}\}$, which appears in Lemma 3.12 (i).

In the same way, we next construct a directed path $X \setminus X^* - X$ that avoids any node in $\hat{\mathcal{D}}$ by adding each element of X^* one by one. Note that each such node can be written as $(X \setminus X^*) \cup E$ for some $E \in 2^{X^*}$ and does not appear in any essential test collection since the only test collection containing the set is the nonessential test collection $\{(X \setminus X^*) \cup E \mid E \in \mathcal{E}\}$, which appears in Lemma 3.12 (ii).

By combining these two directed paths, we obtain a desirable $\emptyset - X$ directed path avoiding any nodes in $\hat{\mathcal{D}}$. ■



Note: The left figure shows the construction of a flow r^1_D in Claim A.4. (Given an incomplete dataset $\rho \in \mathbf{R}_+^{\mathcal{M} \setminus \mathcal{M}^*}$, solid arrows correspond to observable flows and dotted arrows correspond to unobservalbe flows.) The right figure shows the construction of a flow r^2_D in Claim A.5. In the figure $(X^*)^c$ means $X \setminus X^*$.

The next claim shows that for each node D in $\mathcal{C}^*$, there is a flow in which the value of δ is negative on the node D and the values of δ on the other nodes in $\hat{\mathcal{D}}$ are zero. For simplicity, we introduce a notation: Fix $D, E \subseteq X$ such that $D \subsetneq E$. For each directed path Π from D to E in the network defined by (12), define $r^\Pi \in \mathbf{R}^{\{(F, F\cup x) | x \in F \in 2^X\}}$ by

$$r^\Pi(F, F \cup x) = \begin{cases} 1 \text{ if } (F, F \cup x) \text{ is an arc that belongs to } \Pi, \\ 0 \text{ otherwise.} \end{cases} \tag{25}$$

Claim A.6. *Fix $\varepsilon \in (0, 1]$. For any D in $\mathcal{C}^*$, there exists a flow r^1_D such that $\delta(\{D\}) = -\varepsilon$ and $\delta(\{\hat{D}\}) = 0$ for any $\hat{D} \in \hat{\mathcal{D}} \setminus D$. Moreover r^1_D satisfies the three conditions in the Lemma 3.14.*

Proof. Fix $D \in \mathcal{C}^*$. Consider

- an $\emptyset - D$ directed path Π_2 containing going through the node $D \setminus A^*$,
- an $A^* - X$ directed path Π_3 which avoid any nodes in $\hat{\mathcal{D}}$ (Such a path exists because we can take the union of any $A^* - X \setminus X^*$ directed path and any $X \setminus X^* - X$ directed path as in Claim A.3.),
- The directed path Π_4 from A to D which follows the same order as Π_3.

Remember the definition (25). Fix $\varepsilon > 0$ and define $r^1_D \equiv (1-\varepsilon)r^{\Pi_1} + \varepsilon r^{\Pi_2} + \varepsilon r^{\Pi_3} - \varepsilon r^{\Pi_4}$. Note that r^1_D satisfies the three conditions in Lemma 3.14. To confirm the condition (iii) is satisfied it suffices to show that all negative flows are cancelled in the sum $\sum_{E:E\supseteq D} r(E \setminus x, E)$. (All of the negative flows are in Π_4 and are canceled by some flow in Π_3 because Π_4 follows the same order as Π_3.)

Note also that in the flow, $\delta_{r^1_D}(\{A^*\}) = \delta_{r^1_D}(\{D \setminus A^*\}) = \varepsilon$ and $\delta_{r^1_D}(\{X \setminus X^*\}) = -1$. To see $\delta_{r^1_D}(\{D\}) = -\varepsilon$ note that an arc going into D exists and is observable because A^* is not empty and consists of observable alternatives.

Moreover, for all other $\hat{D} \in \hat{\mathcal{D}}$, $\delta_{r^1_D}(\{\hat{D}\}) = 0$. (To see this note that δ is non-zero only when observable inflows are not equal to observable outflows.[34])

Since $A^*, X \setminus X^*, D \setminus A^* \notin \hat{\mathcal{D}}$ by Lemma 3.12, we have $\delta(\{\hat{D}\}) = 0$ for any $\hat{D} \in \hat{\mathcal{D}} \setminus D$. This completes the proof of the claim. ■

The next claim shows that for each node D in $\mathcal{C}^*$, there is a flow from $\emptyset$ to X in which the value of δ is positive on the node D and the values of δ on the other nodes in $\hat{\mathcal{D}}$ is zero.

Claim A.7. *Fix $\varepsilon \in (0,1]$. For any D in $\mathcal{C}^*$, there exists a flow r^2_D such that $\delta_{r^2_D}(\{D\}) = \varepsilon$ and $\delta_{r^2_D}(\{\hat{D}\}) = 0$ for all $\hat{D} \in \hat{\mathcal{D}} \setminus D$. Moreover the flow r^2_D satisfies the all conditions in Lemma 3.14.*

Proof. Choose any directed path from $\emptyset - X$ that goes through nodes A, D, and $(X \setminus X^*) \cup D$. We denote the path by Π_5. Define $r^2_D \equiv (1-\varepsilon)r^{\Pi_1} + \varepsilon r^{\Pi_5}$. Note that r^2_D satisfies the all conditions in Lemma 3.14.

In the flow, $\delta_{r^2_D}(\{A^*\}) = \delta_{r^2_D}((X \setminus X^*) \cup D) = -\varepsilon$ and $\delta_{r^2_D}(\{D\}) = \varepsilon$.

Moreover, for all other $\hat{D} \in \hat{\mathcal{D}}$, $\delta_{r^2_D}(\{\hat{D}\}) = 0$ by the same reaon as the previous claim. Since $A^*, (X \setminus X^*) \cup D \notin \hat{\mathcal{D}}$, we have $\delta_{r^2_D}(\{\hat{D}\}) = 0$ for all $\hat{D} \in \hat{\mathcal{D}} \setminus D$. This completes the proof of the claim. ■

[34] In the figure, this occurs when a dotted line becomes a solid line or vice-versa in the diagram.

Claim A.8. *Fix $\varepsilon \in (0,1]$. There exists a flow $\hat{r}$ such that (i) $-1/(2|\mathcal{C}^*|) = \delta_{\hat{r}}(\mathcal{C}^*) \leq \delta_{\hat{r}}(\mathcal{C})$ for any essential test collection $\mathcal{C}$; (ii) If $\mathcal{C} \not\supseteq \mathcal{C}^*$ then $\delta(\mathcal{C}) \geq 0$.*

Proof. Define $r^3 \equiv \sum_{D \in \mathcal{C}^*} \frac{1}{|\mathcal{C}^*|} r^1(D)$.Then, $\delta_{r^3}(\{D\}) = -\frac{\varepsilon}{|\mathcal{C}^*|}$ for each $D \in \mathcal{C}^*$; moreover, for all $\hat{D} \in \hat{\mathcal{D}} \setminus \mathcal{C}^*$, $\delta_{r^3}(\{\hat{D}\}) = 0$. Define $r^4 \equiv \frac{1}{|\mathcal{C}^*|} r^{\Pi_1} + \frac{|\mathcal{C}^*|-1}{|\mathcal{C}^*|} r^2(A^* \cup X^*)$. Then, $\delta_{r^4}(\{A^* \cup X^*\}) = \frac{|\mathcal{C}^*|-1}{|\mathcal{C}^*|}\varepsilon$; moreover, for all $\hat{D} \in \hat{\mathcal{D}} \setminus \mathcal{C}^*$, $\delta_{r^4}(\{\hat{D}\}) = 0$.[35] Define $\hat{r} = 1/2r^3 + 1/2r^4$. In $\hat{r}$, we have $\delta_{\hat{r}}(\mathcal{C}^*) = \sum_{D \in \mathcal{C}^*} \delta_{\hat{r}}(\{D\}) = \frac{1}{2}\left(-1 + \frac{|\mathcal{C}^*|-1}{|\mathcal{C}^*|}\right)\varepsilon = -\frac{1}{2|\mathcal{C}^*|}\varepsilon$.

Step 1: $\delta_{\hat{r}}(\mathcal{C}^*) \leq \delta_{\hat{r}}(\mathcal{C})$ for any essential test collection $\mathcal{C}$.

Proof. Fix any essential test collection $\mathcal{C}$. We consider the following two cases.

Case 1: There exists $D \in \mathcal{C}$ such that $D \setminus X^* \neq A^*$. (In fact, in this case, by the definition of essential test collection, $D \setminus X^* \neq A^*$ for all $D \in \mathcal{C}$.) Then, $\mathcal{C} \cap \mathcal{C}^* = \emptyset$. Since $\hat{D} \in \hat{\mathcal{D}} \setminus \mathcal{C}^*$, $\delta_{\hat{r}}(\{\hat{D}\}) = 0$, we have $\delta_{\hat{r}}(\mathcal{C}) = 0 \geq \delta(\mathcal{C}^*)$.

Case 2: $D \setminus X^* = A^*$ for all $D \in \mathcal{C}$. Since $\mathcal{C}$ is complete, $\mathcal{C}$ contains $A^* \cup X^*$. Since $\mathcal{C}^*$ contains all $D \in \hat{\mathcal{D}}$ such that $\delta_{\hat{r}}(\{D\}) < 0$ it is clear that $\delta_{\hat{r}}(\mathcal{C}) \geq \delta_{\hat{r}}(\mathcal{C}^*)$. ■

Step 2: If $\mathcal{C} \not\supseteq \mathcal{C}^*$ then $\delta(\mathcal{C}) \geq 0$.

Proof. Suppose that $\mathcal{C} \not\supseteq \mathcal{C}^*$. Then there exists $D^* \in \mathcal{C}^*$ such that $D^* \notin \mathcal{C}$. By Step 1, $\delta_{\hat{r}}(\mathcal{C} \cup \{D^*\}) \geq \delta(\mathcal{C}^*) = -\frac{1}{2|\mathcal{C}^*|}\varepsilon$. Also by definition of $\hat{r}$, $\delta_{\hat{r}}(\{D^*\}) = -\frac{1}{2|\mathcal{C}^*|}\varepsilon$ so $\delta_{\hat{r}}(\mathcal{C}) = \delta_{\hat{r}}(\mathcal{C} \cup \{D^*\}) - \delta_{\hat{r}}(\{D^*\}) \geq 0$. ■

□

We finally prove the statement of the lemma:

Claim A.9. *There exists a flow r^* from $\emptyset$ to X such that $\delta_{r^*}(\mathcal{C}^*) < 0$ and $\delta_{r^*}(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$.*

Proof. For each essential test collection $\mathcal{C}$ such that $\mathcal{C}^* \not\supseteq \mathcal{C}$, choose $D_{\mathcal{C}} \in \mathcal{C} \setminus \mathcal{C}^*$. Let $\mathcal{F}$ be the collection of such $D_{\mathcal{C}}$. (Since the number of test collections is finite, $\mathcal{F}$ is a finite collection.) Define $r^* \equiv \alpha\hat{r} + (1-\alpha)\sum_{D_{\mathcal{C}} \in \mathcal{F}} \frac{1}{|\mathcal{F}|} r^2_{D_{\mathcal{C}}}$. Then for any essential test collection $\mathcal{C}$, $\delta_{r^*}(\mathcal{C}) = \alpha\delta_{\hat{r}}(\mathcal{C}) + (1-\alpha)\sum_{D_{\mathcal{C}} \in \mathcal{F}} \frac{1}{|\mathcal{F}|}\delta_{r^2_{D_{\mathcal{C}}}}(\mathcal{C})$. Since $\delta_{r^2_{D_{\mathcal{C}}}}(D_{\mathcal{C}}) > 0$, there exists α small enough such that for any essential test collection $\mathcal{C}$ such that $\mathcal{C}^* \not\supseteq \mathcal{C}$, we have $\delta_{r^*}(\mathcal{C}) \geq 0$.

Note that $\delta_{r^*}(\mathcal{C}^*) = -\frac{\alpha}{2|\mathcal{C}^*|}\varepsilon$. Note also that $r^2_{D_{\mathcal{C}}}$ does not decrease values of δ for any test collection. Thus by statement (ii) of the previous claim, we have if $\mathcal{C} \not\supseteq \mathcal{C}^*$ then $\delta_{r^*}(\mathcal{C}) \geq 0$. It follows that $\delta_{r^*}(\mathcal{C}^*) < 0$ and $\delta_{r^*}(\mathcal{C}) \geq 0$ for any other essential test collection $\mathcal{C} \neq \mathcal{C}^*$. ■

[35] Note that $A^* \cup X^* \in \mathcal{C}^*$.

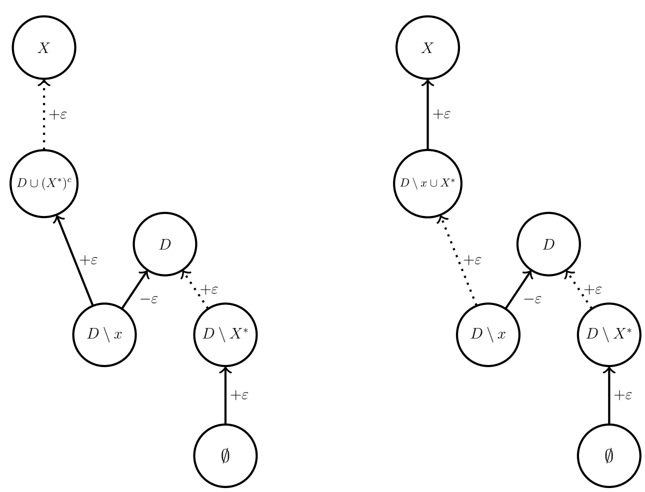
A.8.2 Proof of statement (ii) in Lemma A.3

Choose any arc $(D \setminus x, D)$ with $x \in X \setminus X^*$, $D \setminus x \neq \emptyset$, $D \neq X$. Let $\hat{\mathcal{D}} \equiv \{D \mid$ there exists an essential test collection $\mathcal{C}$ such that $D \in \mathcal{C}\}$. We will consider two cases:

Case 1: $D^c \cap (X \setminus X^*) \neq \emptyset$. Let Π_1 be a $\emptyset$ to X dipath which avoids any nodes in $\hat{\mathcal{D}}$. (Such a dipath exists by Claim A.3 in Section A.8.) Let Π_2 be a dipath from $\emptyset$ to D which passes through $D \cap (X \setminus X^*)$. Let Π_3 be a $D \setminus x$ to X dipath which passes through $D \cup (X \setminus X^*)$ but not D. Such a dipath exists because $D^c \cap (X \setminus X^*) \neq \emptyset$ implies that there exists an observable alternative $y \in D^c \cap (X \setminus X^*)$ and there exists an arc $(D \setminus x, D \cup y \setminus x)$. Fix $\varepsilon > 0$ and define

$$r^* \equiv (1-\varepsilon) r^{\Pi_1} + \varepsilon r^{\Pi_2} - \varepsilon r^{(D \setminus x, D)} + \varepsilon r^{\Pi_3}.$$

By definition, $r^*(D \setminus x, D) < 0$ and $r^*(E \setminus y, E) \geq 0$ for any (E, y) such that $y \notin X^*$ and $(D, x) \neq (E, y)$. Moreover, for any essential test collection $\mathcal{C}$, $\delta_{r^*}(\mathcal{C}) \geq 0$. To see this notice that the flow r^{Π_1} does not change any value of $\delta_r(\mathcal{C})$ for any essential test collection. By the definition of r^*, we have $\delta_{r^*}(\{D \setminus x\}) = 0$ and $\delta_{r^*}(\{D\}) \geq 0$.[36] For all other nodes E, $\delta_{r^*}(\{E\}) = 0$. Thus, we have $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.



Note: The left figure shows the construction of flows r^{Π_2}, r^{Π_3}, and $r^{(D \setminus x, D)}$. Solid arrows correspond to observable flows and dotted arrows correspond to unobservable flows. The right figure shows the construction of flows r^{Π_2}, r^{Π_4}, and $r^{(D \setminus x, D)}$. Solid arrows correspond to observable flows and dotted arrows correspond to unobservable flows.

[36] $\delta_{r^*}(\{D\})$ is either 0 or ε.

Case 2: $D^c \cap (X \setminus X^*) = \emptyset$. This means that D contains all elements in X^*. Notice $(D \setminus x) \cap X^* \neq X^*$ since otherwise $D = X$. So let Π_4 be a $D \setminus x$ to X dipath that passes through $(D \setminus x) \cup X^*$. (Note that the last arc is observable arc $(X \setminus x, X)$, where $x \notin X^*$.) Define

$$r^* \equiv (1-\varepsilon) r^{\Pi_1} + \varepsilon r^{\Pi_2} - \varepsilon r^{(D \setminus x, D)} + \varepsilon r^{\Pi_4}.$$

By definition, $r^*(D \setminus x, D) < 0$ and $r^*(E \setminus y, E) \geq 0$ for any (E, y) such that $y \notin X^*$ and $(D, x) \neq (E, y)$. Moreover, for any essential test collection $\mathcal{C}$, $\delta_{r^*}(\mathcal{C}) \geq 0$. To see this notice (i) $\delta_{r^*}(\{D \setminus x\}) = -\varepsilon$ but $\delta_{r^*}(\{(D \setminus x) \cup X^*\}) = \varepsilon$; (ii) any test collection $\mathcal{C}$ containing $D \setminus x$ contains $(D \setminus x) \cup X^*$. (i) and (ii) implies that the negative value of $\delta_{r^*}(\{D \setminus x\})$ is cancelled by the positive value of $\delta_{r^*}((D \setminus x) \cup X^*)$. For all other nodes E, $\delta_{r^*}(\{E\}) = 0$. Thus, we have $\delta_{r^*}(\mathcal{C}) \geq 0$ for any essential test collection $\mathcal{C}$.

A.8.3 Proof of Corollary A.4

Fix $\mathcal{C}^* \subseteq \mathcal{D}$. By Lemma A.3, there exists $r \in \mathbf{R}^{\{(D \setminus x, D) | x \in D \in 2^X\}}$ that satisfies conditions (a) and (b). By Lemma 3.14, there exists a complete dataset $\hat{\rho} \in \mathbf{R}_+^{\{(D,x) | x \in D \in 2^X\}}$ such that $\sum_{x \in D} \hat{\rho}(D, x) = 1$ for all $D \in 2^X$ and $K(\hat{\rho}, D, x) = r(D \setminus x, D)$ for any (D, x) such that $x \in D \in 2^X$. Let ρ^* be the projection of $\hat{\rho}$ on $\mathcal{M} \setminus \mathcal{M}^*$. In the following, we will show that $\delta_{\rho^*}(\mathcal{C}^*) < 0$ and $\delta_{\rho^*}(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$.

Let $\delta_{\hat{\rho}}$ be the function defined by (13) with respect to the complete dataset $\hat{\rho}$ with $\mathcal{D} = 2^X$ and $X^* = \emptyset$.[37] Let δ_{ρ^*} be the function defined by (13) with respect to the incomplete dataset of ρ^* with given $\mathcal{D}$ and X^*. Remember δ_r is the function defined by (14). Note that for any test collection $\mathcal{C} \subseteq \mathcal{D}$

$$\delta_r(\mathcal{C}) = \delta_{\hat{\rho}}(\mathcal{C}) = \delta_{\rho^*}(\mathcal{C}),$$

where the first equality holds because $K(\hat{\rho}, D, x) = r(D \setminus x, D)$ and the second equality holds because the value of δ does not depend on the values of $\hat{\rho}$ and ρ^* on $\mathcal{M}^*$. Thus, we have $\delta_{\rho^*}(\mathcal{C}^*) < 0$ and $\delta_{\rho^*}(\mathcal{C}) \geq 0$ for all essential test collection $\mathcal{C} \subseteq \mathcal{D}$ except $\mathcal{C}^*$.

[37] That is, $\delta_{\hat{\rho}}(\mathcal{C}) = \left(\sum_{(D,x): D \in \mathcal{C}, D \cup x \notin \mathcal{C}} K(\hat{\rho}, D \cup x, x) - \sum_{(E,y): E \notin \mathcal{C}, E \cup y \in \mathcal{C}} K(\hat{\rho}, E \cup y, y) \right)$ $+ 1\{X \in \mathcal{C}, \emptyset \notin \mathcal{C}\} - 1\{\emptyset \in \mathcal{C}, X \notin \mathcal{C}\}$.

A.9 Proof of Proposition 5.7

For each (E, y) such that $y \in E$, define $a_{(E,y)} \in \mathbf{R}^{\mathcal{D} \times X}$ by

$$a_{(E,y)}(D, x) = \begin{cases} -1 & \text{if } y = x \text{ and } |D \setminus E| \text{ is even,} \\ +1 & \text{if } y = x \text{ and } |D \setminus E| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

For any $\rho \in \mathcal{P}$, we have $a_{(E,y)} \cdot \rho = -K(\rho, E, y)$. For each (E, y), define a sequence (D_i, x_i) so that each (D, x) appears $-a_{(E,y)}(D, x) + 1$ times. Notice that since $a_{(E,y)}(D, x) \in \{-1, 0, 1\}$, we have $-a_{(E,y)}(D, x) + 1 \in \{0, 1, 2\}$. Then we have $\sum_{i=1}^{n} \rho(D_i, x_i) = (-a_{(E,y)} + 1) \cdot \rho = -K(\rho, E, y) + (2^{|X|} - 1)$. So $\sum_{i=1}^{n} \rho^{\succ}(D_i, x_i) = -K(\rho^{\succ}, E, y) + (2^{|X|} - 1)$ for each $\succ \in \mathcal{L}$. Moreover, $\max_{\succ \in \mathcal{L}} \sum_{i=1}^{n} \rho^{\succ}(D_i, x_i) = \max_{\succ \in \mathcal{L}} -K(\rho^{\succ}, E, y) + (2^{|X|} - 1) = -\min_{\succ \in \mathcal{L}} K(\rho^{\succ}, E, y) + (2^{|X|} - 1) = 2^{|X|} - 1$, where the last equality holds because $K(\rho^{\succ}, E, y) = 0$ if $x \succ y$ for some $x \in E$ and $K(\rho^{\succ}, E, y) \geq 0$ for any $\succ \in \mathcal{L}$. Therefore

$$K(\rho, E, y) \geq 0 \Leftrightarrow \max_{\succ \in \mathcal{L}} \sum_{i=1}^{n} \rho^{\succ}(D_i, x_i) \geq \sum_{i=1}^{n} \rho(D_i, x_i) \Leftrightarrow R((D_i, x_i)_{i=1}^{n}, \rho) \geq 0,$$

where the first equivalence holds because $\rho^{\succ}(D_i, x_i) = 1(x_i \succ y_i \forall y_i \in D \setminus x_i)$. Since each (D, x) appears at most twice in the sequence $\{(D_i, x_i)\}$, the nonnegativity of each BM polynomial corresponds to the nonnegativity of a MR polynomial defined with sequence where the same (D, x) appears in the sequence at most twice. This implies statement (i). However, the converse does not hold: Not all of the nonnegativity conditions of a MR polynomial correspond to the nonnegativity conditions of BM polynomials as one can see from the construction above. Thus statement (ii) also holds.

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B Online Appendix

B.1 Random utility polytope

Remark B.1. *For each ranking* $\succ \in \mathcal{L}$ *and* $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$*, define*

$$\rho^{\succ}(D, x) = \begin{cases} 1 & \text{if } x \succ y \text{ for all } y \in D \setminus x; \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

The stochastic choice function $\rho^{\succ}$ *gives probability one to the best alternative* x *in a choice set* D *according to the ranking* $\succ$*. The set of RU-rationalizable datasets is a polytope, that is, co.*$\{\rho^{\succ} \mid \succ \in \mathcal{L}\}$*, where co. denotes the convex hull. Our theorem characterizes all facet defining inequalities of the polytope.*

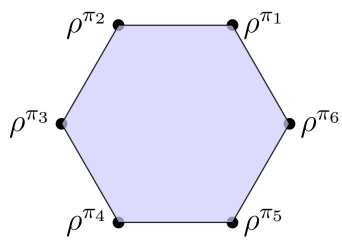

Figure 4: Random utility polytope

The hexagons in Figure 4 illustrates the polytope. Although the geometric intuition is useful, it is important to notice that the figure oversimplifies the reality since the number (i.e., $|X|!$) of vertices and the dimension of a random utility function can be very large.[38]

B.2 Further simplification of bounds when $\mathcal{D} = 2^X \setminus \emptyset$

In this section, we assume that $\mathcal{D} = 2^X \setminus \emptyset$, we provide further simplification of bounds of unosbservavle choice frequencies.

[38]To see why the dimension of a random utility function can be very large, notice that it assigns a number for each pair of $(D, x) \in \mathcal{M} \setminus \mathcal{M}^*$.

Corollary B.2. *Let $\mathcal{D} = 2^X \setminus \emptyset$. For $(D,x) \in \mathcal{M}^*$, the upper bound is obtained by*

$$\overline{\rho}(D,x) = \max_{\{r(D\setminus x, D)\}_{(D,x)\in\mathcal{M}^*}} \sum_{A': A\subseteq A' \subseteq X\setminus X^*} \sum_{E': E\subseteq E'\subseteq X^*} r(A'\cup E' \setminus x, A'\cup E') \tag{27}$$

subject to

$$\sum_{y\in E'} r(A'\cup E'\setminus y, A'\cup E') - \sum_{y\in X^*\setminus E'} r(A'\cup E', A'\cup E'\cup y) = \delta_\rho(A'\cup E') \tag{28}$$

for all $A' \subseteq X \setminus X^*$ and $E' \subseteq X^*$. The lower bound $\underline{\rho}(D,x)$ solves a similar problem with a min replacing the max.

Remark B.3.

- *The form of (28) implies that for $A', A'' \subseteq X \setminus X^*$, $E', E'' \subseteq X^*$, and $y \in E', z \in E''$, if $A' \neq A''$, then variables $r(A'\cup E'\setminus y, A'\cup E')$ and $r(A''\cup E''\setminus z, A''\cup E'')$ are independent; either of them does not restrict the other via the constraints (28).*

- *This means that each constraint can be considered separately.*

- *Therefore, we can optimize the inner sum of (27) separately. That is, the maximum value of the problem is equivalent to the sum of the maximum values of the following problems for all A' such that $A \subseteq A' \subseteq X \setminus X^*$:*

$$\max_{\{r(D\setminus x, D)\}_{(D,x)\in\mathcal{M}^*}} \sum_{E': E\subseteq E'\subseteq X^*} r(A'\cup E'\setminus x, A'\cup E') \tag{29}$$

  subject to (28) for all $E' \subseteq X^*$.

- *A large linear program (27) is now decomposed into smaller problems (29), which improves the computational efficiency, especially when A is large.*