Characterizing the Feasible Payoff Set of OLG Repeated Games*

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June 14, 2023

Abstract

We study the set of feasible payoffs of OLG repeated games. We first provide a complete characterization of the feasible payoffs. Second, we provide a novel comparative statics of the feasible payoff set with respect to players' discount factor and the length of interaction. Perhaps surprisingly, the feasible payoff set becomes *smaller* as the players' discount factor approaches to one.

Keywords: Overlapping generation, repeated games

JEL Classification Numbers: C72, C73

*Chihiro Morooka was supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Number JP22K13360. All errors are ours.

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1 Introduction

In overlapping generation (OLG) repeated games, players play for finite periods and are replaced by their next generation. This class of games has been used to study cooperation among finitely-lived players in long-run organizations (e.g., Hammond (1975) and Cremer (1986)).

In this paper, we study the feasible payoff set of OLG repeated games. In the literature of OLG repeated games, including studies of the folk theorems, the convex hull of the stage game payoffs is mostly used as the feasible payoff set of interest. However, the overlapping structure allows players to achieve average discounted payoffs beyond the convex hull of the static payoffs: Although players share the same discount factor, depending on where they are located in their lifecycle, players discount payoffs differently. Thus, it is not obvious which payoffs are feasible. Our purpose in this paper is to understand how the OLG structure affects what players could obtain.

We study the feasible payoff set when players' discount factor and the period of overlap are fixed, departing from the most studies in the literature, which usually focus on the asymptotic case. On the other hand, as typical in the literature, we focus on "periodic" feasible payoffs in which each generation of the same player plays in the same sequence of actions during their lifetime.[1]

Our first main result concerns a complete characterization of the feasible payoff set of OLG repeated games. We find that it can be characterized by the convex hull of the set of the average discounted payoffs that can be achieved by playing n-length sequences of action profiles, where n is the number of players and each of the action profiles is supposed to be played for T (the interaction length) times consecutively. For such sequences of action profiles, we could calculate the average discounted payoff *as if* players play n-length sequence (rather than nT), while effectively discounting δ^T (rather than δ). Thus, this characterization substantially simplifies the set of action profiles we should consider in obtaining the feasible payoff set. In fact, our characterization allows a closed-form expression of the feasible payoff set given any stage game.

Our second main result is about a comparative statics of the feasible payoff set with respect to δ and T. We find that the feasible payoff set is decreasing (in the set-inclusion sense) in the effective discount factor δ^T. Perhaps surprisingly, this implies that the set is *decreasing* in δ. When the effective discount factor is 1, the feasible payoff set coincides with the convex hull of the static payoffs. When it is close to 0, it is a n-dimensional cube, where for each player the maximum (resp. minimum) feasible payoff coincides with the maximum (resp. minimum) stage payoff. For intermediate effective discount factors, it is a n-dimensional polytope.

[1] More generally, each generation of the same player plays different sequences of action profiles. In this case, each player has an infinite sequence of feasible payoffs. We discuss more about it in Section 6.1.

Related Literature

Previous researches on OLG repeated games have mainly focused on folk-theorem-like approaches as in Kandori (1992) and Smith (1992).[2] Alternatively, in this paper, we study the OLG repeated games with fixed δ and T. By studying the feasible payoff set, we provide a natural benchmark for the equilibrium payoff set which one might be more interested in.

The present paper is also related to the literature of repeated games with differential discounting of players, which has been studied since Lehrer and Pauzner (1999). They study infinitely repeated games between two players who have different discount factors, and show that some payoffs outside the convex hull of the stage game payoffs can be obtained by intertemporal trading of payoffs: The more patient player gives payoffs in early periods to have more in later periods. Allowing differential discounting of players, Sugaya (2015) proves a folk theorem for n-player infinitely repeated games with imperfect public monitoring. Dasgupta and Ghosh (2022) provides a more constructive approach to study feasible and equilibrium payoffs of repeated games with perfect monitoring. On the other hand, Chen (2007) and Chen and Fujishige (2013) study finitely repeated games between two players. These papers examine whether the feasible payoff set becomes larger as the length of the game becomes longer. The latter paper, based on the result of the former, shows that for any two-player stage games, this is indeed the case.[3]

Comparing to the literature of repeated games with differential discounting, in our model of OLG repeated games, players share the *same* discount factor. Nevertheless, players can trade payoffs across periods due to the overlapping generation structure: In a given period, players are located in a different position in their lifecycle ("age"), resulting in different discounting of some future payoffs. Notice that players discount in the same way when they have the same age. In this sense, there is a "symmetricity" in their discounting, which makes our analysis relatively tractable. It results in our characterization of the feasible payoff set, allowing a closed-form expression of the feasible payoff set given any stage game for each discount factor and the length of each generation's lifespan.[4]

The remainder of the paper is organized as follows. In Section 2, we introduce the model of OLG repeated games. In Section 3, we present our first main result which is a complete characterization of the feasible payoff set of OLG games. In Section 4, we

[2]Recently, Morooka (2021) provides an alternative folk theorem with an opposite order of choosing parameters: It shows that if δ is chosen first then T is chosen, any feasible and strictly individually rational payoffs can be achieved by subgame perfect equilibrium payoffs. The feasible payoff set considered is larger than the convex hull of the stage game payoffs.

[3]They leave the question for more general case of arbitrary number of players as open.

[4]In the literature of repeated games with differential discounting, Chen (2007) provides an explicit characterization of the feasible payoffs of finitely repeated games for a specific two-player stage game. Sugaya (2015) provides a recursive characterization of the feasible payoff set of infinitely repeated games for general stage games. Dasgupta and Ghosh (2022) provides several characterizations of the feasible payoff set; in particular, they characterize it when players can have some large discount factors.

<table>
<tr><td>Period</td><td>$1 \sim T$</td><td>$T+1 \sim 2T$</td><td>$2T+1 \sim 3T$</td><td>$3T+1 \sim 4T$</td><td>$4T+1 \sim 5T$</td><td>$5T+1 \sim 6T$</td><td>$6T+1 \sim 7T$</td><td>$\cdots$</td></tr>
<tr><td>A_1</td><td colspan="3">Generation 1</td><td colspan="3">Generation 2</td><td colspan="2">Generation 3 $\cdots$</td></tr>
<tr><td>A_2</td><td>Generation 0</td><td colspan="3">Generation 1</td><td colspan="3">Generation 2</td><td>$\cdots$</td></tr>
<tr><td>A_3</td><td colspan="2">Generation 0</td><td colspan="3">Generation 1</td><td colspan="3">Generation 2 $\cdots$</td></tr>
</table>

Table 1: Structure of OLG repeated game with $n = 3$

provide comparative statics results of the feasible payoff set with respect to δ and T. We provide two examples in Section 5 to illustrate our main results. Section 6 concludes after discussions.

2 Model

2.1 Stage Game

A stage game is defined as a triple $G = (N, (A_i)_i, (u_i)_i)$, where $N = \{1, 2, \cdots, n\}$, for some $n \geq 2$, is the set of one-shot players, A_i is a finite set of pure actions available to player i,[5] and $u_i : \prod_i A_i \to \mathbb{R}$ is player i's one-shot payoff function. Let $A \equiv \prod_{i \in N} A_i$ be the set of action profiles.

Given a set $B \subseteq \mathbb{R}^N$, let $co(B)$ be the convex hull of B. Let V be the feasible set of one-shot payoffs, defined as follows:

$$V \equiv co\{u(a) : a \in A\}.$$

2.2 OLG Repeated Game

Given a stage game G defined above, $\delta \in (0, 1]$ and $T \in \mathbb{N}$, we define the OLG repeated game $OLG(G, \delta, T)$ as follows (also see Table 1):

- In every period $t \in \mathbb{N}$, G is played by n finitely-lived players.
- For $i \in N$ and $d \in \mathbb{N}$, the player with A_i in generation d joins in the game at the beginning of period $(d-1)nT + (i-1)T + 1$, and lives for the following n *overlaps* each of which consists of T periods, until he retires at the end of period $dnT + (i-1)T$. The only exceptions are the players with A_i for $i \in N \setminus \{1\}$ in generation 0, who participates in the game between periods 1 and $(i-1)T$.
- Each player's per-period payoffs are discounted at a common discount factor δ.

[5]As long as a Public Randomizing Device is available, our result by mixed actions is the same with the one by pure actions. Therefore, we only consider pure actions throughout this paper.

When a sequence of actions $(a(t))_{t=1}^{nT} \in A^{nT}$ is played throughout a player's life with A_i, her/his average payoff is as follows:[6]

$$\frac{1}{\sum_{t=1}^{nT} \delta^{t-1}} \sum_{t=1}^{nT} \delta^{t-1} u_i(a(t)).$$

We assume that players can access a Public Randomizing Device (henceforth PRD) uniformly distributed over the unit interval at the beginning of every period.[7] We assume each player can observe the realization of the PRD at each period after her/his birth.

3 A Complete Characterization of the Feasible Payoff Set

We are interested in the set of payoffs that can be achieved by a sequence of action profiles in which every generation of players plays the same sequence of action profiles. We shall provide a complete characterization of this set, which allows us to obtain the feasible payoff set for any given δ and T.

Hereafter, we focus on the payoffs of players in generation $d \geq 1$.[8] For $a^{[nT]} = (a^1, a^2, \cdots, a^{nT}) \in A^{nT}$, we define the average discounted payoff $U_i(a^{[nT]})$ of player i as follows:

$$U_i(a^{[nT]}) := \frac{1}{\sum_{k=1}^{nT} \delta^{k-1}} \left(\sum_{k=1}^{nT} \delta^{k-1} u_i(a^{(i-1)T+k}) \right),$$

where $a^s = a^{s-nT}$ for $s \geq nT + 1$.

In the OLG repeated games, whether a payoff is feasible may depend on the observability of the results of PRDs before this player is born: Between any two overlaps, there is always some player who retires. The most permissible assumption is that players are able to observe all the previous outcomes of PRDs, which results in the following definition.

$$F(\delta, T) = co \left(\bigcup_{a^{[nT]} \in A^{nT}} \{U(a^{[nT]})\} \right).$$

The RHS represents the feasible payoffs that can be achieved by PRD which arises every nT-period, which is the longest given stationary payoffs.

We will show that in fact any payoff in this set can be achieved under a weaker assumption that players only can observe the result of PRDs after they are born.

It is convenient to introduce an additional notation. Given $a^{[n]} = (a^1, a^2, \cdots, a^n) \in A^n$

[6]For the player with A_i for $i \in N \setminus \{1\}$ in generation 0, replace nT with $(i-1)T$.

[7]This assumption is employed also by Chen (2007) and Chen and Fujishige (2013).

[8]For $i \in N$, we refer to any player whose action set is A_i as "player i."

and $i \in N$, we define the value $v_i(a^{[n]})$ as follows:

$$v_i(a^{[n]}) := \frac{1}{\sum_{k=1}^n \delta^{(k-1)T}} \left(\sum_{k=1}^n \delta^{(k-1)T} u_i(a^{i+k-1}) \right),$$

where $a^s = a^{s-n}$ for $s \geq n+1$. Let $v(a^{[n]}) \equiv (v_i(a^{[n]}))_{i \in N}$. That is, $v_i(a^{[n]})$ represents the average discounted payoff of player i from repeatedly playing the same action profile a^k during the k-th overlap:

$$U_i(\underbrace{a^1, \ldots, a^1}_{T \text{ times}}, \ldots, \underbrace{a^n, \ldots, a^n}_{T \text{ times}}) = v_i(a^1, \ldots, a^n).$$

Alternatively, $v_i(a^{[n]})$ can be interpreted as the average discounted payoff over n periods, where the "effective" discount factor is δ^T.

Theorem 1. For any $\delta \in (0, 1]$ and $T \in \mathbb{N}$,

$$F(\delta, T) = co\left(\left\{v(a^{[n]}) : a^{[n]} \in A^n\right\}\right). \tag{1}$$

In words, the feasible payoff set is the convex hull of the average discounted payoffs of length-n sequences of action profiles, each of which is to be played T times. Notice that we could interpret the average discounted payoff of such a sequence of action profiles as the average discounted payoff of length-n sequence of action profiles, discounted by δ^T.

The characterization is useful since it substantially reduces the number of sequences of action profiles we need to consider: Regardless of T, it is sufficient to consider length-n sequences. The result also means that δ and T affect the feasible payoff set only through δ^T. Thus, any (δ, T) and (δ', T') with $\delta^T = \delta'^{T'}$ would result in the same feasible payoff set. For instance, for the Prisoners' Dilemma game, where each player has two actions (so 4 action profiles), the result implies, it is sufficient to consider $4 \times 4 = 16$ sequences of action profiles to obtain the feasible payoff set, regardless of δ and T. In particular, this is true even when T is a large number.

One may wonder whether the RHS of (1) suggests that the feasible payoffs may not be implemented under our assumption on the observability of PRDs. That is, we assumed that a result of a PRD is observed only by the contemporary generation, whereas for each T period (i.e., overlap), there is a player to be replaced; so it might require a stronger informational assumption, for instance, the results of PRDs are observed also by future generations. In the proof, however, we construct a sequence of PRDs each of which is executed every overlap that generates the same average discounted payoff from a sequence of action profiles.

The underlying idea of the proof is as follows: Within an overlap, no player retires so every player discounts their payoff at $t+1$ relatively more than payoff at t in the overlap by δ, although they discount payoffs differently depending on their "age." This allows us to generate the same average discounted payoff for a given sequence of action profiles for a given overlap using a PRD at the beginning of this overlap, where the probability of playing a certain action profile during this overlap is determined in a way to mimic the sum of discounted payoffs whenever this action profile is played in the original sequence of action profiles. On the other hand, across overlaps, some player should retire and for such a player the relative discounting between t and $t+1$ is different, which makes a similar construction of a PRD between overlaps unavailable.

Proof. We show

$$F(\delta, T) \subseteq co\left(\{v(a^{[n]}) : a^{[n]} \in A^n\}\right).$$

Consider an arbitrary sequence $a^{[nT]} = (a^1, \ldots, a^{nT}) \in A^{nT}$. We claim that the same average discounted payoff can be obtained by a convex combination of "n-length action sequences."

To see this, we first claim that for each overlap $k = 1, \ldots n$, the average discounted payoff within the overlap can be achieved by playing constant action profiles resulting from a PRD.

In each k-th overlap, the following PRD is exercised at the beginning of the overlap: An action profile $a \in A$ is randomly drawn with probability $\alpha^k(a)$ defined by

$$\alpha^k(a) := \sum_{t=(k-1)T+1}^{kT} \frac{\delta^{t-1-(k-1)T}}{1 + \cdots + \delta^{T-1}} \mathbf{1}_{\{a^t = a\}}.$$

Clearly, $\sum_{a \in A} \alpha^k(a) = 1$. The players are supposed to play the realized action profile a consecutively until the end of the overlap. To see that this PRD generates the same average discounted payoff as that of $a^{[nT]}$ for each player during the overlap, note that the PRD is constructed in that way so that it mimics discounting. Namely, if $a^t = a$ in $a^{[nT]}$ for some $t \in \{1, 2, \ldots, T\}$, then the probability of playing $(a, \ldots, a)$ increases by $\frac{\delta^{t-1-(k-1)T}}{\sum_{t=1}^{T} \delta^{t-1}}$. This construction is possible because no player retires within the overlap so that every player discounts $t+1$-period payoff by δ times than t-period payoff. This is not the case as long as some player retires and is reborn (i.e., across overlaps).

Lastly, observe that the average discounted payoff of each player from this sequence of RPDs $(\alpha^1, \ldots, \alpha^n)$ is a convex combination of $\{v(a^{[n]}) : a^{[n]} \in A^n\}$, where the weight is $\alpha^1(a^1) \times \cdots \times \alpha^n(a^n)$ for each $(a^1, \ldots, a^n) \in A^n$.

For the converse, observe that each element of $\{v(a^{[n]}) : a^{[n]} \in A^n\}$ is in $F(\delta, T)$ and $F(\delta, T)$ is convex by definition.

□

4 Comparative Statics of the Feasible Payoff Set

In this section, we study comparative statics of the feasible set w.r.t. both T and δ. From Theorem 1 in the previous section, we know that the feasible payoff set depends on δ and T only through δ^T.

4.1 Monotonicity

Our main result in this section asserts that the feasible payoff set is non-increasing in δ^T, implying, perhaps surprisingly, it is *non-increasing* in δ.

Theorem 2. The following hold:

1. For any δ, δ' with $0 < \delta < \delta' \leq 1$, $F(\delta',\ T) \subseteq F(\delta,\ T)$.

2. Given any $\delta \in (0, 1]$, $F(\delta,\ T) \subseteq F(\delta,\ T + 1)$.

The rest of this section devotes to proving Theorem 2.[9] In doing so, we shall also provide a characterization of players' optimal play to maximize the welfare given some weights for players, which might have some independent interest.

Theorem 1 implies that it is without loss of generality to consider a sequence of n pure action profiles for studying the monotonicity. That is, players play the same action profile during an overlap which consists of T periods. In addition, since the feasible payoff set is convex, it is enough to show the maximum "score" increases as δ (resp. T) becomes smaller (resp. larger) for each non-zero direction.

Fix $\lambda \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. For a given $\delta \in (0, 1]$ and $T \in \mathbb{N}$, let $\Delta \equiv \Delta(\delta, T) = \delta^T$. Define λ-weighted welfare as

$$W^*_\lambda(\Delta) := \max_{a^{[n]}=(a^1,\ldots,a^n)\in A^n} W_\lambda(a^{[n]}, \Delta), \tag{2}$$

where

$$W_\lambda(a^{[n]}, \Delta) := \sum_{i=1}^{n} \lambda_i v_i(a^1, \ldots, a^n), \quad \forall a^{[n]} \in A^n.$$

We want to show that $W^*_\lambda(\Delta)$ is non-increasing in Δ (as a result, non-decreasing in T and non-increasing in δ).

Let us introduce a few more notations. Denote the set of optimal solution of (2) by $\mathcal{A}^*_\lambda(\Delta) \subseteq A^n$. Let $\mathcal{U}^*_\lambda(\Delta) = \{(u^k)_k : u^k = u(a^k), a^{[n]} \in \mathcal{A}^*_\lambda(\Delta)\} \subseteq \mathbb{R}^{n\times n}$. That is, the set of sequences of the optimal payoff vectors. Notice that $\mathcal{U}^*_\lambda(\Delta)$ may be a singleton even

[9] The weak set-inclusions in the result cannot be strengthened to be strict. For instance, if the stage game payoff set is an n-dimensional rectangle, the feasible payoff set of the corresponding OLG game would be unchanged with respect to the parameters.

when $\mathcal{A}^*_\lambda(\Delta)$ is not. Lastly, given $u^{[n]} = (u^1, \dots, u^n) \in \mathbb{R}^{n\times n}$, for each $k \in \{1, \dots, n\}$, let

$$w_k(u^{[n]}) := \sum_{i=1}^{n} \lambda_i u_i^{i+k-1},$$

where $u_i^m = u_i^{m-n}$ if $m > n$. That is, $w_k(u^{[n]})$ is the weighted sum of players' payoffs when their "age" is k (i.e., they are in the k-th overlap in their lifetime). For instance, given $u^{[3]} = (u^1, u^2, u^3) \in \mathbb{R}^{3\times 3}$ and $\lambda = (1, 1, 1)$, $w_1(u^{[3]}) = u_1^1 + u_2^2 + u_3^3$, $w_2(u^{[3]}) = u_1^2 + u_2^3 + u_3^1$, $w_3(u^{[3]}) = u_1^3 + u_2^1 + u_3^2$.

Observe that for each $u^{[n]} = (u^1, \dots, u^n) \in \mathcal{U}^*_\lambda(\Delta)$.

$$W^*_\lambda(\Delta) = \frac{\sum_{k=1}^{n} \Delta^{k-1} w_k(u^{[n]})}{\sum_{k=1}^{n} \Delta^{k-1}}. \tag{3}$$

The following lemma is the key to prove the monotonicity.

Lemma 1. Let $\Delta \in (0, 1]$ and $u^{[n]} \in \mathcal{U}^*_\lambda(\Delta)$. For each $m = 1, \dots, n-1$,

$$\sum_{k=1}^{m} \Delta^{k-1} w_k(u^{[n]}) \geq \sum_{k=1}^{m} \Delta^{k-1} w_{n-m+k}(u^{[n]}). \tag{4}$$

In words, the lemma means, at optimum, the payoffs in the earlier stages of a player are higher than those of later stages (in the sense that for any $m \leq n-1$, the first m payoffs should be larger than the last m payoffs). When there are only two players, this reduces to the condition that the payoffs when they are "young" must be larger than those when they are "old" for optimality. When there are more than two players, it is *a priori* not clear what could be the corresponding expression. According to the lemma, for the case of three players, it is $w_1(u^{[3]}) \geq w_3(u^{[3]})$ and $w_1(u^{[3]}) + \Delta w_2(u^{[3]}) \geq w_2(u^{[3]}) + \Delta w_3(u^{[3]})$.

Let us explain the crux of the idea of the proof with 3 players and $\lambda = (1, 1, 1)$. Assume $u^{[3]} = (u^1, u^2, u^3)$ be an optimal sequence of payoff vectors. From optimality of u^1 at $k = 1$, it should be $u_1^1 + \Delta^2 u_2^1 + \Delta u_3^1 \geq u_1^2 + \Delta^2 u_2^2 + \Delta u_3^2$. By multiplying both sides by Δ,

$$\Delta u_1^1 + \Delta^3 u_2^1 + \Delta^2 u_3^1 \geq \Delta u_1^2 + \Delta^3 u_2^2 + \Delta^2 u_3^2.$$

On the other hand, by optimality of u^2 at overlap $k = 2$,

$$\Delta u_1^2 + u_2^2 + \Delta^2 u_3^2 \geq \Delta u_1^1 + u_2^1 + \Delta^2 u_3^1.$$

From the two inequalities, we can conclude $u_2^2 \geq u_2^1$. Intuitively, since the same generation of player 1 and 3 is active both at $k = 1$ and $k = 2$, while player 2 is replaced by the next generation, in order for u^2 to give a larger aggregate payoff at $k = 2$, player 2 should have a larger payoff at u^2 than at u^1. A symmetric argument results in $u_3^3 \geq u_3^2$ and $u_1^1 \geq u_1^3$, and summing them up results in $w_1(u^{[3]}) \geq w_3(u^{[3]})$. Applying a similar argument to "two-overlap apart," we have $\Delta u_2^1 + u_3^1 \leq \Delta u_2^3 + u_3^3$ (as a result, $w_1(u^{[3]}) + \Delta w_2(u^{[3]}) \geq w_2(u^{[3]}) + \Delta w_3(u^{[3]})$).

Proof. Denote the "age" of player i at overlap k by $y_k(i) \in \{1, \dots, n\}$.

Consider overlap $k \in \{1, \dots, n\}$ and $m \in \{1, \dots, n-1\}$. Let $k' = k + m \pmod{n}$. Then, from optimality of u^k at overlap k,

$$\sum_{i=1}^{n} \Delta^{y_k(i)-1} \lambda_i u_i^k \geq \sum_{i=1}^{n} \Delta^{y_k(i)-1} \lambda_i u_i^{k'}.$$

By multiplying both sides by Δ^m,

$$\sum_{i=1}^{n} \Delta^{y_k(i)+m-1} \lambda_i u_i^k \geq \sum_{i=1}^{n} \Delta^{y_k(i)+m-1} \lambda_i u_i^{k'}.$$

Note that $y_{k'}(i) = y_k(i) + m$ if $y_k(i) + m \leq n$; otherwise $y_{k'}(i) = y_k(i) + m - n$.

$$\begin{aligned} &\sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k + \Delta^n \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k \\ &\qquad \geq \sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} + \Delta^n \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} \end{aligned}$$

or equivalently,

$$\begin{aligned} &\sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k - \sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} \\ &\qquad \geq \Delta^n \left(\sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} - \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k \right). \end{aligned} \quad (5)$$

On the other hand, from optimality at overlap k', we have

$$\begin{aligned} &\sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} + \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} \\ &\qquad \geq \sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k + \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k \end{aligned}$$

or

$$\begin{aligned} &\sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k - \sum_{i:y_k(i)+m\leq n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} \\ &\qquad \leq \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^{k'} - \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1} \lambda_i u_i^k. \end{aligned} \quad (6)$$

In order to satisfy both (5) and (6), it must be

$$\sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1}\lambda_i u_i^{k'} \geq \sum_{i:y_k(i)+m>n} \Delta^{y_{k'}(i)-1}\lambda_i u_i^{k}$$

or equivalently,

$$\sum_{i:y_{k'}(i)\in\{1,\ldots,m\}} \Delta^{y_{k'}(i)-1}\lambda_i u_i^{k'} \geq \sum_{i:y_{k'}(i)\in\{1,\ldots,m\}} \Delta^{y_{k'}(i)-1}\lambda_i u_i^{k}.$$

Summing up over $k \in \{1,\ldots,n\}$ both sides, we have the inequality in the statement.

□

Perhaps surprisingly, the following lemma says that the inequality (4) in Lemma 1 is sufficient to prove that the derivative of the aggregate payoff is non-positive.

Lemma 2. For $\Delta \in (0,1)$,

$$\frac{\partial W_\lambda}{\partial \Delta}(a^{[n]},\Delta) \leq 0$$

for any $a^{[n]} \in \mathcal{A}_\lambda^*(\Delta)$.

Proof. See Appendix A.1. □

Proof of Theorem 2 . We shall show that for each $\Delta \in (0,1)$, there exists $a^{[n]} \in \mathcal{A}_\lambda^*(\Delta)$ such that

$$\frac{dW_\lambda^*(\Delta)}{d\Delta} = \frac{\partial W_\lambda}{\partial \Delta}(a^{[n]},\Delta). \tag{7}$$

Once we prove it, Lemma 2 immediately implies the conclusion.

For a generic $\Delta \in (0,1)$, $\mathcal{U}_\lambda^*(\Delta)$ is a singleton. In this case, the λ-weighted discounted sum of $u^{[n]} \in \mathcal{U}_\lambda^*(\Delta)$ forms a "vertex" of the OLG feasible payoff set. Clearly, this means that there exists $\epsilon > 0$ such that for all $\Delta' \in (\Delta-\epsilon,\Delta+\epsilon)$, $u^{[n]} \in \mathcal{U}_\lambda^*(\Delta')$; that is, $W_\lambda^*(\Delta')$ is obtained still from $u^{[n]}$. This implies (7).

Next, consider $\Delta \in (0,1)$ for which $\mathcal{U}_\lambda^*(\Delta)$ contains uncountable elements. In this case, the collection of the λ-weighted discounted sums of the elements of $\mathcal{U}_\lambda^*(\Delta)$ forms an "edge" or "face" of the OLG feasible set. Note that there exists at least one element $u^{[n]} \in \mathcal{U}_\lambda^*(\Delta)$ and $\epsilon > 0$ such that, for all $\Delta' \in (\Delta-\epsilon,\Delta+\epsilon)$ such that $u^{[n]} \in \mathcal{U}_\lambda^*(\Delta')$. Again, this implies (7).

□

4.2 A Limit Characterization of the Feasible Payoff Set

For the asymptotic cases (i.e., $\delta^T \nearrow 1$ or $\delta^T \searrow 0$), we can be more explicit about the shape of the feasible payoff set.

	C	D
C	$1,1$	$-1,2$
D	$2,-1$	$0,0$

Figure 1: The Prisoners' Dilemma

When the players' effective discount factor δ^T is close to 1, there is not much difference in discounting of young and old players. So, the scope of intertemporal trade of payoffs is little and what is the best is to maximize the stage game payoffs for a given welfare weight. On the other hand, when the effective discount factor is close to 0, the difference in discounting of young and old players is large. Hence intertemporal trading can be very helpful. The extreme form of such trading is to maximize the youngest player's (weighted) payoff. We summarize this discussion as a corollary:

Corollary 1. The following hold:

1. For δ^T sufficiently close to 1, the solution of the optimization problem (2) is to play some $a^k \in \arg\max_{a' \in A} \lambda \cdot u(a')$ for each overlap k. As a result, $\lim_{\delta^T \nearrow 1} F(\delta, T) = V$.

2. For δ^T sufficiently close to 0, the solution of the optimization problem (2) is to play $a^k \in \arg\max_{a' \in A} \lambda_{i_k} u_{i_k}(a')$, where $i_k \in N$ is the youngest player in overlap k. As a result, $\lim_{\delta^T \searrow 0} F(\delta, T) = \prod_{i \in N} [\min_{a \in A} u_i(a),\ \max_{a \in A} u_i(a)]$.

Thus, for any $\delta \in (0,1)$, as $T \to \infty$, the feasible set converges to the n-dimensional cube. Similarly, for any T, as $\delta \to 0$, the feasible set converges to the n-dimensional cube.

5 Examples

In this section we illustrate our main results using well-known stage games in the repeated game literature.

5.1 OLG Prisoners' Dilemma

Consider the OLG repeated games with the stage game of Prisoners' Dilemma (see Figure 1). By Theorem 1

$$F(\delta, T) = co\left(\bigcup_{a \in \{CC, DC, DD, CD\}} \{v(a, CC), v(a, DC), v(a, DD), v(a, CD))\}\right), \tag{8}$$

where $v(a^1, a^2) = \left(\frac{u_1(a^1) + \Delta u_1(a^2)}{1+\Delta}, \frac{\Delta u_2(a^1) + u_2(a^2)}{1+\Delta}\right)$ for any $(a^1, a^2) \in A^2$ as previous (see Figure 2).

It is notable that as Δ changes the sequences of action profiles that generate the extreme points of the feasible payoff set may change. For instance, when Δ is large

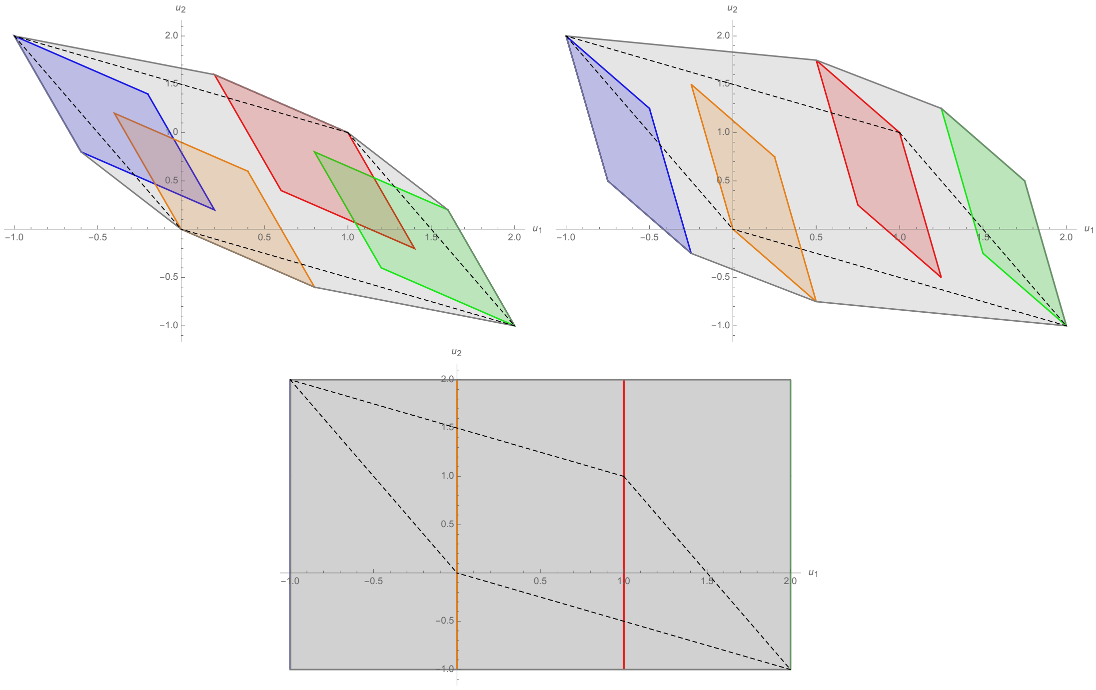

Figure 2: In each figure, the Grey region represents the feasible payoff set (for $\Delta = \frac{2}{3}$, $\Delta = \frac{1}{3}$ and $\Delta \to 0$ clockwise) of the OLG PD game. In each figure, the region surrounded by the dotted lines is the convex hull of the stage game payoffs; the Red region represents the convex hull of the four payoffs, $v(CC, a^2)$, $a^2 \in \{C, D\}^2$ in (8). Similarly, the Green, Orange and Blue represent the counterparts for DC, DD and CD, respectively.

enough (e.g., $\Delta = 2/3$), (CC, CC) yields an extreme point. As Δ becomes smaller (e.g., $\Delta = 1/3$), it is not anymore an extreme point while (DC, CD) becomes a new sequence corresponding one of the extreme points. As Δ becomes even smaller, (CC, CD) is "dominated" by (DC, CD). Intuitively, when Δ is sufficiently small each player should play the action profile that maximizes her/his payoff in order to be on the efficient frontier.

5.2 A 3-player Example from Fudenberg and Maskin (1986)

The second example (see Figure 3) involves three players. Fudenberg and Maskin (1986) used this stage game to show that the folk theorem fails for the standard repeated games with infinitely-lived players. In particular, this stage game does not satisfy the full dimensionality, a sufficient condition of their folk theorem.[10]

Nevertheless we observe that the feasible payoff set of the OLG game exhibits the full dimension for any $\Delta \in (0, 1)$. When $\Delta = 1$, it is the line segment between $(0, 0, 0)$ and $(1, 1, 1)$, which coincides with the convex hull of the stage game payoffs. On the other hand, when $\Delta \in (0, 1)$, it is a polytope with nonempty interior.

[10] Smith (1992) shows that the full-dimensionality is not necessary for his folk theorem for OLG repeated games. Since the folk theorem first chooses T then chooses sufficiently large δ, it concerns the case when δ^T is close to 1. The feasible payoff set in this case is the "smallest" according to our characterization, which is the line segment between $(0, 0, 0)$ and $(1, 1, 1)$.

	A	B
A	$1,1,1$	$0,0,0$
B	$0,0,0$	$0,0,0$

	A	B
A	$0,0,0$	$0,0,0$
B	$0,0,0$	$1,1,1$

Figure 3: The stage game of a 3-player pure coordination game

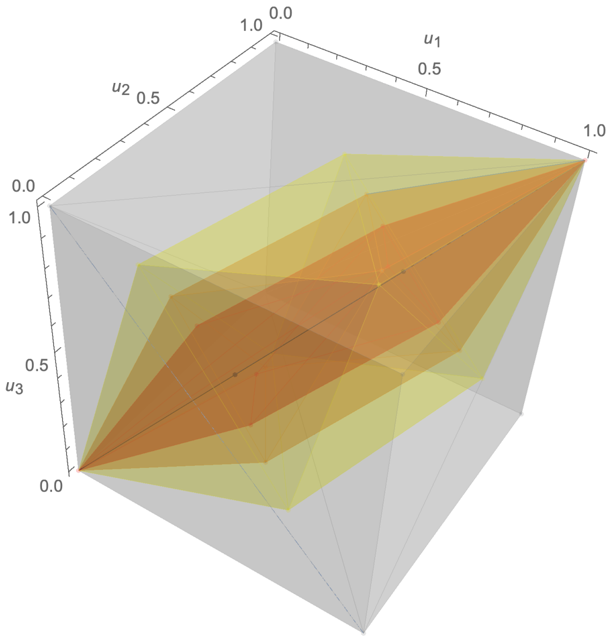

Figure 4: The OLG feasible payoff set of the pure coordination game when $\Delta = 1$ (Black) $\Delta = 2/3$ (Yellow), $\Delta = 1/2$ (Orange), $\Delta = 1/3$ (Red) and $\Delta = 0$ (Gray).

6 Discussion and Conclusion

6.1 Discussion

Thus far we have restricted our attention to periodic feasible payoffs. In this subsection, we discuss how the relaxation of this restriction would affect the monotonicity result in terms of δ. We consider non-stationary sequences of action profiles of players, in which each generation of the same player may play different sequences of action profiles during their lifetime. One way to extend the concept of the monotonicity in this case would be as follows: Given a sequence $(\bar{u}^y(\delta))_{y\in\mathbb{N}}$ of players' average discounted payoffs for each generation $y = 1, 2, \ldots$, which is feasible with discount factor δ, we ask whether the same sequence is feasible with $\delta' < \delta$.

The following example shows that this is not the case: Consider the OLG repeated game with two players with two possible stage game payoffs of $(1, 0)$ and $(0, 1)$. Suppose $T = 1$. Consider the following sequence of payoff vectors for each $t = 1, 2, \ldots$: $(1, 0), (1, 0), (0, 1), (0, 1), \ldots$. That is, the first two periods gives $(1, 0)$, followed by $(0, 1)$ forever. The corresponding average payoff for each generation of player 1 is $\bar{u}_1^1 = 1$ for the first generation, and $\bar{u}_1^y = 0$ for any $y \geq 2$. Observe that player 2 in the first generation,

	A	B
A	$2,1$	$0,0$
B	$0,0$	$1,2$

Figure 5: The Battle of the Sexes

who is born at $t = 2$, obtains the average payoff $\bar{u}_2^1 = \frac{\delta}{1+\delta}$ and $\bar{u}_2^y = 1$ for any $y \geq 2$.

Now consider $\delta' < \delta$. Since $\bar{u}_1^1 = 1$. the first two period must give players $(1,0),(1,0)$. Note that the maximum payoff of player 2 in the first generation is obtained when $(0,1)$ is given at $t = 3$, yielding the average payoff $\frac{\delta'}{1+\delta'}$, which is strictly smaller than $\frac{\delta}{1+\delta}$.

6.2 Concluding Remarks

In the present paper we study the feasible payoff set of OLG repeated games. In our first result, we show that the set can be characterized by a convex combination of the average discounted payoffs of n-period sequences of action profiles, where each of the action profiles is played for T periods consecutively. Our second main result shows that the feasible payoff set is monotonely decreasing in δ and increasing in T.

We note that our monotonicity result of the feasible payoff set can shed also light on the monotonicity of the equilibrium (Nash equilibrium or subgame perfect equilibrium) payoff set: The set of stationary equilibrium (i.e., each generation of a player employs the same strategy) payoffs is, in general, not increasing in δ. This contrasts from the case of repeated games with infinitely-lived players with a PRD (Abreu et al., 1990).[11] As an example, consider the OLG game with the stage game of the Battle of the Sexes, where the coordination gives $(2,1)$ or $(2,1)$ and mis-coordination results in $(0,0)$ (see Figure 5). In this case, the Pareto efficient payoffs, $(2,1)$ and $(1,2)$ can be achieved static Nash equilibria and so any convex combination of playing the two equilibria is also a subgame perfect equilibrium. Our result of the decreasing feasible payoff set then translates into the decreasing efficient equilibrium payoffs as δ increases.

[11] When there is no PRD, the monotonicity might not hold (see Mailath et al. (2002); Yamamoto (2010)).

A Omitted Proofs

A.1 Proof of Lemma 2

Proof. For $a^{[n]} \in \mathcal{A}^*_\lambda(\Delta)$, denote the numerator of $\frac{\partial W_\lambda}{\partial \Delta}(a^{[n]}, \Delta)$ by $\eta(\Delta)$. We will show that $\eta(\Delta) \leq 0$, thereby $\frac{\partial W_\lambda}{\partial \Delta}(a^{[n]}, \Delta) \leq 0$. From (3), observe that, for $u^{[n]} \in \mathcal{U}^*_\lambda(\Delta)$,

$$\begin{aligned}
\eta(\Delta) &= \left(\sum_{k=1}^{n} \frac{d\Delta^{k-1}}{d\Delta} w_k(u^{[n]})\right)\left(\sum_{m=1}^{n} \Delta^{m-1}\right) - \left(\frac{d\sum_{m=1}^{n}\Delta^{m-1}}{d\Delta}\right)\sum_{k=1}^{n}\Delta^{k-1} w_k(u^{[n]}) \\
&= \left(\sum_{k=1}^{n}(k-1)\Delta^{k-2} w_k(u^{[n]})\right)\left(\sum_{m=1}^{n}\Delta^{m-1}\right) - \left(\sum_{k=m}^{n}(m-1)\Delta^{m-2}\right)\sum_{k=1}^{n}\Delta^{k-1} w_k(u^{[n]}) \\
&= \sum_{k=1}^{n}\left((k-1)\Delta^{k-2}\sum_{m=1}^{n}\Delta^{m-1} - \left(\sum_{m=1}^{n}(m-1)\Delta^{m-1}\right)\Delta^{k-2}\right) w_k(u^{[n]}) \\
&= \sum_{k=1}^{n}\left(\sum_{m=1}^{n}(k-m)\Delta^{k+m-3}\right) w_k(u^{[n]}).
\end{aligned}$$

Recall m-th constraint, $m \in \{1, \ldots, n-1\}$, is

$$-\sum_{k=1}^{m}\Delta^{k-1} w_k(u^{[n]}) + \sum_{k=1}^{m}\Delta^{k-1} w_{n-m+k}(u^{[n]}) \leq 0 \quad (\#m)$$

We will show that

$$\eta(\Delta) = \sum_{m=1}^{n-1} p_m \times (\text{LHS of } \#m),$$

where for $1 \leq m \leq n-1$

$$p_m \equiv \Delta^{n-2} + \Delta^{n-3} + \cdots + \Delta^{n-m-1}.$$

Note that as $p_m \geq 0$, this implies $\eta(\Delta) \leq 0$. To see the equality, fix $j \in \{1, \ldots, n\}$. Observe for any constraint $m \geq j$, there exists a non-positive term $-\Delta^{j-1} w_j(u^{[n]})$. And for any constraint m with $j \geq n-m+1$, there exists a non-negative term $\Delta^{j-n+m-1} w_j(u^{[n]})$.

Therefore the coefficient of $w_j(u^{[n]})$ is

$$
\begin{aligned}
&\sum_{m=j}^{n-1} p_m(-\Delta^{j-1}) + \sum_{m=n+1-j}^{n-1} p_m \Delta^{j-n+m-1} \\
&= \sum_{m=j}^{n-1}(\Delta^{n-2} + \cdots + \Delta^{n-m-1})(-\Delta^{j-1}) + \sum_{m=n+1-j}^{n-1}(\Delta^{n-2} + \cdots + \Delta^{n-m-1})\Delta^{j-n+m-1} \\
&= \sum_{m=j}^{n-1}(-\Delta^{n+j-3} - \cdots - \Delta^{n-m+j-2}) + \sum_{m=n+1-j}^{n-1}(\Delta^{j+m-3} + \cdots + \Delta^{j-2}) \\
&= (-\Delta^{n+j-3} - \cdots - \Delta^{n-2}) + (-\Delta^{n+j-3} - \cdots - \Delta^{n-1}) + \cdots + (-\Delta^{n+j-3} - \cdots - \Delta^{j-1}) \\
&+ (\Delta^{n-2} + \cdots + \Delta^{j-2}) + (\Delta^{n-1} + \cdots + \Delta^{j-2}) + \cdots + (\Delta^{j+n-4} + \cdots + \Delta^{j-2}) \\
&= -\Delta^{n+j-3}((n-j) - 0) - \Delta^{n+j-4}((n-j) - 1) - \cdots - \Delta^{n-2}((n-j) - (j-1)) \\
&- \Delta^{n-3}((n-j-1) - (j-1)) - \cdots - \Delta^{j-1}(1 - (j-1)) - \Delta^{j-2}(0 - (j-1)) \\
&= \sum_{m=1}^{n}(j-m)\Delta^{j+m-3}.
\end{aligned}
$$

□

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