A priori bounds on legislative bargaining agreements *

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Abstract

In a workhorse model of legislative bargaining with spatial preferences, I establish easy to compute bounds on all equilibrium acceptable agreements, proposals, and outcomes. The approach constitutes a feasible method to incorporate equilibrium restrictions from the model in correlational and structural empirical studies of legislatures, avoiding the computation of actual equilibria. It also yields a number of theoretical insights on the centrality of equilibrium legislative decisions, the relation of such equilibrium outcomes with social choice solution sets, and the effect of changes on voting and proposal-making rights. These theoretical results highlight the broad conclusion that the proper functioning of democratic institutions is highly contingent on other institutional features besides the assignment of voting rights.

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What policies do legislatures, parliaments, or committees adopt when deciding over a continuum of alternatives? How do these policy outcomes change as the voting rule, rules for the origination of proposals, and other features of the legislative environment vary? Even the first of these questions has no obvious answer. Plott (1967) convincingly showed that a natural candidate set of collective outcomes, the core, exists only under knife-edge conditions in simple majority-rule settings when the policy space comprises more than one dimensions. Richard McKelvey and Norman Schofield (McKelvey (1976, 1979); Schofield (1978)) subsequently showed that in the generic case when such knife-edge conditions are not met, intransitivity of majority preferences afflicts the committee in the most egregious manner, encompassing the entire policy space. These findings posed both a theoretical and a practical challenge for the study of policy-making in committees. On the theoretical side, they forcefully introduced the possibility that preference aggregation in legislatures may generate outcomes that poorly reflect collective preferences and may be subject to normatively unacceptable levels of manipulation by a minority of legislators with procedural prerogatives (McKelvey (1979); Riker (1980)). On the applied side, they left researchers with little guidance as to what to substitute for core policies when specifying likely legislative policy outcomes. Several generalizations of the core were subsequently developed to address the theoretical and practical gap in the literature.[1]

With the subsequent shift in the literature from a cooperative to a non-cooperative game-theoretic approach, the workhorse model to address these questions has been the legislative bargaining model of Baron and Ferejohn (1989). However, most applications using this framework primarily operate in a divide-the-dollar setting (for example, see the recent review by Eraslan and Evdokimov (2019)). Indeed, while the model can certainly accommodate more general spaces of political disagreement (Banks and Duggan (2000, 2006)),

[1]See Austen-Smith and Banks (1999) for a review.

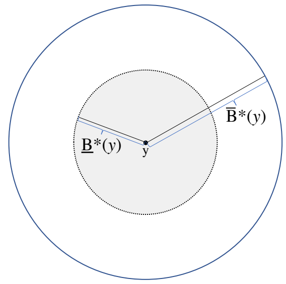
it is rarely analytically tractable with spatial preferences except in special environments.[2] The goal of this study is to develop a computationally feasible method to delineate possible outcomes in the legislative bargaining model with classic spatial preferences, thus providing an inexpensive alternative to the computation of equilibrium for practitioners and, at the same time, allowing a systematic theoretical exploration of the dependence of collective decisions on the allocation of proposal and voting rights and other features of the legislative environment.

I study committees that decide by general voting rules and bargaining protocols, and I allow for large committees, including the possibility of a continuum of members or a separate (possibly continuous) population of proposers. The model captures weighted majority quota rules, multi-cameral legislatures, and veto and veto-override provisions as provided by the US constitution and, in the typical application with finite committees, it encompasses all non-empty, monotonic voting rules. Though some findings apply to a more general domain of preferences, I assume negative quadratic Euclidean preferences, a form prevalent in the literature both in applications of the model but also in empirical work to estimate legislator preferences.

For any point y, I establish inner and outer spherical bounds centered at y on acceptable agreements such that no proposal outside the outer sphere can be approved in any equilibrium, and all proposals within the (possibly empty) inner sphere pass (if proposed) in every equilibrium (Theorem 2). Figure 1 illustrates. These *a priori* restrictions on equilibrium outcomes are straightforward to compute, and can become sharper by combining bounds with different center points y (Corollary 1). To illustrate this approach, I present computations using US Congressional preference data from DW-nominate (Lewis et al. (2021)) in section 4. In that setting involving 537 individual players, I compute and combine bounds for a grid of 40,401 distinct points y in about 2.5 minutes and an average

[2] Primarily one-dimensional instances of the spatial model (e.g., Cardona and Ponsati (2011); Cho and Duggan (2003); Herings and Predtetchinski (2010); Predtetchinski (2011)) or a small number of players in higher dimensions (e.g., Baron (1991)).

Figure 1: Equilibrium bounds



For all y, there exists a ball with radius $\bar{B}^*(y)$ that contains all acceptable agreements in every equilibrium. An open ball with radius $\underline{B}^*(y)$ (possibly zero) contains agreements that are acceptable in every equilibrium.

of 0.0034 seconds per y-specific computed bound.[3] These numerical experiments underscore the practical feasibility of the proposed approach. They also highlight how the predictions of the model differ from static analyses of choice in these settings, and also exhibit some substantively intriguing qualitative properties which invite additional study (due to space limitations, I only briefly discuss these at the end of section 4).

By its make-up, the legislative model I study speaks to the possible decisions legislatures reach, the likely bills different actors pursue, and the likely coalitions that support them. Therefore, the range of possible committee outcomes isolated by the approach pursued in this paper can inform the analysis and interpretation of empirical evidence in a variety of settings, and I briefly highlight a few presently. In experimental settings on spatial committee decision-making originating in the work of Fiorina and Plott (1978), both the preferences and the outcome of the committee interaction are observable, and the computed bounds provide an alternative framework to interpret and evaluate such experimental data, for example, along lines analogous to the work of Bianco et al. (2006). Computations

[3] These computations were performed on a 2017 2.3 GHz Intel Core i5 mac, with 2 cores and 8GB of memory. I wrote the code in MATLAB and did not attempt elaborate vectorization or parallelization. The theory of comparative advantage suggests that an applied analyst with computer expertise can significantly improve over that record.

from US Congressional settings similar to those performed in section 4, where it is assumed that individual preferences (but not necessarily the actual decisions) are observable, can inform theories of Congressional gridlock and/or be evaluated along with empirical measures of legislative productivity as in, for example, Howell et al. (2000) and Clinton and Lapinski (2006). Such an approach would capitalize on consistently stringing together successive Congressional decisions predicted from the model to gauge the degree of shift in policy over successive Congresses. Because the bounds developed in this paper restrict observables across all equilibria through inequality restrictions, the more ambitious lines of empirical application of the model could combine evidence from roll-call votes, bill sponsorship data, and the content of legislative decisions (among others) in order to structurally estimate or set identify model parameters without imposing *ad hoc* equilibrium selection arguments, along the lines pursued in the burgeoning literature on structural estimation of partially identified models (e.g., Tamer (2010) and Pakes et al. (2015)).

In the remainder of this introduction, I first briefly outline how the bounds are established, then summarize the comparative statics questions they help address, and conclude with a review of additional related literature. The analysis builds on two generalizations of the median established in Kalandrakis (2019) and defined for every possible point y: A y-centered P-ball (for *pivotal*) is a ball centered at y such that, for every equilibrium, if all of its members approve some alternative in equilibrium, then some winning coalition approve that alternative in equilibrium. A y-centered W-ball (for *winning*) is a (weakly smaller) ball centered at y such that if some winning coalition approve some alternative in equilibrium, then there exists some member of the y-centered W-ball that also approves it in equilibrium (Lemma 1). When the radius of these generalized median balls is minimized over possible centers y in a unicameral legislature with an odd number of committee members and simple majority rule, these constructs coincide with the classic *yolk* (Ferejohn, McKelvey and Packel (1984); McKelvey (1986)).

Using these generalized medians, I recover a structure of the set of agreements that can pass in equilibrium (Lemma 2) comparing individual payoffs to possible lotteries over agreements induced in any equilibrium. That structure, in turn, allows me to bound the maximum and minimum distance of optimal individual proposals from y as a function of equilibrium-specific bounds centered at y (Lemma 3). These arguments yield inequations for equilibrium-specific bounds and in a final step I rely once more on the generalized medians to relax these inequalities so that they apply to bounds across equilibria (Theorem 1). The resulting equilibrium bounds centered at any y depend indirectly on the voting rule through the radii of the corresponding P- and W-balls, and directly on other model primitives such as the status quo, discount factor, and the distribution of proposers.

I establish comparative statics on the equilibrium bounds with respect to the location of the status quo, showing that they expand as it gets further away from the center of the bounds (Theorem 6). In the discounted case, the inner and outer bound differ by a fixed constant at the limit as the distance of the status quo from the center of the bounds goes to infinity, implying that for bad enough status quo a unique equilibrium in pure strategies prevails if the distribution of proposers has bounded support (Corollary 4). I also study the effect of changes in the voting rule and show that more restrictive (proper) voting rules, such as higher majority quota thresholds in one chamber or the addition of a legislative chamber, have ambiguous effects on the outer bound though the inner bound weakly contracts (Theorem 10). On the one hand, such changes tend to shrink the set of agreements that can pass in any equilibrium. A counter-veiling force emerges, though, because more stringent voting rules expand the set of core points (once the core is non-empty) and may also expand the set of possible equilibria.

What do these bounds imply about the possible or likely location of equilibrium outcomes? First, for any fixed instance of the model, the radius of the outer bound is finite (Theorem 3), therefore social choice induced by the non-cooperative legislative bargaining model cannot wander everywhere in the space of alternatives. This is so *if* the distribution of proposers has finite second moments[4] and in Appendix B I provide an example in which that moment condition is violated and there exists an equilibrium with outcome support on

[4] This restriction is automatically met if the distribution of proposers has finite support.

the entire real line. Furthermore, under the assumed moment restriction on the distribution of proposers, finiteness of the bound also relies on the structure of the set of agreements that can pass in equilibrium induced by quadratic preferences (Lemma 2) which disciplines the distribution of optimal proposals (Lemma 3).

Second, I explore conditions for equilibrium outcomes to be at or near *core* points (if they exist) or *uncovered* points (Fishburn (1977); Miller (1980)). I term these conditions *advocacy* conditions, as they generally amount to the existence or prevalence of proposers that advocate for such outcomes. The weaker of these advocacy conditions are generalizations of conditions of Banks and Duggan (2000, 2006) for core equivalence (Theorem 4, part 2) and require that a unique core is in the support of the distribution of proposers when legislators are perfectly patient. In the general – discounted or not – case, Theorem 5 establishes a lower bound on the probability equilibrium outcomes fall within an envelope containing the uncovered set (or the core), where the bounding probability is equal to the probability of proposers sufficiently close to the center of that envelope.

Thirdly, I show that the outer bound expands as the distribution of proposers piles mass away from its center (Theorem 8, part 1). In the *discounted* case, I show that there is a maximal bound that limits the effect of changes in the distribution of proposers for any fixed discount factor. The radius of this maximal bound coincides with the distance of the center of the bound from the status quo if the bound is centered at a core point, but if the core is empty it increases with the discount factor and, contrary to the expectation that equilibrium outcomes are confined in the center of the policy space, it encompasses the entire policy space at the limit as the discount factor goes to one (Theorem 8, parts 2-3). Furthermore, if the core is empty, then for any arbitrary distance from any point y and sufficiently patient legislators, there exists a distribution of proposers that guarantees the bulk of equilibrium outcomes are farther than that distance (Theorem 8, part 4). In the limit model with perfectly patient legislators, I show that if there exists a(n arbitrary) unique proposer,[5] there exists an equilibrium in which that single proposer can pass her ideal point with probability one if and only if there does not exist a legislator with absolute veto who prefers the status quo more than the ideal point of the proposer (Theorem 9).

In sum, in the spatial legislative bargaining model there is a limit to the effect of proposal rights on equilibrium outcomes for any fixed discount factor less than one and, *if* the advocacy conditions of Theorems 4 or 5 are met, then this model may support the core or its generalizations. But in the absence of a core, manipulation of the rights for the origination of proposals can nullify these centrifugal tendencies as the legislators become patient. Notably, patience does not deliver this result through complex history-dependent equilibrium constructions, and the analysis throughout focuses on equilibria in stationary strategies. These latter findings parallel and complement a number of recent studies on the role of proposal rights in collective decision-making. In the divide-the-dollar setting, Ali, Bernheim and Fan (2019) expand the set of recognition protocols to non-stationary rules and study the role of predictability in allowing a proposer to extract the whole dollar. Kalandrakis (2006) shows that even with stationary recognition protocols (as assumed in this study) all possible equilibrium expected payoffs are attainable for some allocation of proposal rights.

The reliance of the approach pursued in this paper on generalized medians and their connection to the yolk directly links this work with an earlier wave of the literature on collective decision-making motivated by the same broad questions that motivate this study. The yolk first appears in the work of Ferejohn, McKelvey and Packel (1984) who study a dynamic non-equilibrium process of agenda formation that induces a Markov process over policy outcomes. McKelvey (1986) used the yolk in an incisive construction of an outer bound for the uncovered set in the classic spatial model. McKelvey and several authors

[5]A number of papers focus on the special case of a unique proposer in a similar class of bargaining models including Primo (2002) and Duggan and Ma (2018). Kalandrakis (2010), Diermeier and Fong (2012), and Anesi and Duggan (2016) study the special case of a unique proposer in sequential bargaining models with endogenous status quo.

established a connection between the uncovered set and Downsian electoral competition (e.g., Banks, Duggan and LeBreton (2002); Dutta and Laslier (1999); McKelvey (1986)), or certain legislative amendment procedures (McKelvey (1986); Shepsle and Weingast (1984)). Gary Cox (Cox (1987)) generalized McKelvey's conclusion on the central location of the uncovered set to non-Euclidean preferences. Feld, Grofman and Miller (1988) developed *a priori* bounds on the size of the yolk; while Feld, Grofman and Miller (1989) suggest that the ability to manipulate the agenda in committees may be limited when the yolk is small. All of these studies have as common theme a direct or implied restriction on likely social choice outcomes to a centrally located set of the policy space that shrinks as the committee preferences get closer to satisfying conditions for the existence of a core point. As already discussed, there are similarities and important differences between the main thrust of the conclusions in these studies and the information that can be gleaned from the equilibrium bounds established presently.

I conclude this section with some comments on computational aspects of the analysis. An important advantage of the approach of this paper is that it obtains bounds on *all* equilibria at an overall cost that is polynomial in the size of the committee. Therefore, this approach is generally less expensive than computation of even a single equilibrium. Though there are versions of the legislative bargaining model (in the divide-the-dollar case) for which a polynomial time algorithm to compute equilibrium expected payoffs exists (Kalandrakis (2015)), no such algorithm is known to exist for the model with spatial preferences.[6] Outside special cases, computing one equilibrium of this model is considered a hard problem. Furthermore, the proposed approach bounds *all* equilibria, whilst finding *all* Brouwer fixed points is an even harder computational problem (Herings and Peeters (2010); McKelvey and McLennan (1996)). Finally, even if an actual equilibrium is desired, the bounds provide a natural starting point to economize on its computation.

[6]One important difference between these models is that in the divide-the-dollar framework, equilibrium expected payoffs are known to be unique (Eraslan (2002); Eraslan and McLennan (2013)), but payoff uniqueness does not generally hold in the spatial model.

1 Model

Consider policy-making over a D-dimensional policy space, $X = \mathbb{R}^D$, where $D \geq 1$. To allow large committees with a continuum of members let the set of players be $\mathcal{I} = \mathbb{R}^D$. Players are indexed by $\hat{x} \in \mathcal{I}$, and are endowed with a von Neumann-Morgenstern utility function $u : X \times \mathcal{I} \to \mathbb{R}$ over policies, x, that takes the familiar negative quadratic form $u(x; \hat{x}) = -(x - \hat{x})^T \cdot (x - \hat{x})$. Let π be a probability measure on $\mathcal{I}$, and assume that it has finite first and second moments. In each period $t = 1, 2, \dots$ before an agreement is reached, a proposer $\hat{x}$ is realized from distribution π, independently across periods. The proposer offers a proposal z that is put to an up-or-down vote and if the set of legislators $A \subseteq \mathcal{I}$ that approve it satisfies (1) (as explained below), then z is implemented in all remaining periods and the game ends; otherwise, a status quo policy $q \in X$ is implemented in period t and the game moves to period $t+1$ with a new proposer drawn from π, a new proposal, etc., until an agreement is reached. Players discount the future by a common factor $\delta \in [0, 1]$ and player $\hat{x}$'s payoff if agreement x is reached in period t is given by $(1 - \delta^{t-1})u(q; \hat{x}) + \delta^{t-1}u(x; \hat{x})$ (and it is $u(q; \hat{x})$ in the case of perpetual disagreement).

Voting over policies is organized in a finite number of $L, L \geq 1$, legislative chambers indexed by $\ell = 1, \dots, L$, each represented by a pair μ_ℓ, m_ℓ: μ_ℓ is a probability measure over $\mathcal{I}$ which can be construed as the distribution of voting weights in chamber ℓ; $m_\ell \in (0, 1)$ is a threshold of support that is required in chamber ℓ in order for a policy to be approved in that chamber. Proposal x is approved if the set of legislators $A \subseteq \mathcal{I}$ that vote for it are a winning coalition in each chamber

$$\mu_\ell(A) \geq m_\ell, \text{ for all } \ell = 1, \dots, L. \tag{1}$$

A single legislature deciding by majority rule is the special case when $L = 1$ and $m_1 = \frac{1}{2}$.

The equilibrium concept, which is standard in this literature, is subgame perfect Nash in stationary, no-delay,[7] proposal strategies, and voting strategies such that players approve a proposal if and only if they weakly prefer it over their continuation payoff. To formalize, let $\mathcal{P}$ denote the space of Borel probability measures on X. A strategy profile is a pair consisting of a measurable proposal profile $p : \mathcal{I} \to \mathcal{P}$ and a voting profile $A : \mathcal{I} \to 2^X$. Here $p_{\hat{x}} \in \mathcal{P}$ is the probability distribution over proposals when the proposer has ideal point $\hat{x}$ and $A(\hat{x}) \subseteq X$ is the set of proposals that committee member $\hat{x}$ is approving. Let $\sigma = (A, p)$ denote a complete such strategy profile. Given voting strategy profile A the set of agreements that can pass is given (according to (1)) by $A_\sigma = \{x \in X \mid \mu_\ell(\{\hat{x} \mid x \in A(\hat{x})\}) \geq m_\ell \text{ for all } \ell\}$. With proposal strategies that are *no-delay* (see footnote 7), that is, that place probability one on A_σ, the continuation payoff of $\hat{x}$ is given by $v_\sigma(\hat{x}) = (1-\delta)u(q; \hat{x}) + \delta \int \int u(z; \hat{x}) p_{\hat{x}'}(dz)\pi(d\hat{x}')$. Two conditions are *necessary* for such $\sigma = (A, p)$ to be an equilibrium:

$$A(\hat{x}) = \{x \mid u(x; \hat{x}) \geq v_\sigma(\hat{x})\}, \tag{E_v}$$

$$p_{\hat{x}}\left(\arg\max_x \{u(x; \hat{x}) \mid x \in A_\sigma\}\right) = 1. \tag{E_p}$$

While these conditions preclude profitable one-stage deviations, additional conditions must be verified in general to preclude profitable infinite deviations.[8] I will refer to a profile σ that satisfies (E_v) as a *voting equilibrium*. I call σ a *quasi-equilibrium* if it also satisfies (E_p), to emphasize that profitable infinite deviations have not been ruled out (yet). I call a profile

[7] It is well understood that the restriction to no-delay proposal strategies is without loss of generality in this model. With strictly concave payoffs, delay is only possible in knife-edge situations (see Banks and Duggan (2006)), and in those cases such equilibria with delay prevail, they are outcome equivalent to an equilibrium without delay. For the exact same reasons, there is no consequence to the added restriction on voting strategies to accept proposals when indifferent.

[8] This is so even in the discounted case ($\delta < 1$) because we have allowed payoffs to be unbounded. If $\delta < 1$, the bounds established in Section 2 allows us to invoke the one-stage-deviation principle to conclude that every quasi-equilibrium is an equilibrium (Corollary 3).

σ an *equilibrium* when profitable infinite deviations have also been precluded.

A few comments are in order regarding this model setup. First, allowing for a continuum (or even countably infinite) number of players is a significant departure from existing literature. Infinite committees conveniently capture direct democracy settings in which proposals are approved through referendum in a large electorate approximated by a continuum. Furthermore, the support of probability measures $\mu_1, \ldots, \mu_L, \pi$ is unrestricted. It is possible, for example, that the distribution of proposers has full support so that all conceivable positions might be advocated, albeit with vanishing likelihood, in society. Finitely populated legislatures are special cases when each μ_ℓ is a discrete measure with finite support. The set of proposers can be restricted among players with voting rights $(Support(\pi) \subseteq \left(\cup_{\ell=1}^{L} Support(\mu_\ell)\right))$ or we can assume proposals to the legislature arise from outside members such as the electorate, lobbies, interest groups, etc. The only restriction placed on the distribution of proposers, π, is that it has finite first and second moments, a restriction that is automatically met in the finite committee setting.

It is not consequential that we require legislators to have distinct ideal points. We can equivalently capture the case two or more legislators share an ideal point $\hat{x}$ by accordingly increasing the voting weight (and possibly proposal probability) of $\hat{x}$. Under this interpretation, legislators that share an ideal point are restricted to play the same strategy. With regard to voting strategies, this restriction is already imposed broadly in the literature via condition (E_v). The restriction on proposal strategies also does not affect the equilibrium outcome distribution: Every equilibrium such that two or more legislators with identical ideal points employ distinct proposal strategies can be transformed to an equivalent equilibrium in which they share the same proposal strategy. Finally, the restrictions on the voting rule are also minimal. As shown by Taylor and Zwicker (1993) any multi-cameral legislature, along with veto and veto-override provisions similar to those found in the US constitution, are easily accommodated. In fact, *all* monotonic voting rules can be captured if the committee is finite (Taylor and Zwicker (1993)).

With these comments on the model, we can proceed to the analysis. Before doing so, I introduce additional necessary notation. Let a hyperplane $H_{a,c}$ in $\mathbb{R}^D$ be defined by direction $a \in \mathcal{A}$ and level c such that $H_{a,c} = \{x \mid a^T \cdot x = c\}$, where $\mathcal{A} = \{a \in \mathbb{R}^D \mid ||a|| = 1\}$ is the set of unit vectors in $\mathbb{R}^D$. Let $H^+_{a,c} = \{x \mid a^T \cdot x \geq c\}$ and $H^-_{a,c} = \{x \mid a^T \cdot x \leq c\}$ be the two closed half-spaces defined by $H_{a,c}$. If the direction a and level c are not needed in the context, I denote a hyperplane by H, and let H^+ and H^- be the corresponding (arbitrary) two half-spaces. For any $x \in \mathbb{R}^D$, let $\mathcal{B}(x, d) = \{x' \in \mathbb{R}^D \mid d(x, x') \leq d\}$ be the *closed* ball of points within distance d from x, and denote the corresponding open ball by $\mathcal{B}^o(x, d)$.

2 Equilibrium bounds

To illustrate the approach, I first show its incarnation in the special case of the one-dimensional model ($X = \mathbb{R}$), with one chamber ($L = 1$) operating under simple majority rule ($m_1 = \frac{1}{2}$), and a *unique* median committee member at $\hat{x}_m$. In this setting there is a unique equilibrium when $\delta < 1$ and the committee is finite (Cho and Duggan (2003)). Uniqueness of equilibrium aside, for any equilibrium σ the majority-acceptable agreements are given by an interval $A_\sigma = [\hat{x}_m - B_\sigma, \hat{x}_m + B_\sigma]$ for some scalar $B_\sigma \geq 0$. It follows that the distance of $\hat{x}$'s equilibrium proposal from the median is

$$\tilde{d}(B_\sigma, \hat{x}) = \min\{d(\hat{x}_m, \hat{x}), B_\sigma\}. \tag{2}$$

This allows us to express the continuation payoff of the median as

$$v_\sigma(\hat{x}_m) = -\left((1-\delta)(d(\hat{x}_m, q))^2 + \delta \int (\tilde{d}(B_\sigma, \hat{x}))^2 \pi(d\hat{x})\right),$$

and pin down the distance, B_σ, of the farthest alternative from $\hat{x}_m$ the median approves by the indifference condition $u(\hat{x}_m + B_\sigma; \hat{x}_m) = u(\hat{x}_m - B_\sigma; \hat{x}_m) = v_\sigma(\hat{x}_m)$. Therefore, B_σ is a solution for B to

$$B^2 - \left((1-\delta)d(\hat{x}_m, q)^2 + \delta \int (\tilde{d}(B, \hat{x}_m))^2 \pi(d\hat{x})\right) = 0. \tag{3}$$

Generically, equation (3) admits a unique solution, but to cover cases with multiple equilibria, define B^* to be its largest solution. This coincides with the general outer bound $\bar{B}^*(\hat{x}_m)$ I characterize later in this section,[9] that is, in every equilibrium σ, possible equilibrium agreements are confined within $\mathcal{B}(\hat{x}_m, B^*) \supseteq A_\sigma$. What is particularly appealing in this special setting is that the computation involved could not be simpler: B^* solves one equation in one unknown.

Two stumbling blocks stand in the way of obtaining an equation analogous to (3) more generally. First, the set of agreements that can pass in equilibrium does not coincide with the set of agreements acceptable by some decisive player, such as a unique median, whose preferences can be used to determine the bound on equilibrium agreements as in equation (3). In more than one dimensions, such decisive player does not exist generally or even generically. Second, when $D > 1$ the optimal proposals cannot be determined *a priori* as a function of the bound in the sense of equation (2) since the set A_σ is generally not convex and has no standard geometric form. This is so even if equilibrium is unique and in pure strategies, though in more than one dimensions equilibrium need not be unique and may require non-degenerate mixed proposal strategies.

To overcome the first obstacle, I use two generalizations of the median developed in Kalandrakis (2019), and I introduce these concepts here for completeness and to make the analysis self-contained. Define a *pivotal* hyperplane as one that renders one of its half-spaces a smallest such (that is, half-space) winning coalition. Specifically:

Definition 1. *Hyperplane $H_{a,c}$ is* pivotal *if $c = c^*(a)$ where*

$$c^*(a) = \max_c \{c \mid \mu_\ell(\{\hat{x} \mid a^T \cdot \hat{x} \geq c\}) \geq m_\ell \text{ for all } \ell\}.$$

For every $a \in \mathcal{A}$, $H^+_{a,c^*(a)}$ is a pivotal winning half-space*.*

We can now define the two median generalizations for an arbitrary point y. A y-centered P-ball (for pivotal) is a smallest radius ball $\mathcal{B}(y, r)$ that intersects all pivotal hyperplanes,

[9] If there is a unique equilibrium, B^* also coincides with the inner bound $\underline{B}^*(\hat{x}_m)$.

that is, r solves

$$(P_y) \qquad \min_r \{r \mid \mathcal{B}(y,r) \cap H_{a,c^*(a)} \neq \emptyset \text{ for all } a \in \mathcal{A}\}.$$

Analogously define for all y a y-centered W-ball (for winning) as a smallest radius ball $\mathcal{B}(y,r)$ that intersects all pivotal winning half-spaces

$$(W_y) \qquad \min_r \{r \mid \mathcal{B}(y,r) \cap H^+_{a,c^*(a)} \neq \emptyset \text{ for all } a \in \mathcal{A}\}.$$

Denote the minimizer (and attained minimum) of programs (P_y) and (W_y) by $r_p(y)$ and $r_w(y)$. It is easy to see that $r_p(y) \geq r_w(y)$ for all y.

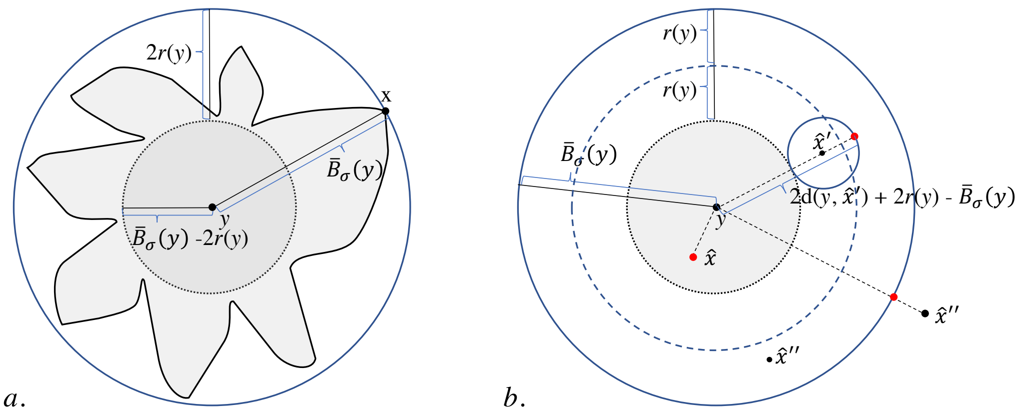
Lemma 1 renders precise the claim that these constructs function as generalized medians: For all y, any agreement that can pass in a voting equilibrium must be weakly preferred by *some* player with ideal point in the y-centered W-ball; and, if all $\hat{x}$ in the y-centered P-ball approve x in a voting equilibrium, then x is approved by some winning coalition in equilibrium.

Lemma 1. *Let σ be a voting equilibrium. For all y and all x*

1. *If $x \in A_\sigma$, then there exists $\hat{x} \in \mathcal{B}(y, r_w(y))$ such that $u(x;\hat{x}) \geq v_\sigma(\hat{x})$.*
2. *If $u(x;\hat{x}) \geq v_\sigma(\hat{x})$ for all $\hat{x} \in \mathcal{B}(y, r_p(y))$, then $x \in A_\sigma$.*

The proofs of all results appear in Appendix A. Building on Lemma 1, the next Lemma ensures that if a policy x can be approved in equilibrium, then for all y, any policy that is closer to y than x by a distance $2(r_p(y) + r_w(y))$ is also acceptable. This result bears a close connection to arguments of McKelvey (1986) for the *winset* under simple majority rule. In the present setting, the argument applies to general voting rules, arbitrary center points y, and proposals. Also, Lemma 1 is not implied by the results in Kalandrakis (2019) as it involves a social preference comparison of proposals against equilibrium continuation lotteries instead of deterministic alternatives. To economize on notation, in what follows I set $r(y) := r_p(y) + r_w(y)$.

Figure 2: Illustration of Lemma 2 and Lemma 3



a. If x is the farthest policy from y that can be approved $(d(y,x) = \bar{B}_\sigma(y))$, then every alternative in the open ball $\mathcal{B}\left(y, \bar{B}_\sigma(y) - 2r(y)\right)$ can also be approved in equilibrium.

b. Proposals by $\hat{x} \in \mathcal{B}\left(y, \bar{B}_\sigma(y) - 2r(y)\right)$ are at their ideal; proposals by $\hat{x}' \in \mathcal{B}\left(y, \bar{B}_\sigma(y) - r(y)\right) \setminus \mathcal{B}\left(y, \bar{B}_\sigma(y) - 2r(y)\right)$ are at maximal distance $2d(y, \hat{x}') + 2r(y) - \bar{B}_\sigma(y)$ from y; proposals by $\hat{x}'' \notin \mathcal{B}\left(y, \bar{B}_\sigma(y) - r(y)\right)$ are no farther than $\bar{B}_\sigma(y)$ from y. Red points indicate most distant possible proposals.

Lemma 2. *For all y, x, and voting equilibrium σ, if $x \in A_\sigma$, then $\mathcal{B}(y, d(y,x) - 2r(y)) \subseteq A_\sigma$.*

Lemma 2 is crucial. For starters, it allows us to address the second difficulty to generalizing the approach illustrated in the one-dimensional model: Instead of an exact distance $\tilde{d}$ of optimal proposals from the median, as in equation (2), Lemma 2 implies upper and lower bounds on the distance of optimal proposals from any point y. I establish these bounds on proposals in the forthcoming Lemma 3, and outline a sketch of the argument for that result in what follows. Specifically, if B is the distance of the furthest policy from y that can be approved, then $\hat{x}$'s equilibrium proposal *cannot be further* than a distance $\bar{d}(B, \hat{x}, y)$ from y, where

$$\bar{d}(B, \hat{x}, y) := \begin{cases} B & \text{if } B < \max\{r(y), d(y, \hat{x})\} + r(y), \\ 2d(y, \hat{x}) + 2r(y) - B & \text{if } \max\{r(y), d(y, \hat{x})\} + r(y) \leq B < d(y_p, \hat{x}) + 2r(y), \\ d(y_p, \hat{x}) & \text{if } B \geq d(y, \hat{x}) + 2r(y). \end{cases}$$

The first case on the right-hand-side is obvious because, by assumption, B is the maximal distance of acceptable proposals from y. The last case follows from the fact that all agreements within $\mathcal{B}(y, B-2r(y))$ are acceptable by Lemma 2 (these points are in the highlighted disk in the center of Figure 2.b), so that an optimizing player $\hat{x} \in \mathcal{B}(y, B-2r(y))$ must be proposing her ideal point. The middle case pertains to proposers with ideal points such as point $\hat{x}'$ of Figure 2.b. Because alternatives in $\mathcal{B}(y, B-2r(y))$ are available to $\hat{x}'$, her proposals cannot be outside the ball centered at her ideal $\hat{x}'$ and tangent to $\mathcal{B}(y, B-2r(y))$ (which has radius $d(y,\hat{x}) - (B-2r(y))$). The farthest point from y in that ball is at a distance $d(y,\hat{x}) + (d(y,\hat{x}) - (B-2r(y)))$, as specified by the function $\bar{d}(B,\hat{x},y)$.

Assume on the other hand that every policy within distance B from y is approved in equilibrium. We can similarly reason that $\hat{x}$'s proposal is *no closer* to y than

$$\underline{d}(B,\hat{x},y) := \min\{B, d(y,\hat{x})\}.$$

In this case, $\hat{x}$ can pass her ideal if $B > d(y,\hat{x})$ and otherwise would not propose anything closer than B to y since a proposal at a distance exactly $d(y,\hat{x}) - B$ from her ideal is available.

To put all these results together, for all y and any quasi-equilibrium σ define

$$\bar{B}_\sigma(y) = \inf\{d \geq 0 \mid A_\sigma \subseteq \mathcal{B}(y,d)\},$$
$$\underline{B}_\sigma(y) = \sup\{d \geq 0 \mid \mathcal{B}^o(y,d) \subseteq A_\sigma\}.$$

$\bar{B}_\sigma(y)$ is the largest distance of acceptable policies from y; and $\underline{B}_\sigma(y)$ is the smallest (infimum) distance from y of non-acceptable agreements. Note that by Lemma 3 we must have $\bar{B}_\sigma(y) - \underline{B}_\sigma(y) \leq 2r(y)$. This inequality can be strict, in general. In the one-dimensional, simple majority rule case with a unique median, $\hat{x}_m$, we have $\bar{B}_\sigma(\hat{x}_m) - \underline{B}_\sigma(\hat{x}_m) = r(\hat{x}_m) = 0$. In Lemma 3, I use these quantities to bound optimal proposals generalizing equation (2) to an upper and lower bound (4).

Lemma 3. *For all y and quasi-equilibrium σ, $p_{\hat{x}}(\{\hat{x}\}) = 1$ for all $\hat{x} \in \mathcal{B}^o(y, \underline{B}_\sigma(y))$ and*

$$\underline{d}(\underline{B}_\sigma(y), \hat{x}, y) \leq d(y, z) \leq \bar{d}(\bar{B}_\sigma(y), \hat{x}, y), \text{ for all } \hat{x} \text{ and all } z \in Support(p_{\hat{x}}). \tag{4}$$

To specify analogues of equation (3) in the general case, define

$$\bar{F}(B, y) = (B - r_w(y))^2 - \left((1-\delta)(d(y, q) + r_w(y))^2 + \delta \int \left(\bar{d}\,(B, \hat{x}, y) + r_w(y)\right)^2 \pi(d\hat{x})\right),$$
$$\underline{F}(B, y) = (B + r_p(y))^2 - \left((1-\delta)(d(y, q) - r_p(y))^2 + \delta \int \left(\underline{d}\,(B, \hat{x}, y) - \min\{B, r_p(y)\}\right)^2 \pi(d\hat{x})\right).$$

Using Lemmas 1-3, we now establish:

Theorem 1. *Let σ be a quasi-equilibrium. For all y, $\bar{B}_\sigma(y)$ and $\underline{B}_\sigma(y)$ are finite and satisfy $\bar{F}(\bar{B}_\sigma(y), y) \leq 0$ and $\underline{F}(\underline{B}_\sigma(y), y) \geq 0$.*

Inequalities $\bar{F}(\bar{B}_\sigma(y), y) \leq 0$ and $\underline{F}(\underline{B}_\sigma(y), y) \geq 0$ must hold for any quasi-equilibrium σ. Therefore, the maximum (minimum) candidate bound that satisfies them define the advertised inner (outer) bound on equilibrium acceptable agreements:

$$\bar{B}^*(y) := \sup\{B \geq 0 \mid \bar{F}(B, y) \leq 0\},$$
$$\underline{B}^*(y) := \inf\{B \geq 0 \mid \underline{F}(B, y) \geq 0\}.$$

With these definitions in place, we can now state the main results of this section:

Theorem 2. *For every quasi-equilibrium σ and every y, $\mathcal{B}^o(y, \underline{B}^*(y)) \subseteq A_\sigma \subseteq \mathcal{B}(y, \bar{B}^*(y))$.*

Note that, unlike equation (3) which centers the motivating one-dimensional bound at the median, Theorems 1 and 2 establish generalizations centered at any, arbitrary, point y. Corollary 1 highlights how this generality can be used to advantage by combining bounds across different y.

Corollary 1. *For every $Y \subseteq \mathbb{R}^D$, every $y_p, y_w \in Y$, and and every quasi-equilibrium σ*

$$\mathcal{B}^o(y_p, \underline{B}^*(y_p)) \subseteq \bigcup_{y \in Y} \mathcal{B}^o(y, \underline{B}^*(y)) \subseteq A_\sigma \subseteq \bigcap_{y \in Y} \mathcal{B}(y, \bar{B}^*(y)) \subseteq \mathcal{B}(y_w, \bar{B}^*(y_w)).$$

Though the set Y in Corollary 1 can be a continuum, the practically relevant scenario is when it is a finite grid covering the space of legislator ideal points. The computations reported later in this paper suggest that this approach is both feasible and recommended.

Theorem 2 and Corollary 1 bound equilibrium acceptable agreements. But, by reasoning as in Lemma 3, these results imply stricter bounds on equilibrium *proposals* and (by the no-delay property) on *outcomes*. To state these more stringent bounds first define

$$\bar{D}_{\hat{x}}(y) := \min\{\bar{B}^*(y), \max\{d(y, \hat{x}), r(y)\} + r(y)\}.$$

$\bar{D}_{\hat{x}}(y) = \max_{B \in [0, \bar{B}^*(y)]} \bar{d}(B, \hat{x}, y)$ and, by Lemma 3 and Theorem 2, it is the maximum distance from y that $\hat{x}$'s proposal can reach. We also define

$$\underline{D}_{\hat{x}}(y) := \begin{cases} \max\{0, d(y, \hat{x}) - \underline{B}^*(y)\} & \text{if } \underline{B}^*(y) > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

When $\underline{B}^*(y) > 0$, $\underline{D}_{\hat{x}}(y)$ is that largest possible distance from her ideal point that $\hat{x}$ could propose, given that alternatives in $\mathcal{B}^o(y, \underline{B}^*(y))$ are available to her. These arguments establish:

Corollary 2. *For every $Y \subseteq \mathbb{R}^D$, every $\hat{x}$, and every quasi-equilibrium σ*

$$\text{Support}(p_{\hat{x}}) \subseteq \left(\bigcap_{y \in Y} \mathcal{B}\left(y, \bar{D}_{\hat{x}}(y)\right) \right) \bigcap \mathcal{B}\left(\hat{x}, \min_{y \in Y} \underline{D}_{\hat{x}}(y)\right).$$

Though $\bar{D}_{\hat{x}}(y)$ can be strictly smaller than $\bar{B}^*(y)$, the likely most dramatic restrictions of this type arise from the second part of Corollary 2, in cases $\mathcal{B}^o(y, \underline{B}^*(y))$ is non-empty.[10]

These results leave open the possibility that solutions to the inequations of Theorem 1 may not exist, or that the outer bound may be infinite, or that either bound may be hard to compute. All are precluded by the next Theorem, which ensures that the bounds are well-defined and provides additional details on their location.

Theorem 3. *For all y, $\bar{B}^*(y), \underline{B}^*(y) \in \mathbb{R}_+$ exist. Moreover,*

1.a *If $(1-\delta)d(y,q)+r_w(y) > 0$, then either $\bar{B}^*(y) = 2r(y)$ and $\bar{F}(2r(y), y) > 0, \bar{F}(B, y) \neq 0$ for all $B > 0$, or $\bar{B}^*(y) \geq 2r_w(y)$ is the unique $B > 0$ that solves $\bar{F}(B, y) = 0$.*

1.b *If $(1-\delta)d(y,q) + r_w(y) = 0$, then*

  i. *If $\delta = 1$, $\bar{F}(B, y) = 0$ for all $B \leq \bar{B}^*(y) = \max\left\{2r_p(y), \sup_B \{B \mid \pi\left(\mathcal{B}(y, B - r_p(y))\right) = 0\}\right\}$.*

  ii. *If $\delta < 1$, $\bar{B}^*(y) = 0$.*

2.a *If $(1-\delta)(d(y,q) - r_p(y))^2 > r_p(y)^2$, then $\underline{B}^*(y) > 0$ uniquely solves $\underline{F}(B, y) = 0$.*

2.b *If $(1-\delta)(d(y,q) - r_p(y))^2 \leq r_p(y)^2$, then $\underline{B}^*(y) = 0$.*

It may be tempting to discount the finding that the outer bound $\bar{B}^*(y)$ is finite as obvious, but this result crucially relies on the structure of the set of acceptable agreements established in Lemma 2 and the finiteness of the second moments of the distribution of proposers π. That the latter assumption cannot be disposed with is shown in Appendix B. One immediate implication of the finiteness of the outer bound $\bar{B}^*(y)$ is that the set of equilibria of the game are identical to the set of equilibria of a game with compact policy space $X' = \mathcal{B}(y, \bar{B}^*(y))$, thus ensuring that in the discounted case conditions (E_v) and (E_p) are also sufficient for equilibrium:

[10] These additional restrictions suggest the possibility of *iterating* the procedure for computing the bounds by adding the new proposal restrictions from Corollary 2 in successive iterations. I leave the details of this promising avenue to tighten the bounds for future work.

Corollary 3. *If $\delta < 1$, every quasi-equilibrium is an equilibrium.*

When Theorem 3 does not yield the bounds in closed form, these can be recovered from the respective equations using bisection becasue, in the relevant range dictated by Theorem 3, they consistently change sign on either side of a unique solution. Therefore, the major computation involved in obtaining these bounds is that of solving (P_y) or (W_y) which is straightforward in the finite legislature case by the arguments in Tovey (1992) (specifically Theorem 1 and Corollary 1, page 266). Appendix D provides details.

3 Implications & constitutional design

How tight are these bounds? What do they imply about the set of equilibrium outcomes and its relation to classic social choice concepts? How do they change with model primitives? I take up these questions in this section. I have broken the discussion in small segments.

Relation to the core & the uncovered set: By Theorem 3, if y is the only acceptable agreement in equilibrium ($\bar{B}^*(y) = 0$), it is necessary that $r_w(y) = 0$ and y is a core point (see Kalandrakis (2019)). The following Theorem elaborates:

Theorem 4. *If $\bar{B}^*(y) = 0$ then y is a core point. If y is a core point, then $\bar{B}^*(y) = 0$ if*

1. *$\delta < 1$ and $q = y$, or,*

2. *$\delta = 1$, y is a unique core point, and $\pi(\mathcal{B}(y, \epsilon)) > 0$ for all $\epsilon > 0$.*

Therefore, we have a unique equilibrium with a policy at a core point if either, players are not perfectly patient but the status quo policy is at that core point (part 1); or, if the core is unique, players are perfectly patient, and there exist proposers to advocate for the core (part 2). This last condition on π is a parallel to the requirement that all players have positive recognition probability in the core equivalence and core selection analysis of Banks and Duggan (2000, 2006) for finite committees, which these results recover and complement.

Instead of requiring the core to be the only equilibrium outcome, we may instead consider whether outcomes near the uncovered set (which reduces to the core when the latter is non-empty) prevail. For all y, the uncovered set is contained in $\mathcal{B}(y, 2r(y))$.[11] Relying on Lemma 3, we establish a lower bound on the probability that equilibrium outcomes fall within this envelope. For the purposes of this result, let λ_σ denote the equilibrium lottery over *outcomes*.

Theorem 5. *For all y and every quasi-equilibrium σ, $\lambda_\sigma\left(\mathcal{B}(y, 2r(y))\right) \geq \pi\left(\mathcal{B}(y, r(y))\right)$.*

Thus, if possible proposers are tightly concentrated within half the distance of the envelope on the uncovered set from its center, then equilibrium outcomes are also concentrated within the generalized uncovered set envelope. Combined with Theorem 4 and the forthcoming Theorem 8, these results generally imply that the prevalence of outcomes in or near these sets generally relies on the degree to which they are advocated by proposers.

Status Quo: A classic insight of political theory is that collective choices differ systematically with the location of the status quo. This is an almost immediate conclusion in static models of choice and Romer and Rosenthal (1978) is an elementary but forceful demonstration of that effect. The following result establishes the effect of the status quo on the equilibrium bounds.

Theorem 6. *For all y, $\bar{B}^*(y)$ and $\underline{B}^*(y)$ weakly increase as $d(y,q)$ increases and, if $\delta < 1$,*

$$\lim_{d(y,q)\to+\infty} \underline{B}^*(y) = +\infty \qquad \textit{and} \qquad \lim_{d(y,q)\to+\infty} \left(\bar{B}^*(y) - \underline{B}^*(y)\right) = \left(1 + \sqrt{1-\delta}\right) r(y).$$

Both the set of possible equilibrium outcomes and the set of outcomes guaranteed to be acceptable in every equilibrium weakly expand as the status quo moves further away from the center y. In the discounted case, these sets converge and cover the entire policy space as the distance of the status quo from y goes to infinity, but the difference between the outer and inner bounds converges to a constant at most twice the sum of the radii of the y centered

[11] This is a generalization of an envelope due to McKelvey (1986) (see Kalandrakis (2019)).

P- and W-balls. Combining Theorem 6 and Lemma 3 we conclude that if the distribution of proposers has bounded support and the status quo is bad enough, then there exists a unique equilibrium in which all proposers propose their ideal.

Corollary 4. *If $\delta < 1$ and $\pi\left(\mathcal{B}(y, \bar{P})\right) = 1$ for some y, $\bar{P} > 0$, then there exists $\bar{Q} > 0$ such that for all $q \notin \mathcal{B}(y, \bar{Q})$ equilibrium is unique and satisfies $p_{\hat{x}}(\{\hat{x}\}) = 1$ for all $\hat{x} \in$ Support(π).*

Equilibrium Uniqueness: Besides Corollary 4, we also deduce a unique equilibrium whenever $\underline{B}^*(y) = \bar{B}^*(y)$, whence it follows that $A_\sigma = \mathcal{B}(y, \bar{B}^*(y))$. Theorem 4 provides one set of sufficient conditions for a unique equilibrium at the core. More generally:

Theorem 7. *If $\underline{B}^*(y) = \bar{B}^*(y)$ then y is a core point and there exists a unique equilibrium, σ, and σ is in pure strategies with $A_\sigma = \mathcal{B}(y, \bar{B}^*(y))$. Furthermore, if y is a core point*

1. *$\underline{B}^*(y) = \bar{B}^*(y) = 0$ in cases 1. and 2. of Theorem 4.*
2. *$\underline{B}^*(y) = \bar{B}^*(y) > 0$ if and only if y is the unique core point and $(1 - \delta)d(y, q) > 0$.*

Thus, in the discounted case, equilibrium is unique when the core is non-empty (not necessarily a singleton) and the status quo is a core point; the equilibrium in those cases puts mass one on the status quo (case 1 of Theorem 4). We also have a unique equilibrium with multiple acceptable agreements if the core is unique and the status quo is different than the core (part 2 of Theorem 7). In the undiscounted case ($\delta = 1$) the conditions replicate those in part 2 of Theorem 4. Part 2 of Theorem 7 extends the uniqueness result of Cho and Duggan (2003) to more than one dimensions and infinite committees.

Proposal rights: While Theorems 2 and 3 assure us that equilibrium outcomes are bounded for any given distribution of proposers, π, the actual bounds generally vary as the allocation of proposal rights changes. Let Π denote the space of all admissible distributions over proposers, that is, those with finite second moments. We can show:

Theorem 8. *For all y:*

1. *$\bar{B}^*(y)$ weakly increases if π changes to π' such that $\pi'(\mathcal{B}(y,d)) \leq \pi(\mathcal{B}(y,d))$ for all d.*

2. *For all $\delta < 1$, there exists $\bar{\bar{B}}^*(\delta, y) \in \mathbb{R}_+$ such that $\max_{\pi\in\Pi} \bar{B}^*(y) = \bar{\bar{B}}^*(\delta, y)$.*

3. *$\bar{\bar{B}}^*(\delta, y) = d(y, q)$ if y is a core point, and $\lim_{\delta\to 1} \bar{\bar{B}}^*(\delta, y) = +\infty$ otherwise.*

4. *If the core is empty, for all $\bar{B} > 0$ and all sufficiently large $\delta \in (\frac{1}{2}, 1)$ there exists π_δ and equilibrium σ_δ with $\lambda_{\sigma_\delta}(X \setminus \mathcal{B}(y, \bar{B})) \geq \frac{2\delta-1}{\delta}$.*

Part 1 of Theorem 8 establishes that as proposal rights are shifted away from any point y, in the sense of first-order-stochastic dominance, then the outer bound on acceptable agreements across equilibria centered at y weakly expands. This might allow for policies to reach arbitrarily distant areas of the policy space, but part 2 of the Theorem bars this possibility in the discounted case: With *stationary* proposal rules,[12] there is a maximal distance from y which equilibrium outcomes cannot exceed, no matter how far proposal rights are shifted away from y. That distance is

(5)
$$\bar{\bar{B}}^*(\delta, y) := \frac{1+\delta}{1-\delta} r_w(y) + \sqrt{(d(y,q) + r_w(y))^2 + \delta \left(\frac{2r_w(y)}{1-\delta}\right)^2},$$

and it increases with the radius of the y-centered W-ball as well as with the distance of the status quo from y. The maximal distance also depends on the discount factor, and parts 3 and 4 of Theorem 8 clarify this dependence: If y is a core point, then the distance is constant and equal to the distance of the status quo from y. This is consistent with the finding in the special environment of Primo (2002). But if y is not a core point, then this maximal outer distance is going to infinity as the discount factor tends to one. While this bound itself does not guarantee that there exist equilibria with outcomes at that distance, part 4 of the

[12] With non-stationary rules, a McKelvey (1976)-like chain of proposers would likely not respect such an outer bound. Nevertheless, the result might hold if an appropriate analogue of the restriction on the second moments of the distribution of proposers is imposed on a Markovian recognition protocol.

Theorem ensures that there exist sequences of equilibria with outcomes whose distance from y goes to infinity as the discount factor goes to one, whenever the core is empty ($r_w > 0$).

These findings paint a mixed picture of the impact of proposal rights on legislative bargaining outcomes. On the positive side, part 2 of Theorem 8 imposes a cap on how much proposal rights can influence collective choice for any *fixed* discount factor less than one. On the other hand, unless the core is empty, this cap may be too permissive and, in fact, it becomes non-binding at the limit as legislators get patient in the strong sense implied by parts 3 and 4 of the Theorem. While Theorem 8 does not fully address what happens in the limit case of perfectly patient legislators, this is rectified in the following Theorem by focusing on the special environment when there exists a unique proposer:

Theorem 9. *Assume $\delta = 1$ and $\pi(\{\hat{x}\}) = 1$ for some $\hat{x}$. Then $\bar{B}^*(y) = \max\{d(y, \hat{x}), r_p(y)\} + r_w(y) + r(y)$ for all y. Furthermore, there exists an equilibrium with Support$(\lambda_\sigma) = \{\hat{x}\}$ if and only if for all $\hat{x}'$ such that $u(\hat{x}; \hat{x}') < u(q; \hat{x}')$, $\mu_\ell(\{\hat{x}'\}) \leq 1 - m_\ell$ for all ℓ.*

Theorem 9 establishes very weak necessary and sufficient conditions for a(n arbitrary) unique proposer to propose and pass her ideal point in an equilibrium. The condition requires that if there is a legislator with absolute veto power, then she prefers the ideal point of the proposer over the status quo. Therefore, if there is no legislator with absolute veto, then for any ideal point for the unique proposer there exists an equilibrium in which she passes her ideal with probability one. This condition is weaker than the sufficient condition of Duggan and Ma (2018) that there does not exist a restricted core point.[13]

Voting Rights: What about the effect of voting rules? These variables enter the equations that define equilibrium outcome bounds through the radii $r_p(y)$ and $r_w(y)$ of the y-centered P-ball and W-ball, and these quantities respond differently to stricter (proper) voting rules: Radius $r_p(y)$ weakly increases (to, say, $r'_p(y)$) while $r_w(y)$ weakly decreases (to $r'_w(y)$) as the voting rule becomes more stringent by either requiring a larger majority quota or the assent

[13] Duggan and Ma (2018) look at limits of equilibria as $\delta \to 1$. Such limit equilibria generally constitute a refined subset of the set of equilibria of that limit undiscounted game.

of an additional chamber. Theorem 10 establishes that the introduction of a more stringent voting rule has ambiguous effect on the outer bound though it weakly contracts both the inner bound and the maximal outer bound of equation (5):

Theorem 10. *Assume $m_{\ell^*} > \frac{1}{2}$ for some ℓ^*. For all y, if*

a. m_ℓ increases for any ℓ, or

b. a chamber $\ell = L+1$ is added with any voting weights μ_{L+1} and threshold $m_{L+1} \in (0,1)$,

1. The inner bound $\underline{B}^*(y)$ weakly decreases.

2. The outer bound $\bar{B}^*(y)$ weakly decreases if $r(y) \geq r'(y)$, and may increase otherwise.

3. If $\delta < 1$, the maximal outer bound $\bar{\bar{B}}^*(\delta, y)$ of Theorem 8 weakly decreases.

The comparative statics of Theorem 10 are mostly intuitive as they imply that more stringent voting rules, requiring higher majority thresholds or the assent of more chambers, limit possible acceptable agreements across equilibria *and* proposer distributions in the discounted case (part 3); and also limit agreements that are guaranteed to be acceptable for any specific parameterization of the model (part 1). But, the outer bound and hence the set of possible equilibrium outcomes may decrease or increase (depending on how much $r_p(y)$ increases relative to the decrease of $r_w(y)$). Though paradoxical at first, this finding is reconcilable in view of the possibility that the set of agreements in any one equilibrium may indeed shrink with more stringent voting rules, *but* the set of equilibria may in fact expand leading to an expansion of the set of possible outcomes *across* equilibria. This is already anticipated by part 1(b)i of Theorem 3: When $\delta = 1$ and $r_w(y) = 0$, more stringent voting rules can lead to an increase in $r(y)$ because $r_p(y) < r'_p(y)$ while $r_w(y) = r'_w(y) = 0$. If, for example, π has full support, then the new outer bound becomes $\bar{B}^* = 2r'_p(y)$ and the increase in the bound mirrors the increase in the radius of the y-centered P-ball.

4 Numerical experiments

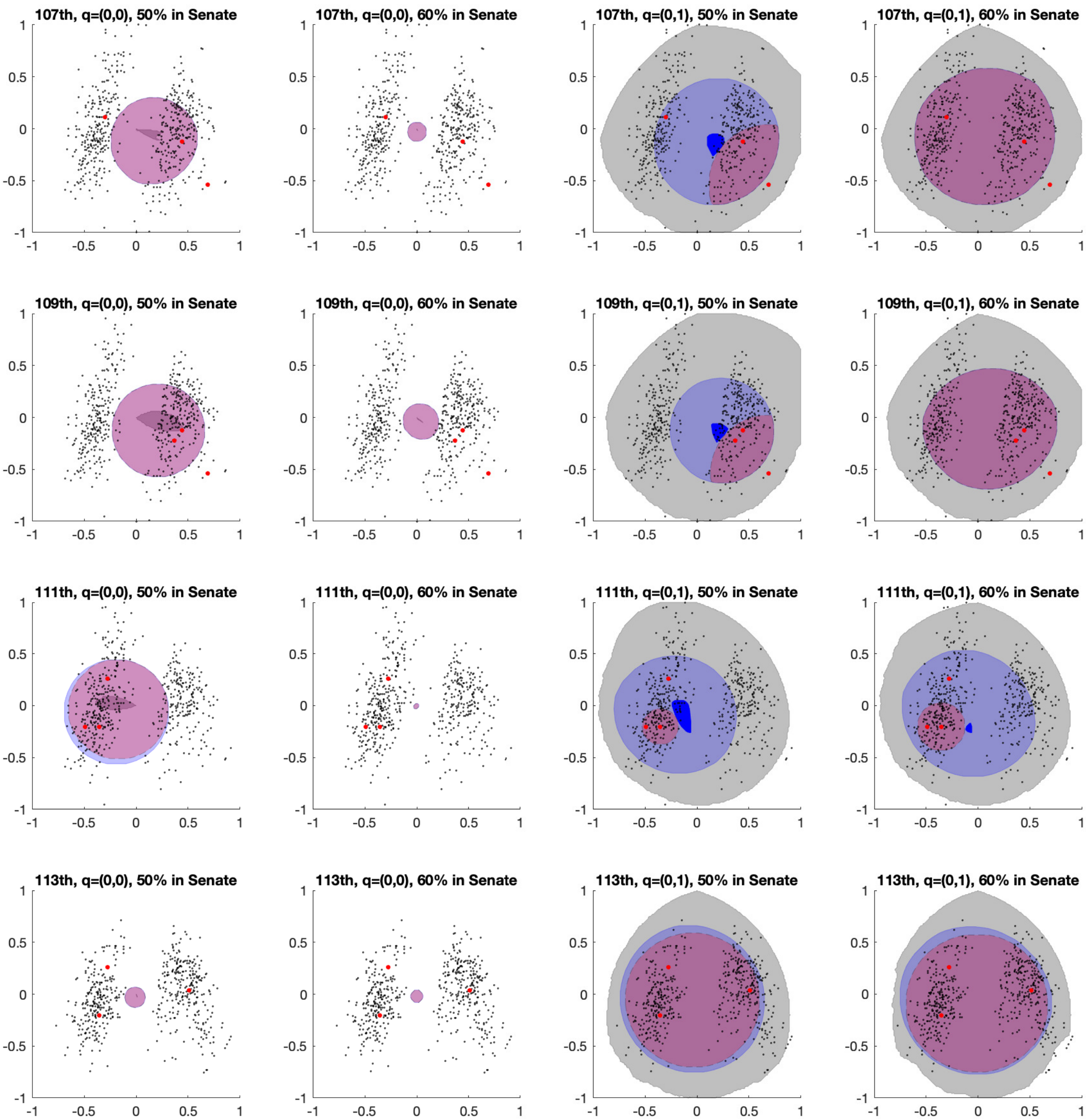
To illustrate the bounds and some of these comparative statics, I present numerical experiments in this section. I focus on the US Congress which, due to size and institutional complexity, provides a challenging environment for this application. I use DW-NOMINATE[14] scores (Lewis et al. (2021)) to locate the ideal points of members in the first Congress of four consecutive presidential terms since 2000. There is divided government in two of these Congresses (107th and 113th). To specify the model, I assume the President proposes with probability $\frac{1}{4}$ and the remainder of proposal-making is equally split between the Senate and the House. Within each chamber, 95% of proposal making goes to the majority party in that chamber. Further, within each party group in each chamber, 95% of the assigned probability goes to the party's leader, and the remainder is equally split among the chamber's remaining party members. These assumptions reflect tight party control on access to the agenda (but are of course arbitrary and should ideally be informed by fitting the model to data). I fix the discount factor to $\delta = 0.9$.

I consider two voting rules, varying the majority threshold for passage in the Senate, one requiring simple majority with a tie-breaking vote by the vice-President, the other requiring 60% filibuster-proof majority. Both rules require passage in the two chambers and the assent of the President or two-thirds majorities without the Presiedent. Finally, I consider one centrally located status quo at $q = (0, 0)$; and one at $q = (0, 1)$ which is extreme without being biased in favor of one or the other party. Additional details can be found in Appendix D. Overall there are four configurations (two status quo and voting rules) for four Congresses, for a total of 16 configurations presented in Figure 3.

The cases with a centrally located status quo are in the left two columns. The bounds are very narrow with filibuster-proof majority in the Senate (second column) and are larger with simple majority. In all of these cases the bounds are larger than the static winset of

[14]Obviously, the payoff function assumed by DW-NOMINATE differs from that in the present model, but this transgression seems acceptable given our purposes in this section.

Figure 3: Equilibrium bounds in the DW-NOMINATE Congress



Computed based on Theorems 1-3 for all y on a grid $Y \subseteq [-1,1] \times [-1,1]$ of 40,401 points.
Light gray: Set of points in the (static) winset of the status quo.
Light blue: Possible acceptable agreements across *all* equilibria (based on Corollary 1).
Dark blue: Agreements acceptable in every equilibrium (based on Corollary 1).
Light red: Possible proposals by the President across all equilibria (based on Corollary 2).
Red points: Ideal points of President, Speaker of the House, and Senate majority leader.

the status quo. They also tend to be slightly larger with unified government (109th and 111th Congresses when simple majority is assumed in the Senate). The figures also display the added restrictions on possible proposals by the President (using Corollary 2), but these restrictions are not very informative over the set of possible acceptable agreements in this setting.

When the status quo is extreme (right two columns) the set of possible acceptable agreements across equilibria expands (as expected by Theorem 6) substantially. Now these sets are smaller than the static winset. Furthermore, in four cases the inner bound of agreements acceptable in every equilibrium is non-empty (107th and 109th Congress with simple majority in the Senate, and 111th Congress). In those cases, the extra restrictions on possible outcomes established in Corollary 2 are substantial, as shown by the highlighted possible proposals by the President. Applied to all proposers, these extra restrictions imply that well over 95% of the set of *outcomes* fall solidly within the Democratic camp in the 111th Congress (or the Republican camp in the 109th Congress), even though the set of agreements possibly acceptable in either Congress encompasses both party camps.

Though an in depth study of all of these questions is outside the scope of this paper, there are three additional patterns worth some discussion: First, with the extreme status quo, the bounds are larger under divided government than under unified government. Second, they also tend to be larger with filibuster-proof majority in the Senate, while the opposite is true with a centrist status quo. These patterns are partly anticipated by Theorem 10. Third, perhaps surprisingly, the core is non-empty in all four Congresses under either voting rule (see Appendix D). These calculations suggest that outcomes near (but not necessarily in the core) tend to prevail when the status quo is centrist,[15] even with proposal-making concentrated to individuals away from the core. But when the status quo is extreme, the spread in agenda-control implied by divided government leads to much larger variation in possible outcomes according to these bounds. Whether these larger bounds arise due to many and variable equilibria coordinating expectations on either side of the spectrum, or few equilibria with

[15] q is not in the core in all cases, so Theorem 4, part 1, does not apply.

large acceptance sets, or whether the bounds are simply too conservative in these cases, remains to be evaluated by either further narrowing these bounds (for example, by iterating their calculation as suggested in footnote 10) or by computing the actual equilibria.

5 Conclusion

I have established an inner and outer envelope on possible equilibrium acceptable agreements and outcomes in the spatial model of legislative bargaining. These bounds are informative theoretically, for example, by recovering and extending several known results for uniqueness and core equivalence of equilibrium and by allowing a number of comparative static analyses. More stringent voting rules across legislative chambers exercise two countervailing forces on these bounds, and are possibly consistent with both smaller and larger sets of possible equilibrium acceptable agreements. For any discount factor less than one, the equilibrium bounds dispel the possibility of arbitrarily distant alternatives prevailing in equilibrium, even if the distribution of possible proposers piles mass away from the center of the policy space. The potential dependence of equilibrium outcomes on the allocation of proposal-making rights becomes more severe as legislators get patient, though, and can take an extreme form at the limit when the legislators are perfectly patient. On the empirical side, these bounds are easy to compute and provide a cheap alternative to computation of actual equilibria, thus opening the possibility for a range of empirical applications informed by the equilibrium constraints from the model.

Appendix

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A Proofs

As a background result, I state without proof a straightforward Lemma that makes use of mean-variance decomposition of expected quadratic payoffs, i.e., for all $\hat{x}$ and for

$$\mathbb{E}(\sigma) := (1-\delta)q + \delta \int \int z p_{\hat{x}}(dz)\pi(d\hat{x}),$$

$$\mathbb{V}(\sigma) := (1-\delta)\left(q - \mathbb{E}(\sigma)\right)^T \cdot \left(q - \mathbb{E}(\sigma)\right) + \delta \int \int \left(z - \mathbb{E}(\sigma)\right)^T \cdot \left(z - \mathbb{E}(\sigma)\right) p_{\hat{x}}(dz)\pi(d\hat{x}),$$

$$v_\sigma(\hat{x}) = u(\mathbb{E}(\sigma); \hat{x}) - \mathbb{V}(\sigma). \tag{6}$$

Lemma A.1. *Consider any x, σ. If $x = \mathbb{E}(\sigma)$, $\{\hat{x} \mid u(x;\hat{x}) \geq v_\sigma(\hat{x})\} = \mathcal{I}$. If $x \neq \mathbb{E}(\sigma)$, there exists hyperplane H perpendicular to the line segment $[x, \mathbb{E}(\sigma)]$, such that (a) $\{\hat{x} \mid u(x;\hat{x}) = v_\sigma(\hat{x})\} = H$, (b) $\{\hat{x} \mid u(x;\hat{x}) > v_\sigma(\hat{x})\} = H^+ \setminus H$, (c) $\{\hat{x} \mid u(x;\hat{x}) < v_\sigma(\hat{x})\} = H^- \setminus H$.*

Proof of Lemma 1. *Part 1.* By Lemma A.1, if $x \in A_\sigma$ there is (at least) a pivotal winning half-space that weakly prefer x. It intersects with $\mathcal{B}(y, r_w(y))$ by definition. Therefore, there exists $\hat{x} \in \mathcal{B}(y, r_w(y))$ such that $u(x;\hat{x}) \geq v_\sigma(\hat{x})$.

Part 2. Assume $u(x;\hat{x}) \geq v_\sigma(\hat{x})$ for all $\hat{x} \in \mathcal{B}(y, r_p(y))$. To get a contradiction, assume that $x \notin A_\sigma$. Then, by Lemma A.1 $x \neq \mathbb{E}(\sigma)$ and there exists a hyperplane $H_{a,c}$ such that $H_{a,c} = \{\hat{x} \mid u(x;\hat{x}) = v_\sigma(\hat{x})\}$ and $H^+_{a,c} = \{\hat{x} \mid u(x;\hat{x}) \geq v_\sigma(\hat{x})\} \neq \mathcal{I}$. By $x \notin A_\sigma$, we must have $\mu_\ell(H^+_{a,c}) < m_\ell$ for some ℓ. There exists a pivotal hyperplane $H_{a,c^*(a)}$ and, by definition, $H_{a,c^*(a)} \cap \mathcal{B}(y, r_p(y)) \neq \emptyset$. But $\mathcal{B}(y, r_p(y)) \subset H^+_{a,c}$, hence it must be that $c^*(a) > c$ and $H^+_{a,c^*(a)} \subset H^+_{a,c}$. This leads to a contradiction as there exists ℓ such that $m_\ell \leq \mu_\ell(H^+_{a,c^*(a)}) \leq \mu_\ell(H^+_{a,c}) < m_\ell$. □

Proof of Lemma 2. Fix y, assume $x \in A_\sigma$, and consider any $x' \in \mathcal{B}(y, d(y,x) - 2r(y))$. We must show $x' \in A_\sigma$. By part 2 of Lemma 1 it suffices to show that $u(x';\hat{x}') \geq v_\sigma(\hat{x}')$, for all $\hat{x}' \in \mathcal{B}(y, r_p(y))$. By Lemma 1 and $x \in A_\sigma$, there exists $\hat{x} \in \mathcal{B}(y, r_w(y))$ such that $u(x;\hat{x}) \geq v_\sigma(\hat{x})$. First we will show that $d(x', \hat{x}') \leq d(x, \hat{x}) - r(y)$ for all $\hat{x}' \in \mathcal{B}(y, r_p(y))$. Indeed, (recalling that $r(y) = r_p(y) + r_w(y)$)

$$
\begin{aligned}
d(x', \hat{x}') &\leq d(y, \hat{x}') + d(x', y) && \text{[triangle inequality]} \\
&\leq d(x', y) + r_p(y) && [\hat{x}' \in \mathcal{B}(y, r_p(y))] \\
&\leq d(x, y) - r_p(y) - 2r_w(y) && [x' \in \mathcal{B}(y, d(x,y) - 2r(y))] \\
&\leq d(x, \hat{x}) + d(\hat{x}, y) - r_p(y) - 2r_w(y) && \text{[triangle inequality]} \\
&\leq d(x, \hat{x}) - r(y) && [\hat{x} \in \mathcal{B}(y, r_w(y))].
\end{aligned}
$$

Now fix arbitrary $\hat{x}' \in \mathcal{B}(y, r_p(y))$; $u(x'; \hat{x}') \geq v_\sigma(\hat{x}')$ follows because we have:

$$\begin{aligned}
u(x';\hat{x}') &= -d(x',\hat{x}')^2 \geq -\left(d(x,\hat{x}) - r(y)\right)^2 && [d(x',\hat{x}') \leq d(x,\hat{x}) - r(y)] \\
&\geq -\left(\sqrt{-v_\sigma(\hat{x})} - r(y)\right)^2 && [u(x;\hat{x}) \geq v_\sigma(\hat{x})] \\
&= -\left(\sqrt{d(\hat{x},\mathbb{E}(\sigma))^2 + \mathbb{V}(\sigma)} - r(y)\right)^2 && [(6)] \\
&\geq -\left(\sqrt{\left(d(\hat{x}',\mathbb{E}(\sigma)) + d(\hat{x},\hat{x}')\right)^2 + \mathbb{V}(\sigma)} - r(y)\right)^2 && [\text{triangle inequality}] \\
&\geq -\left(\sqrt{\left(d(\hat{x}',\mathbb{E}(\sigma)) + d(\hat{x},y) + d(y,\hat{x}')\right)^2 + \mathbb{V}(\sigma)} - r(y)\right)^2 && [\text{triangle inequality}] \\
&\geq -\left(\sqrt{\left(d(\hat{x}',\mathbb{E}(\sigma)) + r(y)\right)^2 + \mathbb{V}(\sigma)} - r(y)\right)^2 && [\ \hat{x} \in \mathcal{B}(y, r_w(y)), \hat{x}' \in \mathcal{B}(y, r_p(y) \\
&= -\left(\sqrt{d(\hat{x}',\mathbb{E}(\sigma))^2 + r(y)^2 + 2r(y)d(\hat{x}',\mathbb{E}(\sigma)) + \mathbb{V}(\sigma)} - r(y)\right)^2 \\
&\geq -\left(\sqrt{-v_\sigma(\hat{x}') + r(y)^2 + 2r(y)\sqrt{-v_\sigma(\hat{x}')}} - r(y)\right)^2 && [\sqrt{-v_\sigma(\hat{x}')} \geq d(\hat{x}',\mathbb{E}(\sigma))] \\
&= -\left(\left(\sqrt{-v_\sigma(\hat{x}')} + r(y)\right) - r(y)\right)^2 = v_\sigma(\hat{x}'). && \square
\end{aligned}$$

Proof of Lemma 3. Fix a quasi-equilibirum σ and y. Consider any $\hat{x} \in \mathcal{B}^o(y, \underline{B}_\sigma(y))$. By definition of $\underline{B}_\sigma(y)$ and A_σ closed, $\hat{x} \in A_\sigma$ so that $\{\hat{x}\} = \arg\max_{x \in A_\sigma} u(x;\hat{x})$, therefore $p_{\hat{x}}(\{\hat{x}\}) = 1$ and $\underline{d}(\underline{B}_\sigma(y), \hat{x}, y) = d(y, \hat{x}) = d(y, z)$ for all $z \in \mathit{Support}(p_{\hat{x}})$. On the other hand, if $d(y, \hat{x}) > \underline{B}_\sigma(y)$ for some $\hat{x}$, then $\max_{x \in A_\sigma} u(x;\hat{x}) \geq \max_{x \in \mathcal{B}(y, \underline{B}_\sigma(y))} u(x;\hat{x}) = -(d(y,\hat{x}) - \underline{B}_\sigma(y))^2$. Therefore, there cannot exist $z \in \mathit{Support}(p_{\hat{x}})$ such that $d(y, z) < \underline{d}(\underline{B}_\sigma(y), \hat{x}, y) = \underline{B}_\sigma(y)$. Combining the two cases we conclude $d(y, z) \geq \underline{d}(\underline{B}_\sigma(y), \hat{x}, y)$ for all $\hat{x}$. To complete the proof of (4), note that the right-hand-side inequality holds trivially when $\bar{B}_\sigma(y) < 2r(y)$ or $\bar{B}_\sigma(y) \leq d(y, \hat{x}) + r(y)$, whence $\bar{d}(\bar{B}_\sigma(y), \hat{x}, y) = \bar{B}_\sigma(y)$. Furthermore, if $\bar{B}_\sigma(y) \geq d(y, \hat{x}) + 2r(y)$, then $\hat{x} \in A_\sigma$ by Lemma 2, therefore $\mathit{Support}(p_{\hat{x}}) = \{\hat{x}\}$ and $\bar{d}(\bar{B}_\sigma(y), \hat{x}, y) = d(y, \hat{x})$. Thus, it remains to consider the case $d(y, \hat{x}) + r(y) \leq \bar{B}_\sigma(y) \leq d(y, \hat{x}) + 2r(y)$ and $\bar{B}_\sigma(y) \geq 2r(y)$. Then, by Lemma 2, $\max_{x \in A_\sigma} u(x;\hat{x}) \geq \max_{x \in \mathcal{B}(y, \bar{B}_\sigma(y) - 2r(y))} u(x;\hat{x}) = -(d(y,\hat{x}) - (\bar{B}_\sigma(y) - 2r(y)))^2$. Now if $d(y, z) > \bar{d}(\bar{B}_\sigma(y), \hat{x}, y) = 2d(y, \hat{x}) + 2r(y) - \bar{B}_\sigma(y)$, we have $u(z;\hat{x}) = -d(z,\hat{x})^2 \leq -(d(y,z) - d(y,\hat{x}))^2 < -(d(y,\hat{x}) -$

$(\bar{B}_\sigma(y) - 2r(y)))^2$, therefore it cannot be that $z \in Support(p_{\hat{x}})$. $\square$

Proof of Theorem 1. Fix a quasi-equilibirum σ and y. We start with showing that both $\bar{B}_\sigma(y), \underline{B}_\sigma(y)$ are finite. If not, since $2r(y) \geq \bar{B}_\sigma(y) - \underline{B}_\sigma(y)$ by Lemma 2, then $\underline{B}_\sigma(y) = +\infty$ and, by Lemma 3, $p_{\hat{x}}(\{\hat{x}\}) = 1$ for all $\hat{x}$. Then

$$v_\sigma(\hat{x}) = (1-\delta)u(q;\hat{x}) + \delta \int u(\hat{x}';\hat{x})\pi(d\hat{x}') > -\infty,$$

for all $\hat{x}$ (because π has finite first and second moments). Furthermore, by Lemma 1, for all $k = 1, 2, \ldots$ there exists $x_k \in A_\sigma$ and $\hat{x}_k \in \mathcal{B}(y, r_w(y))$ such that $d(y, x_k) > k$ and $u(x_k; \hat{x}_k) \geq v_\sigma(\hat{x}_k) > -\infty$. Since $d(x_k, \hat{x}_k) \geq k - r_w(y)$, we have $\lim_{k\to+\infty} u(x_k; \hat{x}_k) = -\infty$, yet $\min\limits_{\hat{x}\in\mathcal{B}(y,r_w(y))} v_\sigma(\hat{x}) > -\infty$, a contradiction. Therefore, both $\bar{B}_\sigma(y), \underline{B}_\sigma(y)$ are finite.

Next, consider $\bar{F}(\bar{B}_\sigma(y), y) \leq 0$. It holds for all $\bar{B}_\sigma(y) \in [0, 2r_w(y)]$, because then

$$(1-\delta)(d(y,q) + r_w(y))^2 + \delta \int \left(\bar{d}\left(\bar{B}_\sigma(y), \hat{x}, y\right) + r_w(y)\right)^2 \pi(d\hat{x}) \geq r_w(y)^2 \geq (\bar{B}_\sigma(y) - r_w(y))^2.$$

To show $\bar{F}(\bar{B}_\sigma(y), y) \leq 0$ for $\bar{B}_\sigma(y) \geq 2r_w(y)$, observe that there exists $x' \in A_\sigma$ such that $d(y, x') = \bar{B}_\sigma(y)$, since A_σ is closed and bounded (because $\bar{B}_\sigma(y)$ is finite). By the first part of Lemma 1 there also exists $\hat{x}' \in \mathcal{B}(y, r_w(y))$ such that $u(x', \hat{x}') \geq v_\sigma(\hat{x}')$. We now have

$$
\begin{aligned}
(\bar{B}_\sigma - r_w(y))^2 &\leq (\bar{B}_\sigma(y) - d(y, \hat{x}'))^2 && [\hat{x}' \in \mathcal{B}(y, r_w(y)), \bar{B}_\sigma(y) \geq r_w(y)] \\
&= (d(y, x') - d(y, \hat{x}'))^2 && [d(y, x') = \bar{B}_\sigma(y)] \\
&\leq d(x', \hat{x}')^2 = -u(x'; \hat{x}') && [d(a, b) \geq | \, d(a, c) - d(c, b) \, |] \\
&\leq -v_\sigma(\hat{x}') && [u(x', \hat{x}') \geq v_\sigma(\hat{x}')] \\
&\leq (1-\delta)(d(\hat{x}', y) + d(y, q))^2 \\
&+ \delta \int \int (d(\hat{x}', y) + d(y, z))^2 p_{\hat{x}}(dz) \pi(d\hat{x}) && [\text{triangle inequality}] \\
&\leq (1-\delta)(r_w(y) + d(y, q))^2 \\
&+ \delta \int \int (r_w(y) + d(y, z))^2 p_{\hat{x}}(dz) \pi(d\hat{x}) && [\hat{x}' \in \mathcal{B}(y, r_w(y))] \\
&\leq (1-\delta)(d(y, q) + r_w(y))^2 \\
&+ \delta \int \left(\bar{d} \left(\bar{B}_\sigma(y), \hat{x}, y \right) + r_w(y) \right)^2 \pi(d\hat{x}) && [(4)].
\end{aligned}
$$

To show $\underline{F}(\underline{B}_\sigma(y), y) \geq 0$, let $\epsilon_k = \frac{1}{k}$. By the definition of $\underline{B}_\sigma(y)$, for all $k = 1, 2, \ldots$, there exists $x_k \in \mathcal{B}(y, \underline{B}_\sigma(y) + \epsilon_k) \setminus \mathcal{B}(y, \underline{B}_\sigma(y))$ such that $x_k \notin A_\sigma$. Since $x_k \notin A_\sigma$, there also exists corresponding $\hat{x}_k \in \mathcal{B}(y, r_p(y))$ such that $u(x_k; \hat{x}_k) < v_\sigma(\hat{x}_k)$, by the contra-positive of the second part of Lemma 1. Now the sequence $(x_k, \hat{x}_k)$ belongs in $(\mathcal{B}(y, \underline{B}_\sigma(y) + 1) \setminus \mathcal{B}^o(y, \underline{B}_\sigma(y))) \times \mathcal{B}(y, r_p(y))$, a compact set, so – by going to a subsequence if necessary – there is a limit $(x_k, \hat{x}_k) \to (x^*, \hat{x}^*) \in (\mathcal{B}(y, \underline{B}_\sigma(y)) \setminus \mathcal{B}^o(y, \underline{B}_\sigma(y))) \times \mathcal{B}(y, r_p(y))$. Since u, v_σ are continuous in $x, \hat{x}$, we conclude

$$
u(x^*; \hat{x}^*) \leq v_\sigma(\hat{x}^*). \tag{7}
$$

We now have

$$\begin{aligned}
(\underline{B}_\sigma(y) + d(y,\hat{x}^*))^2 &= (d(y,x^*) + d(y,\hat{x}^*))^2 && [x^* \in \mathcal{B}(y,\underline{B}_\sigma(y)) \setminus \mathcal{B}^o(y,\underline{B}_\sigma(y))] \\
&\geq d(x^*,\hat{x}^*)^2 && [\text{triangle inequality}] \\
&\geq -v_\sigma(\hat{x}^*) && [(7)] \\
&= (1-\delta)d(\hat{x}^*,q)^2 + \delta \int\int d(\hat{x}^*,z)^2 p_{\hat{x}}(dz)\pi(d\hat{x}) \\
&\geq (1-\delta)\left(d(y,q) - d(y,\hat{x}^*)\right)^2 \\
&+ \delta \int\int \left(d(y,z) - d(y,\hat{x}^*)\right)^2 p_{\hat{x}}(dz)\pi(d\hat{x}) && [d(a,b) \geq \mid d(a,c) - d(c,b) \mid].
\end{aligned}$$

The left-hand-side of the resulting inequality

$$(\underline{B}_\sigma(y) + d(y,\hat{x}^*))^2 - (1-\delta)\left(d(y,q) - d(y,\hat{x}^*)\right)^2 - \delta \int\int \left(d(y,z) - d(y,\hat{x}^*)\right)^2 p_{\hat{x}}(dz)\pi(d\hat{x}) \geq 0,$$

is increasing in $d(y,\hat{x}^*)$ and (because $d(y,\hat{x}^*) \leq r_p(y)$) we obtain

$$(\underline{B}_\sigma(y) + r_p(y))^2 - (1-\delta)\left(d(y,q) - r_p(y)\right)^2 - \delta \int\int \left(d(y,z) - r_p(y)\right)^2 p_{\hat{x}}(dz)\pi(d\hat{x}) \geq 0.$$

Now $\underline{F}(\underline{B}_\sigma(y), y) \geq 0$ follows because

$$\begin{aligned}
\int\int \left(d(y,z) - r_p(y)\right)^2 p_{\hat{x}}(dz)\pi(d\hat{x}) &= \int_{\mathcal{B}^o(y,\underline{B}_\sigma(y))} \left(d(y,\hat{x}) - r_p(y)\right)^2 \pi(d\hat{x}) \\
&+ \int_{\mathcal{I}\setminus\mathcal{B}^o(y,\underline{B}_\sigma(y))} \int \left(d(y,z) - r_p(y)\right)^2 p_{\hat{x}}(dz)\pi(d\hat{x}) \\
&\geq \int_{\mathcal{B}^o(y,\underline{B}_\sigma(y))} \left(d(y,\hat{x}) - r_p(y)\right)^2 \pi(d\hat{x}) \\
&+ \int_{\mathcal{I}\setminus\mathcal{B}^o(y,\underline{B}_\sigma(y))} \min_{z:d(y,z)\geq \underline{d}(\underline{B}_\sigma(y),\hat{x})} \left(d(y,z) - r_p(y)\right)^2 \pi(d\hat{x}). \\
&\geq \int \left(\underline{d}(\underline{B}_\sigma(y),\hat{x},y) - \min\{\underline{B}_\sigma(y), r_p(y)\}\right)^2 \pi(d\hat{x}). \qquad \square
\end{aligned}$$

Proof of Theorem 2. Follows from Theorem 1 and the definition of $\underline{B}^*(y), \bar{B}^*(y)$. $\square$

Proof of Theorem 3. By the Lebesgue dominated convergence Theorem and because π has finite first and second moments,[16] we have

$$\lim_{B\to+\infty}\int\left(\bar{d}(B,\hat{x},y)+r_w(y)\right)^2\pi(d\hat{x})=\int\left(d(y,\hat{x})+r_w(y)\right)^2\pi(d\hat{x})<+\infty,\text{ and}$$

$$\lim_{B\to+\infty}\int\left(\underline{d}\left(B,\hat{x},y\right)-\min\{B,r_p(y)\}\right)^2\pi(d\hat{x})=\int\left(d\left(y,\hat{x},y\right)-r_p(y)\right)^2\pi(d\hat{x})<+\infty.$$

It follows that $\lim_{B\to+\infty}\bar{F}(B,y)=\lim_{B\to+\infty}\underline{F}(B,y)=+\infty$. This ensures that $\underline{B}^*(y)$ (possibly 0) exists. By the definition of $\bar{d}$, $\bar{F}(\cdot,y)$ is continuous in $[0,2r(y))$ and in $[2r(y),+\infty)$, and $\lim_{B\to 2r(y)^+}\bar{F}(B,y)=\bar{F}(2r(y),y)$ and $\lim_{B\to 2r(y)^-}\bar{F}(B,y)\leq\bar{F}(2r(y),y)$, due to the fact that $\lim_{B\to 2r(y)^+}\bar{d}(B,\hat{x},y)=\bar{d}(2r(y),\hat{x},y)\leq 2r(y)=\lim_{B\to 2r(y)^-}\bar{d}(B,\hat{x},y)$ for all $\hat{x}$. Existence of $\bar{B}^*(y)\in\mathbb{R}$ then follows from the above because $(1-\delta)d(y,q)+r_w(y)>0$ implies $\bar{F}(0,y)<0$ in case 1.(a), and $\bar{F}(0,y)=0$ in case 1.(b).

We now proceed to show parts 1.(a)-2.(b), starting with 1.(a). So assume $(1-\delta)d(y,q)+r_w(y)>0$, which implies $\bar{F}(0,y)<0$, therefore $\bar{B}^*(y)>0$. If there does not exist $B>0$ such that $\bar{F}(B,y)=0$, then $\lim_{B\to 2r(y)^-}\bar{F}(B,y)\leq 0<\bar{F}(2r(y),y)$ and $\bar{B}^*(y)=2r(y)$, because $\lim_{B\to+\infty}\bar{F}(B,y)=+\infty$ and $\bar{F}(\cdot,y)$ is continuous in $[0,2r(y))$ and in $[2r(y),+\infty)$. To complete the proof, then, we need to show that if there exists $\bar{B}>0$, $\bar{F}(\bar{B},y)=0$, it is unique. Distinguish two cases:

Case 1. $\bar{B}\in(0,2r(y))$. When $B\in(0,2r(y))$, $\bar{F}(B,y)$ is the quadratic polynomial:

$$(B-r_w(y))^2-(1-\delta)(d(y,q)+r_w(y))^2-\delta(B-r_w(y))^2.$$

Because $\bar{F}(0,y)<0$, $\delta<1$, we can only have at most one root $\bar{B}\in(0,2r(y))$ and $0<\lim_{B\to 2r(y)^-}\bar{F}(B,y)\leq\bar{F}(2r(y),y)$. By Lemma C.2 in Appendix C, there is no solution in $[2r(y),+\infty)$ when $\bar{F}(2r(y),y)>0$.

Case 2. $\bar{B}\in[2r(y),+\infty)$. Then there is no other solution by case 1 and Lemma C.2.

[16] In particular, $(\bar{d}(B,\hat{x},y)+r_w(y))^2\leq(d(y,\hat{x})+2r(y)+r_w(y)))^2$ and $(\underline{d}(B,\hat{x},y)-\min\{B,r_p(y)\})^2\leq(d(y_p(y),\hat{x}))^2$ for all B.

Moving to part 1.(b), observe that $r(y) = r_p(y)$ and

$$\bar{F}(B, y) = \begin{cases} B^2 - \delta B^2 & \text{if } B < 2r_p(y), \\ B^2 - \delta \left(\int_{\mathcal{B}(y, B - r_p(y))} \bar{d}(B, \hat{x}, y)^2 \pi(d\hat{x}) + \int_{\mathcal{I} \setminus \mathcal{B}(y, B - r_p(y))} B^2 \pi(d\hat{x}) \right) & \text{if } B \geq 2r_p(y). \end{cases}$$

When $\delta = 1$, 1.(b)i follows since $\bar{F}(B, y) = 0$ for all $B < 2r_p(y)$, while $B > 2r_p(y)$ satisfies $\bar{F}(B, y) = 0$ if and only if $\pi(\mathcal{B}(y, B - r_p(y))) = 0$. Indeed, $\bar{d}(B, \hat{x}, y) < B$ if and only if $\hat{x} \in \mathcal{B}(y, B - r_p(y))$, $B > 2r_p(y)$, whilst $\bar{F}(B, y) > 0$ for all $B > 2r_p(y)$ such that $\pi(\mathcal{B}(y, B - r_p(y))) > 0$. Similarly, 1.(b)ii, follows when $\delta < 1$, as $\bar{F}(B, y) \geq 0$ and $B = 0$ is the unique solution to $\bar{F}(B, y) = 0$.

Now consider part 2. It is the case that $\underline{F}(0, y) < 0$ if and only if $(1 - \delta)(d(y, q) - r_p(y))^2 > r_p^2(y)$, hence both cases 2.(a) and 2.(b) follow if $\underline{F}$ is strictly increasing when $(1 - \delta)(d(y, q) - r_p(y))^2 > r_p^2(y)$. This is indeed the case, because $\delta < 1$ and the integrand is either constant in B, or (when differentiable w.r.t. B) has derivative smaller in absolute value than $2(B + r_p(y))$. □

Proof of Theorem 4. y is a core point iff $r_w(y) = 0$ (Theorem 4, Kalandrakis (2019)). By Theorem 3, if $\bar{B}^*(y) = 0$ then $r_w(y) = 0$ so y is a core point. Part 1 follows from parts 1(b)ii of Theorem 3; part 2 from part 1(b)i of Theorem 3. □

Proof of Theorem 5. By the definition of $\bar{d}$ we have

$$\begin{aligned} &\lambda_\sigma\left(\mathcal{B}(y, 2r(y))\right) = 1 && \text{if} && \bar{B}^*_\sigma(y) \leq 2r(y), \\ &\lambda_\sigma\left(\mathcal{B}(y, 2r(y))\right) \geq \pi\left(\mathcal{B}(y, \frac{\bar{B}^*_\sigma(y)}{2})\right) && \text{if} && 2r(y) < \bar{B}^*_\sigma(y) \leq 4r(y), \\ &\lambda_\sigma\left(\mathcal{B}(y, 2r(y))\right) \geq \pi\left(\mathcal{B}(y, 2r(y))\right) && \text{if} && \bar{B}^*_\sigma(y) > 4r(y). \end{aligned}$$ □

Proof of Theorem 6. Write $\bar{F}(B, y; q)$, $\underline{F}(B, y; q)$, $\bar{B}^*(y; q)$, and $\underline{B}^*(y; q)$, to make the dependence on q explicit. The result is immediate from the fact that if $d(y, q) < d(y, q')$, then $\bar{F}(B, y; q) \geq \bar{F}(B, y; q')$ so that the largest solution to $\bar{F}(B, y; q) \leq 0$ weakly increases. For the lower bound, the results holds trivially if $\underline{B}^*(y; q) = 0$. If case 2(a) of Theorem 3 applies, then $\underline{F}(B, y; q) \geq \underline{F}(B, y; q')$, and the smallest solution to $\underline{F}(B, y; q) \geq 0$ strictly increases. Thus, if $\delta < 1$ and $d(y, q)$ is large enough we have $\lim_{d(y,q)\to+\infty} \underline{B}^*(y) = +\infty$ and $\bar{F}(\bar{B}^*(y; q), y; q) = 0$ and $\underline{F}(\underline{B}^*(y; q), y; q) = 0$. From these equations we conclude

$$\begin{aligned}\bar{B}^*(y;q) - \underline{B}^*(y;q) = r(y) + &\sqrt{(1-\delta)(d(y,q)+r_w(y))^2 + \delta \int \left(\bar{d}\left(\bar{B}^*(y;q), \hat{x}, y\right) + r_w(y)\right)^2 \pi(d\hat{x})} \\ &- \sqrt{(1-\delta)(d(y,q)-r_p(y))^2 + \delta \int \left(\underline{d}\left(\underline{B}^*(y;q), \hat{x}, y\right) - r_p(y)\right)^2 \pi(d\hat{x})}.\end{aligned}$$

The integrals converge to (finite) constants $\int \left(d(y, \hat{x}) + r_w(y)\right)^2 \pi(d\hat{x})$ and $\int \left(d(y, \hat{x}) - r_p(y)\right)^2 \pi(d\hat{x})$ as $\bar{B}^*(y; q)$ and $\underline{B}^*(y; q)$ go to infinity. The limit of the right-hand-side as $d(y, q) \to +\infty$ is independent of the value of these integrals and equals $r(y) + \sqrt{1-\delta}\, r(y)$. □

Proof of Theorem 7. That the equilibrium is unique and in pure strategies follows trivially from the convexity of the unique possible A_σ. To show that it is necessary for y to be a core point, recall that y is a core point iff $r_w(y) = 0$ and y is a unique core if $r_p(y) = 0$ (Theorem 4, Kalandrakis (2019)). Now assume $\underline{B}^*(y) = \bar{B}^*(y) = B^*$ but $r_w(y) > 0$ to get a contradiction. By Theorem 3, 1(a), it must be that $B^* \geq 2r_w(y) > 0$, therefore it must also be that Theorem 3, 2(a) applies, i.e., $(1-\delta)(d(y, q) - r_p(y))^2 > r_p^2(y)$ and $\underline{F}(B^*, y) = \bar{F}(B^*, y) = 0$, unless $B^* = 2r(y)$ in which case $\underline{F}(B^*, y) = 0 < \bar{F}(B^*, y)$ (by Theorem 3, 1(a)). These inequations then imply

$$\begin{aligned}(B^* - r_w(y))^2 &\geq (1-\delta)(d(y,q)+r_w(y))^2 + \delta \int \left(\bar{d}\left(B^*, \hat{x}, y\right) + r_w(y)\right)^2 \pi(d\hat{x}), \\ (B^* + r_p(y))^2 &= (1-\delta)(d(y,q)-r_p(y))^2 + \delta \int \left(\underline{d}\left(B^*, \hat{x}, y\right) - \min\{B^*, r_p(y)\}\right)^2 \pi(d\hat{x}).\end{aligned}$$

Expanding the squares on the right-hand-side of both inequations, reorganizing, and combining inequations (using $\bar{d}\left(B^*, \hat{x}, y\right) \geq \underline{d}\left(B^*, \hat{x}, y\right)$ for all $\hat{x}$) we conclude

$$(B^* - r_w(y))^2 - r_w^2(y) \geq (B^* + r_p(y))^2 - (1-\delta)r_p^2(y) - \delta \min\{B^*, r_p(y)\}^2$$

which in turn implies

$$-2B^*r_w(y) \geq 2B^*r_p(y) + \delta(r_p^2(y) - \min\{B^*, r_p(y)\}^2).$$

But the left-hand-side is negative and the right-hand-side positive which is impossible. Therefore $r_w(y) = 0$. By the same arguments, it is necessary that y is a unique core in part 2, as we similarly cannot have $\underline{B}^*(y) = \bar{B}^*(y) = B^* > 0$, $r_w(y) = 0$ but $r_p(y) > 0$. For sufficiency in part 2, assume $r_w(y) = r_p(y) = 0$ and $(1-\delta)d(y,q) > 0$. Then $\bar{F}(B^*, y) = \underline{F}(B^*, y)$ for all B^* and $\bar{F}(0, y) = \underline{F}(0, y) = -(1-\delta)(d(y,q))^2 < 0$, therefore $\underline{B}^*(y) = \bar{B}^*(y) > 0$. $\square$

Proof of Theorem 8. *Part 1.* Write $\bar{F}(B, y; \pi)$ to make dependence on π explicit. We have $\bar{F}(B, y; \pi) \geq \bar{F}(B, y; \pi')$, because $\bar{d}$ weakly increases with $d(y, \hat{x})$ so that $\int \left(\bar{d}\left(B, \hat{x}, y\right) + r_w(y)\right)^2 \pi(d\hat{x}) \leq \int \left(\bar{d}\left(B, \hat{x}, y\right) + r_w(y)\right)^2 \pi'(d\hat{x})$.

Part 2. If π is such that $\pi(\mathcal{B}(y, B - r(y))) = 0$, then $\bar{d}(B, \hat{x}, y) = B$ for all $\hat{x}$ in π's support (whether $B > r(y)$ or not). Then $\bar{F}(B, y; \pi) = 0$ takes the form

$$(B - r_w(y))^2 - (1-\delta)(d(y,q) + r_w(y))^2 - \delta \int (B + r_w(y))^2 \pi(d\hat{x}) = 0,$$

which has a unique solution $\bar{\bar{B}}^*(\delta, y)$ given in equation (5). By Theorem 3, for all π such that $\pi(\mathcal{B}(y, \bar{\bar{B}}^*(\delta, y) - r(y))) = 0$, $\bar{B}^*(y) = \bar{\bar{B}}^*(\delta, y)$ is the unique $B > 0$ that solves $\bar{F}(B, y; \pi) = 0$. By part 1, there cannot be a larger $\bar{B}^*(y)$ for some π', because for any π' such that $\pi'(\mathcal{B}(y, \bar{\bar{B}}^*(\delta, y) - r(y))) > 0$, there exists π'' such that $\pi''(\mathcal{B}(y, \bar{\bar{B}}^*(\delta, y) - r(y))) = 0$ and $\pi''(\mathcal{B}(y, d)) \leq \pi'(\mathcal{B}(y, d))$ for all d.

Part 3. Straightforward from equation (5).

Part 4. Fix $\bar{B} > 0$ and let (y_w, r_w) be a solution to (W) of Appendix D. Since the core is empty, $r_w > 0$ (Theorem 4, Kalandrakis (2019)). Consider $\delta \in (\frac{1}{2}, 1)$. Let $\bar{\bar{B}} > \max\{\bar{B} + d(y, y_w), d(y_w, q) + 2r(y_w)\}$. There exist at least two pivotal hyperplanes H_m tangent to $\mathcal{B}(y_w, r_w)$ (otherwise y_w, r_w do not solve (W)). Let the tangency points be

$\hat{x}_m, m = 1, \ldots, M, M \geq 2$,[17] and for each m let $\hat{x}_m^p$ be a point corresponding to $\hat{x}_m$ with the property that $y_w, \hat{x}_m, \hat{x}_m^p$ are collinear, $d(\hat{x}_m^p, y_w) = \bar{\bar{B}}$, and $d(\hat{x}_m^p, \hat{x}_m) = \bar{\bar{B}} - r_w$. Also, let q' be the point with the property that y_w, q, q' are collinear and $d(y_w, q') = d(y_w, q)$. Let λ_δ be a discrete probability distribution with support on $\hat{x}_1^p, \ldots, \hat{x}_M^p$ and expectation y_w. Let π_δ be the compound lottery putting probability $\frac{(1-\delta)}{\delta}$ on q' and $\frac{(2\delta-1)}{\delta}$ on λ_δ. Let σ_δ be such that each of proposers $q', \hat{x}_1^p, \ldots, \hat{x}_m^p$ propose their ideal point. Then, for any $\hat{x}$ we have

$$\begin{aligned} v_\sigma(\hat{x}) &= (1-\delta)u(q;\hat{x}) + \delta\left(\tfrac{1-\delta}{\delta}u(q';\hat{x}) + \tfrac{2\delta-1}{\delta}\sum_m \lambda_\delta(\{\hat{x}_m^p\})u(\hat{x}_m^p;\hat{x})\right) \\ &= u(y_w;\hat{x}) - 2(1-\delta)d(y_w,q)^2 - (2\delta-1)\bar{\bar{B}}^2, \end{aligned}$$

because $\mathbb{E}(\sigma_\delta) = y_w$ and $\mathbb{V}(\sigma_\delta) = 2(1-\delta)d(y_w,q)^2 + (2\delta-1)\bar{\bar{B}}^2$. By Lemma A.1, if $\hat{x}_m$ weakly prefers $\hat{x}_m^p$, then so does everyone in the pivotal winning half-space H_m^+, therefore $\hat{x}_m^p \in A_{\sigma_\delta}$. Indeed, as long as $\delta > \frac{\bar{\bar{B}} - r_w}{\bar{\bar{B}}}$, then for all m:

$$u(\hat{x}_m^p; \hat{x}_m) = -(\bar{\bar{B}} - r_w)^2 > r_w^2 - (2\delta - 1)\bar{\bar{B}}^2 \geq v_\sigma(\hat{x}_m).$$

Now, because $\hat{x}_m^p \in A_{\sigma_\delta}$ and $d(y_w, \hat{x}_m^p) = \bar{\bar{B}} > d(y_w, q) + 2r(y_w)$, we also have $q' \in A_{\sigma_\delta}$ by Lemma 2. Therefore, σ_δ is a quasi-equilibrium that is also an equilibrium by Corollary 3, part 4. Furthermore, for all m, $d(y, \hat{x}_m^p) \geq d(y_w, \hat{x}_m^p) - d(y, y_w) > \bar{B}$. $\square$

Proof of Theorem 9. First, $\bar{B}^*(y) = d(y, \hat{x}) + r(y) + r_w(y)$ uniquely solves $\bar{F}(B, y) = 0$ when $d(y, \hat{x}) \geq r_p(y)$, while if $d(y, \hat{x}) < r_p(y)$, then $\lim_{B \to 2r(y)^-} \bar{F}(B, y) = 0 \leq \bar{F}(2r(y), y) > 0$ and $\bar{B}^*(y) = 2r(y)$ by Theorem 3. Next, we show that there exists a quasi-equilibrium profile σ such that $\hat{x}$ proposes $\hat{x}$ with probability 1 and $A(\hat{x}') = \{x \mid u(x; \hat{x}') \geq u(\hat{x}; \hat{x}')\}$ for all $\hat{x}'$. Trivially, $\hat{x} \in A(\hat{x}')$ for all $\hat{x}'$, therefore $\hat{x} \in A_\sigma$ and σ satisfies (E_v),(E_p). Since proposal $\hat{x}$ is unanimously accepted, voter $\hat{x}' \neq \hat{x}$ has a unilateral infinite deviation that possibly induces a different payoff only if $\hat{x}'$ has absolute veto, i.e., $\mu_\ell(\{\hat{x}''\}) > 1 - m_\ell$ for some ℓ. But

[17] Because y_w, r_w solve (W) the convex hull of these tangency points contains y_w (if there is a continuum of such points, we may select $M = D + 1$ with that property).

such an infinite deviation by vetoer $\hat{x}'$ is profitable if and only if $v_\sigma(\hat{x}') = u(\hat{x}; \hat{x}') < u(q; \hat{x}')$. Thus the stated conditions are indeed necessary and sufficient. □

Proof of Theorem 10. By Theorem 6 in Kalandrakis (2019), $r_w(y)$ weakly decreases and $r_p(y)$ weakly increases with the stated changes in the voting rule. Write $\bar{F}(B, y; r_w(y), r(y))$ and $\underline{F}(B, y; r_p(y))$ to indicate the relevant parameters that enter each inequality. The results follow from the fact that

$$
\begin{aligned}
\bar{F}(B, y; r_w(y), r(y)) &\geq \bar{F}(B, y; r'_w(y), r(y)) && \text{if } r_w(y) < r'_w(y), \\
\bar{F}(B, y; r_w(y), r(y)) &\geq \bar{F}(B, y; r_w(y), r'(y)) && \text{if } r(y) < r'(y), \text{ and,} \\
\underline{F}(B, y; r_p(y)) &\leq \underline{F}(B, y; r'_p(y)) && \text{if } r_p(y) < r'_p(y).
\end{aligned}
$$

The first and last of these inequalities follow directly by taking derivatives (note that $\bar{d}$ is unchanged when $r(y)$ is and $\underline{d}$ does not depend on $r_p(y)$). The second follows because $\bar{d}(B, \hat{x}, y)$ weakly increases with $r(y)$. Parts 1 and 2 now follow from the above inequalities because $\bar{B}^*(y)$ (and $\underline{B}^*(y)$) is the largest (respectively, smallest) solution of $\bar{F}(B, y) \leq 0$ ($\underline{F}(B, y) \geq 0$). Part 3 follows directly from (5). □

B Equilibrium with unbounded support when π does not have finite second moments

Let $D = L = 1$, $\delta \in (0,1)$. Assume μ_1 has finite support with and odd number of $K \geq 3$ atoms $\hat{x}_1, \ldots, \hat{x}_K$ and $\mu_1(\{\hat{x}_k\}) = \frac{1}{K}$. Let π a Student's t distribution with 2 degrees of freedom. The strategy profile σ with $p_{\hat{x}}(\{\hat{x}\}) = 1$ and $A(\hat{x}) = X = \mathbb{R}$ for all $\hat{x}$ is an equilibrium as $v_\sigma(\hat{x}) = -\infty$ for all $\hat{x}$. Conditions (E_v) and (E_p) are obviously met and there are no profitable infinite deviations as no voting atom $\hat{x}_k$ has veto over proposals. Therefore, $\bar{B}^*(y) = \bar{B}_\sigma(y) = +\infty$ for all y.

C Single-crossing of upper bound equation

Lemma C.2. *If $\bar{F}(\bar{B}, y) \geq 0$ for some $\bar{B} \geq 2r(y)$, then there is no $B' > \bar{B}$ such that $\bar{F}(B', y) = 0$.*

Proof. Let $\bar{B} \geq 2r(y)$ be such that $\bar{F}(\bar{B}, y) \geq 0$, or

$$(\bar{B} - r_w(y))^2 - (1-\delta)(d(y,q) + r_w(y))^2 - \delta \int_{\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{d}(\bar{B}, \hat{x}, y) + r_w(y)\right)^2 \pi(d\hat{x})$$
$$-\delta \int_{\mathcal{I}\setminus\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{B} + r_w(y)\right)^2 \pi(d\hat{x}) = \alpha \geq 0.$$

Setting $A = -(1-\delta)(d(y,q)+r_w(y))^2 - \delta \int_{\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{d}(\bar{B}, \hat{x}, y) + r_w(y)\right)^2 \pi(d\hat{x}) - \alpha$, we obtain (because $\delta \int_{\mathcal{I}\setminus\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{B} + r_w(y)\right)^2 \pi(d\hat{x}) = \delta\left(1 - \pi(\mathcal{B}(y, \bar{B} - r(y)))\right)\left(\bar{B} + r_w(y)\right)^2$)

$$\delta\left(1 - \pi(\mathcal{B}(y, \bar{B} - r(y)))\right) = \frac{(\bar{B} - r_w(y))^2 - A}{(\bar{B} + r_w(y))^2} \leq \frac{(\bar{B} - r_w(y))^2}{(\bar{B} + r_w(y))^2}.$$

Now suppose, to get a contradiction, that there exists $B' > \bar{B}$ such that $\bar{F}(B', y) = 0$. Because, $\bar{d}(\bar{B}, \hat{x}, y) \geq \bar{d}(B', \hat{x}, y)$ when $\hat{x} \in \mathcal{B}(y, \bar{B} - r(y))$,[18] it is the case that

$$\int_{\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{d}(\bar{B}, \hat{x}, y) + r_w(y)\right)^2 \pi(d\hat{x}) \geq \int_{\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{d}(B', \hat{x}, y) + r_w(y)\right)^2 \pi(d\hat{x}).$$

[18]Specifically,

$$\begin{aligned}
&\bar{d}(\bar{B}, \hat{x}, y) = d(y, \hat{x}) = \bar{d}(B', \hat{x}, y) && \text{if } d(y,\hat{x}) \leq \bar{B} - 2r(y),\\
&\bar{d}(\bar{B}, \hat{x}, y) = 2d(y,\hat{x}) - \bar{B} + 2r(y) > d(y,\hat{x}) = \bar{d}(B', \hat{x}, y) && \text{if } \bar{B} - 2r(y) < d(y,\hat{x}) \leq \min\{\bar{B} - r(y), B' - 2r(y)\},\\
&\bar{d}(\bar{B}, \hat{x}, y) = 2d(y,\hat{x}) - \bar{B} + 2r(y) > 2d(y,\hat{x}) - B' + 2r(y) = \bar{d}(B', \hat{x}, y) && \text{if } B' - 2r(y) < d(y,\hat{x}) \leq \bar{B} - r(y).
\end{aligned}$$

As a consequence, also using $\bar{F}(\bar{B},y) \geq \bar{F}(B',y) = 0$, we must have

$$(\bar{B} - r_w(y))^2 - \delta \int_{\mathcal{I}\setminus\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{d}(\bar{B},\hat{x},y) + r_w(y)\right)^2 \pi(d\hat{x}) \geq$$
$$(B' - r_w(y))^2 - \delta \int_{\mathcal{I}\setminus\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{d}(B',\hat{x},y) + r_w(y)\right)^2 \pi(d\hat{x}).$$

In turn, because $B' \geq \bar{d}(B',\hat{x},y)$ and $\bar{B} = \bar{d}(\bar{B},\hat{x},y)$ for all $\hat{x} \notin \mathcal{B}(y,\bar{B}-r(y))$, we have

$$(\bar{B}-r_w(y))^2-\delta \int_{\mathcal{I}\setminus\mathcal{B}(y,\bar{B}-r(y))} \left(\bar{B} + r_w(y)\right)^2 \pi(d\hat{x}) \geq (B'-r_w(y))^2-\delta \int_{\mathcal{I}\setminus\mathcal{B}(y,\bar{B}-r(y))} (B' + r_w(y))^2 \pi(d\hat{x}), \text{ or}$$

$$\delta \left(1 - \pi(\mathcal{B}(y,\bar{B}-r(y)))\right) \left((B'+r_w(y))^2 - \left(\bar{B}+r_w(y)\right)^2\right) \geq (B'-r_w(y))^2 - (\bar{B}-r_w(y))^2.$$

Combining inequalities, and using the fact that $B' > \bar{B} \geq 2r(y)$, we have

$$\frac{(\bar{B}-r_w(y))^2}{(\bar{B}+r_w(y))^2} \geq \delta \left(1 - \pi(\mathcal{B}(y,\bar{B}-r(y)))\right) \geq \frac{(B'-r_w(y))^2 - (\bar{B}-r_w(y))^2}{(B'+r_w(y))^2 - \left(\bar{B}+r_w(y)\right)^2} > 0.$$

We now obtain a contradiction, because

$$\begin{aligned}
\frac{(B'+r_w(y))^2 - \left(\bar{B}+r_w(y)\right)^2}{(\bar{B}+r_w(y))^2} \geq \frac{(B'-r_w(y))^2 - (\bar{B}-r_w(y))^2}{(\bar{B}-r_w(y))^2} \quad &\Leftrightarrow \\
\frac{(B'+r_w(y))^2}{(\bar{B}+r_w(y))^2} \geq \frac{(B'-r_w(y))^2}{(\bar{B}-r_w(y))^2} \quad &\Leftrightarrow \\
(B'+r_w(y))\,(\bar{B}-r_w(y)) \geq (B'-r_w(y))(\bar{B}+r_w(y)) \quad &\Leftrightarrow \\
\bar{B}B' - B'r_w(y) + r_w(y)\bar{B} - r_w^2(y) \geq \bar{B}B' + B'r_w(y) - r_w(y)\bar{B} - r_w^2(y) \quad &\Leftrightarrow \\
\bar{B} \geq B'. &
\end{aligned}$$

This completes the proof. $\square$

D Numerical experiments of section 4

In this appendix I add details on the numerical experiments reported in section 4 and discuss how Tovey's algorithm is used to solve programs (W_y) and (P_y). The DW-NOMINATE scores disseminated through voteview Lewis et al. (2021) do not (consistently) provide a score for the vice-president and I assume the vice-president's ideal point coincides with that of the president. These data also may include multiple individuals or scores for a particular House or Senate seat due to party change, death, or resignation. In such cases, I use the last recorded person with available ideology score holding a seat. So (if such changes occur) these calculations primarily pertain to the later portion of each Congressional term.

The computations in section 4 proceed as follows:

1. Solve programs (W_y) and (P_y) for each y.
2. Solve for the bounds $\bar{B}^*(y)$ and $\underline{B}^*(y)$ using $\bar{F}(B, y)$ and $\underline{F}(B, y)$ as detailed in Theorems 1-3. When Theorem 3 does not isolate a closed form solution, the solution is obtained using bisection.
3. Combine the bounds across y according to Corollary 1.
4. Compute the extra bounds on proposals according to Corollary 2.

Steps 2-4 are straightforward, so I further elaborate on step 1. Naturally, what follows draws heavily on Tovey (1992), and I merely highlight the modifications needed in order to solve programs (W_y) and (P_y) in this discussion. Programs (W_y) and (P_y) involve a continuum of possible constraints (in general two possible pivotal hyperplanes for each possible direction $a \in \mathcal{A}$), but Tovey has explicitly studied this type of program as part of his analysis leading to the (more elaborate) algorithm to compute the yolk. The main difference is that Tovey is concerned with median hyperplanes, whereas here we are concerned with pivotal hyperplanes. The relevant results are in Theorem 1 and Corollary 1 of Tovey (1992), but the core idea is easy to convey. In order for a constraint (hyperplane) to be binding for either program, it has to contain at least one ideal point. If not, the hyperplane is clearly not pivotal as we can move it in either direction from y without switching any ideal points on either half-space. In $D = 2$ dimensions, that leaves us with two relevant possibilities (with ideal points in general position, the qualification "(or more)" in the first statement below can be eliminated):

- The binding hyperplane (a line in $D = 2$) has two (or more) ideal points on it.
- The binding hyperplane has only one ideal point on it (Tovey refers to these as "swiveling" hyperplanes).

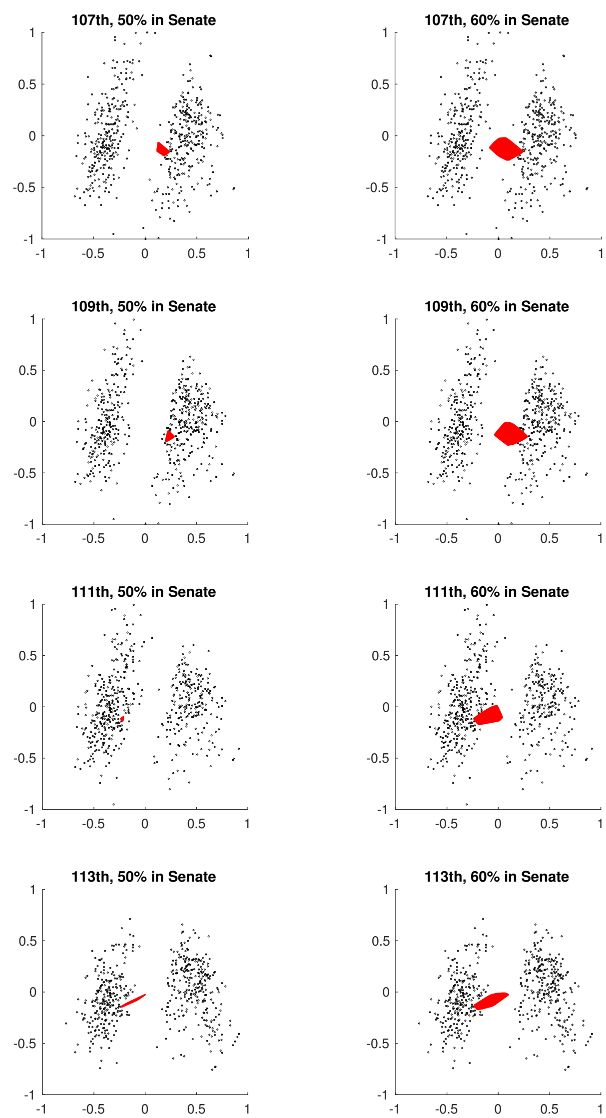
With N players, there are at most $\begin{pmatrix} N \\ 2 \end{pmatrix}$ possible hyperplanes of the first variety. We simply take all pairs of points and recover exactly the corresponding hyperplane parameters (a, c). There are N possibly binding "swiveling" hyperplanes but for programs (W_y) and (P_y) each of those is actually fixed for any specific y by the KKT conditions: If such hyperplane through ideal $\hat{x}$ is a binding constraint, it has to be perpendicular to the line segment $[y, \hat{x}]$. We can then easily compute the hyperplane parameters (a, c) for any y and any one of the N candidate swiveling hyperplanes.

These $\sum_{k=1}^{2} \begin{pmatrix} N \\ k \end{pmatrix}$ hyperplanes are the only relevant (possibly binding) constraints and Tovey refers to them as *determining* hyperplanes. We then retain only those that are *pivotal* by simply checking whether either half-side of them is winning, and whether it ceases to be winning if we remove the points on the hyperplane from that winning coalition. If either criterion fails the hyperplane is not pivotal and these determining hyperplanes can be discarded:[19]

- The solution to (P_y) is determined by the distance to the hyperplane that is farthest from y among the surviving hyperplanes. That distance is $r_p(y)$.

[19]In fact, the pivotality status of the determining hyperplanes with two points on them is independent of y. So, as Tovey discusses later in his paper, we can eliminate those ahead of time to expedite the computation. This is part of the reason we can handle such a large grid in these computations.

Figure D.1: Non-empty core in the DW-NOMINATE Congress



Computed by solving program (W_y) on a grid of 160,801 points. The set of core points (highlighted red) is non-empty in all four Congresses and both voting rules.

To solve (W_y), we further eliminate those among surviving hyperplanes that have only one winning half-space side *and* that side contains y (because any ball centered at y must intersect that winning half-space).

- The solution to (W_y) is determined by the distance to the hyperplane that is farthest from y among this smaller set of surviving hyperplanes. If none survive, then $r_w(y) = 0$ and y is a core point.

Based on such calculations, all Congresses studied in section 4 have a non-empty core for either voting rule, and I plot the set of recovered core points in Figure D.1.

If we wished to center the bounds at only one point, then we could solve two harder optimization problems searching over both a center y and radius r:

$$(P) \qquad \min_{y,r}\{r \mid \mathcal{B}(y,r) \cap H_{a,c^*(a)} \neq \emptyset \text{ for all } a \in \mathcal{A}\},$$

$$(W) \qquad \min_{y,r}\{r \mid \mathcal{B}(y,r) \cap H^+_{a,c^*(a)} \neq \emptyset \text{ for all } a \in \mathcal{A}\}.$$

Tovey's algorithm for computing the yolk (Tovey (1992), Theorem 2 and Corollary 2) builds on the above cited results to show how these more elaborate programs can be solved. When $D = 2$, this algorithm can be used exactly as stated by Tovey (1992) to generate a finite number of candidate solutions for (P). For program (P), the only difference is that we now discard any of these solutions if they involve hyperplanes that are not pivotal, instead of median. For program (W), we also discard those that have a center on the unique winning side of some pivotal binding hyperplane (and if none survive then the radius solution is zero and one of the candidate solutions contains – possibly infinite – center solutions).

Though (P) and (W) are still useful for other purposes, the ease of computing solutions to the simpler programs (W_y) and (P_y), along with the fact that the bounds are sharper when many y-centered bounds are computed, obviate the need to tackle programs

(P) and (W) (in fact, using the grid Y, we recovered many exact solutions for (W), and good approximate solutions for (P) in the application of section 4).

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