Identification and Estimation of Differentiated Products Models without Instruments using Cost Data*

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Abstract

We propose a new methodology for estimating demand and cost functions of differentiated products models when demand and cost data are available. The method deals with the endogeneity of prices to demand shocks and the endogeneity of outputs to cost shocks without any instruments by using cost data. We establish identification, consistency and asymptotic normality of our two-step Sieve Nonlinear Least Squares (SNLLS) estimator for the logit and BLP demand function specifications. Using Monte-Carlo experiments, we show that our method works well in contexts where commonly used instruments are correlated with demand and cost shocks and thus biased. We also apply our method to the estimation of deposit demand in the US banking industry.

Keywords: Differentiated Goods Oligopoly, Instruments, Identification, Cost data.

JEL Codes: C13, C14, L13, L41

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1 Introduction

In this paper, we develop a new methodology for estimating models of differentiated products markets. Our approach requires commonly used demand-side data on products' prices, market shares, observed characteristics, and firm-level cost data. The novelty of our method is that it does not use instrumental variables to deal with the endogeneity of prices to demand shocks in estimating demand, nor the endogeneity of outputs to cost shocks in estimating cost functions. Instead, we use cost data for identification and estimation. Market-level demand and cost data tend to be available for large industries that are subject to regulatory oversight (which often requires firms to report cost data). Examples include banking, telecommunications, and nursing home care. Such major sectors of the economy represent natural settings for the application of our estimator.

Recently, in order to incorporate the rich heterogeneities of agents, aggregate demand or market share equations are becoming more complex and nonlinear, and thus, more instruments and their interactions are needed for the identification of parameters. It is, in general, a challenge to convincingly argue that all these instruments are valid. We show in our Monte-Carlo exercises and an empirical example, how a small subset of invalid instruments can greatly bias parameter estimates in unanticipated directions in nonlinear demand models. In contrast, we find that the cost data based approach that we propose tends to deliver consistent and reasonable parameter estimates. By comparing the IV-based parameter estimates and those obtained by our cost-based approach, our method can provide additional support for the chosen instruments if the estimates are similar.

The frameworks of interest for this paper are the logit and random coefficient logit models of Berry (1994) and Berry et al. (1995) (hereafter, BLP) which have had a substantial impact on empirical research in IO and various other areas of economics.[1] These models incorporate unobserved heterogeneity in product quality and use instruments to deal with the endogeneity of prices to such heterogeneity. As Berry and Haile (2014) and others point out, as long as there are instruments available, demand functions can be identified using market-level data. Popular instruments include cost shifters such as market wages, observed product characteristics of other products in a market ("BLP instruments"), and the price of a given product in other markets ("Hausman instruments"). The attractiveness of this approach is that even in the absence of cost

[1] Leading examples from IO include measuring market power (Nevo (2001)), quantifying welfare gains from new products (Petrin (2002)), and merger evaluation (Nevo (2000)). Applications of these methods to other fields include measuring media slant (Gentzkow and Shapiro (2010)), evaluating trade policy (Berry et al. (1999)), and identifying sorting across neighborhoods (Bayer et al. (2007)).

data, firms' marginal cost functions can be recovered with the assumption that firms set prices to maximize profits given their rivals' prices.[2]

Nonetheless, BLP also propose using cost data, if available, for a variety of purposes, including improving their parameter estimates as well as understanding the relationship between prices and marginal costs. Recently, some researchers have started incorporating cost data as an additional source of identification. For instance, Houde (2012) combines wholesale gasoline prices with first order conditions that characterize stations' optimal pricing strategies to identify stations' marginal cost function. Crawford and Yurukoglu (2012) and Byrne (2015) similarly exploit first order conditions and firm-level cost data to identify the cost functions of cable companies. Some researchers have also used demand and cost data to test assumptions regarding conduct in oligopoly models. See, for instance, Byrne (2015), McManus (2007), Clay and Troesken (2003), Kim and Knittel (2003), and Wolfram (1999). Kutlu and Sickles (2012) estimate market power while allowing for inefficiency in production. Like previous research, these researchers use instrumental variables (IVs) to identify demand in a first step.

We extend the existing research on BLP-type models by developing and formalizing new ways to obtain additional identification with cost data.[3] Our main theoretical finding is that by combining demand and cost data, and by using the restriction that marginal revenue equals marginal cost in equilibrium, one can jointly identify the price coefficients in the BLP demand model and a nonparametric cost function, without using any instruments.[4] The type of cost data we have in mind comes from firms' income statements and balance sheets, among other sources. Such data has been used extensively in a large parallel literature on cost function estimation in empirical IO.[5]

It is important to note that we do not require data on marginal cost or markups nor knowledge of the cost function. If such information were available, it would be straightforward to use the first order condition to identify the price parameters without any instruments. For example, Smith (2004) estimates a demand model using consumer-level choice data for supermarket products.

[2]There has been some research assessing numerical difficulties with the BLP algorithm (Dube et al. (2012) and Knittel and Metaxoglou (2012)), and the use of optimal instruments to help alleviate these difficulties (Reynaert and Verboven (2014)).

[3]At a broader level, our paper shares a common theme with De Loecker (2011), where the usefulness of demand-side data in identifying production functions and measuring productivity is investigated.

[4]Fox et al. (2012) establish identification of the random coefficient discrete choice models with exogenous regressors. Berry and Haile (2014) prove nonparametric identification of a general market share function when the regressors are endogenous but instruments are available.

[5]Numerous studies have used such data to estimate flexible cost functions (e.g., quadratic, translog, generalized Leontief) to identify economies of scale or scope, measure marginal costs, and quantify markups for a variety of industries. For identification, researchers either use instruments for output or argue that output is effectively exogenous from firms' point of view in the market they study.

The author does not, however, have product-level price data. To overcome this missing data problem, the author develops an identification strategy that uses data on national price-cost margins and identifies the price coefficient in the demand model as the one that rationalizes these national margins. Our study differs in that we focus on the more common situation where a researcher has data on prices, aggregate market shares, and total costs, but not marginal costs. We also do not follow the literature on cost function estimation to measure marginal cost as doing so requires use of instruments to deal with the endogeneity of output due to profit maximization by firms.

We note that in the logit as well as BLP demand equations, the inversion procedure in Berry (1994) and BLP enables researchers to substitute away the unobserved product characteristics using observables, and thus, marginal revenue can be expressed as a function of only observables and the parameters. We apply this approach to the cost side as well. Assuming that cost is a function of output, input prices, observed characteristics and a cost shock, we show that when cost data is available, cost (or expected cost conditional on observables) can be used to control for the cost shock. Then, both marginal revenue and marginal cost can be expressed as functions of only observables and parameters. Therefore, no orthogonality conditions between observables and unobservables are needed for identification. Then, the "market structure" variables, such as the observed characteristics of rival firms in BLP, price, market share of rival firms, and market size can be used as sources of variation for the identification of price parameters through the exclusion restriction that they do not enter in the cost function. However, they do not have to be instruments.

Our results then imply that one does not need to use instruments to identify the parameters for endogenous prices in differentiated goods markets, nor to identify the cost function where output is potentially endogenous. Our identification strategy also does not require information on the correlation of the demand and cost shocks, which is used by MacKaye and Miller (2018) to identify the price coefficient of demand, nor do we need orthogonality between observed and unobserved product characteristics.[6]

In our paper, we show that cost data can be used to deal with the endogeneity issue of product prices to demand shocks with relatively weak assumptions. The main requirement we have on the nonparametric cost function is that it is strictly increasing in output and cost shock

[6] Petrin and Seo (2016) propose an identification and estimation scheme that allows for observed and unobserved characteristics in the demand equation to be endogenously determined. They skillfully exploit the optimal choice of observed characteristics to create additional moments. However, in the BLP random coefficient model of demand, the number of first order conditions is less than the number of parameters. Therefore, additional moment restrictions are required.

and in addition marginal cost is strictly increasing in the cost shock. We also allow for cost data with measurement error, random component of fixed cost as well as systematic over/under reporting by firms.

We do not need any variation in market size to identify and estimate the BLP-demand model. For logit demand, we need it, but such variation does not need to be exogenous, that is, orthogonal to other variables that determine the equilibrium outcomes. In our Monte-Carlo simulations, we show that our methodology yields consistent estimates even when market size is correlated with demand and cost shocks, as well as input prices.

We also show that we can identify price coefficients without any instruments, even if the true market size is unobservable, as long as it is a function of observables. We follow Bresnahan and Reiss (1990) partly in that we assume, as they do, that the variables determining market size do not enter the cost function. However, we do not exclude these variables from the demand function as they do. We believe that our exclusion restriction is more reasonable because typically, the determinants of market size are demographics, which are likely to affect demand but do not affect cost directly.

We also highlight a Curse of Dimensionality problem in the nonparametric estimation of the expected cost conditional on observables that likely makes an estimator based on the direct application of our identification results impractical. This motivates our two-step Sieve Non-Linear Least Squares (SNLLS) estimator, which does not suffer from the dimensionality problem. This is essentially because we use marginal revenue, which is a parametric function, to control for the cost shock, rather than expected cost conditional on observables. This estimator is semi-parametric in that it assumes parametric logit or BLP demand and a nonparametric cost function. We also show how this estimator can be adapted to accommodate various data and specification issues that arise in practice. These include endogenous product characteristics, imposing restrictions on cost functions such as homogeneity of degree one in input prices, dealing with the difference between accounting cost and economic cost, missing cost data for some products or firms, and multi-product firms.

Through a set of Monte-Carlo experiments for the BLP demand model, we illustrate how our estimator delivers consistent parameter estimates when the demand shock is not only correlated with the equilibrium price and output, but also with the cost shock, input prices, and market size, observed characteristics of rival products and when the cost shock is correlated with market size as well. In such a setting there are no valid instruments to account for price endogeneity. In particular, market size cannot work as an exogenous variation for the supply side, and the orthogonality between the demand and cost shocks cannot be used as a moment restriction for consistent estimation of price parameters. Hence, the IV estimates are shown to be biased. Our Monte-Carlo experiments also show that no variation in market size is needed for identifying the BLP price parameters.

We then apply our methodology to the estimation of deposit demand in the US banking industry. We find that our method works well. The magnitude of the coefficient estimate on deposit interest rate is smaller than the ones obtained in the existing literature such as Dick (2008) and Ho and Ishii (2012). Further, we find that the IV-based method yields a negative coefficient on deposit interest rate whereas ours is positive which is what one would expect.

This paper is organized as follows. In Section 2, we specify the differentiated products model that we adopt from the literature and review the IV based estimation approach in the literature. In Section 3, we study identification when demand and cost data are available and present our formal identification results. In Section 4, we propose the two-step SNLLS estimator and analyze its large sample properties. Section 5 contains a Monte-Carlo study that illustrates the effectiveness of our estimator in environments where standard approaches to demand estimation yield biased results. In Section 6 we apply our methodology to the estimation of deposit demand in the banking industry. In Section 7 we conclude. The appendix contains several proofs and further details of the deposit demand estimation exercise.

2 Differentiated products models and IV estimation

In this section, we describe the standard differentiated products model that we adopt including some of the assumptions and provide an overview of IV estimation of the demand and supply side. For more details, see Berry (1994), Berry et al. (1995), Nevo (2001) and others. Most features of the model we discuss here are carried over to the next section where we explain our cost data-based identification strategy.

2.1 Differentiated products discrete choice demand models

In the standard model, consumer i in market m gets the following utility from consuming one unit of product j:

$$u_{ijm} = \mathbf{x}_{jm}\boldsymbol{\beta} + p_{jm}\alpha + \xi_{jm} + \epsilon_{ijm},$$

where $\mathbf{x}_{jm}$ is a $1 \times K$ vector of observed product characteristics, p_{jm} is price, ξ_{jm} is the unobserved product quality (or demand shock) that is known to both consumers and firms but unknown to researchers, and ϵ_{ijm} is an idiosyncratic taste shock. Let the demand parameter vector be $\boldsymbol{\theta} = \left[\alpha, \boldsymbol{\beta}'\right]'$ where $\boldsymbol{\beta}$ is a $K \times 1$ vector.

It is assumed that there are $m = 1 \ldots M$ isolated markets that have respective market sizes Q_m.[7] Each market has $j = 0 \ldots J_m$ products whose aggregate demand across individuals is

$$q_{jm} = s_{jm} Q_m,$$

where q_{jm} denotes output and s_{jm} denotes market share. In the case of the Berry (1994) logit demand model, ϵ_{ijm} is assumed to have a logit distribution. Then, the aggregate market share for product j in market m is,

$$s_{jm}(\boldsymbol{\theta}) \equiv s_j\left(\mathbf{p}_m, \mathbf{X}_m, \boldsymbol{\xi}_m; \boldsymbol{\theta}\right) = \frac{\exp\left(\mathbf{x}_{jm}\boldsymbol{\beta} + p_{jm}\alpha + \xi_{jm}\right)}{\sum_{k=0}^{J_m} \exp\left(\mathbf{x}_{km}\boldsymbol{\beta} + p_{km}\alpha + \xi_{km}\right)} = \frac{\exp\left(\delta_{jm}\right)}{\sum_{k=0}^{J_m} \exp\left(\delta_{km}\right)}, \tag{1}$$

where $\mathbf{p}_m = [p_{0m}, p_{1m}, ..., p_{J_m m}]'$ is a $(J_m + 1) \times 1$ vector,

$$\mathbf{X}_m = \begin{bmatrix} \mathbf{x}_{0m} \\ \mathbf{x}_{1m} \\ \vdots \\ \mathbf{x}_{J_m m} \end{bmatrix}$$

is a $(J_m + 1) \times K$ matrix, $\boldsymbol{\xi}_m = [\xi_{0m}, \xi_{1m}, ..., \xi_{J_m m}]'$ is a $(J_m + 1) \times 1$ vector, and

$$\delta_{jm} \equiv \mathbf{x}_{jm}\boldsymbol{\beta} + p_{jm}\alpha + \xi_{jm}, \tag{2}$$

is the "mean utility" of product j in market m. Using this definition, we can express market share in Equation (1) as $s_j\left(\boldsymbol{\delta}(\boldsymbol{\theta})\right) \equiv s_j\left(\mathbf{p}_m, \mathbf{X}_m, \boldsymbol{\xi}_m; \boldsymbol{\theta}\right)$ where $\boldsymbol{\delta}(\boldsymbol{\theta}) = [\delta_{0m}(\boldsymbol{\theta}), \delta_{1m}(\boldsymbol{\theta}), \ldots, \delta_{J_m m}(\boldsymbol{\theta})]'$.

Good $j = 0$ is labeled the "outside good" or "no purchase option" that corresponds to not buying any of the $j = 1, \ldots, J_m$ goods. This good's product characteristics, price, and demand shock are normalized to zero (i.e., $\mathbf{x}_{0m} = \mathbf{0}$, $p_{0m} = 0$, and $\xi_{0m} = 0$ for all m), which implies

$$\delta_{0m}(\boldsymbol{\theta}) = 0. \tag{3}$$

This normalization, together with the logit assumption for the distribution of ϵ_{ijm}, identifies the level and scale of utility.

[7] With panel data the m index corresponds to a market-period.

In BLP, one allows the price coefficient and coefficients on the observed characteristics to be different for different consumers. Specifically, α has a distribution function $F_\alpha(.;\boldsymbol{\theta}_\alpha)$, where $\boldsymbol{\theta}_\alpha$ is the parameter vector of the distribution, and similarly, $\boldsymbol{\beta}$ has a distribution function $F_{\boldsymbol{\beta}}(.;\boldsymbol{\theta}_{\boldsymbol{\beta}})$ with parameter vector $\boldsymbol{\theta}_{\boldsymbol{\beta}}$. The probability with which a consumer with coefficients α and $\boldsymbol{\beta}$ purchases product j is identical to that provided by the market share formula in Equation (1). The aggregate market share of product j is obtained by integrating over the distributions of α and $\boldsymbol{\beta}$,

$$s_j(\mathbf{p}_m, \mathbf{X}_m, \boldsymbol{\xi}_m; \boldsymbol{\theta}) = \int_\alpha \int_{\boldsymbol{\beta}} \frac{\exp(\mathbf{x}_{jm}\boldsymbol{\beta} + p_{jm}\alpha + \xi_{jm})}{\sum_{k=0}^{J_m} \exp(\mathbf{x}_{km}\boldsymbol{\beta} + p_{km}\alpha + \xi_{km})} dF_{\boldsymbol{\beta}}(\boldsymbol{\beta}; \boldsymbol{\theta}_{\boldsymbol{\beta}})\, dF_\alpha(\alpha; \boldsymbol{\theta}_\alpha), \tag{4}$$

where $\boldsymbol{\theta} = \left[\boldsymbol{\theta}'_\alpha, \boldsymbol{\theta}'_{\boldsymbol{\beta}}\right]'$. Letting μ_α to be the mean of α and $\boldsymbol{\mu}_{\boldsymbol{\beta}}$ the mean of $\boldsymbol{\beta}$, the mean utility is defined to be

$$\delta_{jm} \equiv \mathbf{x}_{jm}\boldsymbol{\mu}_{\boldsymbol{\beta}} + p_{jm}\mu_\alpha + \xi_{jm}, \tag{5}$$

with $\delta_{0m} = 0$ for the outside good.

2.1.1 Recovering demand shocks

For each market $m = 1, \ldots M$, researchers are assumed to have data on prices $\mathbf{p}_m$, market shares $\mathbf{s}_m = [s_{0m}, s_{1m}, ..., s_{J_m m}]'$ and observed product characteristics $\mathbf{X}_m$ for all firms in the market. Given $\boldsymbol{\theta}$ and this data, one can solve for the vector $\boldsymbol{\delta}_m$ through market share inversion. That is, if we denote $s_j(\boldsymbol{\delta}_m(\boldsymbol{\theta}); \boldsymbol{\theta})$ to be the market share of firm j predicted by the model, market share inversion involves obtaining $\boldsymbol{\delta}_m$ by solving the following set of J_m equations,

$$s_j(\boldsymbol{\delta}_m(\boldsymbol{\theta}), j; \boldsymbol{\theta}) - s_{jm} = 0, \text{ for } j = 0, \ldots, J_m, \tag{6}$$

and therefore,

$$\boldsymbol{\delta}_m(\boldsymbol{\theta}) = \mathbf{s}^{-1}(\mathbf{s}_m; \boldsymbol{\theta}). \tag{7}$$

The vector of mean utilities that solves these equations perfectly aligns the model's predicted market shares to those observed in the data.

In the logit model, Berry (1994) shows we can easily recover mean utilities for product j using its market share and the share of the outside good as $\delta_{jm}(\boldsymbol{\theta}) = \log(s_{jm}) - \log(s_{0m})$, $j = 1, \ldots, J_m$. In the random coefficient model, there is no such closed-form formula for market share inversion. Instead, BLP propose a contraction mapping algorithm that recovers the unique $\delta_{jm}(\boldsymbol{\theta})$ that solves Equation (7) under some regularity conditions. In both cases, δ_{0m} is normalized to 0.

With the mean utilities and parameters in hand, one can recover the structural demand shocks straightforwardly from Equation (2) for the logit demand and Equation (5) for the BLP demand.

2.1.2 IV estimation of demand

A simple regression of Equation (2) or (5) with $\delta_{jm}(\boldsymbol{\theta})$ being the dependent variable and $\mathbf{x}_{jm}$ and p_{jm} being the regressors would yield a biased estimate of the price coefficient. This is because firms likely set higher prices for products with higher unobserved product quality, which creates a correlation between p_{jm} and ξ_{jm}, violating the OLS orthogonality condition $E[\xi_{jm}p_{jm}] = 0$. Researchers use a variety of demand instruments to overcome this issue. In particular, researchers construct a GMM estimator for $\boldsymbol{\theta}$ by assuming the following population moment conditions are satisfied at the true value of the demand parameters $\boldsymbol{\theta}_0$:

$$E[\xi_{jm}(\boldsymbol{\theta}_0)\,\mathbf{z}_{jm}] = \mathbf{0}$$

where $\mathbf{z}_{jm}$ is an $L \times 1$ vector of instruments that is correlated with $\mathbf{x}_{jm}$. Also, instruments are required to satisfy the exclusion restriction that at least one variable in $\mathbf{z}_{jm}$ is not contained in $\mathbf{x}_{jm}$.

2.2 Cost Function and Supply

For each product j in market m, in addition to the data related to demand explained above, researchers observe output q_{jm} (hence, market size $Q_m = q_{jm}/s_{jm}$ as well), $L \times 1$ vector of input price $\mathbf{w}_{jm}$ and cost C_{jm}. The observed cost C_{jm} is assumed to be a function of output, input prices $\mathbf{w}_{jm}$, observed product characteristics $\mathbf{x}_{jm}$ and a cost shock υ_{jm}. That is,

$$C_{jm} = C\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, \upsilon_{jm}; \boldsymbol{\tau}\right),$$

where $\boldsymbol{\tau}$ is a parameter vector. $C()$ is assumed to be strictly increasing and continuously differentiable in output and cost shock.

Assuming that there is one firm for each product, firm j's profit function is as follows:

$$\pi_{jm} \quad = \quad p_{jm} \times q_{jm} - C\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, \upsilon_{jm}; \boldsymbol{\tau}\right).$$

Let MR_{jm}, be the marginal revenue of firm j in market m. BLP assume that firms act as differentiated products Bertrand price competitors. Therefore, the optimal price and quantity of product j in market m are determined by the first order condition (F.O.C.) that equates marginal revenue and marginal cost:

$$\underbrace{MR_{jm} = \frac{\partial p_{jm} q_{jm}}{\partial q_{jm}} = p_{jm} + s_{jm} \left[\frac{\partial s_j \left(\mathbf{p}_m, \mathbf{X}_m, \boldsymbol{\xi}_m; \boldsymbol{\theta} \right)}{\partial p_{jm}} \right]^{-1}}_{MR_{jm}} = MC_{jm} = \underbrace{\frac{\partial C \left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, \upsilon_{jm}; \boldsymbol{\tau} \right)}{\partial q_{jm}}}_{MC_{jm}}. \tag{8}$$

Note that given the market share inversion in Equation (6), and the specification of mean utility $\boldsymbol{\delta}_m$, $\boldsymbol{\xi}_m$ is a function of $(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m)$ and $\boldsymbol{\theta}$. Therefore, marginal revenue of firm j in market m, MR_{jm} in Equation (8) can be written as a function of observables and parameters as follows:

$$MR_{jm} \equiv MR_j \left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \boldsymbol{\theta} \right), \tag{9}$$

where $MR_j \left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \boldsymbol{\theta} \right)$ is the j th element of the vector of marginal revenue functions in market m, denoted by $\mathbf{MR} \left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \boldsymbol{\theta} \right)$. Equations (8) and (9) imply that demand parameters can potentially be identified if there is data on marginal cost[8] or even without such data, if the cost function is known or can be estimated and its derivative with respect to output can be taken. Berry et al. (1995) assume that marginal cost is log-linear in output and observed product characteristics, i.e., $MC_{jm} = exp \left(\mathbf{w}_{jm} \boldsymbol{\gamma}_w + q_{jm} \gamma_q + \upsilon_{jm} \right)$ (see their Equation 3.6). They then use instruments to deal with the endogeneity of output with cost shocks and of prices to demand shocks. As long as the parametric specification of the supply side is accurate and there are enough instruments for identification, the demand side and F.O.C. based orthogonality conditions are sufficient for identifying demand parameters. We in contrast assume cost to be a nonparametric function of output, input prices, observed product characteristics and the cost shock, and use observed cost to control for the cost shock. This point is further explained in Section 3.

2.2.1 Cost function estimation

As with demand estimation, one can recover unobserved cost shocks through inversion:

$$C_{jm} = C \left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, \upsilon_{jm}; \boldsymbol{\tau} \right) \Rightarrow \upsilon_{jm} = \upsilon \left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, C_{jm}; \boldsymbol{\tau} \right). \tag{10}$$

[8] Genesove and Mullin (1998) use data on marginal cost to estimate the conduct parameters of the homogeneous goods oligopoly model.

Like demand estimation, there are important endogeneity concerns with standard approaches to estimating cost functions. Specifically, output q_{jm} is endogenously determined by profit-maximizing firms as in Equation (8), and is potentially negatively correlated with the cost shock υ_{jm}. That is, all else equal, less efficient firms tend to produce less. In dealing with this issue, researchers have traditionally focused on selected industries where endogeneity can be ignored, or used instruments for output.

The IV approach to cost function estimation typically uses excluded demand shifters as instruments. Denoting this vector of cost instruments by $\widetilde{\mathbf{z}}_{jm}$, one can estimate $\boldsymbol{\tau}$ assuming that the following population moments are satisfied at the true value of the cost parameters $\boldsymbol{\tau}_0$: $E\left[\upsilon_{jm}\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, C_{jm}; \boldsymbol{\tau}_0\right)\widetilde{\mathbf{z}}_{jm}\right] = \mathbf{0}$.

3 Identification using cost data

In this section, we present our methodology for dealing with the endogeneity issues in identification mentioned above. We propose using cost data in addition to demand data to identify price parameters. We do so by using the control function approach. That is, given output, input prices and observed product characteristics, we use the observed cost to control for the cost shock. We focus primarily on the BLP random coefficients model but also use logit as a simple example to illustrate our identification strategy. Instead of Equation (10), we use a nonparametric cost function. We elaborate on our demand and cost structure further below.

To begin with, we assume that market size is observable. In Subsection 3.3, we demonstrate how our methodology can be modified to the case where market size is not observed, and thus needs to be estimated.

3.1 Main Assumptions

We first state all the main assumptions for our methodology. Most of these assumptions are standard as discussed in the previous section or simply describe the environment our methodology is applicable to. For each market in the population, we attach a unique positive real number m as an identifier. Then, we assume $m \in \mathcal{M}$, where $\mathcal{M}$ is the set of all market identifiers, and is an uncountable subset of R_+.

Assumption 1 *Data Requirements: Researchers have data on outputs, product prices, market shares, input prices, observed product characteristics, and total costs of firms.*

Note that market size can be derived from data on outputs and market shares. Thus, we need to assume observability of only two of these three variables. In contrast to BLP, we require data on total costs of firms. But we do not need data on marginal cost.

Assumption 2 *Isolated Markets: Outputs, market shares, prices and costs in market m are functions of variables in market m .*

Assumption 3 *Common Input Prices within Markets: Input price $\mathbf{w}_{jm} = \mathbf{w}_m$ for all j, m.*

We make this assumption to show that we do not need within-market variation in input prices. That is, relaxing it makes it easier for our methodology to work. The assumption is reasonable as usually there is little within-market variation in input prices in the data.

Assumption 4 *BLP demand: Market share s_{jm} is specified as in Equation (4). The distributions of α and each element of $\boldsymbol{\beta}$ are assumed to be independently normal, i.e., $\alpha \sim N\left(\mu_\alpha, \sigma_\alpha^2\right)$, $\beta_k \sim N\left(\mu_{\boldsymbol{\beta}k}, \sigma_{\boldsymbol{\beta}k}^2\right)$, $k = 1, \ldots, K$. Further, $\mu_{\boldsymbol{\beta}k} = 0$, $k = 1, \ldots, K$; $\mu_\alpha < 0$.*

Assumption 5 *Equilibrium Concept: Bertrand-Nash equilibrium holds in each market. That is, for any $j = 1, \ldots, J_m$, firm j in market m chooses its price p_{jm} to equalize marginal revenue and marginal cost, given market size Q_m and prices of other firms in the same market $\mathbf{p}_{-j,m}$.*[9]

The next assumption describes the support of variables that determine the equilibrium outcomes in market m. Let the set of these variables be denoted by $\mathbf{V}_m$. Then $\mathbf{V}_m \equiv (Q_m, \mathbf{w}_m, \mathbf{X}_m, \boldsymbol{\xi}_m, \boldsymbol{\upsilon}_m)$, and let $\mathbf{V} \equiv \{V_m\}_{m \in \mathcal{M}}$. Let $\mathbf{V} \setminus w_{lm}$ to be the set $\mathbf{V}$ without the element $\mathbf{w}_{lm}$ for any $l = 1, 2, \ldots, L$. For other elements of $\mathbf{V}$, the set $\mathbf{V}$ without the element is similarly defined. The assumption imposes substantially weaker restrictions on the support of the variables in $\mathbf{V}$ than is typical in the literature. In particular, it imposes minimal restrictions on the joint distribution of these variables as stated below.

Assumption 6 *Support of $\mathbf{V}$: The support of Q_m conditional on $\mathbf{V} \setminus Q_m$ can be any nonempty subset of R_+ for all m. The support of $\mathbf{w}_{lm}$ conditional on $\mathbf{V} \setminus \mathbf{w}_{lm}$ is R_+ for all l, m; the support of x_{kjm} conditional on $\mathbf{V} \setminus x_{kjm}$ is either R or R_+ for all k, j, m; and the support of ξ_{jm} conditional on $\mathbf{V} \setminus \xi_{jm}$ is R. Finally, the support of υ_{jm} conditional on $\mathbf{V} \setminus \upsilon_{jm}$ is R_+.*

[9]Note that we have assumed this for expositional purposes only. It is not required for identification. MR is a one-to-one function of MC in equilibrium, and not necessarily equal to MC, we can identify the price parameters. This makes our framework applicable to firms that are under government regulation and firms under organizational incentives or behavioral aspects that prevent them from setting $MR = MC$.

Assumption 6 ensures that the variables in $\mathbf{V}$ are not subject to any orthogonality conditions, which typically restrict the moments of a subset of the unobserved variables $(\boldsymbol{\xi}_m, \boldsymbol{\upsilon}_m)$ conditional on the other variables to be zero. In other words, we do not require them to be econometrically exogenous, and thus, Assumption 6 removes the validity of any conventional instruments.

Note that we do not impose any assumptions on the support of market size other than that it is nonempty and positive. For logit, we require the conditional support to be R_+ since as we show later, market size variation is needed for the identification of the price parameters of logit but not for BLP.

The next two assumptions are about our nonparametric cost function.[10]

Assumption 7 *Properties of the Cost Function: Let C^*_{jm} denote true cost. Then,*

$$C^*_{jm} \equiv C^v\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \upsilon_{jm}\right) + e_f\left(\mathbf{w}_m, \mathbf{x}_{jm}, \upsilon_{jm}\right) + \varsigma_{jm}, \tag{11}$$

where $C^v()$ is the variable cost component, which is a continuous function of q, $\mathbf{w}$, $\mathbf{x}$ and υ, strictly increasing, and continuously differentiable in q and υ, and marginal cost is strictly increasing in υ; $e_f()$ is the deterministic component of fixed cost, a continuous function of $\mathbf{w}$, $\mathbf{x}$ and υ and increasing in υ. The fixed cost shock ς is i.i.d., with mean zero and independent of $\mathbf{V}_m$. Further, for any $q > 0$, $\mathbf{w} > \mathbf{0}$ and $\mathbf{x} \in \mathcal{X}$, where $\mathcal{X}$ is the support of $\mathbf{x}$,

$$lim_{\upsilon \searrow 0} \frac{\partial C^v\left(q, \mathbf{w}, \mathbf{x}, \upsilon\right)}{\partial q} = 0, \quad lim_{\upsilon \nearrow \infty} \frac{\partial C^v\left(q, \mathbf{w}, \mathbf{x}, \upsilon\right)}{\partial q} = \infty.$$

Assumption 8 *Measurement Error in Cost: Let C_{jm} be the observed cost. Then,*

$$C_{jm} = C^*_{jm} + e_{me}\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}\right) + \nu_{jm}, \tag{12}$$

where $e_{me}()$ is a continuous function and ν_{jm} is i.i.d. with mean 0 and independent of $\mathbf{V}_m$ and the fixed cost shock $\boldsymbol{\varsigma}_m$.[11]

The assumption implies that the measurement error in cost is $e_{me}\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}\right) + \nu_{jm}$, where $e_{me}()$ is the deterministic component.[12]

[10] Note that we assume that market share s_{jm} does not enter in the cost function. This restriction rules out situations in which firms with high market shares have buying power in the input market.

[11] We can also include $\mathbf{x}_{me,jm}$, a vector of additional variables that determine the deterministic component of the measurement error as well as fixed cost. However, we omit these for the sake of expositional simplicity.

[12] Note that we could also allow for systematic misreporting of true costs. For example, if $\nu(C^*)$ is the systematic component of the reported true cost, then, if firms report costs truthfully but with an error, then $\nu(C^*) = C^*$. Alternatively, if firms systematically under-report their true costs, then we could consider a specification like $\nu(C^*) = \nu C^*$ where $0 < \nu < 1$. Over-reporting could be captured by the same specification with $\nu > 1$.

Using Equations (11) and (12), we obtain

$$C_{jm} = C^{v}\left(q_{jm}, \mathbf{w}_{m}, \mathbf{x}_{jm}, \upsilon_{jm}\right) + e_{f}\left(\mathbf{w}_{jm}, \mathbf{x}_{jm}, \upsilon\right) + e_{me}\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}\right) + \nu_{jm} + \varsigma_{jm}.$$

In the next subsection, we show that cost data together with other data and our assumptions enables us to identify the parameters of the distribution of the random coefficients on price $(\mu_{\alpha}, \sigma_{\alpha})$, as well as σ_{β_k}, $k = 1, \ldots, K$. Note that we can only identify $(\mu_{\alpha 0}, \sigma_{\alpha 0}, \boldsymbol{\sigma}_{\boldsymbol{\beta} 0})$ and $\xi_{0jm} + \mathbf{x}_{jm}\boldsymbol{\mu}_{\beta 0}$ in the absence of further restrictions imposed on the model. However, this information is sufficient to identify marginal revenue, and thus, markup, which is of primary interest of most empirical exercises in IO. The additional orthogonality assumption $E\left(\mathbf{x}_{jm}\xi_{jm}\right) = 0$ identifies $\boldsymbol{\mu}_{\beta 0}$. For the logit model of demand, we identify the price coefficient α_0 and $\xi_{0jm} + \mathbf{x}_{jm}\boldsymbol{\beta}_0$. From now on, except when notified otherwise, we will denote the vector of true parameters $\boldsymbol{\theta}_{c0}$ we identify to be α_0 for the logit demand specification and $\left(\mu_{\alpha 0}, \sigma_{\alpha 0}, \boldsymbol{\sigma}'_{\boldsymbol{\beta} 0}\right)$ for the BLP specification. More generally, we denote by $\boldsymbol{\theta}_{c0}$ the demand parameters that we identify.

3.2 The Main Result

In this subsection, we derive our main theoretical result, namely that given our assumptions, parameters of the BLP model, that is, $\boldsymbol{\theta}_{c0} = \left(\mu_{\alpha 0}, \sigma_{\alpha 0}, \boldsymbol{\sigma}'_{\boldsymbol{\beta} 0}\right)$, are identified.

We use cost data to control for the cost shock. Since we allow for measurement error in the cost data, the first step in proving our results is to show that the expected cost conditional on observables contains only deterministic components of the cost function, which is done in Lemma 1. We then provide an intuitive explanation of our identification strategy using the logit example and then develop two equivalent definitions of identification using cost data. The first definition (Definition 1) is based on the first order condition (Equation (8)) and highlights the sources of variation behind identification as well as the exclusion restrictions. However, since the control function approach leads to marginal cost becoming an unspecified function of output, input prices, observed product characteristics and the deterministic component of cost, it is difficult to implement and thus we develop a second definition of identification (Definition 2) based on pairing of firms that have the same output, input prices and observed characteristics. We show that the two definitions are equivalent and then specify a condition on marginal revenue that together with our assumptions ensures identification in the BLP model.

Lemma 1 *Let Assumptions 1-3 and 7-8 be satisfied, and let the marginal revenue function be specified as in Equation (9). Further, let $\mathcal{R} = \{\mathbf{p}, \mathbf{s}, \mathbf{X}, q, \mathbf{w}, Q, j\}$, and let $\widetilde{C}$ be the deterministic* *component of cost. Then, for any firm j in the population with $\mathcal{R}$ being observable,*

$$E\left[C\middle|\left(\widetilde{q}=q,\widetilde{\mathbf{w}}=\mathbf{w},\widetilde{\mathbf{p}}=\mathbf{p},\widetilde{\mathbf{s}}=\mathbf{s},\widetilde{\mathbf{X}}=\mathbf{X},j\right)\right]=C^{v}\left(q,\mathbf{w},\mathbf{x},\upsilon\right)+e_{f}\left(\mathbf{w},\mathbf{x},\upsilon\right)+e_{me}\left(q,\mathbf{w},\mathbf{x}\right)\equiv\widetilde{C}.$$

Proof. From Assumption 7, marginal cost is strictly increasing in υ, and given any $(q,\mathbf{w},\mathbf{x})$ in the population, the support of marginal cost is R_{+}. Therefore, given Assumption 5, for any observation $\mathcal{R}=\{\mathbf{p},\mathbf{s},\mathbf{X},q,\mathbf{w},Q,j\}$ in the population, there exists a unique υ such that

$$MR_{j}\left(\mathbf{p},\mathbf{s},\mathbf{X};\boldsymbol{\theta}_{c0}\right)=MC\left(q,\mathbf{w},\mathbf{x},\upsilon\right). \tag{13}$$

Because Equation (13) determines a unique υ given $(q,\mathbf{w},\mathbf{x})$, υ is a function of $(q,\mathbf{w},\mathbf{x},MR_{j}\left(\mathbf{p},\mathbf{s},\mathbf{X};\boldsymbol{\theta}_{c0}\right))$. Thus, Assumptions 7 and 8 result in

$$\begin{aligned}
&E\left[C\middle|\left(\widetilde{q}=q,\widetilde{\mathbf{w}}=\mathbf{w},\widetilde{\mathbf{p}}=\mathbf{p},\widetilde{\mathbf{s}}=\mathbf{s},\widetilde{\mathbf{X}}=\mathbf{X},j\right)\right]\\
&=E\Big[C^{v}\left(q,\mathbf{w},\mathbf{x},\upsilon\left(q,\mathbf{w},\mathbf{x},MR_{j}\left(\mathbf{p},\mathbf{s},\mathbf{X};\boldsymbol{\theta}_{c0}\right)\right)\right)+e_{f}\left(\mathbf{w},\mathbf{x},\upsilon\left(q,\mathbf{w},\mathbf{x},MR_{j}\left(\mathbf{p},\mathbf{s},\mathbf{X};\boldsymbol{\theta}_{c0}\right)\right)\right)\\
&+e_{me}\left(q,\mathbf{w},\mathbf{x}\right)+\varsigma+\nu\Big|\left(\widetilde{q}=q,\widetilde{\mathbf{w}}=\mathbf{w},\widetilde{\mathbf{p}}=\mathbf{p},\widetilde{\mathbf{s}}=\mathbf{s},\widetilde{\mathbf{X}}=\mathbf{X},j\right)\Big]\\
&=C^{v}\left(q,\mathbf{w},\mathbf{x},\upsilon\right)+e_{f}\left(\mathbf{w},\mathbf{x},\upsilon\right)+e_{me}\left(q,\mathbf{w},\mathbf{x}\right)
\end{aligned}$$

■

In this subsection, we call a variable observable if it is directly observable in the population or can be recovered as the expectation of a directly observable variable conditional on other directly observable variables. Hence, from now, we call the deterministic component of cost as cost whenever there won't be any confusion. $\widetilde{C}$ is observed because it is the conditional expectation of observed cost conditional on other observed data.

Before providing formal results, we outline the logic of our identification argument. In particular, we first explain how we remove the need for instruments to deal with the endogeneity of the supply shock. We use the following three equations for identification. For firm j in market m, they are:

$$s_{jm}=\frac{q_{jm}}{Q_{m}}, \tag{14}$$

$$MR_{j}\left(\mathbf{p}_{m},\mathbf{s}_{m},\mathbf{X}_{m};\boldsymbol{\theta}_{c0}\right)=MC\left(q_{jm},\mathbf{w}_{m},\mathbf{x}_{jm},\upsilon_{jm}\right), \tag{15}$$

$$\widetilde{C}_{jm}=C^{v}\left(q_{jm},\mathbf{w}_{m},\mathbf{x}_{jm},\upsilon_{jm}\right)+e_{f}\left(\mathbf{w}_{m},\mathbf{x}_{jm},\upsilon_{jm}\right)+e_{me}\left(q_{jm},\mathbf{w}_{m},\mathbf{x}_{jm}\right). \tag{16}$$

The market size Q_m is subsumed in market share $\mathbf{s}_m$ and q_{jm} through Equation (14). The first order condition (15) holds in equilibrium because we assume in Assumption 5 that firms choose prices to equate marginal revenue to marginal cost.

If we had data on marginal cost, i.e., MC_{jm}, then, we could just use Equation (15) for identification of $\boldsymbol{\theta}_{c0}$. By substituting MC_{jm} into the RHS, we would have a function of only observables. In the logit model, Equation (15) then becomes

$$p_{jm} + \frac{1}{(1 - s_{jm})\,\alpha_0} = MC_{jm},$$

and the price coefficient α_0 can be identified as

$$\alpha_0 = \frac{1}{(1 - s_{jm})\,(MC_{jm} - p_{jm})}.$$

However, marginal cost is generally not observable. Then, one could consider estimating the cost function $\widetilde{C}_{jm} \equiv C\,(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \upsilon_{jm})$, and taking its derivative with respect to output to derive the marginal cost. However, as discussed in the previous section, this strategy runs into the potential endogeneity issue of output being correlated with the cost shock and thus requires the use of instruments which we propose to avoid with our methodology.[13]

Instead, we use cost data to control for the cost shock. That is, we invert the cost function in Equation (16) to derive

$$\upsilon_{jm} = \upsilon\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right). \tag{17}$$

Such inversion is possible because of Assumption 7 stating that given output, input price and observed characteristics, the deterministic component of cost is a strictly increasing function of the cost shock. After substituting Equation (17) into Equation (15), F.O.C. becomes:

$$MR_j\,(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \boldsymbol{\theta}_{c0}) = \psi\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right), \tag{18}$$

where ψ is an unspecified function. Note that there are no unobservables in Equation (18) that may be correlated with the observables and create endogeneity problems.

We now define identification based on this F.O.C, letting firm jm denote firm j in market m.

Definition 1 *Identification by F.O.C.: Let the marginal revenue function be specified as in*

[13] Such a derivative would also include the derivative of the deterministic component of the measurement error with respect to output, which should not be part of the marginal cost.

Equation (9). Then, the true parameter vector $\boldsymbol{\theta}_{c0}$ is identified if the following two statements hold only at $\boldsymbol{\theta}_{c0}$:

1. *For any firm jm in the population, marginal revenue is positive and can be expressed as a function of only $\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right)$.*

2. *Given $(q, \mathbf{w}, \mathbf{x})$, this function is one-to-one in $\widetilde{C}$.*

The function of $\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right)$ in the above definition corresponds to $\psi\,()$ in Equation (18). From Assumption 7, we know that given $(q, \mathbf{w}, \mathbf{x})$, $\widetilde{C}$ and marginal cost are strictly increasing in the cost shock υ, which implies that, given $(q, \mathbf{w}, \mathbf{x})$, ψ is strictly increasing and thus one-to-one in $\widetilde{C}$. Our definition implies that only at the true parameter vector $\boldsymbol{\theta}_{c0}$, given $(q, \mathbf{w}, \mathbf{x})$, each value of $\widetilde{C}_{jm}$ is associated with a unique marginal revenue. Intuitively this means that for firms with the same output, input prices and observed characteristics, having equal observed cost is equivalent to having equal marginal revenue only at the true parameter vector.

The sources of variation we use for identification of $\boldsymbol{\theta}_{c0}$ are similar to the ones in the literature. In the logit model, as we explain below, these are market size Q_m and price p_{jm}, and in BLP, additionally we can use price, market share and observed characteristics of the rival firms. All these sources of variation reflect the "market structure" and appear in the marginal revenue function but not in the marginal cost function ψ (see Equation (18)). The difference from the literature is that these variables do not need to be instruments, i.e. orthogonal to the cost shock υ_{jm} because we have already controlled for it using $\widetilde{C}_{jm}$. Thus, our identification strategy is based on the exclusion restrictions that market structure variables do not enter the cost function directly.

To illustrate how Equation (18) identifies $\boldsymbol{\theta}_{c0}$, we use the logit model, where the parameter we identify is the price coefficient, i.e, $\boldsymbol{\theta}_{c0} = \alpha_0$. Then, using Equation (14), Equation (18) can be written as:

$$MR_j\left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \alpha_0\right) = p_{jm} + \frac{1}{\left(1 - q_{jm}/Q_m\right)\alpha_0} = \psi\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right). \tag{19}$$

Conditions 1 and 2 of Definition 1 are clearly satisfied for α_0. Next, we consider any $\alpha \neq \alpha_0$.

Then,

$$
\begin{aligned}
& MR_j\left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \alpha_0\right) \\
= \quad & p_{jm} + \frac{1}{\left(1 - q_{jm}/Q_m\right)\alpha_0} = p_{jm} + \frac{1}{\left(1 - q_{jm}/Q_m\right)\alpha} + \frac{1}{\left(1 - q_{jm}/Q_m\right)}\left(\frac{1}{\alpha_0} - \frac{1}{\alpha}\right) \\
= \quad & MR_j\left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \alpha\right) + \frac{1}{\left(1 - q_{jm}/Q_m\right)}\left(\frac{1}{\alpha_0} - \frac{1}{\alpha}\right)
\end{aligned}
$$

Substituting into (19), we obtain

$$
\begin{aligned}
& MR_j\left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m; \alpha\right) \\
= \quad & p_{jm} + \frac{1}{\left(1 - q_{jm}/Q_m\right)\alpha} = \psi\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right) - \frac{1}{\left(1 - q_{jm}/Q_m\right)}\left(\frac{1}{\alpha_0} - \frac{1}{\alpha}\right) \\
\equiv \quad & \widetilde{\psi}(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}, Q_m, \alpha).
\end{aligned}
\tag{20}
$$

Note that $\widetilde{\psi}$ includes market size as an argument, violating Condition 1 of Definition 1 for $\alpha \neq \alpha_0$. Thus, α_0 is identified. Note also that if we do not have any variation in market size, i.e., $Q_m = \overline{Q}$, then $\widetilde{\psi}$ remains a function of $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$. Furthermore, if $Q_m = \overline{Q}$, in Equation (20), $\widetilde{C}$ enters in ψ and we know that given $(q, \mathbf{w}, \mathbf{x})$, ψ is a one-to-one function of $\widetilde{C}$, and thus, $\widetilde{\psi}$ satisfies both Conditions 1 and 2 for $\alpha \neq \alpha_0$. Hence, the true price coefficient cannot be identified. Thus, for the logit model, our identification strategy requires variation in market size. Price variation is also needed unless $1/\alpha_0$ is zero, otherwise Equation (19) would fail to hold. This becomes transparent later in this subsection.

However, dealing with unspecified functions ψ and $\widetilde{\psi}$ makes the identification analysis complex and unintuitive. This is because for each parameter $\boldsymbol{\theta}_c$, we need to evaluate whether marginal revenue at $\boldsymbol{\theta}_c$ is a function of only $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$. Instead, in our analysis, we use an alternative equivalent way of proving identification which we call the pairing approach. This approach lets us focus on the marginal revenue side. The only role of cost data and the marginal cost function is to identify the following two sets of pairs of firms: pairs of firms in different markets that have the same true marginal revenue, and pairs that have different true marginal revenues. Then, from these two sets of pairs, we proceed to identify the price coefficient by using only the demand side. We illustrate this approach for the logit model first. As we will see, the pairing approach provides us with the exact sources of variation needed to identify the price parameter in the logit model, namely, market size and price of the firm's product.

More specifically, we "fix" the variables in the marginal cost function by finding a pair of firms

$(jm, j^\dagger m^\dagger)$ in the data that have the same output, same input price, same observed characteristics and the same cost: $\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right) = \left(q_{j^\dagger m^\dagger}, \mathbf{w}_{m^\dagger}, \mathbf{x}_{j^\dagger m^\dagger}, \widetilde{C}_{j^\dagger m^\dagger}\right)$ but $(p_{jm}, Q_m) \neq (p_{j^\dagger m^\dagger}, Q_{m^\dagger})$. Then, from Definition 1,

$$\psi\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right) = \psi\left(q_{j^\dagger m^\dagger}, \mathbf{w}_{m^\dagger}, \mathbf{x}_{j^\dagger m^\dagger}, \widetilde{C}_{j^\dagger m^\dagger}\right),$$

and thus, using Equation (19), at α_0,

$$\begin{aligned} & p_{jm} + \frac{1}{(1 - q_{jm}/Q_m)\,\alpha_0} = \psi\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \widetilde{C}_{jm}\right) \\ = \;& \psi\left(q_{j^\dagger m^\dagger}, \mathbf{w}_{m^\dagger}, \mathbf{x}_{j^\dagger m^\dagger}, \widetilde{C}_{j^\dagger m^\dagger}\right) = p_{j^\dagger m^\dagger} + \frac{1}{\left(1 - q_{j^\dagger m^\dagger}/Q_{m^\dagger}\right)\alpha_0}. \end{aligned} \tag{21}$$

In the above equation, ψ is eliminated and what remains is the equality of the true marginal revenue of two firms that have different prices and market shares.

Then, α_0 can be identified straightforwardly from the above marginal revenue equality as follows:

$$\alpha_0 = -\frac{1}{p_{jm} - p_{j^\dagger m^\dagger}}\left[\frac{1}{(1 - q_{jm}/Q_m)} - \frac{1}{\left(1 - q_{j^\dagger m^\dagger}/Q_{m^\dagger}\right)}\right]. \tag{22}$$

Note that since $q_{jm} = q_{j^\dagger m^\dagger}$, if we assume constant market size, the term in the bracket is always zero, and thus, $\alpha_0 \neq 0$ cannot be identified. Furthermore, without variation in price, RHS is either not bounded or not well defined. Therefore, identification using pairing requires variation in both price and market size. We also show next that given $q_{jm} = q_{j^\dagger m^\dagger}$, $\mathbf{w}_m = \mathbf{w}_{m^\dagger}$ and $\mathbf{x}_{jm} = \mathbf{x}_{j^\dagger m^\dagger}$, there exist (Q_m, ξ_{jm}) and $\left(Q_{m^\dagger}, \xi_{j^\dagger m^\dagger}\right)$ that generate the above prices p_{jm}, $p_{j^\dagger m^\dagger}$ and market shares s_{jm}, $s_{j^\dagger m^\dagger}$. First, choose $Q_m = q_m/s_{jm}$, $Q_{m^\dagger} = q_{m^\dagger}/s_{j^\dagger m^\dagger}$. This is feasible because from Assumption 6, the conditional support of market size Q is R^+. Also, using Equation (1), and the normalization of Equation (3), choose ξ_{jm}, $\xi_{j^\dagger m^\dagger}$ from the conditional support of R as:

$$\begin{aligned} \xi_{jm} &= \mathbf{x}_{jm}\boldsymbol{\beta}_0 + p_{jm}\alpha_0 - (ln\,(s_{jm}) - ln\,(s_{0m}))\,, \\ \xi_{j^\dagger m^\dagger} &= \mathbf{x}_{j^\dagger m^\dagger}\boldsymbol{\beta}_0 + p_{j^\dagger m^\dagger}\alpha_0 - \left(ln\left(s_{j^\dagger m^\dagger}\right) - ln\,(s_{0m^\dagger})\right). \end{aligned}$$

It is clear from the discussion above that in logit, identification of the price coefficient is based on the true marginal revenue equality for a pair of firms that has the same output, input prices, observed product characteristics and cost. Indeed, Equations (21) and (22) indicate that we only need to consider a single pair of firms to prove identification. In BLP however, marginal revenue is a complex, non-linear function of parameters and we cannot show analytically that the demand parameters $\boldsymbol{\theta}_{c0}$ are identified or that the marginal revenue equality for one pair of firms generates a unique set of true parameters. Instead, we exploit the information contained in the data about the set of pairs whose two firms have the same output, input prices and observed characteristics. We divide these firm-pairs into two subsets of pairs according to whether the two firms within a pair have equal observed cost or not. The group of pairs whose firms have the same observed cost must have the same true marginal revenue while the opposite is true for the other group. We use this insight to formulate a condition on the marginal revenue function that is sufficient for identifying $\boldsymbol{\theta}_{c0}$.

We now reformulate our identification definition in terms of pairing.

Definition 2 *Identification by Pairing: Let the marginal revenue function be specified as in Equation (9). We say that $\boldsymbol{\theta}_{c0}$ is identified if the following holds only for $\boldsymbol{\theta}_c = \boldsymbol{\theta}_{c0}$:*

1 *For any firm jm in the population,*

$$MR_j\left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m, \boldsymbol{\theta}_c\right) > 0, \tag{23}$$

2 *Given any two firms $jm \neq j^{\dagger}m^{\dagger}$ in the population with $\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}\right) = \left(q_{j^{\dagger}m^{\dagger}}, \mathbf{w}_{m^{\dagger}}, \mathbf{x}_{m^{\dagger}}\right)$,*

$$MR_j\left(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m, \boldsymbol{\theta}_c\right) = MR_{j^{\dagger}}\left(\mathbf{p}_{m^{\dagger}}, \mathbf{s}_{m^{\dagger}}, \mathbf{X}_{m^{\dagger}}, \boldsymbol{\theta}_c\right),$$

if and only if

$$\widetilde{C}_{jm} = \widetilde{C}_{j^{\dagger}m^{\dagger}}.$$

It is straightforward to see that the two definitions are equivalent. Both definitions require positivity of true marginal revenue and a one-to-one relationship between the observable cost and true marginal revenue given output, input prices and observed characteristics, and violation of at least one of these conditions if the parameter is not the true one. The pairwise approach to identification is simpler than the first definition because we no longer have to deal with the unspecified marginal cost function. Instead we identify the true parameter vector by examining all pairs in which the two firms have the same output, input prices and observed characteristics and checking strict positivity of marginal revenue and whether the within-pair equality between the two observed costs and between the two marginal revenues hold simultaneously at a candidate parameter vector. If any marginal revenue is nonpositive or if simultaneity does not hold for some of these pairs, i.e., if for some pairs, only the costs are equal but not the marginal revenues, or vice versa, then the candidate parameter vector cannot be the true one. Another way to interpret this definition is to ask whether at the candidate parameter vector, all the marginal revenues in the population are nonnegative, and the set of pairs of firms that have the same output, input prices, observed characteristics and the observed cost is the same as the set of pairs of firms that have the same output, input prices, observed characteristics and marginal revenue. If not, then the parameter is not the true one. We can also state it more formally by defining the set $\mathcal{S}$ to be the set of pairs whose two firms have the same output, input prices and observed characteristics, and letting $\mathcal{C} \subset \mathcal{S}$ be the subset of $\mathcal{S}$ whose two firms have the same cost, and $\mathcal{MR}(\boldsymbol{\theta}_c) \subset \mathcal{S}$ be the subset of $\mathcal{S}$ whose two firms have the same marginal revenue. Then, Definition 2 states that $\boldsymbol{\theta}_{c0}$ is identified if for any $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$, either positivity in Equation (23) is violated for some firm jm in the population, or

$$\mathcal{C} = \mathcal{MR}(\boldsymbol{\theta}_{c0}) \neq \mathcal{MR}(\boldsymbol{\theta}_c).$$

or both.

Lemma 2 *Definitions 1 and 2 are equivalent.*

The proof is in the Appendix.

We next state a condition on the demand model that together with our assumptions is sufficient for indentification of the demand parameters. We need this condition because the information we can use from data on cost and the assumptions on properties of the cost function are not sufficient to identify the true marginal revenue. Among the pairs of firms that have the same output, input prices and observed characteristics, the cost data allows us to identify the subset of pairs whose two firms have the same true marginal revenue (i.e. have the same cost), and the subset of pairs whose two firms have different true marginal revenues (i.e. different costs). The additional source of information that identifies the true marginal revenue and $\boldsymbol{\theta}_{c0}$ needs to come from the functional form of the demand model. It needs to be such that only the true parameter can exactly replicate the two subsets described above that are identified by the data. Given the assumptions stated in Subsection 3.1, we prove that the condition below is sufficient for identification of the true parameter $\boldsymbol{\theta}_{c0}$.

Condition 1 *Let the marginal revenue function be specified as in Equation (9). Let* $\mathcal{D} = \{\mathbf{p}, \mathbf{s}, \mathbf{X}\}$ *and* $\mathcal{D}^{\dagger} = \{\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}\}$ *be two sets of vectors of prices, market shares and the matrix of observed product characteristics with* J *and* $J^{\dagger}$ *rows, and let* $\boldsymbol{\theta}_{c0}$ *be the true parameter* *vector. Then, for any given $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$, either positivity in Equation (23) is violated for a firm jm in the population; or the following statement holds, or both. There exist $\mathcal{D}$ and $\mathcal{D}^\dagger$ that satisfy the following properties: for a row j in $\mathcal{D}$ and a row $j^\dagger$ in $\mathcal{D}^\dagger$, any reordering of the rows in set $\mathcal{D}^\dagger$, $\mathcal{D} \neq \mathcal{D}^\dagger$ and*

1. $p_l > 0$, $0 < s_l < 1$ *for* $l = 1, ..., J$ *and* $p_l^\dagger > 0$, $0 < s_l^\dagger < 1$, *for* $l = 1, .., J^\dagger$, *and* $0 < \sum_{l=1}^{J} s_l < 1$, $0 < \sum_{l=1}^{J^\dagger} s_l^\dagger < 1$.

2. $\mathbf{x}_j = \mathbf{x}_{j^\dagger}^\dagger$.

3. *Either* $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_c) = MR_{j^\dagger}\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \mathbf{X}^\dagger, \boldsymbol{\theta}_c\right)$ *or* $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = MR_{j^\dagger}\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \mathbf{X}^\dagger, \boldsymbol{\theta}_{c0}\right)$ *but not both.*

Condition 1 is the restatement of Definition 2 of identification without any restrictions on the cost side. That is, Condition 1 applied to the population is equivalent to requiring that for any $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$, either for some firm in the population, marginal revenue is nonpositive at $\boldsymbol{\theta}_c$ or within $\mathcal{S}$, the set of pairs whose two firms have the same positive marginal revenue under $\boldsymbol{\theta}_{c0}$ and the corresponding set of pairs under $\boldsymbol{\theta}_c$ cannot be equal (that is, $\mathcal{MR}(\boldsymbol{\theta}_{c0}) \neq \mathcal{MR}(\boldsymbol{\theta}_c)$), or both. In the lemma below we show that given our assumptions, Condition 1 is sufficient for identification because for any pairs of observed demand variables $(\mathbf{p}, \mathbf{s}, \mathbf{X}, j)$ and $\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \mathbf{X}^\dagger, j^\dagger\right)$ satisfying $\mathbf{x}_j = \mathbf{x}_{j^\dagger}$, we can always find two firms in the population that have the same output, input prices and observed characteristics and these demand variables.

Lemma 3 *Suppose Assumptions 1-3, 5-8 and Condition 1 are satisfied. Then, $\boldsymbol{\theta}_{c0}$ is identified according to Definition 2 of identification.*

Proof. First, consider $\boldsymbol{\theta}_{c0}$. Then, given Assumptions 5 and 7, in the population, marginal revenue is positive at $\boldsymbol{\theta}_{c0}$. Also, given Assumptions 2-3, 5-8, Equation (18) holds, and from Equation (18), for any $\left(\mathcal{D}, q, \mathbf{w}, \mathbf{x}, \widetilde{C}, j\right)$ and $\left(\mathcal{D}^\dagger, q, \mathbf{w}, \mathbf{x}, \widetilde{C}^\dagger, j^\dagger\right)$ in the population with $\widetilde{C} = \widetilde{C}^\dagger$,

$$MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = \psi\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right) = \psi\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}^\dagger\right) = MR_{j^\dagger}\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \mathbf{X}^\dagger, \boldsymbol{\theta}_{c0}\right).$$

Similarly, if $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = MR_{j^\dagger}\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \mathbf{X}^\dagger, \boldsymbol{\theta}_{c0}\right)$, then, given $(q, \mathbf{w}, \mathbf{x})$, from Assumptions 5-7, there exists a unique cost shock υ such that

$$MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = MR_{j^\dagger}\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \mathbf{X}^\dagger, \boldsymbol{\theta}_{c0}\right) = MC(q, \mathbf{w}, \mathbf{x}, \upsilon).$$

Therefore, $\widetilde{C} = \widetilde{C}^{\dagger}$. Hence, for any pair of firms with the same $(q, \mathbf{w}, \mathbf{x})$, $\widetilde{C} = \widetilde{C}^{\dagger}$ if and only if $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_{c0})$. Therefore, $\boldsymbol{\theta}_{c0}$ satisfies the conditions for identification of Definition 2.

We now consider any $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$. Suppose there exists $\left(\mathcal{D}, q, \mathbf{w}, \mathbf{x}, \widetilde{C}, j\right)$ in the population satisfying $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_c) \leq 0$. Then, $\boldsymbol{\theta}_c$ violates the first condition of Definition 2. Next, we consider the case where any $\left(\mathcal{D}, q, \mathbf{w}, \mathbf{x}, \widetilde{C}, j\right)$ in the population satisfies $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_c) > 0$.

We analyze the two cases of Condition 1 separately.

Case 1. Suppose $\mathcal{D}$, j and $\mathcal{D}^{\dagger}$, $j^{\dagger}$ satisfy $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_c) = MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_c)$ but $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) \neq MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_{c0})$. Then, for any $(q, \mathbf{w}, \mathbf{x})$, from Assumption 7, there exist υ, $\upsilon^{\dagger}$ in the population such that $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = MC(q, \mathbf{w}, \mathbf{x}, \upsilon)$, $MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_{c0}) = MC(q, \mathbf{w}, \mathbf{x}, \upsilon^{\dagger})$. Because marginal cost is strictly increasing in the cost shock, this implies that $\upsilon \neq \upsilon^{\dagger}$, and since the deterministic component of cost is increasing in the cost shock, $\widetilde{C} \neq \widetilde{C}^{\dagger}$. Therefore, in this case, $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_c) = MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_c)$ but for $(q, \mathbf{w}, \mathbf{x}) = (q^{\dagger}, \mathbf{w}^{\dagger}, \mathbf{x}^{\dagger})$, $\widetilde{C} \neq \widetilde{C}^{\dagger}$.

Case 2. Suppose $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_{c0})$ but $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_c) \neq MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_c)$. Then, for any $(q, \mathbf{w}, \mathbf{x})$, from Assumption 7, there exists υ such that $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_{c0}) = MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_{c0}) = MC(q, \mathbf{w}, \mathbf{x}, \upsilon)$. Therefore, both firms have the same cost shock, and thus, $\widetilde{C} = \widetilde{C}^{\dagger}$. Therefore, in this case, for $(q, \mathbf{w}, \mathbf{x}) = (q^{\dagger}, \mathbf{w}^{\dagger}, \mathbf{x}^{\dagger})$, $\widetilde{C} = \widetilde{C}^{\dagger}$ but $MR_j(\mathbf{p}, \mathbf{s}, \mathbf{X}, \boldsymbol{\theta}_c) \neq MR_{j^{\dagger}}(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}, \boldsymbol{\theta}_c)$.

If marginal revenue is positive at $\boldsymbol{\theta}_c$ for all firms in the population, either Case 1 or Case 2 holds.

Together, we have shown that $\boldsymbol{\theta}_{c0}$ is identified. ■

Note that in proving the lemmas above or in Condition 1, we did not need to make any assumptions about independence of any of the variables from each other, except for the fixed cost shock and the random component of the measurement error of the cost. That is, the lemmas go through and Condition 1 can be satisfied regardless of possible correlation across input prices, variable cost shock, observed characteristics, unobserved product characteristics and market size, within markets or across markets. For example, a positive correlation between market size and the demand/cost shock can arise as larger market size may induce firms to invest in higher quality or more advertising, which improves unobserved product quality but increases cost. But this does not break our identification strategy.[14] Thus, these findings illustrate that given cost

[14]Later, we show that for the identification of the parameters of the distribution of the BLP price coefficients, market size does not need to have any variation. In our Monte-Carlo analysis, we provide a scenario where market size is correlated with demand shock. Results demonstrate consistency of our estimator. Furthermore, notice that

data, one does not need any IV- or orthogonality assumptions.[15]

Also note that in the market share specification, there are no moment restrictions on the unobserved characteristics, and thus, they can contain market-level fixed effects. In particular, consider the BLP specification with market-level fixed effects where

$$\begin{aligned} u_{ij} &= \mathbf{x}_j \boldsymbol{\beta}_i + p_j \alpha_i + \xi_{jm} + \epsilon_{ij}, \\ \xi_{jm} &= \xi_{f,m} + \widetilde{\xi}_{jm}, \\ E\left[\widetilde{\xi}_{jm} \middle| \mathbf{x}_{jm}, \xi_{f,m}\right] &= 0, \end{aligned}$$

where we denote $\xi_{f,m}$ to be the market m specific heterogeneity. Because we do not use such moment conditions for identification, those fixed effects do not prevent us from identifying and consistently estimating the BLP parameters $\boldsymbol{\theta}_{c0} = (\mu_{\alpha 0}, \sigma_{\alpha 0}, \boldsymbol{\sigma}_{\beta 0})$.

Our main identification result is stated in the following proposition, with the proof in the appendix:

Proposition 1 *Suppose Assumptions 1-8 are satisfied. Then, the BLP coefficients* $\boldsymbol{\theta}_{c0} = (\mu_{\alpha 0}, \sigma_{\alpha 0}, \boldsymbol{\sigma}_{\beta 0})$ *are identified.*

Note that the identification of $\boldsymbol{\theta}_{c0}$ in the BLP demand model holds even without any variation in market size across markets. To see why, consider the identification using the pairing approach. That is, suppose we find a pair firms jm and $j^{\dagger}m^{\dagger}$ that have the same marginal revenue. Then, if the demand specification is logit, without market size variation, they have the same market share, and Equation (22) tells us that we cannot identify the price coefficient. On the other hand, under BLP, even though the same market size leads to the pair of firms having the same market share, these firms can have different price effect on own market share due to differences in prices, market shares and observed product characteristics of rival firms, and thus, different prices in the relationship.

More formally, those two firms satisfy

$$s_{jm} = s_{j^{\dagger}m^{\dagger}},\ MR_{jm} = MR_{j^{\dagger}m^{\dagger}},\ \frac{\partial s_{jm}}{\partial p_{jm}} \neq \frac{\partial s_{j^{\dagger}m^{\dagger}}}{\partial p_{j^{\dagger}m^{\dagger}}}.$$

an important component of it is the F.O.C. Equation (15). Any violation of this F.O.C. may result in $\boldsymbol{\theta}_{c0}$ not being identified. An example would be if higher prices and more advertising spending signal product quality, as in the model of Milgrom and Roberts (1986).

[15] It is important to note that each pair of observations satisfying Condition 1 can be generated from different equilibria. Since the observables $\left\{q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right\}$ uniquely determine the pair of firms that have the same cost shock υ, and the marginal cost, the above procedure identifies the true price coefficient even when multiple equilibria exist.

Therefore,

$$p_{jm} = MR_{jm} - \left[\frac{\partial s_{jm}}{\partial p_{jm}}\right]^{-1} s_{jm} \neq MR_{j^\dagger m^\dagger} - \left[\frac{\partial s_{j^\dagger m^\dagger}}{\partial p_{j^\dagger m^\dagger}}\right]^{-1} s_{j^\dagger m^\dagger} = p_{j^\dagger m^\dagger}.$$

Thus, the relationship

$$p_{jm} - p_{j^\dagger m^\dagger} = -s_{jm}\left[\left(\frac{\partial s_{jm}}{\partial p_{jm}}\right)^{-1} - \left(\frac{\partial s_{j^\dagger m^\dagger}}{\partial p_{j^\dagger m^\dagger}}\right)^{-1}\right]$$

identifies the parameters.[16]

3.3 Identification of Unobserved Market Size

We now consider the case where market size Q_m is not observed, and thus needs to be estimated. This is an important issue in the empirical IO literature. Because market participation is unobserved, it is often hard for researchers to measure the total number of participants of a market without any arbitrariness.

We follow Bresnahan and Reiss (1991) and specify the market size as follows:

$$ln\left(Q_m\right) = \lambda_{c0} + \mathbf{z}_m \boldsymbol{\lambda}_{\mathbf{z}0}, \tag{24}$$

where $\mathbf{z}_m$ is a $1 \times K_{\mathbf{z}}$ vector of observables in market m, and $\boldsymbol{\lambda}_{\mathbf{z}0} = (\lambda_{\mathbf{z}01}, \ldots, \lambda_{\mathbf{z}0K_{\mathbf{z}}})$.

Then, the true market share of firm j in market m, denoted by

$$s^*_{jm} \equiv \frac{q_{jm}}{exp\left(\lambda_{c0} + \mathbf{z}_m \boldsymbol{\lambda}_{\mathbf{z}0}\right)}, \tag{25}$$

is unobservable. Bresnahan and Reiss (1991) and other literature on this issue assume that variables that determine market size are not included in the market share equation. However, we do not impose such a restriction since one can convincingly argue that demographic variables determine not only market size but also consumer demand. Thus, the modified utility function

[16] The exclusion restriction for the logit model is that marginal revenue only depends on own price and own market share. That is, unobserved product characteristics of firms and prices of rival firms in a market do not enter directly in the marginal revenue equation of any given firm: these variables only enter indirectly through the market share function. For the BLP demand, we have similar exclusion restrictions at high prices. That is, if we let p_{jm} be own price, the exclusion restriction we use is that at high prices, in the 2nd term of the marginal revenue function, own price only enters through $p_{jm} - p_{j-1,m}$ where $p_{j-1,m}$ is the next highest price in the market), and $p_{jm} - p_{j+1,m}$ where $p_{j+1,m}$ is the next lowest price in the market. For details, see the appendix.

for individual i in market m consuming product j is

$$u_{ijm} = \mathbf{x}_{jm}\boldsymbol{\beta}_{\mathbf{x}} + \mathbf{z}_m\boldsymbol{\beta}_{\mathbf{z}} + p_{jm}\alpha + \xi_{jm} + \epsilon_{ijm}. \tag{26}$$

On the other hand, following the literature, we assume that the variables determining market size are not included in the cost function. This assumption is reasonable as demographic variables usually do not enter the production function. Then, we maintain the original exclusion restrictions that market structure variables only enter in the marginal revenue function but not in the marginal cost function. Therefore, the identification procedure is the same as before.

We prove identification for the logit demand model here, and for the BLP model in the appendix.

First, note that since market share $\mathbf{s}$ and market size Q are unobserved, and market size is a function of $\mathbf{z}$ in Equation (25), marginal cost is a function of $(\mathbf{p}, \mathbf{z}, \mathbf{q}, \mathbf{X})$ instead of $(\mathbf{p}, \mathbf{s}, \mathbf{X})$. Therefore, the first order condition is modified to be

$$MR_j\left(\mathbf{p}, \mathbf{z}, \mathbf{q}, \mathbf{X}; \boldsymbol{\theta}_{c0}\right) = MC\left(q, \mathbf{w}, \mathbf{x}, \upsilon\right),$$

where $\boldsymbol{\theta}_{c0}$ now includes the price coefficient and the parameters of the market size equation, i.e. $\boldsymbol{\theta}_{c0} = (\alpha_0, \lambda_{c0}, \boldsymbol{\lambda}_{\mathbf{z}0})$. Then, the cost shock can be expressed as follows:

$$\upsilon = \upsilon\left(q, \mathbf{w}, \mathbf{x}, MR_j\left(\mathbf{p}, \mathbf{z}, \mathbf{q}, \mathbf{X}; \boldsymbol{\theta}_{c0}\right)\right).$$

Furthermore, instead of $\mathcal{R} = \{\mathbf{p}, \mathbf{s}, \mathbf{X}, \mathbf{q}, \mathbf{w}, j\}$, now we have $\mathcal{R} = \{\mathbf{p}, \mathbf{X}, \mathbf{q}, \mathbf{w}, \mathbf{z}, j\}$ with which we derive the deterministic component of cost given below:

$$E\left[C\middle|\left(\widetilde{\mathbf{q}} = \mathbf{q}, \widetilde{\mathbf{w}} = \mathbf{w}, \widetilde{\mathbf{p}} = \mathbf{p}, \widetilde{\mathbf{z}} = \mathbf{z}, \widetilde{\mathbf{X}} = \mathbf{X}, j\right)\right] = C^v\left(q, \mathbf{w}, \mathbf{x}, \upsilon\right) + e_f\left(\mathbf{w}, \mathbf{x}, \upsilon\right) + e_{me}\left(q, \mathbf{w}, \mathbf{x}\right) \equiv \widetilde{C}.$$

Then, as before, we form pairs of firms that have the same $(q, \mathbf{w}, \mathbf{x})$ and the same $\widetilde{C}$. Through these pairs, we identify the parameters $\boldsymbol{\theta}_{c0}$ using the following two restrictions in the cost function and the marginal revenue function of the logit demand model. 1) p and $\mathbf{z}$ do not enter in the cost function. 2) Given q, variation in $\mathbf{z}$ changes the market share only through the market size equation (24), not through the utility function in Equation (26).

Furthermore, for any pair of firms jm and $j^\dagger m^\dagger$ in the above set, we restrict their prices to be equal, i.e., $p_{jm} = p_{j^\dagger m^\dagger}$ as well. Then, since the two firms in the pair have the same marginal revenue, we derive

$$p_{jm} + \frac{1}{\left(1 - s^*_{jm}\right)\alpha_0} = p_{j^\dagger m^\dagger} + \frac{1}{\left(1 - s^*_{j^\dagger m^\dagger}\right)\alpha_0}.$$

It follows that within each pair, the true market shares must be equal. Thus, using Equation (25), we obtain

$$\begin{aligned} ln\left(s^*_{jm}\right) &= ln\left(q_{jm}\right) - ln\left(Q_m\right) = ln\left(q_{jm}\right) - \lambda_{c0} - \mathbf{z}_m \boldsymbol{\lambda}_{\mathbf{z}0} \\ &= ln\left(s^*_{j^\dagger m^\dagger}\right) = ln\left(q_{j^\dagger m^\dagger}\right) - \lambda_{c0} - \mathbf{z}_{m^\dagger} \boldsymbol{\lambda}_{\mathbf{z}0}, \end{aligned}$$

which results in

$$\left(\mathbf{z}_m - \mathbf{z}_{m^\dagger}\right)\boldsymbol{\lambda}_{\mathbf{z}0} = 0. \tag{27}$$

By using Equation (27), we first show that the vector $\boldsymbol{\lambda}_{\mathbf{z}0}$ is identified up to a multiplicative constant. That is, $\widehat{\boldsymbol{\lambda}}_{\mathbf{z}0} \equiv \boldsymbol{\lambda}_{\mathbf{z}0}/\lambda_{\mathbf{z}01}$ is identified. We do so by finding $k = 1, \ldots, K_{\mathbf{z}}$ pairs of firms in different markets $j^{(k)}m^{(k)}$ and $j^{\dagger(k)}m^{\dagger(k)}$ satisfying $q_{j^{(k)}m^{(k)}} = q_{j^{\dagger(k)}m^{\dagger(k)}}$, $\mathbf{w}_{m^{(k)}} = \mathbf{w}_{m^{\dagger(k)}}$, $\mathbf{x}_{j^{(k)}m^{(k)}} = \mathbf{x}_{j^{\dagger(k)}m^{\dagger(k)}}$, $p_{j^{(k)}m^{(k)}} = p_{j^{\dagger(k)}m^{\dagger(k)}}$ and $\widetilde{C}_{j^{(k)}m^{(k)}} = \widetilde{C}_{j^{\dagger(k)}m^{\dagger(k)}}$, but $\mathbf{z}_{m^{(k)}} \neq \mathbf{z}_{m^{\dagger(k)}}$. If we assume that conditional on $(\boldsymbol{\xi}_m, \boldsymbol{\upsilon}_m, \mathbf{w}_m, \mathbf{X}_m)$, the support of $\mathbf{z}_m$ is $R^{K_{\mathbf{z}}}$, then the space spanned by $\mathbf{z}_{m^{(k)}} - \mathbf{z}_{m^{\dagger(k)}}$, $k = 1, \ldots K_z$, subject to the restriction of Equation (27), has rank $K_z - 1$. Therefore, under our normalization, the equations

$$\left(\mathbf{z}_{m^{(k)}} - \mathbf{z}_{m^{\dagger(k)}}\right)\widehat{\boldsymbol{\lambda}}_{\mathbf{z}0} = 0,\ k = 1, \ldots, K_z$$

identify $\widehat{\boldsymbol{\lambda}}_{\mathbf{z}0} \equiv \boldsymbol{\lambda}_{\mathbf{z}0}/\lambda_{\mathbf{z}01}$.

Next, we focus on the identification of λ_{c0} and $\lambda_{\mathbf{z}01}$ by setting $z_{mk} = 0$ for $k = 2, \ldots K_{\mathbf{z}}$. That is, only one variable z_{m1} determines market size. We then proceed by considering two pairs of firms $k = 1, 2$ where $\mathbf{w}_{m^{(k)}} = \mathbf{w}_{m^{\dagger(k)}} = \mathbf{w}$, $\mathbf{x}_{m^{(k)}} = \mathbf{x}_{m^{\dagger(k)}} = \mathbf{x}$, $\widetilde{C}_{j^{(k)}m^{(k)}} = \widetilde{C}_{j^{\dagger(k)}m^{\dagger(k)}}$ and for small $\Delta z > 0$, $z_{m^{(k)}1} = 0$, $z_{m^{\dagger(k)}1} = \Delta z$. Output is different across the two pairs, that is, $q_{j^{(1)}m^{(1)}} = q_{j^{\dagger(1)}m^{\dagger(1)}} = q$, $q_{j^{(2)}m^{(2)}} = q_{j^{\dagger(2`)}m^{\dagger(2)}} = q'$ for $q' \neq q$. Note we do not put any restrictions on prices within the pairs. Then, these two pairs identify λ_{c0} regardless of the value of α_0. More concretely, using

$$s^*_{j^{(1)}m^{(1)}} = \frac{q}{exp\left(\lambda_{c0}\right)},\ s^*_{j^{\dagger(1)}m^{\dagger(1)}} = \frac{q}{exp\left(\lambda_{c0} + \Delta z \lambda_{\mathbf{z}01}\right)},$$

and Equation (22), we have for pair 1,

$$p_{j^{(1)}m^{(1)}} - p_{j^{\dagger(1)}m^{\dagger(1)}} = -\frac{1}{\alpha_0}\left[\frac{q}{exp\left(\lambda_{c0}\right) - q} - \frac{q}{exp\left(\lambda_{c0} + \Delta z \lambda_{\mathbf{z}01}\right) - q}\right] \equiv \Delta p\left(q, 0, \Delta z\right).$$

Note that

$$\frac{q}{exp\left(\lambda_{c0} + \Delta z \lambda_{\mathbf{z}01}\right) - q} \approx \frac{q}{exp\left(\lambda_{c0}\right) - q} - \frac{q}{\left(exp\left(\lambda_{c0}\right) - q\right)^2}\left[exp\left(\lambda_{c0} + \Delta z \lambda_{\mathbf{z}01}\right) - exp\left(\lambda_{c0}\right)\right],$$

and since we can find q such that $q \neq exp\left(\lambda_{c0}\right)$,

$$\Delta p\left(q, 0, \Delta z\right) = p_{j^{(1)}m^{(1)}} - p_{j^{\dagger(1)}m^{\dagger(1)}} \approx -\frac{1}{\alpha_0}\frac{q}{\left(exp\left(\lambda_{c0}\right) - q\right)^2} exp\left(\lambda_{c0}\right)\Delta z \lambda_{\mathbf{z}01} \neq 0,$$

holds and is bounded.

Next, we do the same with the second pair with $q' \neq q$ where $q' \neq exp\left(\lambda_{c0}\right)$ as well. Letting $B\left(q, q', 0, \Delta z\right) \equiv \Delta p\left(q, 0, \Delta z\right)/\Delta p\left(q', 0, \Delta z\right)$, we have

$$B\left(q, q', 0, \Delta z\right) \equiv \frac{\Delta p\left(q, 0, \Delta z\right)}{\Delta p\left(q', 0, \Delta z\right)} \approx \frac{q}{q'}\left[\frac{exp\left(\lambda_{c0}\right) - q'}{exp\left(\lambda_{c0}\right) - q}\right]^2 = \frac{q}{q'}\left[1 + \frac{q - q'}{exp\left(\lambda_{c0}\right) - q}\right]^2,$$

which identifies λ_{c0}. Then, to identify $\lambda_{\mathbf{z}01}$, we do the same with two new pairs having $z_{m1} = z$, $z_{m^{\dagger}1} = z + \Delta z$, and everything else defined in the same manner as for the first two pairs, and, given λ_{c0}, we do similar calculations as before to derive

$$B\left(q, q', z, \Delta z\right) \equiv \frac{\Delta p\left(q, z, \Delta z\right)}{\Delta p\left(q', z, \Delta z\right)} \approx \frac{q}{q'}\left[\frac{exp\left(\lambda_{c0} + z\lambda_{\mathbf{z}01}\right) - q'}{exp\left(\lambda_{c0} + z\lambda_{\mathbf{z}01}\right) - q}\right]^2 = \frac{q}{q'}\left[1 + \frac{q - q'}{exp\left(\lambda_{c0} + z\lambda_{\mathbf{z}01}\right) - q}\right]^2.$$

Since λ_{c0} is already identified, the above equation identifies $\lambda_{\mathbf{z}01}$.

Later, we also show that in our Monte-Carlo results, even when market size is unobserved, we can identify the price coefficients and the parameters of the market size equation well.

4 Estimation

In practice, an estimator that directly applies the parametric identification results in Subsection 3.2 will likely suffer from a Curse of Dimensionality. To implement such an estimator, one would need to obtain the deterministic component of cost by deriving the nonparametric estimate of the conditional mean cost as below:

$$\widetilde{C}_{jm} = E\left[C \middle| \widetilde{q} = q_{jm}, \widetilde{\mathbf{w}} = \mathbf{w}_m, \widetilde{\mathbf{p}} = \mathbf{p}_m, \widetilde{\mathbf{s}} = \mathbf{s}_m, \widetilde{\mathbf{X}} = \mathbf{X}_m, j\right].$$

Furthermore, we need to find pairs with $q_{jm} \approx q_{j^\dagger m^\dagger}$, $\mathbf{x}_{jm} \approx \mathbf{x}_{j^\dagger m^\dagger}$, $\mathbf{w}_m \approx \mathbf{w}_{m^\dagger}$ and $\widetilde{C}_{jm} \approx \widetilde{C}_{j^\dagger m^\dagger}$. For most markets of interest, $\mathbf{X}_m$ will contain some product characteristics across a non-negligible number of firms. This makes the dimensionality problem potentially quite severe.

Because of this dimensionality issue, we construct an estimator that exploits the parametric marginal revenue in such a way that the calculation of conditional mean cost is no longer required. This estimator conditions on marginal revenue, which is a parametric function of the observables, rather than the conditional expected cost.

We propose to embed the estimation of demand parameters in the estimation of the deterministic component of cost $\widetilde{C}_{jm}$. To overcome the problem of a possible correlation between the cost shock υ_{jm} and output q_{jm}, we argue below that given q_{jm}, $\mathbf{w}_m$, and $\mathbf{x}_{jm}$, we can use marginal revenue $MR_j(\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m, \boldsymbol{\theta}_c)$ to control for υ_{jm} as long as the demand parameter vector $\boldsymbol{\theta}_c$ equals the vector of true values $\boldsymbol{\theta}_{c0}$. The lemma below formalizes this control function idea.

Lemma 4 *Suppose that Assumptions 3, 5-8 are satisfied. Then,* $\widetilde{C}_{jm} = \varphi(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_{c0}))$ *for firm j in market m with observables* $\{\mathbf{p}_m, \mathbf{s}_m, \mathbf{X}_m, q_{jm}, \mathbf{w}_m, j\}$, *where φ is a function that is strictly increasing and continuous in marginal revenue.*

Proof. By Assumption 7, we can invert the the marginal cost function with respect to to cost shock υ such that $\upsilon = \upsilon(q, \mathbf{w}, \mathbf{x}, MC)$. We can use this function to control for υ. Then, the deterministic component of cost becomes

$$\widetilde{C} = C^v(q, \mathbf{w}, \mathbf{x}, \upsilon) + e_f(\mathbf{w}, \mathbf{x}, \upsilon) + e_{me}(q, \mathbf{w}, \mathbf{x}) = \varphi(q, \mathbf{w}, \mathbf{x}, MC),$$

where φ is an increasing and continuous function of MC by Assumptions 7 and 8. Because Equation (15) holds, i.e., $MR = MC$, at the true parameter vector $\boldsymbol{\theta}_{c0}$,

$$\widetilde{C} = \varphi(q, \mathbf{w}, \mathbf{x}, MC) = \varphi(q, \mathbf{w}, \mathbf{x}, MR) \tag{28}$$

and the claim holds. ■

We call the function $\varphi(q, \mathbf{w}, \mathbf{x}, MR)$ the pseudo-cost function. Notice that in this proof, we invert the marginal cost function in Equation (15) rather than the cost function in Equation (16) in Subsection 3.2 to derive υ. That is, we exploit the first order condition at the true parameter vector, and thereby use marginal revenue to control for the cost shock. By doing so, we avoid the Curse of Dimensionality in estimation mentioned above in using $\widetilde{C}$ to control for υ. An equivalent way to explain what we are doing is that we invert the first order condition Equation (18), which is the basis of our identification strategy, with respect to MR and $\widetilde{C}$, given q,w and x. Because Definition 1 guarantees invertibility at the true parameter $\boldsymbol{\theta}_0$, the pseudo-cost function is derived from our identification definition.

4.1 Two-step Sieve Non-linear least squares (SNLLS) estimator

Using the above lemma, we construct an estimator that is based on the control function approach. In the first step, it selects parameters $\boldsymbol{\theta}_c$ to fit the pseudo-cost function to the cost data using a nonparametric sieve regression (see Chen (2007) and Bierens (2014)). The following assumption formalizes this:

Assumption 9 *φ can be expressed as a linear function of an infinite sequence of polynomials.*

$$\varphi\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right)\right) = \sum_{l=1}^{\infty} \gamma_{l0} \psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right)\right), \tag{29}$$

where $\psi_1(\cdot), \psi_2(\cdot), \ldots$ are the basis functions for the sieve and $\gamma_1, \gamma_2, \ldots$ is a sequence of their coefficients, satisfying $\sum_{l=1}^{\infty} |\gamma_{l0}| < \infty$.[17]

Our estimator is based on Equation (29). It is useful to introduce some additional notation before formally defining the estimator and its sample analog. Let M be the number of markets in the sample, and L_M an integer that increases with M. For some bounded but sufficiently large constant $T > 0$, let $\Gamma_k(T) = \{\pi_k \boldsymbol{\gamma} : \|\pi_k \boldsymbol{\gamma}\| \leq T\}$ where π_k is the operator that applies to an infinite sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=1}^{\infty}$, replacing γ_n, $n > k$ with zeros. That is, for $n \leq k$, $\pi_k \gamma_n = \gamma_n$, and for $n > k$, $\pi_k \gamma_n = 0$. The norm $\|\mathbf{x}\|$ is defined as $\|\mathbf{x}\| = \sqrt{\sum_{k=1}^{\infty} x_k^2}$.

We now present our main result on estimation. The proof is in the appendix.

[17] Suppose the vector $(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm})$ belongs to a compact finite-dimensional Euclidean space, $\mathcal{W}$. Then, if $\varphi(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm})$ is a continuous function on $\mathcal{W}$, from the Stone-Weierstrass Theorem, it follows that the function can be approximated arbitrarily well by a polynomial function of a sufficiently higher order. However, the polynomials may not converge absolutely, so we still need to assume absolute convergence.

Proposition 2 *Suppose Assumptions 1-9 are satisfied. Then*

$$[\boldsymbol{\theta}_{c0}, \boldsymbol{\gamma}_0] = argmin_{(\boldsymbol{\theta},\gamma)\in\Theta_c\times\Gamma} E\left[C_{jm} - \sum_{l=1}^{\infty}\gamma_l\psi_l\Big(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c)\Big)\right]^2, \tag{30}$$

where $\Gamma = lim_{M\to\infty}\Gamma_{L_M}(T)$*; and equation (30) identifies* $\boldsymbol{\theta}_{c0}$*.*

Note that we do not require the sieve function $\sum_{l=1}^{\infty}\gamma_l\psi_l(q, \mathbf{w}, \mathbf{x}, MR_{jm}(\boldsymbol{\theta}_c))$ to be one-to-one with respect to $MR_{jm}(\boldsymbol{\theta}_c)$ given $(q, \mathbf{w}, \mathbf{x})$. As shown in the proof, we rely on the functional form of the BLP demand to do so.

Our SNLLS (Sieve-NLLS) approach deals with issues of endogeneity by adopting a control function approach for the unobserved cost shock υ_{jm}. With our estimator, the right-hand side of Equation (30) is minimized only when parameters are at their true value $\boldsymbol{\theta}_{c0}$ so that the computed marginal revenue equals the true marginal revenue, and thus works as a control function for the supply shock υ_{jm}. If $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$, then using the false marginal revenue adds noise, which increases the the sum of squared residuals in Equation (30). This can be seen from the following: because ν_{jm} and ς_{jm} are independent to the other observed variables,

$$\begin{aligned}
& E\left[\left(C_{jm} - \sum_{l=1}^{\infty}\gamma_l\psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c)\right)\right)^2\right] \\
= \; & E\left[\left(C_{jm} - \widetilde{C}_{jm} + \widetilde{C}_{jm} - \sum_{l=1}^{\infty}\gamma_l\psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c)\right)\right)^2\right] \\
\geq \; & E\left[(\nu_{jm} + \varsigma_{jm})^2\right] = \sigma_\nu^2 + \sigma_\varsigma^2,
\end{aligned} \tag{31}$$

Equation (31) holds with equality if and only if $\boldsymbol{\theta}_c = \boldsymbol{\theta}_{c0}$. This is because of the definition of the pseudo-cost function in Equation (28). Thus, the true demand parameter $\boldsymbol{\theta}_{c0}$ can be obtained as a by-product of this control function approach.[18]

We let the sample of M markets be the M random draw of the population, and denote market m to be the mth random draw from the population. The sample analog of Equation (30), given a sample of M markets is:

$$\left[\widehat{\boldsymbol{\theta}}_{cM}, \widehat{\boldsymbol{\gamma}}_M\right] = argmin_{(\boldsymbol{\theta}_c,\gamma)\in\Theta_c\times\Gamma_{L_M}(T)} \frac{1}{\sum_{m=1}^{M} J_m}\sum_{m=1}^{M}\sum_{j=1}^{J_m}\left[C_{jm} - \sum_{l=1}^{L_M}\gamma_l\psi_l\Big(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c)\Big)\right]^2. \tag{32}$$

[18] After estimating the marginal revenue function, we can recover the cost function. The details are discussed in the appendix.

The set $\Gamma_{L_M}(T)$ makes clear the fact that the complexity of the sieve is increasing in the sample's number of markets.

In the actual estimation exercise, the objective function can be constructed in the following 2 steps.

Step 1: Given a candidate parameter vector $\boldsymbol{\theta}_c$, derive the marginal revenue $MR_{jm}(\boldsymbol{\theta}_c)$ for each j, m, $j = 1, ..., J_m$, $m = 1, ..., M$.

Step 2: Derive the estimates of $\widehat{\gamma}_l$, $l = 1, ..., L_M$ by OLS, where the dependent variable is C_{jm} and the RHS variables are $\psi_l(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c))$, $l = 1, ..., L_M$. Then, construct the objective function, which is the average of squared residuals

$$Q_M(\boldsymbol{\theta}_c) = \frac{1}{\sum_{m=1}^{M} J_m} \sum_{m=1}^{M} \sum_{j=1}^{J_m} \left[C_{jm} - \sum_{l=1}^{L_M} \widehat{\gamma}_l \psi_l(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c)) \right]^2 .$$

We choose $\boldsymbol{\theta}_c$ that minimizes the objective function $Q_M(\boldsymbol{\theta}_c)$. In sum, we search for the price parameters in an outer loop and find the best fitting cost function on an inner loop for each candidate set of demand parameters. We use the Newton search algorithm to find the solution.

Note that practitioners need to be careful in the nonparametric estimation of the pseudo-cost function if $\mathbf{x}_{jm}$ includes discrete variables, just like in any cases where nonparametric estimation involves both discrete and continuous variables.

In the second step, to identify $\boldsymbol{\beta}$ for the logit model and $\boldsymbol{\mu}_{\boldsymbol{\beta}}$ for BLP, we include additional moment conditions in our estimator that leverage the (common) assumption that $E[\boldsymbol{\xi}_{jm}|\mathbf{x}_{jm}] = 0$. Then, after obtaining $\widehat{\boldsymbol{\theta}}_{cM}$, we can recover $\widehat{\boldsymbol{\delta}}_M$ by inversion, and we can simply estimate $\widehat{\boldsymbol{\beta}}_M$ for logit or $\widehat{\boldsymbol{\mu}}_{\boldsymbol{\beta}M}$ for BLP simply by OLS as follows.

$$\widehat{\boldsymbol{\beta}}_M = \left(\sum_{m=1}^{M} \mathbf{X}_m' \mathbf{X}_m \right)^{-1} \sum_{m=1}^{M} \mathbf{X}_m' \left(\widehat{\boldsymbol{\delta}}_m - \mathbf{p}_m \widehat{\alpha}_M \right) \ \text{or} \ \widehat{\boldsymbol{\mu}}_{\beta M} = \left(\sum_{m=1}^{M} \mathbf{X}_m' \mathbf{X}_m \right)^{-1} \sum_{m=1}^{M} \mathbf{X}_m' \left(\widehat{\boldsymbol{\delta}}_m - \mathbf{p}_m \widehat{\mu}_{\alpha M} \right) \tag{33}$$

Equations (32) and (33) constitute our two-step SNLLS estimator for parameters $\boldsymbol{\theta} = (\alpha, \boldsymbol{\beta})$ for logit demand and $\boldsymbol{\theta} = (\mu_\alpha, \sigma_\alpha, \boldsymbol{\mu}_{\boldsymbol{\beta}}, \boldsymbol{\sigma}_{\boldsymbol{\beta}})$ for BLP demand.[19] After estimating the demand

[19] Note that if the exclusion restriction is not met, we can still obtain consistent parameter estimate of $\boldsymbol{\beta}$ (or $\boldsymbol{\mu}_{\boldsymbol{\beta}}$) if we assume that the observed characteristics of other products $\mathbf{X}_{-jm}$ and ξ_{jm} are uncorrelated. Then, we can use $\mathbf{X}_{-jm}$ as instruments for $\mathbf{x}_{jm}$. In contrast, the literature uses these variables as instruments for both p_{jm} and $\mathbf{x}_{jm}$. Berry et al. (1995) use the sum of product characteristics over other firms as instruments for p_{jm}. In that case, if no other instruments are available, only functional form restrictions identify the coefficients of p_{jm} and $\mathbf{x}_{jm}$. It is also important to recall that even if $\boldsymbol{\beta}$ (or $\boldsymbol{\mu}_{\boldsymbol{\beta}}$) cannot be consistently estimated, in our procedure $\boldsymbol{\theta}_c$ is still estimated consistently, and so are marginal revenue and profit margin.

parameters and the semiparametric pseudo-cost function, we can recover the nonparametric cost function as well. We present the details of this in the appendix.

4.2 Further specification and data issues

We have thus far worked with the standard differentiated products model of Berry (1994) and BLP. Depending on the empirical context, however, a number of specification and data-related issues can potentially arise. In this subsection, we list some empirical settings in which our estimator can be adapted by modifying the SNLLS part of the objective function in Equation (32). The details are in the appendix. They are:

1. *Economic versus accounting cost*: With only minor modifications to our estimation procedure, we can consistently estimate the parameters even if the cost data in accounting statements do not reflect the economic cost.

2. *Endogenous product characteristics*: We can deal with the case where firms also choose product characteristics by including the additional first order conditions in our estimator.

3. *Cost function restrictions*: We can incorporate the restriction that the cost function satisfies homogeneity of degree one in input price. Incorporating such a restriction in the estimation procedure has the benefit of reducing the dimensionality of the nonparametric pseudo-cost function.

4. *Missing cost data*: Because the SNLLS part of our estimator does not involve any orthogonality conditions, and because the random components of the measurement error of cost and fixed cost are assumed to be i.i.d, choosing only those firms for which cost data is available will not result in selection bias in estimation. It is important to notice, however, that we still need demand-side data for all firms in the same market to compute marginal revenue.

5. *Multi product firms*: Even though firms produce multiple products, in most accounting statements, only the total cost of all products is reported. In such a case, with logit or BLP functional form restrictions on market share functions and additional reasonable restrictions on the cost function, we can still estimate the parameters of the model.

4.3 Large sample properties

In the appendix, we prove consistency and asymptotic normality of our estimator. These proofs are based on the asymptotic analysis of sieve estimators by Bierens (2014).

4.4 Bootstrap procedure for calculating the standard errors

In this section, we propose a bootstrap procedure for deriving the standard errors of $\widehat{\boldsymbol{\theta}}_{cM}$. In equilibrium models, bootstrapping by resampling the demand shocks ξ_{jm} and supply shocks υ_{jm} is computationally demanding because the equilibrium prices $\mathbf{p}_m$ and the market shares $\mathbf{s}_m$ need to be recomputed for each market m. Instead, we could follow Fu and Wolpin (2018) and others and conduct nonparametric bootstrap where we would resample market outcomes $(\mathbf{X}_m, \mathbf{p}_m, \mathbf{s}_m, \mathbf{w}_m, \mathbf{q}_m, \mathbf{C}_m)$, $m = 1, \ldots, M$ and estimate based on the resampled market data. Then, one does not need to recompute the equilibrium prices and market shares. However, the results may be subject to small sample issues due to the relatively small number of markets. Furthermore, the bootstrapped parameter estimates would likely be affected by the additional variation from the resampled $\mathbf{X}_m$ and $\mathbf{w}_m$ as well, which could overestimate the standard errors. In addition, if the demand and supply shocks, $\mathbf{X}_m$ and $\mathbf{w}_m$ are correlated across markets, then this correlation needs to be dealt with in resampling.

In our bootstrap, we instead resample $\nu_{jm} + \varsigma_{jm}$ to reconstruct the cost data and then, reestimate the parameters. The procedure is valid since we assume that $\nu_{jm} + \varsigma_{jm}$ is independent of other variables, which we leave unchanged. We describe the procedure below.

Step 1 Estimate the parameters $\widehat{\boldsymbol{\theta}}_{cM}^{(1)}$ and $\widehat{\boldsymbol{\gamma}}_M^{(1)}$ using $C_{jm}, \mathbf{x}_{jm}, s_{jm}, p_{jm}, q_{jm}, \mathbf{w}_m$, $j = 1, \ldots, J_m$, $m = 1, \ldots, M$.

Step 2 Derive the residuals

$$\left(\widehat{\nu + \varsigma}\right)_{jm} = C_{jm} - \sum_{l=1}^{L_M} \widehat{\gamma}_{lM}^{(1)} \psi_l \left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\widehat{\boldsymbol{\theta}}_{cM}^{(1)}\right)\right).$$

Step 3 Resample with replacement from $\left\{\left(\widehat{\nu + \varsigma}\right)_{jm}, j = 1, \ldots, J_m, m = 1, \ldots, M\right\}$ to generate $\left\{\left(\widetilde{\nu + \varsigma}\right)_{jm}, j = 1, \ldots, J_m, m = 1, \ldots, M\right\}$.

Step 4 Generate the bootstrapped cost

$$\widehat{C}_{jm} = \sum_{l=1}^{L_M} \widehat{\gamma}_{lM}^{(1)} \psi_l \left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\widehat{\boldsymbol{\theta}}_{cM}^{(1)}\right)\right) + \left(\widetilde{\nu + \varsigma}\right)_{jm}.$$

Step 5 Go back to Step 1 with $\widehat{C}_{jm}$ instead of C_{jm}, and reestimate to derive $\widehat{\boldsymbol{\theta}}_{cM}^{(2)}$, $\widehat{\boldsymbol{\gamma}}_M^{(2)}$ using $\widehat{C}_{jm}, \mathbf{x}_{jm}, s_{jm}, p_{jm}, q_{jm}, \mathbf{w}_m$, $j = 1, \ldots, J_m$, $m = 1, \ldots, M$.

Repeat the above steps $M_B - 1$ times to derive $\boldsymbol{\theta}_{cM}^{(l_B)}$, $l_B = 1, \ldots, M_B$ and report standard errors from the M_B bootstrapped parameter estimates.

5 Monte-Carlo experiments

This section presents results from a series of Monte-Carlo experiments that highlight the finite sample performance of our estimator. To generate samples, we use the following random coefficients logit demand model:

$$s_{jm}(\boldsymbol{\theta}) = \int_\alpha \int_\beta \frac{\exp(\mathbf{x}_{jm}\beta + p_{jm}\alpha + \xi_{jm})}{\sum_{j=0}^{J_m} \exp(\mathbf{x}_{jm}\beta + p_{jm}\alpha + \xi_{jm})} \frac{1}{\sigma_\alpha}\phi\left(\frac{\alpha - \mu_\alpha}{\sigma_\alpha}\right) \frac{1}{\sigma_\beta}\phi\left(\frac{\beta - \mu_\beta}{\sigma_\beta}\right) d\alpha d\beta, \quad (34)$$

where we set the number of product characteristics K to be 1, and $\phi()$ to be the density of the standard normal distribution. We assume that each market has four firms, each producing one product (e.g., $J_m = J = 4$). Hence consumers in each market have a choice of $j = 1, \ldots, 4$ differentiated products or not purchasing any of them ($j = 0$).

On the supply-side, we assume firms compete on prices a la differentiated products Bertrand competition, use labor and capital inputs in production and have a Cobb-Douglas production function. Given output, input prices $\mathbf{w} = [w, r]'$ (w is the wage and r is the rental rate of capital), total cost and marginal cost functions are specified as[20]

$$C(q, w, r, x, \upsilon) = x\left[\frac{w^{\alpha_c} r^{\beta_c}}{B}\left(\left(\frac{\beta_c}{\alpha_c}\right)^{\alpha_c} + \left(\frac{\alpha_c}{\beta_c}\right)^{\beta_c}\right)\upsilon q\right]^{\frac{1}{\alpha_c+\beta_c}}$$

$$MC(q, w, r, x, \upsilon) = x\left[\frac{w^{\alpha_c} r^{\beta_c}}{B}\left(\left(\frac{\beta_c}{\alpha_c}\right)^{\alpha_c} + \left(\frac{\alpha_c}{\beta_c}\right)^{\beta_c}\right)\upsilon\right]^{\frac{1}{\alpha_c+\beta_c}} \frac{1}{\alpha_c + \beta_c} q^{\frac{1}{\alpha_c+\beta_c}-1}.$$

Notice that in the above specification, the cost function is homogeneous of degree one in input prices.[21]

To create our Monte-Carlo samples, we generate wage, rental rate, variable cost shock, market

[20] In our Monte-Carlo, we assume away the deterministic components of the fixed cost and the measurement error.

[21] The cost function given the Cobb-Douglas production technology is defined as

$$C(q, w, r, x, \upsilon) = argmin_{L,K} wL + rK \quad subject\ to\ q = B\upsilon^{-1} L^{\alpha_c} K^{\beta_c}/x.$$

size Q_m, and observable product characteristics x_{jm} as follows:

$$w_m \sim i.i.d.TN\left(\mu_w, \sigma_w\right), \quad e.g., w_m = \mu_w + \sigma_w \varrho_{wm}, \quad \varrho_{wm} \sim i.i.d.TN\left(0,1\right).$$

$$r_m \sim i.i.d.TN\left(\mu_r, \sigma_r\right), \quad e.g., r_m = \mu_r + \sigma_r \varrho_{rm}, \quad \varrho_{rm} \sim i.i.d.TN\left(0,1\right).$$

$$Q_m \sim i.i.d.U\left(Q_L, Q_H\right).$$

$$x_{jm} \sim i.i.d.TN\left(\mu_x, \sigma_x\right), \quad e.g., x_{jm} = \mu_x + \sigma_x \varrho_{xjm}, \quad \varrho_{xjm} \sim i.i.d.TN\left(0,1\right).$$

$TN\left(0,1\right)$ is the truncated standard normal distribution, where we truncate both upper and lower 0.82 percentiles. $U\left(Q_L, Q_H\right)$ is the uniform distribution with lower bound of Q_L and upper bound of Q_H. Furthermore, we specify the variable cost shock as follows:

$$\upsilon_{jm} = \mu_\upsilon + \sigma_\upsilon \varrho_{\upsilon jm} + \zeta_Q \Phi^{-1}\left(\delta + (1.0 - 2\delta)\frac{Q_m - Q_L}{Q_H - Q_L}\right), \quad \varrho_{\upsilon jm} \sim i.i.d.TN\left(0,1\right).$$

For transforming the uniformly distributed market size shock to truncated normal distribution, we use small positive $\delta = 0.025$ for truncation. We truncate the distribution of the shocks to ensure that the true cost function is positive and bounded given the parameter values of the cost function we set (which will be discussed later). We let the cost shock υ_{jm} be positively correlated with the market size shock, i.e., we set ζ_Q to be 0.2.

Importantly, we specify the unobserved characteristics so as to allow for correlation between ξ_{jm} and input prices, the cost shock, market size and the observed characteristics of the products other than j in market m denoted by $xo_{jm} \equiv (1/3)\sum_{l \neq j} \varrho_{xlm}$. Specifically, we set:

$$\xi_{jm} = \delta_0 + \delta_\xi \varrho_{\xi jm} + \delta_w \varrho_{wm} + \delta_r \varrho_{rm} + \delta_\upsilon \varrho_{\upsilon jm} + \delta_Q \Phi^{-1}\left(\delta + (1.0 - 2\delta)\frac{Q_m - Q_L}{Q_H - Q_L}\right) + \delta_{xo} xo_{jm},$$

where ϱ_ξ is the idiosyncratic component of the demand shock. We set $\delta_l = \frac{1}{2\sqrt{6}}$ for $l \in \{\xi, w, r, \upsilon, Q, xo\}$.

By construction, neither input prices, nor observed characteristics of other products can be used as valid instruments for prices in demand estimation. Furthermore, since both demand and variable cost shocks are correlated with market size, one cannot use the variation of market size as an instrument for prices, or for output in the cost function estimation discussed in Subsection 2.2. We let the sum of their random terms be distributed $TN\left(0, \sqrt{Var\left(\nu + \varsigma\right)}\right)$ where $\sqrt{Var\left(\nu + \varsigma\right)} = \sqrt{\sigma_\nu^2 + \sigma_\varsigma^2} = 0.2$.

To solve for the equilibrium price, quantity, and market share for each oligopoly firm, we use the golden section search on price.[22]

Table 1 summarizes the parameter setup of the Monte-Carlo experiments. Table 2 presents sample statistics from the simulated data of 1600 market-firm observations (there are 400 local markets). Note that $\sigma_{\nu+\varsigma}$ is about seven percent of the total cost. The parameter estimates of $\boldsymbol{\theta}_c = (\mu_\alpha, \sigma_\alpha, \sigma_\beta)$ are obtained by the following minimization algorithm:

$$\left[\widehat{\boldsymbol{\theta}}_M, \widehat{\boldsymbol{\gamma}}_M\right] = argmin_{(\boldsymbol{\theta}_c,\gamma)\in\Theta_c\times\Gamma_{k_M}(T)}\Big[\frac{1}{\sum_{m=1}^{M} J_m}\sum_{jm}\left[\frac{C_{jm}}{r_m} - \sum_l \gamma_l\psi_l\left(q_{jm}, \frac{w_m}{r_m}, \mathbf{x}_{jm}, \frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)}{r_m}\right)\right]^2.$$

In this pseudo-cost function, we exploit the homogeneity of degree one property of the cost function. For a detailed discussion, see the appendix. We then recover $\boldsymbol{\delta}$ by inversion and in the 2nd stage, we estimate the parameter μ_β as follows:

$$\widehat{\mu}_{\beta M} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\left(\widehat{\boldsymbol{\delta}}_M - \mathbf{p}\widehat{\mu}_{\alpha M}\right).$$

In Table 3, we present the Monte-Carlo results for our two-step estimator. We report the mean, standard deviation, and square root of the mean squared errors (RMSE) of the parameter estimates from 100 Monte-Carlo simulation/estimation replications. From the table, we see that as sample size increases, the standard deviation and the RMSE of the parameter estimates decrease. The results highlight the consistency of our estimator. It is noteworthy that means of the estimates are quite close to their true values even with a small sample size of 200. Furthermore, since the estimated parameter values are close to their true values, the standard deviations and RMSEs are close to each other as well. Overall, these Monte-Carlo results demonstrate the validity of our approach.[23] In Table 4, we present the results where we allow for the observed characteristics x to be correlated with the unobserved characteristics ξ. That is,

$$\xi_{jm} = \delta_0+\delta_\xi\varrho_{\xi jm}+\delta_w\varrho_{wm}+\delta_r\varrho_{rm}+\delta_\upsilon\varrho_{\upsilon jm}+\delta_Q\Phi^{-1}\left(\delta + (1.0-2\delta)\frac{Q_m - Q_L}{Q_H - Q_L}\right)+\delta_{xo}xo_{jm}+\delta_x\varrho_{xjm}, \tag{35}$$

where $\delta_0 = 4.0$, $\delta_\xi = \delta_w = \delta_r = \delta_\upsilon = \delta_Q = \delta_{xo} = \delta_x = 1/\left(2\sqrt{7}\right)$. Now, by construction, no observed variable of the firm can be used as a valid instrument for its prices in demand estimation. We can see in Table 4 that the parameter vector $\boldsymbol{\theta}_c$ is consistently estimated. On

[22] The algorithm for finding equilibria in oligopoly markets is available upon request.

[23] Results with $\sigma_{\nu+\varsigma}$ larger than 0.2 are similar to the ones presented, but with larger standard deviations and RMSEs.

Table 1: Monte Carlo Parameter Values

Parameter	Description	Value
(a) Demand-side parameters		
μ_α	Price coef. mean	2.0
σ_α	Price coef. std. dev.	0.5
μ_β	Product characteristic coef. mean	1.0
σ_β	Product characteristic coef. std. dev.	0.2
μ_X	Product characteristic mean	3.0
σ_X	Product characteristic std. dev.	1.0
δ_0	Unobserved product quality mean	4.0
δ_ξ	Unobserved product quality std. dev.	0.5
Q_L	Lower bound on market size	5.0
Q_H	Upper bound on market size	10.0
(b) Supply-side parameters		
α_c	Labor coef. in Cobb-Douglas prod. fun.	0.4
β_c	Capital coef. in Cobb-Douglas prod. fun.	0.4
μ_w	Wage mean	1.0
σ_w	Wage std. dev.	0.2
μ_r	Rental rate mean	1.0
σ_r	Rental rate std. dev.	0.2
μ_v	Cost shock mean	0.3
σ_v	Cost shock std. dev.	0.1
J	Number of firms in each market	4
B	Scaling factor for output in the cost function	1.0
(c) Cost measurement error		
$\sigma_{\nu+\varsigma}$	Measurement std. dev.	0.2
(d) Correlation parameters with unobservables ξ_{jm} and v_{jm}		
δ_{xo}	ξ_{jm} and X_{-jm} correlation	$1/(2\sqrt{6})$
δ_w	ξ_{jm} and w_m correlation	$1/(2\sqrt{6})$
δ_r	ξ_{jm} and r_m correlation	$1/(2\sqrt{6})$
δ_v	ξ_{jm} and v_{jm} correlation	$1/(2\sqrt{6})$
δ_Q	ξ_{jm} and Q_m correlation	$1/(2\sqrt{6})$
ζ_Q	v_{jm} and Q_m correlation	$1/(2\sqrt{6})$

the other hand, μ_β is estimated to be around 1.2, much higher than the true coefficient 1.0. The upward bias is due to the positive correlation between the demand shock ξ_{jm} and the random term of the observed characteristics ϱ_{xjm} as specified in Equation (35). However, since rest of the parameters are estimated consistently, the markup of the firms can still be recovered consistently.

In Table 5, we report the results where we set the variation in market size to be zero. We can see that overall, means of the parameter estimates become closer to the true values, and the standard deviations and RMSEs become smaller as sample size increases. In the Monte-Carlo results with the sample size of 200, one out of a hundred simulation/estimation exercises did not converge, so we removed it, and took the sample statistics over 99 parameter estimates. By comparing the results in Table 3, we can see that the standard deviations and the RMSEs are higher than the ones where we had variation in market size. We conclude that even though the variation in market size is not needed, it helps in improving the accuracy of the estimators.

Table 2: Sample Statistics from Simulated Data

Variable	Description	Mean	Std. Dev.
p_m	Price	4.104	1.239
x_{jm}	Product characteristic	1.704	0.465
ξ_m	Unobserved product quality	4.008	0.436
s_{jm}	Market share	0.191	0.091
q_{jm}	Output	1.395	0.662
C_{jm}	Total cost	2.814	1.005
w_m	Wage	1.007	0.183
r_m	Rental Rate	1.001	0.195

Notes: Sample statistics from simulated data from a Monte Carlo sample with 400 markets, $J = 4$ firms per market, and 1600 observations.

Table 3: SNLLS Estimator of Random Coefficient Demand Parameters
(Product Characteristic x_{jm} and Unobserved Product Quality ξ_{jm} Uncorrelated)

			(a) Price coefficients parameters						
			$\hat{\mu}_\alpha$			$\hat{\sigma}_\alpha$			
Markets	Sample Size	Polynomials	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	CPU Minutes
50	200	144	-2.178	0.823	0.838	0.455	0.215	0.218	274.0
100	400	171	-2.195	0.539	0.571	0.547	0.178	0.183	788.5
200	800	204	-2.004	0.184	0.183	0.506	0.094	0.093	1795.4
400	1600	256	-2.020	0.124	0.125	0.504	0.053	0.053	2278.4
True Value			-2.000			0.500			
			(b) Product characteristic coefficients parameters						
			$\hat{\mu}_\beta$			$\hat{\sigma}_\beta$			
Markets	Sample Size	Polynomials	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	Obj. Fun.
50	200	144	1.130	0.586	0.598	0.258	0.284	0.289	1.392D-2
100	400	171	1.069	0.272	0.280	0.265	0.213	0.222	2.294D-2
200	800	204	0.988	0.118	0.118	0.204	0.085	0.084	2.944D-2
400	1600	256	1.007	0.086	0.086	0.207	0.056	0.056	3.365D-2
True Value			1.000			0.200			

Notes: Monte-carlo experiment results based on calibration described in panels (a)-(d) of Table 1. CPU minutes is the average estimation time in minutes across the Monte Carlo simulations. Measurement error in cost data has a standard deviation of $\sigma_{\nu+\varsigma} = 0.2$ which is approximately seven percent of mean total cost.

Next, we consider the case where market size is not observable, and needs to be estimated. We specify market size as follows:

$$Q^*_m = \lambda_0 + \lambda_1 z_m,$$

where Q^*_m is the unobserved market size, and we set $z_m = Q_m$, and $\lambda_0 = 0$, $\lambda_1 = 1$. Then, the true market share vector is $\mathbf{s}_m = \mathbf{q}_m / Q^*_m$. We keep the BLP market share equation as specified in Equation (34) except that market size is unobservable and therefore, parameters λ_0 and λ_1 need to be jointly estimated. Note that market shares $\mathbf{s}_m$ are unobservable as well. In Table 6, panel (a), we present the statistics of the parameter estimates that were generated from 100 repeated simulation/estimation exercises, based on the model used in Table 3 with unobservable

Table 4: SNLLS Estimator of Random Coefficient Demand Parameters
(Product Characteristic x_{jm} and Unobserved Product Quality ξ_{jm} Correlated)

			(a) Price coefficients parameters						
			$\hat{\mu}_\alpha$			$\hat{\sigma}_\alpha$			
Markets	Sample Size	Polynomials	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	CPU Minutes
50	200	144	-2.204	0.954	0.971	0.508	0.196	0.195	762.2
100	400	171	-2.071	0.272	0.279	0.517	0.117	0.118	1526.7
200	800	204	-2.012	0.150	0.150	0.500	0.066	0.065	3189.4
400	1600	256	-2.006	0.088	0.088	0.502	0.034	0.034	5770.0
True Value			-2.000			0.500			

			(b) Product characteristic coefficients parameters						
			$\hat{\mu}_\beta$			$\hat{\sigma}_\beta$			
Markets	Sample Size	Polynomials	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	Obj. Fun.
50	200	144	1.315	0.661	0.729	0.231	0.256	0.256	1.392D-2
100	400	171	1.222	0.158	0.272	0.213	0.111	0.111	2.320D-2
200	800	204	1.197	0.090	0.216	0.202	0.078	0.078	2.953D-2
400	1600	256	1.190	0.056	0.198	0.198	0.052	0.052	3.375D-2
True Value			1.000			0.200			

Notes: Monte-carlo experiment results based on calibration described in panels (a)-(c) of Table 1 with ξ_{jm} distributed according to Equation (35) with $\delta_\xi = \delta_w = \delta_r = \delta_\upsilon = \delta_Q = \delta_{xo} = \delta_x = 1/\left(2\sqrt{7}\right)$. CPU minutes is the average estimation time in minutes across the Monte Carlo simulations. Measurement error in cost data has a standard deviation of $\sigma_{\nu+\varsigma} = 0.2$ which is approximately seven percent of mean total cost.

market size. In addition, in panel (b), we report the results where we set $x_{jm} = \Phi^{-1}(z_m)$ in Equation (34). As we can see, in both cases, means of the parameter estimates are close to the true values.

In Table 7, we compare the estimated parameters using our two-step SNLLS method with the standard IV approach using instruments that are commonly used in the literature. These are: wage, rental rate and observed product characteristics of own and rival firms and their interactions. Results show that in a parameter setup where the instruments are invalid, while our two-step SNLLS estimates are consistent, the IV estimates of the demand parameters are biased.

In the first row (SNLLS 1) of Table 7, we show results of our estimator when the demand shock is orthogonal to the other variables, (e.g., $\delta_\xi = 0.5$, $\delta_w = \delta_r = \delta_\upsilon = \delta_Q = \delta_{xo} = 0$), so that the instruments are valid. As we can see, the two-step SNLLS estimated coefficients are close to the true values, as are the IV estimates presented in the third row (IV1). However, the standard deviations of the IV estimates of σ_α and σ_β are higher than those of the two-step SNLLS estimates. That is, higher order interactions of the instruments may be needed to estimate σ_α and σ_β as accurately as the two-step SNLLS ones. Next, in the fourth row (IV2), we show the statistics of the IV estimates when the input prices are not valid instruments. That is, we set

Table 5: SNLLS Estimator of Random Coefficient Demand Parameters
(No Variation in Market Size)

			(a) Price coefficients parameters						
			$\hat{\mu}_\alpha$			$\hat{\sigma}_\alpha$			
Markets	Sample Size	Polynomials	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	CPU Minutes
50	200	144	-2.430	1.797	1.839	0.432	0.291	0.298	295.1
100	400	171	-2.205	0.460	0.501	0.529	0.178	0.180	670.0
200	800	204	-2.024	0.225	0.225	0.501	0.098	0.097	1226.8
400	1600	256	-2.024	0.115	0.117	0.500	0.052	0.052	2125.0
True Value			-2.000			0.500			

			(b) Product characteristic coefficients parameters						
			$\hat{\mu}_\beta$			$\hat{\sigma}_\beta$			
Markets	Sample Size	Polynomials	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	Obj. Fun.
50	200	144	1.389	1.596	1.635	0.208	0.206	0.205	2.537D-2
100	400	171	1.104	0.307	0.323	0.235	0.181	0.183	2.328D-2
200	800	204	1.007	0.136	0.136	0.212	0.112	0.112	2.970D-2
400	1600	256	1.015	0.084	0.085	0.205	0.060	0.060	3.390D-2
True Value			1.000			0.200			

Notes: Monte-carlo experiment results based on calibration described in panels (a)-(d) of Table 1 except we restrict $Q_L = Q_H$ so that there is no variation in market size. CPU minutes is the average estimation time in minutes across the Monte Carlo simulations. Measurement error in cost data has a standard deviation of $\sigma_{\nu+\varsigma} = 0.2$ which is approximately seven percent of mean total cost.

the demand shock and the correlation between the demand shock and other variables[24] to be $\delta_\xi = 1/\left(2\sqrt{1.08}\right)$, $\delta_v = \delta_Q = \delta_{x0} = 0$, and $\delta_w = \delta_r = 0.2\delta_\xi$. We can see that while the two-step SNLLS estimates in the second row (SNLLS2) are close to the true values, the IV estimated μ_α has an upward bias. The positive direction of bias is to be expected because ξ in Equation (5) is set up to be positively correlated with the instruments. Notice also that the coefficient estimate on the observed characteristic is downwardly biased, and the heterogeneity parameter of price effect, σ_α is upwardly biased.

Next, in row IV3, we present the IV results where the rival firms' observed product characteristics $\mathbf{X}_{-jm}$ are correlated with own unobserved characteristics ξ_{jm}. That is, we set $\delta_{xo} = \frac{1}{10\sqrt{1.04}}, \delta_\xi = \frac{1}{2\sqrt{1.04}}, \delta_w = \delta_r = \delta_v = \delta_Q = 0$. Hence, the observed characteristics of rival firms cannot be used as instruments for own price. Results show that the IV-estimated μ_α again has a positive bias. The parameter μ_β is again estimated with a negative bias, and so is σ_α, unlike results in Table 3, where we show the two-step SNLLS estimator delivers consistent parameter estimates even if the demand shock is correlated with rival product characteristics.

Finally, in row IV4, we report the results where all the instruments considered here are positively correlated with the demand shock. Again, we have an upward bias in the IV-estimated

[24] In all the subsequent analysis where we allow correlation between the demand shocks and the other variables, these correlations are set to be smaller than the ones used for the SNLLS estimates. We also conducted the Monte-Carlo experiments with larger correlations, but faced numerical difficulties during the IV estimation.

Table 6: SNLLS Estimator of Random Coefficient Demand Parameters (Unobservable Market Size)

		(a) z_m not in market share function			*(b) z_m in market share function*		
Parameter	True Value	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE
$\hat{\mu}_\alpha$	2.000	-2.025	0.125	0.127	-1.964	0.130	0.134
$\hat{\sigma}_\alpha$	0.500	0.510	0.043	0.041	0.506	0.050	0.050
$\hat{\mu}_\beta$	1.000	1.005	0.100	0.100	0.996	0.254	0.253
$\hat{\sigma}_\beta$	0.200	0.208	0.057	0.057	0.199	0.073	0.073
$\hat{\lambda}_0$	0.000	-0.034	0.249	0.250	0.002	0.096	0.096
$\hat{\lambda}_1$	1.000	1.020	0.073	0.075	1.060	0.136	0.148

Notes: Monte-carlo experiment results based on calibration described in panels (a)-(d) of Table 1 except we assume market size is unobserved and distributed according to equation (42) in the paper, where we set $\lambda_0 = 0$ and $\lambda_1 = 1$ in generating Monte Carlo samples. All results are based on samples with 500 markets, sample size of 2000 market-firm observations, with 256 polynomials used in estimation. CPU minutes is the average estimation time in minutes across the Monte Carlo simulations. Measurement error in cost data has a standard deviation of $\sigma_{\nu+\varsigma} = 0.2$ which is approximately seven percent of mean total cost.

μ_α and downward bias in the estimates of σ_α, μ_β and σ_β.

Overall, we conclude that our two-step SNLLS estimator provides unbiased parameter estimates even in situations where the commonly-used instruments are invalid and thus the IV estimates are biased. In addition, our two-step SNLLS estimator performs well even when market size is not observable, and the variable that determines market size is correlated with the demand shock or enters directly in the market share equation. Furthermore, with similar convergence criteria, the CPU minutes required for the two-step SNLLS estimator are less than for the IV estimates. Thus, we tentatively conclude that our sieve-estimation procedure does not impose excessive computational burden.

6 Empirical application to U.S. banking industry

We next apply our method to the actual data on banks and depository institutions to estimate the demand for deposits. We estimate a slightly different version of the demand model estimated by Dick (2008). In particular, we assume that each consumer has one unit to deposit. The indirect utility function of individual i putting his/her deposits in bank j in market m is specified as:

$$u_{ijm} = \mathbf{x}_{jm}\boldsymbol{\beta} + r_{djm}\alpha + \xi_{jm} + \epsilon_{ijm},$$

where $\mathbf{x}_{jm}$ is a vector of observed characteristics of bank j in market m, which consists of log of number of its branches, log of number of markets served and log of one plus bank age; r_{djm} is the deposit interest rate of bank j in market m net of the service charge, and ξ_{jm} is its unobserved

Table 7: SNLLS and IV Estimators of Random Coefficient Demand Parameters (Variation in Market Size)

(a) Price coefficients parameters

	$\hat{\mu}_\alpha$			$\hat{\sigma}_\alpha$			
Experiment	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	CPU Minutes
SNLLS1	-2.025	0.085	0.088	0.505	0.036	0.036	9400.8
SNLLS2	-2.034	0.087	0.092	0.507	0.033	0.034	7932.0
IV1	-1.978	0.086	0.089	0.475	0.069	0.073	11571.3
IV2	-1.614	0.088	0.395	0.609	0.051	0.120	13607.4
IV3	-1.453	0.078	0.552	0.157	0.055	0.347	11793.1
IV4	-1.278	0.086	0.727	0.394	0.054	0.119	12875.1
True Value	-2.000			0.500			

(b) Product characteristic coefficients parameters

	$\hat{\mu}_\beta$			$\hat{\sigma}_\beta$			
Experiment	Mean	Std. Dev.	RMSE	Mean	Std. Dev.	RMSE	Obj. Fun.
SNLLS1	1.011	0.048	0.048	0.200	0.041	0.041	3.487D-2
SNLLS2	1.016	0.052	0.054	0.204	0.043	0.042	3.504D-2
IV1	0.982	0.052	0.055	0.197	0.147	0.146	8.344D-4
IV2	0.620	0.046	0.383	0.206	0.148	0.148	1.498D-3
IV3	0.775	0.030	0.227	0.203	0.111	0.111	9.492D-4
IV4	0.533	0.035	0.468	0.159	0.120	0.126	1.203D-3
True Value	1.000			0.200			

Notes: Monte-carlo experiment results based on calibration described in panels (a)-(d) of Table 1 with variations in experimental designs described at the end of this table's note. All results are based on Monte Carlo samples with 500 markets, sample size of 2000 market-firm observations, with 256 polynomials used in estimation. CPU minutes is the average estimation time in minutes across the Monte Carlo simulations. Measurement error in cost data has a standard deviation of $\sigma_{\nu+\varsigma} = 0.2$ which is approximately seven percent of mean total cost.

Monte Carlo experiment designs:

SNLLS1, IV1: instruments are valid, $\delta_w = \delta_r = \delta_v = \delta_Q = \delta_{xo} = 0$

SNLLS2, IV2: input prices correlated with demand shock: $\delta_\xi = \delta_w = \delta_r = \frac{1}{2\sqrt{3}}$, $\delta_v = \delta_Q = \delta_{xo} = 0$

IV3: rival product observed characteristic is correlated with demand shock, $\delta_\xi = \frac{1}{2\sqrt{1.04}}$, $\delta_{xo} = \frac{1}{10\sqrt{1.04}}$, $\delta_w = \delta_r = \delta_v = \delta_Q = 0$

IV4: input price, variable cost shock, market size, and rival product observed characteristics are correlated with the demand shock, $\delta_\xi = \frac{1}{2\sqrt{1.20}}$, $\delta_w = \delta_r = \delta_v = \delta_Q = \delta_{xo} = \frac{1}{10\sqrt{1.20}}$

characteristics. Finally, ϵ_{ijm} is the random residual term in the utility function, which is assumed to be i.i.d. Extreme-Value distributed.

Then, the market share of deposits for bank j in market m is

$$s_{jm} = \int_\alpha \int_{\boldsymbol{\beta}} \frac{\exp\left(\mathbf{x}_{jm}\boldsymbol{\beta} + r_{djm}\alpha + \xi_{jm}\right)}{\left[1 + \sum_{k=1}^{J} \exp\left(\mathbf{x}_{km}\boldsymbol{\beta} + r_{dkm}\alpha + \xi_{km}\right)\right]} \left[\prod_{l=1}^{K} \frac{1}{\sigma_{\beta l}} \phi\left(\frac{\beta_l}{\sigma_{\beta l}}\right) d\beta_l\right] \frac{1}{\sigma_\alpha} \phi\left(\frac{\alpha - \mu_\alpha}{\sigma_\alpha}\right) d\alpha,$$

where ϕ is the density function of the standard normal distribution. We can rewrite the above equation by applying the change of variables as follows:

$$s_{jm} = \int_\alpha \int_{\boldsymbol{\beta}} \frac{exp\left(\left(\mathbf{x}_{jm} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}}\right)\boldsymbol{\beta} + r_{djm}\left(\sigma_\alpha \alpha + \mu_\alpha\right) + \xi_{jm}\right)}{1 + \sum_k exp\left(\left(\mathbf{x}_{km} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}}\right)\boldsymbol{\beta} + r_{dkm}\left(\sigma_\alpha \alpha + \mu_\alpha\right) + \xi_{km}\right)} \phi(\boldsymbol{\beta})\, d\boldsymbol{\beta}\, \phi(\alpha)\, d\alpha,$$

and the market share of the outside option is

$$s_{0m} = \int_\alpha \int_{\boldsymbol{\beta}} \frac{exp\left(\xi_{0m}\right)}{1 + \sum_k exp\left(\left(\mathbf{x}_{km} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}}\right) \boldsymbol{\beta} + r_{dkm}\left(\sigma_\alpha \alpha + \mu_\alpha\right) + \xi_{km}\right)} \phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta} \phi\left(\alpha\right) d\alpha.$$

Note that we expect the sign of μ_α to be positive because consumers prefer higher deposit interest rate. We follow Dick (2008) and let the outside option be depositing in credit unions.

Notice that since individuals receive deposit interest rate on their deposits, banks need to loan out or invest the deposits to earn any revenues. Thus, we assume revenue to be $\left(r_{jm} - r_{djm}\right) q_{jm}$, where r_{jm} is the interest rate earned by bank j in market m. In the empirical example, the market size Q_m is the total number of deposits (including credit unions). We set the interest rate to be $r_{jm} = r$, where r is the interest rate on the government treasury notes in January 2002.[25] Then, the marginal revenue of deposit is

$$MR_{jm}\left(\boldsymbol{\theta}\right) = \left(r - r_{djm}\right) - s_{jm} \left[\frac{\partial s_j\left(\mathbf{r}_{dm}, \mathbf{X}_m, \boldsymbol{\xi}_m; \boldsymbol{\theta}\right)}{\partial r_{djm}}\right]^{-1},$$

where

$$\begin{aligned} & \frac{\partial s_j\left(\mathbf{r}_{dm}, \mathbf{X}_m, \boldsymbol{\xi}_m; \boldsymbol{\theta}\right)}{\partial r_{djm}} \\ = & \int_\alpha \int_{\boldsymbol{\beta}} \left[\frac{exp\left(\left(\mathbf{x}_{jm} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}}\right) \boldsymbol{\beta} + r_{djm}\left(\sigma_\alpha \alpha + \mu_\alpha\right) + \xi_{jm}\right)}{\left[1 + \sum_k exp\left(\left(\mathbf{x}_{km} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}}\right) \boldsymbol{\beta} + r_{dkm}\left(\sigma_\alpha \alpha + \mu_\alpha\right) + \xi_{km}\right)\right]^2} \right. \\ & \left. \times \left[1 + \sum_{k \neq j} exp\left(\left(\mathbf{x}_{km} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}}\right) \boldsymbol{\beta} + r_{dkm}\left(\sigma_\alpha \alpha + \mu_\alpha\right) + \xi_{km}\right)\right]\right] \left(\sigma_\alpha \alpha + \mu_\alpha\right) \phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta} \phi\left(\alpha\right) d\alpha. \end{aligned}$$

We define the variable cost of a bank to be the total annual labor cost. That is, we assume we can measure the variable cost, and thus do not need to specify the fixed cost. We also assume away the deterministic measurement error component of the cost data. We define the cost function as $C\left(q_{jm}, w_m, \mathbf{x}_{jm}, \upsilon_{jm}\right)$, where q_{jm} is the total deposits of bank j in market m,

[25]There are three possible choices of variables for the interest rate r_{jm}. One could use the interest rate on assets such as government bonds, loan interest rate, or a basket of the rate of returns on loans and other financial assets. Note that setting the interest rate to be the loan interest rate would raise an additional endogeneity issue of bank lending. Since one of the important goals of this empirical analysis is to demonstrate the validity of our estimator, we decided to focus on the deposit side so that our results are comparable to the literature on deposit demand estimation. We use the interest rate on the government treasury notes in January 2002, since one can reasonably assume it to be exogenous. We believe that an interesting future direction of research would be an empirical analysis of the banking industry where both deposits and loans are endogenous. To the best of our knowledge, structural empirical analysis of banking that includes both deposits and loans (or investments) is scarce in the literature.

w_m is the wage level in market m, and υ_{jm} is the cost shock of bank j in market m. Since the cost function is homogeneous of degree one in wage, we can rewrite it as follows:

$$C\left(q_{jm}, w_m, \mathbf{x}_{jm}, \upsilon_{jm}\right) = w_m \varphi\left(q_{jm}, \mathbf{x}_{jm}, \frac{MR_{jm}}{w_m}\right) = w_m \sum_l \gamma_l \psi_l\left(q_{jm}, \mathbf{x}_{jm}, \frac{MR_{jm}}{w_m}\right),$$

where ψ_l, $l = 1, \ldots$ are the basis functions of the sieve and γ_l are the coefficients. Hence, we estimate $\boldsymbol{\theta}_{c0}$ by minimizing the following objective function:

$$\frac{1}{\sum J_m} \sum_{j,m} \left[\frac{C_{jm}}{w_m} - \sum_{l=1}^{\Gamma_{L_M}(T)} \gamma_l \psi_l\left(q_{jm}, \mathbf{x}_{jm}, \frac{MR_j\left(\mathbf{r}_{dm}, r, \mathbf{s}_m, \mathbf{X}_m, \boldsymbol{\theta}_c\right)}{w_m}\right)\right]^2. \tag{36}$$

with respect to both $\boldsymbol{\theta}_c$ and $\boldsymbol{\gamma}$. In the second step, we estimate $\boldsymbol{\mu_\beta}$ by OLS, where the dependent variable is $\delta_{jm}\left(\widehat{\boldsymbol{\theta}}_c\right) - r_{djm}\widehat{\mu}_\alpha$ and the vector of independent variables is $\mathbf{x}_{jm}$. We obtain $\delta_{jm}\left(\widehat{\boldsymbol{\theta}}_c\right)$ using the inversion algorithm explained earlier. That is, the equation we estimate is:

$$\delta_{jm}\left(\widehat{\boldsymbol{\theta}}_c\right) - r_{djm}\widehat{\mu}_\alpha = \mathbf{x}_{jm}\boldsymbol{\mu_\beta} + \xi_{jm}.$$

We calculate the standard errors of the parameter estimates of $\widehat{\boldsymbol{\theta}}_c$ by bootstrap, i.e., by resampling the residuals in Equation (36).

We use data on banks in the year 2002 from similar sources as Dick (2008). That is, we obtained the branch level information, such as the number of branches, the total deposits for each branch and the branch location from the FDIC (Federal Deposit Insurance Corporation). We collected data on bank characteristics from the balance sheet and income statements of banks in the UBPR (Uniform Bank Performance Report) from the FFIEC (Federal Financial Institutions Examination Council). Information on county level weekly wage is obtained from the annualized data of Quarterly Census of Employment and Wages (QCEW) data files managed by the Bureau of Labor Statistics. The definition of a market is either metropolitan statistical areas (MSA's) or counties if the area is not included in any of the MSA's. In the appendix, we provide the sample statistics and discuss some additional data and estimation issues.

By closely inspecting the data, we noticed that in some markets, credit unions seem to be effectively nonexistent as an outside option. Therefore, we removed the markets in which market share of credit unions is less than 1%. Furthermore, we use the cost data of only those banks that operate in a single market. We did so because banks who operate in multiple markets may not exercise third degree discrimination, which then violates Assumption 5. Finally, for the sake of reducing the computational burden, we restrict the sample to markets where there are 40 banks or less. In the final sample, the number of banks whose cost data we use is 2067, whereas the number of all banks is 3230. That is, about two thirds of the banks are single-market banks. As we can see in the sample statistics and the results in Table 9, the number of banks that operate only in a single market is high enough for identification of $\boldsymbol{\theta}_c$. It is important to remember that $\boldsymbol{\theta}_c$ is identified based on the assumption of independence of the measurement error to other variables. Therefore, using cost data of only those banks that serve one market does not result in any selection bias for estimation of $\boldsymbol{\theta}_c$, as long as the data of all banks are available for the variables in the marginal revenue function. For the estimation of $\boldsymbol{\mu}_{\boldsymbol{\beta}}$, selection matters, and thus we use all banks in the data.

Note that since we only use those banks that operate in a single market for cost estimation, we do not include the number of markets served in the cost function.

Table 8: Parameter Estimates for Deposit Demand Model

		(a) Two-Step SNLLS		*(b) IV-GMM*	
Variable	Parameter	Estimate	Std. Err.	Estimate	Std. Err.
Deposit interest rate	μ_α	31.920***	(0.633)	-208.800***	(1.767)
	σ_α	9.829E-3	(0.031)	2.697***	(0.026)
Log(Number of branches)	μ_{β_1}	4.256***	(0.053)	2.067	(12.310)
	σ_{β_1}	0.336***	(0.032)	9.646	(16.350)
Log(Number of markets)	μ_{β_2}	0.554***	(0.021)	0.086	(1.694)
	σ_{β_2}	5.044E-4	(6.200E-3)	1.481E-3	(1.477E-3)
Log(Bank age + 1)	μ_{β_3}	4.802***	(0.045)	1.356	(8.027)
	σ_{β_3}	2.892E-3	(0.011)	0.113	(0.264)

Notes: Bootstrap standard errors in parentheses. $^*p < 0.1$ $^{**}p < 0.01$. $^{***}p < 0.001$

We present our results in Table 8. In panel (a), we report the results where we use the two-step SNLLS procedure. In panel (b), we report the results where we use the IV-GMM procedure. We used regional wage, regional housing price index, observed own product characteristics (log number of branches, log number of markets, log age plus one) and those of rival banks in the market, and the interactions of those variables as instruments.

Our estimated price coefficient is around 32 and the average price elasticity is 1.64. The proportion of banks whose elasticity is less than one is 4 %. In contrast the IV-estimated price coefficient on deposit interest rate in Dick (2008) ranges from 54.19 to 100.23, depending on the inclusion of the bank/market/state fixed effects. A possible reason for this difference could be that Dick (2008) uses banks in MSAs which are predominantly urban, whereas we also include rural banks in our data. Indeed, our estimated price effects are closer to the ones in Ho and Ishii (2012), who also include rural markets in their analysis and find that the own-price elasticity is smaller in rural markets. The price elasticity is expected to be lower in rural markets, where the distance to branches of other banks is likely to be greater, and our data includes rural markets as well.

We see in panel (b) of the table that the IV estimated price coefficient is negative and significant. It is unintuitive because it implies that a higher deposit interest rate reduces deposits. Furthermore, the IV estimated parameters $(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \boldsymbol{\sigma}_{\boldsymbol{\beta}})$ are all insignificant,[26] whereas in panel (a), the two-step SNLLS estimated coefficients on observed characteristics are all positive, as is intuitive, and significant.

7 Conclusion

We have developed a new methodology for estimating demand and cost parameters of a differentiated products oligopoly model. The method uses data on prices, market shares, and product characteristics, and some data on firms' costs. Using these data, our approach identifies demand parameters in the presence of price endogeneity, and a nonparametric cost function in the presence of output endogeneity without any instruments. That is, demand and variable cost shocks do not need to be uncorrelated with demand shifters, cost shifters or market size, and demand shocks can be correlated with the observed characteristics of other products. Also, demand and variable cost shocks are allowed to be uncorrelated with each other. Moreover, our method can accommodate measurement error and fixed cost in cost data, endogenous product characteristics, multi-product firms, difference between accounting and economic costs, and some non-profit maximizing firms. In addition, we allow market size to be unobservable, and show that even without conventional exclusion restrictions on the variables determining demand and market size, we are able to identify and recover the unobserved market size, and consistently estimate the demand parameters.

In our empirical application, we use data on the banking industry to compare our estimated price coefficient of deposit interest rate to the one in the literature estimated using IVs. Our results indicate that cost data identifies the demand parameters well. In contrast, studies such as Dick (2008), Ho and Ishii (2012) and others use a large number of instruments (often 20 or more) for estimating the demand parameters. The validity of all these instruments is often quite

[26]We have tried various versions of setups and instruments, both logit and BLP demand. What we find is that among the cases we tried, only the logit specification with a relatively small number of instruments resulted in the price coefficients with the plausible sign. What we infer from those results is: Since BLP has more parameters that need to be estimated than the logit specification, it requires more instruments, and in this particular case, some of the instruments or some of the polynomials of the instruments are invalid.

difficult to assess.

The small bootstrapped standard errors, especially for $\boldsymbol{\theta}_c$ estimate in our banking application imply that the cost data and the nonparametric pseudo-cost function provide strong identification restrictions to control for endogeneity. This is also consistent with the favorable small sample Monte-Carlo results provided earlier. In many situations in empirical work, researchers do not have enough identification power from instruments to have their estimated coefficients to be significant. Even in such cases, the cost-based estimation method could provide significant parameter estimates. Then, our method has the potential to work well as a complement to the IV based approach. As we have seen, both methodologies use similar variation in the data. The input price, which is used as an instrument also appears as one of the variables in the cost function in the cost-based approach. The main differences between the IV approach and our cost-based approach are: 1) in the cost-based approach, such variation in the data is more explicitly modeled, which may improve efficiency, 2) unlike the IV approach, such variation does not need to be exogenous, and of course, 3) in our approach, unobservable market size can be identified and estimated without strong exclusion restrictions, providing some guidance on the specification of markets in the IV approach, and 4) the cost based approach requires cost data.

Our estimation strategy also presents an alternative tool for anti-trust authorities since they have the power to subpoena detailed cost data from firms for merger evaluation. Fundamental to the predictions from merger simulations based on the standard IV approach (Nevo (2001)) is the estimated demand elasticity and inferred marginal costs from the supply-side first order conditions of the structural model. The demand elasticity and nonparametric cost estimates based on our instrument-free approach can yield a complementary set of estimates and predictions regarding the welfare effects of proposed mergers when reliable instruments are scarce, or there are differences in opinions among the parties on the validity of the instruments.

Our estimation procedure requires marginal revenue to equal marginal cost. We believe that a fruitful direction of future research would be to make the method applicable to situations where marginal revenue fails to be equal to marginal cost. Examples include firms facing capacity constraints, or when firms' decisions include dynamic considerations.

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A Appendix

A.1 Proof of Lemma 2

Proof. Definition 2 says that $\boldsymbol{\theta}_{c0}$ is identified if $\mathcal{C} = \mathcal{MR}(\boldsymbol{\theta}_{c0}) \neq \mathcal{MR}(\boldsymbol{\theta}_c)$ for $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$.

If $\boldsymbol{\theta}_{c0}$ is identified according to Definition 1, $\mathcal{C} = \mathcal{MR}(\boldsymbol{\theta}_{c0})$. On the other hand, for $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$, $\mathcal{C} \neq \mathcal{MR}(\boldsymbol{\theta}_c)$ because at least one of the two following cases holds for $\boldsymbol{\theta}_c$. 1) Marginal revenue at $\boldsymbol{\theta}_c$ cannot be expressed as a function of $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$. Then, there exists a pair of firms that have the same $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$ but different marginal revenues. 2) Marginal revenue at $\boldsymbol{\theta}_c$ can be expressed as a function of $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$ but that function is not one to one in $\widetilde{C}$. Then, there is a pair of firms that have the same $(q, \mathbf{w}, \mathbf{x})$, different $\widetilde{C}$'s but the same marginal revenues for $\boldsymbol{\theta}_c$. In both cases 1) and 2), there is at least one pair that belongs either to group $\mathcal{C}$ or to group $\mathcal{MR}(\boldsymbol{\theta}_c)$ but not to both. Therefore, $\boldsymbol{\theta}_{c0}$ is identified according to Definition 2.

Next, suppose $\boldsymbol{\theta}_{c0}$ is identified according to Definition 2. Then, $\mathcal{C} = \mathcal{MR}(\boldsymbol{\theta}_{c0})$. Therefore, in the population mapping from $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$ to marginal revenue is a function, and given $(q, \mathbf{w}, \mathbf{x})$, the function from $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$ to marginal revenue is one to one. On the other hand, for $\boldsymbol{\theta}_c \neq \boldsymbol{\theta}_{c0}$, $\mathcal{C} \neq \mathcal{MR}(\boldsymbol{\theta}_c)$. Then, at least one of the following three cases need to hold for $\boldsymbol{\theta}_c$. 1) Under $\boldsymbol{\theta}_c$, there exists pairs of firms that have the same $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$ but different marginal revenues. Then, marginal revenue cannot be expressed as a function of $\left(q, \mathbf{w}, \mathbf{x}, \widetilde{C}\right)$, violating Condition 1 of Definition 1. 2) Under $\boldsymbol{\theta}_c$, there exists a pair of firms that have the same $(q, \mathbf{w}, \mathbf{x})$ and the same marginal revenue but different $\widetilde{C}$'s, which violates Condition 2 of Definition 1. 3) Under $\boldsymbol{\theta}_c$ there exist a pair of firms that have the same $(q, \mathbf{w}, \mathbf{x})$ and the same marginal revenue but they are nonpositive, which again violates Condition 1 of Definition 1. Therefore, $\boldsymbol{\theta}_{c0}$ is identified according to Definition 1. ■

A.2 Identification of the random coefficient BLP model with endogeneity. Proof of Proposition 1.

We prove that given Assumptions 1-8, the BLP demand function satisfies Condition 1. Then, from Lemma 3, identification follows.

Consider two firms j and $j^{\dagger}$ in different markets. The first market has demand side observables $(\mathbf{X}, \mathbf{p}, \mathbf{s})$ where $\mathbf{X}$ is a $J \times K$ matrix of observed product characteristics, $\mathbf{p}$ is the $J \times 1$ price vector and $\mathbf{s}$ is the $J \times 1$ market share vector. Let $\left(\mathbf{X}^{\dagger}, \mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}\right)$ denote the corresponding observables for the second market where the number of products/firms is $J^{\dagger}$. We start by assuming that these two firms satisfy 1 and 2 of Condition 1. We then prove that among those pairs, there exists one that satisfies 3 of Condition 1. We denote $\Phi\left(\right)$ to be the standard normal distribution function and $\phi\left(\right)$ to be its density function. Then, the market shares of firms j and $j^{\dagger}$ are respectively:

$$
\begin{aligned}
s_j &= \int_{\alpha}\int_{\boldsymbol{\beta}} \frac{\exp\left(\mathbf{x}_j\boldsymbol{\beta}+p_j\alpha+\xi_{j,0}\right)}{\left[1+\sum_{l=1}^{J}\exp\left(\mathbf{x}_l\boldsymbol{\beta}+p_l\alpha+\xi_{l,0}\right)\right]}\left[\prod_{k=1}^{K}\frac{1}{\sigma_{\beta 0k}}\phi\left(\frac{\beta_k-\mu_{\beta 0k}}{\sigma_{\beta 0k}}\right)d\beta_k\right]\frac{1}{\sigma_{\alpha 0}}\phi\left(\frac{\alpha-\mu_{\alpha 0}}{\sigma_{\alpha 0}}\right)d\alpha \\
s^{\dagger}_{j^{\dagger}} &= \int_{\alpha}\int_{\boldsymbol{\beta}} \frac{\exp\left(\mathbf{x}^{\dagger}_{j^{\dagger}}\boldsymbol{\beta}+p^{\dagger}_{j^{\dagger}}\alpha+\xi^{\dagger}_{j^{\dagger},0}\right)}{\left[1+\sum_{l=1}^{J^{\dagger}}\exp\left(\mathbf{x}^{\dagger}_l\boldsymbol{\beta}+p^{\dagger}_l\alpha+\xi^{\dagger}_{l,0}\right)\right]}\left[\prod_{k=1}^{K}\frac{1}{\sigma_{\beta 0k}}\phi\left(\frac{\beta_k-\mu_{\beta 0k}}{\sigma_{\beta 0k}}\right)d\beta_k\right]\frac{1}{\sigma_{\alpha 0}}\phi\left(\frac{\alpha-\mu_{\alpha 0}}{\sigma_{\alpha 0}}\right)d\alpha.
\end{aligned}
$$

Now, denote, $\eta_{\alpha 0}=\mu_{\alpha 0}/\sigma_{\alpha 0}$. Then, by a change of variables such that $\alpha^{*}=\alpha/\sigma_{\alpha 0}-\eta_{\alpha 0}$ and $\beta^{*}_j=\beta_j/\sigma_{\beta 0j}-\mu_{\beta 0k}/\sigma_{\beta 0j}$, we obtain

$$
\begin{aligned}
s_j &= \int_{\alpha^{*}}\int_{\boldsymbol{\beta}^{*}} \frac{\exp\left(\left(\mathbf{x}_j\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}^{*}+p_j\sigma_{\alpha 0}\left(\alpha^{*}+\eta_{\alpha 0}\right)+\mathbf{x}_j\boldsymbol{\mu}_{\boldsymbol{\beta}0}+\xi_{j,0}\right)}{\left[1+\sum_{l=1}^{J}\exp\left(\left(\mathbf{x}_l\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}^{*}+p_l\sigma_{\alpha 0}\left(\alpha^{*}+\eta_{\alpha 0}\right)+\mathbf{x}_l\boldsymbol{\mu}_{\boldsymbol{\beta}0}+\xi_{l,0}\right)\right]}\phi\left(\boldsymbol{\beta}^{*}\right)d\boldsymbol{\beta}^{*}\phi\left(\alpha^{*}\right)d\alpha^{*} \\
s^{\dagger}_{j^{\dagger}} &= \int_{\alpha^{*}}\int_{\boldsymbol{\beta}^{*}} \frac{\exp\left(\left(\mathbf{x}^{\dagger}_{j^{\dagger}}\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}^{*}+p^{\dagger}_{j^{\dagger}}\sigma_{\alpha 0}\left(\alpha^{*}+\eta_{\alpha 0}\right)+\mathbf{x}^{\dagger}_{j^{\dagger}}\boldsymbol{\mu}_{\boldsymbol{\beta}0}+\xi^{\dagger}_{j^{\dagger},0}\right)}{\left[1+\sum_{l=1}^{J^{\dagger}}\exp\left(\left(\mathbf{x}^{\dagger}_l\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}^{*}+p^{\dagger}_l\sigma_{\alpha 0}\left(\alpha^{*}+\eta_{\alpha 0}\right)+\mathbf{x}^{\dagger}_{l^{\dagger}}\boldsymbol{\mu}_{\boldsymbol{\beta}0}+\xi^{\dagger}_{l,0}\right)\right]}\phi\left(\boldsymbol{\beta}^{*}\right)d\boldsymbol{\beta}^{*}\phi\left(\alpha^{*}\right)d\alpha^{*},
\end{aligned}
$$

where $\mathbf{x}_j\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}=\left(x_{j1}\sigma_{\boldsymbol{\beta}01},\ldots,x_{jK}\sigma_{\boldsymbol{\beta}0K}\right)$, and $\phi\left(\boldsymbol{\beta}^{*}\right)\equiv\prod_{k=1}^{K}\phi\left(\beta_k\right)$ is the joint standard normal density function. We then denote α^{*} to be α and $\boldsymbol{\beta}^{*}$ to be $\boldsymbol{\beta}$, and $\mathbf{x}_j\boldsymbol{\mu}_{\boldsymbol{\beta}}+\xi_j$ to be ξ_j (similarly for $\mathbf{x}^{\dagger}_{j^{\dagger}}\boldsymbol{\mu}_{\boldsymbol{\beta}}+\xi^{\dagger}_{j^{\dagger}}$ to be $\xi^{\dagger}_{j^{\dagger}}$). Then,

$$
\begin{aligned}
s_j &= \int_{\alpha}\int_{\boldsymbol{\beta}} \frac{\exp\left(\left(\mathbf{x}_j\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}+p_j\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\xi_{j,0}\right)}{\left[1+\sum_{l=1}^{J}\exp\left(\left(\mathbf{x}_l\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}+p_l\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\xi_{l,0}\right)\right]}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha \\
s^{\dagger}_{j^{\dagger}} &= \int_{\alpha}\int_{\boldsymbol{\beta}} \frac{\exp\left(\left(\mathbf{x}^{\dagger}_{j^{\dagger}}\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}+p^{\dagger}_{j^{\dagger}}\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\xi^{\dagger}_{j^{\dagger},0}\right)}{\left[1+\sum_{l=1}^{J^{\dagger}}\exp\left(\left(\mathbf{x}^{\dagger}_l\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}+p^{\dagger}_l\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\xi^{\dagger}_{l,0}\right)\right]}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha
\end{aligned}
$$

Notice now that the parameters that enter in the market share function are $\left(\eta_{\alpha},\sigma_{\alpha},\boldsymbol{\sigma}_{\boldsymbol{\beta}}\right)$, which corresponds to $\boldsymbol{\theta}_c=\left(\mu_{\alpha},\sigma_{\alpha},\boldsymbol{\sigma}_{\boldsymbol{\beta}}\right)$. Then,

$$
\begin{aligned}
&\frac{\partial s_j}{\partial p_j} = \\
&\int_{\alpha}\int_{\beta}\frac{\exp\left(\left(\mathbf{x}_j\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}+p_j\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\xi_{j,0}\right)\left[1+\sum_{l\neq j}\exp\left(\left(\mathbf{x}_l\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}+p_l\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\xi_{l,0}\right)\right]}{\left[1+\sum_{l=1}^{J}\exp\left(\left(\mathbf{x}_l\circ\boldsymbol{\sigma}_{\boldsymbol{\beta}0}\right)\boldsymbol{\beta}+p_l\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\xi_{l,0}\right)\right]^2} \\
&\times\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\boldsymbol{\phi}\left(\boldsymbol{\alpha}\right)d\alpha
\end{aligned}
$$

and a similar expression holds for $\partial s^{\dagger}_{j^{\dagger}}/\partial p_{j^{\dagger}}$.

Note that if we let

$$\mathbf{\Sigma}_{\boldsymbol{\beta}} = \begin{bmatrix} \sigma_{\boldsymbol{\beta}1} & 0 & \dots & 0 \\ 0 & \sigma_{\boldsymbol{\beta}2} & & 0 \\ & & \ddots & \\ 0 & \dots & 0 & \sigma_{\boldsymbol{\beta}K} \end{bmatrix},$$

then $(\mathbf{x}_j \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0})\,\boldsymbol{\beta} = \mathbf{x}_j \mathbf{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}$. For the sake of simplicity, we choose two markets in which prices of firms in each market are different from each other. Further, let $p_l < p_{l+1}$ for $l = 1, \ldots, J$, and similarly for the second market. Also, let $\mathbf{v}_{i,0} \equiv \boldsymbol{\xi}_{i,0}/\sigma_{\alpha 0}$.

Note that the market share function can be expressed as the aggregate of individuals with different α, $\boldsymbol{\beta}$ and the i.i.d. extreme value distributed preference shock $\boldsymbol{\epsilon}$ as below. That is, consider an individual with specific α, $\boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$ values who chooses product $i = 0, \ldots, J$, (where 0 is no purchase) if

$$\mathbf{x}_i \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + p_i \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \sigma_{\alpha 0} v_{i,0} + \epsilon_i \geq \mathbf{x}_l \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + p_l \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \sigma_{\alpha 0} v_{l,0} + \epsilon_l,\ l \neq i,$$

where $\mathbf{x}_0 = 0$, $p_0 = 0$ $v_{0,0} = 0$ for normalization. From now on, we set the price vector to be $\gamma\mathbf{p}$, where $\gamma > 0$ is a scalar. Since we only consider cases where γ is large for the identification of $\eta_{\alpha 0}$, we set $\gamma > \underline{\gamma}$ for some $\underline{\gamma} > 0$. Furthermore, we define $\mathbf{v}_0\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) \equiv \boldsymbol{\xi}\left(\mathbf{X}, \gamma\mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)/\left(\gamma\sigma_{\alpha 0}\right)$. Then,

$$\begin{aligned} s_i \;=\; & \int_{\alpha}\int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}}\prod_{l\neq i} I\Big(\mathbf{x}_i \mathbf{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_i \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0} v_{i,0}(\gamma, .) + \epsilon_i \\ & \geq \mathbf{x}_l \mathbf{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_l \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0} v_{l,0}(\gamma, .) + \epsilon_l\Big) f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha, \end{aligned}$$

where $f\left(\boldsymbol{\epsilon}\right) \equiv \prod_{l=1}^{J} g\left(\epsilon_l\right)$ is the joint distribution of $\boldsymbol{\epsilon}$, with $g\left(\right)$ being the extreme value density function. As $\gamma \to \infty$, Then, dividing the integrand by γ, and holding s_i fixed, as $\gamma \to \infty$[27], we

[27] Note that prices $\gamma\mathbf{p}$ and market shares $\mathbf{s}$ are observed, and the unobserved product characteristics $\boldsymbol{\nu}$ is implied through the functional relationship, given $\mathbf{X}$ and the true parameters. Given our assumptions, we can find demand shocks that generate $\gamma\mathbf{p}$ and $\mathbf{s}$, for arbitrarily large γ.

obtain

$$
\begin{aligned}
s_i &= lim_{\gamma\to\infty}\int_\alpha\int_{\boldsymbol\beta}\int_{\boldsymbol\epsilon}\prod_{l\neq i} I\left(\frac{\mathbf{x}_i}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol\beta 0}\boldsymbol\beta + p_i\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\sigma_{\alpha 0}v_{i,0}(\gamma,.)+\frac{\epsilon_i}{\gamma}\right.\\
&\geq \left.\frac{\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol\beta 0}\boldsymbol\beta + p_l\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\sigma_{\alpha 0}v_{l,0}(\gamma,.)+\frac{\epsilon_l}{\gamma}\right) f\left(\boldsymbol\epsilon\right)d\boldsymbol\epsilon\phi\left(\boldsymbol\beta\right)d\boldsymbol\beta\phi\left(\alpha\right)d\alpha,
\end{aligned}
$$

and $\mathbf{x}_l/\gamma\to\mathbf{0}$, $l=1,\ldots,J$, $\boldsymbol\epsilon/\gamma\to\mathbf{0}$ for any $\boldsymbol\epsilon\in R^J$. Therefore, for a fixed vector $\widetilde{\boldsymbol v}_0$, the following holds:

$$
\begin{aligned}
& I\left(\frac{\mathbf{x}_i}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol\beta 0}\boldsymbol\beta + p_i\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\sigma_{\alpha 0}\widetilde{v}_{i,0}+\frac{\epsilon_i}{\gamma}\geq\frac{\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol\beta 0}\boldsymbol\beta + p_l\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\sigma_{\alpha 0}\widetilde{v}_{l,0}+\frac{\epsilon_l}{\gamma}\right)\\
\to\ & I\left(p_i\left(\alpha+\eta_{\alpha 0}\right)+\widetilde{v}_{i,0}\geq p_l\left(\alpha+\eta_{\alpha 0}\right)+\widetilde{v}_{l,0}\right)\ as\ \gamma\to\infty.
\end{aligned}
$$

Furthermore, $\left|\prod_{l\neq i}I\left(\right)\right|\leq 1$ and $\int_\alpha\int_{\boldsymbol\beta}\int_{\boldsymbol\epsilon}\left[\prod_{l\neq i}1\right]f\left(\boldsymbol\epsilon\right)d\boldsymbol\epsilon\phi\left(\boldsymbol\beta\right)d\boldsymbol\beta\phi\left(\alpha\right)d\alpha=1$. Therefore, from the Dominated Convergence Theorem,

$$
\begin{aligned}
s_i &= \int_\alpha\int_{\boldsymbol\beta}\int_{\boldsymbol\epsilon}\prod_{l\neq i} I\left(p_i\left(\alpha+\eta_{\alpha 0}\right)+\widetilde{v}_{i,0}\geq p_l\left(\alpha+\eta_{\alpha 0}\right)+\widetilde{v}_{l,0}\right)f\left(\boldsymbol\epsilon\right)d\boldsymbol\epsilon\phi\left(\boldsymbol\beta\right)d\boldsymbol\beta\phi\left(\alpha\right)d\alpha\\
&= \int_\alpha\prod_{l\neq i} I\left(p_i\left(\alpha+\eta_{\alpha 0}\right)+\widetilde{v}_{i,0}\geq p_l\left(\alpha+\eta_{\alpha 0}\right)+\widetilde{v}_{l,0}\right)\phi\left(\alpha\right)d\alpha\\
&\equiv \widetilde{s}_i\left(\mathbf{p},\widetilde{\mathbf{v}}_0,\boldsymbol\theta_{c0}\right),
\end{aligned}
$$

and

$$
s_0=\int_\alpha\prod_{l=1}^{J} I\left(p_l\left(\alpha+\eta_{\alpha 0}\right)+\widetilde{v}_{l,0}\leq 0\right)\phi\left(\alpha\right)d\alpha,
$$

where $\widetilde{\mathbf{v}}_0$ satisfies $\widetilde{s}\left(\mathbf{p},\widetilde{\mathbf{v}}_0,\boldsymbol\theta_0\right)=\mathbf{s}$.

Next, we prove that such $\widetilde{\mathbf{v}}_0$ exists and is unique, as long as we normalize $\widetilde{v}_{0,0}$ to be zero. In particular, we prove that given $\widetilde{v}_{0,0}=0$, $p_0=0$, $\widetilde{v}_{i,0}$, $i>0$ can be derived recursively as follows.

$$
\widetilde{v}_{i+1,0}=\widetilde{v}_{i,0}+\left(p_i-p_{i+1}\right)\left[\eta_{\alpha 0}+\Phi^{-1}\left(\sum_{l\leq i}s_l\right)\right].
$$

First, let $\widetilde{\mathbf{v}}_0$ to be an arbitrary $J\times 1$ vector. Denote

$$
a_{i,l}=\frac{\widetilde{v}_{i,0}-\widetilde{v}_{l,0}}{p_i-p_l},\ l\neq i
$$

Since we assumed $p_i < p_{i+1}$ for $i = 0, \ldots, J-1$, i is chosen if

$$\alpha + \eta_{\alpha 0} + a_{i,l} \geq 0 \quad if \quad i > l$$
$$\alpha + \eta_{\alpha 0} + a_{i,l} < 0 \quad if \quad i < l.$$

Then, for $i \geq 1$,

$$\begin{aligned} I\left(choose\ i\right) &= \prod_{l=0}^{i-1} I\left(\alpha \geq -\eta_{\alpha 0} - a_{i,l}\right) \prod_{l=i+1}^{J} I\left(\alpha < -\eta_{\alpha 0} - a_{i,l}\right) \\ &= I\left(\alpha \geq max_{l<i}\left\{-\eta_{\alpha 0} - a_{i,l}\right\}\right) I\left(\alpha < min_{l>i}\left\{-\eta_{\alpha 0} - a_{i,l}\right\}\right). \end{aligned}$$

Therefore, if we denote $\underline{a}_i = min_{l<i}\left\{a_{i,l}\right\}$ $\overline{a}_i = max_{l>i}\left\{a_{i,l}\right\}$,

$$\widetilde{s}_i\left(\mathbf{p}, \widetilde{\mathbf{v}}_0, \boldsymbol{\theta}_0\right) = \int_\alpha I\left(-\eta_{\alpha 0} - \underline{a}_i \leq \alpha < -\eta_{\alpha 0} - \overline{a}_i\right) \phi\left(\alpha\right) d\alpha.$$

Furthermore,

$$I\left(choose\ 0\right) = I\left(\alpha < min_{l>0}\left\{-\eta_{\alpha 0} - a_{0,l}\right\}\right).$$

Therefore, as before,

$$\widetilde{s}_0\left(\mathbf{p}, \widetilde{\mathbf{v}}_0, \boldsymbol{\theta}_0\right) = \int_\alpha I\left(-\eta_{\alpha 0} - \underline{a}_0 \leq \alpha < -\eta_{\alpha 0} - \overline{a}_0\right) \phi\left(\alpha\right) d\alpha,$$

where $\underline{a}_0 = \infty$, $\overline{a}_0 = max_{l>0}\left\{a_{0,l}\right\}$. Similarly,

$$\widetilde{s}_J = \int_\alpha I\left(-\eta_{\alpha 0} - \underline{a}_J \leq \alpha < -\eta_{\alpha 0} - \overline{a}_J\right) \phi\left(\alpha\right) d\alpha,$$

where $\underline{a}_J = min_{l<J}\left\{a_{J,l}\right\}$, $\overline{a}_J = -\infty$.

Given $s_i > 0$ for any $i = 0, \ldots, J$ (2 of Condition 1), $-\underline{a}_i < -\overline{a}_i$ holds. Furthermore, note that

$$\underline{a}_{i+1} = min_{l<i+1}\left\{a_{i+1,l}\right\} \leq a_{i+1,i} = a_{i,i+1} \leq max_{l>i}\left\{a_{i,l}\right\} = \overline{a}_i. \tag{37}$$

Therefore, $-\overline{a}_i \leq -\underline{a}_{i+1}$ $i = 0, \ldots, J-1$. Now, suppose $-\overline{a}_i < -\underline{a}_{i+1}$. Then, $\ldots - \overline{a}_{i-1} \leq -\underline{a}_i < -\overline{a}_i < -\underline{a}_{i+1} < -\overline{a}_{i+1} \leq -\underline{a}_{i+2} \ldots$. It implies that an individual with $\alpha \in \left(-\eta_{\alpha 0} - \overline{a}_i, -\eta_{\alpha 0} - \underline{a}_{i+1}\right)$ will not choose any of the available product choices $i \in \{0, 1, \ldots, J\}$ including the no-purchase option 0, which is a contradiction. Therefore, $-\overline{a}_i = -\underline{a}_{i+1}$ holds. Furthermore, Equation (37) and $\overline{a}_i = \underline{a}_{i+1}$ imply $\overline{a}_i = \underline{a}_{i+1} = a_{i,i+1}$. Therefore,

$$s_0 = \widetilde{s}_0\left(\mathbf{p}, \widetilde{\mathbf{v}}_0, \boldsymbol{\theta}_0\right) = \int_\alpha I\left(\alpha \leq -\eta_{\alpha 0} - \frac{\widetilde{v}_{1,0}}{p_1}\right)\phi\left(\alpha\right)d\alpha = \Phi\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{1,0}}{p_1}\right) \tag{38}$$

and for $i \geq 1$,

$$\begin{aligned} s_i &= \widetilde{s}_i\left(\mathbf{p}, \widetilde{\mathbf{v}}_0, \boldsymbol{\theta}_0\right) \equiv \int_\alpha I\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{i-1,0} - \widetilde{v}_{i,0}}{p_{i-1} - p_i} \leq \alpha \leq -\eta_{\alpha 0} - \frac{\widetilde{v}_{i,0} - \widetilde{v}_{i+1,0}}{p_i - p_{i+1}}\right)\phi\left(\alpha\right)d\alpha \\ &= \Phi\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{i,0} - \widetilde{v}_{i+1,0}}{p_i - p_{i+1}}\right) - \Phi\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{i-1,0} - \widetilde{v}_{i,0}}{p_{i-1} - p_i}\right) \end{aligned}$$

and

$$s_J = 1 - \Phi\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{J-1,0} - \widetilde{v}_{J,0}}{p_{J-1} - p_J}\right) \tag{39}$$

Therefore, from Equations (38) to (39), we can derive

$$\Phi\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{i,0} - \widetilde{v}_{i+1,0}}{p_i - p_{i+1}}\right) = \sum_{l \leq i} s_l.$$

Thus,

$$\widetilde{v}_{i+1,0} = \widetilde{v}_{i,0} + \left(p_i - p_{i+1}\right)\left[\eta_{\alpha 0} + \Phi^{-1}\left(\sum_{l \leq i} s_l\right)\right], \tag{40}$$

As $\widetilde{v}_{0,0}$ is normalized to be 0, $\widetilde{v}_{i+1,0}$, $i \geq 0$ can be recursively derived as follows:

$$\widetilde{v}_{i+1,0} = \sum_{k=0}^{i}\left(p_k - p_{k+1}\right)\left[\eta_{\alpha 0} + \Phi^{-1}\left(\sum_{l \leq k} s_l\right)\right]. \tag{41}$$

Therefore, we have proven that $\widetilde{\mathbf{v}}_0$ satisfying $\widetilde{s}\left(\mathbf{p}, \widetilde{\mathbf{v}}_0, \boldsymbol{\theta}_0\right) = \mathbf{s}$ exists and is unique.

Next, we show that $lim_{\gamma\to\infty}\mathbf{v}_0\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)=\widetilde{\mathbf{v}}_0$. Now,

$$
\begin{aligned}
&\int_\alpha\int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}}\prod_{l\neq i} I\left[\left(p_i-p_l\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)\geq-\frac{\mathbf{x}_i-\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}-\frac{\epsilon_i-\epsilon_l}{\gamma}\right.\\
&\left.-\,\sigma_{\alpha 0}\left(v_{i,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)-v_{l,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)\right)\right]f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha\\
=\;&\int_\alpha\int_{\boldsymbol{\beta}\in\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^K}\int_{\boldsymbol{\epsilon}\in\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^J}\prod_{l\neq i} I\left[\left(p_i-p_l\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)\geq-\frac{\mathbf{x}_i-\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}-\frac{\epsilon_i-\epsilon_l}{\gamma}\right.\\
&\left.-\sigma_{\alpha 0}\left(v_{i,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)-v_{l,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)\right)\right]f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha\\
&+\int_\alpha\int_{\boldsymbol{\beta}\notin\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^K}\int_{\boldsymbol{\epsilon}\in\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^J}\prod_{l\neq i} I\left[\left(p_i-p_l\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)\geq-\frac{\mathbf{x}_i-\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}-\frac{\epsilon_i-\epsilon_l}{\gamma}\right.\\
&\left.-\sigma_{\alpha 0}\left(v_{i,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)-v_{l,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)\right)\right]f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha\\
&+\int_\alpha\int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}\notin\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^J}\prod_{l\neq i} I\left[\left(p_i-p_l\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)\geq-\frac{\mathbf{x}_i-\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}-\frac{\epsilon_i-\epsilon_l}{\gamma}\right.\\
&\left.-\sigma_{\alpha 0}\left(v_{i,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)-v_{l,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)\right)\right]f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha. \qquad (42)
\end{aligned}
$$

Then, for $\boldsymbol{\beta}\in\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^K$and $\boldsymbol{\epsilon}\in\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^J$, and for $A\equiv\max_{k,i,l}\left|\left[-\left[\mathbf{x}_i-\mathbf{x}_l\right]\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\right]_k\right|K+2$

$$
\begin{aligned}
\left|-\frac{\mathbf{x}_i-\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}-\frac{\epsilon_i-\epsilon_l}{\gamma}\right| &\leq \frac{1}{\gamma}\max_{k,i,l}\left|\left[-\left[\mathbf{x}_i-\mathbf{x}_l\right]\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\right]_k\right|K\sup_{\boldsymbol{\beta}\in\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^K}\left|\beta_k\right|+\frac{1}{\gamma}\sup_{\boldsymbol{\epsilon}\in\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^J}\left|\epsilon_i-\epsilon_l\right|\\
&= \frac{1}{\sqrt{\gamma}}\left[\max_{k,i,l}\left|\left[-\left[\mathbf{x}_i-\mathbf{x}_l\right]\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\right]_k\right|K+2\right]=\frac{A}{\sqrt{\gamma}}.
\end{aligned}
$$

Furthermore, the 2nd and 3rd terms of Equation (42) can be made arbitrarily small by choosing γ to be arbitrarily large. Therefore, for any $\delta>0$, we can choose γ to be sufficiently large such that $\delta>A/\left(\sigma_{\alpha 0}\sqrt{\gamma}\right)$ and

$$
\begin{aligned}
&\int_\alpha\int_{\boldsymbol{\beta}\notin\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^K}\int_{\boldsymbol{\epsilon}}\prod_{l\neq i} I\Big[\left(p_i-p_l\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)\geq-\frac{\mathbf{x}_i-\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}-\frac{\epsilon_i-\epsilon_l}{\gamma}\\
&-\sigma_{\alpha 0}\left(v_{i,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)-v_{l,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)\right)\Big]f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha\\
\leq\;&\int_\alpha\int_{\boldsymbol{\beta}\notin\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^K}\int_{\boldsymbol{\epsilon}}f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha\leq\frac{\delta}{2},\\
&\int_\alpha\int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}\notin\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^K}\prod_{l\neq i} I\Big[\left(p_i-p_l\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)\geq-\frac{\mathbf{x}_i-\mathbf{x}_l}{\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}-\frac{\epsilon_i-\epsilon_l}{\gamma}\\
&-\sigma_{\alpha 0}\left(v_{i,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)-v_{l,0}\left(\gamma,\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)\right)\Big]f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha\\
\leq\;&\int_\alpha\int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}\notin\left[-\sqrt{\gamma},\sqrt{\gamma}\right]^J}f\left(\boldsymbol{\epsilon}\right)d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right)d\boldsymbol{\beta}\phi\left(\alpha\right)d\alpha\leq\frac{\delta}{2}. \qquad (43)
\end{aligned}
$$

Because $\delta > A/\left(\sigma_{\alpha 0}\sqrt{\gamma}\right)$, for $\boldsymbol{\beta} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^K$and $\boldsymbol{\epsilon} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^J$,

$$\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\frac{\mathbf{x}_i - \mathbf{x}_l}{\sigma_{\alpha 0}\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} - \frac{\epsilon_i - \epsilon_l}{\sigma_{\alpha 0}\gamma} - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right)$$

implies

$$\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\delta - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right).$$

Therefore,

$$\begin{aligned}
& \int_\alpha \int_{\boldsymbol{\beta}} \int_{\boldsymbol{\epsilon}} \prod_{l \neq i} I\Big[\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\delta \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right)\Big] f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha \\
\geq \; & \int_\alpha \int_{\boldsymbol{\beta} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^K} \int_{\boldsymbol{\epsilon} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^K} \prod_{l \neq i} I\Big[\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\delta \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right)\Big] f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha \\
\geq \; & \int_\alpha \int_{\boldsymbol{\beta} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^K} \int_{\boldsymbol{\epsilon} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^K} \prod_{l \neq i} I\Big[\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\frac{\mathbf{x}_i - \mathbf{x}_l}{\sigma_{\alpha 0}\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} - \frac{\epsilon_i - \epsilon_l}{\sigma_{\alpha 0}\gamma} \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right)\Big] f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha. \qquad (44)
\end{aligned}$$

Then, using Equations (43) and (44), we obtain

$$\begin{aligned}
& \int_\alpha \prod_{l \neq i} I\Big[\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\delta \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right)\Big] \phi\left(\alpha\right) d\alpha + \delta \\
\geq \; & \int_\alpha \int_{\boldsymbol{\beta}} \int_{\boldsymbol{\epsilon}} \prod_{l \neq i} I\Big[\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\frac{\mathbf{x}_i - \mathbf{x}_l}{\sigma_{\alpha 0}\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} - \frac{\epsilon_i - \epsilon_l}{\sigma_{\alpha 0}\gamma} \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right)\Big] f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha = s_i.
\end{aligned}$$

Similarly, for $\boldsymbol{\beta} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^K$and $\boldsymbol{\epsilon} \in \left[-\sqrt{\gamma}, \sqrt{\gamma}\right]^J$

$$\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq \delta - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right)$$

implies

$$\left(p_i - p_l\right)\left(\alpha + \eta_{\alpha 0}\right) \geq -\frac{\mathbf{x}_i - \mathbf{x}_l}{\sigma_{\alpha 0}\gamma}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} - \frac{\epsilon_i - \epsilon_l}{\sigma_{\alpha 0}\gamma} - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right).$$

Therefore, we obtain

$$\begin{aligned}
& \int_{\alpha} \prod_{l \neq i} I \Big[\left(p_i - p_l\right) \left(\alpha + \eta_{\alpha 0}\right) \geq \delta \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right) \Big] \phi\left(\alpha\right) d\alpha - \delta \\
\leq \; & \int_{\alpha} \int_{\boldsymbol{\beta}} \int_{\boldsymbol{\epsilon}} \prod_{l \neq i} I \Big[\left(p_i - p_l\right) \left(\alpha + \eta_{\alpha 0}\right) \geq - \frac{\mathbf{x}_i - \mathbf{x}_l}{\sigma_{\alpha 0} \gamma} \boldsymbol{\Sigma}_{\beta 0} \boldsymbol{\beta} - \frac{\epsilon_i - \epsilon_l}{\sigma_{\alpha 0} \gamma} \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right) \Big] f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon} \phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta} \phi\left(\alpha\right) d\alpha = s_i.
\end{aligned}$$

Then, if we let

$$\begin{aligned}
\overline{s}_i \equiv \; & \int_{\alpha} \prod_{l \neq i} I \Big[\left(p_i - p_l\right) \left(\alpha + \eta_{\alpha 0}\right) \geq -\delta \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right) \Big] \phi\left(\alpha\right) d\alpha \\
\underline{s}_i \equiv \; & \int_{\alpha} \prod_{l \neq i} I \Big[\left(p_i - p_l\right) \left(\alpha + \eta_{\alpha 0}\right) \geq \delta \\
& - \left(v_{i,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)\right) \Big] \phi\left(\alpha\right) d\alpha,
\end{aligned}$$

$$\underline{s}_i - \delta \leq s_i \leq \overline{s}_i + \delta. \tag{45}$$

Also, from continuity of $\overline{s}_i$ and $\underline{s}_i$ in δ, we obtain

$$\left|\overline{s}_i - \underline{s}_i\right| \leq \varpi_i, \tag{46}$$

where ϖ_i can be made arbitrarily small by making δ sufficiently small, which can be done by making γ sufficiently large. Then, from the Triangle Inequality and Equations (45) and (46),

$$\left|\overline{s}_i - s_i\right| \leq \left|\overline{s}_i + \delta - s_i\right| + \delta \leq \left|\overline{s}_i + \delta - \left(\underline{s}_i - \delta\right)\right| + \delta \leq \left|\overline{s}_i - \underline{s}_i\right| + 3\delta \leq \varpi_i + 3\delta. \tag{47}$$

Hence, $\left|\overline{s}_i - s_i\right|$ can be made arbitrarily small by making γ sufficiently large. Similarly, $\left|s_i - \underline{s}_i\right| \leq \left|s_i + \delta - \underline{s}_i\right| + \delta$, and since $\underline{s}_i - \delta \leq s_i \leq \overline{s}_i + \delta$,

$$\overline{s}_i + \delta - \left(\underline{s}_i - \delta\right) \geq s_i + \delta - \underline{s}_i \geq 0.$$

Thus,

$$|s_i - \underline{s}_i| \leq |s_i + \delta - \underline{s}_i| + \delta \leq |\overline{s}_i + \delta - (\underline{s}_i - \delta)| + \delta \leq |\overline{s}_i - \underline{s}_i| + 3\delta \leq \varpi_i + 3\delta. \tag{48}$$

Furthermore, we can use similar arguments that resulted in Equations (40) and (41) to derive

$$\begin{aligned}(p_i - p_{i+1}) \left[\eta_{\alpha 0} + \frac{\delta}{p_i - p_{i+1}} + \Phi^{-1} \left(\sum_{l \leq i} (\overline{s}_l + \delta) \right) \right] &\leq v_{i+10}(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}) - v_{i,0}(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}) \\ &\leq (p_i - p_{i+1}) \left[\eta_{\alpha 0} - \frac{\delta}{p_i - p_{i+1}} + \Phi^{-1} \left(\sum_{l \leq i} (\underline{s}_l - \delta) \right) \right],\end{aligned}$$

where δ is chosen to be a small enough positive number to satisfy

$$\delta < min \left\{ \frac{1 - \sum_{l \leq J-1} \overline{s}_l}{J}, min_l \{\underline{s}_l\} \right\}.$$

Then, from Inequalities (47) and (48), for sufficiently small $\delta > 0$, both $0 < \sum_{l \leq i} (\overline{s}_l + \delta) < 1$ and $0 < \sum_{l \leq i} (\underline{s}_l - \delta) < 1$ hold for any $i = 0, \ldots, J-1$.[28]

Therefore, for any $i = 0, \ldots, J-1$,

$$\begin{aligned}&\sum_{k=0}^{i} (p_k - p_{k+1}) \left[\eta_{\alpha 0} + \frac{\delta}{p_i - p_{i+1}} + \Phi^{-1} \left(\sum_{l \leq k} (\overline{s}_l + \delta) \right) \right] \\ \leq\; & v_{i+10}(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}) \leq \sum_{k=0}^{i} (p_k - p_{k+1}) \left[\eta_{\alpha 0} - \frac{\delta}{p_i - p_{i+1}} + \Phi^{-1} \left(\sum_{l \leq k} (\underline{s}_l - \delta) \right) \right]\end{aligned}$$

and since

$$\begin{aligned}\sum_{k=0}^{i} (p_k - p_{k+1}) \left[\eta_{\alpha 0} + \frac{\delta}{p_i - p_{i+1}} + \Phi^{-1} \left(\sum_{l \leq k} (\overline{s}_l + \delta) \right) \right] &\rightarrow \sum_{k=0}^{i} (p_k - p_{k+1}) \left[\eta_{\alpha 0} + \Phi^{-1} \left(\sum_{l \leq k} s_l \right) \right], \\ \sum_{k=0}^{i} (p_k - p_{k+1}) \left[\eta_{\alpha 0} - \frac{\delta}{p_i - p_{i+1}} + \Phi^{-1} \left(\sum_{l \leq k} (\underline{s}_l - \delta) \right) \right] &\rightarrow \sum_{k=0}^{i} (p_k - p_{k+1}) \left[\eta_{\alpha 0} + \Phi^{-1} \left(\sum_{l \leq k} s_l \right) \right],\end{aligned}$$

as $\gamma \rightarrow \infty$, we have proven the claim.

By taking the derivative of the limit of the market share function of good j obtained above

[28] Note that from Inequality (47), $lim_{\delta \searrow 0} \overline{\mathbf{s}} = \mathbf{s}$, therefore, $lim_{\delta \searrow 0} \left[1 - \sum_{l \leq J-1} \overline{s}_l \right] = 1 - \sum_{l \leq J-1} s_l > 0$ and the definition of $\underline{\mathbf{s}}$ imply $\delta > 0$.

(see Equations (38) to (39)) with respect to its price, we obtain

$$\frac{\partial \widetilde{s}_j}{\partial p_j} = \phi\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{j,0} - \widetilde{v}_{j+1,0}}{p_j - p_{j+1}}\right)\frac{\widetilde{v}_{j,0} - \widetilde{v}_{j+1,0}}{\left(p_j - p_{j+1}\right)^2} + \phi\left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{j-1,0} - \widetilde{v}_{j,0}}{p_{j-1} - p_j}\right)\frac{\widetilde{v}_{j-1,0} - \widetilde{v}_{j,0}}{\left(p_j - p_{j-1}\right)^2}.$$

Next, we show that limit of the derivative of the market share function is the same as derived above. Differentiating the market share function given $\mathbf{v}_0$, we obtain

$$\begin{aligned}
\frac{\partial s_j}{\partial p_j} &= \frac{\partial}{\partial p_j}\int_\alpha\int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}}\prod_{l\neq j} I\left(\mathbf{x}_j\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_j\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0}v_{j,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) + \epsilon_j\right. \\
&\quad \left.\geq \mathbf{x}_l\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_l\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0}v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) + \epsilon_l\right) f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha \\
&= \int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}}\frac{\partial}{\partial p_j}\int_\alpha\prod_{l\neq j} I\left(\mathbf{x}_j\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_j\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0}v_{j,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) + \epsilon_j\right. \\
&\quad \left.\geq \mathbf{x}_l\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_l\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0}v_{l,0}\left(\gamma, \mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) + \epsilon_l\right)\phi\left(\alpha\right) d\alpha f\left(\boldsymbol{\epsilon}\right) d\boldsymbol{\epsilon}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}
\end{aligned}$$

First, since the function $\prod_{l\neq j} I\left(\right)$ is nonnegative and the measure space of $\left(\alpha, \boldsymbol{\beta}, \boldsymbol{\epsilon}\right)$ is σ-finite, from Tonelli's Theorem, we can interchange the order of integrals. Next, we prove that the integral and the derivative above can be interchanged. We first focus on the function that is inside the integral over $\boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$, which is,

$$\begin{aligned}
G\left(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \gamma, \boldsymbol{\beta}, \boldsymbol{\epsilon}, \boldsymbol{\theta}_{c0}\right) &\equiv \int_\alpha\prod_{l\neq j} I\left(\mathbf{x}_j\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_j\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0}v_{j,0} + \epsilon_j\right. \\
&\geq \left.\mathbf{x}_l\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + \gamma p_l\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma\sigma_{\alpha 0}v_{l,0} + \epsilon_l\right)\phi\left(\alpha\right) d\alpha.
\end{aligned}$$

It suffices to show that there exists a function $H\left(\right)$ such that

$$\left|\frac{\partial G\left(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \gamma, \boldsymbol{\beta}, \boldsymbol{\epsilon}, \boldsymbol{\theta}_{c0}\right)}{\partial p_j}\right| \leq H\left(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \gamma, \boldsymbol{\beta}, \boldsymbol{\epsilon}, \boldsymbol{\theta}_{c0}\right),$$

where $\int_{\boldsymbol{\beta}} \int_{\boldsymbol{\epsilon}} H(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \gamma, \boldsymbol{\beta}, \boldsymbol{\epsilon}, \boldsymbol{\theta}_{c0}) f(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon} \phi(\boldsymbol{\beta}) d\boldsymbol{\beta} < \infty$. We show it as follows: First,

$$
\begin{aligned}
& G(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \gamma, \boldsymbol{\beta}, \boldsymbol{\epsilon}, \boldsymbol{\theta}_{c0}) \\
= & \int_\alpha \prod_{p_j - p_l > 0} I\left(\alpha \geq -\eta_{\alpha 0} - \frac{v_{j,0} - v_{l,0}}{p_j - p_l} - \frac{(\mathbf{x}_j - \mathbf{x}_l)}{\gamma(p_j - p_l)\sigma_{\alpha 0}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} - \frac{\epsilon_j - \epsilon_l}{\gamma(p_j - p_l)\sigma_{\alpha 0}}\right) \\
& \times \prod_{p_j - p_l < 0} I\left(\alpha \leq -\eta_{\alpha 0} - \frac{v_{j,0} - v_{l,0}}{p_j - p_l} - \frac{(\mathbf{x}_j - \mathbf{x}_l)}{\gamma(p_j - p_l)\sigma_{\alpha 0}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} - \frac{\epsilon_j - \epsilon_l}{\gamma(p_j - p_l)\sigma_{\alpha 0}}\right) \phi(\alpha) d\alpha \\
= & \int_\alpha I\left(Max_{\{l: p_j - p_l > 0\}} \left\{-\eta_{\alpha 0} - \frac{v_{j,0} - v_{l,0}}{p_j - p_l} - \frac{(\mathbf{x}_j - \mathbf{x}_l)}{\gamma(p_j - p_l)\sigma_{\alpha 0}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} - \frac{\epsilon_j - \epsilon_l}{\gamma(p_j - p_l)\sigma_{\alpha 0}}\right\} \leq \alpha \right. \\
\leq & \left. Min_{\{l: p_j - p_l < 0\}} \left\{-\eta_{\alpha 0} - \frac{v_{j,0} - v_{l,0}}{p_j - p_l} - \frac{(\mathbf{x}_j - \mathbf{x}_l)}{\gamma(p_j - p_l)\sigma_{\alpha 0}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} - \frac{\epsilon_j - \epsilon_l}{\gamma(p_j - p_l)\sigma_{\alpha 0}}\right\}\right) \phi(\alpha) d\alpha.
\end{aligned}
\tag{49}
$$

Now, if we let

$$
\begin{aligned}
& \underline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0}) \\
\equiv & Max_{\{l: p_j - p_l > 0\}} \left\{-\eta_{\alpha 0} - \frac{v_{j,0} - v_{l,0}}{p_j - p_l} - \frac{(\mathbf{x}_j - \mathbf{x}_l)}{\gamma(p_j - p_l)\sigma_{\alpha 0}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} - \frac{\epsilon_j - \epsilon_l}{\gamma(p_j - p_l)\sigma_{\alpha 0}}\right\}
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
& \overline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0}) \\
\equiv & Min_{\{l: p_j - p_l < 0\}} \left\{-\eta_{\alpha 0} - \frac{v_{j,0} - v_{l,0}}{p_j - p_l} - \frac{(\mathbf{x}_j - \mathbf{x}_l)}{\gamma(p_j - p_l)\sigma_{\alpha 0}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} - \frac{\epsilon_j - \epsilon_l}{\gamma(p_j - p_l)\sigma_{\alpha 0}}\right\},
\end{aligned}
$$

then

$$
\begin{aligned}
(49) \quad = & \int_\alpha I\left(\underline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0}) \leq \alpha \leq \overline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0})\right) \phi(\alpha) d\alpha \\
= & \left[\Phi\left(\overline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0})\right) - \Phi\left(\underline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0})\right)\right] \\
& \times I\left[\underline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0}) < \overline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0})\right].
\end{aligned}
$$

Thus, taking the derivative of (49) with respect to p_j given $\mathbf{v}_0$ being constant, we obtain

$$
\begin{aligned}
& \frac{\partial}{\partial p_j} \int_\alpha I\left(\underline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0}) \leq \alpha \leq \overline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0})\right) \phi(\alpha) d\alpha \\
= & \left[\phi(\overline{A}) \frac{\partial \overline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0})}{\partial p_j} - \phi(\underline{A}) \frac{\partial \underline{A}(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0})}{\partial p_j}\right] I\left[\underline{A} < \overline{A}\right]
\end{aligned}
$$

Now, let

$$\underline{\mathcal{L}}^*(j) \equiv argmax_{\{l:p_j-p_l>0\}} \left\{ -\eta_{\alpha 0} - \frac{v_{j,0}-v_{l,0}}{p_j-p_l} - \frac{(\mathbf{x}_j-\mathbf{x}_l)}{\gamma(p_j-p_l)\sigma_{\alpha 0}} \mathbf{\Sigma}_{\beta 0}\boldsymbol{\beta} - \frac{\epsilon_j-\epsilon_l}{\gamma(p_j-p_l)\sigma_{\alpha 0}} \right\}.$$

Then, $\underline{\mathcal{L}}^*(j)$ may have multiple elements. But the values of $\boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$ that results in multiple elements have measure zero, and other than those cases, $\underline{\mathcal{L}}^*(j)$ is a singleton. We denote the singleton element to be $\underline{l}^*(j)$. We also define similarly $\overline{\mathcal{L}}^*(j)$ as follows

$$\overline{\mathcal{L}}^*(j) \equiv argmin_{\{l:p_j-p_l<0\}} \left\{ -\eta_{\alpha 0} - \frac{v_{j,0}-v_{l,0}}{p_j-p_l} - \frac{(\mathbf{x}_j-\mathbf{x}_l)}{\gamma(p_j-p_l)\sigma_{\alpha 0}} \mathbf{\Sigma}_{\beta 0}\boldsymbol{\beta} - \frac{\epsilon_j-\epsilon_l}{\gamma(p_j-p_l)\sigma_{\alpha 0}} \right\}.$$

and $\overline{l}^*(j)$ accordingly. Then,

$$\frac{\partial \underline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0})}{\partial p_j} = \frac{v_{j,0}-v_{\underline{l}^*(j),0}}{\left(p_j-p_{\underline{l}^*(j)}\right)^2} + \frac{1}{\gamma}\frac{\left(\mathbf{x}_j-\mathbf{x}_{\underline{l}^*(j)}\right)}{\left(p_j-p_{\underline{l}^*(j)}\right)^2\sigma_{\alpha 0}}\mathbf{\Sigma}_{\beta}\boldsymbol{\beta} + \frac{1}{\gamma}\frac{\epsilon_j-\epsilon_{\underline{l}^*(j)}}{\left(p_j-p_{\underline{l}^*(j)}\right)^2\sigma_{\alpha 0}}$$

except for the values of $\boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$ when $\underline{\mathcal{L}}^*(j)$ contains more than one element. Then, because $\gamma > \underline{\gamma}$,

$$\begin{aligned} & \left| \frac{\partial \underline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0})}{\partial p_j} \right| \\ \leq \; & \max_{l:p_j-p_l>0} \frac{|v_{j,0}-v_{l,0}|}{(p_j-p_l)^2} + \max_{k,l:p_j-p_l>0} \left| \frac{[(\mathbf{x}_j-\mathbf{x}_l)\mathbf{\Sigma}_{\beta 0}]_k}{\underline{\gamma}(p_j-p_l)^2\sigma_{\alpha 0}} \right| \sum_k |\beta|_k + \max_{l:p_j-p_l>0} \frac{|\epsilon_j-\epsilon_l|}{\underline{\gamma}(p_j-p_l)^2\sigma_{\alpha 0}} \end{aligned} \tag{51}$$

Since $\boldsymbol{\beta}$, $\boldsymbol{\epsilon}$ are i.i.d. normally distributed, the RHS of the above inequality is integrable with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$ for any $\underline{\gamma} > 0$. Similar results hold for $\partial \overline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_0)/\partial p_j$. Therefore, the derivative and the integral are interchangeable. Hence,

$$\begin{aligned} & \frac{\partial}{\partial p_j}\int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}}\int_{\alpha} I\left(\underline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}) \leq \alpha \leq \overline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0})\right)\phi(\alpha)\,d\alpha f(\boldsymbol{\epsilon})\,d\boldsymbol{\epsilon}\phi(\boldsymbol{\beta})\,d\boldsymbol{\beta} \\ = \; & \int_{\boldsymbol{\beta}}\int_{\boldsymbol{\epsilon}}\left[\phi(\overline{A})\frac{\partial\overline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0})}{\partial p_j} - \phi(\underline{A})\frac{\partial\underline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0})}{\partial p_j}\right] \\ & \times I\left[\underline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}) < \overline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0})\right] f(\boldsymbol{\epsilon})\,d\boldsymbol{\epsilon}\phi(\boldsymbol{\beta})\,d\boldsymbol{\beta}. \end{aligned}$$

Now, from Equation (50), because $lim_{\gamma\to\infty}\mathbf{v}_0(\mathbf{X},\gamma\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}) = \widetilde{\mathbf{v}}_0$,

$$lim_{\gamma\to\infty}\underline{A}(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}) = max_{l:p_j-p_l>0}\left\{-\eta_{\alpha 0} - \frac{\widetilde{v}_{j,0}-\widetilde{v}_{l,0}}{p_j-p_l}\right\} = -\eta_{\alpha 0} - \underline{a}_j,$$

and similarly,

$$lim_{\gamma\to\infty}\overline{A}\left(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}\right)=max_{l:p_j-p_l<0}\left\{-\eta_{\alpha 0}-\frac{\widetilde{v}_{j,0}-\widetilde{v}_{l,0}}{p_j-p_l}\right\}=-\eta_{\alpha 0}-\overline{a}_j.$$

Hence,

$$\underline{\mathcal{L}}^*\left(j\right)\to argmax_{\{l:p_j-p_l>0\}}\left\{-\eta_{\alpha 0}-\frac{\widetilde{v}_{j,0}-\widetilde{v}_{l,0}}{p_j-p_l}\right\}=j-1\ as\ \gamma\to\infty.$$

Hence,

$$\begin{aligned}&I\left(\underline{A}\left(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}\right)<\overline{A}\left(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}\right)\right)\\ \to\quad &I(-\underline{a}_j<-\overline{a}_j)=1\ as\ \gamma\to\infty\end{aligned}$$

Therefore,

$$\frac{\partial\underline{A}\left(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}\right)}{\partial p_j}\to-\frac{\partial}{\partial p_j}\left[\frac{\widetilde{v}_{j-1,0}-\widetilde{v}_{j,0}}{p_{j-1}-p_j}\right]\ as\ \gamma\to\infty.$$

Similarly,

$$\overline{\mathcal{L}}^*\left(j\right)\to argmin_{\{l:p_j-p_l<0\}}\left\{-\eta_{\alpha 0}-\frac{\widetilde{v}_{j,0}-\widetilde{v}_{l,0}}{p_j-p_l}\right\}=j+1\ as\ \gamma\to\infty.$$

Hence,

$$\frac{\partial\overline{A}\left(\mathbf{X},\mathbf{p},\mathbf{v}_0,\boldsymbol{\beta},\boldsymbol{\epsilon},j,\gamma,\boldsymbol{\theta}_{c0}\right)}{\partial p_j}\to-\frac{\partial}{\partial p_j}\left[\frac{\widetilde{v}_{j,0}-\widetilde{v}_{j+1,0}}{p_j-p_{j+1}}\right]\ as\ \gamma\to\infty.$$

Therefore, as $\gamma\to\infty$,

$$\begin{aligned}&\frac{\partial}{\partial p_j}\int_\alpha\prod_{l=0,l\neq j}^{J}I\left(\mathbf{x}_j\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}+\gamma p_j\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\gamma\sigma_{\alpha 0}v_{j,0}+\epsilon_j\right.\\ \geq\quad&\left.\mathbf{x}_l\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta}+\gamma p_l\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\gamma\sigma_{\alpha 0}v_{l,0}+\epsilon_l\right)\phi\left(\alpha\right)d\alpha\\ \to\quad&\phi\left(-\eta_{\alpha 0}-\frac{\widetilde{v}_{j,0}-\widetilde{v}_{j+1,0}}{p_j-p_{j+1}}\right)\frac{\widetilde{v}_{j,0}-\widetilde{v}_{j+1,0}}{\left(p_j-p_{j+1}\right)^2}+\phi\left(-\eta_{\alpha 0}-\frac{\widetilde{v}_{j-1,0}-\widetilde{v}_{j,0}}{p_{j-1}-p_j}\right)\frac{\widetilde{v}_{j-1,0}-\widetilde{v}_{j,0}}{\left(p_{j-1}-p_j\right)^2}.\end{aligned}$$

Furthermore, Equation (51) shows that the conditions for the Dominated Convergence Theorem holds so that the limit and the integral can be interchanged to derive

$$
\begin{aligned}
& lim_{\gamma \to \infty} \int_{\boldsymbol{\beta}} \int_{\boldsymbol{\epsilon}} \left[\frac{\partial}{\partial p_j} \int_{\alpha} \prod_{l=0, l \neq j}^{J} I \left(\mathbf{x}_j \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} + \gamma p_j \sigma_{\alpha 0} \left(\alpha + \eta_{\alpha 0} \right) + \gamma \sigma_{\alpha 0} v_{j,0} \left(\gamma \right) + \epsilon_j \right. \right. \\
\geq \quad & \left. \mathbf{x}_l \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} + \gamma p_l \sigma_{\alpha 0} \left(\alpha + \eta_{\alpha 0} \right) + \gamma \sigma_{\alpha 0} v_{l,0} + \epsilon_l \right) \phi \left(\alpha \right) d\alpha] \, f \left(\boldsymbol{\epsilon} \right) d\boldsymbol{\epsilon} \phi \left(\boldsymbol{\beta} \right) d\boldsymbol{\beta} \\
= \quad & \int_{\boldsymbol{\beta}} \int_{\boldsymbol{\epsilon}} lim_{\gamma \to \infty} \left[\phi \left(\overline{A} \right) \frac{\partial \overline{A} \left(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0} \right)}{\partial p_j} - \phi \left(\underline{A} \right) \frac{\partial \underline{A} \left(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0} \right)}{\partial p_j} \right] \\
& \times I \left[\underline{A} \left(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0} \right) < \overline{A} \left(\mathbf{X}, \mathbf{p}, \mathbf{v}_0, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \gamma, \boldsymbol{\theta}_{c0} \right) \right] f \left(\boldsymbol{\epsilon} \right) d\boldsymbol{\epsilon} \phi \left(\boldsymbol{\beta} \right) d\boldsymbol{\beta}. \\
= \quad & \phi \left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{j,0} - \widetilde{v}_{j+1,0}}{p_j - p_{j+1}} \right) \frac{\widetilde{v}_{j,0} - \widetilde{v}_{j+1,0}}{\left(p_j - p_{j+1} \right)^2} + \phi \left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{j-1,0} - \widetilde{v}_{j,0}}{p_{j-1} - p_j} \right) \frac{\widetilde{v}_{j-1,0} - \widetilde{v}_{j,0}}{\left(p_{j-1} - p_j \right)^2}.
\end{aligned}
$$

Therefore, we have shown that the derivative of the limiting market share function is the same as the limit of its derivative.

We now argue that $\eta_{\alpha 0}$ is identified from the equality of marginal revenues at the limit. From what we have derived, the following holds

$$
\begin{aligned}
& lim_{\gamma \to \infty} \frac{MR_j \left(\mathbf{X}, \gamma \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0} \right)}{\gamma} \\
& = lim_{\gamma \to \infty} \frac{1}{\gamma} \left[\gamma p_j + \left[\frac{\partial s_j \left(\mathbf{X}, \gamma \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0} \right)}{\partial \left(\gamma p_j \right)} \right]^{-1} s_j \right] = lim_{\gamma \to \infty} \frac{1}{\gamma} \left[\gamma p_j + \gamma \left[\frac{\partial s_j \left(\mathbf{X}, \gamma \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0} \right)}{\partial p_j} \right]^{-1} s_j \right] \\
& = p_j + \left[\phi \left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{j,0} - \widetilde{v}_{j+1,0}}{p_j - p_{j+1}} \right) \frac{\widetilde{v}_{j,0} - \widetilde{v}_{j+1,0}}{\left(p_j - p_{j+1} \right)^2} + \phi \left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{j-1,0} - \widetilde{v}_{j,0}}{p_{j-1} - p_j} \right) \frac{\widetilde{v}_{j-1,0} - \widetilde{v}_{j,0}}{\left(p_{j-1} - p_j \right)^2} \right]^{-1} s_j \\
= \quad & p_j + \left[-\phi \left(\Phi^{-1} \left(\sum_{l \leq j} s_l \right) \right) \frac{\Phi^{-1} \left(\sum_{l \leq j} s_l \right) + \eta_{\alpha 0}}{p_j - p_{j+1}} - \phi \left(\Phi^{-1} \left(\sum_{l \leq j-1} s_l \right) \right) \frac{\Phi^{-1} \left(\sum_{l \leq j-1} s_l \right) + \eta_{\alpha 0}}{p_{j-1} - p_j} \right]^{-1} s_j,
\end{aligned}
\tag{52}
$$

where $\widetilde{v}_{00} = 0$, $p_0 = 0$, and

$$
\Phi \left(-\eta_{\alpha 0} - \frac{\widetilde{v}_{i,0} - \widetilde{v}_{i+1,0}}{p_i - p_{i+1}} \right) = \sum_{l \leq i} s_l.
$$

We can similarly write down the limit of the marginal revenue function of the second market.

Note that we can ensure that the derivative of the market share function with respect to price is negative by first choosing $\mathbf{s}$ and $\mathbf{s}^{\dagger}$ so that $\sum_{l \leq j} s_l$ and $\sum_{l \leq j^{\dagger}} s_l^{\dagger}$ are small enough that $\Phi^{-1} \left(\sum_{l \leq j} s_l \right) + \eta_{\alpha 0} < 0$ as well as $\Phi^{-1} \left(\sum_{l \leq j^{\dagger}} s_l^{\dagger} \right) + \eta_{\alpha 0} < 0$.

We now show that there exist $\mathbf{p}$, $\mathbf{p}^{\dagger}$, $\mathbf{s}$, $\mathbf{s}^{\dagger}$ and $j \in \{1, \ldots J-1\}$, $j^{\dagger} \in \left\{1, \ldots, J^{\dagger} - 1\right\}$ such that

$$lim_{\gamma\to\infty}\frac{MR_j\left(\mathbf{X},\gamma\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)}{\gamma}=lim_{\gamma\to\infty}\frac{MR_{j^\dagger}\left(\mathbf{X}^\dagger,\gamma\mathbf{p}^\dagger,\mathbf{s}^\dagger,\boldsymbol{\theta}_{c0}\right)}{\gamma}. \tag{53}$$

The equality above holds for some $(\mathbf{p},\mathbf{s})$ and $\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger\right)$ because if it does not hold, then we can adjust $p_j-p^\dagger_{j^\dagger}$ while keeping p_j-p_{j+1}, $j=1,\ldots,J-1$, $p^\dagger_{j^\dagger}-p^\dagger_{j^\dagger+1}$, $j^\dagger=1,\ldots,J^\dagger-1$ constant to ensure equality. Note that we can keep $\mathbf{s}$, $\mathbf{s}^\dagger$ the same as well by adjusting $\widetilde{\mathbf{v}}_0$. Further, we can make p_j and $p^\dagger_{j^\dagger}$ large enough so that marginal revenues are positive.

We next claim that Equation (53) identifies $\eta_{\alpha0}$. To see why, consider the case where $p_j=p^\dagger_{j^\dagger}$. Then, from Equation (53), we derive

$$\eta_{\alpha0}=-\frac{\left(s^\dagger_{j^\dagger}/s_j\right)\left(C\left(\mathbf{p},\mathbf{s},j,j+1\right)+C\left(\mathbf{p},\mathbf{s},j-1,j\right)\right)-C\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,j^\dagger,j^\dagger+1\right)-C\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,j^\dagger-1,j^\dagger\right)}{\left(s^\dagger_{j^\dagger}/s_j\right)\left(B\left(\mathbf{p},\mathbf{s},j,j+1\right)+B\left(\mathbf{p},\mathbf{s},j-1,j\right)\right)-B\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,j^\dagger,j^\dagger+1\right)-B\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,j^\dagger-1,j^\dagger\right)},$$

where $B\left(\mathbf{p},\mathbf{s},j,j+1\right)\equiv\phi\left(\Phi^{-1}\left(\sum_{l\le j}s_l\right)\right)/\left(p_j-p_{j+1}\right)$,

$C\left(\mathbf{p},\mathbf{s},j,j+1\right)\equiv\phi\left(\Phi^{-1}\left(\sum_{l\le j}s_l\right)\right)\Phi^{-1}\left(\sum_{l\le j}s_l\right)/\left(p_j-p_{j+1}\right)$ and these expressions for the second market are similarly defined. We can identify $\eta_{\alpha0}$ as long as we can find $(\mathbf{p},\mathbf{s})$ and $\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger\right)$ such that the denominator is nonzero. Such price-market share combinations exist. For example, set $j=j^\dagger$, $\mathbf{s}=\mathbf{s}^\dagger$, $p_{j-1}-p_j=p^\dagger_{j^\dagger-1}-p^\dagger_{j^\dagger}$, $p_j-p_{j+1}\neq p^\dagger_{j^\dagger}-p^\dagger_{j^\dagger+1}$, and $\Phi^{-1}\left(\sum_{l\le j}s_l\right)=-\eta_{\alpha0}$. Note that since $\mathbf{s}=\mathbf{s}^\dagger$, market size variation is not needed for identification of $\eta_{\alpha0}$.

Note also that while the above equality holds for some $(\mathbf{p},\mathbf{s})$, $\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger\right)$ and identifies $\eta_{\alpha0}$, the price vectors used are not the actual prices which are infinite.

Next, we show that the equality holds also for large γ, and identifies $\eta_{\alpha0}$ as follows. Let

$$\begin{aligned}&A\left(\mathbf{p},\mathbf{s},\eta_{\alpha0},j\right)\equiv lim_{\gamma\to\infty}\frac{MR_j\left(\mathbf{X},\gamma\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)}{\gamma}\\ =\;& p_j+\left[-\phi\left(\Phi^{-1}\left(\sum_{l\le j}s_l\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j}s_l\right)+\eta_{\alpha0}}{p_j-p_{j+1}}-\phi\left(\Phi^{-1}\left(\sum_{l\le j-1}s_l\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j-1}s_l\right)+\eta_{\alpha0}}{p_{j-1}-p_j}\right]^{-1}s_j.\end{aligned}$$

Then, for any $\eta_\alpha\neq\eta_{\alpha0}$, $A\left(\mathbf{p},\mathbf{s},\eta_\alpha,j\right)\neq A\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\eta_\alpha,j\right)$. Now, consider small $\delta>0$ such that $\delta\le\left|A\left(\mathbf{p},\mathbf{s},\eta_\alpha,j\right)-A\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\eta_\alpha,j\right)\right|$. Now, choose $\mathbf{p}'$ such that $p'_i=p_i+\delta$ for $i=1,\ldots,J$. Then, because $p'_i-p'_{i+1}=p_i-p_{i+1}$ for $i=1,\ldots,J-1$,

$$A\left(\mathbf{p}',\mathbf{s},\eta_{\alpha0},j\right)-A\left(\mathbf{p},\mathbf{s},\eta_{\alpha0},j\right)=\delta,\ A\left(\mathbf{p}',\mathbf{s},\eta_\alpha,j\right)-A\left(\mathbf{p},\mathbf{s},\eta_\alpha,j\right)=\delta.$$

Similarly, we choose prices $\mathbf{p}''$ such that $p_i'' = p_i - \delta$ for $i = 1, \ldots, J-1$. Then,

$$A\left(\mathbf{p}'', \mathbf{s}, \eta_{\alpha 0}, j\right) - A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha 0}, j\right) = -\delta,\ A\left(\mathbf{p}'', \mathbf{s}, \eta_{\alpha}, j\right) - A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha}, j\right) = -\delta.$$

For any $\delta > 0$, for a sufficiently large $\gamma > 0$,

$$\left|\frac{MR_j\left(\mathbf{X}, \gamma\mathbf{p}', \mathbf{s}, \boldsymbol{\theta}_{c0}\right)}{\gamma} - A\left(\mathbf{p}', \mathbf{s}, \eta_{\alpha 0}, j\right)\right| < \frac{\delta}{3},\ \left|\frac{MR_j\left(\mathbf{X}^\dagger, \gamma\mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{c0}\right)}{\gamma} - A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha 0}, j\right)\right| < \frac{\delta}{3},$$

$$\left|\frac{MR_j\left(\mathbf{X}, \gamma\mathbf{p}'', \mathbf{s}, \boldsymbol{\theta}_{c0}\right)}{\gamma} - A\left(\mathbf{p}'', \mathbf{s}, \eta_{\alpha 0}, j\right)\right| < \frac{\delta}{3},$$

$$\left|\frac{MR_j\left(\mathbf{X}, \gamma\mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c}\right)}{\gamma} - A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha}, j\right)\right| < \frac{\delta}{3},\ \left|\frac{MR_j\left(\mathbf{X}^\dagger, \gamma\mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{c}\right)}{\gamma} - A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha}, j\right)\right| < \frac{\delta}{3},$$

Then,

$$\frac{MR_i\left(\mathbf{X}, \gamma\mathbf{p}', \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - MR_i\left(\mathbf{X}^\dagger, \gamma\mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{c0}\right)}{\gamma} > \frac{\delta}{3} > 0$$

$$\frac{MR_i\left(\mathbf{X}^\dagger, \gamma\mathbf{p}'', \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - MR_i\left(\mathbf{X}^\dagger, \gamma\mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{c0}\right)}{\gamma} < -\frac{\delta}{3} < 0$$

Therefore, from continuity of the marginal revenue function, it follows from the Intermediate Value Theorem that there exists $\widetilde{\mathbf{p}}$ such that $|\widetilde{\mathbf{p}} - \mathbf{p}| < \delta$, and

$$\frac{MR_i\left(\mathbf{X}, \gamma\widetilde{\mathbf{p}}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) - MR_i\left(\mathbf{X}^\dagger, \gamma\mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{c0}\right)}{\gamma} = 0.$$

Furthermore, from the Triangle Inequality,

$$\begin{aligned}
& \left|A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha}, j\right) - A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha}, j\right)\right| \\
\leq\ & \left|A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha}, j\right) - A\left(\widetilde{\mathbf{p}}, \mathbf{s}, \eta_{\alpha}, j\right)\right| + \left|A\left(\widetilde{\mathbf{p}}, \mathbf{s}, \eta_{\alpha}, j\right) - A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha}, j\right)\right| \\
\leq\ & \left|A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha}, j\right) - A\left(\widetilde{\mathbf{p}}, \mathbf{s}, \eta_{\alpha}, j\right)\right| + \left|A\left(\widetilde{\mathbf{p}}, \mathbf{s}, \eta_{\alpha}, j\right) - \frac{MR_j\left(\mathbf{X}, \gamma\widetilde{\mathbf{p}}, \mathbf{s}, \boldsymbol{\theta}_{c}\right)}{\gamma}\right| \\
& + \left|\frac{MR_j\left(\mathbf{X}, \gamma\mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{c}\right)}{\gamma} - A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha}, j\right)\right| + \frac{\left|MR_j\left(\mathbf{X}, \gamma\widetilde{\mathbf{p}}, \mathbf{s}, \boldsymbol{\theta}_{\boldsymbol{c}}\right) - MR_j\left(\mathbf{X}, \gamma\mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{\boldsymbol{c}}\right)\right|}{\gamma}
\end{aligned}$$

Therefore, for sufficiently small $\delta > 0$,

$$
\begin{aligned}
&\frac{\left|MR_j\left(\mathbf{X}, \gamma\widetilde{\mathbf{p}}, \mathbf{s}, \boldsymbol{\theta_c}\right) - MR_j\left(\mathbf{X}, \gamma\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \boldsymbol{\theta_c}\right)\right|}{\gamma} \\
&\geq \left|A\left(\mathbf{p}, \mathbf{s}, \eta_\alpha, j\right) - A\left(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \eta_\alpha, j\right)\right| - \left|A\left(\mathbf{p}, \mathbf{s}, \eta_\alpha, j\right) - A\left(\widetilde{\mathbf{p}}, \mathbf{s}, \eta_\alpha, j\right)\right| - \frac{2}{3}\delta \\
&> 0
\end{aligned}
$$

This is because $\left|A\left(\mathbf{p}, \mathbf{s}, \eta_\alpha, j\right) - A\left(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \eta_\alpha, j\right)\right| > 0$ from identification, and from continuity of $A\left(\mathbf{p}, \mathbf{s}, \eta_\alpha, j\right)$ with respect to $\mathbf{p}$, $\left|A\left(\mathbf{p}, \mathbf{s}, \eta_\alpha, j\right) - A\left(\widetilde{\mathbf{p}}, \mathbf{s}, \eta_\alpha, j\right)\right|$ can be made arbitrarily small for sufficiently small $\delta > 0$. Given the assumptions, we can find firms in the population in different markets m and m' satisfying $\left(\mathbf{X}_m, \mathbf{p}_m, \mathbf{s}_m\right) = \left(\mathbf{X}, \gamma\widetilde{\mathbf{p}}, \mathbf{s}\right)$, $\left(\mathbf{X}_{m'}, \mathbf{p}_{m'}, \mathbf{s}_{m'}\right) = \left(\mathbf{X}^{\dagger}, \gamma\widetilde{\mathbf{p}}^{\dagger}, \mathbf{s}^{\dagger}\right)$. Therefore, $\eta_{\alpha 0}$ is identified.

We next show that the market size is identified. Recall that the equation for market size is $log\left(Q\right) = \lambda_0 + \mathbf{z}\boldsymbol{\lambda}_{z0}$. Let $\mathbf{s}$ be the true market share, i.e. $ln\left(s_{im}\right) = ln\left(q_{im}\right) - ln\left(Q_m\right) = ln\left(q_{im}\right) - \lambda_0 - \mathbf{z}\boldsymbol{\lambda}_{z0}$. Then, the recovered market share satisfies $\widehat{\mathbf{s}} = \chi\left(\mathbf{z}\right)\mathbf{s}$, where $\chi\left(\mathbf{z}\right) \equiv exp\left[\left(\lambda_0 + \mathbf{z}\boldsymbol{\lambda}_{z0}\right) - \left(\widehat{\lambda} + \mathbf{z}\widehat{\boldsymbol{\lambda}}_z\right)\right]$. Then, if $\left(\widehat{\lambda}, \widehat{\boldsymbol{\lambda}}_z\right) = \left(\lambda_0, \boldsymbol{\lambda}_{z0}\right)$, $\chi\left(\mathbf{z}\right) = 1$ for all $\mathbf{z}$. On the other hand, if $\left(\widehat{\lambda}, \widehat{\boldsymbol{\lambda}}_z\right) \neq \left(\lambda_0, \boldsymbol{\lambda}_{z0}\right)$, then for some $\mathbf{z}$, we have $\chi\left(\mathbf{z}\right) \neq 1$. We show below that $\chi\left(\mathbf{z}\right) = 1$ for all $\mathbf{z}$.

To do so, we choose two firms with $\mathbf{z} = \mathbf{z}^{\dagger}$ so that $\widehat{\mathbf{s}} = \chi\mathbf{s}$, $\widehat{\mathbf{s}}^{\dagger} = \chi\mathbf{s}^{\dagger}$. Then, expected cost conditional on observables when true market share is observed is the same for these two firms if and only if it is also the same when the true market share is not observed. That is,

$$
E\left[C\middle|\left(\widetilde{q} = q, \widetilde{\mathbf{w}} = \mathbf{w}, \widetilde{\mathbf{p}} = \mathbf{p}, \widetilde{\mathbf{s}} = \widehat{\mathbf{s}}, \widetilde{\mathbf{X}} = \mathbf{X}, j\right)\right] = E\left[C\middle|\left(\widetilde{q} = q, \widetilde{\mathbf{w}} = \mathbf{w}, \widetilde{\mathbf{p}} = \mathbf{p}^{\dagger}, \widetilde{\mathbf{s}} = \widehat{\mathbf{s}}^{\dagger}, \widetilde{\mathbf{X}} = \mathbf{X}^{\dagger}, j^{\dagger}\right)\right] \tag{54}
$$

if and only if

$$
E\left[C\middle|\left(\widetilde{q} = q, \widetilde{\mathbf{w}} = \mathbf{w}, \widetilde{\mathbf{p}} = \mathbf{p}, \widetilde{\mathbf{s}} = \mathbf{s}, \widetilde{\mathbf{X}} = \mathbf{X}, j\right)\right] = E\left[C\middle|\left(\widetilde{q} = q, \widetilde{\mathbf{w}} = \mathbf{w}, \widetilde{\mathbf{p}} = \mathbf{p}^{\dagger}, \widetilde{\mathbf{s}} = \mathbf{s}^{\dagger}, \widetilde{\mathbf{X}} = \mathbf{X}^{\dagger}, j^{\dagger}\right)\right]
$$

Therefore, from Lemma 1, for those two firms with $q = q^{\dagger}$, $\mathbf{w} = \mathbf{w}^{\dagger}$ and $\mathbf{x} = \mathbf{x}^{\dagger}$, we know that Equation (54) holds if and only if

$$
MR_j\left(\mathbf{p}, \mathbf{s}, \mathbf{X}; \boldsymbol{\theta}_{c0}\right) = MR_{j^{\dagger}}\left(\mathbf{p}^{\dagger}, \mathbf{s}^{\dagger}, \mathbf{X}^{\dagger}; \boldsymbol{\theta}_{c0}\right).
$$

Then, we prove that Condition 1 holds for $\boldsymbol{\theta}'_{c0} \equiv \left(\chi_0, \eta_{\alpha 0}\right)$, i.e., we essentially prove joint identification of $\chi_0 = 1$ and $\eta_{\alpha 0}$. That is, we show that for any $\eta_{\alpha 0} < 0$ and $\eta_\alpha < 0$ such that

$\eta_\alpha \neq \eta_{\alpha 0}$, for $\chi \neq 1$, there exist $(\mathbf{p}, \mathbf{s})$ and $\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger\right)$ such that for $\mathbf{s} = \chi_0 \mathbf{s}$ and $\mathbf{s}^\dagger = \chi_0 \mathbf{s}^\dagger$, $\widehat{\mathbf{s}} = \chi \mathbf{s}$ and $\widehat{\mathbf{s}}^\dagger = \chi \mathbf{s}^\dagger$,

$$\begin{aligned} A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha 0}, j\right) &= A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha 0}, j^\dagger\right) \\ & \quad \textit{and either } A\left(\mathbf{p}, \widehat{\mathbf{s}}, \eta_\alpha, j\right) \neq A\left(\mathbf{p}^\dagger, \widehat{\mathbf{s}}^\dagger, \eta_\alpha, j^\dagger\right) \\ & \quad \textit{or max}\left\{\sum_{l \leq j} \widehat{s}_l, \sum_{l \leq j^\dagger} \widehat{s}_l^\dagger\right\} > 1 \textit{ or both.} \end{aligned}$$

On the other hand, if $\chi = 1$, then $\widehat{\mathbf{s}}^\dagger = \mathbf{s}^\dagger$, and since in this case, $\eta_{\alpha 0}$ is identified, $A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha 0}, j\right) = A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha 0}, j^\dagger\right)$ for any $(\mathbf{p}, \mathbf{s})$ if and only if $A\left(\mathbf{p}, \widehat{\mathbf{s}}, \eta_{\alpha 0}, j\right) = A\left(\mathbf{p}^\dagger, \widehat{\mathbf{s}}^\dagger, \eta_{\alpha 0}, j^\dagger\right)$.

Now, consider the case $\chi \neq 1$. Let p_{j+1}, $p_{j^\dagger+1}$ satisfy

$$p_j - p_{j+1} \to 0,\ p_{j^\dagger}^\dagger - p_{j^\dagger+1}^\dagger \to 0.$$

Now, choose $\mathbf{s}$ and $\mathbf{s}^\dagger$ such that $\sum_{l \leq j} s_l = \sum_{l \leq j^\dagger} s_l^\dagger = \Phi\left(-\eta_{\alpha 0}\right)$. Then

$$\begin{aligned} & -\phi\left(\Phi^{-1}\left(\sum_{l \leq j} s_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l \leq j} s_l\right) + \eta_{\alpha 0}}{p_j - p_{j+1}} - \phi\left(\Phi^{-1}\left(\sum_{l \leq j-1} s_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l \leq j-1} s_l\right) + \eta_{\alpha 0}}{p_{j-1} - p_j} \\ = & -\phi\left(\Phi^{-1}\left(\sum_{l \leq j-1} s_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l \leq j-1} s_l\right) + \eta_{\alpha 0}}{p_{j-1} - p_j} < 0. \end{aligned}$$

The negative sign holds because $\sum_{l \leq j^\dagger - 1} s_l^\dagger < \Phi\left(-\eta_{\alpha 0}\right)$ and $\eta_{\alpha 0} < 0$. Hence, $p_j > A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha 0}, j\right)$. Then, choose $s_{j^\dagger}^\dagger = s_j$ such that $\sum_{l \leq j-1} s_l = \sum_{l \leq j^\dagger - 1} s_l^\dagger$, but $p_{j-1}^\dagger - p_j^\dagger \neq p_{j^\dagger-1}^\dagger - p_{j^\dagger}^\dagger$. Thus, $A\left(\mathbf{p}, \mathbf{s}, \eta_{\alpha 0}, j\right) = A\left(\mathbf{p}^\dagger, \mathbf{s}^\dagger, \eta_{\alpha 0}, j^\dagger\right)$, implies $p_j \neq p_j^\dagger$.

Now, let, $\widehat{\mathbf{s}} = \chi \mathbf{s}$, $\widehat{\mathbf{s}}^\dagger = \chi \mathbf{s}^\dagger$. Then, if $max\left\{\sum_{l \leq j} \widehat{s}_l, \sum_{l \leq j^\dagger} \widehat{s}_l^\dagger\right\} < 1$, and $\Phi^{-1}\left(\sum_{l \leq j} \widehat{s}_l\right) + \eta_\alpha \neq 0$, then,

$$\begin{aligned} & \left[-\phi\left(\Phi^{-1}\left(\sum_{l \leq j} \widehat{s}_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l \leq j} \widehat{s}_l\right) + \eta_\alpha}{p_j - p_{j+1}} - \phi\left(\Phi^{-1}\left(\sum_{l \leq j-1} \widehat{s}_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l \leq j-1} \widehat{s}_l\right) + \eta_\alpha}{p_{j-1} - p_j}\right]^{-1} \\ & \times \widehat{s}_j \to 0 \end{aligned}$$

as $p_j - p_{j+1} \to 0$, and the same holds for firm $j^\dagger$. Therefore, $A\left(\mathbf{p}, \widehat{\mathbf{s}}, \eta_\alpha, j\right) \to p_j$, and $A\left(\mathbf{p}^\dagger, \widehat{\mathbf{s}}^\dagger, \eta_\alpha, j^\dagger\right) \to p_{j^\dagger}^\dagger$. Then, since $p_j \neq p_j^\dagger$, for sufficiently small $\left|p_j - p_{j+1}\right|$ and $\left|p_{j^\dagger}^\dagger - p_{j^\dagger+1}^\dagger\right|$, $A\left(\mathbf{p}, \widehat{\mathbf{s}}, \eta_\alpha, j\right) \neq A\left(\mathbf{p}^\dagger, \widehat{\mathbf{s}}^\dagger, \eta_\alpha, j^\dagger\right)$. If $max\left\{\sum_{l \leq j^\dagger} \widehat{s}_l, \sum_{l \leq j^\dagger} \widehat{s}_l^\dagger\right\} \geq 1$, then either $\Phi^{-1}\left(\sum_{l \leq j} \widehat{s}_l\right)$ or $\Phi^{-1}\left(\sum_{l \leq j^\dagger} \widehat{s}_l^\dagger\right)$

or both are not defined.

Next, consider the case where for $\sum_{l\le j} s_l = \sum_{l\le j^\dagger} s_l^\dagger = \Phi(-\eta_{\alpha 0})$, $\sum_{l\le j} \widehat{s}_l = \sum_{l\le j} \widehat{s}_l^\dagger = \Phi(-\eta_\alpha)$ holds. Then choose $\mathbf{p}$ and $\mathbf{p}^\dagger$ such that $p_j - p_{j+1} = p_{j^\dagger}^\dagger - p_{j^\dagger+1}^\dagger$ and $p_{j-1} - p_j = p_{j^\dagger-1}^\dagger - p_{j^\dagger}^\dagger$. Then,

$$-\phi\left(\Phi^{-1}\left(\sum_{l\le j^\dagger} s_l^\dagger\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j^\dagger} s_l^\dagger\right)+\eta_{\alpha 0}}{p_{j^\dagger}^\dagger - p_{j^\dagger+1}^\dagger} - \phi\left(\Phi^{-1}\left(\sum_{l\le j^\dagger-1} s_l^\dagger\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j^\dagger-1} s_l^\dagger\right)+\eta_{\alpha 0}}{p_{j^\dagger-1}^\dagger - p_{j^\dagger}^\dagger}$$

$$= -\phi\left(\Phi^{-1}\left(\sum_{l\le j^\dagger-1} s_l^\dagger\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j^\dagger-1} s_l^\dagger\right)+\eta_{\alpha 0}}{p_{j^\dagger-1}^\dagger - p_{j^\dagger}^\dagger}.$$

Next, choose $p_j^\dagger$ such that

$$p_j^\dagger = p_j - \left[\phi\left(\Phi^{-1}\left(\sum_{l\le j-1} s_l\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j-1} s_l\right)+\eta_{\alpha 0}}{p_{j-1} - p_j}\right]^{-1} s_j$$

$$+\left[\phi\left(\Phi^{-1}\left(\sum_{l\le j^+-1} s_l^\dagger\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j^\dagger-1} s_l^\dagger\right)+\eta_{\alpha 0}}{p_{j^\dagger-1}^\dagger - p_{j^\dagger}^\dagger}\right]^{-1} s_{j^\dagger}^\dagger.$$

Then, for $\chi < 1$, since Φ is normally distributed,

$$\left[\phi\left(\Phi^{-1}\left(\chi\sum_{l\le j-1} s_l\right)\right)\frac{\Phi^{-1}\left(\chi\sum_{l\le j-1} s_l\right)+\eta_{\alpha}}{p_{j-1} - p_j}\right]^{-1} \chi s_j$$

$$-\left[\phi\left(\Phi^{-1}\left(\sum_{l\le j-1} s_l\right)\right)\frac{\Phi^{-1}\left(\sum_{l\le j-1} s_l\right)+\eta_{\alpha}}{p_{j-1} - p_j}\right]^{-1} s_j$$

can be made arbitrarily large by making $\sum_{l\le j-1} s_l$ to be sufficiently small. Then, $A(\mathbf{p}, \widehat{\mathbf{s}}, \eta_\alpha, j) \neq A\left(\mathbf{p}^\dagger, \widehat{\mathbf{s}}^\dagger, \eta_\alpha, j^\dagger\right)$. Similar arguments can be made for $\chi > 1$. Therefore, we have proven the identification of $(\chi_0, \eta_{\alpha 0})$ using the limiting marginal revenue functions. As we argued for identification of $\eta_{\alpha 0}$, while the argument above uses infinite prices, we can show that it works approximately for sufficiently large prices.

We next show that $\sigma_{\alpha 0}$ is identified. Since we have already shown that $\eta_{\alpha 0}$ and $\chi_0 = 1$ are identified, we assume we can recover the true market shares $\mathbf{s}$ and $\mathbf{s}^\dagger$, and only focus on identifying $(\sigma_\alpha, \boldsymbol{\sigma_\beta})$ of $\boldsymbol{\theta}_c = (\eta_{\alpha 0}, \sigma_\alpha, \boldsymbol{\sigma_\beta})$.

Below, we start by assuming that $\boldsymbol{\theta}_c$ is not identified. Then, from Condition 1, there exists

$\boldsymbol{\theta}_c = (\eta_{\alpha 0}, \sigma_\alpha, \boldsymbol{\sigma}_{\boldsymbol{\beta}}) \neq \boldsymbol{\theta}_{c0}$ such that for any $(\mathbf{X}, \mathbf{p}, \mathbf{s}, j)$ and $\left(\mathbf{X}^\dagger, \mathbf{p}^\dagger, \mathbf{s}^\dagger, j^\dagger\right)$, that satisfies $\mathbf{x}_j = \mathbf{x}^\dagger_{j^\dagger}$, both

$$MR\left(\mathbf{X}, \mathbf{p}, \mathbf{s}, j, \boldsymbol{\theta}_{c0}\right) = MR\left(\mathbf{X}^\dagger, \mathbf{p}^\dagger, \mathbf{s}^\dagger, j^\dagger, \boldsymbol{\theta}_{c0}\right) > 0,\ MR\left(\mathbf{X}, \mathbf{p}, \mathbf{s}, j, \boldsymbol{\theta}_{c}\right) = MR\left(\mathbf{X}^\dagger, \mathbf{p}^\dagger, \mathbf{s}^\dagger, j^\dagger, \boldsymbol{\theta}_{c}\right) > 0 \tag{55}$$

hold. Then, we demonstrate a contradiction.

Let $\gamma \equiv \sigma_{\alpha 0}/\sigma_\alpha$ and

$$\boldsymbol{\Gamma}_{\boldsymbol{\beta}} \equiv \begin{bmatrix} \gamma_{\boldsymbol{\beta}1} & 0 & \dots & 0 \\ 0 & \gamma_{\boldsymbol{\beta}2} & & 0 \\ & & \ddots & \\ 0 & \dots & 0 & \gamma_{\boldsymbol{\beta}K} \end{bmatrix},$$

where $\gamma_{\boldsymbol{\beta}l} \equiv \sigma_{\beta 0l}/\sigma_{\beta l}, l = 1, \dots, K$. Notice that $\gamma_{\boldsymbol{\beta}l} > 0$ for all l and $\gamma > 0$. We suppose that either $\boldsymbol{\Gamma}_{\boldsymbol{\beta}} \neq I$ or $\gamma \neq 1$ or both, and show that for some $(\mathbf{X}, \mathbf{p}, \mathbf{s}, j)$ and $\left(\mathbf{X}^\dagger, \mathbf{p}^\dagger, \mathbf{s}^\dagger, j^\dagger\right)$, Equation (55) is not satisfied. We consider the following cases in the order given: 1) $\gamma < 1,\ max_l\left\{\gamma_{\boldsymbol{\beta}l}\right\} \leq 1$; 2) $\gamma > 1,\ min_l\left\{\gamma_{\boldsymbol{\beta}l}\right\} \geq 1$; 3)$\gamma < 1,\ max_l\left\{\gamma_{\boldsymbol{\beta}l}\right\} > 1$; 4) $\gamma > 1,\ min_l\left\{\gamma_{\boldsymbol{\beta}l}\right\} < 1$; 5) $\gamma = 1$, $max_l\left\{\gamma_{\boldsymbol{\beta}l}\right\} > 1$; and 6) $\gamma = 1,\ min_l\left\{\gamma_{\boldsymbol{\beta}l}\right\} < 1$.

Case 1) $\gamma < 1,\ max_l\left\{\gamma_{\boldsymbol{\beta}l}\right\} \leq 1$. Choose $(\mathbf{X}, \mathbf{p}, \mathbf{s}, j)$ and $\left(\mathbf{X}^\dagger, \mathbf{p}^\dagger, \mathbf{s}^\dagger, j^\dagger\right)$ with $s_j \neq s^\dagger_{j^\dagger}$. Denote $\mathbf{X}^{(n)} \equiv \mathbf{X}\boldsymbol{\Gamma}^n_{\boldsymbol{\beta}}$, $\mathbf{p}^{(n)} \equiv \gamma^n \mathbf{p}$. Then, $\mathbf{X}^{(0)} \equiv \mathbf{X}$, $\mathbf{p}^{(0)} \equiv \mathbf{p}$, $lim_{n\to\infty}\mathbf{p}^{(n)} = \mathbf{0}$. Let $\widetilde{\mathbf{X}} \equiv lim_{n\to\infty}\mathbf{X}^{(n)}$. Hence, for firms $j = 1, \dots J$ and characteristics $k = 1, \dots, K$, $\widetilde{x}_{j,k} = x_{j,k}$ if $\gamma_{\boldsymbol{\beta}k} = 1$ and $\widetilde{x}_{j,k} = 0$ if $\gamma_{\boldsymbol{\beta}k} < 1$. Using $\sigma_{\alpha 0} = \gamma\sigma_\alpha$, we obtain

$$\begin{aligned} s_j\left(\mathbf{X}^{(0)}, \mathbf{p}^{(0)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_{c0}\right) &= \int_\alpha\int_{\boldsymbol{\beta}} \frac{\exp\left(\mathbf{x}_j\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + p_j\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \xi_{j,0}\right)}{\left[1 + \sum_{l=1}^J \exp\left(\mathbf{x}_l\boldsymbol{\Sigma}_{\boldsymbol{\beta}0}\boldsymbol{\beta} + p_l\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \xi_{l,0}\right)\right]}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha \\ &= \int_\alpha\int_{\boldsymbol{\beta}} \frac{\exp\left(\mathbf{x}_j\boldsymbol{\Gamma}_{\boldsymbol{\beta}}\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\boldsymbol{\beta} + \gamma p_j\sigma_{\alpha}\left(\alpha + \eta_{\alpha 0}\right) + \xi_{j,0}\right)}{\left[1 + \sum_{l=1}^J \exp\left(\mathbf{x}_l\boldsymbol{\Gamma}_{\boldsymbol{\beta}}\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\boldsymbol{\beta} + \gamma p_l\sigma_{\alpha}\left(\alpha + \eta_{\alpha 0}\right) + \xi_{l,0}\right)\right]}\phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta}\phi\left(\alpha\right) d\alpha \\ &= s_j\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_{c}\right), \end{aligned}$$

$$
\begin{aligned}
& \frac{\partial s_j}{\partial p_j^{(0)}}\left(\mathbf{X}^{(0)}, \mathbf{p}^{(0)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_{c0}\right) \\
= & \int_\alpha \int_\beta \frac{\exp\left(\mathbf{x}_j \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + p_j \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \xi_{j,0}\right)\left[1 + \sum_{l \neq j} \exp\left(\mathbf{x}_l \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + p_l \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \xi_{l,0}\right)\right]}{\left[1 + \sum_{l=1}^J \exp\left(\mathbf{x}_l \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + p_l \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \xi_{l,0}\right)\right]^2} \\
& \times \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) \phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta} \phi\left(\alpha\right) d\alpha \\
= & \gamma \int_\alpha \int_\beta \frac{\exp\left(\mathbf{x}_j \boldsymbol{\Gamma}_{\boldsymbol{\beta}} \boldsymbol{\Sigma}_{\boldsymbol{\beta}} \boldsymbol{\beta} + \gamma p_j \sigma_\alpha\left(\alpha + \eta_{\alpha 0}\right) + \xi_{j,0}\right)}{\left[1 + \sum_{l=1}^J \exp\left(\mathbf{x}_l \boldsymbol{\Gamma}_{\boldsymbol{\beta}} \boldsymbol{\Sigma}_{\boldsymbol{\beta}} \boldsymbol{\beta} + \gamma p_l \sigma_\alpha\left(\alpha + \eta_{\alpha 0}\right) + \xi_{l,0}\right)\right]^2} \\
& \times \left[1 + \sum_{l \neq j} \exp\left(\mathbf{x}_l \boldsymbol{\Gamma}_{\boldsymbol{\beta}} \boldsymbol{\Sigma}_{\boldsymbol{\beta}} \boldsymbol{\beta} + \gamma p_l \sigma_\alpha\left(\alpha + \eta_\alpha\right) + \xi_{l,0}\right)\right] \sigma_\alpha\left(\alpha + \eta_{\alpha 0}\right) \phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta} \phi\left(\alpha\right) d\alpha \\
= & \gamma \frac{\partial s_j}{\partial p_j^{(1)}}\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_c\right),
\end{aligned}
$$

and similarly for $s_{j^\dagger}^\dagger$, $\frac{\partial s_{j^\dagger}^\dagger}{\partial p_{j^\dagger}}$. Thus,

$$
s_j = s_j\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_c\right), \ s_{j^\dagger}^\dagger = s_{j^\dagger}\left(\mathbf{X}^{\dagger(1)}, \mathbf{p}^{\dagger(1)}, \boldsymbol{\xi}_0^\dagger, \boldsymbol{\theta}_c\right), \ s_j \neq s_{j^\dagger}^\dagger,
$$

and if marginal revenues are equal for firms with $(\mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0})$ and $\left(\mathbf{X}, \mathbf{p}^\dagger, \mathbf{s}^\dagger, \boldsymbol{\theta}_{c0}\right)$, then they are also equal for $\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \mathbf{s}, \boldsymbol{\theta}_c\right)$ and $\left(\mathbf{X}^{\dagger(1)}, \mathbf{p}^{\dagger(1)}, \mathbf{s}^\dagger, \boldsymbol{\theta}_c\right)$, because,

$$
\begin{aligned}
& p_j^{(0)} + \left[\frac{\partial s_j}{\partial p_j^{(0)}}\right]^{-1}\left(\mathbf{X}^{(0)}, \mathbf{p}^{(0)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_{c0}\right) s_j\left(\mathbf{X}^{(0)}, \mathbf{p}^{(0)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_{c0}\right) \\
= & \gamma^{-1} p_j^{(1)} + \gamma^{-1}\left[\frac{\partial s_j}{\partial p_j^{(1)}}\right]^{-1}\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_c\right) s_j\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0, \boldsymbol{\theta}_c\right) \\
= & p_{j^\dagger}^\dagger + \left[\frac{\partial s_{j^\dagger}}{\partial p_{j^\dagger}^{(0)}}\right]^{-1}\left(\mathbf{X}^{\dagger(0)}, \mathbf{p}^{\dagger(0)}, \boldsymbol{\xi}_0^\dagger, \boldsymbol{\theta}_{c0}\right) s_{j^\dagger}\left(\mathbf{X}^{\dagger(0)}, \mathbf{p}^{\dagger(0)}, \boldsymbol{\xi}_0^\dagger, \boldsymbol{\theta}_{c0}\right) \\
= & \gamma^{-1} p_{j^\dagger}^{\dagger(1)} + \gamma^{-1}\left[\frac{\partial s_{j^\dagger}}{\partial p_{j^\dagger}^{(1)}}\right]^{-1}\left(\mathbf{X}^{\dagger(1)}, \mathbf{p}^{\dagger(1)}, \boldsymbol{\xi}_0^\dagger, \boldsymbol{\theta}_c\right) s_j\left(\mathbf{X}^{\dagger(1)}, \mathbf{p}^{\dagger(1)}, \boldsymbol{\xi}_0^\dagger, \boldsymbol{\theta}_c\right). \qquad (56)
\end{aligned}
$$

If $\boldsymbol{\theta}_c$ is not identified, then Equation (56) implies that there exist $\boldsymbol{\xi}_0^{(1)}$ and $\boldsymbol{\xi}_0^{\dagger(1)}$ that satisfy

$$
s_j = s_j\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0^{(1)}, \boldsymbol{\theta}_{c0}\right), \ s_{j^\dagger}^\dagger = s_{j^\dagger}\left(\mathbf{X}^{\dagger(1)}, \mathbf{p}^{\dagger(1)}, \boldsymbol{\xi}_0^{\dagger(1)}, \boldsymbol{\theta}_{c0}\right),
$$

$$p_j^{(1)} + \left[\frac{\partial s_j \left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0^{(1)}, \boldsymbol{\theta}_{c0} \right)}{\partial p_j^{(1)}} \right]^{-1} s_j \left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}_0^{(1)}, \boldsymbol{\theta}_{c0} \right)$$

$$= \quad p_{j^\dagger}^{\dagger(1)} + \left[\frac{\partial s_{j^\dagger} \left(\mathbf{X}^{\dagger(1)}, \mathbf{p}^{\dagger(1)}, \boldsymbol{\xi}_0^{\dagger(1)}, \boldsymbol{\theta}_{c0} \right)}{\partial p_{j^\dagger}^{(1)}} \right]^{-1} s_{j^\dagger}^{\dagger} \left(\mathbf{X}^{\dagger(1)}, \mathbf{p}^{\dagger(1)}, \boldsymbol{\xi}_0^{\dagger(1)}, \boldsymbol{\theta}_{c0} \right) \geq 0.$$

That is, if $\boldsymbol{\theta}_c$ is not identified, then there exist unobserved product characteristics $\boldsymbol{\xi}_0^{(1)}$ and $\boldsymbol{\xi}_0^{(1)}$ such that market shares remain the same and marginal revenues are equal at the true parameter vector.

Using the same logic, we define $\boldsymbol{\xi}_0^{(n)}$ to satisfy $s_j = s_j \left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \boldsymbol{\xi}_0^{(n)}, \boldsymbol{\theta}_{c0} \right)$ and $\widetilde{\boldsymbol{\xi}}_0 = lim_{n \to \infty} \boldsymbol{\xi}_0^{(n)}$ which satisfies $\mathbf{s} = \mathbf{s} \left(\widetilde{\mathbf{X}}, \mathbf{0}, \widetilde{\xi}_0, \boldsymbol{\theta}_{c0} \right)$ because of the Implicit Function Theorem. Then, as $n \to \infty$,

$$\begin{aligned}
& \frac{\partial s_j \left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \boldsymbol{\xi}_0^{(n)}, \boldsymbol{\theta}_{c0} \right)}{\partial p_j^{(n)}} \\
= \quad & \int_\alpha \int_\beta \frac{\exp \left(\mathbf{x}_j^{(n)} \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + p_j^{(n)} \sigma_{\alpha 0} \left(\alpha + \eta_{\alpha 0} \right) + \xi_{j,0}^{(n)} \right)}{\left[1 + \sum_{l=1}^J \exp \left(\mathbf{x}_l^{(n)} \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + p_l^{(n)} \sigma_{\alpha 0} \left(\alpha + \eta_{\alpha 0} \right) + \xi_{l,0}^{(n)} \right) \right]^2} \\
& \times \left[1 + \sum_{l \neq j} \exp \left(\mathbf{x}_l^{(n)} \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + p_l^{(n)} \sigma_{\alpha 0} \left(\alpha + \eta_{\alpha 0} \right) + \xi_{l,0}^{(n)} \right) \right] \\
& \times \sigma_{\alpha 0} \left(\alpha + \eta_{\alpha 0} \right) \phi \left(\alpha \right) \phi \left(\boldsymbol{\beta} \right) d\boldsymbol{\beta} d\alpha \\
\to \quad & \int_\alpha \int_\beta \frac{\exp \left(\widetilde{\mathbf{x}}_j \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + \widetilde{\xi}_{j,0} \right) \left[1 + \sum_{l \neq j} \exp \left(\widetilde{\mathbf{x}}_l \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + \widetilde{\xi}_{l,0} \right) \right]}{\left[1 + \sum_{l=1}^J \exp \left(\widetilde{\mathbf{x}}_l \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + \widetilde{\xi}_{l,0} \right) \right]^2} \\
& \times \sigma_{\alpha 0} \left(\alpha + \eta_{\alpha 0} \right) \phi \left(\alpha \right) \phi \left(\boldsymbol{\beta} \right) d\boldsymbol{\beta} d\alpha \\
= \sigma_{\alpha 0} \eta_{\alpha 0} \quad & \int_\beta \frac{\exp \left(\widetilde{\mathbf{x}}_j \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + \widetilde{\xi}_{j,0} \right) \left[1 + \sum_{l \neq j} \exp \left(\widetilde{\mathbf{x}}_l \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + \widetilde{\xi}_{l,0} \right) \right]}{\left[1 + \sum_{l=1}^J \exp \left(\widetilde{\mathbf{x}}_l \boldsymbol{\Sigma}_{\boldsymbol{\beta}0} \boldsymbol{\beta} + \widetilde{\xi}_{l,0} \right) \right]^2} \phi \left(\boldsymbol{\beta} \right) d\boldsymbol{\beta} < 0
\end{aligned}$$

because $\eta_{\alpha 0} < 0$. Therefore,

$$p_j^{(n)} + \left[\frac{\partial s_j \left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \boldsymbol{\xi}_0^{(n)}, \boldsymbol{\theta}_{c0} \right)}{\partial p_j^{(n)}} \right]^{-1} s_j < 0$$

for sufficiently large n, which contradicts positivity of the marginal revenue.

Case 2) $\gamma > 1$, $min_l \left\{ \gamma_{\boldsymbol{\beta} l} \right\} \geq 1$ Let $\widetilde{\gamma} \equiv 1/\gamma$, $\widetilde{\gamma}_{\boldsymbol{\beta} l} \equiv 1/\gamma_{\boldsymbol{\beta} l}$. Then $\sigma_\alpha = \widetilde{\gamma} \sigma_{\alpha 0}$. Similarly, define $\mathbf{X}^{(n)} \equiv \mathbf{X} \widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{\beta}}^n$, $\mathbf{p}^{(n)} \equiv \widetilde{\gamma}^n \mathbf{p}$. Then, $lim_{n \to \infty} \mathbf{p}^{(n)} = \mathbf{0}$. Let $\widetilde{\mathbf{X}} \equiv lim_{n \to \infty} \mathbf{X}^{(n)}$. Then, $\widetilde{x}_{j,l} = x_{j,l}$ if

$\gamma_{\boldsymbol{\beta} l} = 1$ and $\widetilde{x}_{j,l} = 0$ if $\gamma_{\boldsymbol{\beta} l} > 1$. Then, using the same steps as in Case 1), but starting with the parameter vector $\boldsymbol{\theta}_c$, we find that $s_j\left(\mathbf{X}^{(0)}, \mathbf{p}^{(0)}, \boldsymbol{\xi}, \boldsymbol{\theta}_c\right) = s_j\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}, \boldsymbol{\theta}_{c0}\right)$. To see this, note that:

$$
\begin{aligned}
s_j\left(\mathbf{X}^{(0)}, \mathbf{p}^{(0)}, \boldsymbol{\xi}, \boldsymbol{\theta}_c\right) &= \int_\alpha \int_{\boldsymbol{\beta}} \frac{\exp\left(\mathbf{x}_j^{(0)} \boldsymbol{\Sigma}_{\boldsymbol{\beta}} \boldsymbol{\beta} + p_j^{(0)} \sigma_\alpha (\alpha + \eta_{\alpha 0}) + \xi_j\right)}{\left[1 + \sum_{l=1}^J \exp\left(\mathbf{x}_l^{(0)} \boldsymbol{\Sigma}_{\boldsymbol{\beta}} \boldsymbol{\beta} + p_l^{(0)} \sigma_\alpha (\alpha + \eta_{\alpha 0}) + \xi_l\right)\right]} \phi(\boldsymbol{\beta})\, d\boldsymbol{\beta} \phi(\alpha)\, d\alpha \\
&= \int_\alpha \int_{\boldsymbol{\beta}} \frac{\exp\left(\mathbf{x}_j^{(0)} \widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{\beta}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} + \widetilde{\gamma} p_j^{(0)} \sigma_{\alpha 0} (\alpha + \eta_{\alpha 0}) + \xi_j\right)}{\left[1 + \sum_{l=1}^J \exp\left(\mathbf{x}_l^{(0)} \widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{\beta}} \boldsymbol{\Sigma}_{\boldsymbol{\beta} 0} \boldsymbol{\beta} + \widetilde{\gamma} p_l^{(0)} \sigma_{\alpha 0} (\alpha + \eta_{\alpha 0}) + \xi_l\right)\right]} \phi(\boldsymbol{\beta})\, d\boldsymbol{\beta} \phi(\alpha)\, d\alpha \\
&= s_j\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}, \boldsymbol{\theta}_{c0}\right).
\end{aligned}
$$

It then follows that $\frac{\partial s_j}{\partial p_j^{(0)}}\left(\mathbf{X}^{(0)}, \mathbf{p}^{(0)}, \boldsymbol{\xi}, \boldsymbol{\theta}_c\right) = \widetilde{\gamma} \frac{\partial s_j}{\partial p_j^{(1)}}\left(\mathbf{X}^{(1)}, \mathbf{p}^{(1)}, \boldsymbol{\xi}, \boldsymbol{\theta}_{c0}\right)$ and similarly for $s^\dagger_{j^\dagger}$, $\frac{\partial s^\dagger_{j^\dagger}}{\partial p^{(0)}_{j^\dagger}}$. Then by applying the non-identification condition as in the case above, and then iterating, we conclude that

$$
p_j^{(n)} + \left[\frac{\partial s_j\left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \boldsymbol{\xi}^{(n)}, \boldsymbol{\theta}_c\right)}{\partial p_j^{(n)}}\right]^{-1} s_j < 0,
$$

which is a contradiction to positivity of marginal revenue. Thus $\boldsymbol{\theta}_{c0}$ is identified.

3) $\gamma < 1$, $max_l \{\gamma_{\boldsymbol{\beta} l}\} > 1$. Assume for some $k \in \{1, \ldots, K\}$, $\gamma_{\boldsymbol{\beta} k} > \gamma_{\boldsymbol{\beta} l}$ for any $l \neq k$.

Then, as $n \to \infty$, $x_{i,l}^{(n)} / \gamma_{\boldsymbol{\beta} k}^n \to 0$ for $l \neq k$ and $x_{i,k}^{(n)} / \gamma_{\boldsymbol{\beta} k}^n = x_{ik}$ for all $i = 1, \ldots, J$, $\mathbf{p}^{(n)} / \gamma_{\boldsymbol{\beta} k}^n \to 0$. Now, let $u_{i,0}^{(0)} = \boldsymbol{\xi}_{i,0} / \sigma_{\boldsymbol{\beta} 0k}$, and $u_{i,0}^{(n)}$ be such that $s_i = s_i\left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \gamma_{\boldsymbol{\beta} k}^n \sigma_{\boldsymbol{\beta} 0k} \boldsymbol{u}_0^{(n)}, \boldsymbol{\theta}_{c0}\right)$. For the sake of simplicity, we assume that the k^{th} observed characteristics of firms in a market are different from each other. Now, wlog. order the firms such that $x_{1,k} < x_{2,k} \ldots < x_{J,k}$ and $x_{1,k}^\dagger < x_{2,k}^\dagger \ldots < x_{J^\dagger,k}^\dagger$. Then, denoting $\boldsymbol{\Sigma}_{\boldsymbol{\beta} 0,-k,-k}$ to be the submatrix of $\boldsymbol{\Sigma}_{\boldsymbol{\beta} 0}$ that does not include the kth row and the kth column, we get,

$$
\begin{aligned}
& \int_{\beta_k} \prod_{l \neq j}^{J} I\left(\left(x_{l,k}^{(n)} - x_{j,k}^{(n)}\right)\sigma_{\boldsymbol{\beta}0k}\beta_k + \left(\mathbf{x}_{l,-k}^{(n)} - \mathbf{x}_{j,-k}^{(n)}\right)\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\right. \\
& \left. + \left(p_l^{(n)} - p_j^{(n)}\right)\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(u_{l,0}^{(n)} - u_{j,0}^{(n)}\right) + \epsilon_l - \epsilon_j \leq 0\right)\phi\left(\beta_k\right)d\beta_k \\
= & \int_{\beta_k} \prod_{l \neq j}^{J} I\left(\left(x_{l,k} - x_{j,k}\right)\sigma_{\boldsymbol{\beta}0k}\beta_k + \frac{\mathbf{x}_{l,-k}^{(n)} - \mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\right. \\
& \left. + \left(\frac{p_l^{(n)} - p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}}\right)\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \sigma_{\boldsymbol{\beta}0k}(u_{l,0}^{(n)} - u_{j,0}^{(n)}) + \frac{\epsilon_l - \epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}} \leq 0\right)\phi\left(\beta_k\right)d\beta_k \\
= & \int_{\beta_k} \prod_{x_{l,k} - x_{j,k} > 0}^{J} I\left(\beta_k \leq -\frac{\mathbf{x}_{l,-k}^{(n)} - \mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\right. \\
& \left. - \left(\frac{p_l^{(n)} - p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right)\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) - \frac{u_{l,0}^{(n)} - u_{j,0}^{(n)}}{\left(x_{l,k} - x_{j,k}\right)} - \frac{\epsilon_l - \epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right\}\Bigg) \\
& \prod_{x_{l,k} - x_{j,k} < 0}^{J} I\left(\beta_k \geq -\frac{\mathbf{x}_{l,-k}^{(n)} - \mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\right. \\
& \left. - \left(\frac{p_l^{(n)} - p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right)\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) - \frac{u_{l,0}^{(n)} - u_{j,0}^{(n)}}{\left(x_{l,k} - x_{j,k}\right)} - \frac{\epsilon_l - \epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right\}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{\beta_k} I\left(\beta_k \leq Min_{x_{l,k} - x_{j,k} > 0}\left\{-\frac{\mathbf{x}_{l,-k}^{(n)} - \mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\right.\right. \\
& \left.\left. - \left(\frac{p_l^{(n)} - p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right)\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) - \frac{u_{l,0}^{(n)} - u_{j,0}^{(n)}}{\left(x_{l,k} - x_{j,k}\right)} - \frac{\epsilon_l - \epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right\}\right) \\
& \times I\left(\beta_k \geq Max_{x_{l,k} - x_{j,k} < 0}\left\{-\frac{\mathbf{x}_{l,-k}^{(n)} - \mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k}\boldsymbol{\beta}_{-k}\right.\right. \\
& \left.\left. - \left(\frac{p_l^{(n)} - p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right)\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) - \frac{u_{l,0}^{(n)} - u_{j,0}^{(n)}}{\left(x_{l,k} - x_{j,k}\right)} - \frac{\epsilon_l - \epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right\}\right)\phi\left(\beta_k\right)d\beta_k.
\end{aligned}
$$

Note that $\frac{\mathbf{x}_{l,-k}^{(n)} - \mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k} \to 0$, $\left(\frac{p_l^{(n)} - p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)}\right)\sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) \to 0$ and $\frac{\epsilon_l - \epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k} - x_{j,k}\right)} \to 0$ as $n \to \infty$. Hence, using earlier arguments, from continuity, there exists $\widetilde{\mathbf{u}}_0$ that satisfies

$$
s_0 = \Phi\left(-\frac{\widetilde{u}_{1,0}}{x_{1,k}}\right),\ s_i = \Phi\left(-\frac{\widetilde{u}_{i,0} - \widetilde{u}_{i+1,0}}{\left(x_{i,k} - x_{i+1,k}\right)}\right) - \Phi\left(-\frac{\widetilde{u}_{i,0} - \widetilde{u}_{i-1,0}}{\left(x_{i,k} - x_{i-1,k}\right)}\right), i = 1, \ldots, J.
$$

It then follows that,

$$-\frac{\widetilde{u}_{i,0}-\widetilde{u}_{i+1,0}}{(x_{i,k}-x_{i+1,k})}=\Phi^{-1}\left(\sum_{l\leq i}s_l\right).$$

Thus, we can derive $\widetilde{u}_{i,0}$ recursively, as earlier. Furthermore, following the similar arguments as before, we can show that $lim_{n\to\infty}u_{i,0}^{(n)}=\widetilde{u}_{i,0}$ for $i=1,\ldots,J$.

Next, we consider the derivative of the market share with respect to the price. We define $a_{i,l}\equiv\frac{\widetilde{u}_{i,0}-\widetilde{u}_{l,0}}{(x_{i,k}-x_{l,k})}$ and $\overline{a}_i$ and $\underline{a}_i$ similarly as well. Then,

$$\ldots<\underline{a}_{i+1}=\overline{a}_i=\frac{\widetilde{u}_{i,0}-\widetilde{u}_{i+1,0}}{(x_{i,k}-x_{i+1,k})}<\underline{a}_i=\overline{a}_{i-1}=\frac{\widetilde{u}_{i,0}-\widetilde{u}_{i-1,0}}{(x_{i,k}-x_{i-1,k})}<\ldots<\frac{\widetilde{u}_{1,0}}{x_{1,k}}$$

, and as before, we derive

$$\begin{aligned}
&\frac{\partial}{\partial p_j^{(n)}}\int_{\beta_k}\prod_{,l\neq j}^{J}I\left(\left(x_{l,k}^{(n)}-x_{j,k}^{(n)}\right)\sigma_{\boldsymbol{\beta}0k}\beta_k+\left(\mathbf{x}_{l,-k}^{(n)}-\mathbf{x}_{j,-k}^{(n)}\right)\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\right.\\
&\left.+\left(p_l^{(n)}-p_j^{(n)}\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(u_{l,0}^{(n)}-u_{j,0}^{(n)}\right)+\epsilon_l-\epsilon_j\leq 0\right)\phi\left(\beta_k\right)d\beta_k\\
=\;&\frac{\partial}{\partial p_j^{(n)}}\int_{\beta_k}\prod_{l\neq j}^{J}I\left(\left(x_{l,k}-x_{j,k}\right)\sigma_{\boldsymbol{\beta}0k}\beta_k+\frac{\mathbf{x}_{l,-k}^{(n)}-\mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\right.\\
&\left.+\left(\frac{p_l^{(n)}-p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}}\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)+\sigma_{\boldsymbol{\beta}0k}(u_{l,0}^{(n)}-u_{j,0}^{(n)})+\frac{\epsilon_l-\epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}}\leq 0\right)\phi\left(\beta_k\right)d\beta_k. \qquad (57)
\end{aligned}$$

Now, let

$$\begin{aligned}
&B\left(\mathbf{p}^{(n)},\mathbf{u}_0^{(n)},n,\mathbf{X},\Gamma_{\boldsymbol{\beta}},\boldsymbol{\beta},\boldsymbol{\epsilon},j,\boldsymbol{\theta}_{c0}\right)\\
\equiv\;&-\frac{\mathbf{x}_{l,-k}^{(n)}-\mathbf{x}_{j,-k}^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k}-x_{j,k}\right)}\boldsymbol{\Sigma}_{\boldsymbol{\beta}0-k,-k}\boldsymbol{\beta}_{-k}\\
&-\left(\frac{p_l^{(n)}-p_j^{(n)}}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k}-x_{j,k}\right)}\right)\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)-\frac{u_{l,0}^{(n)}-u_{j,0}^{(n)}}{\left(x_{l,k}-x_{j,k}\right)}-\frac{\epsilon_l-\epsilon_j}{\gamma_{\boldsymbol{\beta}k}^{n}\sigma_{\boldsymbol{\beta}0k}\left(x_{l,k}-x_{j,k}\right)}.
\end{aligned}$$

Then let

$$\begin{aligned}
&\underline{B}\left(\mathbf{p}^{(n)},\mathbf{u}_0^{(n)},n,\mathbf{X},\Gamma_{\boldsymbol{\beta}},\boldsymbol{\beta},\boldsymbol{\epsilon},j,\boldsymbol{\theta}_{c0}\right)\\
\equiv\;&Min_{x_{l,k}-x_{j,k}>0}B\left(\mathbf{p}^{(n)},\mathbf{u}_0^{(n)},n,\mathbf{X},\Gamma_{\boldsymbol{\beta}},\boldsymbol{\beta},\boldsymbol{\epsilon},j,\boldsymbol{\theta}_{c0}\right)
\end{aligned}$$

and

$$
\begin{aligned}
& \overline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right) \\
\equiv \quad & Max_{x_{lk}-x_{jk}<0} B\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{\partial \underline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, , n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)}{\partial p_j^{(n)}} &= \frac{\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)}{\gamma_{\boldsymbol{\beta} k}^n \sigma_{\boldsymbol{\beta} 0 k}\left(x_{l_1,k}-x_{j,k}\right)}, \\
\frac{\partial \overline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)}{\partial p_j^{(n)}} &= \frac{\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)}{\gamma_{\boldsymbol{\beta} k}^n \sigma_{\boldsymbol{\beta} 0 k}\left(x_{l_2,k}-x_{j,k}\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
l_1 &= argmin_{l \neq j, x_{l,k}-x_{j,k}>0} B\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right) \\
l_2 &= argmax_{l \neq j, x_{l,k}-x_{j,k} \leq 0} B\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)
\end{aligned}
$$

except for the case where l_1 and l_2 have multiple values, which we ignore because those situations occur with probability zero. Therefore,

$$
\begin{aligned}
(57) \quad = \quad & \frac{\partial}{\partial p_j^{(n)}} \int_{\beta_k} I\left(\beta_k \leq \underline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)\right) \\
& \times I\left(\beta_k \geq \overline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)\right) \phi\left(\beta_k\right) d\beta_k \\
= \quad & \left[\phi\left(\underline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)\right) \frac{\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)}{\gamma_{\boldsymbol{\beta} k}^n \sigma_{\boldsymbol{\beta} 0 k}\left(x_{l_1,k}-x_{j,k}\right)}\right. \\
& \left.-\phi\left(\overline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)\right) \frac{\sigma_{\alpha 0}\left(\alpha+\eta_{\alpha 0}\right)}{\gamma_{\boldsymbol{\beta} k}^n \sigma_{\boldsymbol{\beta} 0 k}\left(x_{l_2,k}-x_{j,k}\right)}\right] \\
& \times I\left(\underline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right) > \overline{B}\left(\mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, n, \mathbf{X}, \Gamma_{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, j, \boldsymbol{\theta}_{c0}\right)\right) \quad (58)
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\gamma^{n}_{\boldsymbol{\beta}k}\times(58)\\
\to&\phi\left(lim_{n\to\infty}\left\{-\frac{u^{(n)}_{l_1,0}-u^{(n)}_{j,0}}{(x_{l_1,k}-x_{j,k})}\right\}\right)\frac{\sigma_{\alpha0}\left(\alpha+\eta_{\alpha0}\right)}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_1,k}-x_{j,k}\right)}-\phi\left(lim_{n\to\infty}\left\{-\frac{u^{(n)}_{l_2,0}-u^{(n)}_{j,0}}{(x_{l_2k}-x_{j,k})}\right\}\right)\frac{\sigma_{\alpha0}\left(\alpha+\eta_{\alpha0}\right)}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_2,k}-x_{j,k}\right)}\\
&\times I\left(lim_{n\to\infty}\left\{-\frac{u^{(n)}_{l_1,0}-u^{(n)}_{j,0}}{(x_{l_1,k}-x_{j,k})}\right\}>lim_{n\to\infty}\left\{-\frac{u^{(n)}_{l_2,0}-u^{(n)}_{j,0}}{(x_{l_2,k}-x_{j,k})}\right\}\right)\\
=&\phi\left(-\frac{\widetilde{u}_{j+1,0}-\widetilde{u}_{j,0}}{(x_{j+1,k}-x_{j,k})}\right)\frac{\sigma_{\alpha0}\left(\alpha+\eta_{\alpha0}\right)}{\sigma_{\boldsymbol{\beta}0k}\left(x_{j+1,k}-x_{j,k}\right)}-\phi\left(-\frac{\widetilde{u}_{j-1,0}-\widetilde{u}_{j,0}}{(x_{j-1,k}-x_{j,k})}\right)\frac{\sigma_{\alpha0}\left(\alpha+\eta_{\alpha0}\right)}{\sigma_{\boldsymbol{\beta}0k}\left(x_{j-1,k}-x_{j,k}\right)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\gamma^{n}_{\boldsymbol{\beta}k}\frac{\partial s_j}{\partial p^{(n)}_j} \quad\to\quad & \int_{\boldsymbol{\beta}_{-k}}\int_{\alpha}\left[\phi\left(-\frac{\widetilde{u}_{j+1,0}-\widetilde{u}_{j,0}}{(x_{j+1,k}-x_{jk})}\right)\frac{\sigma_{\alpha0}\left(\alpha+\eta_{\alpha0}\right)}{\sigma_{\boldsymbol{\beta}0k}\left(x_{j+1,k}-x_{j,k}\right)}\right.\\
&\left.-\phi\left(-\frac{\widetilde{u}_{j-1,0}-\widetilde{u}_{j,0}}{(x_{j-1,k}-x_{j,k})}\right)\frac{\sigma_{\alpha0}\left(\alpha+\eta_{\alpha0}\right)}{\sigma_{\boldsymbol{\beta}0k}\left(x_{j-1,k}-x_{j,k}\right)}\right]\phi\left(\alpha\right)d\alpha\phi\left(\boldsymbol{\beta}_{-k}\right)d\boldsymbol{\beta}_{-k}\\
= \quad & \phi\left(-\frac{\widetilde{u}_{j+1,0}-\widetilde{u}_{j,0}}{(x_{j+1,k}-x_{j,k})}\right)\frac{\sigma_{\alpha0}\eta_{\alpha0}}{\sigma_{\boldsymbol{\beta}0k}\left(x_{j+1,k}-x_{j,k}\right)}-\phi\left(-\frac{\widetilde{u}_{j-1,0}-\widetilde{u}_{j,0}}{(x_{j-1,k}-x_{j,k})}\right)\frac{\sigma_{\alpha0}\eta_{\alpha0}}{\sigma_{\boldsymbol{\beta}0k}\left(x_{j-1,k}-x_{j,k}\right)}<0
\end{aligned}
$$

because $x_{j+1,k}-x_{j,k}>0$, $x_{j-1,k}-x_{j,k}<0$, $\eta_{\alpha0}<0$. Therefore, for sufficiently large n,

$$
\begin{aligned}
& p^{(n)}_j+\left[\frac{\partial s_j}{\partial p^{(n)}_j}\right]^{-1}\left(\mathbf{X}^{(n)},\mathbf{p}^{(n)},\mathbf{u}^{(n)}_0,\boldsymbol{\theta}_0\right)s_j\\
= \quad & \gamma^{n}p_j+\gamma^{n}_{\boldsymbol{\beta}k}\left[\gamma^{n}_{\boldsymbol{\beta}k}\frac{\partial s_j}{\partial p^{(n)}_j}\left(\mathbf{X}^{(n)},\mathbf{p}^{(n)},\mathbf{u}^{(n)}_0,\boldsymbol{\theta}_0\right)\right]^{-1}s_j<0
\end{aligned}
$$

because $\gamma<1<\gamma_{\boldsymbol{\beta}k}$, which contradicts positivity of marginal revenue.

Next, we modify the above case to allow $\gamma_{\boldsymbol{\beta}l}$ to be the same for $l\in\mathcal{K}\subset\{1,2,\ldots,K\}$. Denote

$\gamma_{\boldsymbol{\beta}\mathcal{K}}$ as $\gamma_{\boldsymbol{\beta}\mathcal{K}} \equiv \gamma_{\boldsymbol{\beta}l}$ for $l \in \mathcal{K}$. Let $\mathbf{x}_{j,\mathcal{K}}$ and $\mathbf{x}_{j,\mathcal{K}^c}$ be defined accordingly. Then, letting $k \in \mathcal{K}$,

$$
\begin{aligned}
s_j &= \int_\alpha \int_{\boldsymbol{\beta}} \int_\epsilon s_j \left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, \alpha, \boldsymbol{\beta}, \boldsymbol{\theta}_0\right) f(\epsilon) d\epsilon \phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta} \phi(\alpha) d\alpha \\
&= \int_\alpha \int_{\boldsymbol{\beta}} \int_\epsilon \prod_{l \neq j}^{J} I\left(\left(\mathbf{x}_{l,\mathcal{K}}^{(n)} - \mathbf{x}_{j,\mathcal{K}}^{(n)}\right) \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}} \boldsymbol{\beta}_{\mathcal{K}} + \left(\mathbf{x}_{l,\mathcal{K}^c}^{(n)} - \mathbf{x}_{j,\mathcal{K}^c}^{(n)}\right) \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}^c} \boldsymbol{\beta}_{\mathcal{K}^c}\right. \\
&\quad \left. + \left(p_l^{(n)} - p_j^{(n)}\right) \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \gamma_{\boldsymbol{\beta}\mathcal{K}}^n \sigma_{\boldsymbol{\beta}0k} \left(u_{l,0}^{(n)} - u_{j,0}^{(n)}\right) + \epsilon_l - \epsilon_j \leq 0\right) \phi\left(\beta\right) d\beta d\alpha \\
&= \int_\alpha \int_{\boldsymbol{\beta}} \int_\epsilon \prod_{l \neq j}^{J} I\left(\left(\mathbf{x}_{l,\mathcal{K}} - \mathbf{x}_{j,\mathcal{K}}\right) \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}} \boldsymbol{\beta}_{\mathcal{K}} + \frac{\mathbf{x}_{l,\mathcal{K}^c}^{(n)} - \mathbf{x}_{j,\mathcal{K}^c}^{(n)}}{\gamma_{\boldsymbol{\beta}\mathcal{K}}^n} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}^c} \boldsymbol{\beta}_{\mathcal{K}^c}\right. \\
&\quad \left. + \left(\frac{p_l^{(n)} - p_j^{(n)}}{\gamma_{\boldsymbol{\beta}\mathcal{K}}^n}\right) \sigma_{\alpha 0}\left(\alpha + \eta_{\alpha 0}\right) + \sigma_{\boldsymbol{\beta}0k}(u_{l,0}^{(n)} - u_{j,0}^{(n)}) + \frac{\epsilon_l - \epsilon_j}{\gamma_{\boldsymbol{\beta}\mathcal{K}}^n} \leq 0\right) f(\epsilon) d\epsilon \phi\left(\boldsymbol{\beta}\right) d\boldsymbol{\beta} \phi(\alpha) d\alpha
\end{aligned}
$$

Using earlier arguments, it is straightforward to show that,

$$
s_j \to \int_{\boldsymbol{\beta}_{\mathcal{K}}} \prod_{l \neq j} I\left(\left(\mathbf{x}_{l,\mathcal{K}} - \mathbf{x}_{j,\mathcal{K}}\right) \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}} \boldsymbol{\beta}_{\mathcal{K}} + \sigma_{\boldsymbol{\beta}0k}\left(\widetilde{u}_{l,0} - \widetilde{u}_{j,0}\right) \leq 0\right) \phi\left(\boldsymbol{\beta}_{\mathcal{K}}\right) d\boldsymbol{\beta}_{\mathcal{K}}
$$

Similarly, we can show the following:

$$
\begin{aligned}
\gamma_{\boldsymbol{\beta}\mathcal{K}}^n \frac{\partial s_j}{\partial p_j^{(n)}} \quad \to \quad & \eta_{\alpha 0} \sigma_{\alpha 0} \int_{\boldsymbol{\beta}_{\mathcal{K}-k}} \left[\phi\left(-\frac{\mathbf{x}_{l_1,\mathcal{K}-k} - \mathbf{x}_{j,\mathcal{K}-k}}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_1,k} - x_{j,k}\right)} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}-k} \boldsymbol{\beta}_{\mathcal{K}-k} - \frac{\widetilde{u}_{l_1,0} - \widetilde{u}_{j,0}}{\left(x_{l_1,k} - x_{j,k}\right)}\right) \frac{1}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_1,k} - x_{j,k}\right)}\right. \\
& \left. -\phi\left(-\frac{\mathbf{x}_{l_2,\mathcal{K}-k} - \mathbf{x}_{j,\mathcal{K}-k}}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_2,k} - x_{j,k}\right)} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}-k} \boldsymbol{\beta}_{\mathcal{K}-k} - \frac{\widetilde{u}_{l_2,0} - \widetilde{u}_{j,0}}{\left(x_{l_2,k} - x_{j,k}\right)}\right) \frac{1}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_2,k} - x_{j,k}\right)}\right] \\
& \times I\left(-\frac{\mathbf{x}_{l_1,\mathcal{K}-k} - \mathbf{x}_{j,\mathcal{K}-k}}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_1,k} - x_{j,k}\right)} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}-k} \boldsymbol{\beta}_{\mathcal{K}-k} - \frac{\widetilde{u}_{l_1,0} - \widetilde{u}_{j,0}}{\left(x_{l_1,k} - x_{j,k}\right)}\right. \\
& \left. > -\frac{\mathbf{x}_{l_2,\mathcal{K}-k} - \mathbf{x}_{j,\mathcal{K}-k}}{\sigma_{\boldsymbol{\beta}0k}\left(x_{l_2,k} - x_{j,k}\right)} \circ \boldsymbol{\sigma}_{\boldsymbol{\beta}0\mathcal{K}-k} \boldsymbol{\beta}_{\mathcal{K}-k} - \frac{\widetilde{u}_{l_2,0} - \widetilde{u}_{j,0}}{\left(x_{l_2,k} - x_{j,k}\right)}\right) \phi\left(\boldsymbol{\beta}_{\mathcal{K}-k}\right) d\boldsymbol{\beta}_{\mathcal{K}-k} < 0,
\end{aligned}
$$

where l_1 and l_2 are defined as earlier. Therefore, for sufficiently large n,

$$
\begin{aligned}
p_j^{(n)} + \left[\frac{\partial s_j}{\partial p_j^{(n)}}\left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, \boldsymbol{\theta}_0\right)\right]^{-1} s_j &= \gamma^n p_j + \gamma_{\boldsymbol{\beta}\mathcal{K}}^n \left[\gamma_{\boldsymbol{\beta}\mathcal{K}}^n \frac{\partial s_j}{\partial p_j^{(n)}}\left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, \boldsymbol{\theta}_0\right)\right]^{-1} s_j \\
&= \gamma^n p_j + \gamma_{\boldsymbol{\beta}\mathcal{K}}^n \left[\gamma_{\boldsymbol{\beta}\mathcal{K}}^n \frac{\partial s_j}{\partial p_j^{(n)}}\left(\mathbf{X}^{(n)}, \mathbf{p}^{(n)}, \mathbf{u}_0^{(n)}, \boldsymbol{\theta}_0\right)\right]^{-1} s_j < 0
\end{aligned}
$$

which contradicts positivity of marginal revenue.

Case 4) $\gamma > 1$, $min_l \{\gamma_{\boldsymbol{\beta}l}\} < 1$, As before, let $\widetilde{\gamma}_{\boldsymbol{\beta}l} \equiv 1/\gamma_{\boldsymbol{\beta}l}$, $\widetilde{\gamma} \equiv 1/\gamma$. Then $\widetilde{\gamma} < 1$, $max_l \{\widetilde{\gamma}_{\boldsymbol{\beta}l}\} > 1$, and we can proceed as in Case 3).

Case 5) $\gamma = 1$, $max_l \{\gamma_{\boldsymbol{\beta}l}\} > 1$. We can show that this leads to contradiction as in Case 3).

Case 6) $\gamma = 1$, $min_l \{\gamma_{\boldsymbol{\beta} l}\} < 1$. Let $\widetilde{\gamma}_{\boldsymbol{\beta} l} \equiv 1/\gamma_{\boldsymbol{\beta} l}$, $\widetilde{\gamma} \equiv 1/\gamma$. Then, $\widetilde{\gamma} = 1$, $max_l \{\widetilde{\gamma}_{\boldsymbol{\beta} l}\} > 1$, and from Case 5), this leads to contradiction.

The results of Case 1) and Case 3) imply that $\gamma < 1$ leads to a contradiction. Similarly, the results of Case 2) and Case 4) imply that $\gamma > 1$ leads to a contradiction as well. Therefore, it follows that $\gamma = 1$ is the only possibility. Then, from Cases 5) and 6) it follows that $\gamma = 1$ and both $max_l \{\gamma_{\boldsymbol{\beta} l}\} \leq 1$ and $min_l \{\gamma_{\boldsymbol{\beta} l}\} \geq 1$ need to hold. Therefore, $\gamma_{\boldsymbol{\beta} l} = 1$ for $l = 1, \ldots K$ and therefore, $\boldsymbol{\sigma}_{\boldsymbol{\beta} 0}$ and $\sigma_{\alpha 0}$ are identified.

A.3 Identification of the SNLLS Estimator

Proof of Proposition 2:

For notational convenience, we denote

$$\psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right), \boldsymbol{\gamma}\right) \equiv \sum_{l=1}^{\infty} \gamma_l \psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right)\right).$$

Recall that, from Equation (31),

$$E\left[\left(C_{jm} - \psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), \boldsymbol{\gamma}_0\right)\right)^2\right] = \sigma_\nu^2 + \sigma_\varsigma^2,$$

because

$$C_{jm} = \psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), \boldsymbol{\gamma}_0\right) + \varsigma_{jm} + \nu_{jm}$$

and ς_{jm}, ν_{jm} are assumed to be i.i.d. distributed and independent from $(q_{jm}, \mathbf{w}_{jm}, \mathbf{X}_m, \mathbf{p}_m, \mathbf{s}_m)$. Then, for any $(\boldsymbol{\theta}_c, \boldsymbol{\gamma})$

$$\begin{aligned}
& E\left[\left(C_{jm} - \psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right), \boldsymbol{\gamma}\right)\right)^2\right] \\
= \; & E\left[\left(\psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), \boldsymbol{\gamma}_0\right) - \psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right), \boldsymbol{\gamma}\right)\right)^2\right] \\
& +2E\Big[\left(\psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), \boldsymbol{\gamma}_0\right) - \psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right), \boldsymbol{\gamma}\right)\right)\left(\nu_{jm} + \varsigma_{jm}\right)\Big] \\
& +E\left(\nu_{jm} + \varsigma_{jm}\right)^2 \\
= \; & E\left[\left(\psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), \boldsymbol{\gamma}_0\right) - \psi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right), \boldsymbol{\gamma}\right)\right)^2\right] + \sigma_\nu^2 + \sigma_\varsigma^2 \\
\geq \; & \sigma_\nu^2 + \sigma_\varsigma^2.
\end{aligned}$$

Therefore, if $(\boldsymbol{\theta}_{c*}, \boldsymbol{\gamma}_*)$ satisfies

$$[\boldsymbol{\theta}_{c*}, \boldsymbol{\gamma}_*] = argmin_{(\boldsymbol{\theta},\gamma)\in\Theta_c\times\Gamma} E\left[C_{jm} - \sum_{l=1}^{\infty} \gamma_l \psi_l \left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right)\right)\right]^2,$$

then,

$$\psi\left(q_j, \mathbf{w}_j, \mathbf{x}_{jm}, MR_j\left(\boldsymbol{\theta}_{c*}\right), \boldsymbol{\gamma}_*\right) = \psi\left(q_j, \mathbf{w}_j, \mathbf{x}_{jm}, MR_j\left(\boldsymbol{\theta}_{c0}\right), \boldsymbol{\gamma}_0\right) \tag{59}$$

needs to be satisfied for all $(q_j, \mathbf{w}_j, \mathbf{X}_m, \mathbf{p}_m, \mathbf{s}_m)$ in the population. Now, by assumption, given $(q_j, \mathbf{w}_j, \mathbf{x}_{jm})$, $\psi(, \boldsymbol{\gamma}_0)$ is continuous and strictly increasing in $MR_j(\boldsymbol{\theta}_{c0})$. Then, we show that given $(q_j, \mathbf{w}_j, \mathbf{x}_{jm})$, $\psi(, \boldsymbol{\gamma}_*)$, is also continuous and strictly monotone function of $MR_j(\theta_{c*})$. As before, wlog, we only consider markets where firms have different prices, and let the numbering of firms be ordered by price in a market. That is, the lowest price firm is $j = 1$ and the highest price firm is $j = J_m$. Now, consider the firm j such that $2 \leq j \leq J_m - 1$. We use Equation (52). That is,

$$\begin{aligned}
&\overline{MR}_j\left(\mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) \equiv lim_{\gamma\to\infty} \frac{MR_j\left(\mathbf{X}, \gamma\mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right)}{\gamma} \\
= \;& p_j + \left[-\phi\left(\Phi^{-1}\left(\sum_{l\leq j} s_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l\leq j} s_l\right) + \eta_{\alpha 0}}{p_j - p_{j+1}} - \phi\left(\Phi^{-1}\left(\sum_{l\leq j-1} s_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l\leq j-1} s_l\right) + \eta_{\alpha 0}}{p_{j-1} - p_j}\right]^{-1} s_j.
\end{aligned}$$

Because of Assumption 6, given $\mathbf{X}$, $\mathbf{p}$, there exists $\mathbf{s}$ such that $\sum_{l\leq j} s_l = \Phi(-\eta_{\alpha 0})$. Then,

$$\overline{MR}_j\left(\mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) = p_j - \left[\phi\left(\Phi^{-1}\left(\sum_{l\leq j-1} s_l\right)\right) \frac{\Phi^{-1}\left(\sum_{l\leq j-1} s_l\right) + \eta_{\alpha 0}}{p_{j-1} - p_j}\right]^{-1} s_j.$$

Let

$$A_j \equiv \frac{s_j}{\phi\left(\Phi^{-1}\left(\sum_{l\leq j-1} s_l\right)\right)\left[\Phi^{-1}\left(\sum_{l\leq j-1} s_l\right) + \eta_{\alpha 0}\right]}.$$

Then, $A_j < 0$, and $\overline{MR}_j\left(\mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) = 0$ implies

$$p_j = \frac{A_j}{A_j + 1} p_{j-1} > p_{j-1}$$

if $A < -1$, which holds for sufficiently small $\sum_{l\leq j-1} s_l$. Now, consider $j = 2$. Let $\underline{\mathbf{p}}$ be such that for some $\underline{p}_1 > 0$, $\underline{p}_2 = \frac{A_2}{A_2+1}\underline{p}_1$, $\underline{p}_3 < \ldots < \underline{p}_J$ such that $\overline{MR}_j\left(\mathbf{X}, \mathbf{p}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) > 0$ for $j > 2$. Then, $\overline{MR}_2\left(\mathbf{X}, \underline{\mathbf{p}}, \mathbf{s}, \boldsymbol{\theta}_{c0}\right) = 0$. Note that by letting $p_j(y) \equiv \underline{p}_j + y$, $j = 1, \ldots, J$, for $y \geq 0$ and by increasing y, we can continuously increase $\overline{MR}_2\left(\mathbf{X},\mathbf{p},\mathbf{s},\boldsymbol{\theta}_{c0}\right)$ from zero to an arbitrarily large number, say $B>0$. Then, by construction, for sufficiently large $\gamma>0$, $MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c0}\right)$ is increasing in $y>0$ with range being $[0,\gamma B]$. Therefore, $\psi\left(q_2,\mathbf{w}_2,\mathbf{x}_{2m},MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c0}\right),\boldsymbol{\gamma}_0\right)$ is also strictly increasing and continuous in y. Because $\psi\left(q_2,\mathbf{w}_2,\mathbf{x}_{2m},MR,\boldsymbol{\gamma}_*\right)$ is continuous in MR, in order the Equation (59) to hold, $MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)$ needs to be continuous in y as well.

Now, suppose there exists a segment $\left[\underline{y},\overline{y}\right]$ where $MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)$ is constant. Then, for any two values $y\in\left[\underline{y},\overline{y}\right]$ and $y'\in\left[\underline{y},\overline{y}\right]$, satisfying $y>y'$, $MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)=MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y'\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)$ implies

$$\psi\left(q_2,\mathbf{w}_2,\mathbf{x}_{2m},MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right),\boldsymbol{\gamma}_*\right)=\psi\left(q_2,\mathbf{w}_2,\mathbf{x}_{2m},MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y'\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right),\boldsymbol{\gamma}_*\right)$$

whereas

$$\psi\left(q_2,\mathbf{w}_2,\mathbf{x}_{2m},MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c0}\right),\boldsymbol{\gamma}_0\right)>\psi\left(q_2,\mathbf{w}_2,\mathbf{x}_{2m},MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y'\right),\mathbf{s},\boldsymbol{\theta}_{c0}\right),\boldsymbol{\gamma}_0\right),$$

which contradicts Equation (59). Next, consider the case where $MR_2\left(\mathbf{X},\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)$ is strictly increasing in $y\in\left[\underline{y},\widehat{y}\right]$ and strictly decreasing in $y\in[\widehat{y},\overline{y}]$. Then, from continuity, for $y<\widehat{y}$ sufficiently close to $\widehat{y}$, $MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(\overline{y}\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)<MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)<MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(\widehat{y}\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)$ holds. Therefore, from the intermediate value function, there exist $y'\in[\widehat{y},\overline{y}]$ such that $MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y'\right),\mathbf{s},,\boldsymbol{\theta}_{c*}\right)=MR_2\left(\mathbf{X},\gamma\mathbf{p}\left(y\right),\mathbf{s},,\boldsymbol{\theta}_{c*}\right)$. Again, this contradicts $\psi\left(,\boldsymbol{\gamma}_0\right)$ being strictly increasing in MR. Similar argument can be made for the case where $MR_2\left(\mathbf{X},\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)$ is strictly decreasing in $y\in\left[\underline{y},\widehat{y}\right]$ and strictly increasing in $y\in[\widehat{y},\overline{y}]$. Thus, we have shown that $MR_2\left(\mathbf{X},\mathbf{p}\left(y\right),\mathbf{s},\boldsymbol{\theta}_{c*}\right)$ is strictly monotone in y, and therefore, from Equation (59), $\psi\left(,\boldsymbol{\gamma}_*\right)$, is a strictly monotone function of $MR_j\left(\boldsymbol{\theta}_{c*}\right)$ for $j=1,\ldots,J$.

Recall that in Proposition 1 we have shown that the BLP demand function satisfies Condition 1. From Condition 1, there exist two firms with $\left(\mathbf{p},\mathbf{s},\mathbf{X},q,\mathbf{w},j\right)$ and $\left(\mathbf{p}^{\dagger},\mathbf{s}^{\dagger},\mathbf{X}^{\dagger},q,\mathbf{w},j^{\dagger}\right)$ and

$$MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c0}\right)=MR_{j^{\dagger}}\left(\mathbf{p}^{\dagger},\mathbf{s}^{\dagger},\mathbf{X}^{\dagger},\boldsymbol{\theta}_{c0}\right).$$

Then,

$$\psi\left(q,\mathbf{w},\mathbf{x},MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c0}\right),\boldsymbol{\gamma}_0\right)=\psi\left(q,\mathbf{w},\mathbf{x},MR_{j^{\dagger}}\left(\mathbf{p}^{\dagger},\mathbf{s}^{\dagger},\mathbf{X}^{\dagger},\boldsymbol{\theta}_{c0}\right),\boldsymbol{\gamma}_0\right),$$

which implies, from Equation (59)

$$\psi\left(q,\mathbf{w},\mathbf{x},MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c*}\right),\boldsymbol{\gamma}_*\right)=\psi\left(q,\mathbf{w},\mathbf{x},MR_{j\dagger}\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\mathbf{X}^\dagger,\boldsymbol{\theta}_{c*}\right),\boldsymbol{\gamma}_*\right).$$

Then, from strict monotonicity of the function $\psi\left(,\boldsymbol{\gamma}_*\right)$ with respect to MR,

$$MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c*}\right)=MR_{j\dagger}\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\mathbf{X}^\dagger,\boldsymbol{\theta}_{c*}\right).$$

On the other hand, suppose

$$MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c0}\right)\neq MR_{j\dagger}\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\mathbf{X}^\dagger,\boldsymbol{\theta}_{c0}\right).$$

Then,

$$\psi\left(q,\mathbf{w},\mathbf{x},MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c0}\right),\boldsymbol{\gamma}_0\right)\neq\psi\left(q,\mathbf{w},\mathbf{x},MR_{j\dagger}\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\mathbf{X}^\dagger,\boldsymbol{\theta}_{c0}\right),\boldsymbol{\gamma}_0\right),$$

which implies, from Equation (59)

$$\psi\left(q,\mathbf{w},\mathbf{x},MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c*}\right),\boldsymbol{\gamma}_*\right)\neq\psi\left(q,\mathbf{w},\mathbf{x},MR_{j\dagger}\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\mathbf{X}^\dagger,\boldsymbol{\theta}_{c*}\right),\boldsymbol{\gamma}_*\right),$$

which implies

$$MR_j\left(\mathbf{p},\mathbf{s},\mathbf{X},\boldsymbol{\theta}_{c*}\right)\neq MR_{j\dagger}\left(\mathbf{p}^\dagger,\mathbf{s}^\dagger,\mathbf{X}^\dagger,\boldsymbol{\theta}_{c*}\right).$$

Because BLP satisfies Condition 1, $\boldsymbol{\theta}_{c*}=\boldsymbol{\theta}_{c0}$.

A.4 Semi-Parametric Cost Function Estimation

The cost function can be recovered from the pseudo-cost function estimate in three steps.

Step 1

Suppose that we already estimated $\widehat{\psi}\left(q,\mathbf{w},\mathbf{x},MR,\widehat{\gamma}_M\right)$. We then nonparametrically estimate marginal cost for a given point $(q,\mathbf{w},\mathbf{x},C)$ as follows,

$$\begin{aligned}\widehat{MC}\left(q,\mathbf{w},\mathbf{x},C\right) &= \sum_{jm}MR_j\left(\mathbf{X}_m,\mathbf{p}_m,\mathbf{s}_m,\widehat{\boldsymbol{\theta}}_{cM}\right)\\ &\quad\times W_{\mathbf{h}}\left(q-q_{jm},\mathbf{w}-\mathbf{w}_{jm},\mathbf{x}-\mathbf{x}_{jm},C-\widehat{\psi}\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\widehat{\boldsymbol{\theta}}_{cM}\right),\widehat{\gamma}_M\right)\right),\end{aligned}$$

where $\boldsymbol{\theta}_{cM}$ is the vector of estimated demand parameters, and the weight function is

$$\begin{aligned} & W_{\mathbf{h}}\left(q-q_{jm}, \mathbf{w}-\mathbf{w}_m, \mathbf{x}-\mathbf{x}_{jm}, C-\widehat{\psi}_{jm}\right) \\ = \; & \frac{K_{h_q}\left(q-q_{jm}\right) K_{\mathbf{h_w}}\left(\mathbf{w}-\mathbf{w}_m\right) K_{\mathbf{h_x}}\left(\mathbf{x}-\mathbf{x}_{jm}\right) K_{h_C}\left(C-\widehat{\psi}_{jm}\right)}{\sum_{kl} K_{h_q}\left(q-q_{kl}\right) K_{\mathbf{h_w}}\left(\mathbf{w}-\mathbf{w}_l\right) K_{\mathbf{h_x}}\left(\mathbf{x}-\mathbf{x}_{kl}\right) K_{h_C}\left(C-\widehat{\psi}_{kl}\right)}. \end{aligned}$$

where $\mathbf{h} \equiv (h_q, \mathbf{h_w}, \mathbf{h_x}, h_C)$, and $K_{h_q}()$, $K_{\mathbf{h_w}}()$, $K_{\mathbf{h_x}}()$ and $K_{h_C}()$ are kernels of q, $\mathbf{w}$, $\mathbf{x}$, and C with bandwidths h_q, $\mathbf{h_w}$, $\mathbf{h_x}$ and h_C, respectively.

Step 2

Start with a vector of input price, observed product characteristics, output and the conditional expected cost $\left(\mathbf{w}, \mathbf{x}, \overline{q}, \overline{C}\right)$. Then, there exists a variable cost shock $\overline{\upsilon}$ that corresponds to $\widehat{MC}\left(\overline{q}, \mathbf{w}, \mathbf{x}, \overline{C}\right) = MC\left(\overline{q}, \mathbf{w}, \mathbf{x}, \overline{\upsilon}\right)$. Notice that we cannot derive the value of $\overline{\upsilon}$ because we have not constructed the cost function yet. For small Δq, the cost estimate for output $\overline{q}+\Delta q$, input price $\mathbf{w}$ and the same variable cost shock $\overline{\upsilon}$ is

$$\widehat{C}\left(\overline{q}+\Delta q, \mathbf{w}, \mathbf{x}, \overline{\upsilon}\right) = \overline{C} + \widehat{MC}\left(\overline{q}, \mathbf{w}, \mathbf{x}, \overline{C}\right)\Delta q.$$

Then, from the consistency of the marginal revenue estimator (which we will prove later) and the Taylor series expansion,

$$\widehat{C}\left(\overline{q}+\Delta q, \mathbf{w}, \mathbf{x}, \overline{\upsilon}\right) = C\left(\overline{q}+\Delta q, \mathbf{w}, \mathbf{x}, \overline{\upsilon}\right) + O\left(\left(\Delta q\right)^2\right) + o_p\left(1\right)\Delta q.$$

At iteration $k>1$, given $\widehat{C}_{k-1} = \widehat{C}\left(\overline{q}+\left(k-1\right)\Delta q, \mathbf{w}, \overline{\upsilon}\right)$

$$\widehat{C}\left(\overline{q}+k\Delta q, \mathbf{w}, \mathbf{x}, \overline{\upsilon}\right) = \widehat{C}_{k-1} + \widehat{MC}\left(\overline{q}+\left(k-1\right)\Delta q, \mathbf{w}, \mathbf{x}, \widehat{C}_{k-1}\right)\Delta q.$$

Then, from Taylor expansion, we know that for any $k>0$,

$$\widehat{C}\left(\overline{q}+k\Delta q, \mathbf{w}, \mathbf{x}, \overline{\upsilon}\right) = C\left(\overline{q}+k\Delta q, \mathbf{w}, \mathbf{x}, \overline{\upsilon}\right) + O\left(k\left(\Delta q\right)^2\right) + k o_p\left(1\right)\Delta q$$

Thus, we can derive the approximate cost function for given input price $\mathbf{w}$ and quantity q.

Step 3

Next, we derive the cost function for different input price. First, we derive the nonparametric estimate of the input demand. Denote $\mathbf{l}(q, \mathbf{w}, \mathbf{x}, C)$ to be the vector of input demand given output q, input prices $\mathbf{w}$, observed characteristics $\mathbf{x}$ and cost C. Then, its nonparametric estimate is:

$$\widehat{\mathbf{l}}(q, \mathbf{w}, \mathbf{x}, C) = \sum_{jm} \mathbf{l}_{jm} W_{\mathbf{h}} \left(q - q_{jm}, \mathbf{w} - \mathbf{w}_{jm}, \mathbf{x} - \mathbf{x}_{jm}, C - \widehat{\psi} \left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm} \left(\widehat{\boldsymbol{\theta}}_M \right), \widehat{\gamma}_M \right) \right).$$

where $\mathbf{l}_{jm}$ is the vector of inputs of firm j in market m. Notice that from Shepard's Lemma,

$$\mathbf{l} = \frac{\partial C (q, \mathbf{w}, \mathbf{x}, \overline{v})}{\partial \mathbf{w}}.$$

Start, as before, with $\overline{q}$, $\mathbf{w}$, and $\overline{C}$. Next, we derive the cost for the output $\overline{q}$, $\mathbf{w} + \Delta\mathbf{w}$ for small $\Delta\mathbf{w}$ that has the same cost shock $\overline{v}$. Then:

$$\widehat{C}_1 = \widehat{C} (\overline{q}, \mathbf{w} + \Delta\mathbf{w}, \mathbf{x}, \bar{v}) = \overline{C} + \widehat{\mathbf{l}} (\overline{q}, \mathbf{w}, \mathbf{x}, \overline{C}) \Delta\mathbf{w} + O \left(\left(\|\Delta\mathbf{w}\|^2 \right) \right) + o_p (1) \|\Delta\mathbf{w}\|.$$

At iteration $k > 1$, given $\widehat{C}_{k-1} = \widehat{C} (\overline{q}, \mathbf{w} + (k-1) \Delta\mathbf{w}, \mathbf{x}, \overline{v})$

$$\widehat{C} (\overline{q}, \mathbf{w} + k\Delta\mathbf{w}, \mathbf{x}, \overline{v}) = \widehat{C}_{k-1} + \widehat{\mathbf{l}} \left(\overline{q}, \mathbf{w} + (k-1) \Delta\mathbf{w}, \mathbf{x}, \widehat{C}_{k-1} \right) \Delta\mathbf{w}$$

By iterating this, we can derive the approximated cost function, which satisfies

$$\widehat{C} (\overline{q}, \mathbf{w} + k\Delta\mathbf{w}, \mathbf{x}, \overline{v}) = C (\overline{q}, \mathbf{w} + k\Delta\mathbf{w}, \mathbf{x}, \overline{v}) + O \left(\left(k \|\Delta\mathbf{w}\|^2 \right) \right) + k o_p (1) \|\Delta\mathbf{w}\|$$

for any $k > 0$.

A.5 Further specification and data issues

A.5.1 Economic versus accounting cost

The cost data we envision using comes from accounting statements of firms.[29] Such data do not necessarily reflect the economic cost that the firm considers in making input and output choices. More concretely, we may not be appropriately taking into account the opportunity cost of the resources that are used in purchasing the necessary input to produce output. Fortunately, from

[29] Indeed, accounting data are typically used in the literature that estimates cost functions to evaluate market power, measure economies of scale or scope, and so on.

accounting statements, we may be able to obtain information on other activities that the firm may be pursuing in addition to the production of output. For example, we may find details on firms' financial investments including their rate of return.[30] Suppose that the real return on a dollar of a financial investment is r_{jm}. Then, the opportunity cost of production is $p_{jm}r_{jm}$, and the firm will produce and sell output until marginal revenue equals marginal cost that incorporates this cost, i.e.,

$$MR_{jm}\left(\boldsymbol{\theta}_c\right) = MC\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, \upsilon_{jm}\right) + p_{jm}r_{jm}.$$

Substituting this into our estimator, we obtain the modified SNLLS part as follows:

$$\frac{1}{\sum_m J_m}\sum_{j,m}\left[C_{jm} - \psi\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right) - p_{jm}r_{jm}, \boldsymbol{\gamma}\right)\right]^2.$$

That is, as long as we can obtain information on the financial opportunities that the firm has other than production, we can incorporate them into our estimator. Then, the estimator will not be subject to bias even if the cost data we use corresponds to accounting costs.

A.5.2 Endogenous product characteristics.

If firms strategically choose prices *and* product characteristics, then elements of $\mathbf{X}_m$ will be correlated with the demand shock ξ_{jm}. To accommodate endogenous product characteristics, researchers have recently started estimating BLP models that include first order conditions for optimal prices and product characteristics.[31] In particular, Petrin and Seo (2016) use the first order conditions of product choice and additional panel data moment restrictions to identify the price coefficient. The endogeneity of product characteristics does not prevent us from identifying the price coefficient, and thus, markups, as discussed before. It is an issue only if researchers want to estimate the coefficients of the observed characteristics.

To deal with endogenous product characteristics, we follow a strategy that is similar to Petrin and Seo (2016). We differ from them by using cost data instead of instruments. We modify the cost function as below:

$$C\left(q, \mathbf{x}, \mathbf{w}, \upsilon, \boldsymbol{\upsilon}_{\mathbf{x}}\right),$$

[30] In the application of our estimator to the U.S. banking industry, such information is readily available. Most large industries like banking that are subject to some form of regulatory oversight are likely to report such data.

[31] See, for example, Chu (2010), Fan (2013), and Byrne (2015). For an excellent overview of the empirical literature on endogenous product characteristics, see Crawford (2012). It is worth noting that all these applications maintain the static decision-making assumption of BLP - firms are allowed to adjust their product characteristics period-by-period but are not forward-looking in doing so. A recent paper by Gowrisankaran and Rysman (2012) develops and estimates a dynamic version of a differentiated products oligopoly model, the solution of which is computationally extremely burdensome.

where $\boldsymbol{\upsilon}_{\mathbf{x}}$ corresponds to the shock that affects the production of observed characteristics. The additional F.O.C. for optimal product characteristics would then be

$$MR_{\mathbf{x},jm}(\boldsymbol{\theta}_0) = MC_{\mathbf{x}_{jm}}(q_{jm}, \mathbf{x}_{jm}, \mathbf{w}_m, \upsilon_{jm}, \boldsymbol{\upsilon}_{\mathbf{x},jm}),$$

where $MR_{\mathbf{x},jm}$ is the vector of marginal revenues of firm j in market m of the product characteristics choice. Then, the SNLLS part would be modified as

$$\frac{1}{\sum_m J_m} \sum_{j,m} \left[C_{jm} - \sum_l \gamma_l \psi_l \left(q_{jm}, \mathbf{x}_{jm}, \mathbf{w}_m, MR_{jm}(\boldsymbol{\theta}), MR_{\mathbf{x},jm}(\boldsymbol{\theta}) \right) \right]^2,$$

where MR_{jm} is, as before, marginal revenue with respect to output.

A.5.3 Cost function restrictions.

So far, in the estimation exercise, we have not imposed any assumptions on the shape of the pseudo-cost function φ except that it is a smooth function of output, input price, observed product characteristics and marginal revenue. The cost function that is recovered is not restricted to have properties such as a positive slope, homogeneity of degree one in input prices, nor convexity in output or marginal cost to increase with cost shock.

Imposing the restriction of homogeneity in input prices in estimation is straightforward. If the cost function is homogeneous of degree one in input price, so is the marginal cost function. Then, for an input price w_1, which is the first element of the vector $\mathbf{w}$ of input prices,

$$C(q, \mathbf{w}, \mathbf{x}, \upsilon) = w_1 C(q, \frac{\mathbf{w}}{w_1}, \mathbf{x}, \upsilon)$$

and

$$MC(q, \mathbf{w}, \mathbf{x}, \upsilon) = w_1 \frac{\partial C(q, \mathbf{w}/w_1, \mathbf{x}, \upsilon)}{\partial q}.$$

We can thus modify the SNLLS component of our pseudo-cost estimator to impose the homogeneity restriction as follows,

$$\frac{1}{\sum_m J_m} \sum_{j,m} \left[\frac{C_{jm}}{w_{1m}} - \sum_l \gamma_l \psi_l \left(q_{jm}, \frac{\mathbf{w}_{-1,m}}{w_{1m}}, \mathbf{x}_{jm}, \frac{MR_{jm}(\boldsymbol{\theta}_c)}{w_{1m}} \right) \right]^2,$$

where $\mathbf{w}_{-1,m} = (w_{2m}, \ldots, w_{Lm})$ is the vector of input prices except w_{1m}.

A.5.4 Missing cost data and multi product firms

Until now we have assumed that cost data are available for all firms in the sample. However, it could very well be the case that we observe costs only for some firms and not others. Even in this case, we can estimate the structural parameters consistently by constructing the SNLLS part using only those firms for which we have cost data. Because the SNLLS part of our estimator does not involve any orthogonality conditions, and because the random components of the fixed cost and the measurement error of cost are assumed to be i.i.d, choosing firms in this way for estimation will not result in selection bias. It is important to notice, however, that we still need demand-side data for all firms in the same market to compute marginal revenue. Luckily, such demand-side data tends to be available to researchers for many industries. Allowing for missing cost data is important in reducing the concern about reliability of cost data. That is, in practice, researchers can go over the accounting cost data carefully and simply remove the cost data that have anomalies.

A more difficult situation would be when firms produce multiple products, but only the total cost of all products is observable in the data. To address this issue, we follow the setup of BLP closely. Let $\mathcal{F}_f$ be the set of product-market combinations that is produced by firm f. That is, $\mathcal{F}_f = \{(j,m) : I_f(j,m) = 1\}$ where $I_f(j,m) = 1$ if the j^{th} product in market m is produced by firm f and 0 if otherwise. Let $\mathbf{q}_f$ be the vector of outputs of products produced by firm f. That is, $\mathbf{q}_f = \left(q_{j_1,m_1}, q_{j_2,m_2}, \ldots, q_{j_{n_f},m_{n_f}}\right)$ and n_f is the number of product-market combinations produced by firm f. In our vector notation, we follow the convention of ordering the products in the same market and the products in different markets. That is, if the firm f has two different products j and j' with $j < j'$ in the same market m, then q_{jm} comes before $q_{j'm}$ in the vector $\mathbf{q}_f$. If the firm f has two different products j in market m and j' in different market m' with $m < m'$, then q_{jm} comes before $q_{j'm'}$ in the vector $\mathbf{q}_f$. Similarly, let $\mathbf{X}_f$ be the $(K \times n_f)$ matrix of observable product characteristics and $\boldsymbol{\upsilon}_f$ be the vector of cost shocks of products belonging to $\mathcal{F}_f$. Similarly, denote $\mathbf{mc}_f$, $\mathbf{p}_f$ and $\mathbf{s}_f$ to be the vectors of marginal costs, prices and market shares of products belonging to $\mathcal{F}_f$. Input prices are denoted by the matrix $\mathbf{w}_f$ with L input prices for each market the firm operates in. Then, we can modify the cost function of firm f as

$$C\left(\mathbf{q}_f, \mathbf{w}_f, \mathbf{X}_f, \boldsymbol{\upsilon}_f\right).$$

Then, the F.O.C. for profit maximization for product and market $(j,m) \in \mathcal{F}_f$ is:

$$s_{jm} + \sum_{(r',m') \in \mathcal{F}_f} \left(p_{r',m'} - mc_{r',m'}\right) \frac{\partial s_{r',m'}}{\partial p_{jm}} = 0.$$

Hence, we can write

$$\mathbf{mc}_f = \mathbf{p}_f + \mathbf{\Delta}_f^{-1} \mathbf{s}_f,$$

where row h and column i of the matrix $\mathbf{\Delta}_f$ is $[\mathbf{\Delta}_f]_{hi} = \partial s_{j_h,m_h} / \partial p_{j_i,m_i}$. Since markets are isolated, $\partial s_{j_h,m_h} / \partial p_{j_i,m_i} = 0$ for $m_h \neq m_i$.

Then, the SNLLS component can be modified as follows:

$$\frac{1}{F} \sum_f \left[C_f - \sum_l \gamma_{lf} \psi_{lf} \left(\mathbf{q}_f, \mathbf{w}_f, \mathbf{X}_f, \mathbf{p}_f + \mathbf{\Delta}_f^{-1} \left(\mathbf{X}_f, \mathbf{p}_f, \mathbf{s}_f, \boldsymbol{\theta}_0\right) \mathbf{s}_f \left(\mathbf{X}_f, \mathbf{p}_f, \mathbf{s}_f, \boldsymbol{\theta}_0\right) \right) \right]^2,$$

where F is the number of firms; ψ_{lf} is the lth polynomial for firm f and γ_{lf} is the corresponding coefficient. These basis polynomials vary with firms because they depend on the number of products produced by each firm. Estimation can proceed as long as the number of product-market combination for a firm is not too large. Otherwise, we would face a Curse of Dimensionality issue in estimation.

If the number of products n_f is large, one could consider imposing more structure on the cost function. For example, we could specify the total cost of firm f as the sum of the costs of all the products that it produces. That is:

$$\begin{aligned} C_f &= \sum_i C\left(q_{j_i,m_i}, \mathbf{w}_{m_i}, \mathbf{X}_{j_i,m_i}, \upsilon_{j_i,m_i}\right) + \nu_f \\ &= \sum_i \sum_l \gamma_l \psi_l \left(q_{j_i,m_i}, \mathbf{w}_{m_i}, \mathbf{X}_{j_i,m_i}, \mathbf{p}_{j_i,m_i} + \left[\mathbf{\Delta}_f^{-1} \left(\mathbf{X}_f, \mathbf{p}_f, \mathbf{s}_f, \boldsymbol{\theta}_0\right) \mathbf{s}_f \left(\mathbf{X}_f, \mathbf{p}_f, \mathbf{s}_f, \boldsymbol{\theta}_0\right) \right]_i \right) + \nu_f \end{aligned}$$

. Hence, the SNLLS component of our estimator can be modified as follows,

$$\frac{1}{F} \sum_f \left[C_f - \sum_i \sum_l \gamma_l \psi_l \left(q_{j_i,m_i}, \mathbf{w}_{m_i}, \mathbf{X}_{j_i,m_i}, \mathbf{p}_{j_i,m_i} + \left[\mathbf{\Delta}_f^{-1} \left(\mathbf{X}_f, \mathbf{p}_f, \mathbf{s}_f, \boldsymbol{\theta}_0\right) \mathbf{s}_f \left(\mathbf{X}_f, \mathbf{p}_f, \mathbf{s}_f, \boldsymbol{\theta}_0\right) \right]_i \right) \right]^2.$$

Note if we were to use the data on the total costs of the firms that operate in multiple markets, they need to practice third-degree price discrimination. Then, the question arises as to what to do if they don't. In that case, we don't use their cost data and only use the cost data from firms that either operate in a single market or exercise third-degree price discrimination if they operate in multiple markets. Such choice of firms whose cost data we use does not result in any sample selection issues because the selection does not alter the conditional distribution of the measurement error.

A.6 Large Sample Properties

In this section we show that the estimator is consistent and asymptotically normal. Notice that in our sample, we have oligopolistic firms in the same market. Because of strategic interaction, equilibrium prices and outputs of the firms in the same market are likely to be correlated. To avoid the difficulty arising from such within-market correlation, our consistency proof will primarily exploit the large number of isolated markets, with the assumption that wages, unobserved product quality and variable cost shocks are independent across markets

The assumption of independence of variables across markets is employed for simplicity. We leave the asymptotic analysis with some across market dependence for future research. For Strong Law of Large Numbers under weaker assumptions, see Andrews (1988) and the related literature. As we have discussed earlier, those assumptions are not required for identification. Without loss of generality, we assume that in each market, the number of firms is J. Notice that in our estimation procedure, we have two steps: one that involves the difference between the cost in the data and the nonparametrically approximated pseudo-cost function to identify $\boldsymbol{\theta}_c$ (which is α for the Berry logit model and $(\mu_\alpha, \sigma_\alpha)$ and σ_β for the BLP random coefficient logit model) and the second step is the OLS estimation based on the orthogonality condition $\boldsymbol{\xi}_m \perp \mathbf{X}_m$ which identifies $\boldsymbol{\beta}$ for the Berry logit model and $\boldsymbol{\mu}_\beta$ for the BLP model. We start with denoting $\boldsymbol{\theta} = (\boldsymbol{\theta}_\beta, \boldsymbol{\theta}_c)$, where $\boldsymbol{\theta}_\beta$ is the vector of parameters estimated in the second step. In our proof, for the pseudo-cost function part, we follow Bierens (2014) closely.

For the description of our objective function, let $\mathbf{y}_m = \left(\mathbf{q}_m, vec\left(\mathbf{W}_m\right)', \mathbf{C}_m, vec\left(\mathbf{X}_m\right)', vec\left(\mathbf{p}_m\right)', vec\left(\mathbf{s}_m\right)'\right)'$, $\mathbf{z}_m = \left(\mathbf{q}_m, vec\left(\mathbf{W}_m\right)', vec\left(\mathbf{X}_m\right)', vec\left(\mathbf{p}_m\right)', vec\left(\mathbf{s}_m\right)'\right)'$, where $\mathbf{C}_m = (C_{1m}, C_{2m}, ..., C_{Jm})'$, $\mathbf{W}_m = (\mathbf{w}_{1m}, \mathbf{w}_{2m}, ..., \mathbf{w}_{jm})'$.

For the sake of analytical simplicity, we derive the asymptotic results for a slightly modified component of the objective function for firm j in market m, which is

$$\widehat{f}_j\left(\mathbf{y}_m, \boldsymbol{\theta}_c, t, b\right) \equiv \left[C_{jm} - \varphi\left(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_c\right)\right)\right]^2 \frac{h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)}{\frac{1}{MJ}\sum_{m=1}^{M}\sum_{j=1}^{J} h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)}.$$

We set the function $h\left(x\right) \geq 0$ to be twice differentiable; equal to 1 for $x \leq 0$; decreasing in x

for $x > 0$ and equal to zero for $x \geq 4$. Since the objective function is zero for $MR_{jm}(\boldsymbol{\theta}_c) \geq t + 4b$, and the rest of the variables in the objective function are assumed to be bounded, it follows that the objective function is bounded, which simplifies the proofs. For large $t > 0$, the modified objective function is practically equivalent to the original objective function $[C_{jm} - \varphi(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c))]^2$.

We next specify $h()$. Let

$$\begin{aligned} \widetilde{g}(u) &\equiv \left[\frac{1}{2} - \frac{1}{12}u^3\right] I(0 \leq u \leq 1) + \left[1 - \frac{1}{2}u - \frac{1}{12}(2-u)^3\right] I(1 < u \leq 2) \\ g(u) &= \widetilde{g}(0) + \widetilde{g}(u) I(0 \leq u \leq 2) - \widetilde{g}(4-u) I(2 < u \leq 4) \end{aligned}$$

and let

$$h(u) \equiv I(u < 0) + g(u) I(0 \leq u \leq 4).$$

Then, h is a continuous function and

$$h(u) = \begin{cases} 1 & \textit{if } u \leq 0 \\ \frac{11}{12} & \textit{if } u = 1 \\ \frac{1}{2} & if\ u = 2 \\ \frac{1}{12} & if\ u = 3 \\ 0 & if\ u \geq 4. \end{cases}$$

Also,

$$\widetilde{g}'(u) \equiv -\frac{1}{4}u^2 I(0 \leq u \leq 1) + \left[-\frac{1}{2} + \frac{1}{4}(2-u)^2\right] I(1 < u \leq 2),$$

and h' is a continuous function with

$$h'(u) = \begin{cases} 0 & \textit{if } u \leq 0 \\ -\frac{1}{4} & \textit{if } u = 1 \\ -\frac{1}{2} & if\ u = 2 \\ -\frac{1}{4} & if\ u = 3 \\ 0 & if\ u \geq 4. \end{cases}$$

Similarly,

$$\widetilde{g}''(u) \equiv -\frac{1}{2}uI(0 \leq u \leq 1) - \left[\frac{1}{2}(2-u)\right] I(1 < u \leq 2),$$

and h'' is a continuous function with

$$h''(u) = \begin{cases} 0 & \textit{if } u \leq 0 \\ -\frac{1}{2} & \textit{if } u = 1 \\ 0 & if\ u = 2 \\ -\frac{1}{2} & if\ u = 3 \\ 0 & if\ u \geq 4. \end{cases}$$

Now, define

$$\widetilde{f}_j(\mathbf{y}_m, \boldsymbol{\theta}_c, t, b) \equiv f_j(\mathbf{y}_m, \boldsymbol{\theta}_c)\, \widetilde{w}_{jm}(MR_{jm}(\boldsymbol{\theta}_c), t, b),$$

where

$$f_j(\mathbf{y}_m, \boldsymbol{\theta}_c) \equiv [C_{jm} - \varphi(q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm}(\boldsymbol{\theta}_c))]^2$$

and

$$\widetilde{w}_{jm}(MR_{jm}(\boldsymbol{\theta}_c), t, b) \equiv \frac{h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)}{E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}$$

for $t > 0$, and $b > 0$. We also define

$$A \equiv \frac{E\left[\frac{1}{b}h'\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}{E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]^2}.$$

Then,

$$\frac{\partial \widetilde{w}_{jm}(MR_{jm}, t, b)}{\partial MR} = \frac{\frac{1}{b}h'\left(\frac{MR_{jm}-t}{b}\right)}{E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]} - Ah\left(\frac{MR_{jm}-t}{b}\right).$$

Let

$$\begin{aligned} B &= E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right] \\ \frac{\partial \widetilde{w}_{jm}(MR_{jm}, t, b)}{\partial MR \partial MR} &= \frac{1}{B}\left[\frac{1}{b^2}h''\left(\frac{MR_{jm}-t}{b}\right)\right] - \frac{\frac{1}{b}h'\left(\frac{MR_{jm}-t}{b}\right)}{B^2}\frac{\partial B}{\partial MR} - \frac{\partial}{\partial MR}Ah\left(\frac{MR_{jm}-t}{b}\right). \end{aligned}$$

Then, let

$$\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}, t, b) \equiv \frac{1}{J}\sum_{j=1}^{J} f_j(\mathbf{y}_m, \boldsymbol{\chi})\, \widetilde{w}_{jm}(MR_{jm}(\boldsymbol{\theta}_c), t, b), \tag{60}$$

and $Q(\boldsymbol{\chi}) = E\left[\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}, t, b)\,|\mathbf{z}_m\right]$, where $\boldsymbol{\chi} = (\boldsymbol{\theta}_c', \boldsymbol{\gamma}')' = \{\chi_n\}_{n=1}^{\infty}$, with

$$\chi_n = \begin{cases} \theta_{cn} & \text{for } n = 1, ..., p, \\ \gamma_{n-p} & \text{for } n \geq p+1. \end{cases}$$

where p is the number of parameters in $\boldsymbol{\theta}_c$. Denote, as before, $\boldsymbol{\chi}_0 = (\boldsymbol{\theta}_{c0}', \boldsymbol{\gamma}_0')'$ to be the true parameter vector. In both the SNP logit estimation exercise of Bierens (2014) and ours, the polynomials are functions of parametric functions. In our model, the parameters $\boldsymbol{\theta}_c$ are identified. Let $\boldsymbol{\theta}_c \in \Theta_c$ be compact and let the parameter space $\Xi = \Theta_c \times \Gamma$, $(\boldsymbol{\theta}_c, \boldsymbol{\gamma}) \in \Xi$ satisfy

$$\Xi = \{\times_{n=1}^{\infty}[-\overline{\chi}_n, \overline{\chi}_n]\}$$

where $\overline{\chi}_n$, $n = 1, \ldots$ are a priori chosen positive sequence satisfying $\sum_{n=1}^{\infty} \overline{\chi}_n^2 < \infty$, $sup_{n \geq p} |\chi_n| / \overline{\chi}_n \leq 1$. We also assume that the parameter space is endowed with the metric $d(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) \equiv \|\boldsymbol{\chi}_1 - \boldsymbol{\chi}_2\|$, where $\|\boldsymbol{\chi}\| = \sqrt{\sum_{k=1}^{\infty} \chi_k^2}$. Let $\boldsymbol{\chi}_0$ be the vector of true parameters. Define

$$\Xi_k = \begin{cases} \Theta_c & \text{for } k \leq p, \\ \Theta_c \times \Gamma_{k-p}(T) & \text{for } k \geq p+1, \end{cases}$$

where $k \in \mathbb{N}$, $\Gamma_k(T) = \{\pi_k \boldsymbol{\gamma} : \|\pi_k \boldsymbol{\gamma}\| \leq T\}$, and π_k is the operator that applies to an infinite sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=1}^{\infty}$, replacing all the γ_n's for $n > k$ with zeros. Suppose further that $\boldsymbol{\chi}_0 \in \Xi$. Furthermore, since we assume that firms choose prices so that the marginal revenue equals marginal cost, we restrict the parameter space Θ_c so that any $\boldsymbol{\theta}_c \in \Theta_c$ satisfies $MR(\mathbf{X}, \mathbf{p}, \mathbf{s}, j, \boldsymbol{\theta}_c) \geq 0$ for all $(\mathbf{X}, \mathbf{p}, \mathbf{s}, j)$ in the population.

In proving consistency, for the sake of simplicity, we add the following assumptions:

Assumption 10 *$(\mathbf{w}_m, \mathbf{X}_m, Q_m, \boldsymbol{\xi}_m, \boldsymbol{\upsilon}_m, \boldsymbol{\varsigma}_m, \boldsymbol{\nu}_m)$, $m = 1, \ldots, M$ are i.i.d. across markets.*

Define $\widetilde{\mathbf{d}}_m \equiv (Q_m, vec(\mathbf{w}_m)', vec(\boldsymbol{\xi}_m)', vec(\boldsymbol{\upsilon}_m)', vec(\mathbf{X}_m)')$.

Assumption 11 *The support of $\widetilde{\mathbf{d}}_m$, denoted as $\mathcal{D}_d$, is compact, i.e., $\mathcal{D}_d \equiv [\underline{Q}, \overline{Q}] \times [\underline{w}, \overline{w}]^L \times [-\overline{\xi}, \overline{\xi}]^J \times [\underline{\upsilon}, \overline{\upsilon}]^J \times [-\overline{x}, \overline{x}]^{JK}$.*

Assumption 12 *The parameter space Ξ is compact. The true parameter $\boldsymbol{\chi}_0$ is in the interior of Ξ.*

$\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}, \tau, b)$, $\mathbf{y}_m$, $m = 1, \ldots, M$, $\boldsymbol{\chi}_0$ and Ξ satisfy the assumptions that are similar to the Assumption 4.1 of Bierens (2014). Some inequalities are reversed because in our case, our estimator is derived by minimizing the objective function, whereas in Bierens (2014), it is based on maximization.

(a) $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_M$ are i.i.d. The support of $\mathbf{y}_m$ is contained in an open set $\mathcal{Y}$ of the Euclidean space.

(b) Ξ is a metric space with metric $d(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)$.

(c) For each $\boldsymbol{\chi} \in \Xi$, $\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}, \tau, b)$ is a Borel measurable real function of $\mathbf{y}_m$.

(d) $\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}, \tau, b)$ is a.s. continuous in $\boldsymbol{\chi} \in \Xi$.

(e) There exists a non-negative Borel measurable real function $\underline{f}(\mathbf{y})$ such that $E\left[\underline{f}(\mathbf{y}_m)\right] > -\infty$ and $\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}, \tau, b) > \underline{f}(\mathbf{y}_m)$ for all $\boldsymbol{\chi} \in \Xi$.

(f) There exists an element $\boldsymbol{\chi}_0 \in \Xi$ such that $Q(\boldsymbol{\chi}) > Q(\boldsymbol{\chi}_0)$ for all $\boldsymbol{\chi} \in \Xi \backslash \{\boldsymbol{\chi}_0\}$, and $Q(\boldsymbol{\chi}_0) < \infty$.

(g) There exists an increasing sequence of compact subspaces Ξ_k in Ξ such that $\boldsymbol{\chi}_0 \in \overline{\bigcup_{k=1}^{\infty} \Xi_k} = \overline{\Xi} \subset \Xi$.

(h) Each sieve space Ξ_k is isomorph to a compact subset of a Euclidean space.

(i) Each sieve space Ξ_k contains an element $\boldsymbol{\chi}_k$ such that, $lim_{k \to \infty} E\left[\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}_k, \tau, b)\right] = E\left[\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}_0, \tau, b)\right]$.

(j) The set $\Xi_\infty = \left\{\boldsymbol{\chi} \in \Xi : E\left[\widetilde{f}(\mathbf{y}_m, \boldsymbol{\chi}, \tau, b)\right] = \infty\right\}$ does not contain an open ball.

Lemma 5 *Assumptions (a)$\sim$(j) hold.*

Proof. First, we show that assumption (a) holds. First, since market size is bounded above by $\overline{Q}$, $q_{jm} \leq \overline{Q}$ for any j, m by definition of market share $s_{jm} = q_{jm}/Q_m < 1$, and thus, $q_{jm} < Q_m \leq \overline{Q}$. From Assumption 7, the cost function is continuously differentiable in output. Therefore, $MC(q_{jm}, \mathbf{w}_m, \mathbf{X}_m, \upsilon_{jm})$ is also bounded. Suppose $p_{jm} \to \infty$. Then, if all other prices are bounded, since $\boldsymbol{\xi}_m$ is bounded, $s_{jm} \to Prob(\alpha : \alpha + \eta_{\alpha 0} > 0) = \Phi(\eta_{\alpha 0}) < 1/2$. Therefore, $lim_{p_{jm} \to \infty} p_{jm} s_{jm} \to \infty$. Now, consider for a small $\delta > 0$, $s'_{jm} = \Phi(\eta_{\alpha 0}) + \delta$, and p'_{jm} corresponding to it. Then, because $lim_{p_{jm} \to \infty} s_{jm} = \Phi(\eta_{\alpha 0}) > 0$ and $lim_{p_{jm} \to \infty} p_{jm} s_{jm} \to \infty$, there exists $(\widetilde{s}_{jm}, \widetilde{p}_{jm})$ such that $\widetilde{s}_{jm} < s'_{jm}$ and $\widetilde{p}_{jm} > p'_{jm}$, and $p'_{jm} s'_{jm} - \widetilde{p}_{jm} \widetilde{s}_{jm} < 0$. Therefore, from the mean value theorem, there exists $s^*_{jm} \in \left(\widetilde{s}_{jm}, s'_{jm}\right)$ and the corresponding p^*_{jm} such that

$$lim_{\widetilde{p}\to\infty}\frac{p'_{jm}s'_{jm}-\widetilde{p}_{jm}\widetilde{s}_{jm}}{s'_{jm}-\widetilde{s}_{jm}}=\frac{\partial p^*_{jm}s^*_{jm}}{\partial s_{jm}}=-\infty.$$

On the other hand, consider $s'_{jm}=\Phi\left(\eta_{\alpha 0}\right)-\delta$ for a small $\delta>0$, and p'_{jm} corresponding to it. Then, because $lim_{p_{jm}\to\infty}s_{jm}=\Phi\left(\eta_{\alpha 0}\right)>0$ and $lim_{p_{jm}\to\infty}p_{jm}s_{jm}\to\infty$, there exists $(\widetilde{s}_{jm},\widetilde{p}_{jm})$ such that $\widetilde{s}_{jm}>s'_{jm}$ and $\widetilde{p}_{jm}>p'_{jm}$, and $\widetilde{p}_{jm}\widetilde{s}_{jm}-p'_{jm}s'_{jm}>0$ and can be made arbitrarily large by choosing arbitrarily high $\widetilde{p}_{jm}$. Therefore,

$$lim_{\widetilde{p}\to\infty}\frac{\widetilde{p}_{jm}\widetilde{s}_{jm}-p'_{jm}s'_{jm}}{\widetilde{s}_{jm}-s'_{jm}}=\infty.$$

Therefore, from the mean value theorem, there exists $s^*_{jm} \in \left(s'_{jm}, \widetilde{s}_{jm}\right)$ and the corresponding p^*_{jm} such that

$$lim_{\widetilde{p}\to\infty}\frac{\widetilde{p}_{jm}\widetilde{s}_{jm}-p'_{jm}s'_{jm}}{\widetilde{s}_{jm}-s'_{jm}}=\frac{\partial p^*_{jm}s^*_{jm}}{\partial s_{jm}}=\infty.$$

From the boundedness of the marginal cost function, we conclude from continuity of the market share function that for sufficiently large p_J, marginal revenue is either negative or higher than the marginal cost. We can show that similar results occur if the price goes to infinity for multiple firms as well. Therefore, we conclude that prices are bounded. Furthermore, $s_{jm}\in(0,1)$. Lastly, $C_{jm}\in(-\infty,\infty)$. Therefore, (a) holds.

Assumption (b) is also satisfied with $d\left(\right)$ being the Euclidean metric. (c), (d) are also satisfied given the Borel measurability and the continuity of the market share function and the deterministic component of the cost $C^v\left(q,\mathbf{w},\mathbf{x},\upsilon\right)+e_f\left(\mathbf{w},\mathbf{x},\upsilon\right)+e_{me}\left(q,\mathbf{w},\mathbf{x}\right)$, and the Borel measurability of the random component of the fixed cost and the measurement error, $\nu_{jm}+\varsigma_{jm}$.

Assumption (e) is satisfied because $\widetilde{f}_j\left(\mathbf{y},\chi\right)\geq 0$ for any $(\mathbf{y},\boldsymbol{\chi})$ if we set $\underline{f}\left(\boldsymbol{y}\right)\equiv -1$, from Equation (60). Assumption (f) follows from the identification of $\boldsymbol{\chi}_0$ in Proposition 2 as follows:

$$\begin{aligned}&E\left[\left[C_{jm}-\psi\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\boldsymbol{\theta}_c\right),\boldsymbol{\gamma}\right)\right]^2\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)\right]\\ \geq\ &E\left[\left(\nu_{jm}+\eta_{jm}\right)^2\right]\end{aligned}$$

where strict inequality holds if $(\boldsymbol{\theta}_c,\boldsymbol{\gamma})\neq(\boldsymbol{\theta}_{c0},\boldsymbol{\gamma}_0)$ for sufficiently large t, $\overline{\xi}$ and $\overline{x}$, so that our identification proof goes through.

Assumptions (g) and (h) hold from Assumption 9 (Equation (29)). Next, we consider Assumption (i). We set $\boldsymbol{\chi}_k$ to be $\boldsymbol{\chi}_k = \pi_k \boldsymbol{\chi}$ and $\chi_l = \theta_{c0l}$ for $l = 1, \ldots, p$. Furthermore,

$$\begin{aligned}
& E\left[\widetilde{f}_j\left(\mathbf{y}_m, \boldsymbol{\chi}_{0k}, t, b\right)\right] \\
\leq \; & E\left[v_{jm}^2 + \eta_{jm}^2\right] + E\left[\left(\sum_{l=p+1}^{\infty} \gamma_{0l}\psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_{c0}\right)\right)\right)^2 w_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), t, b\right)\right] \\
\leq \; & E\left[v_{jm}^2 + \eta_{jm}^2\right] + E\left[\sup_{l \geq p+1} \psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_0\right)\right)^2 w_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), t, b\right)\right]\left(\sum_{l=p+1}^{\infty} |\gamma_{0l}|\right)^2 \\
\leq \; & E\left[v_{jm}^2 + \eta_{jm}^2\right] + E\left[\sup_{l \geq p+1} \psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_0\right)\right)^2 w_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), t, b\right)\right]\left(\sum_{l=p+1}^{\infty} |\gamma_{0l}|\right)^2,
\end{aligned}$$

and the RHS is uniformly bounded. Thus, by taking expectation, we obtain

$$E\left[\widetilde{f}_j\left(\mathbf{y}_m, \boldsymbol{\chi}_{0k}, t, b\right)\right] \leq E\left[v_{jm}^2 + \eta_{jm}^2\right] + E\left[\sup_{l \geq p+1} \psi_l\left(q_{jm}, \mathbf{w}_m, \mathbf{x}_{jm}, MR_{jm}\left(\boldsymbol{\theta}_0\right)\right)^2 w_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_{c0}\right), t, b\right)\right] < \infty.$$

Therefore, from the Dominated Convergence theorem, $E\left[\widetilde{f}_j\left(\mathbf{y}_m, \boldsymbol{\chi}_{0k}, t, b\right)\right] \to E\left[\widetilde{f}_j\left(\mathbf{y}_m, \boldsymbol{\chi}_0, t, b\right)\right]$ as $k \to \infty$. Hence, (i) is satisfied.

Furthermore, by construction, $\widetilde{f}\left(\mathbf{y}_m, \boldsymbol{\chi}_k, t, b\right)$ is uniformly bounded, and hence,

$$\Xi_\infty = \left\{\boldsymbol{\chi} \in \Xi : E\left[\widetilde{f}\left(\mathbf{y}_m, \boldsymbol{\chi}, t, b\right)\right] = \infty\right\}$$

is an empty set, and assumption (j) is satisfied. ■

Therefore, the assumptions that are equivalent of Assumption 4.1 of Bierens (2014) are satisfied. Then, consider the ϵ-neighborhood of a parameter vector $\boldsymbol{\chi}_* \in \Xi$. Then, because $\mathbf{y}_m$ is i.i.d, for $\boldsymbol{\chi}$ in the ϵ neighborhood of $\boldsymbol{\chi}_*$, the random variable $\inf_{\boldsymbol{\chi} \in \Xi, d(\boldsymbol{\chi}_*, \boldsymbol{\chi}) < \epsilon} \widetilde{f}_j\left(\mathbf{y}_m, \boldsymbol{\chi}, t, b\right)$ is also i.i.d. Thus, we can use the Kolmogorov's LLN. To do so, we need to show that one can take the expectation of $\inf_{\boldsymbol{\chi}_* \in \Xi, d(\boldsymbol{\chi}, \boldsymbol{\chi}_*) < \epsilon} \widetilde{f}_j\left(\mathbf{y}_m, \boldsymbol{\chi}_*, t, b\right)$ over $\mathbf{y}_m$, i.e., that it is integrable over $\mathbf{y}_m$. Since, by construction, $\widetilde{f}\left(\mathbf{y}_m, \boldsymbol{\chi}_k, t, b\right)$ is nonnegative and uniformly bounded, given any $\boldsymbol{\chi} \in \Xi$, for any $\epsilon > 0$,

$$E\left[\left|\inf_{\boldsymbol{\chi}_* \in \Xi, d(\boldsymbol{\chi}, \boldsymbol{\chi}_*) < \epsilon} \widetilde{f}_j\left(\mathbf{y}_m, \boldsymbol{\chi}_*, t, b\right)\right|\right] < \infty$$

Hence, from Kolmogorov's LLN,

$$\frac{1}{M}\sum_{m=1}^{M}\inf_{\boldsymbol{\chi}_*\in\Xi,d(\boldsymbol{\chi},\boldsymbol{\chi}_*)<\epsilon}\widetilde{f}_j\left(\mathbf{y}_m,\boldsymbol{\chi}_*,t,b\right)\rightarrow_{a.s.}E\left[\inf_{\boldsymbol{\chi}_*\in\Xi,d(\boldsymbol{\chi},\boldsymbol{\chi}_*)<\epsilon}\widetilde{f}_j\left(\mathbf{y}_m,\boldsymbol{\chi}_*,t,b\right)\right]\tag{61}$$

as $M\rightarrow\infty$. Now, let

$$\widehat{f}\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right)\equiv\frac{1}{J}\sum_{j=1}^{J}f_j\left(\mathbf{y}_m,\boldsymbol{\chi}\right)w_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)$$

where

$$\begin{aligned}
w_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right) &\equiv \frac{h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)}{[MJ]^{-1}\sum_{m=1}^{M}\sum_{j=1}^{J}h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)}\\
&= \frac{E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}{[MJ]^{-1}\sum_{m=1}^{M}\sum_{j=1}^{J}h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)}\times\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)
\end{aligned}$$

Now, let

$$\widehat{E}_M\left[h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)\right]\equiv\frac{1}{MJ}\sum_{m=1}^{M}\sum_{j=1}^{J}h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)$$

and let

$$A_M\equiv\frac{E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}{\widehat{E}_M\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}.$$

By applying the Kolmogorov Law of Large Numbers, as $M\rightarrow\infty$,

$$\widehat{E}_M\left[h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)\right]\stackrel{a.s.}{\rightarrow}E\left[h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)\right]$$

For sufficiently large $t>0$, $E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)|\mathbf{z}_m\right]>0$, thus,

$$A_M\stackrel{a.s.}{\rightarrow}1.\tag{62}$$

Then, since

$$\widehat{f}\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right)=A_M\times\widetilde{f}\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right)$$

Therefore, from Equation (61) and (62), for any $\epsilon > 0$

$$\frac{1}{M}\sum_{m=1}^{M}\inf_{\boldsymbol{\chi}_*\in\boldsymbol{\Theta}_c\times\boldsymbol{\Gamma},d(\boldsymbol{\chi},\boldsymbol{\chi}_*)<\epsilon}\widehat{f}\left(\mathbf{y}_m,\boldsymbol{\chi}_*,t,b\right)\overset{a.s.}{\to}E\left[\inf_{\boldsymbol{\chi}_*\in\boldsymbol{\Theta}_c\times\boldsymbol{\Gamma},d(\boldsymbol{\chi},\boldsymbol{\chi}_*)<\epsilon}\widehat{f}\left(\mathbf{y}_m,\boldsymbol{\chi}_*,t,b\right)\right].$$

Therefore, Theorem 4.1 of Bierens holds. That is, $plim_{M\to\infty}d\left(\boldsymbol{\chi}_M,\boldsymbol{\chi}_0\right)=0$ if and only if for any compact subset Ξ_c of Ξ with $\boldsymbol{\chi}_0$ in its interior, $lim_{M\to\infty}Pr\left(\boldsymbol{\chi}_{bM}\in\Xi_c\right)=1$.

Next, we explain how we set up the parameter space so that Assumption 4.2 of Bierens (2014) is satisfied. That is,

Assumption 13 *(Assumption 4.2, Bierens (2014)) Either*

(a) $\overline{\Xi}=\overline{\cup_{n=1}^{\infty}\Xi_n}$ *is compact itself, or*

(b) *There exists a compact set* Ξ_c *containing* $\boldsymbol{\chi}_0$ *such that* $Q\left(\boldsymbol{\chi}_0\right)<E\left[\inf_{\boldsymbol{\chi}\in\overline{\Xi}\setminus\Xi_c}\widetilde{f}\left(\mathbf{y}_m,\boldsymbol{\chi},\tau,b\right)\right]<\infty$.

Assumption 12 and (g) guarantee (a). Then, we can prove consistency using an equivalent of Theorem 4.2 of Bierens (2014). That is,

Theorem 1 *(Theorem 4.2, Bierens (2014)) Under assumptions that are equivalent to the Assumptions 4.1 and 4.2 of Bierens (2014),* $plim_{M\to\infty}d\left(\boldsymbol{\chi}_M,\boldsymbol{\chi}_0\right)=0$.

Therefore, we proved consistency of our estimator.

Next, we prove the asymptotic normality of our estimator. First, we prove that $\partial A_M/\partial\theta_{cl}\overset{a.s.}{\to}0$ and $\partial^2 A_M/\partial\theta_{ck}\partial\theta_{cl}\overset{a.s.}{\to}0$. Now, let

$$a_M\left(\boldsymbol{\theta}_c\right)\equiv log\left(E\left[h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)\right]\right)-log\left(\widehat{E}_M\left[h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)\right]\right).$$

Furthermore, since $h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)$ satisfies the conditions required for interchanging the integration and derivatives, we do so and apply the Kolmogorov's Law of Large Numbers to derive

$$\widehat{E}_M\left[\frac{\partial}{\partial\theta_{cl}}h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)\right]\overset{a.s.}{\to}E\left[\frac{\partial}{\partial\theta_{cl}}h\left(\frac{MR_{jm}\left(\boldsymbol{\theta}_c\right)-t}{b}\right)\right],\ l=1,\ldots,p,$$

$$\frac{\partial}{\partial\theta_{cl}}a_M\left(\boldsymbol{\theta}_c\right)=\frac{E\left[\frac{\partial}{\partial\theta_{cl}}h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}{E\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}-\frac{\widehat{E}_M\left[\frac{\partial}{\partial\theta_{cl}}h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}{\widehat{E}_M\left[h\left(\frac{MR_{jm}(\boldsymbol{\theta}_c)-t}{b}\right)\right]}\overset{a.s.}{\to}0.$$

Since,

$$\frac{\partial}{\partial\theta_{cl}}a_M\left(\boldsymbol{\theta}_c\right)=\frac{1}{A_M}\frac{\partial A_M}{\partial\theta_{cl}},\ and\ A_M\overset{a.s.}{\to}1,\ \frac{\partial A_M}{\partial\theta_{cl}}\overset{a.s.}{\to}0.$$

By the same logic, it is straightforward to show

$$\frac{\partial}{\partial\theta_{ck}\theta_{cl}} a_M(\boldsymbol{\theta}_c) \quad \overset{a.s.}{\rightarrow} \quad 0,$$

and thus,

$$\frac{\partial^2 A_M}{\partial\theta_{ck}\partial\theta_{cl}} \overset{a.s.}{\rightarrow} 0.$$

We next discuss how we either impose or derive Assumption 6.1 of Bierens (2014).

Assumption 14 *(Assumption 6.1, Bierens (2014))*

(a) *Parameter space Ξ is endowed with the norm*

$$\|\boldsymbol{\chi}\|_r = \sum_{n=1}^{\infty} n^r |\chi_n|$$

and the associated metric $d(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = \|\boldsymbol{\chi}_1 - \boldsymbol{\chi}_2\|_r$.

(b) *The true parameter $\boldsymbol{\chi}_0 = \{\chi_{0,n}\}_{n=1}^{\infty}$ satisfies $\|\boldsymbol{\chi}_0\|_r < \infty$.*

(c) *There exists $k \in \mathbb{N}$ such that for any $n \geq k$ $\boldsymbol{\chi}_{0,n} = \pi_n \boldsymbol{\chi}_0 \in \Xi_n^{Int}$, where Ξ_n^{Int} is the interior of the sieve space Ξ_n.*

(d) *$f(\mathbf{y}_m, \boldsymbol{\chi}, \tau, b)$ is a.s. twice continuously differentiable in an open neighborhood of $\boldsymbol{\chi}_0$.*

We assume (a) and (b). For (c), as explained earlier, we follow Bierens (2014) and construct the compact parameter space so that (c) is satisfied. The logit and BLP marginal revenues are twice differentiable and the cost function is also assumed to be twice continuously differentiable. Furthermore, $\widetilde{w}_{jm}(MR_{jm}, t, b)$ is also constructed to be twice continuously differentiable. Therefore, (d) holds.

It is straightforward to show that Assumption 6.2 of Bierens (2014) is satisfied given Assumption 14 and the earlier consistency proof. That is,

Assumption 15 *For any subsequence $k = k_M$ of the sample size M satisfying $k_M \to \infty$ as $M \to \infty$, $plim_{M\to\infty} ||\boldsymbol{\chi}_{k_M} - \boldsymbol{\chi}_0||_r = 0$.*

Then, Bierens (2014) proves in Lemma 6.1 that there exists a subsequence $K_n \leq n$, $\lim_{n\to\infty} K_n = \infty$ such that

$$\lim_{M\to\infty} Pr\left[\sum_{m=1}^{M} \nabla_k \widetilde{f}\left(\mathbf{y}_m, \boldsymbol{\chi}_{n_M}, t, b\right) = 0 \ for \ k = 1, \ldots, K_{n_M}\right] = 1$$

Furthermore, we also assume:

Assumption 16 *(a) :* $E\left[(\nu+\zeta)^4\right]<\infty$

First, we set $r=2$. Then, we prove that Assumption 6.3 of Bierens (2014) is satisfied. That is,

Lemma 6 *(Assumption 6.3, Bierens (2014))There exists a nonnegative integer $r_0<r$ such that the following local Lipschitz conditions hold for all positive integer $l\in\mathbb{N}$ we have*

$$E\left\|\nabla_l\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_0,t,b\right)-\nabla_l\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0,k},t,b\right)\right\|\leq C_l\left\|\boldsymbol{\chi}_0-\boldsymbol{\chi}_{0,k}\right\|_{r_0}$$

where $\nabla_l\widetilde{f}\left(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b\right)=\partial\widetilde{f}\left(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b\right)/\partial\boldsymbol{\chi}_{0,l}$, $\sum_{l=1}^{\infty}2^{-l}C_l<\infty$ and the sieve order $k=k_M$ is chosen such that

$$\lim_{M\to\infty}\sqrt{M}\sum_{n=k_M+1}^{\infty}n^{r_0}\left|\boldsymbol{\chi}_{0,n}\right|=0.$$

Proof. We choose $k_M>p$ for any $M>0$.

$$\begin{aligned}\widetilde{f}_j\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right) &= f_j\left(\mathbf{y}_m,\boldsymbol{\theta}_c\right)\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)\\ &= \left[C_{jm}-\psi\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\boldsymbol{\theta}_c\right),\boldsymbol{\gamma}\right)\right]^2\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)\end{aligned}$$

For $k>p$, $l>p$,

$$\nabla_k\widetilde{f}_j\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right)=-2\left[C_{jm}-\psi\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\boldsymbol{\theta}_c\right),\boldsymbol{\gamma}\right)\right]\psi_{k-p}\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right) \tag{63}$$

$$E\left[\left|\nabla_{l,k}\widetilde{f}\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right)\right|\right]=E\left[2\left|\psi_{l-p}\psi_{k-p}\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)\right|\right]<B. \tag{64}$$

for sufficiently large $B>0$.

For $k>p$, $l\leq p$,

$$\begin{aligned}\nabla_l\widetilde{f}_j\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right) &= \Big[-2\left[C_{jm}-\psi\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\boldsymbol{\theta}_c\right),\boldsymbol{\gamma}\right)\right]\\ &\quad\times\psi_{MR}\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\boldsymbol{\theta}_c\right),\boldsymbol{\gamma}\right)\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)\\ &\quad+\left[C_{jm}-\psi\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\boldsymbol{\theta}_c\right),\boldsymbol{\gamma}\right)\right]^2\frac{\partial}{\partial MR}\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)\Big]\times\frac{\partial MR_{jm}\left(\boldsymbol{\theta}_c\right)}{\partial\theta_{cl}}\end{aligned} \tag{65}$$

$$\nabla_{k,l}\widetilde{f}_j\left(\mathbf{y}_m,\boldsymbol{\chi},t,b\right)=\left[2\psi_{MR}\psi_{k-p}\widetilde{w}_{jm}-2A_1\psi_{MR,k-p}\widetilde{w}_{jm}-2A_1\psi_{k-p}\frac{\partial}{\partial MR}\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_c\right),t,b\right)\right]\times\frac{\partial MR_{jm}\left(\boldsymbol{\theta}_c\right)}{\partial\theta_{cl}}$$

where $A_1\equiv C_{jm}-\psi\left(q_{jm},\mathbf{w}_{jm},\mathbf{x}_{jm},MR_{jm}\left(\boldsymbol{\theta}_c\right),\boldsymbol{\gamma}\right)$.

Since $\psi\left(,MR_{jm}\left(\boldsymbol{\theta}_{c0}\right)\right)$, $\psi\left(,MR_{jm}\left(\boldsymbol{\theta}_{c}\right),\boldsymbol{\gamma}\right)$, $\psi_{MR}\left(MR_{jm}\left(\boldsymbol{\theta}_{c}\right),\boldsymbol{\gamma}\right)$, $\psi_{MR,k-p}\left(,MR_{jm}\left(\boldsymbol{\theta}_{c}\right),\boldsymbol{\gamma}\right)$, $\widetilde{w}_{jm}$ and $\partial\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_{c}\right),t,b\right)/\partial MR$, $\partial MR_{jm}\left(\boldsymbol{\theta}_{c}\right)/\partial\theta_{cl}$ are uniformly bounded over the compact domain,

$$E\left[\left|\nabla_{l,k}\widetilde{f}\left(\mathbf{y}_{m},\boldsymbol{\chi},t,b\right)\right|\right]<B$$

for sufficiently large $B>0$.

Therefore, from the mean value theorem, for $k>p$,

$$E\left\|\nabla_{l}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0},t,b\right)-\nabla_{l}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0,k},t,b\right)\right\|\leq B\sum_{n=k+1}^{\infty}\left|\chi_{0n}\right|<B\left\|\boldsymbol{\chi}_{0}-\boldsymbol{\chi}_{0,k}\right\|_{r_{0}}$$

and claim holds. ■

Next, Assumption 6.4 of Bierens (2014) requires that for all $k\in\mathbb{N}$, $E\left[\nabla_{k}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0},t,b\right)\right]=0$. This holds because of the F.O.C. for the parameter value $\boldsymbol{\chi}_{0}$.

We show next that Assumption 6.5 of Bierens (2014): $\sum_{j=1}^{\infty}j2^{-j}E\left[\left(\nabla_{j}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0},t,b\right)\right)^{2}\right]<\infty$, holds. From Equations (63) and (65), since for $l>0$, $\psi\left(,MR_{jm}\left(\boldsymbol{\theta}_{c0}\right)\right)$, $\psi\left(,MR_{jm}\left(\boldsymbol{\theta}_{c}\right),\boldsymbol{\gamma}\right)$, $\psi_{MR}\left(MR_{jm}\left(\boldsymbol{\theta}_{c}\right),\boldsymbol{\gamma}\right)$, $\psi_{MR,l-p}\left(,MR_{jm}\left(\boldsymbol{\theta}_{c}\right),\boldsymbol{\gamma}\right)$ for $l>p$, $\widetilde{w}_{jm}$ and $\partial\widetilde{w}_{jm}\left(MR_{jm}\left(\boldsymbol{\theta}_{c}\right),t,b\right)/\partial MR$, $\partial MR_{jm}\left(\boldsymbol{\theta}_{c}\right)/\partial\theta_{cl}$ are uniformly bounded over the compact domain, for $j>0$, $\nabla_{j}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0},t,b\right)$ is uniformly bounded over the compact domain. Therefore,

$$E\left[\left[\nabla_{l}\widetilde{f}\left(\mathbf{y}_{m},\boldsymbol{\chi},t,b\right)\right]^{2}\right]<B$$

for sufficiently large $B>0$, and claim is satisfied.

Next, we prove that (a)-(c) of Assumption 6.6 of Bierens (2014) is satisfied.

Lemma 7 *(Assumption 6.6, Bierens (2014)) For some* $\tau>0$,

(a) $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(jk\right)^{-2-\tau}E\left[\left|\nabla_{j,k}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0},t,b\right)\right|\right]<\infty$, *where* $\nabla_{j,k}\widetilde{f}\left(\mathbf{y}_{m},\boldsymbol{\chi}_{0},t,b\right)=\partial^{2}\widetilde{f}\left(\mathbf{y}_{m},\boldsymbol{\chi}_{0},t,b\right)/\left(\partial\boldsymbol{\chi}_{0,j}\partial\boldsymbol{\chi}_{0,k}\right)$.

(b) $\lim_{\epsilon\downarrow0}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(jk\right)^{-2-\tau}E\left[sup_{\left\|\boldsymbol{\chi}-\boldsymbol{\chi}_{0}\right\|_{r}\leq\epsilon}\left|\nabla_{j,k}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi},t,b\right)-\nabla_{j,k}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0},t,b\right)\right|\right]=0$.

(c) *For at least one pair of positive integers* l,n, $E\left[\nabla_{l,p+n}\widetilde{f}\left(\mathbf{y},\boldsymbol{\chi}_{0},t,b\right)\right]\neq0$.

Proof. (a): From the proof of Lemma 6, for $j>p$, $k>p$, and for $j\leq p$, $k>p$ and $j>p$, $k\leq p$

$$E\left[\left|\nabla_{j,k}\widetilde{f}\left(\mathbf{y}_{m},\boldsymbol{\chi},t,b\right)\right|\right]<B$$

for sufficiently large $B > 0$. For $k \leq p$, $l \leq p$,

$$\begin{aligned}
\nabla_l \widetilde{f}_j (\mathbf{y}_m, \boldsymbol{\chi}, t, b) &= [-2 [C_{jm} - \psi (q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm} (\boldsymbol{\theta}_c), \boldsymbol{\gamma})] \\
&\quad \times \psi_{MR} (q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm} (\boldsymbol{\theta}_c), \boldsymbol{\gamma}) \widetilde{w}_{jm} (MR_{jm} (\boldsymbol{\theta}_c), t, b) \\
&\quad \left. + [C_{jm} - \psi (q_{jm}, \mathbf{w}_{jm}, \mathbf{x}_{jm}, MR_{jm} (\boldsymbol{\theta}_c), \boldsymbol{\gamma})]^2 \frac{\partial}{\partial MR} \widetilde{w}_{jm} (MR_{jm} (\boldsymbol{\theta}_c), t, b) \right] \times \frac{\partial MR_{jm} (\boldsymbol{\theta}_c)}{\partial \theta_{cl}} \\
&= \left[-2A_1 \psi_{MR} \widetilde{w}_{jm} + A_1^2 \frac{\partial}{\partial MR} \widetilde{w}_{jm} \right] \frac{\partial MR_{jm} (\boldsymbol{\theta}_c)}{\partial \theta_{cl}}
\end{aligned}$$

$$\begin{aligned}
\nabla_{k,l} \widetilde{f} (\mathbf{y}_m, \boldsymbol{\chi}, t, b) &= \left[-2\psi_{MR}^2 \widetilde{w}_{jm} + A_1 \psi_{MR,MR} \widetilde{w}_{jm} + A_1 \psi_{MR} \frac{\partial}{\partial MR} \widetilde{w}_{jm} \right] \frac{\partial MR (\boldsymbol{\theta}_c)}{\partial \theta_{cl}} \\
&\quad + \left[-2A_1 \psi_{MR} \frac{\partial}{\partial MR} \widetilde{w}_{jm} + A_1^2 \frac{\partial^2}{\partial MR \partial MR} \widetilde{w}_{jm} \right] \frac{\partial MR (\boldsymbol{\theta}_c)}{\partial \theta_{cl}} \\
&\quad + \left[-2A_1 \psi_{MR} \widetilde{w}_{jm} + A_1^2 \frac{\partial}{\partial MR} \widetilde{w}_{jm} \right] \frac{\partial^2 MR_{jm} (\boldsymbol{\theta}_c)}{\partial \theta_{cl} \partial \theta_{cl}}
\end{aligned}$$

Once again, by earlier arguments, it follows that,

$$E \left[\left| \nabla_{k,l} \widetilde{f} (\mathbf{y}_m, \boldsymbol{\chi}, t, b) \right| \right] = 2E [|B|] < \infty.$$

Therefore, (a) holds.

Note that all the terms of $\nabla_{k,l} \widetilde{f} (\mathbf{y}_m, \tau, b, \boldsymbol{\chi})$ are continuous for $\|\boldsymbol{\chi} - \boldsymbol{\chi}_0\|_r \leq \epsilon$ for sufficiently small $\epsilon > 0$. Furthermore, it is uniformly continuous because the parameters belong to a compact set. Therefore, (b) holds.

(c) Suppose that for $k > p$ and $k = l$, from Equation (64),

$$E \left[\nabla_{k,l} \widetilde{f}_j (\mathbf{y}_m, \boldsymbol{\chi}, t, b) \right] = E \left[2\psi_{k-p}^2 \widetilde{w}_{jm} (MR_{jm} (\boldsymbol{\theta}_c), t, b) \right] \neq 0$$

Therefore, (c) holds. ■

Now, let $\eta_k (u)$s be orthogonal weight functions on $[0, 1]$ such that $\sigma_k = \int_0^1 \eta_k (u)^2 du$ converges fast enough to zero as k goes to infinity. Further, n in $\boldsymbol{\chi}_n$ denotes the number of parameters, including the coefficients on sieve polynomials, so that $\chi_k = 0$ for all $k > n$. K_n used below is the subsequence defined in Lemma 6.1. of Bierens (2014).

$$\begin{aligned}
\widetilde{U}(u) &\equiv \left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]\eta_k(u)\,, \quad \widehat{U}(u)\equiv\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\widehat{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]\eta_k(u)=A_M\widetilde{U}(u)\\
\widetilde{W}_n(u) &\equiv \sum_{k=1}^{K_n}\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\nabla_k\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_n,t,b)\right]\eta_k(u)\,,\\
\widehat{W}_n(u) &\equiv \sum_{k=1}^{K_n}\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\nabla_k\widehat{f}(\mathbf{y}_m,\boldsymbol{\chi}_n,t,b)\right]\eta_k(u)\\
\widetilde{V}_n(u) &\equiv \sum_{k=1}^{K_n}\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\left(\nabla_k\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)-\nabla_k\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_{0n},t,b)\right)\right]\eta_k(u)\,,\\
\widehat{V}_n(u) &\equiv \sum_{k=1}^{K_n}\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\left(\nabla_k\widehat{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)-\nabla_k\widehat{f}(\mathbf{y}_m,\boldsymbol{\chi}_{0n},t,b)\right)\right]\eta_k(u)\\
\widetilde{Z}_n(u) &\equiv \sum_{k=1}^{K_n}\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\nabla_k\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]\eta_k(u)\,,\\
\widehat{Z}_n(u) &\equiv \sum_{k=1}^{K_n}\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\nabla_k\widehat{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]\eta_k(u)\\
\widetilde{b}_{l,n}(u) &\equiv -\sum_{k=1}^{K_n}\left[\frac{1}{M}\sum_{m=1}^{M}\nabla_{k,l}\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_{0n}+\lambda_k(\boldsymbol{\chi}_n-\boldsymbol{\chi}_{0n}),t,b)\right]\eta_k(u)\\
\widehat{b}_{l,n}(u) &\equiv -\sum_{k=1}^{K_n}\left[\frac{1}{M}\sum_{m=1}^{M}\nabla_{k,l}\widehat{f}(\mathbf{y}_m,\boldsymbol{\chi}_{0n}+\lambda_k(\boldsymbol{\chi}_n-\boldsymbol{\chi}_{0n}),t,b)\right]\eta_k(u)
\end{aligned}$$

where $\eta_k(u)$s are orthogonal weight functions on $[0,1]$ such that $\sigma_k=\int_0^1\eta_k(u)^2\,du$ converges fast enough to zero as k goes to infinity. Note that in this case, n denotes the number of parameters, including sieve polynomials. Then,

$$\sum_{l=1}^{n}\widetilde{b}_{l,n}(u)\sqrt{M}(\chi_{n,l}-\chi_{0,l})=\widetilde{Z}_n(u)-\widetilde{W}_n(u)-\widetilde{V}_n(u)\,,$$

where, from Lemma 1, Bierens (2014), $sup_{0\le u\le 1}\left|\widetilde{W}_{n_N}(u)\right|=o_p(1)$ and $sup_{0\le u\le 1}\left|\widetilde{V}_{n_N}(u)\right|=o_p(1)$ because of Assumption 6.3 of Bierens (2014).

Next, we state Lemma 6.2 of Bierens (2014).

Lemma 8 *Under Assumption 6.4 and 6.5,* $\widetilde{Z}_n\Rightarrow Z$ *on* $[0,1]$ *where* Z *is a mean zero Gaussian* *process with covariance function*

$$\begin{aligned}\Gamma(u_1,u_2) &= E[Z(u_1)Z(u_2)]\\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} E\left[\nabla_k \widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\nabla_l \widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\eta_k(u_1)\eta_l(u_2)\right].\end{aligned}$$

Moreover,

$$sup_{0\le u_1\le 1, 0\le u_2\le 1}|\Gamma(u_1,u_2)|<\infty$$

and thus, Assumptions 6.1-6.5 imply that

$$\sum_{l=1}^{n}\widetilde{b}_{l,n}(u)\sqrt{M}(\chi_{n,l}-\chi_{0,l})\Rightarrow Z(u).$$

Now, $\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)$ is i.i.d. distributed with mean $E\left[\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]$ and variance

$$E\left[\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)-E\left[\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]^2\right]<\infty.$$

Thus, from the Central Limit Theorem, we obtain

$$\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\Rightarrow N\left(E\left[\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right],Var\left(\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right)\right).$$

Now,

$$\widehat{Z}_n(u)=A_M\widetilde{Z}_n+\left[\sum_{k=1}^{p}\frac{\partial A_M}{\partial\theta_{ck}}\right]\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]\eta_k(u)\Rightarrow Z,$$

because $A_M\overset{a.s.}{\to}1$, $\partial A_M/\partial\theta_{ck}\overset{a.s.}{\to}0$, $A_M\widetilde{Z}_n\Rightarrow Z$, $\left[\sum_{k=1}^{p}\frac{\partial A_M}{\partial\theta_{ck}}\right]\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_0,t,b)\right]\eta_k(u)\overset{P}{\to}0$. Similarly, we derive $sup_{0\le u\le 1}\left|\widehat{W}_{n_N}(u)\right|=o_p(1)$ and $sup_{0\le u\le 1}\left|\widehat{V}_{n_N}(u)\right|=o_p(1)$.

Therefore, $\widehat{Z}_n$ also satisfies Lemma 6.2. Furthermore,

$$\begin{aligned}&\sum_{l=1}^{n}\widehat{b}_{l,n}(u)\sqrt{M}(\chi_{n,l}-\chi_{0,l})\\ =\;& A_M\sum_{l=1}^{n}\widetilde{b}_{l,n}(u)\sqrt{M}(\chi_{n,l}-\chi_{0,l})-\sum_{l=1}^{p}\frac{\partial A_M}{\partial\theta_{cl}}\widetilde{b}_n(u)\sqrt{M}(\chi_{n,l}-\chi_{0,l})\\ &-\sum_{l=1}^{p}\sum_{k=1}^{p}\frac{\partial A_M}{\partial\theta_{cl}\partial\theta_{ck}}\left[\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\widetilde{f}(\mathbf{y}_m,\boldsymbol{\chi}_{0n}+\lambda_k(\boldsymbol{\chi}_n-\boldsymbol{\chi}_{0n}),t,b)\right]\eta_k(u)\sqrt{M}(\chi_{n,k}-\chi_{0,k})\\ &\Rightarrow Z(u),\end{aligned}$$

because the 2nd converges to zero in distribution and so does the 3rd term because of $\partial^2 A_M/\partial\theta_{ck}\partial\theta_{cl} \overset{a.s.}{\to} 0$.

Then, given Assumptions 6.1-6.5, we have

$$\sum_{l=1}^{n} \widehat{b}_{l,n}(u)\sqrt{M}\left(\chi_{n,l}-\chi_{0,l}\right) \Rightarrow Z(u).$$

Next, we decompose this equation as follows.

$$\begin{aligned} & \left(\widehat{b}_{1,n}(u),\ldots,\widehat{b}_{p,n}(u)\right)\sqrt{M}\left(\boldsymbol{\theta}_{cn}-\boldsymbol{\theta}_{c0}\right)+\sum_{l=1}^{n-p}\widehat{b}_{l+p,n}(u)\sqrt{M}\left(\gamma_{n,l}-\gamma_{0,l}\right) \\ = \;& \widehat{Z}_n(u)-\widehat{W}_n(u)-\widehat{V}_n(u) \Rightarrow Z(u) \end{aligned}$$

Now, as in Bierens (2014), let

$$\widehat{\boldsymbol{a}}_n(u) = \left(\widehat{a}_{1,n}(u),\widehat{a}_{2,n}(u),\ldots,\widehat{a}_{p,n}(u)\right)'$$

be the residual of the following projection.

$$\left(\widehat{b}_{1,n}(u),\ldots,\widehat{b}_{p,n}(u)\right)' = A\left(\widehat{b}_{p+1,n}(u),...,\widehat{b}_{n,n}(u)\right)' + \widehat{\boldsymbol{a}}_n(u).$$

Then, one can show that

$$\int_0^1 \widehat{\boldsymbol{a}}_n(u)\widehat{\boldsymbol{a}}_n(u)'\,du\sqrt{M}\left(\boldsymbol{\theta}_{cn}-\boldsymbol{\theta}_{c0}\right) = \int_0^1 \widehat{\boldsymbol{a}}_n(u)\left(\widehat{Z}_n(u)-\widehat{W}_n(u)-\widehat{V}_n(u)\right)du,$$

where $\widehat{\boldsymbol{a}}_n(u)\widehat{\boldsymbol{a}}_n(u)'$ is a p by p matrix, and $\boldsymbol{\theta}_{cn}-\boldsymbol{\theta}_{c0}$ a p by 1 vector.

Then, from Lemma 6.3, Bierens (2014),

$$plim_{M\to\infty}\int_0^1 \widehat{\boldsymbol{a}}_n(u)\widehat{\boldsymbol{a}}_n(u)'\,du = \int_0^1 \boldsymbol{a}(u)\boldsymbol{a}(u)'\,du$$

where $a(u)$ is the residual of the following projection exercise

$$b(u) = \left(b_1(u),...,b_p(u)\right)' = A\left(b_{p+1}(u),...,b_\infty(u)\right)' + a(u),$$

where $\boldsymbol{b}(u) \in L_0^2(0,1)$ satisfies

$$\left\|\widehat{b}_{l,n} - b_l\right\| = \sqrt{\int_0^1 \left(\widehat{b}_{l,n}(u) - b_l(u)\right)^2 du} = o_p(1)$$

for $l = 1, \ldots, p$ and

$$\sum_{l=p+1}^{n} \rho_l \left\|\widehat{b}_{l,n} - b_l\right\| = o_p(1)$$

and

$$\liminf_{n\to\infty} \left\|\sum_{l=p+1}^{n} \rho_l b_l\right\| > 0$$

We impose the Assumptions 6.7 of Bierens (2014), which is:

Assumption 17 *Assumption 6.7, Bierens (2014): Let*

$$B_{k,l} = \begin{bmatrix} E\left[\nabla_{1,1}\widehat{f}(\mathbf{y}, \boldsymbol{\chi}_0, t, b)\right] & \ldots & E\left[\nabla_{1,n}\widehat{f}(\mathbf{y}, \boldsymbol{\chi}_0, t, b)\right] \\ \vdots & \ddots & \vdots \\ E\left[\nabla_{j,1}\widehat{f}(\mathbf{y}, \boldsymbol{\chi}_0, t, b)\right] & \ldots & E\left[\nabla_{j,n}\widehat{f}(\mathbf{y}, \boldsymbol{\chi}_0, t, b)\right] \end{bmatrix}.$$

$rank(B_{k,k}) = k$ for each $k \geq p$.

Next, we impose Assumption 6.8:

$$\lim_{n\to\infty} \inf_{k\to\infty} det\left(L_p^{(n,k)}\right) > 0$$

where $L^{(n,k)}$ satisfies $\Phi_k^{1/2} B_{n,k} = Q_{k,n} L^{(n,k)}$, and $\Phi_k = diag\left(2^{-2}, 2^{-4}, \ldots, 2^{-2k}\right)$, $Q_{k,n}$ is a $k \times n$ orthogonal matrix and $L_p^{(n,k)}$ is the upper-left $p \times p$ block of the triangular matrix $L^{(n,k)}$. Then, Lemma 6.4 in Bierens (2014) holds, and

$$\mathbf{F} = \int_0^1 a(u)\, a(u)'\, du$$

is a full rank matrix, thus, invertible. Then,

$$\sqrt{M}\left(\boldsymbol{\theta}_{cM} - \boldsymbol{\theta}_{c0}\right) \xrightarrow{d} N_p\left(\mathbf{0}, \mathbf{F}^{-1}\boldsymbol{\Upsilon}\mathbf{F}'^{-1}\right),$$

where

$$\boldsymbol{\Upsilon} = \int_0^1 \int_0^1 a(u_1)\, \boldsymbol{\Gamma}(u_1, u_2)\, a(u_2)\, du_1 du_2.$$

Then, for the logit model, $\boldsymbol{\theta}_{c0} = \{\alpha\}$ and for the BLP model, $\boldsymbol{\theta}_{c0} = \{\mu_\alpha, \sigma_\alpha, \sigma_{\beta k}, k = 1, \ldots, K\}$. Then, because of consistency, $plim_{M\to\infty}\widehat{\boldsymbol{\theta}}_{cM} = \boldsymbol{\theta}_0$. Hence, $\widehat{\boldsymbol{\delta}}_{mM} = s^{-1}\left(\mathbf{s}_m, \widehat{\boldsymbol{\theta}}_{cM}\right) \to_p \boldsymbol{\delta}_{m0}$ for each $m = 1, \ldots, M$ and therefore, given the orthogonality assumption $E\left[\mathbf{X}_m \boldsymbol{\xi}_m\right] = 0$, given conditions for the consistency of the OLS estimator for the logit model

$$\widehat{\boldsymbol{\beta}}_M = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\left[\widehat{\boldsymbol{\delta}}_M - \mathbf{p}\widehat{\alpha}\right] \to_p \boldsymbol{\beta}_0,$$

or

$$\widehat{\boldsymbol{\mu}}_{\boldsymbol{\beta}M} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\left[\widehat{\boldsymbol{\delta}}_M - \mathbf{p}\widehat{\mu}_\alpha\right] \to_p \boldsymbol{\mu}_{\boldsymbol{\beta}0},$$

as $M \to \infty$. Therefore, $\widehat{\boldsymbol{\xi}}_M \to_p \boldsymbol{\xi}_0$. Therefore, if we impose standard assumptions to ensure $plim_{M\to\infty}\frac{1}{M}\mathbf{X}'\mathbf{X} = E\left(\mathbf{X}_m'\mathbf{X}_m\right)$, and $E\left(\mathbf{X}_m'\mathbf{X}_m\right)$ being nonsingular, $plim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\mathbf{X}_m'\left(\widehat{\boldsymbol{\xi}_m'\boldsymbol{\xi}_m}\right)\mathbf{X}_m = E\left[\mathbf{X}_m'\boldsymbol{\Sigma}_0\mathbf{X}_m\right]$, then

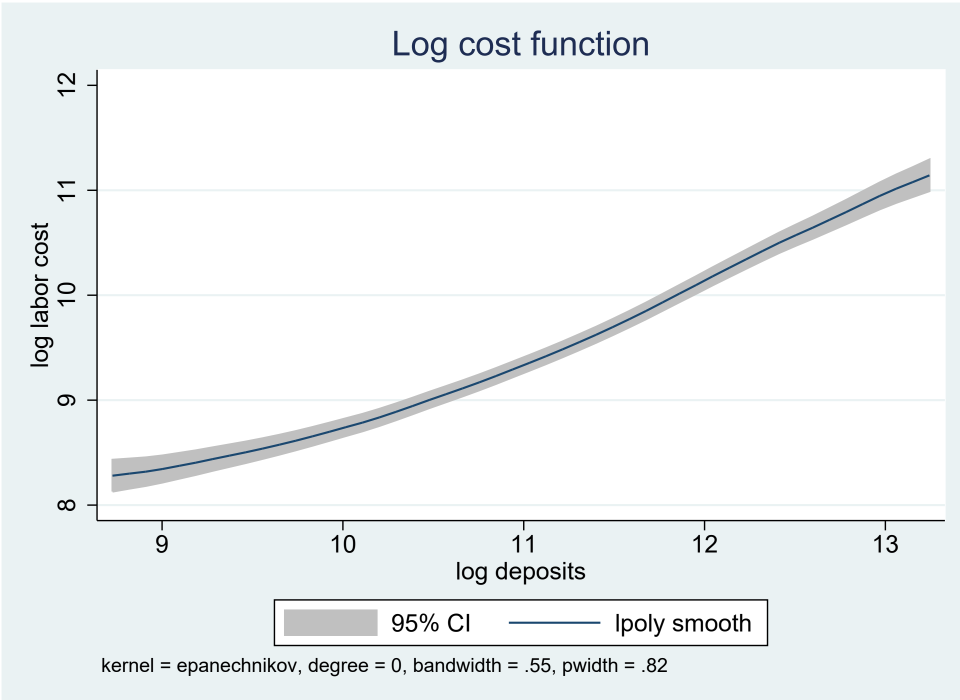
$$\sqrt{M}\left(\widehat{\boldsymbol{\mu}}_{\boldsymbol{\beta}} - \boldsymbol{\mu}_{\boldsymbol{\beta}0}\right) \to_d N\left(0, E\left(\mathbf{X}_m'\mathbf{X}_m\right)^{-1} E\left[\mathbf{X}_m'\boldsymbol{\Sigma}_0\mathbf{X}_m\right] E\left(\mathbf{X}_m'\mathbf{X}_m\right)^{-1}\right)$$

as $M \to \infty$.

G: Details on the empirical application on the banking sector

We use the bank level total deposits as the output and the bank level employee salaries as the variable cost. We derive the deposit interest rate by calculating the deposit interest payment per bank level total deposit minus per deposit service fee. Market level weekly wage is used as the input price. The controls we use are the log of population per branch, log of number of markets served and the log of bank age plus one. We construct the bank level dataset of these variables by merging the data from various sources. We obtain the bank level total deposits and the number of branches for each market from Federal Deposit Insurance Corporation (FDIC), employee salaries, deposit and loan interest rates at the bank level from the balance sheet information reported to the Federal Financial Institutions Examination Council (FFIEC). Information on county level weekly wage and county level population is obtained from the Census and the Bureau of Labor Statistics. We define markets as Metropolitan Statistical Areas (MSA's) for urban areas and counties for rural areas.

By closely inspecting the data, we noticed that in many markets, credit unions seem to be effectively nonexistent as an outside option. We made a judicious choice of removing the markets whose market share of credit unions is less than 1 %. More concretely, we start with



16417 banks in 2325 markets. After removing the markets that have missing data on banks and credit unions, we have 11647 banks in 1117 markets. Then, removing the markets with small share of credit unions, we are left with 10513 banks in 954 markets. Finally, for the sake of reducing the computational burden, we only selected markets where the total number of banks are no more than 40. Then, we are left with 8155 banks in 914 markets.

In Figure 3, we show the kernel fitted relationship between log deposits and log of salaries divided by weekly wage, where deposits and total salaries are in millions of dollars. As we can see, overall, the relationship between log deposits and log variable cost is increasing. In Table 9, we report the sample statistics.

Table 9: Sample Statistics

Description	Mean	Std.dev
(a) Bank and market data		
Bank deposits (in million \$)	182.2	735.4
Bank deposit market share	0.094	0.114
Bank total salaries (in million \$)	455.9	1396.0
Outside option deposit market share	0.153	0.131
Deposit interest rate	0.025	7.484E-3
No. of banks per market	14.08	9.534
No. of markets per bank	2.428	6.751
Number of branches per market	4.137	7.835
Bank age	75.68	47.08
2002 Jan. treasury note interest rate	0.055	0
Housing price index	139.4	14.02
(b) Sample		
Sample size	8155	
No. of banks	3230	
No. of markets	914	
No. of single market banks	2067	