A New Folk Theorem in OLG Games

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Abstract

In this study, we analyze the model of general n-person overlapping generations games. It is assumed that players strictly discount the future at a common rate and that the dimension of feasible one-shot payoffs with individual rationality is equal to n. In contrast to previous researches, this study shows that players can obtain payoffs outside the feasible set of one-shot games.

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1 Introduction

Overlapping generations (OLG) games have been widely studied to investigate economic organizations whose members change over time. In OLG games, players in the same generation interact for a sufficiently long time, and then are replaced sequentially by successors in the next generation. Kandori (1992) first proved the folk theorem in general n-person OLG games, which stated that the players in each generation can obtain any payoffs in V^*, the convex combination of feasible one-shot payoffs with individual rationality. Following him, Smith (1992) proved several versions of folk theorems with stronger results.

In the analyses of Kandori (1992), Smith (1992), and subsequent studies on OLG games, the characterization of equilibrium set of payoffs is restrictive. They showed that players can obtain any payoffs in V^* without discounting, and that the result is robust against low discounting. However, this does not mean that players cannot attain payoffs outside V^*: when they discount the future, intertemporal trades of payoffs using the difference of time preferences can be possible.

Based on this issue, we recently have shown in Morooka (2020) that in 2-person OLG games with discounting, players can obtain payoffs outside V^*. In the present paper, we extend the result of Morooka

(2020) to general n-person OLG games. Our result shows that under the full dimensionality of V^*, any payoffs in the cubic hull of V^*, that is, any payoffs in the smallest n-dimensional cube that contains V^*, are obtainable by the players in each generation.

In order to get the intuition of our result, we consider the one-shot game illustrated in Figure 1. In our model of overlapping generations, each player lives for $3T$ periods, and his first and second opponents are replaced by corresponding new players in T and $2T$ periods, respectively, after his entrance, as depicted in Figure 2. Players are assumed to strictly discount the future at a common rate δ which is less than 1. When each player takes Y in the first T periods, and X in the next $2T$ periods, an equilibrium is formed with an average payoff of $\frac{3(1-\delta^T)}{1-\delta^{3T}}$ which is sufficiently close to 3 as the player's continuation payoffs diminish as periods pass. As a result, players' payoffs in each generation are almost $(3,\ 3,\ 3)$, which Pareto-dominate $(1,\ 1,\ 1)$ and are outside V^*.

The main difference between Morooka (2020) and this work is in rewarding players after punishments. In 2-person games, mutual minimaxing is available for punishment against deviations by one player. When there are 3 or more players in one-shot games, they may want to quit punishment because their payoffs during punishment are less than their minimax payoffs. Players must be rewarded with appropriate payoffs after the punishment as an incentive, which requires the full dimensionality of feasible one-shot payoffs with individual rationality.

In order to obtain this result, players' strict discount is essential. In our study, the rate of discounting has two different roles, which can be seen as a trade-off. First, players must be so patient that do not deviate, the argument of which is seen everywhere in the literature on repeated games. At this point, we do not want players to discount the future. The second role, which is our original one, is that it must diminish the payoffs in players' later days. If players do not discount the future, they fail the intertemporal trades of payoffs.

It must also be noted that there is an order in the choice of parameters; we must fix the discount first, and then choose players' lifespan, depending on the discount. This is because, as the discount rate approaches 1, we need a longer lifespan in order for players' continuation payoffs after they become older to diminish sufficiently. This order is inverse to that of the non-uniform OLG folk theorem in Smith (1992), where he first fixes players' lifespan, allowing players to punish deviations strictly under no discount, and then chooses the discount with which the punishments are still available.

In relation to infinitely repeated games, our logic is compared to that of Lehrer and Pauzner (1999). They studied two-person infinitely repeated games with different discounting between players, and showed that players can obtain equilibrium payoffs outside the set of feasible one-shot payoffs V. In their model, the player who evaluates future payoffs at a higher rate gives payoffs to his opponent with less patience first, and is rewarded later. This allows players to obtain higher payoffs outside V. In Lehrer and Pauzner (1999), however, there is an asymmetricity of payoffs between players. In our model, on the other hand, all players' payoffs can be raised at no cost.

We also show that players cannot obtain even better payoffs outside the set characterized in this research. The intuitive reason is as follows. When a player obtains a high feasible payoff without individual rationality, he has already obtained it in his early life. Therefore, at least one other player's payoff early in her life must be less than the minimax value. As a result, whatever she gets thereafter, her payoff throughout her life is also less than the minimax value.

We define the model and prove the main theorem in Section 2. In Section 3, we show the impossibility for players to obtain payoffs outside the characterized set. Section 4 concludes.

2 Model and Results

Let $N = \{1, 2, \cdots, n\}$ for $n \geq 2$ represent the set of players in a one-shot game. For $i \in N$, let A_i be the set of i's actions and let $A = \prod_{i \in N} A_i$. Let $g_i : \triangle(A) \to \mathbb{R}$ be the one-shot payoff function of i, where $\triangle(A)$ is the set of players' mixed (but not correlated) actions. Let $G = G(N,\ A,\ g)$ be the one-shot game considered. Without loss of generality, we assume that each player's minimax payoff in G is 0[i]. Let $V = co(g(A))$ be the set of feasible payoffs in G. The set of feasible and individually rational payoffs is then defined as $V^* = V \cap \mathbb{R}^{n+}$. We assume the full dimensionality, that is, $\dim V^* = n$.

In order to rigorously characterize the set of equilibrium payoffs in our result, consider the following value r_i for $i \in N$:

$$r_i = \max_{v \in V^*} v_i.$$

[i] Note that in order to minimax a player, some of his opponents may have to take a mixed action.

This value is the best feasible and individually rational payoff for player i. We then define the *open cubic hull* of V^*, say $C(V^*)$, as follows:

$$C(V^*) = \prod_{i \in N} (0,\ r_i).$$

For example, in the game shown in Figure 1, $r_1 = r_2 = r_3 = 3$ and $C(V^*) = (0,\ 3)^3$.

We now turn to construct our model of OLG games, OLG$(G;\ \delta,\ T)$, as follows. Let T be a natural number. Each player lives for nT periods: the player in k'th generation with action A_i lives between period $(k-1)nT + (i-1)T + 1$ and $knT + (i-1)T$. Exclusively, the player in 0'th generation with action A_i for $i \neq 1$ lives between period 1 and $(i-1)T$. The case of $n = 3$ is shown in Figure 2.

In our model, each player's future payoffs are discounted at a common rate $\delta \leq 1$[ii]. It is assumed that players can use the public randomizing device each period, and that the monitoring is perfect[iii]. When the sequence of actions $\{a(t)\}_{t=1}^{nT}$ is played throughout the life of a player with A_i (except in the case of players in 0'th generation), his average payoff is as follows:[iv]:

$$\frac{1}{\sum_{t=1}^{nT} \delta^{t-1}} \sum_{t=1}^{nT} \delta^{t-1} g_i(a(t)).$$

The following result holds.

Theorem 1 (The Folk Theorem in discounted n-person OLG Games). For every $v \in C(V^*)$, and for every $\epsilon > 0$, there is $\delta_0 \in (0,\ 1)$ such that if $\delta \in [\delta_0,\ 1)$, then for every T sufficiently large, there is a subgame perfect equilibrium in OLG$(G;\ \delta,\ T)$, where the players in each generation[v] obtain average payoffs ϵ-close to v. This vector in particular need not be feasible.

Before the formal proof, we see in detail that in the OLG model of the following stage game in Figure 3, each player can obtain $v_i \fallingdotseq 3$ when δ is sufficiently close to 1. In the Pareto frontier of this game,

[ii] Although our result does not hold with $\delta = 1$, it is convenient to consider this case in order to construct the strict punishments under no discount which is still available with $\delta < 1$.

[iii] In our model, players can only observe realized pure actions.

[iv] For players in 0'th generation, please replace nT by their lifespans.

[v] Players in 0'th generation with shorter lifespans are excluded from this result.

the sum of payoffs is 7, which is less than $\sum_i v_i = 9$. In this game, Y is the strongly dominant action for all players. However, in order to generate efficient (resp. minimaxing) outcome, two players must take X (resp. Z) enduring payoff -1, which is less than the minimax value. Moreover, the only one Nash profile $(Y,\ Y,\ Y)$ is not only Pareto-inferior but also gives considerably high payoffs to players, so it cannot be used both on the path and for punishments. It is assumed that in each period, the nature draws one integer from 1 to 10 with equal probability, and that players commonly observe it. Consider the following strategy.

If there is no deviation, players play the following path strategy:

·When a player exits, he takes Y and his opponents take X.

·In all the other periods, players play the action which gives 3 to the youngest player and gives 2 to the other players. This payoff is generated when players play $(Y,\ X,\ X)$ (which yields 9 to the youngest player and -1 to his opponents) with probability 2/5, $(X,\ Y,\ X)$ with probability 3/10, and $(X,\ X,\ Y)$ with probability 3/10.

If someone deviates, players play one from the following punishments. The punishments are also applied to the deviations from themselves.

·(Punishment 1) If there are at least 7 periods left between the next period and the oldest player's exit, the deviator plays Y and his opponents play Z in the next 3 periods. Then, in the following 3 periods, players play the action which gives 1 to the deviator and gives 3 to his opponents. The payoffs $(1,\ 3,\ 3)$ are generated when players play $(Y,\ X,\ X)$ with probability 1/5, $(X,\ Y,\ X)$ with probability 2/5, $(X,\ X,\ Y)$ with probability 2/5. After that, they return to the path.

·(Punishment 2) If the oldest player deviates and there are at most 6 periods left between the next period and his exit, he plays Y and his opponents play Z until he exits. After that, players return to the path.

·(Punishment 3) If a younger player deviates and there are at most

6 periods left between the next period and the oldest player's exit, players play $(Y,\ Y,\ Y)$ until the exit of the oldest player. After the exit, players play the action which gives 1 to the deviator and gives 3 to his opponents for 21 periods. Then, they return to the path.

We show that the above strategy forms an equilibrium under $\delta \risingdotseq 1$. First, observe that players' minimum one-shot payoff is -1 and that their maximum one-shot gain from deviations is 1, by the construction of equilibrium strategy. Consider the deviation of one player from the path when there are at least 7 periods left between the next period and the oldest player's exit. If he does not deviate, his payoff in the current period and next 3 periods is almost at least $-1+1\times 3=2$ with δ sufficiently close to 1. If he deviates, it reduces to at most $1+0\times 3=1$ by Punishment 1.

Second, consider the deviation of one player from the minimaxing when there are at least 7 periods left between the next period and the oldest player's exit. If he does not deviate, his payoff in the current period and next 6 periods is at least $-3+3\times 3-1=5$. If he deviates, it reduces to at most $1+0\times 3+1\times 3=4$ by Punishment 1.

Third, consider the oldest player's deviation right before his exit. When there are 3 periods left between the next period and his exit, his payoff from the current period on is at least $-1\times 3+9=6$ without deviations. And when there are at least 6 periods left before his exit, he can get at least 6 on the path. If he deviates, it reduces to at most 1 by Punishment 2.

Fourth, consider a younger player's deviation right before the oldest player's exit. When there are 6 periods left between the next period and the oldest player's exit, his payoff from the current period to the 21st period after the oldest player's exit is at least $-7+2\times 21=35$ if he does not deviate. If he deviates, it reduces to at most $1+2\times 6+1\times 21=34$ by Punishment 3.

Finally, we calculate each player's equilibrium payoff. For each player, he gets 3 in every period for the first $T-1$ periods of his life, and thereafter, he gets at least -1 in each period. Therefore, when T is sufficiently large and players discount the future at least a little, players' continuation payoff after becoming older diminishes and their average payoff, $\frac{3(1-\delta^{T-1})-(\delta^{T-1}-\delta^{3T})}{1-\delta^{3T}}$, is almost 3.

Now, we proceed to formally prove the result. We generalize the punishments against the deviations described above.

Proof. Fix $v = (v_1^1, \cdots, v_n^n) \in C(V^*)$ and $\epsilon > 0$. By the definition of $C(V^*)$, there exists an $(n-1)$-dimensional vector v_{-i}^i for $i \in N$ which satisfies $v^i = (v_i^i,\ v_{-i}^i) \in \text{int}.V^*$. Let a^i be a (correlated) profile with one-shot payoffs v^i. Because $\dim V^* = n$, there exists a (correlated) profile $d^{j,i}$ with one-shot payoffs $g(d^{j,i}) \in V^*$ satisfying $g_i(d^{j,i}) < g_i(d^{j,k})$ (no dev. from minimax k) and $g_i(d^{j,i}) < v_i^j$ (no dev. to $d^{j,i}$) for $i,\ j \in N$ and $k \neq i$[vi]. The profile $d^{j,i}$ can be viewed as a "reward" after minimaxing the deviator with A_i, when the player with A_j is the youngest. We also denote the minimaxing profile against i, the profile which gives the maximum payoff to i, and a one-shot Nash profile, as m^i, b^i, and e, respectively.

We also define $r^{j,i,T_2}(\{a(t)\}_{t=1}^{T_1})$ for $i, j \in N$, T_1, $\{a(t)\}_{t=1}^{T_1} \in A^{T_1}$, δ and T_2, as a (correlated) profile, which gives a one-shot payoff $g_k(d^{j,i}) - \frac{\sum_{t=1}^{T_1} \delta^{t-1} g_k(a(t))}{\delta^{T_1} \sum_{t=1}^{T_2} \delta^{t-1}}$ to the player with A_k for $k \neq i$ and $g_i(d^{j,i})$ to the player with A_i. Given T_1, such a profile exists when δ and T_2 are sufficiently large. According to this profile, each player with A_k is indifferent among all actions during minimaxing to the player with A_i, because $\sum_{t=1}^{T_1} \delta^{t-1} g_k(a(t)) + \delta^{T_1} \sum_{t=1}^{T_2} \delta^{t-1} g_k(r^{j,i,T_2}(\{a(t)\}_{t=1}^{T_1})) = \delta^{T_1} \sum_{t=1}^{T_2} \delta^{t-1} g_k(d^{j,i})$ holds.

The game starts with playing a^1 in period 1. The parameters Q, M, S, and P are natural numbers and determined later in this order for which the following strategy profile is subgame perfect. The profile consists of six steps. We also need the public randomizing device for Steps 1, 3, 4, and 5 in order to let players play correlated actions.

When there is no deviation, players play the following path strategy.

Step 1. When there are at least S periods left between the next period and the oldest player's exit, players play a^i, when the player with A_i is the youngest.

Step 2. When there are at most $S-1$ periods left between the next period and the oldest player's exit, players play b^i, when the player with A_i is the oldest.

[vi]We can indeed find such $(d^{j,i})_{i,j \in N}$. To see this, choose any $i, j \in N$. There exists a vector on the line connecting the origin and v^j, which is included in int.$V^* \setminus \{v^j\}$. We can choose c^j as a profile of actions giving such a vector. Then, because of the full dimensionality condition, we can choose a vector in int.V^* such that only the i'th component is less than $g_i(c^j)$, with remaining other components being the same as $g(c^j)$. We can choose $d^{j,i}$ as giving such a vector.

After someone unilaterally deviates, players play the following punishments. The punishments are also applied to the deviations from themselves. Right after the punishments, players return to the path. Suppose that the deviator has the action set A_i.

Step 3. When there are at least $Q+M+S$ periods left between the next period and the oldest player's exit, players play m^i for Q periods, and then, after $\{a(t)\}_{t=1}^{Q}$ is observed, play $r^{j,i,M}(\{a(t)\}_{t=1}^{Q})$ for M periods, when the player with A_j is the youngest.

Step 4. When the deviation is by a younger player and there are at most $Q+M+S-1$ periods left between the next period and the oldest opponent's exit, players play e until the opponent's exit. After that, when $\{a(t)\}_{t=1}^{Q+M+S-1}$ is observed and the player with A_j is the youngest, players play $r^{j,i,P}(\{a(t)\}_{t=1}^{Q+M+S-1})$ for P periods.

Step 5. When the deviation is by the oldest player and there are at most $Q+M+S-1$ periods left between the next period and his exit, players play m^i until his exit. After that, when $\{a(t)\}_{t=1}^{Q+M+S-1}$ is observed, players play $r^{i,i,P}(\{a(t)\}_{t=1}^{Q+M+S-1})$ for P periods.

When players play this strategy profile, a^i is played for $T-S$ periods on the equilibrium path when the player with A_i is the youngest, which guarantees that his average payoff is almost equal to v_i^i under the positive rate of discounting and sufficiently long lifespan. In the remainder of this section, it is proved that the above strategy profile forms a subgame perfect equilibrium for δ sufficiently close to 1 when we choose the length of punishments appropriately. The methodology is to provide strictly positive penalties for any unilateral deviation from each step when $\delta = 1$. Then, by continuity of payoffs, there is some $\delta_0 \in (0,\ 1)$ for which all penalties are still positive for $\delta \in [\delta_0,\ 1)$. For each step, the "worst-case scenario" is concerned, where the incentive to deviate is greatest.

We define the following variables: $u_0 = \min_{i,j,k \in N}\{v_i^j,\ \frac{1}{2}g_i(d^{j,k})\}$, $\beta = \max_{i\in N, a\in A} g_i(a)$, $\beta_0 = \min_{i\in N}\max_{a\in A} g_i(a)$, $\omega = \min_{i\in N, a\in A} g_i(a)$. Suppose that the deviator has the action set A_i.

- Deviations from Steps 1 and 3 when there are at least $Q+M+S$

periods left between the next period and the oldest player's exit:
Suppose that no player is to be minimaxed in the current period. The deviator gets at most β immediately, and at most 0 in the following Q periods by Step 3. When he does not deviate, his payoff through these $1+Q$ periods is at least u_0. We can avoid the deviation when we choose Q and M satisfying the following equations for $\{a(t)\}_{t=1}^{Q} \in A^Q$:

$$(1+Q)u_0 > \beta,$$

$$g_k(d^{j,i}) - \frac{\sum_{t=1}^{Q} g_k(a(t))}{M} > \frac{1}{2} g_k(d^{j,i}) \ for \ i,j,k \in N,$$

$$g_k(d^{j,i}) - \frac{\sum_{t=1}^{Q} g_k(a(t))}{M} > g_k(d^{j,k}) \ for \ i,j \in N \ and \ k \neq i, \ and$$

$$the \ profile \ r^{j,i,M}(\{a(t)\}_{t=1}^{Q}) \ exists \ for \ i,j \in N.$$

·Deviations by the oldest player when there are at most $Q+M+S-1$ periods left between the next period and his exit:
Suppose that the oldest player deviates from the beginning of Step 3 and there are $Q+M+S-1$ periods left between the next period and his exit. He gets at most β immediately, and at most 0 in the following $Q+M+S-1$ periods because of Step 5. When he does not deviate, his payoff through these $Q+M+S$ periods is at least $(Q+M)\omega+S\beta_0$. We can avoid the deviation when we choose S satisfying the following equations:

$$(Q+M)\omega + S\beta_0 > \beta.$$

·Deviations by a younger player when there are at most $Q+M+S-1$ periods left between the next period and the oldest player's exit:
Suppose that the player deviates from the beginning of Step 3 and there are $Q+M+S-1$ periods left between the next period and the exit of the oldest player with A_j. He gets at most β immediately and at most $(Q+M+S-1)\beta$ in the following $Q+M+S-1$ periods. After the oldest player's exit, his payoff in the following P periods is reduced to $Pg_i(r^{j,i,P}(\{a(t)\}_{t=1}^{Q+M+S-1}))$ for the observed pure

actions $\{a(t)\}_{t=1}^{Q+M+S-1}$ because of Step 4. When he does not deviate, he gets at least $(Q+M+S)\omega$ in the following $Q+M+S$ periods and at least Pv_i^j after the entry of younger opponent. We can avoid the deviation when we choose P satisfying the following equations for $\{a(t)\}_{t=1}^{Q+M+S-1} \in A^{Q+M+S-1}$:

$$g_k(d^{j,i}) - \frac{\sum_{t=1}^{Q+M+S-1} g_k(a(t))}{P} > \frac{1}{2} g_k(d^{j,i}) \ for \ i, j, k \in N,$$

$$g_k(d^{j,i}) - \frac{\sum_{t=1}^{Q+M+S-1} g_k(a(t))}{P} > g_k(d^{j,k}) \ for \ i, j \in N \ and \ k \neq i,$$

$$the \ profile \ r^{j,i,P}(\{a(t)\}_{t=1}^{Q+M+S-1}) \ exists \ for \ i, j \in N, \ and$$

$$(Q+M+S)\omega + Pv_i^j$$

$$> (Q+M+S)\beta + Pg_i(r^{j,i,P}(\{a(t)\}_{t=1}^{Q+M+S-1})) \ for \ i, j \in N.$$

Therefore, under no discount, deviations from any steps strictly decrease deviators' payoffs, or do not change their payoffs by $r(\cdot)$, which means that the above strategy profile forms a subgame perfect equilibrium for some $\delta_0 \in (0,\ 1)$ and $\delta \in [\delta_0,\ 1)$. Fix such a δ. Note that the parameters Q, M, S, and P are independent of the players' lifespan. We then choose T_0 which satisfies the following equations:

$$T_0 > P + Q + M + S,$$

$$\frac{1}{1-\delta^{nT_0}} \{(1-\delta^{T_0-S})v_i^i + (\delta^{T_0-S} - \delta^{nT_0})\omega\} > v_i^i - \epsilon, \ and$$

$$\frac{1}{1-\delta^{nT_0}} \{(1-\delta^{T_0-S})v_i^i + (\delta^{T_0-S} - \delta^{nT_0})\beta\} < v_i^i + \epsilon \ for \ i \in N.$$

In the last two equations, each LHS is the minimum and maximum of each player's average payoff on the path, respectively. These values converge to v_i^i for any $\delta < 1$ when $T_0 \to \infty$. Set $T \geq T_0$. Then, the average payoff of each player with A_i is almost equal to v_i^i within the distance of ϵ. $\square$

3 Impossibility of Further Extension of Equilibrium Payoff Set

In this section, we show that players cannot obtain payoffs outside $C(V^*)$, as the following Lemma 1.

Lemma 1. $\forall \epsilon > 0$, $\forall \delta \in (0,\ 1)$, $\forall v \in \mathbb{R}^n \setminus C(V^*)$ which satisfies $\inf_{w \in C(V^*)} ||w - v|| > 2\epsilon$, $\exists i \in N$, $\exists T_0$, $\forall T \geq T_0$, for any subgame perfect equilibrium in OLG(G; δ, T), the difference between v_i and the equilibrium payoff of the player with A_i is at least ϵ.

Proof. Take any $\epsilon > 0$, $\delta \in (0,\ 1)$, and $v \in \mathbb{R}^n \setminus C(V^*)$. We only consider the case, where $v \in V$ and $\max_{w \in V^*} w_i + 2\epsilon < v_i$ for some $i \in N$ holds. There exists at least one other player whose payoff is less than the minimax value. Therefore, there exists a $\gamma < 0$ which satisfies the following condition:

$$\gamma = \max_{j \neq i} u_j \ \textit{subject to}\ u \in V,\ u_i \geq \max_{w \in V^*} w_i + \frac{1}{2}\epsilon,\ \textit{and}\ u_j < 0.$$

For each T, take any sequence $\{w(t)\}_{t=1}^{nT}$ in $\mathbb{R}^n$ which satisfies the following conditions:

$$w(t) \in V\ \textit{for all}\ t \in \{1, \cdots, nT\},\ \textit{and}$$

$$\max_{w \in V^*} w_i + \epsilon < \frac{\sum_{t=1}^{nT} \delta^{t-1} w_i(t)}{1 + \cdots + \delta^{nT-1}}.$$

When a player with A_i wants to get v_i within the error of ϵ, one sequence which satisfies these conditions must be realized.

Then, there exists a T_0 and for all $T \geq T_0$ and $\{w(t)\}_{t=1}^{nT}$ which satisfies the above conditions, the following inequations hold:

$$\max_{w \in V^*} w_i + \frac{1}{2}\epsilon < \frac{\sum_{t=1}^{T_0} \delta^{t-1} w_i(t)}{1 + \cdots + \delta^{T_0-1}},\ \textit{and}$$

$$\sum_{t=1}^{T_0} \delta^{t-1}\gamma + \delta^{T_0} \frac{\beta}{1-\delta} < 0.$$

Here, the first inequation means that the player with A_i has already received the payoff almost equal to v_i in his first T_0 periods of life, and the second inequation means that whatever happens after T_0, one other player's payoff throughout her life is less then her minimax value.

Therefore, because V is convex and $w(t) \in V$ holds for each t, $\frac{\sum_{t=1}^{T_0} \delta^{t-1} w_i(t)}{1+\cdots+\delta^{T_0-1}} \in V$ holds. By the definition of γ, there exists a $j \neq i$, and $\frac{\sum_{t=1}^{T_0} \delta^{t-1} w_j(t)}{1+\cdots+\delta^{T_0-1}} \leq \gamma$ holds. As a result, the payoff of the player with A_j is at most $\sum_{t=1}^{T_0} \delta^{t-1}\gamma + \frac{\delta^{T_0}\beta}{1-\delta} < 0$, which means that she can improve her payoff by taking the best response and hence that the sequence of payoffs $\{w(t)\}_{t=1}^{nT}$ with the payoff v_i to the player with A_i cannot be obtained in equilibria of OLG(G; δ, T). $\square$

4 Conclusion

In this paper, we have analyzed the model of n-person OLG games. Under the condition of full dimensionality, we have seen that players can obtain average payoffs outside the convex hull of one-shot payoffs. One remaining question is how the result will change when the stage game does not have the full dimensionality. Even in such a case, players may still be able to obtain payoffs outside the feasible set of one-shot payoffs. Consider the pure coordination game in Figure 4. Clearly, the dimension of this game is 1. For example, when players play Y in the first T periods and X in the following $2T$ periods, one player attains almost 1 and his opponents attain almost 0. As a result, payoffs in the cube $(0,\ 1)^3$ are attainable in equilibria of the OLG game. We will study further whether players can still attain payoffs outside one-shot feasible payoffs in other games without full dimensionality.

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Mat: ***X***	*X*	*Y*
X	1	0, 3, 0
Y	3, 0, 0	0

Mat: ***Y***	*X*	*Y*
X	0, 0, 3	0
Y	0	0

Figure 1: Stage game 1

Period	$1\sim T$	$T+1\sim 2T$	$2T+1\sim 3T$	$3T+1\sim 4T$	$4T+1\sim 5T$	$5T+1\sim 6T$	$6T+1\sim 7T$	$7T+1\sim 8T$	$8T+1\sim 9T$	⋯
Row	1st Generation			2nd Generation			3rd Generation			⋯
Col	(0th)	1st Generation			2nd Generation			3rd Generation		
Mat	(0th)		1st Generation			2nd Generation			3rd ⋯	

Figure 2: Structure of overlapping generations in the case of n=3

Mat: ***X***	*X*	*Y*	*Z*
X	−1	−1, 9, −1	−1
Y	9, −1, −1	1, 1, −1	1, −1, −1
Z	−1	−1, 1, −1	−1

Mat: ***Y***	*X*	*Y*	*Z*
X	−1, −1, 9	−1, 1, 1	−1, −1, 1
Y	1, −1, 1	2	1, −1, 1
Z	−1, −1, 1	−1, 1, 1	−1, −1, 0

Mat: ***Z***	*X*	*Y*	*Z*
X	−1	−1, 1, −1	−1
Y	1, −1, −1	1, 1, −1	0, −1, −1
Z	−1	−1, 0, −1	−1

Figure 3: Stage game 2

Mat: ***X***	*X*	*Y*
X	0	
Y		

Mat: ***Y***	*X*	*Y*
X	0	0
Y	0	1

Figure 4: Pure coordination game