Theory, Identification, and Estimation for Scoring Auctions*

Makoto HANAZONO†, Yohsuke HIROSE‡,
Jun NAKABAYASHI§, and Masanori TSURUOKA¶

November 28, 2016

Preliminary and incomplete

Abstract

This paper offers an analytical framework for the scoring auction. We first characterize a symmetric monotone equilibrium in the scoring auction. We then propose a method to semiparametrically identify the joint distribution of the bidder's multidimensional signal from scoring auction data. Our approach allows for a broad class of scoring rules in settings with multidimensional signals. Finally, using our analytical framework, we conduct an empirical experiment to estimate the impacts of the change of auction formats and scoring rules. The data on scoring auctions are from public procurement auctions for construction projects in Japan.

Key words: scoring auctions, identification, structural estimation
JEL classification: C13, D44, L70

*We thank Masaki Aoyagi, Alina Arefeva, Christian Essling, Masahisa Fujita, Chiaki Hara, Fumio Hayashi, Atsushi Kajii, Ronald Harstad, Chi-Hsiang Liu, Hiroshi Ohashi, Janne Tukiainen, and Christine Zulehner for helpful comments. This study has been supported by JSPS KAKENHI (Grant Number 24530195, 26590034), 2013 Japan Construction Information Center Foundation (JACIC) Research Grant, and the Joint Usage/Research Center at ISER in Osaka University. All remaining errors are our own.

†School of Economics, Nagoya University, Chikusa, Nagoya 464-8601, Japan. Email: hanazono@soec.nagoya-u.ac.jp

‡Department of Economics, Meiji Gakuin University, 1-2-37 Shirokanedai Minato, Tokyo, 108-8636, Japan. E-mail: yhirose@eco.meijigakuin.ac.jp

§Faculty of Economics, Kindai University, Kowakae 3-4-1, Higashi-Osaka, 577-8502, Japan. E-mail: nakabayashi.1@eco.kindai.ac.jp.

¶Department of Economics, Yokohama National University, 79-3 Tokiwadai, Hodogaya-ku, Yokohama, Kanagawa, 240-8501, Japan. E-mail: matrok0603@gmail.com

1 Introduction

Over centuries, procurement agencies have dominantly used low-price auctions to allocate contracts. An important feature of the mechanism in practice is that the awarded supplier performs the contracted tasks, taking the contract design and specification including delivery schedule and quantity as given. The separation of the design phase from the task performing phase helps preventing favouritism in awarding process and promoting supplier competition. At the same time, the separation results in a conflict of incentives between procurement buyers and contractors; lowest responsive bidders may not very much have concerns on (ex post) cost overrun, delay, quality deterioration, etc. Resulting contracts were thus not necessarily welfare maximizing, or most *valuable for money* (VFM), for taxpayers.

Toward a more VFM contract, procurement agencies in a last few decades have introduced alternative mechanisms in which not only price but also other non-monetary factors are assessed in awarding process. The *scoring auction* – i.e., a form of multidimensional bidding – is a typical practice. By year 2007, more than a half of the State Departments of Transportation for instance had used the scoring auction.[1]

In the scoring auction, suppliers are requested to bid price and non-price proposals. The auctioneer preannounces a scoring rule that specifies the way to rank suppliers based upon their multidimensional bid as well as their attributes such as past performance and experience. Through more comprehensive comparison of proposals and attributes of suppliers, the scoring auction allows the auctioneer to obtain greater welfare than with price-only auctions (e.g., Bichler, 2000; Milgrom, 2004; Asker and Cantillon, 2008; and Lewis and Bajari, 2011).

This paper offers a framework to analyze the scoring auction both theoretically and econometrically. We construct a theoretical model of the scoring auction in which there are $n \geqq 2$ risk-neutral suppliers, each of which has a smooth and convex cost function parameterized with a $(K \geqq 1)$-dimensional signal. After drawing a signal, each supplier submits an $(L \geqq 2)$-dimensional bid, and the lowest-score bidder wins. We characterize the equilibrium of the scoring auction. Then, we propose a semiparametric identification method of the scoring auction model, where the bidder cost function is known up to the

[1] In Japan, more than 98 percent of construction projects by the Ministry of Land, Infrastructure, and Transportation were auctioned off with the scoring auction in fiscal year 2014. See *2014 Annual Report on the use of Sogo-Hyoka Nyusatsu*:`http://www.nilim.go.jp/lab/peg/siryou/20160708_s_hinkakukon/H26_20160301_siryou40315_s160708.pdf`

K-dimensional signal.

Our contribution is threefold. First, we demonstrate the existence of a unique monotone equilibrium in the scoring auction. A key to our approach is to break down the multidimensional bidding process into two steps: each bidder selects a profit maximizing price-quality pair for a given targeted score, and then the auction as bidding a score occurs. The approach allows us to characterize a scoring auction equilibrium with minimum restriction on primitives, being free from the specifics of scoring rules – quasilinear (QL) – and of the cost function – additively separable in quality and type, etc.[2] Consequently, our model covers a wide range of scoring auction settings seen in practice.[3] Furthermore, our approach is applicable to more general forms of multidimensional bidding, such as the fixed-price best-proposal auction by Thiel (1988) and the unit-price auction by Bajari et al. (2014).

Our analysis also uncovers a close link between the problem in the scoring auction and that in the auction with risk-averse bidders. Regarding the value function of the optimal price- and quality-choice problem as the bidder utility, we identify that the utility function is generally nonlinear in its score.[4] The first-order condition for optimal score bidding thus has a similar structure as that seen in Maskin and Riley (1984). While this finding provides an intuitive view to equilibrium properties of the scoring auction, we are still left the nontrivial problem regarding the second-order condition – for the existence of a monotone equilibrium – in settings with multidimensional bids and types. We propose a sufficient primitive condition for the single-crossing property in the scoring auction model. With this property, we show the uniqueness of a monotone equilibrium in the scoring auction.[5]

[2]Existing literature deal with the optimization problem on multidimensional strategy and type spaces by using *pseudotype* with which the problem can be reduced in to a unidimensional auction problem (e.g., Asker and Cantillon, 2008). A crucial assumption this approach relies on is that the scoring function be quasilinear or is a monotone transformation thereof. In practice, auctioneers commonly use non-QL scoring rules, e.g., Albano, Dini and Zampino (2009). Obviously, a monotone transformation converts some but not all non-QL functions to a QL form.

[3]In real-world procurement auctions, a much wider variety of scoring rules than what has been examined by theory are adopted. For example, many state departments of transportation (DOTs) in the United States, including those in Alaska, Colorado, Florida, Michigan, North Carolina, and South Dakota, have adopted the "adjusted bid," in which a score is equal to the price bid divided by the quality bid (Molenaar and Yakowenko, 2007). The Department of Health and Aging in Australia also employs a price-per-quality-ratio awarding rule for contracts that need to achieve better returns on public investment (The Department of Health and Ageing, Australia, 2011). In addition, most public procurement contracts in Japan are allocated to the bidder with the highest price-per-quality bid ratio. In Japan, the Ministry of Economy, Trade, and Industry uses a QL scoring rule with a reserve price.

[4]The analogy in models of multidimensional bidding and Maskin and Riley (1984) was first discussed by Thiel (1988).

[5]McAdams (2003) is the first study that shows the equilibrium existence in games with incomplete information where types and actions are multidimensional. More recently, Reny (2011) shows the equilibrium

Second, we examine identification of the scoring auction model. Specifically, we propose a necessary and sufficient condition on bidder cost functions for the identification of the multidimensional signal distribution from scoring auction data. To the best of our knowledge, our analysis is the first to propose the condition.

To make our argument precise, let $\boldsymbol{\theta}$, $\mathbf{q}^*$, and $\mathbf{b}^*$ denote the bidder's K-dimensional latent signal, the observed $(L-1)$-dimensional equilibrium quality bid, and an L-dimensional vector including the equilibrium price bid and the distribution of equilibrium scores. Let $C(\mathbf{q}, \boldsymbol{\theta})$ and $C_{q^\ell}(\mathbf{q}, \boldsymbol{\theta})$ with $\ell = 1, \ldots, L-1$ denote the bidder's cost function and its marginal cost with respect to the ℓth dimensional quality, q^ℓ, and suppose that vector function $A(\boldsymbol{\theta}; \mathbf{q}) := (C(\mathbf{q}, \boldsymbol{\theta}), C_{q^1}(\mathbf{q}, \boldsymbol{\theta}), \ldots, C_{q^{L-1}}(\mathbf{q}, \boldsymbol{\theta}))^{\mathrm{T}}$ is well defined and smooth. We show that the bidder's system of best-response functions can be rearranged to an L-dimensional nonlinear system, $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$. We then propose a necessary and sufficient condition under which $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible with respect to $\boldsymbol{\theta}$ for any $\mathbf{q}$. The condition – i.e., each dimension of $\boldsymbol{\theta}$ has a unique impact on $A(\boldsymbol{\theta}; \mathbf{q})$ – is satisfied if the cost function is additively separable.[6] By exploiting the global inverse function theorem, we then show that the condition is equivalent to the existence of a unique solution to the nonlinear system.

Given that the approach relies on the bidder's best response, our procedure is a natural extension of the structural estimation method of auctions by Guerre, Perrigne and Vuong (2000). At the same time, given that it relies on the invertibility of the system of functions, our approach is related in spirit to the literature on identification for simultaneous equation systems, which exploits the global invertibility of real functions (e.g., Matzkin (2008)). Global invertibility – as in Beckert and Blundell (2008) and Berry, Gandhi and Haile (2013) – plays the central role in demand estimation analyses, as well. Both papers provide economically interpretable sufficient conditions (e.g., connected substitutes) for the global invertibility of a demand system. In our analysis, the condition for the invertibility is interpreted as relatively strong separability in the cost function.

Our final contribution is to provide an empirical analysis based on the structural method we propose. We estimate the bidder's multidimensional signal using the scoring auction data. The data are from bid results of public procurement auctions for construction projects in Japan, where score is given by the weighted sum of non-price attributes divided by price

existence in settings with multidimensional types and actions. Other studies on equilibrium existence frequently referred to in the auction literature include Lebrun (1996) and Athey (2001).

[6] Note that the condition implies that $K \leqq L$.

(Price-per-Quality Ratio scoring rule). A series of counterfactual analyses suggest that the changes in auction formats (i.e., first-score (FS) vs second-score (SS) auctions) and scoring rules have limited impact on utilities of both the procurement buyer and suppliers (e.g., .7 percent improvement of the buyer's utility for each of both changes). The gain from using the scoring auction instead of conventional price-only auctions has, on average, greater impact on both buyer and contractors' welfare (e.g., 1.0 to 8.7 percent improvement of the buyer's welfare). The numbers vary depending on the quality standards set in price-only auctions, which are supposed to be chosen by the contractors in the scoring auction. The results suggest that a procurer can obtain an almost equivalent (slightly lower) gain with the use of a price-only auction with a well-designed fixed quality standard.

The reminder of this article is organized as follows. Section 2 gives related literature. Section 3 describes the model of scoring auctions. Section 4 gives the equilibrium analysis. Section 5 discusses the identification of the distribution of bidders' cost schedule parameters. Section 6 conducts empirical examinations using our structural estimation method, and Section 6 concludes.

2 Related literature

Che (1993) gives the seminal analysis on scoring auctions. Branco (1997) relaxes the assumption that the bidder's private signal be independent, and Asker and Cantillon (2008) extend Che (1993) to settings in which bidders' signals are multidimensional. More recently, Wang and Liu (2014) and Dastidar (2014) provide theoretical analyses that relax the assumption that the scoring rule is quasilinear.

Our paper is also related to the analysis on multidimensional bidding in which bidders submit non-price attributes only. Thiel (1988) analyzes the Fixed-Price Best Proposal (FxPBP) auction in which the auctioneer predetermines the winner's payment and the winner is the bidder that proposes the best quality offer. He shows that the multidimensional auction can be collapsed into a single-dimensional bidding problem – i.e., bidding the auctioneer's utility – and that the bidder's payoff upon winning (utility) is nonlinear in its single-dimensional bid. Bajari, Houghton and Tadelis (2014) analyze the multidimensional unit-price auction, in which the bidder submits a vector of unit prices, each corresponding to an item of the procurement contract. The winner is the bidder whose weighted sum of all the itemized bids is the lowest.[7] Regarding the itemized bids as non-price attributes, they

[7] Athey and Levin (2001) also study the multidimensional unit-price auction.

analyze this auction by using a scoring auction model in which (i) score is an aggregation of non-price attributes and (ii) payment (price) is a function of non-price attributes. By the similar reason as Thiel (1988), the bidder utility is nonlinear in its strategy.

A large body of literature examines auctions in which the price is not the sole determinant of the winner. In this literature, a set of papers analyzes multidimensional bidding where non-price attributes are bidder characteristics – what bidders cannot choose at bidding (e.g., Marion, 2007; Krasnokutskaya and Seim, 2011; Krasnokutskaya, Song and Tang, 2013; and Mares and Swinkels, 2014). If the way to rank multidimensional bids is public, the situation can fall into our framework.[8] As multidimensional bidding outside of our framework, Asker and Cantillon (2008) discuss multidimensional bidding in which bidders choose non-price attributes, but the auctioneer keeps the scoring rule secret at bidding (i.e., menu auctions and beauty contests). Che (1993) and Asker and Cantillon (2010) study optimal design in multidimensional bidding given that the auctioneer has a quasilinear preference.

Our paper is also related to the literature on identification of the auction model, such as Athey and Haile (2002) and Athey and Haile (2007), and the literature on the structural estimation method of auctions, including Paarsch (1992); Laffont, Ossard and Vuong (1995); Guerre et al. (2000); and Krasnokutskaya (2011).[9] Given the multidimensionality in bidder private information, the structural estimation of the scoring auction model has a challenge similar to that of the structural estimation of auctions with risk-averse bidders, as in Guerre, Perrigne and Vuong (2009), Campo, Guerre, Perrigne and Vuong (2011), etc.[10] Similar to these analyses, our approach exploits a parametric assumption on the cost function to address this challenge.

As for empirical analyses, Lewis and Bajari (2011) is the first structural analysis on scoring auctions. They found that the A+B bidding (a scoring auction) in which the non-price attribute is the number of days to complete the project, improves taxpayers' welfare by approximately 19%.[11] Iimi (2016) and Koning and van de Meerendonk (2014) offer analyses based on the reduced-form approach. So far, structural analyses have relied on

[8] If each bidder observes the identity of its competitors and their characteristics, bidders are ex ante asymmetric.

[9] See Paarsch and Hong (2006) and Athey and Haile (2007) for book-length surveys. For a more recent survey, see, e.g., Hickman, Hubbard and Sağlam (2012).

[10] More recent papers include Campo (2012) and Fang and Tang (2014).

[11] Nakabayashi (2013) and Takahashi (2014) also perform structural analyses to investigate the impacts of small-business set asides and uncertainty of evaluating non-price attributes in design competition in procurement settings.

restriction either on the scoring rule (i.e., the QL form, as in Lewis and Bajari, 2011) or on the cost function (e.g., an additively separable form in Takahashi (2014)). Neither of these analyses discusses conditions under which the multidimensional type is identifiable in the scoring auction.

3 The Model of Scoring Auctions

A buyer would like to procure an item through competitive bidding by $n \geqq 2$ risk-neutral ex ante symmetric suppliers. Based on the knowledge of n, each supplier submits a price bid $p \in \mathbb{R}_+$, as well as an $(L-1)$-dimensional quality bid $\mathbf{q} = (q^1, \ldots, q^{L-1}) \in [\underline{q}^1, \bar{q}^1] \times \cdots \times [\underline{q}^{L-1}, \bar{q}^{L-1}] \equiv \mathcal{Q} \subset \mathbb{R}^{L-1}$. A scoring function, $S(p, \mathbf{q}) : \mathbb{R}^L \to \mathbb{R}$, is common knowledge, mapping the L-dimensional bid into a score. Let $\mathcal{S} := \{S(p, \mathbf{q}) | p \in \mathbb{R}_+, \mathbf{q} \in \mathcal{Q}\}$ denote the feasible set of score.

Let $\boldsymbol{\theta} \in \boldsymbol{\Theta} := [\underline{\theta}^0, \bar{\theta}^0] \times \cdots \times [\underline{\theta}^{K-1}, \bar{\theta}^{K-1}] \subset \mathbb{R}^K$ denote a K-dimensional private signal that each supplier obtains prior to bidding. The supplier's cost to perform the procurement contract promising quality, $\mathbf{q}$, is given by the cost function $C(\mathbf{q}, \boldsymbol{\theta})$ defined on $\mathcal{Q} \times \boldsymbol{\Theta}$. Let $C_{q^\ell}(\mathbf{q}, \boldsymbol{\theta})$ and $C_{\theta^k}(\mathbf{q}, \boldsymbol{\theta})$ denote the partial derivative of $C(\mathbf{q}, \boldsymbol{\theta})$ with respect to q^ℓ and θ^k with $\ell = 1, \ldots, L-1$ and $k = 0, \ldots, K-1$.

Two scoring-auction formats are considered in the analysis: the first-score (FS) auction and the second-score (SS) auctions. Let $(p_i, \mathbf{q}_i)$ denote supplier i's multidimensional bid in the scoring auction with $i = 1, \ldots n$, and let $(p^{post}, \mathbf{q}^{post})$ be the contracted payment and quality. The bidder with the lowest $S(p, \mathbf{q})$ wins, and only the winning bidder performs the contract and receives a payment. Let $s_{(j)}$ denote the jth-lowest score in the auction with $j = 1, \ldots, n$.

In the FS auction, the contract payment and quality are equal to the winning bidder's bid – i.e., $(p^{post}, \mathbf{q}^{post}) = (p_i, \mathbf{q}_i)$ if i wins. In the SS auction, the successful bidder can choose $(p^{post}, \mathbf{q}^{post})$ ex post such that $S(p^{post}, \mathbf{q}^{post}) = s_{(2)}$.

Then, bidder i's problems in the FS and SS auctions are given as follows:

$$\max_{p_i,\mathbf{q}_i} \left[p^{post} - C(\mathbf{q}^{post}, \boldsymbol{\theta}_i)\right] \Pr\{\text{win}|S(p_i, \mathbf{q}_i)\}, \tag{FS}$$
$$\text{subject to } (p^{post}, \mathbf{q}^{post}) = (p_i, \mathbf{q}_i) \text{ if } i \text{ wins.}$$

$$\max_{p_i,\mathbf{q}_i} E_{s_{(2)}}\left[\max_{p^{post},\mathbf{q}^{post}} \left\{p^{post}-C(\mathbf{q}^{post}, \boldsymbol{\theta}_i)\middle|\, S(p^{post}, \mathbf{q}^{post})=s_{(2)}\right\}\middle| \text{win}\right] \Pr\{\text{win}|S(p_i, \mathbf{q}_i)\}. \tag{SS}$$

Throughout the paper, we impose the following four assumptions:

Assumption 1 (Regularity)**.**

(i) The scoring rule, $S(p, \mathbf{q})$, is sufficiently smooth with $S_p(p, \mathbf{q}) > 0$ and $S_{q^\ell}(p, \mathbf{q}) < 0$ for all $\ell = 1, \ldots, L-1$.

(ii) The bidder's cost function, $C(\mathbf{q}, \boldsymbol{\theta})$, is weakly convex and twice-continuously differentiable for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. In addition, $C_{q^\ell}(\mathbf{q}, \boldsymbol{\theta}) \geqq 0$, $C_{\theta^k}(\mathbf{q}, \boldsymbol{\theta}) > 0$, and all are bounded for all $\ell = 1, \ldots, L-1$ and $k = 0, \ldots, K-1$.

(iii) The signal, $\boldsymbol{\theta}$, is distributed independently and identically according to a publicly known joint density, $f(\boldsymbol{\theta})$, which is continuous, strictly positive, and bounded for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Assumption 1 is a set of regularity conditions. Condition (i) assumes that the score function is monotone. Condition (ii) implies that the bidder's total and marginal costs are smooth and increasing in $\boldsymbol{\theta}$, and (iii) assumes independent private values. While assuming independence of $\boldsymbol{\theta}$ across suppliers, we allow for correlation of $\boldsymbol{\theta}$ across dimension.

Given Assumption 1-(i), the scoring function is invertible with respect to p. It follows that, for all $s = S(p, \mathbf{q}) \in \mathcal{S}$,

$$P(s, \mathbf{q}) = p \tag{1}$$

is well defined for any $\mathbf{q}$.[12] Moreover, because $S(p, \mathbf{q})$ is sufficiently smooth, $P(s, \mathbf{q})$ is a sufficiently smooth function.[13]

[12] Note that Function $P(s, \mathbf{q})$ is discussed in Asker and Cantillon (2008), as denoted by $\Psi(\mathbf{Q}, t^w)$, where $\mathbf{Q}$ and t^w correspond to $\mathbf{q}$ and s in our model.

[13] The partial derivatives of $P(s, \mathbf{q})$ with respect to s and q^ℓ are given by $P_s(s, \mathbf{q}) = 1/S_p(p, \mathbf{q})$ and $P_{q^\ell}(s, \mathbf{q}) = -S_{q^\ell}(p, \mathbf{q})/S_p(p, \mathbf{q})$, respectively, for all $\ell = 1, \ldots, L-1$.

Assumption 2 (Interior Solution)**.** For all $p \in \mathbb{R}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,

(i)

$$\left(-\frac{S_{q^\ell q^m}(p,\mathbf{q})S_p(p,\mathbf{q}) - S_{q^\ell}(p,\mathbf{q})S_{pq^m}(p,\mathbf{q})}{S_p(p,\mathbf{q})^2} - C_{q^\ell q^m}(\mathbf{q},\boldsymbol{\theta})\right)_{\ell,m=1,\ldots,L-1}$$

is negative definite for all $\mathbf{q} \in \mathcal{Q}$*;*

(ii) *for all* $\ell = 1, \ldots, L-1$*,*

(a)

$$\left(-\frac{S_{q^\ell}(p,\mathbf{q})}{S_p(p,\mathbf{q})} - C_{q^\ell}(\mathbf{q},\boldsymbol{\theta})\right)_{\ell=1,\ldots,L-1} > 0$$

at $\underline{q}^\ell$*,*

(b)

$$\left(-\frac{S_{q^\ell}(p,\mathbf{q})}{S_p(p,\mathbf{q})} - C_{q^\ell}(\mathbf{q},\boldsymbol{\theta})\right)_{\ell=1,\ldots,L-1} < 0$$

at $\bar{q}^\ell$*.*

The assumption simply gives technical conditions needed to ensure that the bidder's problem in choosing an optimal $\mathbf{q}$ is strictly concave and has an interior solution; condition (i) states that the bidder payoff upon winning, i.e., $P(s,\mathbf{q}) - C(\mathbf{q},\boldsymbol{\theta})$, is strictly convex in $\mathbf{q}$, and (ii) states that the marginal payoff, i.e., $P_{q^\ell} - C_{q^\ell} \equiv -S_{q^\ell}/S_p - C_{q^\ell}$, eventually becomes negative as $\mathbf{q}$ rises. In Section 5, we will discuss that the interior solution is necessary for the identification of the bidders' K-dimensional signals. Note that, given Assumption 1, Assumption 2 is generally satisfied unless the auctioneer uses a reserve price or quality bounds.

The condition is, of course, sufficient. Hence, an interior solution is guaranteed if we relax condition (i) such that the marginal profit is constant if $\mathbf{q}$ is very irrelevant point in the domain of $C(\mathbf{q},\boldsymbol{\theta})$ such as $\mathbf{q}$ such that $C_{q^\ell} = 0$ or $P(s,\mathbf{q}) - C(\mathbf{q},\boldsymbol{\theta}) < 0$.

Example: We briefly discuss a set of scoring rules used in practice and see that these satisfy both Assumptions 1 and 2. Let $V(\cdot)$ and $W(\cdot)$ be strictly increasing, smooth, and concave functions, and let $\mathbf{q}^b = (q^1, \ldots, q^\ell)$ and $\mathbf{q}^c = (q^{\ell+1}, \ldots, q^{L-1})$ for some $\ell \in \{1, \ldots L-2\}$. Then, examples of the scoring functions are given as follows:

$$S^{\text{QL}}(p, \mathbf{q}) = p - V(\mathbf{q}), \qquad \text{(Quasilinear, QL)}$$
$$S^{\text{PQR}}(p, \mathbf{q}) = p/V(\mathbf{q}), \qquad \text{(Price-per-Quality Ratio, PQR)}$$
$$S^{\text{ABC}}(p, \mathbf{q}) = (p - V(\mathbf{q}^b))/W(\mathbf{q}^c). \qquad \text{(Hybrid, ABC)}$$

As mentioned earlier, the lowest-score bidder wins. Note that the allocation, payment, and payoffs are all invariant to a monotone transformation of the scoring function. Hence, $S(p, \mathbf{q}) = \log(p - V(\mathbf{q}^b)) - \log W(\mathbf{q}^c)$ is, for instance, equivalent to the ABC rule in terms of outcome. Note also that no monotone function makes any pair of these scoring rules equivalent in outcome.

Both QL and PQR are frequently used in practice (e.g., the US states DOT in Delaware, Idaho, Oregon, Massachusetts, Utah, and Virginia for QL and Alaska, Colorado, Florida, Michigan, North Carolina, and South Dakota for PQR).[14] The last one (ABC) is discussed in a report by the Federal Highway Administration (FHWA), called "Multiparameter Bidding (A+B+C)" (California Department of Transportation, Caltrans, 2007). Component (B), $\mathbf{q}^b$, includes the number of days required to complete the project, and (C) component, $\mathbf{q}^c$, include the contractor's past performance. By adjusting A+B bids by C when determining the winner, the scoring rule favors contractors with higher quality levels on past projects, giving more chance to win in the current auction.

Now, let $s = S(p, \mathbf{q})$. Then, $P(s, \mathbf{q})$ is given by

$$P^{\text{QL}}(s, \mathbf{q}) = s + V(\mathbf{q}), \qquad \text{(QL)}$$
$$P^{\text{PQR}}(s, \mathbf{q}) = sV(\mathbf{q}), \qquad \text{(PQR)}$$
$$P^{\text{ABC}}(s, \mathbf{q}) = V(\mathbf{q}^b) + sW(\mathbf{q}^c). \qquad \text{(ABC)}$$

under these scoring functions. These three scoring rules satisfy Assumptions 1 and 2 for any convex cost function.

Assumption 3 (Single Crossing)**.** The cost function is a quasilinear form:

$$C(\mathbf{q}, \boldsymbol{\theta}) = \theta^0 + C^0(\mathbf{q}, \theta^1, \ldots, \theta^{K-1}).$$

[14]See Molenaar and Yakowenko (2007) for more details.

Assumption 3 is a sufficient primitive condition for the single-crossing property or, equivalently, strictly increasing differences of the bidder's objective function (Lemma 1). Note that the condition is for simplicity, keeping the number of type-space dimensions equal to K. In fact, we do not need the condition if we augment the type space so that an additional dimension plays the same role as θ^0.[15] With the single-crossing property, we demonstrate that the FS auction has a unique pure monotone equilibrium (Section 4.3).[16] Note that the assumption is not required for the equilibrium analysis of the SS auction or of the FS auction with a QL scoring rule.

Assumption 4 (Identification)**.** *For each $\boldsymbol{\theta}$ in the interior of $\boldsymbol{\Theta}$ and for each* $\mathbf{q}$ in $\mathcal{Q}$*, there exists a K-dimensional nonsingular matrix, $\boldsymbol{\Gamma}$, with which, for all $k = 0, 1, \ldots, K-1$, $\widetilde{C}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) := \widetilde{C}(\mathbf{q}, \boldsymbol{\Gamma}\boldsymbol{\theta}) = C(\mathbf{q}, \boldsymbol{\theta})$ satisfies*

$$|\widetilde{C}_{\widetilde{\theta}^k}(\mathbf{q}, \widetilde{\boldsymbol{\theta}})| > \sum_{m \neq k} |\widetilde{C}_{\widetilde{\theta}^m}(\mathbf{q}, \widetilde{\boldsymbol{\theta}})| \tag{2}$$

or

$$|\widetilde{C}_{q^\ell \widetilde{\theta}^k}(\mathbf{q}, \widetilde{\boldsymbol{\theta}})| > \sum_{m \neq k} |\widetilde{C}_{q^\ell \widetilde{\theta}^m}(\mathbf{q}, \widetilde{\boldsymbol{\theta}})| \tag{3}$$

for some $\ell \in \{1, \ldots, L-1\}$.

The assumption addresses that, with an appropriate bijective transformation of $\boldsymbol{\theta}$, each dimension of $\boldsymbol{\theta}$ has a unique set of marginal or total costs on which it has a stronger impact than the sum of all other dimensions of $\boldsymbol{\theta}$. An additively-separable cost function satisfies the condition, i.e., $C_{q^\ell \theta^k} = 0$ for all $k \neq \ell$. Moreover, many non-additively-separable cost functions satisfies the condition, including the Cobb-Douglas form.[17] Figure 1 illustrates the cost function that violates Assumption 4. The assumption is satisfied if and only if the Jacobian matrix of the $L-1$-dimensional vector:

$$A(\boldsymbol{\theta}; \mathbf{q}) := (C(\mathbf{q}, \boldsymbol{\theta}), C_{q^1}(\mathbf{q}, \boldsymbol{\theta}), \ldots, C_{q^{L-1}}(\mathbf{q}, \boldsymbol{\theta}))^{\mathrm{T}}$$

[15] Appendix Appendix C gives a brief discussion on this point.

[16] The existing approach can show only the existence, but not the uniqueness, of a monotone equilibrium in the FS auction. See, e.g., Athey (2001) and McAdams (2003).

[17] See Appendix B for examples of cost functions that satisfy our settings.

is full column rank (Lemma 3).[18] This implies that the cost function fails to satisfy the assumption if $K > L$.

[Figure 1 about here.]

Note that, under Assumption 3, the Jacobian matrix of $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible if and only if the Jacobian matrix of $\nabla_{\mathbf{q}} C(\mathbf{q}, \boldsymbol{\theta})$ is full column rank. Hence, if the cost function satisfies Assumption 3, we need neither condition (2) nor $k = 0$ in expression (3) of Assumption 4.

4 Equilibrium Analysis

In this section, we examine the equilibrium of the scoring auction by replicating with an alternative game (called the score-bidding game). The key to our approach is that we decompose the multidimensional bidding process into two steps: the selection of a profit maximizing price-quality pair for a given targeted score and the auction as bidding a score. The approach simplifies the equilibrium analysis of the scoring auction with minimum restrictions on primitives in settings with multidimensional types. Moreover, our approach uncovers a close connection of the scoring auction model to the auction with non-risk-neutral bidders. We then show the uniqueness of a symmetric isotone equilibrium in the FS auction (i.e., monotone in each dimension of multidimensional types).

4.1 An Outcome-equivalent Score-bidding Game

Bidder i's problems (FS) and (SS) are equivalently expressed by the following two-step maximization:

$$\max_{s_i} \left[\max_{p^{post}, \mathbf{q}^{post}} \left\{ p^{post} - C(\mathbf{q}^{post}, \boldsymbol{\theta}_i) \middle| S(p^{post}, \mathbf{q}^{post}) = s_i \right\} \right] \Pr(s_i \leqq \min_{j \neq i} s_j), \tag{FS'}$$

$$\max_{s_i} E_{s_{(2)}} \left[\max_{p^{post}, \mathbf{q}^{post}} \left\{ p^{post} - C(\mathbf{q}^{post}, \boldsymbol{\theta}_i) \middle| S(p^{post}, \mathbf{q}^{post}) = s_{(2)} \right\} \middle| \text{win} \right] \Pr(s_i \leqq \min_{j \neq i} s_j). \tag{SS'}$$

[18] If $\boldsymbol{\Gamma}$ is a diagonal matrix, the assumption implies that the Jacobian of $A(\boldsymbol{\theta}; \mathbf{q})$ is a quasi-dominant diagonal (q.d.d.) matrix discussed in McKenzie (1960). Because Assumption 4 is weaker than the condition for the Jacobian matrix to be q.d.d, Assumption 4 is not only sufficient but also necessary for a full column rank Jacobian of $A(\boldsymbol{\theta}; \mathbf{q})$ in our setting in which $A(\cdot)$ is differentiable.

Obviously, the outcome of this game is equivalent to that of the original scoring auction game. We call the alternative game the *score-bidding game*.

To solve the two-step optimization problem backward, let us first examine the value function of the second-step maximization:

$$u(s,\boldsymbol{\theta}) := \max_{\mathbf{q}^{post}} P(s,\mathbf{q}^{post}) - C(\mathbf{q}^{post},\boldsymbol{\theta}),$$

where $s = s_i(= s_{(1)})$ in the FS auction and $s = s_{(2)}$ in the SS auction. We call $u : \mathcal{S}\times\boldsymbol{\Theta} \to \mathbb{R}$ the *induced utility function*. Note that $u(s,\boldsymbol{\theta})$ is the amount of the payment bidder type $\boldsymbol{\theta}$ earns when winning. Because Assumption 2 guarantees a unique interior solution to the second-step optimization problem, $u(s,\boldsymbol{\theta})$ is well defined. Moreover, $u(s,\boldsymbol{\theta})$ is sufficiently smooth but generally nonlinear.[19]

By using $u(\cdot)$, we can then rewrite bidder i's first-step problems as:

$$\max_{s_i\in\mathcal{S}} u(s_i,\boldsymbol{\theta}_i)\Pr\{\text{win}|s_i\}, \tag{FS''}$$

$$\max_{s_i\in\mathcal{S}} u(s_{(2)},\boldsymbol{\theta}_i)\Pr\{\text{win}|s_i\}. \tag{SS''}$$

The expressions indicate an analogy of the scoring auction to the auction with non-risk-averse bidders examined e.g., by Maskin and Riley (1984). In the scoring auction, $u(s,\boldsymbol{\theta})$ is induced as the value function of the second-step maximization. This contrasts $u(s,\boldsymbol{\theta})$ from the bidder's utility function in Maskin and Riley (1984) who give this exogenously. Later, we show that the sufficient smoothness and the log-supermodularity of $u(s,\boldsymbol{\theta})$ are keys to the existence of a monotone equilibrium.

To conclude this subsection, we discuss how the bidder chooses $\mathbf{q}^{post}$ in the second-step maximization. Let us define

$$\mathbf{q}(s,\boldsymbol{\theta}) := \arg\max_{\mathbf{q}^{post}} \left\{ P(s,\mathbf{q}^{post}) - C(\mathbf{q}^{post},\boldsymbol{\theta}) \middle| \, s \right\}.$$

[19]Because both $P(s,\mathbf{q})$ and $C(\mathbf{q},\boldsymbol{\theta})$ are smooth, we have $P_s(\cdot) = 1/S_p(\cdot) > 0$ and $C_{\theta^k} > 0$. Therefore, the partial derivatives of $u(s,\boldsymbol{\theta})$ exist as:

$$\begin{aligned} u_s(s,\boldsymbol{\theta}) = &\quad P_s(s,\mathbf{q}(s,\boldsymbol{\theta})) > 0, \\ u_{\theta^k}(s,\boldsymbol{\theta}) = &\quad -C_{\theta^k}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) < 0 \text{ for all } k = 0,\ldots,K-1. \end{aligned}$$

In addition, $u(\cdot)$ is twice-continuously differentiable. This is because (i) both $P(\cdot)$ and $C(\cdot)$ are twice-continuously differentiable; (ii) applying the implicit function theorem to (4), we see that $\mathbf{q}(\cdot)$ is differentiable.

The first-order condition (FOC) gives:

$$\nabla_{\mathbf{q}} P(s, \mathbf{q}(s, \boldsymbol{\theta})) = \begin{bmatrix} P_{q^1}(s, \mathbf{q}(s, \boldsymbol{\theta})) \\ \vdots \\ P_{q^{L-1}}(s, \mathbf{q}(s, \boldsymbol{\theta})) \end{bmatrix} = \begin{bmatrix} C_{q^1}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) \\ \vdots \\ C_{q^{L-1}}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) \end{bmatrix} = \nabla_{\mathbf{q}} C(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}). \quad (4)$$

This suggests that the bidder chooses **q** so that its marginal revenue matches the marginal cost given a target score s.

The previous example of scoring rules provides an intuition of the score bidding game – a supplier's competition for a cost reimbursement contract. Under the PQR scoring rule where $P(s, \mathbf{q}) = sV(\mathbf{q})$, suppliers compete in terms of the unit price of $V(\mathbf{q})$. The winner is reimbursed for each unit of $V(\mathbf{q})$ at s and chooses quality such that the marginal cost equals the unit price of quality. Under the QL contract, the reimbursement consists of two parts: (i) each unit of $V(\mathbf{q})$ at a fixed price (equal to one), and (ii) a lump-sum subsidy, s. The bidder who requests the minimum lump-sum subsidy wins.

The interpretation also makes the distinction clear between non-QL and QL scoring rules. Under the non-QL scoring rule, suppliers are price makers – i.e., the unit price of quality depends on s. Then, as price rises, (i) the inframarginal profit rises linearly and (ii) there is an extramarginal profit due to the adjustment of **q** under the PQR scoring rule. Thus, the induced utility is convex in s. On the other hand, suppliers are price takers under the QL scoring rule, where the unit price of **q** is fixed. This makes $u(s, \boldsymbol{\theta})$ linear in s. The difference of the curvature of $u(s, \boldsymbol{\theta})$ generally induces the difference in the expected scores between the FS and SS auctions under the non-QL scoring rules.

4.2 Equilibrium in the Second-Score Auction

In the Second-Score (SS) bidding game, there exists a dominant strategy equilibrium, $\sigma_{\text{II}} : \boldsymbol{\Theta} \to \mathcal{S}$, such that bidder i with $i = 1, \ldots, n$ chooses $z(\boldsymbol{\theta}_i)$:

$$\sigma_{\text{II}}(\boldsymbol{\theta}) = \min_{\mathbf{q} \in \mathcal{Q}} S(C(\mathbf{q}, \boldsymbol{\theta}), \mathbf{q}) =: z(\boldsymbol{\theta}).$$

where $z(\boldsymbol{\theta})$ denotes the break-even score, i.e., the bidder's minimum possible score subject to non-negative payoffs. By construction, $z(\boldsymbol{\theta})$ satisfies $u(z(\boldsymbol{\theta}), \boldsymbol{\theta})$. It is easy to see that

$z(\boldsymbol{\theta})$ is well defined.[20] Moreover, $z(\boldsymbol{\theta})$ is strictly increasing and smooth.[21] If the scoring rule is QL, $z(\boldsymbol{\theta})$ is equivalent to the bidder's *pseudotype* discussed in Asker and Cantillon (2008).

By (4), the optimal quality choice in the SS bidding game is given by

$$\mathbf{q}^z(\boldsymbol{\theta}) := \{\mathbf{q} | \nabla_{\mathbf{q}} P(z(\boldsymbol{\theta}), \mathbf{q}) = \nabla_{\mathbf{q}} C(\mathbf{q}, \boldsymbol{\theta})\}.$$

Assumption 2 ensures the uniqueness of $\mathbf{q}^z(\boldsymbol{\theta})$. Therefore, the equilibrium in the SS auction is summarized as follows:

Proposition 1. *In the SS auction, there exists a dominant strategy equilibrium in which bidder i submits $(p^*(\boldsymbol{\theta}_i), \mathbf{q}^*(\boldsymbol{\theta}_i))$ such that*

$$(p^*(\boldsymbol{\theta}_i), \mathbf{q}^*(\boldsymbol{\theta}_i)) = (P(z(\boldsymbol{\theta}_i), \mathbf{q}^z(\boldsymbol{\theta}_i)), \mathbf{q}^z(\boldsymbol{\theta}_i)),$$

where $z(\boldsymbol{\theta}_i)$ and $\mathbf{q}^z(\boldsymbol{\theta}_i)$ are given by

$$u(z(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i) = 0,$$
$$\nabla_{\mathbf{q}} P(z(\boldsymbol{\theta}_i), \mathbf{q}^z(\boldsymbol{\theta}_i)) = \nabla_{\mathbf{q}} C(\mathbf{q}^z(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i).$$

Proof. See Appendix C. □

The winner determines its ex post payment and quality, $(p^{post}, \mathbf{q}^{post})$, such that $S(p^{post}, \mathbf{q}^{post})$ matches the second-lowest score.

4.3 Equilibrium in the First-Score Auction

Suppose that a symmetric pure monotone equilibrium strategy: $\sigma_{\text{I}} : \boldsymbol{\Theta} \to \mathcal{S}$ exists in the first-score (FS) bidding game. If all bidders play σ_{I}, then bidder i's equilibrium multidi-

[20] Because $\mathcal{Q}$ is a compact set, the Weierstrass Theorem implies the existence of the minimum.

[21] To show the smoothness, recall that $z(\cdot)$ satisfies $u(z(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Therefore, by the implicit function theorem, $z_{\theta^k}(\boldsymbol{\theta}) := \partial z(\boldsymbol{\theta}) / \partial \theta^k$ exists locally for all $k = 0, \ldots, K-1$ in the interior of $\boldsymbol{\Theta}$. To show the strict increase, take the derivative on both sides of expression $u(z(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0$ with respect to θ^k. Then, we have

$$u_s(z(\boldsymbol{\theta}), \boldsymbol{\theta}) z_{\theta^k}(\boldsymbol{\theta}) + u_{\theta^k}(z(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0.$$

Given that $u_{\theta^k} < 0$ and that $u_s > 0$, we have $z_{\theta^k}(\boldsymbol{\theta}) > 0$.

mensional bid in the original FS auction is given by:

$$(p^*(\boldsymbol{\theta}_i), \mathbf{q}^*(\boldsymbol{\theta}_i)) = (P(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \mathbf{q}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)), \mathbf{q}(\sigma_{\mathrm{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)) \,. \tag{5}$$

Suppose that all bidders except for i follow σ_{I}. Let $G(s)$ and $g(s)$ denote the distribution and density of the score by bidder i's rival. Then, for bidder i, (FS") is given by

$$\max_{s_i \in \mathcal{S}} \pi(s_i, \boldsymbol{\theta}_i) = u(s_i, \boldsymbol{\theta}_i)[1 - G(s_i)]^{n-1}. \tag{6}$$

The following lemma demonstrates that Assumption 3 is a sufficient primitive condition for the *log-supermodularity* of $u(s, \boldsymbol{\theta})$.

Lemma 1 (Single-crossing property)**.** *Assumption 3 implies that there exists a linear transformation* $\mathbf{M}$*, with which* $\tilde{u}(s, \tilde{\boldsymbol{\theta}}) := \tilde{u}(s, \mathbf{M}\boldsymbol{\theta}) \equiv u(s, \boldsymbol{\theta})$ *satisfies the following sorting property:*

$$\frac{\partial}{\partial \theta^k} \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} < 0 \textit{ for all } k = 0, \ldots, K-1.$$

Proof. See Appendix D. □

The idea of the proof is that we align the type space in the way that kth dimension of the new type space is given by the composition of θ^0 and θ^k for all $k = 1, \ldots, L-1$. Specifically, the above inequality is reexpressed as:

$$\frac{1}{u_s(s, \boldsymbol{\theta})} \left[\frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} \left(\sum_{\ell=1}^{L-1} -C_{q^\ell \theta^k}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) q_s^\ell(s, \boldsymbol{\theta}) \right) + C_{\theta^k}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) \right] > 0.$$

The inequality does not hold in general because C_{q^ℓ} may rise much faster than C as $\boldsymbol{\theta}$ rises along kth dimension of its type space with $k = 1, \ldots, K-1$. If the cost function satisfies the assumption, then we find a sufficiently large number, M_k, with which, by redefining the type space as $\tilde{\theta}^0 = \theta^0 - \sum_{k=1}^{K-1} M_k \theta^k$ and as $\tilde{\theta}^k = \theta^k$ for all $k = 1, \ldots, K-1$, C can rise sufficiently faster than C_{q^ℓ} on the new type space. In what follows, we slightly abuse the notation of $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}$ is an element of the new type space so that $u(s, \boldsymbol{\theta})$ is log-supermodular.

The log-supermodularity of $u(s, \boldsymbol{\theta})$ implies the *single-crossing property* – i.e., $\partial^2 \pi(s, \boldsymbol{\theta}) / \partial s \partial \theta^k \geqq 0$ for all $\ell = 0, \ldots, L-1$.[22] With the single-crossing property, McAdams (2003) has shown

[22]In our situation, $u(s, \boldsymbol{\theta})$ has its cross-partial derivative. Therefore, given (Guess), the log-

the existence of a pure monotone equilibrium. We here demonstrate that a monotone equilibrium is unique as well:

Proposition 2. *The symmetric pure monotone equilibrium strategy, $\sigma_{\text{I}}(\cdot)$, is unique. Moreover, $\sigma_{\text{I}}(\boldsymbol{\theta}_i)$ and $\mathbf{q}(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)$ satisfy the L-dimensional system of best-response functions:*

$$\frac{1 - G(\sigma_{\text{I}}(\boldsymbol{\theta}_i))}{(n-1)g(\sigma_{\text{I}}(\boldsymbol{\theta}_i))} = \frac{u(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)}{u_s(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)}, \tag{7}$$

$$\nabla_{\mathbf{q}} P(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \mathbf{q}(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i)) = \nabla_{\mathbf{q}} C(\mathbf{q}(\sigma_{\text{I}}(\boldsymbol{\theta}_i), \boldsymbol{\theta}_i), \boldsymbol{\theta}_i). \tag{8}$$

It is easy to see that expression (8) is given by expression (4) with $s = \sigma_{\text{I}}(\boldsymbol{\theta}_i)$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_i$. Hence, in what follows, we show that $\sigma_{\text{I}}(\boldsymbol{\theta}_i)$ in (7) exists uniquely.

We first show that $\sigma(\cdot)$ is a unique solution to bidder problem (6) for some $G(\cdot)$. Let us denote by $\bar{s} = \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} z(\boldsymbol{\theta})$, the score from which the least efficient bidder obtains zero profits. We then guess that $G(s)$ satisfies

$$\begin{cases} G(s) \text{ is strictly increasing and continuously differentiable;} \\ G(\bar{s}) = 1. \end{cases} \tag{Guess}$$

The single-crossing property is well known to ensure the pseudoconcavity of the bidder's objective function with respect to s_i.[23] Therefore, if $G(\cdot)$ satisfies (Guess), expression (7) is sufficient for the unique global maximum. Moreover, the solution is strictly increasing in $\boldsymbol{\theta}$.[24]

To conclude the proof, we demonstrate that $G(\cdot)$ exists uniquely in the following two steps. First, we construct a differential equation with respect to $G(\cdot)$ that is consistent with the FOC expressed in (7). Next, we show that the differential equation has a unique solution. Note that the FOC implies that all bidders that choose the same score in equilibrium have an identical value of $u(\cdot)/u_s(\cdot)$. As shown below, this implies that the distributions of the score and $s - u(s, \cdot)/u_s(s, \cdot)$ have an equivalence in equilibrium. The differential equation we construct is based on the equivalence of these two distributions so that it is consistent with the FOC.

To show the equivalence of the two distributions, let us denote the cumulative distribu-

supermodularity of $u(s, \boldsymbol{\theta})$ is equivalent to the single-crossing property of the bidder's interim expected profit, $\pi(s_i, \boldsymbol{\theta}_i)$.

[23]See, e.g., Matthews (1995).

[24]In Online Appendix I, we verify these points.

tion of $s - u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})$ given s by:

$$\xi(\kappa; s) := \Pr\left\{ s - \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} \leqq \kappa \right\}. \tag{9}$$

Let $s_i^* = \sigma(\boldsymbol{\theta}_i)$ and $\kappa_i^* := s_i^* - u(s_i^*, \boldsymbol{\theta}_i)/u_s(s_i^*, \boldsymbol{\theta}_i)$. An important feature of the FOC is that, for any bidder that also chooses s_i^* in equilibrium, the value of $s_i^* - u(s_i^*, \cdot)/u_s(s_i^*, \cdot)$ is equal to κ_i^* as well.

Given the log-supermodularity of $u(\cdot)$ and the monotonicity of $\sigma_{\mathrm{I}}(\cdot)$, this feature implies that for any $\boldsymbol{\theta}$, $\sigma_{\mathrm{I}}(\boldsymbol{\theta}) < s_i^*$ if and only if $s^* - u(s^*, \boldsymbol{\theta})/u_s(s^*, \boldsymbol{\theta}) < \kappa^*$.[25] Thus, we have

$$\Pr\{s_i^* - u(s_i^*, \boldsymbol{\theta})/u_s(s_i^*, \boldsymbol{\theta}) \leqq \kappa_i^*\} = \Pr\{\sigma_{\mathrm{I}}(\boldsymbol{\theta}) \leqq s_i^*\}.$$

The expression shows the equivalence in the distributions of score and $s - u(s, \cdot)/u_s(s, \cdot)$ in equilibrium.

Eliminating subscript i and using the notation defined above, we rewrite the above expression as:

$$\xi\left(s^* - \frac{u(s^*, \boldsymbol{\theta})}{u_s(s^*, \boldsymbol{\theta})}; s^*\right) = G(s^*)$$

subject to $s^* = \sigma_{\mathrm{I}}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$. Then, plugging expression (7) to substitute out $u(\cdot)/u_s(\cdot)$, we obtain:

$$\xi\left(s - \frac{1 - G(s)}{(n-1)g(s)}; s\right) = G(s) \tag{10}$$

subject to $s = \sigma_{\mathrm{I}}(\boldsymbol{\theta})$. Given that $\xi(\cdot)$ is well defined for all $s \in \mathcal{S}$, the expression constitutes a differential equation with respect to $G(s)$. This implies that if $G(s)$ is the equilibrium score distribution, it must be the solution to the differential equation.

Next, we show that the differential equation has a unique solution. Let $\underline{\kappa}(s) := \min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} s - u(s, \boldsymbol{\theta})/u_s(s, \underline{\boldsymbol{\theta}})$ and $\bar{\kappa}(s) := \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} s - u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})$. Because $u(s, \boldsymbol{\theta})$ is log-supermodular, $\xi(\underline{\kappa}(s); s) = 0$ and $\xi(\bar{\kappa}(s); s) = 1$ for any s. It is easy to see that $\xi(\kappa; s)$ is continuous and

[25] Let $\boldsymbol{\Theta}_i^*$ is the set of bidders that submit s_i^*. Note that for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_i^*$, $s_i^* - u(s_i^*, \boldsymbol{\theta})/u_s(s_i^*, \boldsymbol{\theta}) = \kappa_i^*$. Then, let $\boldsymbol{\theta}'$ is a bidder type that chooses a strictly lower score than s_i^*. Given the monotonicity of $\sigma_{\mathrm{I}}(\cdot)$, there exists a bidder type, $\boldsymbol{\theta} \in \boldsymbol{\Theta}_i^*$, such that $\boldsymbol{\theta}' \leq \boldsymbol{\theta}$. Therefore, by the log-supermodularity of $u(\cdot)$, $s_i^* - u(s_i^*, \boldsymbol{\theta}')/u_s(s_i^*, \boldsymbol{\theta}') < s_i^* - u(s_i^*, \boldsymbol{\theta})/u_s(s_i^*, \boldsymbol{\theta})$. Conversely, let $\boldsymbol{\theta}''$ denote a bidder whose value of $s_i^* - u(s_i^*, \cdot)/u_s(s_i^*, \cdot)$ is strictly lower than κ_i^*. By the log-supermodularity, there is a bidder type $\boldsymbol{\theta} \in \boldsymbol{\Theta}_i^*$ such that $\boldsymbol{\theta}'' \leq \boldsymbol{\theta}$. Then, by the monotonicity of $\sigma_{\mathrm{I}}(\cdot)$, $\sigma_{\mathrm{I}}(\boldsymbol{\theta}'') < s_i^*$.

strictly increasing in κ.[26] Therefore, $\xi : [\underline{\kappa}(s), \bar{\kappa}(s)] \to [0,1]$ is bijective given s. It follows that $\xi(\kappa; s)$ is invertible with respect to κ for all $s \in \mathcal{S}$. Then, let $\kappa(x, s)$ denote the inverse of $\xi(\kappa; s)$ with respect to κ for some $x = \xi(\kappa; s) \in [0,1]$.

Note that $\kappa(x, s)$ is bounded, because $\kappa(x, s) \in [\underline{\kappa}(s), \bar{\kappa}(s)]$ by definition. Moreover, the next lemma shows that $\kappa(x, s)$ is smooth in s for any $x \in [0,1]$. These ensure that $\kappa(\cdot, s)$ is Lipschitz continuous with respect to s.

Lemma 2 (Differentiability of $\kappa(x, s)$)**.**
For any $x \in [0,1]$, $\kappa(x, s)$ is differentiable with respect to s in the interior of $\mathcal{S}$.

Proof. See Appendix E. □

In Appendix F, we show that differential equation (10) has a unique solution with boundary condition $G(\bar{s}) = 1$. In the appendix, we also demonstrate that $G(s)$ is strictly increasing, which implies that $G(s)$ satisfies (Guess). Recall that $\sigma_{\mathrm{I}}(\boldsymbol{\theta})$ is the unique solution to (7) given $G(\cdot)$. Hence, we conclude that $\sigma_{\mathrm{I}}(\cdot)$ is the unique symmetric pure monotone equilibrium.

4.4 Equilibrium Features of the Scoring Auction Model

We now demonstrate that $\kappa(x, s)$ is an extension of the *pseudotype* discussed in Asker and Cantillon (2008) to settings with non-QL scoring rules. Define

$$\boldsymbol{\Theta}(s, x) = \left\{ \boldsymbol{\theta} \in \boldsymbol{\Theta} \,\middle|\, \xi\left(s - \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})}; s \right) = x \right\}, \tag{11}$$

representing the set of bidder types for whom the value of $s - u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})$ is identical and the cumulative distribution of $s - u(s, \cdot)/u_s(s, \cdot)$ is equal to x. Let $y(x) = G^{-1}(x)$.

[26]Note that $\xi(\kappa; s)$ is reexpressed as

$$\xi(\kappa; s) = \int_{\underline{\kappa}(s)}^{\kappa} \int_{\{\tilde{\boldsymbol{\theta}} | s - u(s, \tilde{\boldsymbol{\theta}})/u_s(s, \tilde{\boldsymbol{\theta}}) = \hat{\kappa}\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} d\hat{\kappa}.$$

Taking the derivative with respect to κ, we have

$$\frac{\partial}{\partial \kappa} \xi(\kappa; s) = \int_{\{\tilde{\boldsymbol{\theta}} | s - u(s, \tilde{\boldsymbol{\theta}})/u_s(s, \tilde{\boldsymbol{\theta}}) = \kappa\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} > 0.$$

Furthermore, this is bounded for any $s \in \mathcal{S}$ and $\kappa \in [\underline{\kappa}(s), \bar{\kappa}(s)]$, because $f(\boldsymbol{\theta})$ is bounded for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Given that $\xi(\cdot)$ is a smooth function, it is continuous.

Then, from expression (10), we derive an equivalent expression of the FOC of the FS bidding game as

$$y(x)'(1-x)^{n-1}-(n-1)(1-x)^{n-2}y(x)=-(n-1)(1-x)^{n-2}\kappa(x,y(x)).$$

Note that x represents a set of types whose equilibrium score is equal to $y(x)$. Integrating both sides gives a familiar form:

$$\sigma_{\mathrm{I}}(\boldsymbol{\theta})=\int_x^1\frac{(n-1)(1-\tilde{x})^{n-2}}{(1-x)^{n-1}}\kappa(\tilde{x},y(\tilde{x}))d\tilde{x}\ \text{ subject to } x=G(\sigma_{\mathrm{I}}(\boldsymbol{\theta})). \tag{12}$$

The optimal strategy in the FS auction is to choose the score equal to the conditional expectation of the second-lowest bidder's $\kappa(x,s)$.

Note that if the scoring rule is QL, we see that $u_s(s,\boldsymbol{\theta})=1$ and that $u(s,\boldsymbol{\theta})=s+V(\mathbf{q}^z(\boldsymbol{\theta}))$. Therefore, for any $s\in\mathcal{S}$ and $x\in[0,1]$, we have

$$\kappa(x,s)\equiv s-u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})=C(\mathbf{q}^z(\boldsymbol{\theta}),\boldsymbol{\theta})-V(\mathbf{q}^z(\boldsymbol{\theta}))\equiv z(\boldsymbol{\theta})$$

for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}(x,s)$. This suggests that $\kappa(x,s)$ in the QL scoring rule is the pseudotype in Asker and Cantillon (2008). Under the QL scoring rule, x is the cumulative distribution of $z(\boldsymbol{\theta})$ – i.e., $\Pr(z(\boldsymbol{\theta}))=x$.

Che (1993) and Asker and Cantillon (2008) discussed that the pseudotype is a productivity measure. We show that so is $\kappa(x,s)$ by referring to the PQR scoring rule (an example of non-QL scoring rules) discussed above. If the scoring rule is PQR, $\kappa(x,s)=C(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})/V(\mathbf{q}(s,\boldsymbol{\theta}))$, i.e., the average cost (AC). If the scoring rule is QL, $\kappa(x,s)=C(\mathbf{q}^z(\boldsymbol{\theta}),\boldsymbol{\theta})-V(\mathbf{q}^z(\boldsymbol{\theta}))$, representing the minimum lump-sum subsidy required for the bidder to achieve a nonnegative profit. Then, $\kappa(x,s)$ under the QL rule is interpreted as the bidder's net cost (NC). It is easy to see that, in both cases, the lowest-κ supplier is awarded.

Note that each supplier chooses s strictly higher than its minimum AC or NC in the FS competition. Because the unit price of $V(\mathbf{q})$ is fixed under the QL scoring rule, the quality level chosen is the one that minimizes NC under the QL rule. Under the PQR scoring rule, on the other hand, $\mathbf{q}(s,\boldsymbol{\theta})$ is greater than the efficient scale. This suggests that the score will be strictly greater than the lowest rival's minimum AC.

In the SS competition, the amount of cost reimbursement (under the QL rule, the lump-sum part only) supplier i receives is independent of s_i. Hence, it is dominant for suppliers to choose their minimum NC or AC.[27] As shown in the next subsection, the price-making behavior in the FS auction causes inequivalence in the expected winning score between the FS and SS auctions.

4.4.1 A Discussion on Expected Score Ranking

Our two-step approach also gives a great view on the comparison of the expected scores across auction formats; the expected score ranking depends on the curvature of the (induced) utility function as seen in Maskin and Riley (1984). However, comparison of the expected scores involves the following two difficulties. First, score is an ordinary measure. Hence, the ranking is subject to the curvature of a nonlinear monotone transform of the scoring rule, yet the outcome of the scoring auction, e.g., the contracted price and quality, is invariant to any monotone transform. Second, the buyer may not reveal its true true preference through the scoring rule. Therefore, our argument begins by setting aside welfare implication. Then, if we focus on a linear scoring rule, i.e., s is linear in p, we obtain the following observation.

Observation Suppose that Assumptions 1 and 2 are satisfied and that the scoring rule is linear in price. Then, the expected score is weakly lower in the SS auction than in the FS auction. Moreover, the expected scores are the same in the FS and SS auctions if and only if the scoring rule is QL.

A formal discussion is given in Appendix G, which demonstrates that the expected score in the FS auction is higher than in the SS auction if the bidder induced utility is convex in s. Note that the observation is robust to the extent that the bidder optimization for non-price attributes is unrestricted.

While limited, the observation provides a welfare implication. If the buyer has a cardinal preference and reveals its true preference through the scoring function, then the expected score represents the buyer utility. On top of these assumptions on buyer preference, if the buyer utility is linear in price, our observation implies that the buyer has an advantage in the use of the SS rather than FS auction. The result indicates that the buyer has a strict advantage in the use of the SS rather than the FS auction if the scoring rule represents the buyer's true preference and if the scoring rule is non-QL, e.g., PQR and (A+B+C).

[27] In the contract, the winner chooses $\mathbf{q}(s_{(2)}, \boldsymbol{\theta})$ which makes κ greater than the efficient scale.

The observation also uncovers a salient feature of multidimensional bidding. A key to the result above is that the choice in multidimensional attributes is made by the utility recipient. Therefore, if the choice is restricted or made by other players than the utility recipient, the expected score ranking may flip. An example for the latter situation is examined by Hansen (1988) who analyzes a homogeneous multiple-object procurement auction in which risk-neutral bidders submit a unit price of the item. The quantity produced is determined by the auctioneer's demand function. Given that the buyer's preference is quasilinear, he shows that the buyer has an advantage in the use of the first-price instead of the second-price auction. For the former case, we show in the next section that binding constraints in the second-step maximization makes the induced utility function convex with a series of examples.

4.4.2 Reserve price, quality constraints, etc.

So far, we have focused on the case in which bidders can choose non-price attributes without any restrictions (Assumption 2). We now relax the assumption to illustrate that our two-step approach works in many cases. Moreover, we show that our approach is applicable to more general settings of multidimensional bidding than scoring.

Let us consider the value function of the following constraint maximization problem:

$$u(s, \boldsymbol{\theta}) = \max_{\mathbf{q}} p - C(\mathbf{q}, \boldsymbol{\theta}) \text{ subject to } \quad S(p, \mathbf{q}) = s, \ h(p, \mathbf{q}) \geqq 0, \ H(p, \mathbf{q}) = 0 \tag{13}$$

where $h(p, \mathbf{q})$ and $H(p, \mathbf{q})$ are, respectively, inequality and equality constraints. To the extent that $u(s, \boldsymbol{\theta})$ in (13) is well defined and sufficiently smooth, our discussion so far is applicable.[28]

For instance, Thiel (1988) examined the Fixed–Price Best–Proposal auction in which bidders submit $\mathbf{q}$ only. The winner receives a fixed amount of payment, $\bar{p}$, which is preannounced by the auctioneer. Given that $S(p, \mathbf{q})$ is quasiconcave in his model, the induced utility is well defined by (13) with $H(p, \mathbf{q}) \equiv \bar{p} - p$ and $h(p, \mathbf{q})$ being degenerated. More-

[28] As discussed in Section 5, a corner solution may entail nonidentification of the bidder's multidimensional signal. Furthermore, additional conditions may be needed to assure differentiability and Lipschitz continuity of $\kappa(\cdot)$. Assuming the single-dimensional assumption, we have shown that $\kappa(\cdot)$ is Lipschitz continuous unless two or more inequality constraints bind simultaneously. This is because $u_s(s, \cdot)$ may jump at the score for which two constraints bind at the same time. The proof is given upon request.

over, the second derivative of $u(s, \boldsymbol{\theta})$ is:

$$u_{ss}(s, \boldsymbol{\theta}) = -\sum_{\ell=1}^{L-1}\sum_{m=1}^{L-1} C_{q^\ell q^m}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) q_s^\ell(s, \boldsymbol{\theta}) q_s^m(s, \boldsymbol{\theta}) - \sum_{\ell=1}^{L-1} C_{q^\ell}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) q_{ss}^\ell(s, \boldsymbol{\theta}). \tag{14}$$

The first term is strictly negative because $C(\mathbf{q}, \boldsymbol{\theta})$ is strictly convex. It is easily shown that the second term is nonpositive if $S(p, \mathbf{q})$ is weakly concave in **q**.[29] Thus, $u(s, \boldsymbol{\theta})$ is concave in s.

In procurement settings, buyers may set a reserve price in the scoring auction. The induced utility is given by (13) with $h(p, \mathbf{q}) \equiv \bar{p} - p \geqq 0$ and $H(p, \mathbf{q})$ being degenerated. As before, the induced utility is concave if the reserve price binds. This implies that even if the auction uses the QL scoring rule, the expected score equivalence fails between the FS and SS auction – i.e., the FS auction more likely yields a lower score.

Bajari et al. (2014) examined a unit price auction in which each bidder submits an $(L-1)$-dimensional unit price, which corresponds to submitting **q** in the scoring auction. Bidders do not bid p. Instead, p is given by an additional constraint: $p = P(\mathbf{q})$, which is concave in **q** in their setting. Therefore, the scoring rule is given by $S(P(\mathbf{q}), \mathbf{q}) = \tilde{S}(\mathbf{q})$. The second-step maximization is thus described as $\max_{\mathbf{q}} p - C(\boldsymbol{\theta})$ subject to $\tilde{S}(\mathbf{q}) = s$ and $p = P(\mathbf{q})$. Given the concavity of $P(\cdot)$, u_{ss} is negative in their model.

In a homogeneous multiple-object procurement auction, risk-neutral bidders submit a unit price of the item. If the awarded supplier chooses the quantity level, the bidder's problem is equivalent to the scoring auction with the PQR scoring rule, where the bidder submits unit price s. Hansen (1988) instead examines the auction in which the auctioneer chooses quantity procured, following its demand schedule. The multiobject auction can be analyzed by setting, $S(p, q) = p/q$ and $H(p, q) \equiv D(q) - p/q$, where $D(q)$ is a decreasing function.

[29] See Online Appendix II for the proof.

5 Structural estimation of the scoring auction model

5.1 Outline

In this section, we demonstrate that the K-dimensional i.i.d. signal is identified from L-dimensional bids if Assumptions 1 through 4 hold and $K \leqq L$. The key to identification is the invertibility of $A(\boldsymbol{\theta}; \mathbf{q}) \equiv (C, C_{q^1}, \ldots, C_{q^{L-1}})$ with respect to $\boldsymbol{\theta}$. We show that Assumption 4 is a necessary and sufficient condition for the global invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$.

5.2 Identification of the multidimensional signal in FS and SS auctions

We first examine the FS auction. Let $(p^*, \mathbf{q}^*)$ denote an observed multidimensional bid, and let s^* denote the associated score, given by $s^* = S(p^*, \mathbf{q}^*)$. Suppose that p^* and $\mathbf{q}^*$ are generated by equilibrium strategy $\sigma_{\text{I}}(\cdot)$, as discussed in (5). Then, s^* satisfies the FOC, (7), as

$$\frac{1 - G(s^*)}{(n-1)g(s^*)} = \frac{u(s^*, \boldsymbol{\theta})}{u_s(s^*, \boldsymbol{\theta})}.$$

Given that the observed quality bid satisfies $\mathbf{q}^* = \mathbf{q}(s^*, \boldsymbol{\theta})$, we have $u(s^*, \boldsymbol{\theta}) = P(s^*, \mathbf{q}^*) - C(\mathbf{q}^*, \boldsymbol{\theta})$ and $u_s(s^*, \boldsymbol{\theta}) = P_s(s^*, \mathbf{q}^*)$. Then, we rearrange the FOC as:

$$C(\mathbf{q}^*, \boldsymbol{\theta}) = p^* - P_s(s^*, \mathbf{q}^*)\frac{1 - G(s^*)}{(n-1)g(s^*)}. \tag{15}$$

Moreover, $\mathbf{q}^* = \mathbf{q}(s^*, \boldsymbol{\theta})$ satisfies (4) such that

$$\nabla_{\mathbf{q}} C(\mathbf{q}^*, \boldsymbol{\theta}) = \nabla_{\mathbf{q}} P(s^*, \mathbf{q}^*). \tag{16}$$

Then, from equations (15) and (16), we have the following system of nonlinear equations:

$$A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*, \text{ where} \tag{17}$$

$$A(\boldsymbol{\theta}; \mathbf{q}^*) = \begin{bmatrix} C(\mathbf{q}^*, \boldsymbol{\theta}) \\ \nabla_{\mathbf{q}} C(\mathbf{q}^*, \boldsymbol{\theta}) \end{bmatrix}; \quad \mathbf{b}^* = \begin{bmatrix} p^* - P_s(s^*, \mathbf{q}^*)(1 - G(s^*))/(n-1)g(s^*) \\ \nabla_{\mathbf{q}} P(s^*, \mathbf{q}^*) \end{bmatrix}. \tag{18}$$

Given that bidders follow a strictly increasing strategy $\sigma_{\text{I}}(\cdot)$, $b^0 \equiv p - P_s(s, \mathbf{q})(1 - G(s))/(n-1)g(s)$ is monotone in s given $\mathbf{q}$. This implies that $\mathbf{b}$ and s are one-to-one with each other. In other words, for any observables $(p^*, \mathbf{q}^*)$, $\mathbf{b}^*$ is uniquely given. Therefore, the monotonicity of b^0 gives a refutable restriction on the distribution of s in the FS scoring auction model.

The case of the SS auction is analogous; suppose that each bidder's multidimensional bid, $(p^*, \mathbf{q}^*)$, is generated by the equilibrium strategy $\sigma_{\text{II}}(\cdot)$.[30] Then, the associated score, $s^* = S(p^*, \mathbf{q}^*)$, is given by

$$s^* = \sigma_{\text{II}}(\boldsymbol{\theta}) = z(\boldsymbol{\theta}),$$

where p^* and $\mathbf{q}^*$ are given by

$$p^* = P(s^*, \mathbf{q}^*), \text{ and} \tag{19}$$

$$\nabla_{\mathbf{q}} C(\mathbf{q}^*, \boldsymbol{\theta}) = \nabla_{\mathbf{q}} P(s^*, \mathbf{q}^*). \tag{20}$$

Then, the system of nonlinear equations is given as $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$ with

$$A(\boldsymbol{\theta}; \mathbf{q}^*) = \begin{bmatrix} C(\mathbf{q}^*, \boldsymbol{\theta}) \\ \nabla_{\mathbf{q}} C(\mathbf{q}^*, \boldsymbol{\theta}) \end{bmatrix}; \quad \mathbf{b}^* = \begin{bmatrix} p^* \\ \nabla_{\mathbf{q}} P(s^*, \mathbf{q}^*) \end{bmatrix}. \tag{21}$$

Now, we discuss our approach to identification. First, $P(\cdot)$ is a known function. For the FS auction, $g(\cdot)$ and $G(\cdot)$ can be obtained from observations on $s^* = S(p^*, \mathbf{q}^*)$. Hence, all elements of $\mathbf{b}^*$ can be evaluated from the observed multidimensional bid, $(p^*, \mathbf{q}^*)$, in both FS and SS cases. Furthermore, $C(\mathbf{q}, \boldsymbol{\theta})$ is known except for $\boldsymbol{\theta}$. In other words, only $\boldsymbol{\theta}$ is the unknown element in the nonlinear system, $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$. Therefore, $\boldsymbol{\theta}$ is identified from observations if function $A(\boldsymbol{\theta}; \mathbf{q})$ is invertible with respect to $\boldsymbol{\theta}$. More specifically, if $A(\boldsymbol{\theta}; \mathbf{q})$ is invertible with respect to $\boldsymbol{\theta}$, we can recover $\boldsymbol{\theta}$ as the unique solution to the nonlinear system. In what follows, we give a formal argument for this by demonstrating that the nonlinear system has a unique solution.

The unique solution to the nonlinear system is shown by the global inverse function theorem.[31] In our situation, we have to show the following conditions: (i) $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\mathbf{q} \in \mathcal{Q}$; (ii) $A(\boldsymbol{\theta}; \mathbf{q})$ is a proper mapping for any $\boldsymbol{\theta}$ and $\mathbf{q}$; and (iii) $\boldsymbol{\Theta}$ is arcwise connected, and the image of $A(\boldsymbol{\theta}; \mathbf{q})$ is simply connected for all $\mathbf{q} \in \mathcal{Q}$.

The following lemma demonstrates that Assumption 4 is equivalent to the local invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$.

[30] In the SS auction, the winner also chooses $(p^{post}, \mathbf{q}^{post})$, which is also observable. In this analysis, we ignore the effect of these additional observations on identification.

[31] See Ambrosetti and Prodi (1995) for more details.

Lemma 3. *Suppose that $K \leqq L$. Then, the cost function, $C(\mathbf{q}, \boldsymbol{\theta})$, satisfies Assumption 4 if and only if the Jacobian matrix of $A(\boldsymbol{\theta}; \mathbf{q}) = (C(\mathbf{q}, \boldsymbol{\theta}), \nabla_{\mathbf{q}} C(\mathbf{q}, \boldsymbol{\theta})^{\mathrm{T}})^{\mathrm{T}}$ with respect to $\boldsymbol{\theta}$ is full column rank for all $\boldsymbol{\theta}$ in the interior of Θ and $\mathbf{q} \in \mathcal{Q}$.*

Proof. See Appendix H. □

We then have a proposition regarding the global invertibility of $A(\boldsymbol{\theta}; \mathbf{q})$. Given the local invertibility of $A(\cdot)$, our proof focuses on demonstrating the remaining conditions, (ii) and (iii), for the global inverse function theorem.

Proposition 3. *Suppose that $K \leqq L$. Then, under Assumptions 1 through 4, vector-valued function $A(\boldsymbol{\theta}; \mathbf{q})$ is globally invertible with respect to $\boldsymbol{\theta}$ for all $\mathbf{q} \in \mathcal{Q}$.*

Proof. See Appendix I. □

The following corollary is an immediate consequence of Proposition 3.

Corollary 1 (Identification)**.** *Under Assumptions 1 through 4, the bidder's K-dimensional signal is identified from L-dimensional bid samples.*

Several remarks are in order. First, we briefly discuss the case in which Assumption 4 is not satisfied. If the cost function exhibits the rank-deficient Jacobian matrix of $A(\boldsymbol{\theta}; \mathbf{q})$ at $\mathbf{q}^*$, then the impact of a dimension of $\boldsymbol{\theta}$ on the marginal (and total) costs is identical to that of another dimension or a linear combination of a set of other dimensions of $\boldsymbol{\theta}$ at $\mathbf{q}^*$. If K-dimensional $\boldsymbol{\theta}$ exhibits such dependence in the cost function, then two different bidder types $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$ have the same total and marginal costs at $\mathbf{q}^*$ – i.e., $A(\boldsymbol{\theta}; \mathbf{q}^*) = A(\boldsymbol{\theta}'; \mathbf{q}^*)$. Given that the optimal choice in s depends solely on $\boldsymbol{\theta}$ in the scoring auction, these two bidders are observationally equivalent, as their score and quality are identical to s^* and $\mathbf{q}^*$, respectively. Therefore, the multidimensional signal is not identified.

Second, we make a note on identification when the second-step maximization has a corner solution. For instance, expression (16) becomes inequality if a quality upper bound binds:

$$\nabla_{\mathbf{q}} C(\mathbf{q}^*, \boldsymbol{\theta}) \geqq \nabla_{\mathbf{q}} P(s^*, \mathbf{q}^*). \tag{16$'$}$$

That is, one obtains $A(\boldsymbol{\theta}; \mathbf{q}^*) \geqq \mathbf{b}^*$, suggesting that $\boldsymbol{\theta}$ is not identified. In this case, one may need to exploit additional observations or constraints on primitives for identification or to use partial identification.

Finally, we explore the specification test for the cost function. Suppose that the researcher uses a cost function, $\widehat{C}(\mathbf{q}, \boldsymbol{\theta})$, that may not be the true cost function, $C(\mathbf{q}, \boldsymbol{\theta})$. Given that the observation of the scoring auction data is L-dimensional, one needs additional variations in data to identify signals of $(L+1)$ or higher dimensions. This, in turn, implies that there is no way to test the cost function with L-dimensioning bid data only.

Several ways have been proposed to obtain additional dimensions of information, such as exogenous variations in the scoring rules and in the number of bidders.[32] However, in Appendix J, we show that at least the exogenous variation in the number of bidders does not help to test the cost function if the scoring rule is QL.

Note that, while the cost function may be testable with non-QL scoring rule, it generally has some limitations, as discussed in Athey and Haile (2007). For instance, the alternative hypothesis is that some component of the specification is incorrect. A failure of the test may indicate the presence of unobserved heterogeneity, risk aversion, non-equilibrium bidding behavior, etc.

5.3 Estimation for the distribution of $\boldsymbol{\theta}$

Let T be the number of scoring auction samples, each indexed by $t = 1, \ldots, T$. Let $\hat{\boldsymbol{\theta}}_{i,t} = (\hat{\theta}^0_{i,t}, \ldots, \hat{\theta}^{K-1}_{i,t})$ with $K \leqq L$ be the solution to $A(\boldsymbol{\theta}; \mathbf{q}^*) = \mathbf{b}^*$, where $\mathbf{b}^*$ is given by (18) and (21) for the FS and SS auctions, respectively.

In the FS auction, both $G(s)$ and $g(s)$ are estimated by the standard kernel estimator. Auction-specific heterogeneities, such as the number of bidders, properties of the item to be purchased, etc., are controlled; let n_t and $\mathbf{x}_t = (x^1_t, \ldots, x^d_t)$ denote the number of bidders and the covariates of auction t, respectively. Let $g(s, n, \mathbf{x})$ denote the joint density function of s, n, and $\mathbf{x}$. Then, the kernel estimator for $G(s, n, \mathbf{x}) := \int_{-\infty}^{s} g(v, n, \mathbf{x}) dv$ is provided by

$$\hat{G}(s, n, \mathbf{x}) = \frac{1}{T h_{G_n} h^d_{G_x}} \sum_{t=1}^{T} \frac{1}{n_t} \sum_{i=1}^{n_t} \mathbf{1}(s_{i,t} \leqq s) K_G \left(\frac{n - n_t}{h_{G_n}}, \frac{x_1 - x_{1,t}}{h_{G_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{G_x}} \right), \tag{22}$$

where $\mathbf{1}(\cdot)$ is an indicator function, K_G is a kernel with a bounded support, and h_{G_n} and

[32] The idea to exploit a variation in the scoring rule is seen in Asker and Cantillon (2008). For more detailed arguments on the use of a variation in the number of bidders, see Athey and Haile (2002, 2007).

h_{G_x} are bandwidths. Similarly, the kernel density estimator for $g(s, n, \mathbf{x})$ is given by

$$\hat{g}(s,n,\mathbf{x}) = \frac{1}{Th_s h_{g_n} h_{g_x}^d} \sum_{t=1}^{T} \frac{1}{n_t} \sum_{i=1}^{n_t} K_g \left(\frac{s - s_{i,t}}{h_s}, \frac{n - n_t}{h_{g_n}}, \frac{x_1 - x_{1,t}}{h_{g_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{g_x}} \right), \tag{23}$$

where K_g is a kernel with a bounded support and h_s, h_{g_n}, and h_{g_x} are bandwidths. In practice, the discrete variables, such as the number of bidders and the maximum quality level, are smoothed out in the way that Li and Racine (2006) discuss.

Corollary 1 suggests that $\hat{\boldsymbol{\theta}}_{i,t}$ is recovered in both FS and SS auctions. The estimation for $F(\boldsymbol{\theta}, \mathbf{x}) := \int_{-\infty}^{\theta^0} \cdots \int_{-\infty}^{\theta^{K-1}} f(\boldsymbol{\tau}, \mathbf{x}) d\tau^0 \cdots d\tau^{K-1}$ is given by the standard kernel method:

$$\hat{F}(\boldsymbol{\theta}, \mathbf{x}) = \frac{1}{Th_{F_x}^d} \sum_{t=1}^{T} \sum_{i=1}^{n_t} \mathbf{1}(\boldsymbol{\theta} \leq \boldsymbol{\theta}_{i,t}) K_F \left(\frac{x_1 - x_{1,t}}{h_{F_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{F_x}} \right),$$

where K_F is a kernel with a bounded support, and h_{F_x} is a bandwidth. Similarly, the kernel density estimator for the joint density function of $\boldsymbol{\theta}$ and the covariate vector $\mathbf{x}$ is given by

$$\hat{f}(\boldsymbol{\theta}, \mathbf{x}) = \frac{1}{Th_{f_0} \cdots h_{f_{K-1}} h_{f_x}^d} \sum_{t=1}^{T} K_f \left(\frac{\theta^0 - \theta_{i,t}^0}{h_{f_0}}, \ldots, \frac{\theta^{K-1} - \theta_{i,t}^{K-1}}{h_{f_{K-1}}}, \frac{x_1 - x_{1,t}}{h_{f_x}}, \cdots, \frac{x_d - x_{d,t}}{h_{f_x}} \right),$$

where K_f is a kernel with bounded support, and h_{f_0},..., $h_{f_{K-1}}$, and h_{f_x} are bandwidths.

6 An Empirical experiment

6.1 Data and Institution

The data used in our analysis contain the bid results of the procurement auctions for civil engineering projects conducted from January 2010 through August 2014 by the Ministry of Land, Infrastructure, and Transportation (MLIT) in Japan. The data include project names, dates of auctions, engineers' estimates, scoring auctions or not, and submitted bids with the bidder's identity. The ministry let 18,183 civil engineering projects during the study period. The projects cost approximately 750 billion yen a year, which accounts for approximately four percent of the public construction investment in Japan.

Among these, 6,610 projects were allocated through the scoring auction in which bidders were asked to submit a technical proposal.[33] After removing samples with only one bidder and possibly misrecorded auctions, we are left 5,142 scoring auction samples.[34]

Table 1 reports the sample statistics. The mean of winners' bids and engineers' estimated prices were approximately 423 or 477 million yen, respectively. The quality bids ranged from approximately 130 through 200. The score is calculated as the quality bid divided by the price bid (Inverse PQR). The bidder with the highest score wins the project. To control for project size heterogeneity, we report as the *score* the observed score multiplied by the engineer's estimate for each auction. In each auction, approximately ten firms participated, on average.

[Table 1 about here.]

While the quality-bid point is given by a weighted sum of all non-price attributes, including noise level, completion time, and bidder experience, our data records the (aggregated) quality-bid point. The lower bound of the point is 100 for all auctions, and the upper bound is 150 to 200, depending on the auction. The bidder proposing nothing has a quality bid equal to 100. Table 2 reports the sample statistics by upper bound in quality.

[Table 2 about here.]

6.2 Specifications

6.2.1 Percentage bids

Let $(B_{i,t}, q_{i,t})$ and $\bar{B}_t$ denote the raw values of bidder i's price–quality pair and the engineer's estimate of scoring auction $t \in T$. Under the inverse PQR scoring rule, the actual score is given by $q_{i,t}/B_{i,t}$. In our analysis, we use the normalized price, $p_{i,t} = B_{i,t}/\bar{B}_t$, in replace with $B_{i,t}$ to control for project size heterogeneity. Let q^{post} and p^{post} denote the contracted quality and normalized price. We assume that the buyer's utility from auction t

[33] There are three types of scoring auctions: Technical Proposal Type (*Kodo Gijutsu Teian Gata*); Regular Type (*Hyojun Gata*); and Simple Type (*Kan-i Gata*). We use Technical Proposal Type and Regular Type. In the Simple Type, bidders are not asked to turn in any proposal; instead, the buyer evaluates the bidder's past experience and the technology levels as non-price attributes. Hence, we removed these auctions from our samples. The Simple Type is used for relatively smaller projects.

[34] Misrecorded auctions include those in which quality or price bids are too low or too high (outside of the feasible level for the quality bid or less than 10% or greater than 200% of the engineer's estimate for the price bid).

is represented by:

$$w_t = \frac{p_t^{post}}{q_t^{post}}. \tag{24}$$

For estimation purposes, we use the inverse of the score as bidder i's score.[35] The associated scoring rule is given by $S(p_{i,t}, q_{i,t}) = p_{i,t}/q_{i,t}$. Figure 2 shows the histogram of $s_{i,t}$ for the auction samples.

[Figure 2 about here.]

Finally, while we control the heterogeneity in project size by normalizing the price bid with the engineer's estimates, our sample still involves heterogeneity in the number of bidders and the quality upper bound. Thus, the covariate x that we use is the quality upper bound.

6.2.2 Cost function

Given the data, we assume that $L = K = 2$. We use the following polynomial cost function:

$$C(q, \boldsymbol{\theta}) = \begin{cases} (q + \theta^1)^\beta + \theta^0 & \text{if } q > -\theta^1 \\ \theta^0 & \text{otherwise,} \end{cases} \tag{25}$$

where β is 2, 3, or 4 for the robustness of our analysis.

6.3 Estimation of $\boldsymbol{\theta}$

Let $s_{i,t} = S(p_{i,t}, q_{i,t})$. Given that $P = p_{i,t}$ and $P_s = q_{i,t}$ under the PQR scoring rule, we have $\mathbf{b} = \left(p_{i,t} - q_{i,t}(1 - \hat{G}(s_{i,t}, n_t, x_t))/[(n_t - 1)\hat{g}(s_{i,t}, n_t, x_t)], s_{i,t}\right)^{\mathrm{T}}$. Hence, we have

$$\left(\hat{\theta}^0, \hat{\theta}^1\right) = \left(p_{i,t} - q_{i,t}\frac{1}{n_t - 1}\frac{1 - \hat{G}(s_{i,t}, n_t, x_t)}{\hat{g}(s_{i,t}, n_t, x_t)} - \left(\frac{s_{i,t}}{\beta}\right)^{\frac{\beta}{\beta-1}}, \left(\frac{s_{i,t}}{\beta}\right)^{\frac{1}{\beta-1}} - q_{i,t}\right).$$

For estimating $\hat{G}$ and $\hat{g}$, we use the triweight kernel:

$$K(u) = \frac{35}{32}(1 - u^2)^3 \mathbf{1}(|u| < 1).$$

[35] Recall that the outcome of the scoring auction is invariant to any monotone transformation of the scoring rule.

As usual, the bandwidths h_s and h_x are given by the so-called rule of thumb; $h_s = \eta_s(\sum_{t=1}^{T} n_t)^{-1/5}$ and $h_x = \eta_x(\sum_{t=1}^{T} n_t)^{-1/5}$, where $\eta_s = 1.06\hat{\rho}_s$ and $\eta_x = 1.06\hat{\rho}_x$, respectively. Both $\hat{\rho}_s$ and $\hat{\rho}_x$ are sample standard deviations of the normalized scoring bids and the observed covariate, respectively. The following figures are the estimated joint density functions assuming that the cost function is the quadratic polynomial ($\beta = 2$). Axes x (horizontal) and y (depth) represent θ^0 and θ^1, respectively.

[Figure 3 about here.]

6.4 Counterfactual analyses

6.4.1 Second-price vs. FS auctions

One of the appeals of scoring auctions is that both the auctioneer and the bidders increase welfare from a more complete comparison of suppliers' attributes.(See, e.g., Milgrom (2004).) Our first empirical examination, thus, measures the gains from the use of scoring auctions.

We create a series of counterfactual second-price auctions, in each of which the quality level is fixed at $\bar{q} = 110$, 120, and 130. Using the estimated cost functions, we point-estimate bidders' costs at any quality standard, $\bar{q}$. We then select the second-lowest cost as $\hat{p}_t^{post}$, the contract price of the counterfactual second-price auctions. The buyer's utility in the price-only auction is given according to (24) as $w_t = \hat{p}_t^{post}/\bar{q}_t$. Because the bidder's cost functions are differentiated by $\beta = 2$, 3, and 4, fifteen types of counterfactual second-price auctions are generated.

Table 3 compares the buyer's utilities in the observed FS auction versus a series of counterfactual price-only auctions. The buyer's expected gains depend crucially on $\bar{q}$. While the buyer's utilities would drop by more than seven percent if $\bar{q} = 110$, the drop would be trivial (merely .96 percent, for instance, if $\bar{q} = 120$ with the Quadratic cost function). Considering the buyer's costly process of evaluating the quality bids in the scoring auction, the results indicate that a simple low-price auction still performs well, as long as the buyer can appropriately design the quality standard of the price-only auction.

Table 4 reports the winning bidders' (normalized) expected payoffs, which is computed by taking the average of the estimated (nominal) payoff divided by the engineer's estimate for each auction. The results show that the bidder's payoff also varies, depending on the quality standard in the price-only auction. Note that the positive relationship between payoffs and quality standards is due to greater information rents left over to bidders, as suggested by Che (1993).

[Table 3 about here.]

[Table 4 about here.]

The results do not take into account the bidder's participation decision. If bidders earn more in the scoring auction than the price-only auction, scoring auctions can encourage bidders' participation. On the other hand, if the bid preparation costs are significantly greater in a scoring auction than in a price-only auction, then participation is discouraged. Given the buyer's small gain from the scoring auction in comparison to the price-only auction with $\bar{q} = 160$, our results suggest that a price-only auction with an appropriate $\bar{q}$ is still a good mechanism to allocate the government contract.

6.4.2 SS vs. FS auctions

We test Proposition 4: the introduction of the SS auction lowers the expected s. We create counterfactual SS auction samples from the estimated parameters, $\hat{\boldsymbol{\theta}}_{i,t}$. Then, we measure the difference between FS and SS auctions regarding the buyer's and bidders' utilities and the contract quality level.[36]

Table 5 reports the buyer's expected utility, $E(w_t)$. The SS auction lowers the values by .71 to .72 percent, which is in line with the theoretical prediction. Note that the variance is larger in the SS auction, which some buyers may not prefer.

Table 6 reports the expectation of the contracted quality. The quality level declines, on average, by approximately .05 to .06 percents if SS auctions were used. This suggests that the higher expected s is due to excessive quality proposal in the FS auction. In fact, Table 7 shows that bidders earn larger payoffs in the FS auction, on average (about 2.3 percent). This suggests that, while the FS auction would result in a higher expected score (or, equivalently, lower buyer utilities), the drawback can be remedied by more intensified competition as the FS auction is more profitable for bidders under the PQR scoring rule.

[Table 5 about here.]

[Table 6 about here.]

[Table 7 about here.]

[36] The way to generate counterfactual SS auction samples is available in Online Appendix III.

6.4.3 QL vs. PQR rules

Finally, we examine the impact of the change in the scoring rule. The QL rules that we consider are given as

$$S(p, q) = p - \phi(\beta)q, \tag{26}$$

for some $\phi > 0$. We choose $\phi(\beta) \approx .0058$, for example, for the auctions with the quality upper bound is equal to 160. Using $\phi(\beta)$, we predict the expected winning score in the QL scoring auction, which is given by the mean of the second-lowest pseudotype due to the expected score equivalence. In Online Appendix IV, we show the way to generate the counterfactual QL auction samples.

Table 8 reports the buyer's gain from the counterfactual QL scoring auctions. In all cases, utilities rise by .7 percent, on average. Note that standard deviations are larger in our counterfactual QL scoring auctions because we use the SS auction to generate the QL scoring auction samples.

Table 9 shows the winning bidder's profits. The profits drop by about 3.8 to 4.0 percent. This suggests that, using an appropriate QL scoring rule, the buyer can extract more rents from bidders.

[Table 8 about here.]

[Table 9 about here.]

Table 10 compares the contracted quality levels in the observed FS auction and in simulated QL scoring auctions. The quality bids rise by approximately .04 to .06 percent under the well-designed QL scoring rule. This suggests that the QL scoring rule can limit the winner's informational rent while promoting higher quality proposals.

[Table 10 about here.]

7 Conclusion

In this research, we provide a method to analyze the scoring auction theoretically and econometrically. Allowing a broad class of scoring rules, we demonstrate the existence and the characterization of a symmetric monotone equilibrium of the scoring auction. Based on our theoretical model, we then examine identification of the scoring auction model. Furthermore, we take our framework to the scoring auction data to quantify the impact of the use of the scoring auction and the change in design of scoring auctions.

We restrict attention to the independent scoring rule, in which the bidder's score depends only on his or her price and quality bids. In practice, score may depend on other bidders' price and quality bids as well (an interdependent scoring rule). Albano et al. (2009) suggest that the interdependent scoring rule may cause a significant efficiency loss, (approximately 11 percent in their estimation). Interesting future research may be to analyse the scoring auction with an interdependent scoring rule. A counterfactual analysis would quantify the expected score difference between the FS and SS auctions with an interdependent scoring rule.

References

Albano, Gian Luigi, Federico Dini, and Roberto Zampino, "Bidding for Complex Projects: Evidence from the Acquisitions of IT Services," *Lecture Notes in Computer Science*, 2009, *5693*, 353–363.

Ambrosetti, Antonio and Giovanni Prodi, *A primer of nonlinear analysis* number 34, Cambridge University Press, 1995.

Asker, John and Estelle Cantillon, "Properties of Scoring Auctions," *RAND Journal of Economics*, 2008, *39* (1), 69–85.

_ **and** _ , "Procurement when Price and Quality Matter," *RAND Journal of Economics*, 2010, *41* (1), 1–34.

Athey, Susan, "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, July 2001, *69* (4), 861–889.

_ **and Jonathan D Levin**, "Information and Competition in US Forest Service Timber Auctions," *Journal of Political Economy*, 2001, *109* (2), 375–417.

_ **and Philip A Haile**, "Identification of standard auction models," *Econometrica*, 2002, *70* (6), 2107–2140.

_ **and** _ , "Nonparametric approaches to auctions," *Handbook of econometrics*, 2007, *6*, 3847–3965.

Bajari, Patrick, Stephanie Houghton, and Steven Tadelis, "Bidding for Incomplete Contracts: An Empirical Analysis of Adaptation Costs," *American Economic Review*, April 2014, *104* (4), 1288–1319.

Beckert, Walter and Richard Blundell, "Heterogeneity and the non-parametric analysis of consumer choice: conditions for invertibility," *The Review of Economic Studies*, 2008, *75* (4), 1069–1080.

Berry, Steven, Amit Gandhi, and Philip Haile, "Connected substitutes and invertibility of demand," *Econometrica*, 2013, *81* (5), 2087–2111.

Bichler, Martin, "An experimental analysis of multi-attribute auctions," *Decision Support Systems*, 2000, *29* (3), 249–268.

Branco, Fernando, "The Design of Multidimensional Auctions," *RAND Journal of Economics*, 1997, *28* (1), 63–81.

California Department of Transportation, Caltrans, *Innovative Procurement Practices. Alternative Procurement and Contracting Methods Tasks 3.2 and 3.3*, Trauner Consulting Services, 2007.

Campo, Sandra, "Risk aversion and asymmetry in procurement auctions: Identification, estimation and application to construction procurements," *Journal of Econometrics*, 2012, *168* (1), 96–107.

_ , Emmanuel Guerre, Isabelle Perrigne, and Quang Vuong, "Semiparametric estimation of first-price auctions with risk-averse bidders," *The Review of Economic Studies*, 2011, *78* (1), 112–147.

Che, Yeon-Koo, "Design Competition through Multidimensional Auctions," *RAND Journal of Economics*, Winter 1993, *24* (4), 668–680.

Dastidar, Krishnendu Ghosh, "Scoring auctions with non-quasilinear scoring rules," ISER Discussion Paper 0902, Institute of Social and Economic Research, Osaka University June 2014.

Fang, Hanming and Xun Tang, "Inference of biddersf risk attitudes in ascending auctions with endogenous entry," *Journal of Econometrics*, 2014, *180* (2), 198–216.

Guerre, Emmanuel, Isabelle Perrigne, and Quang Vuong, "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 2000, *68* (3), 525–574.

_ , _ , and _ , "Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions," *Econometrica*, 07 2009, *77* (4), 1193–1227.

Hansen, Robert G., "Auctions with Endogenous Quantity," *RAND Journal of Economics*, Spring 1988, *19* (1), 44–58.

Hickman, Brent R, Timothy P Hubbard, and Yiğit Sağlam, "Structural econometric methods in auctions: A guide to the literature," *Journal of Econometric Methods*, 2012, *1* (1), 67–106.

Iimi, Atsushi, "Multidimensional Auctions for Public Energy Efficiency Projects: Evidence from Japanese Esco Market," *Review of Industrial Organization*, 2016, pp. 1–24.

Koning, Pierre and Arthur van de Meerendonk, "The impact of scoring weights on price and quality outcomes: An application to the procurement of Welfare-to-Work contracts," *European Economic Review*, 2014, *71* (C), 1–14.

Krasnokutskaya, Elena, "Identification and estimation of auction models with unobserved heterogeneity," *The Review of Economic Studies*, 2011, *78* (1), 293–327.

_ and Katja Seim, "Bid Preference Programs and Participation in Highway Procurement Auctions," *American Economic Review*, October 2011, *101* (6), 2653–86.

_ , Kyungchul Song, and Xun Tang, "The Role of Quality in Service Markets Organized as Multi-Attribute Auctions," 2013.

Laffont, Jean-Jacques, Herve Ossard, and Quang Vuong, "Econometrics of First-Price Auctions," *Econometrica*, July 1995, *63* (4), 953–80.

Lebrun, Bernard, "Existence of an Equilibrium in First Price Auctions," *Economic Theory*, 1996, *7* (3), pp. 421–443.

Lewis, Gregory and Patrick Bajari, "Procurement Contracting With Time Incentives: Theory and Evidence," *The Quarterly Journal of Economics*, 2011, *126* (3), 1173–1211.

Li, Qi and Jeffrey Scott Racine, *Nonparametric Econometrics: Theory and Practice*, Vol. 1 of Economics Books, Princeton University Press, 2006.

Mares, Vladimir N. and Jeroen Swinkels, "Comparing First and Second Price Auctions with Multiple Insiders and Outsiders," *International Journal of Game Theory*, 2014, *43*, 487–514.

Marion, Justin, "Are bid preferences benign? The effect of small business subsidies in highway procurement auctions," *Journal of Public Economics*, August 2007, *91* (7-8), 1591–1624.

Maskin, Eric and John Riley, "Optimal Auctions with Risk Averse Buyers," *Econometrica*, 1984, *52* (6), pp. 1473–1518.

Matthews, Steven A., "A Technical Primer on Auction Theory I: Independent Private Values," Discussion Papers 1096, Northwestern University, Center for Mathematical Studies in Economics and Management Science May 1995.

Matzkin, Rosa L., "Identification in Nonparametric Simultaneous Equations Models," *Econometrica*, 2008, *76* (5), pp. 945–978.

McAdams, David, "Isotone Equilibrium in Games of Incomplete Information," *Econometrica*, 2003, *71* (4), 1191–1214.

McKenzie, Lionel W., "Matrices with Quasi-Dominant Diagonals and Economic Theory," in Kenneth Joseph Arrow, ed., *Mathematical Methods in the Social Sciences*, Stanford University Press, 1960, pp. 47–62.

Milgrom, Paul, *Putting Auction Theory to Work*, Campridge University Press, 2004.

Molenaar, Keith R. and Gerald Yakowenko, *Alternative Project Delivery, Procurement, and Contracting Methods for Highways*, American Society of Civil Engineers, 2007.

Nakabayashi, Jun, "Small business set-asides in procurement auctions: An empirical analysis," *Journal of Public Economics*, 2013, *100* (C), 28–44.

Paarsch, Harry J., "Deciding between the common and private value paradigms in empirical models of auctions," *Journal of Econometrics*, 1992, *51* (1-2), 191–215.

_ **and Han Hong**, *An Introduction to the Structural Econometrics of Auction Data*, Vol. 1 of MIT Press Books, The MIT Press, June 2006.

Reny, Philip J, "On the Existence of Monotone Pure-Strategy Equilibria in Bayesian Games," *Econometrica*, 2011, *79* (2), 499–553.

Takahashi, Hidenori, "Strategic Design under Uncertain Evaluations: Theory and Evidence from Design-Build Auctions," Technical Report 2014.

The Department of Health and Ageing, Australia, "Tender Evaluation Plan," 2011. `http://www.health.gov.au/internet/main/publishing.nsf/Content/205B1A69101B75C3CA257909000720F1/$File/FOI%20264_1011%20doc%2013.pdf`.

Thiel, Stuart E., "Multidimensional auctions," *Economics Letters*, 1988, *28* (1), 37–40.

Wang, Mingxi and Shulin Liu, "Equilibrium bids in practical multi-attribute auctions," *Economics Letters*, 2014, *123* (3), 352–355.

Appendix A Augmentation of Type Space Dimension

For simplicity we focus on the case in which the original type space is one dimensional: $\theta^1 \in \mathbf{\Theta}$. Suppose that $C(q, \theta^1)$ satisfies Assumptions 1 and 2. Then, define an alternative cost function:

$$C^+(q, \boldsymbol{\theta}) = C(q, \theta^1) + M_1\theta^1 + \theta^0,$$

where $\boldsymbol{\theta} = (\theta^0, \theta^1)$ and

$$M_1 = \max_{s,\theta^1} \left| -\frac{u(s,\theta^1)}{u_s(s,\theta^1)} C_{q\theta^1}(q(s,\theta^1),\theta^1) q_s(s,\theta^1) \right|.$$

Note that for all q and θ^1, we have

$$\begin{aligned} C(q,\theta^1) &= C^+(q,\boldsymbol{\theta}), \\ C_q(q,\theta^1) &= C_q^+(q,\boldsymbol{\theta}), \\ C_{qq}(q,\theta^1) &= C_{qq}^+(q,\boldsymbol{\theta}) \end{aligned}$$

if $\theta^0 + M_1\theta^1 = 0$. Moreover, $C^+(\cdot)$ satisfies Assumption 1 because we have

$$\begin{aligned} C^+_{\theta^1}(q,\boldsymbol{\theta}) &= C_{\theta^1}(q,\theta^1) + M_1 > 0, \\ C^+_{q\theta^1}(q,\boldsymbol{\theta}) &= C_{q\theta^1}(q,\theta^1) \geqq 0. \end{aligned}$$

Therefore, if θ^0 is distributed subject to $\theta^0 + M_1\theta^1 = 0$, the model based on cost function $C^+(q,\boldsymbol{\theta})$ is equivalent to the original scoring auction model. We can thus adopt the alternative cost function, $C^+(\cdot)$, to analyze an equilibrium without any loss.

We show that the single-crossing property holds under cost function $C^+(q,\boldsymbol{\theta})$: Define $u^+(s,\boldsymbol{\theta}) := \max_q P(s,q) - C^+(q,\boldsymbol{\theta})$. Note that

$$\begin{aligned} u^+(s,\boldsymbol{\theta}) &= u(s,\theta^1), \\ u^+_s(s,\boldsymbol{\theta}) &= u_s(s,\theta^1)(= P_{sq}(s,\cdot)), \\ q(s,\boldsymbol{\theta}) &= q(s,\theta^1) \end{aligned}$$

if $\theta^0 + M_1\theta^1 = 0$. Therefore, we have

$$\begin{aligned} &-\frac{\partial}{\partial\theta^1}\frac{u^+(s,\boldsymbol{\theta})}{u^+_s(s,\boldsymbol{\theta})} \\ &= \frac{1}{u^+_s(s,\boldsymbol{\theta})}\left[-\frac{u^+(s,\boldsymbol{\theta})}{u^+_s(s,\boldsymbol{\theta})}C^+_{q\theta^1}(q(s,\boldsymbol{\theta}),\boldsymbol{\theta})q_s(s,\boldsymbol{\theta}) + C^+_{\theta^1}(q(s,\boldsymbol{\theta}),\boldsymbol{\theta})\right] \\ &= \frac{1}{u_s(s,\boldsymbol{\theta})}\left[-\frac{u(s,\theta^1)}{u_s(s,\theta^1)}C_{q\theta^1}(q(s,\theta^1),\theta^1)q_s(s,\theta^1) + M_1 + C_{\theta^1}(q(s,\theta^1),\theta^1)\right] \\ &\geqq \frac{1}{u_s(s,\theta^1)}C_{\theta^1}(q(s,\theta^1),\theta^1) \\ &> 0 \end{aligned}$$

if $\theta^0 + M_1\theta^1 = 0$. Note that $q_s(s,\boldsymbol{\theta}) = q_s(s,\theta^1)$ holds because

$$\begin{aligned} q_s(s,\boldsymbol{\theta}) &= P_{sq}(s,q(s,\boldsymbol{\theta}))/(C_{qq}(q(s,\boldsymbol{\theta}),\boldsymbol{\theta}) - P_{qq}(s,q(s,\boldsymbol{\theta}))), \\ &= P_{sq}(s,q(s,\theta^1))/(C_{qq}(q(s,\theta^1),\theta^1) - P_{qq}(s,q(s,\theta^1))), \\ &= q_s(s,\theta^1) \end{aligned}$$

if $\theta^0 + M_1\theta^1 = 0$.

Appendix B Identifiable Cost Functions (Example)

We provide several examples of cost functions that satisfy Assumption 4.

Additively Separable Functions

We first consider an additively separable cost function:

$$C(\mathbf{q}, \boldsymbol{\theta}) \equiv \theta^0 + \sum_{\ell=1}^{L-1} c^\ell(q^\ell, \theta^\ell), \tag{A-1}$$

with $c^\ell_{\theta^\ell}(\cdot) > 0$ for all $k = 0, \ldots, L-1$ and $\ell = 1, \ldots, L-1$. Because $c^0(\theta^0)$ and $c^\ell(q^\ell, \theta^\ell)$ are continuous functions of q^ℓ and θ^ℓ, they attain maximum and minimum values on $\mathcal{Q} \times \boldsymbol{\Theta}$. Define M and m as follows:

$$M = \sum_{\ell=1}^{L-1} \max_{q^\ell, \theta^\ell} |c^\ell_{\theta^\ell}(q^\ell, \theta^\ell)| + 1, \quad m = \min_{\theta^0} |c^0_{\theta^0}(\theta^0)|.$$

Then, this cost function satisfies Assumption 4 with

$$\boldsymbol{\Gamma}^{-1} = \begin{pmatrix} M/m & \mathbf{0} \\ \mathbf{0} & I_{L-1} \end{pmatrix},$$

where I_{L-1} is the identity matrix of size $L-1$.

A Cobb-Douglas form with fixed cost

Consider the following cost function:

$$C(\mathbf{q}, \boldsymbol{\theta}) = \theta^0 + (q^1)^{\theta^1} \cdots (q^{L-1})^{\theta^{L-1}}.$$

In this example, we assume that $q^\ell > 1$ and $\theta^\ell > 0$ for each $\ell \in \{0, ..., L-1\}$.

The determinant of Jacobian matrix of $C(\mathbf{q}, \boldsymbol{\theta})$, $\det J_{\boldsymbol{\theta}}$ is

$$\det J_{\boldsymbol{\theta}} = \det \begin{pmatrix} c^0_{\theta^0}(\theta^0) & \prod_{\ell=1}^{L}(q^\ell)^{\theta^\ell} \ln q^1 & \cdots & \prod_{\ell=1}^{L}(q^\ell)^{\theta^\ell} \ln q^L \\ 0 & (1+\theta^1 \ln q^1) f_1(\mathbf{q}, \boldsymbol{\theta}) & \cdots & \theta^1 \ln q^L f_1(\mathbf{q}, \boldsymbol{\theta}) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \theta^L \ln q^1 f_L(\mathbf{q}, \boldsymbol{\theta}) & \cdots & (1+\theta^L \ln q^L) f_L(\mathbf{q}, \boldsymbol{\theta}) \end{pmatrix}$$
$$\propto \det \begin{pmatrix} 1+\theta^1 \ln q^1 & \theta^1 \ln q^2 & \cdots & \theta^1 \ln q^L \\ \vdots & \ddots & \ddots & \vdots \\ \theta^L \ln q^1 & \cdots & \theta^L \ln q^{L-1} & 1+\theta^L \ln q^L, \end{pmatrix}$$

where $f_k(\mathbf{q}, \boldsymbol{\theta}) = (q^1)^{\theta^1} \cdots (q^{k-1})^{\theta^{k-1}} (q^k)^{\theta^k - 1} (q^{k+1})^{\theta^{k+1}} (q^L)^{\theta^L}$. We show that $\det J_{\boldsymbol{\theta}}$ is non-singular. Consider the following matrix:

$$A \equiv \begin{pmatrix} \frac{1}{\ln q^1} + \theta^1 & \theta^1 & \cdots & \theta^1 \\ \vdots & \ddots & \ddots & \vdots \\ \theta^L & \cdots & \theta^L & \frac{1}{\ln q^L} + \theta^L, \end{pmatrix}.$$

Suppose that $\boldsymbol{a}_1, ..., \boldsymbol{a}_L$ are linearly dependent, where $\boldsymbol{a}_i$ is i-th column vector of A. Then, there exists $(y_1, ..., y_L)' \neq \mathbf{0}$ such that

$$y_1 \boldsymbol{a}_1 + \cdots y_L \boldsymbol{a}_L = \mathbf{0}.$$

Since $\boldsymbol{a}_i = \boldsymbol{\theta} + \frac{1}{\ln q^i} \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the unit vector whose i-th element is one, we have

$$-\boldsymbol{\theta} \sum_{i=1}^{L} y_i = \begin{pmatrix} \frac{y_1}{\ln q^1} \\ \vdots \\ \frac{y_L}{\ln q^L} \end{pmatrix}.$$

Therefore, $y_i = -\theta_i \ln q^i \sum_{i=1}^{L} y_i$ holds for each $i \in \{1, ..., L\}$.[37] Suppose $\sum_{i=1}^{L} y_i < 0$. Then, y_i must be positive for each $i \in \{1, ..., L\}$ since $\ln q^i > 0$ and $\theta^i > 0$. Therefore, $\sum_{i=1}^{L} y_i > 0$. Analogously, suppose $\sum_{i=1}^{L} y_i > 0$. Then, y_i must be negative for each $i \in \{1, ..., L\}$. Therefore, $\sum_{i=1}^{L} y_i < 0$. The assumption of linearly dependence causes a contradiction. Thus, $\boldsymbol{a}_1, ..., \boldsymbol{a}_L$ are linearly independent. In other words, A is non-singular.

[37] Note that $\sum_{i=1}^{L} y_i$ is not zero. If $\sum_{i=1}^{L} y_i = 0$, $y_i = 0$ for each $i \in \{1, ..., L\}$. This result contradicts $(y_1, ..., y_L)' \neq \mathbf{0}$.

Since $\det J_{\boldsymbol{\theta}} = \prod_{l=1}^{L} \frac{1}{\ln q^l} \det A$, $J_{\boldsymbol{\theta}}$ is also non-singular. Note that when the Jacobian matrix of $\tilde{C}(\mathbf{q}, \tilde{\boldsymbol{\theta}})$, Assumption 4 is satisfied. Therefore, the cost function in this example satisfies Assumption 4 as desired.

Appendix C Proof of Proposition 1

Proof. Fix a bidder with type $\boldsymbol{\theta}$ and let $\Psi(s_{(2)})$ be an arbitrary distribution of the lowest rival's score. Then, the bidder's optimal choice in s is given by the solution of

$$\max_{s} \int_{s}^{\infty} u(\tau, \boldsymbol{\theta}) d\Psi(\tau).$$

Note that $u(s, \boldsymbol{\theta})$ is strictly increasing in s and $u(z(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0$. Hence, $u(s, \boldsymbol{\theta}) \lesseqqgtr 0$ for $s \lesseqqgtr z(\boldsymbol{\theta})$, respectively. Therefore, for any s,

$$\int_{z(\boldsymbol{\theta})}^{\infty} u(\tau, \boldsymbol{\theta}) d\Psi(\tau) - \int_{s}^{\infty} u(\tau, \boldsymbol{\theta}) d\Psi(\tau) = \int_{z(\boldsymbol{\theta})}^{s} u(\tau, \boldsymbol{\theta}) d\Psi(\tau) \geqq 0.$$

This shows that bidding $z(\boldsymbol{\theta})$ is optimal for the bidder. Since $\Psi(s_{(2)})$ is arbitrary, bidding $z(\boldsymbol{\theta})$ is a weakly dominant strategy for all types. $\square$

Appendix D Proof of Lemma 1

Proof. Let $\widetilde{\boldsymbol{\Theta}}$ denote the bidder's original type space, and let $\tilde{\boldsymbol{\theta}}$ denote an element of the set. In addition, let $\widetilde{C}(\mathbf{q}, \tilde{\boldsymbol{\theta}})$ denote the bidder cost function that satisfies Assumptions 1 through 3, and let $\tilde{u}(s, \tilde{\boldsymbol{\theta}})$ and $\mathbf{q}(s, \tilde{\boldsymbol{\theta}})$ denote the bidder induced utility and the optimal choice in quality in the second-step optimization, respectively.

Then, $\tilde{u}(s, \tilde{\boldsymbol{\theta}})$ is supermodular if and only if

$$\begin{aligned} & - \frac{\partial}{\partial \tilde{\theta}^k} \frac{\tilde{u}(s, \tilde{\boldsymbol{\theta}})}{\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})} \\ & = \frac{1}{\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})} \left[\frac{\tilde{u}(s, \tilde{\boldsymbol{\theta}})}{\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})} \left(\sum_{\ell=1}^{L-1} -\widetilde{C}_{q^\ell \tilde{\theta}^k}(\mathbf{q}(s, \tilde{\boldsymbol{\theta}}), \tilde{\boldsymbol{\theta}}) q_s^\ell(s, \tilde{\boldsymbol{\theta}}) \right) + \widetilde{C}_{\tilde{\theta}^k}(\mathbf{q}(s, \tilde{\boldsymbol{\theta}}), \tilde{\boldsymbol{\theta}}) \right] > 0. \end{aligned} \quad \text{(A-2)}$$

The inequality does not hold in general, because the sign of the first term inside the square bracket can be negative. This means that the bidder's strategy may not be monotone, i.e., the score chosen by the bidder in the first-step optimization may not rise as its type rises along any dimension of its type space. Nonetheless, we show below that Assumption 3 always gives a way to *align* the type space appropriately so that the bidder's action is monotone in any dimension of its type space.

Let us then define

$$M_k = \max_{s,\tilde{\boldsymbol{\theta}}} \left| \frac{\tilde{u}(s,\tilde{\boldsymbol{\theta}})}{\tilde{u}_s(s,\tilde{\boldsymbol{\theta}})} \left(\sum_{\ell=1}^{L-1} -\widetilde{C}_{q^\ell \tilde{\theta}^k}(\mathbf{q}(s,\tilde{\boldsymbol{\theta}}),\tilde{\boldsymbol{\theta}}) q_s^\ell(s,\tilde{\boldsymbol{\theta}}) \right) \right| \tag{A-3}$$

for all $k = 1, \dots, K-1$ and consider a K times K nonsingular matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & -M_1 & -M_2 & \cdots & -M_{K-1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

If we transform $\tilde{\boldsymbol{\theta}}$ by $\mathbf{M}$, we have

$$\mathbf{M}\tilde{\boldsymbol{\theta}} = \begin{pmatrix} \tilde{\theta}^0 - \sum_{k=1}^{K-1} M_k \tilde{\theta}^k \\ \tilde{\theta}^1 \\ \vdots \\ \tilde{\theta}^{K-1} \end{pmatrix}.$$

Given that $\widetilde{C}(\mathbf{q},\cdot)$ is quasilinear, $\widetilde{C}(\mathbf{q},\mathbf{M}\tilde{\boldsymbol{\theta}})$ is well defined as:

$$\widetilde{C}(\mathbf{q},\mathbf{M}\tilde{\boldsymbol{\theta}}) = \widetilde{C}(\mathbf{q},\tilde{\boldsymbol{\theta}}) - \sum_{k=1}^{K-1} M_k \tilde{\theta}^k. \tag{A-4}$$

Let $\boldsymbol{\theta} = \mathbf{M}\tilde{\boldsymbol{\theta}}$. We then denote by $\boldsymbol{\Theta}$ a new type space defined as

$$\boldsymbol{\Theta} := [\min\theta^0, \max\theta^0] \times \cdots \times [\min\theta^{K-1}, \max\theta^{K-1}], \tag{A-5}$$

Given that $M_k > 0$ for all $k = 1, \dots, K-1$, we have $\min\theta^0 < \min\tilde{\theta}^0$ and $\max\theta^0 = \max\tilde{\theta}^0$. Moreover, we have $[\min\theta^k, \max\theta^k] = [\min\tilde{\theta}^k, \max\tilde{\theta}^k]$ for all $k = 1, \dots, K-1$.

Hence, we observe that

$$\boldsymbol{\Theta} = [\min \theta^0, \max \tilde{\theta}^0] \times [\min \tilde{\theta}^1, \max \tilde{\theta}^1] \times \cdots \times [\min \tilde{\theta}^{K-1}, \max \tilde{\theta}^{K-1}],$$

meaning that $\boldsymbol{\Theta}$ is obtained by stretching $\widetilde{\boldsymbol{\Theta}}$ along its zero dimension downward. Note that $\boldsymbol{\Theta}$ is a superset of $\mathbf{M}\widetilde{\boldsymbol{\Theta}}$. On the other hand, because $\widetilde{C}(\mathbf{q}, \cdot)$ is quasilinear, $\widetilde{C}(\mathbf{q}, \boldsymbol{\theta})$ is well defined for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ (even for any $\boldsymbol{\theta} \in \boldsymbol{\Theta} \backslash \mathbf{M}\widetilde{\boldsymbol{\Theta}}$).

Now, we define a cost function on the new type space as:

$$C(\mathbf{q}, \boldsymbol{\theta}) := \widetilde{C}(\mathbf{q}, \mathbf{M}^{-1}\boldsymbol{\theta}) + \sum_{k=1}^{K-1} M_k \theta^k.$$

Note that, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $C(\mathbf{q}, \boldsymbol{\theta})$ is well defined. Moreover, because $\widetilde{C}(\mathbf{q}, \mathbf{M}\tilde{\boldsymbol{\theta}})$ satisfies (A-4), we have

$$C(\mathbf{q}, \mathbf{M}\tilde{\boldsymbol{\theta}}) = \widetilde{C}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) \tag{A-6}$$

for all $\tilde{\boldsymbol{\theta}} \in \widetilde{\boldsymbol{\Theta}}$.

Then, the Jacobian matrix of $(C(\mathbf{q}, \boldsymbol{\theta}), C_{q^1}(\mathbf{q}, \boldsymbol{\theta}), \ldots, C_{q^{L-1}}(\mathbf{q}, \boldsymbol{\theta}))^{\mathrm{T}}$ with respect to $\boldsymbol{\theta}$ is given by:

$$\begin{aligned}
\mathbf{J}(\boldsymbol{\theta}; \mathbf{q}) &= \widetilde{\mathbf{J}}_{\tilde{\boldsymbol{\theta}}}(\tilde{\boldsymbol{\theta}}; \mathbf{q})\mathbf{M}^{-1}, \\
&= \begin{bmatrix}
1 & M_1 + \widetilde{C}_{\tilde{\theta}^1}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) & \cdots & M_{K-1} + \widetilde{C}_{\tilde{\theta}^{K-1}}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) \\
0 & \widetilde{C}_{q^1\tilde{\theta}^1}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) & \cdots & \widetilde{C}_{q^1\tilde{\theta}^{K-1}}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) \\
\vdots & \vdots & \ddots & \vdots \\
0 & \widetilde{C}_{q^{L-1}\tilde{\theta}^1}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) & \cdots & \widetilde{C}_{q^{L-1}\tilde{\theta}^{K-1}}(\mathbf{q}, \tilde{\boldsymbol{\theta}}).
\end{bmatrix}.
\end{aligned}$$

Let $u(s, \boldsymbol{\theta})$ denote the induced utility defined on the new type space as

$$u(s, \boldsymbol{\theta}) := \max_{\mathbf{q} \in \mathcal{Q}} P(s, \mathbf{q}) - C(\mathbf{q}, \boldsymbol{\theta}).$$

Note that $u(s, \boldsymbol{\theta}) \equiv \tilde{u}(s, \tilde{\boldsymbol{\theta}})$. This implies that $u_s(s, \boldsymbol{\theta}) = u_s(s, \boldsymbol{\theta})$. Moreover, we see that

$$\arg\max_{\mathbf{q}} P(s, \mathbf{q}) - C(\mathbf{q}, \boldsymbol{\theta}) = \arg\max_{\mathbf{q}} P(s, \mathbf{q}) - \widetilde{C}(\mathbf{q}, \tilde{\boldsymbol{\theta}}) \equiv \mathbf{q}(s, \tilde{\boldsymbol{\theta}}).$$

Thus, we denote by $\mathbf{q}(s, \boldsymbol{\theta}) = \mathbf{q}(s, \tilde{\boldsymbol{\theta}})$ the solution to the second-step optimization.

Then, we see that $u(s, \boldsymbol{\theta})$ is log-supermodular:

$$
\begin{aligned}
& -\frac{\partial}{\partial \theta^k} \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} \\
&= \frac{1}{u_s(s, \boldsymbol{\theta})} \left[\frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} \left(\sum_{\ell=1}^{L-1} -C_{q^\ell \theta^k}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) q_s^\ell(s, \boldsymbol{\theta}) \right) + C_{\theta^k}(\mathbf{q}(s, \boldsymbol{\theta}), \boldsymbol{\theta}) \right] \\
&= \frac{1}{\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})} \left[\frac{\tilde{u}(s, \tilde{\boldsymbol{\theta}})}{\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})} \left(\sum_{\ell=1}^{L-1} -\widetilde{C}_{q^\ell \tilde{\theta}^k}(\mathbf{q}(s, \tilde{\boldsymbol{\theta}}), \tilde{\boldsymbol{\theta}}) q_s^\ell(s, \tilde{\boldsymbol{\theta}}) \right) + M_k + \widetilde{C}_{\tilde{\theta}^k}(\mathbf{q}(s, \tilde{\boldsymbol{\theta}}), \tilde{\boldsymbol{\theta}}) \right] \\
&\geqq \frac{\widetilde{C}_{\tilde{\theta}^k}(\mathbf{q}(s, \tilde{\boldsymbol{\theta}}), \tilde{\boldsymbol{\theta}})}{\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})} \\
&> 0
\end{aligned}
$$

for all $k = 1, \ldots, K-1$, and

$$
-\frac{\partial}{\partial \theta^0} \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} = \frac{\widetilde{C}_{\tilde{\theta}^0}(\mathbf{q}(s, \tilde{\boldsymbol{\theta}}), \tilde{\boldsymbol{\theta}})}{\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})} > 0.
$$

□

Appendix E Proof of Lemma 2

Proof. Let $\widetilde{\boldsymbol{\Theta}}$ and $\tilde{\boldsymbol{\theta}}$ denote the original type space and an element in the set, respectively, as in Appendix D. In addition, let $\widetilde{C}(\mathbf{q}, \tilde{\boldsymbol{\theta}})$ denote the bidder cost function that satisfies Assumptions 1 through 3, and let $\tilde{u}(s, \tilde{\boldsymbol{\theta}})$ and $\mathbf{q}(s, \tilde{\boldsymbol{\theta}})$ denote the bidder induced utility and the optimal choice in quality in the second-step optimization, respectively. We then denote by $\widetilde{\boldsymbol{\Theta}}(s, x)$ the set of bidder types for whom the value of $s - \tilde{u}(s, \tilde{\boldsymbol{\theta}})/\tilde{u}_s(s, \tilde{\boldsymbol{\theta}})$ is identical and the cumulative distribution of $s - u(s, \cdot)/u_s(s, \cdot)$ is equal to x:

$$
\widetilde{\boldsymbol{\Theta}}(s, x) = \left\{ \tilde{\boldsymbol{\theta}} \in \widetilde{\boldsymbol{\Theta}} \,\middle|\, \xi\left(s - \frac{u(s, \tilde{\boldsymbol{\theta}})}{u_s(s, \tilde{\boldsymbol{\theta}})}; s \right) = x \right\}, \tag{A-7}
$$

Then, define a type space: $\boldsymbol{\Theta} := [\min \theta^0, \max \tilde{\theta}^0] \times [\min \tilde{\theta}^1, \max \tilde{\theta}^1] \times \cdots \times [\min \theta^{K-1}, \max \tilde{\theta}^{K-1}]$ as in (A-5), where $\min \theta^0 = \min(\tilde{\theta}^0 - \sum_{k=1}^{K-1} M_k \tilde{\theta}^k)$ and M_k is given by (A-3). Then, type space $\boldsymbol{\Theta}$ is a superset of $\widetilde{\boldsymbol{\Theta}}$. It follows that $f(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta} \backslash \widetilde{\boldsymbol{\Theta}}$.

Now we define

$$\mathbf{\Theta}^{+}(x,s)=\left\{\boldsymbol{\theta}\in\mathbf{\Theta}\,\middle|\,\kappa(x,s)=s-\frac{u(s,\boldsymbol{\theta})}{u_s(s,\boldsymbol{\theta})}\right\}. \tag{A-8}$$

The set collects a set of types in $\mathbf{\Theta}$ for whom the value of $s-u(s,\cdot)/u_s(s,\cdot)$ is identical and the cumulative distribution is equal to x, regardless of whether the realization of the type is positive or zero. Hence, $\mathbf{\Theta}^{+}(x,s)$ expands the notion of $\widetilde{\mathbf{\Theta}}(x,s)$ to the new type space, $\mathbf{\Theta}$, defined in (A-5). Specifically, $\mathbf{\Theta}^{+}(x,s)$ includes $\widetilde{\mathbf{\Theta}}(x,s)$ because $\kappa(x,s)\equiv s-u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})$ if $\boldsymbol{\theta}\in\widetilde{\mathbf{\Theta}}(x,s)$. Set $\mathbf{\Theta}^{+}(x,s)$ also includes $\boldsymbol{\theta}\in\mathbf{\Theta}\backslash\widetilde{\mathbf{\Theta}}$ such that $s-u(s,\tilde{\boldsymbol{\theta}})/u_s(s,\tilde{\boldsymbol{\theta}})=\kappa(x,s)$. As seen in Appendix D, $u(s,\cdot)/u_s(s,\cdot)$ is continuous and strictly monotone in $\boldsymbol{\theta}$. Hence, $\mathbf{\Theta}(x,s)$ is nonempty.

We first show that for all $x\in[0,1]$ and for all $s_1,s_2\in\mathcal{S}$,

$$\mathbf{\Theta}^{+}(x,s_1)\cap\mathbf{\Theta}^{+}(x,s_2)$$

is nonempty.

If $s_1=s_2$, the statement is true trivially. To see the case that $s_1\neq s_2$, suppose, by contradiction, that $\mathbf{\Theta}^{+}(x,s_1)\cap\mathbf{\Theta}^{+}(x,s_2)$ is empty if $s_1\neq s_2$. Then, define $L_{\mathbf{\Theta}}(x,s)=\{\boldsymbol{\theta}|\kappa(x,s)\geqq s-u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta}))\}$. That is, $L_{\mathbf{\Theta}}(x,s)$ denotes the lower contour set of $\boldsymbol{\theta}$ such that, for all $\boldsymbol{\theta}\in L_{\mathbf{\Theta}}(x,s)$, $s-u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})$ is less than or equal to $\kappa(x,s)$. Note that the log-supermodularity of $u(s,\boldsymbol{\theta})$ implies that if $\boldsymbol{\theta}$ is in $L_{\mathbf{\Theta}}(x,s)$, so is any $\boldsymbol{\theta}'\leq\boldsymbol{\theta}$. More formally, ($\boldsymbol{\theta}\in L_{\mathbf{\Theta}}(x,s)$ and $\boldsymbol{\theta}'\leq\boldsymbol{\theta}$) $\Rightarrow$ ($\boldsymbol{\theta}'\in L_{\mathbf{\Theta}}(x,s)$) for all s and x.[38]

Note that $(\min\theta^0,\ldots,\min\theta^{K-1})$ is a common element for both $L_{\mathbf{\Theta}}(x,s_1)$ and $L_{\mathbf{\Theta}}(x,s_2)$. Therefore, if $\mathbf{\Theta}^{+}(x,s_1)\cap\mathbf{\Theta}^{+}(x,s_2)$ is empty, then the $L_{\mathbf{\Theta}}(x,s_1)$ is either the strict subset or superset of $L_{\mathbf{\Theta}}(x,s_2)$:

$$\begin{aligned}&\textbf{Case 1:}\ \ L_{\mathbf{\Theta}}(x,s_1)\subsetneqq L_{\mathbf{\Theta}}(x,s_2),\ \text{or}\\&\textbf{Case 2:}\ \ L_{\mathbf{\Theta}}(x,s_1)\supsetneqq L_{\mathbf{\Theta}}(x,s_2),\end{aligned}$$

where $\supsetneqq$ and $\subsetneqq$ denote strict superset and strict subset.

Consider Case 1. Note that sets $L_{\mathbf{\Theta}}(x,s_1)\cap\widetilde{\mathbf{\Theta}}$ and $L_{\mathbf{\Theta}}(x,s_2)\cap\widetilde{\mathbf{\Theta}}$ are not identical. If they are, $\mathbf{\Theta}^{+}(x,s_1)$ and $\mathbf{\Theta}^{+}(x,s_2)$ would have a common element. Hence, $L_{\mathbf{\Theta}}(x,s_1)\cap\widetilde{\mathbf{\Theta}}\subsetneqq$

[38]This means that $\mathbf{\Theta}(x,s)$ is the frontier of $L_{\mathbf{\Theta}}(x,s)$ as being similar in spirit to the production possibility frontier of the production set in the firm theory. The log-supermodularity of $u(s,\boldsymbol{\theta})$ plays the same role as the free-disposal assumption.

$L_{\mathbf{\Theta}}(x, s_2) \cap \widetilde{\mathbf{\Theta}}$. It follows that $(L_{\mathbf{\Theta}}(x, s_2) \backslash L_{\mathbf{\Theta}}(x, s_1)) \cap \widetilde{\mathbf{\Theta}}$ is nonempty. Then, we have

$$
\begin{aligned}
x &= \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_2)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_1)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} + \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_2) \backslash L_{\mathbf{\Theta}}(x, s_1)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= x + \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_2) \backslash L_{\mathbf{\Theta}}(x, s_1)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&> x.
\end{aligned}
$$

The last inequality holds because $(L_{\mathbf{\Theta}}(x, s_2) \backslash L_{\mathbf{\Theta}}(x, s_1)) \cap \widetilde{\mathbf{\Theta}}$ is nonempty and because $f(\boldsymbol{\theta}) > 0$ for all $\boldsymbol{\theta} \in (L_{\mathbf{\Theta}}(x, s_2) \backslash L_{\mathbf{\Theta}}(x, s_1)) \cap \mathbf{\Theta}$. Therefore, we have a contradiction.

Obtaining a contradiction in Case 2 is analogous. In Case 2, $L_{\mathbf{\Theta}}(x, s_2)$ is a strict subset of $L_{\mathbf{\Theta}}(x, s_1)$. Then, we have a contradiction:

$$
\begin{aligned}
x &= \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_1)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_2)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} + \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_1) \backslash L_{\mathbf{\Theta}}(x, s_2)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&= x + \int_{\{\tilde{\boldsymbol{\theta}} \in L_{\mathbf{\Theta}}(x, s_1) \backslash L_{\mathbf{\Theta}}(x, s_2)\}} f(\tilde{\boldsymbol{\theta}}) d\tilde{\boldsymbol{\theta}} \\
&> x.
\end{aligned}
$$

Therefore, for all $x \in [0, 1]$ and for all $s_1, s_2 \in \mathcal{S}$, there exists $\boldsymbol{\theta} \in \mathbf{\Theta}^+(x, s_1) \cap \mathbf{\Theta}^+(x, s_2)$.

Next, we show the differentiability of $\kappa(\cdot)$. Given the previous result, we see that there exists $\boldsymbol{\theta} \in \mathbf{\Theta}^+(x, s) \cap \mathbf{\Theta}^+(x, s + h)$ for some $h \in \mathbb{R}$ such that $s + h \in \mathcal{S}$. Hence, there exists $\boldsymbol{\theta}$ that satisfies

$$
\begin{aligned}
\kappa(x, s) &\equiv s - \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})}, \text{ and} \\
\kappa(x, s + h) &\equiv s + h - \frac{u(s + h, \boldsymbol{\theta})}{u_s(s + h, \boldsymbol{\theta})}
\end{aligned}
$$

simultaneously for some $h \neq 0$. Hence, we have

$$
\frac{\kappa(x, s + h) - \kappa(x, s)}{h} = \frac{[s + h - u(s + h, \boldsymbol{\theta})/u_s(s + h, \boldsymbol{\theta})] - [s - u(s, \boldsymbol{\theta})/u_s(s, \boldsymbol{\theta})]}{h}.
$$

Given that $u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})$ is differentiable with respect to s, the mean-value theorem implies that there exists $s' \in [s, s+h]$ such that

$$\frac{[s+h-u(s+h,\boldsymbol{\theta})/u_s(s+h,\boldsymbol{\theta})]-[s-u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})]}{h} = 1 - \frac{\partial}{\partial s}\frac{u(s',\boldsymbol{\theta})}{u_s(s',\boldsymbol{\theta})}.$$

In the limit of $h \to 0$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}^+(x,s) \cap \boldsymbol{\Theta}^+(x,s+h)$ still exists that satisfies this equality. Therefore, we have

$$\begin{aligned}\frac{d}{ds}\kappa(x,s) &= \lim_{h\to 0}\frac{[s+h-u(s+h,\boldsymbol{\theta})/u_s(s+h,\boldsymbol{\theta})]-[s-u(s,\boldsymbol{\theta})/u_s(s,\boldsymbol{\theta})]}{h} \\ &= 1 - \frac{\partial}{\partial s}\frac{u(s,\boldsymbol{\theta})}{u_s(s,\boldsymbol{\theta})}.\end{aligned}$$

This shows that $\kappa(x,s)$ is differentiable with respect to s for all $x \in [0,1]$ and for all s in the interior of $\mathcal{S}$.

□

Appendix F Existence of a unique solution to (10)

Proof. Rewrite expression (10):

$$\frac{1-G(s)}{(n-1)g(s)} = s - \kappa(G(s),s). \tag{A-9}$$

If $G(s)$ is strictly increasing, $G(s)$ has its inverse. Then, let $y(\cdot)$ denote the inverse of $G(\cdot)$. Then, from (A-9), we obtain an ordinary differential equation:

$$\left\{\; y'(x) = \frac{n-1}{1-x}\zeta(x,y(x)) \text{ with } y(1) = \bar{s}. \right. \tag{A-10}$$

where $\zeta(x,y(x)) = y(x) - \kappa(x,y(x))$.

The reason that the ordinary differential equation has a unique solution is given as follows.

Let $s^r < \bar{s} \equiv z(\bar{\boldsymbol{\theta}})$, and let $x^r = \Pr\{z(\boldsymbol{\theta}) \leqq s^r\}$. Then, consider the following differential equation

$$y'(x) = \frac{n-1}{1-x}\zeta(x,y(x)) \text{ with } y(x^r) = s^r. \tag{A-11}$$

First, the right-hand side is differentiable with respect to s for all $x \in [0, x^r]$. This is because, by Lemma 2, $\kappa(x; s)$ is differentiable with respect to s for all $x \in [0, x^r]$. Second, $\kappa(x; s)$ is bounded for all $x \in [0, x^r]$ and $s \in \mathcal{S}$. These ensure that the right-hand side of (A-11) is Lipschitz continuous with respect to $s \in \mathcal{S}$ for any $x \in [0, x^r]$. Then, the standard argument of the ordinary differential equation applies to see that $y(\cdot)$ is a unique solution to (A-11).

It is easy to see that $y'(x^r) = 0$ because $\kappa(x^r, s^r) = \underline{t}(s^r) = 0$. In addition, $y'(\cdot)$ is strictly positive and bounded for all $x \in [0, x^r]$. It follows that $G(\cdot) = y^{-1}(\cdot)$ satisfies (Guess). Thus, there uniquely exists $G(\cdot)$ that satisfies (Guess) and (A-9).

Note that this argument holds even if s^r is equal to $\bar{s}$, i.e., the least efficient supplier, $\bar{\boldsymbol{\theta}}$ is indifferent between bidding and staying out. In this case, the right-hand side of (A-11) goes to infinity as $x \to 1$ for some s and fails to meet the Lipschitz condition.[39] The differential equation has multiple solutions for three initial values of $y(1)$: $-\infty$, $\bar{s}$, and ∞. If $y(1)$ is negative infinite, then $y(x)$ is decreasing at some x close to 1. Hence, it is not a monotone equilibrium. If $y(1)$ is positive infinite, $y(x)$ is not an equilibrium given that s^r is finite. □

Appendix G An Analysis on the Expected Score Ranking

Observe the second derivative of $u(s, \boldsymbol{\theta})$ with respect to s as

$$u_{ss}(s, \boldsymbol{\theta}) = P_{ss}(s, \mathbf{q}) + \sum_{\ell=1}^{L-1} P_{sq^\ell}(s, \mathbf{q}(s, \boldsymbol{\theta})) q_s^\ell(s, \boldsymbol{\theta}). \tag{A-12}$$

The first term is zero if score is linear in price.[40] The term is positive or negative if the marginal contribution of price to score is increasing or decreasing. Thus, the term is considered to capture cardinality of score. Suppose, for instance, that a buyer favors a substantial quality improvement over deep price discounts. If the buyer uses a scoring function with $S_{pp} > 0$ (and $S_{pq^\ell} < 0$), this term is negative. If a buyer transforms a scoring function with a nonlinear monotone function, the sign of this term may flip.

A noticeable feature of the scoring auction is that the sign of the second term is always nonnegative. More specifically, if the second-step maximization is interior, the term is

[39] The argument follows Matthews (1995).

[40] The primitive expression of this term is: $P_{ss}(s, \mathbf{q}) = -S_{pp}(P(s, \mathbf{q}), \mathbf{q}) / S_p(P(s, \mathbf{q}), \mathbf{q})^3$. Hence, the term is zero if score is linear in price, i.e., $S_{pp} = 0$.

given by[41]

$$\sum_{\ell=1}^{L-1} P_{sq^\ell}(s,\mathbf{q}(s,\boldsymbol{\theta}))q_s^\ell(s,\boldsymbol{\theta})$$
$$= -\sum_{\ell=1}^{L-1}\sum_{m=1}^{L-1}\left[P_{q^\ell q^m}(s,\mathbf{q}(s,\boldsymbol{\theta})) - C_{q^\ell q^m}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})\right] q_s^m(s,\boldsymbol{\theta})q_s^\ell(s,\boldsymbol{\theta}) \geqq 0.$$

The inequality is due to the strict concavity of $P(s,\mathbf{q}) - C(\mathbf{q},\boldsymbol{\theta})$ in $\mathbf{q}$ in the interior of $\mathcal{Q}$.

The term captures the extent to which the optimal adjustment in **q** contributes to the increase in the induced utility. This term is nonnegative if the bidder can optimize non-price attributes without any restriction. A special case arises in the QL scoring rule (i.e., the scoring rule with $P_{sq^\ell} = 0$) in which the optimal **q** is constant for any choice in s – i.e., $(q_s^1, q_s^2, \ldots, q_s^{L-1})^{\mathrm{T}} = \mathbf{0}$. Therefore, the term is zero if and only if $P_{sq^\ell} = 0$ for all ℓ.[42]. For any scoring rule in which $(q_s^1, q_s^2, \ldots, q_s^{L-1})^{\mathrm{T}} \neq \mathbf{0}$, we see that this term is strictly positive.

We then claim the following proposition, focusing on the scoring rule with $S_{pp} = 0$.

Proposition 4 (Expected Score ranking)**.** *Suppose that the score function is linear in p. Then, the expected score in the FS auction is weakly larger than that in the SS auction. Moreover, the expected scores are identical if and only if the scoring rule is QL.*

Proof. Using (12), we observe the expected score in the FS auction as:

$$\begin{aligned}E[\sigma_{\mathrm{I}}(\boldsymbol{\theta}_{(1)})] &= \int_0^1 \left\{\int_x^1 \frac{(n-1)(1-\tilde{x})^{n-2}}{(1-x)^{n-1}}\kappa(\tilde{x}, y(\tilde{x}))d\tilde{x}\right\}(1-x)^{n-1}dx \\ &= \int_0^1 \int_x^1 (n-1)(1-\tilde{x})^{n-2}\kappa(\tilde{x}, y(\tilde{x}))d\tilde{x}dx\end{aligned}$$

where $y(x) = G^{-1}(x)$.

To examine the expected score in the SS auction, consider the cumulative distribution

[41] Differentiating $P_{q^\ell}(s,\mathbf{q}(s,\boldsymbol{\theta})) - C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) = 0$ with respect to s gives

$$P_{sq^\ell}(s,\mathbf{q}(s,\boldsymbol{\theta})) + \sum_{m=1}^{L-1}\left[P_{q^\ell q^m}(s,\mathbf{q}(s,\boldsymbol{\theta})) - C_{q^\ell q^m}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta})\right] q_s^m(s,\boldsymbol{\theta}) = 0 \qquad \text{(A-13)}$$

for all $\ell = 1, \ldots, L-1$. Multiplying by $q_s^\ell(s,\boldsymbol{\theta})$ and summing over $\ell = 1, \ldots L-1$ gives the expression.

[42] (If part) Obvious. (Only if part) Given that the Hessian matrix of $P(s,\mathbf{q}) - C(\mathbf{q},\boldsymbol{\theta})$ is full-rank, $(q_s^1, q_s^2, \ldots, q_s^{L-1})^{\mathrm{T}}$ that satisfies (A-13) is zero if $(P_{sq^1}, P_{sq^2}, \ldots, P_{sq^{L-1}}) = \mathbf{0}$.

function of $z(\boldsymbol{\theta})$:

$$\psi(z) = \int_{\{\boldsymbol{\theta}|z(\boldsymbol{\theta})\leqq z\}} f(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

Let $x = \psi(z)$. Given that $z(\boldsymbol{\theta})$ is continuous and that $f(\boldsymbol{\theta}) > 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $\psi(z)$ is continuous and strictly increasing. Hence, $\psi^{-1}(x)$ exists. Now, let $z(x) := \psi^{-1}(x)$.

Note that x is uniformly distributed on $[0, 1]$.[43] Then, the expected score in the SS auction is

$$\begin{aligned} E[\sigma_{\text{II}}(\boldsymbol{\theta}_{(2)})] &= \int_0^1 \left\{ \int_x^1 \frac{(n-1)(1-\tilde{x})^{n-2}}{(1-x)^{n-1}} z(\tilde{x})d\tilde{x} \right\} (1-x)^{n-1}dx. \\ &= \int_0^1 \int_x^1 (n-1)(1-\tilde{x})^{n-2} z(\tilde{x})d\tilde{x}dx. \end{aligned}$$

Hence, $E[\sigma_{\text{I}}(\boldsymbol{\theta}_{(1)})] \lesseqqgtr E[\sigma_{\text{II}}(\boldsymbol{\theta}_{(2)})]$ if and only if $\kappa(x, y(x)) \lesseqqgtr z(x)$.

Then, consider the following two sets of types for some $x \in [0, 1]$:

$$\begin{aligned} \boldsymbol{\Theta}(x, y(x)) &= \left\{ \boldsymbol{\theta} \middle| \xi \left(y(x) - \frac{u(y(x), \boldsymbol{\theta})}{u_s(y(x), \boldsymbol{\theta})}; y(x) \right) = x \right\}, \\ \boldsymbol{\Theta}(x, z(x)) &= \left\{ \boldsymbol{\theta} \left| \psi\left(z(x)\right) = x \right. \right\}. \end{aligned}$$

Recall that $z(\boldsymbol{\theta})$ is continuous and strictly increasing (Subsection 4.2). Hence, by the same token as in the proof of Lemma 2, we find a bidder type: $\boldsymbol{\theta} \in \boldsymbol{\Theta}(x, y(x)) \cap \boldsymbol{\Theta}(x, z(x))$.

Given the fact that the second term in expression (A-12) is nonnegative, $u(s, \boldsymbol{\theta})$ is weakly convex in s if $S_{pp}(\cdot) = 0$. Then, we have

$$u(s, \boldsymbol{\theta}) = u(s, \boldsymbol{\theta}) - u(z(\boldsymbol{\theta}), \boldsymbol{\theta}) \leqq u_s(s, \boldsymbol{\theta})(s - z(\boldsymbol{\theta})).$$

Thus, for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}(x, y(x)) \cap \boldsymbol{\Theta}(x, z(x))$, we have

$$\kappa(x, y(x)) \equiv s - \frac{u(s, \boldsymbol{\theta})}{u_s(s, \boldsymbol{\theta})} \geqq z(\boldsymbol{\theta}) \equiv z(x).$$

Hence, the expected score is not less in the FS than in the SS auction. Note that $u_{ss}(\cdot) = 0$ for all s and $\boldsymbol{\theta}$ if and only if $P_{sq}(\cdot) = 0$ for all s and $\mathbf{q}$, i.e., the QL scoring rule. $\square$

[43] Because $z(x)$ is strictly increasing, the cumulative distribution of $z(\cdot)$ at $z(x)$ is equal to x. By construction, $\psi(z(x)) = \psi(\psi^{-1}(x)) = x$. This implies that the cumulative distribution function of x is x.

Appendix H Proof of Lemma 3

Case in which $K = L$

Proof. First, we show that Assumption 4 implies that $A(\boldsymbol{\theta}; \mathbf{q})$ is locally invertible for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\mathbf{q} \in \mathcal{Q}$.

Fix $\mathbf{q}$ in $\mathcal{Q}$. Let $\boldsymbol{\Gamma}$ denote a nonsingular matrix, and let $\widetilde{\boldsymbol{\theta}} := \boldsymbol{\Gamma}\boldsymbol{\theta}$. Let $\mathrm{int}\boldsymbol{\Theta}$ denote the interior of $\boldsymbol{\Theta}$. Then, there exists $\epsilon > 0$ such that $B_\epsilon(\theta) := \{\boldsymbol{\theta}' | d(\boldsymbol{\theta}', \boldsymbol{\theta}) < \epsilon\}$ is in $\mathrm{int}\boldsymbol{\Theta}$. Then, $\boldsymbol{\Gamma} B_\epsilon(\boldsymbol{\theta})$ is also open in $\boldsymbol{\Theta}$. Then, for any $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$, let $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q})$ denote the Jacobian matrix of $A(\boldsymbol{\theta}; \mathbf{q})$ with respect to $\boldsymbol{\theta}$; namely:

$$\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q}) = \begin{bmatrix} C_{\theta^0}(\mathbf{q}, \boldsymbol{\theta}) & C_{\theta^1}(\mathbf{q}, \boldsymbol{\theta}) & \cdots & C_{\theta^{K-1}}(\mathbf{q}, \boldsymbol{\theta}) \\ C_{q^1\theta^0}(\mathbf{q}, \boldsymbol{\theta}) & C_{q^1\theta^1}(\mathbf{q}, \boldsymbol{\theta}) & \cdots & C_{q^1\theta^{K-1}}(\mathbf{q}, \boldsymbol{\theta}) \\ \vdots & \vdots & \ddots & \vdots \\ C_{q^{L-1}\theta^0}(\mathbf{q}, \boldsymbol{\theta}) & C_{q^{L-1}\theta^1}(\mathbf{q}, \boldsymbol{\theta}) & \cdots & C_{q^{L-1}\theta^{K-1}}(\mathbf{q}, \boldsymbol{\theta}) \end{bmatrix}.$$

Then, $\widetilde{C}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) := \widetilde{C}(\mathbf{q}, \boldsymbol{\Gamma}^{-1}\widetilde{\boldsymbol{\theta}})$ is well defined in the neighborhood of $\widetilde{\boldsymbol{\theta}}$.

Now, let $\widetilde{A}(\widetilde{\boldsymbol{\theta}}; \mathbf{q}) := (\widetilde{C}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}), \widetilde{C}_{q^1}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}), \ldots, \widetilde{C}_{q^{L-1}}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}))^{\mathrm{T}}$. Then, the Jacobian matrix of $\widetilde{A}(\widetilde{\boldsymbol{\theta}}; \mathbf{q})$ at $\boldsymbol{\theta}$ is given by

$$\begin{aligned} \widetilde{\mathbf{J}}_{\widetilde{\boldsymbol{\theta}}}(\widetilde{\boldsymbol{\theta}}; \mathbf{q}) &= \mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q})\boldsymbol{\Gamma}^{-1}, \\ &= \begin{bmatrix} \widetilde{C}_{\widetilde{\theta}^0}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) & \widetilde{C}_{\widetilde{\theta}^1}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) & \cdots & \widetilde{C}_{\widetilde{\theta}^{K-1}}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) \\ \widetilde{C}_{q^1\widetilde{\theta}^0}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) & \widetilde{C}_{q^1\widetilde{\theta}^1}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) & \cdots & \widetilde{C}_{q^1\widetilde{\theta}^{K-1}}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{C}_{q^{L-1}\widetilde{\theta}^0}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) & \widetilde{C}_{q^{L-1}\widetilde{\theta}^1}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) & \cdots & \widetilde{C}_{q^{L-1}\widetilde{\theta}^{K-1}}(\mathbf{q}, \widetilde{\boldsymbol{\theta}}) \end{bmatrix}. \end{aligned}$$

Then, $\widetilde{\mathbf{J}}_{\widetilde{\boldsymbol{\theta}}}(\widetilde{\boldsymbol{\theta}}; \mathbf{q})$ is a strictly diagonally dominant matrix by Assumption 4. Therefore, $\widetilde{\mathbf{J}}_{\widetilde{\boldsymbol{\theta}}}(\widetilde{\boldsymbol{\theta}}; \mathbf{q})$ is nonsingular by the Levy-Desplanques theorem. Note that $\boldsymbol{\Gamma}^{-1}$ is nonsingular by definition. Thus, $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q})$ is also nonsingular.

The above argument holds for all $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$ and for all $\mathbf{q} \in \mathcal{Q}$. Therefore, $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q})$ is nonsingular for all $\boldsymbol{\theta} \in \mathrm{int}\boldsymbol{\Theta}$ and for all $\mathbf{q} \in \mathcal{Q}$.

Second, we show that the nonsingularity of $\mathbf{J}_{\boldsymbol{\theta}}$ implies Assumption 4. Suppose that $\mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q})$ is nonsingular for all $\boldsymbol{\theta}$ in the interior of $\boldsymbol{\Theta}$ and $\mathbf{q} \in \mathcal{Q}$. Then, set $\boldsymbol{\Gamma} := \mathbf{J}_{\boldsymbol{\theta}}(\boldsymbol{\theta}; \mathbf{q})$ for all $\mathbf{q}$ and $\boldsymbol{\theta}$ so that we have $\mathbf{J}_{\boldsymbol{\theta}}\boldsymbol{\Gamma}^{-1} = I_L$. Then, Assumption 4 is satisfied.

$\square$

Case in which $K < L$

Proof. As before, we first show that Assumption 4 implies the local invertibility of $A(\cdot)$. Let $\mathbf{D}$ denote an $L \times (L-K)$ matrix such that its ℓth column with $\ell = \{K, \ldots, L-1\}$ is given by

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \sum_{k=0}^{K-1} C_{q^K \theta^k} + \epsilon & 0 & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{K-1} C_{q^{K+1} \theta^k} + \epsilon & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \sum_{k=0}^{K-1} C_{q^{L-1} \theta^k} + \epsilon \end{bmatrix},$$

with some $\epsilon > 0$. Then, for some $\mathbf{z} \in \mathbb{R}^{L-K}$, define $\widehat{A}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q}) := A(\boldsymbol{\theta}; \mathbf{q}) + \mathbf{Dz}$. Now, let $\widehat{\mathbf{J}}_{(\boldsymbol{\theta}, \mathbf{z})}$ denote the Jacobian matrix of $\widehat{A}$. Then, for any $(\boldsymbol{\theta}, \mathbf{0}) \in \text{int}\boldsymbol{\Theta} \times \mathbb{R}^{L-K}$, $\widehat{\mathbf{J}}_{(\boldsymbol{\theta}, \mathbf{z})}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q})$ is a full-rank matrix. Hence, applying the local inverse function theorem, we have the inverse of $\widehat{A}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q})$. Let $\widehat{A}^{-1}(\cdot)$ denote the inverse and T denote an operator that trims $L-K$ elements of an L-dimensional vector from the end. Then, because $\widehat{A}^{-1} := \left(\widehat{A}_0^{-1}, \cdots, \widehat{A}_{K-1}^{-1}, \widehat{A}_K^{-1}, \cdots, \widehat{A}_{L-1}^{-1}\right)$, we have $A^{-1} \equiv (\widehat{A}_0^{-1}, \cdots, \widehat{A}_{K-1}^{-1}) = T\widehat{A}^{-1}(\cdot)$. Note that we have $(\boldsymbol{\theta}, \mathbf{0}) = \widehat{A}^{-1}(A(\boldsymbol{\theta}; \mathbf{q}); \mathbf{q})$. Therefore,

$$\begin{aligned} \boldsymbol{\theta} &= T(\boldsymbol{\theta}, \mathbf{0}) \\ &= T(\widehat{A}^{-1}(A(\boldsymbol{\theta}; \mathbf{q}); \mathbf{q})) \\ &= A^{-1}(A(\boldsymbol{\theta}; \mathbf{q}); \mathbf{q}). \end{aligned}$$

Hence, $A(\cdot)$ is locally invertible for all $\boldsymbol{\theta} \in \Theta$ and for all $\mathbf{q} \in \mathcal{Q}$.

Next, we show the converse. Suppose that $A(\cdot)$ is locally invertible for any $\boldsymbol{\theta}$ and $\mathbf{q}$. Then, from the above argument, $\widehat{A}(\boldsymbol{\theta}, \mathbf{z}; \mathbf{q}) := A(\boldsymbol{\theta}; \mathbf{q}) + \mathbf{Dz}$ has the nonsingular Jacobian matrix for any $(\boldsymbol{\theta}, \mathbf{0})$ and for any $\mathbf{q}$. Let $\widehat{\mathbf{J}}_{(\boldsymbol{\theta}, \mathbf{0})}(\boldsymbol{\theta}, \mathbf{0}; \mathbf{q})$ denote the Jacobian matrix. Then, set $\boldsymbol{\Gamma} = \widehat{\mathbf{J}}$ so that we have $\widehat{\mathbf{J}}\boldsymbol{\Gamma}^{-1} = I_L$. Then, Assumption 4 is satisfied. $\square$

Appendix I Proof of Proposition 3

Proof. By the global inverse function, $A(\boldsymbol{\theta};\mathbf{q})$ is globally invertible if, for all $\mathbf{q}\in\mathcal{Q}$,

1. $A(\boldsymbol{\theta};\mathbf{q})$ is locally invertible and its inverse function is continuous;

2. $A(\boldsymbol{\theta};\mathbf{q})$ is proper;

3. $\boldsymbol{\Theta}$ is arcwise connected, and the image of $A(\boldsymbol{\theta},\mathbf{q})$ is simply connected.

In Lemma 3, we have shown that $A(\boldsymbol{\theta};\mathbf{q})$ is locally invertible for all $\boldsymbol{\theta}\in\mathrm{int}\boldsymbol{\Theta}$ (where $\mathrm{int}\boldsymbol{\Theta}$ denotes the interior of $\boldsymbol{\Theta}$) and $\mathbf{q}\in\mathcal{Q}$. We now show that $A(\boldsymbol{\theta};\mathbf{q})$ is locally invertible for any $\boldsymbol{\theta}\in\partial\boldsymbol{\Theta}$. (where $\partial\boldsymbol{\Theta}:=\boldsymbol{\Theta}\setminus\mathrm{int}\boldsymbol{\Theta}$, the boundary of $\boldsymbol{\Theta}$.) Let $B_\epsilon(\tilde{\boldsymbol{\theta}}):=\{\boldsymbol{\theta}|d(\tilde{\boldsymbol{\theta}},\boldsymbol{\theta})<\epsilon\}$ denote the ϵ-neighborhood of $\tilde{\boldsymbol{\theta}}$. Suppose that $A(\tilde{\boldsymbol{\theta}};\mathbf{q})$ is not locally injective at $\tilde{\boldsymbol{\theta}}\in\partial\boldsymbol{\Theta}$. Then, for a fixed $\epsilon>0$, there exists $\boldsymbol{\theta}\in B_\epsilon(\tilde{\boldsymbol{\theta}})\cap\boldsymbol{\Theta}$ such that $A(\tilde{\boldsymbol{\theta}};\mathbf{q})=A(\boldsymbol{\theta};\mathbf{q})$. Let $\delta:=d(\tilde{\boldsymbol{\theta}},\boldsymbol{\theta})$. Then, for a fixed $\epsilon'\in(0,\delta)$, there exists $\boldsymbol{\theta}'\in B_{\epsilon'}(\tilde{\boldsymbol{\theta}})\cap\boldsymbol{\Theta}$ such that $A(\tilde{\boldsymbol{\theta}};\mathbf{q})=A(\boldsymbol{\theta}';\mathbf{q})$. Since $\boldsymbol{\theta}\notin B_{\epsilon'}(\tilde{\boldsymbol{\theta}})$, we have $\boldsymbol{\theta}\neq\boldsymbol{\theta}'$ and $A(\boldsymbol{\theta};\mathbf{q})=A(\boldsymbol{\theta}';\mathbf{q})$. However, by Assumption 4, $A(\boldsymbol{\theta};\mathbf{q})$ is injective for all $\boldsymbol{\theta}\in\mathrm{int}\boldsymbol{\Theta}$. Thus, there exists $\epsilon>0$, such that $A(\tilde{\boldsymbol{\theta}};\mathbf{q})\neq A(\boldsymbol{\theta};\mathbf{q})$ for all $\boldsymbol{\theta}\in B_\epsilon(\tilde{\boldsymbol{\theta}})\cap\boldsymbol{\Theta}$. Then, $A(\cdot;\cdot)$ is locally invertible with respect to $\boldsymbol{\theta}$ at $\tilde{\boldsymbol{\theta}}\in\partial\boldsymbol{\Theta}$ with the inverse $A^{-1}(\cdot;\cdot):A(B_\epsilon(\tilde{\boldsymbol{\theta}})\cap\boldsymbol{\Theta};\mathbf{q})\to B_\epsilon(\tilde{\boldsymbol{\theta}})\cap\boldsymbol{\Theta}$, where $A(B_\epsilon(\tilde{\boldsymbol{\theta}})\cap\boldsymbol{\Theta};\mathbf{q}):=\{A(\boldsymbol{\theta};\mathbf{q})|\boldsymbol{\theta}\in B_\epsilon(\tilde{\boldsymbol{\theta}})\cap\boldsymbol{\Theta}\}$.

Next, we show that $A^{-1}(\cdot)$ is continuous at any boundary points $\tilde{\boldsymbol{\theta}}\in\partial\boldsymbol{\Theta}$. Take a sufficiently small $\epsilon>0$. Let $\bar{\delta}:=\sup_{d(\tilde{\boldsymbol{\theta}},\boldsymbol{\theta})<\epsilon}d(A(\tilde{\boldsymbol{\theta}};\mathbf{q}),A(\boldsymbol{\theta};\mathbf{q}))$. Then, since $A(\cdot)$ is continuous at $\tilde{\boldsymbol{\theta}}\in\partial\boldsymbol{\Theta}$, for any $\delta\in(0,\bar{\delta})$, there is an $\epsilon'\in(0,\epsilon)$ such that if $d(\tilde{\boldsymbol{\theta}},\boldsymbol{\theta})<\epsilon'$, we have $d(A(\tilde{\boldsymbol{\theta}};\mathbf{q}),A(\boldsymbol{\theta};\mathbf{q}))<\delta$. Furthermore, by the definition of δ, if $d(A(\tilde{\boldsymbol{\theta}};\mathbf{q}),A(\boldsymbol{\theta};\mathbf{q}))<\delta$, then $d(\tilde{\boldsymbol{\theta}},\boldsymbol{\theta})<\epsilon$ holds. Therefore, if $d(A(\tilde{\boldsymbol{\theta}};\mathbf{q}),A(\boldsymbol{\theta};\mathbf{q}))<\delta$, we have

$$\begin{aligned} d(A^{-1}(A(\tilde{\boldsymbol{\theta}})),A^{-1}(A(\boldsymbol{\theta}))) &= d(\tilde{\boldsymbol{\theta}},\boldsymbol{\theta}) \quad \text{(since } A(\cdot,\cdot) \text{ is locally invertible)} \\ &< \epsilon. \end{aligned}$$

Note that $A(\cdot,\cdot)$ is bijective with respect to $\boldsymbol{\theta}$ in a neighborhood of $\tilde{\boldsymbol{\theta}}$. Therefore, for any point $\mathbf{b}$ in a neighborhood of $A(\tilde{\boldsymbol{\theta}};\mathbf{q})$, there exists $\boldsymbol{\theta}$ such that $\mathbf{b}=A(\boldsymbol{\theta};\mathbf{q})$. Thus, $A^{-1}(\cdot,\cdot)$ is continuous, as required.

Second, we show that $A(\boldsymbol{\theta};\mathbf{q})$ is a proper map for all $\mathbf{q}\in\mathcal{Q}$. That is, we show that for any compact subset $Y\in\{A(\boldsymbol{\theta};\mathbf{q})|\boldsymbol{\theta}\in\boldsymbol{\Theta})\}$, the inverse image of Y, $A^{-1}(Y;\mathbf{q}):=\{\boldsymbol{\theta}\in\boldsymbol{\Theta}|A(\boldsymbol{\theta};\mathbf{q})\in Y\}$, is also compact. Since $A(\boldsymbol{\theta};\mathbf{q})$ is continuous, $A(Y;\mathbf{q})$ is also closed for any closed set Y. Furthermore, $\boldsymbol{\Theta}$ is bounded. Therefore, by the definition of inverse image, $A^{-1}(Y;\mathbf{q})$ is a subset of $\boldsymbol{\theta}$. Therefore, $A^{-1}(Y;\mathbf{q})$ is bounded for all $\mathbf{q} \in \mathcal{Q}$. Thus, $A(\boldsymbol{\theta};\mathbf{q})$ is a proper map.

Finally, we show that i) domain $\boldsymbol{\Theta}$ is arcwise connected and that ii) image $A(\boldsymbol{\Theta};\mathbf{q}) := \{A(\boldsymbol{\theta};\mathbf{q}|\boldsymbol{\theta} \in \boldsymbol{\Theta})\}$ is simply connected. We show i) first. In our model, $\boldsymbol{\Theta}$ is a Cartesian product of simply connected interval $[\underline{\theta}^k, \bar{\theta}^k]$ for all $k = 0, \dots, K-1$. Thus, $\boldsymbol{\Theta}$ is obviously arcwise connected. Next, we show ii). Since $C(\mathbf{q},\boldsymbol{\theta})$ and $C_{q^\ell}(\mathbf{q},\boldsymbol{\theta})$ are continuous, images $C(\mathbf{q},\boldsymbol{\Theta}) := \{C(\mathbf{q},\boldsymbol{\theta}|\boldsymbol{\theta} \in \boldsymbol{\Theta})\}$ and $C_{q^\ell}(\mathbf{q},\boldsymbol{\Theta}) := \{C_{q^\ell}(\mathbf{q},\boldsymbol{\theta}|\boldsymbol{\theta} \in \boldsymbol{\Theta})\}$ are simply connected for $\ell = 1, \dots, L-1$. Thus, image $A(\boldsymbol{\Theta};\mathbf{q})$ is also simply connected. $\square$

Appendix J A Test for the Cost Function

First, we define exogenous variation in the number of bidders:

Definition 1 (Athey and Haile (2007))**.** *A bidding environment has exogenous variation in the number of bidders if, for all n', n'' such that $n' < n'' \leqq n$, $F(\cdot;n')$ is identical to $F(;n'')$.*

Then, consider the case in which the econometrician seeks to estimate $\boldsymbol{\theta}$ by using a cost function, $\widehat{C}(\mathbf{q},\boldsymbol{\theta})$, that differs from the true cost function – i.e., $\widehat{C}(\mathbf{q},\boldsymbol{\theta}) \neq C(\mathbf{q},\boldsymbol{\theta})$ for some $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\mathbf{q}$. Then, let $\mathbf{b}^*(\boldsymbol{\theta},n) = \{p^*(\boldsymbol{\theta},n), \hat{G}^*(\boldsymbol{\theta},n), \hat{g}^*(\boldsymbol{\theta},n)\}$ and $\mathbf{q}^*(\boldsymbol{\theta},n)$ denote observations implied by the bidder with type $\boldsymbol{\theta}$, given that the number of bidders in the auction is n. Let $\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta},\widehat{C},n)$ denote the estimate. Then, the following two estimates:

$$\begin{aligned}\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta},\widehat{C},n') &\equiv A^{-1}(\mathbf{b}^*(\boldsymbol{\theta},n');\mathbf{q}^*(\boldsymbol{\theta},n'),\widehat{C}) \text{ and}\\ \widehat{\boldsymbol{\theta}}(\boldsymbol{\theta},\widehat{C},n'') &\equiv A^{-1}(\mathbf{b}^*(\boldsymbol{\theta},n'');\mathbf{q}^*(\boldsymbol{\theta},n''),\widehat{C})\end{aligned}$$

generally differ for some or all $\boldsymbol{\theta}$. Then, values of $\widehat{F}(\boldsymbol{\theta})$ – i.e., the distribution of $\widehat{\boldsymbol{\theta}}$ – generally differ depending on n, which could give a testable implication because the true distribution of $\boldsymbol{\theta}$ is identical for all n.

In the following, we show that the test does not function if the scoring rule is QL. The observation of bidder type $\boldsymbol{\theta}$ in the scoring auction with n bidders implies that

$$C(\mathbf{q}^*(\boldsymbol{\theta}),\boldsymbol{\theta}) = p^*(\boldsymbol{\theta},n) - \frac{1-\hat{G}^*(\boldsymbol{\theta},n)}{(n-1)\hat{g}^*(\boldsymbol{\theta},n)}, \tag{A-14}$$

$$C_{q^\ell}(\mathbf{q}^*(\boldsymbol{\theta}),\boldsymbol{\theta}) = P_{q^\ell}(\mathbf{q}^*(\boldsymbol{\theta})) \text{ with } \ell = 1,\dots,L-1. \tag{A-15}$$

We write $\mathbf{q}^*(\boldsymbol{\theta})$ instead of $\mathbf{q}^*(\boldsymbol{\theta}, n)$ because $\mathbf{q}^*(\cdot)$ is identical for all n under the QL scoring rule. Given that $C(\mathbf{q}^*(\boldsymbol{\theta}), \boldsymbol{\theta})$ and $C_{q^\ell}(\mathbf{q}^*(\boldsymbol{\theta}), \boldsymbol{\theta})$ with $\ell = 1, \ldots, L-1$ are all identical for any n, the right-hand side in expression (A-14) is constant for any n. Then, $\boldsymbol{\theta}$ is recovered as:

$$\hat{\boldsymbol{\theta}}(\boldsymbol{\theta}, \widehat{C}) = A^{-1}(\mathbf{b}^*(\boldsymbol{\theta}); \mathbf{q}^*(\boldsymbol{\theta}), \widehat{C}), \tag{A-16}$$

for all $\widehat{C}(\cdot)$ that satisfies Assumptions 1 through 4. It is easy to see that $\widehat{F}(\cdot; n') = \widehat{F}(\cdot; n'')$. This implies that the scoring auction model does not give a refutable restriction on observations under the exogenous variation in the number of bidders.

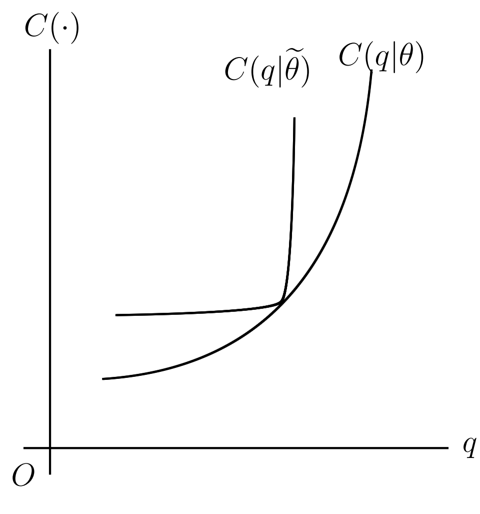

Figure 1: Case in which Assumption 4 does not hold.

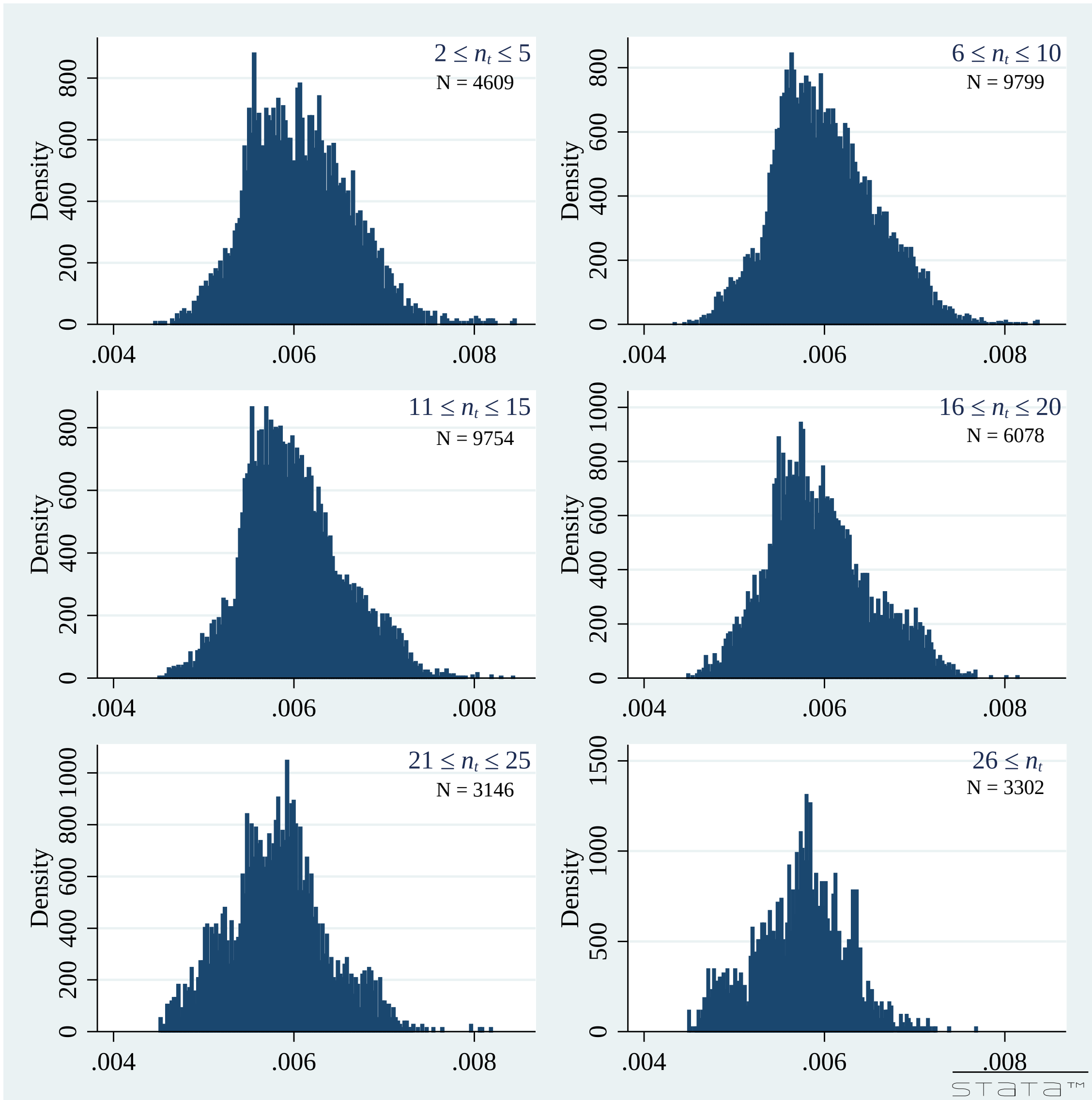

Figure 2: Distribution of Normalized Score for the set of auctions with the number of bidders ranging from 2 through 5 and from 6 through 10 (top row), from 11 through 15 and from 16 through 20 (middle row), and from 21 through 25 and 26 or greater (bottom row).

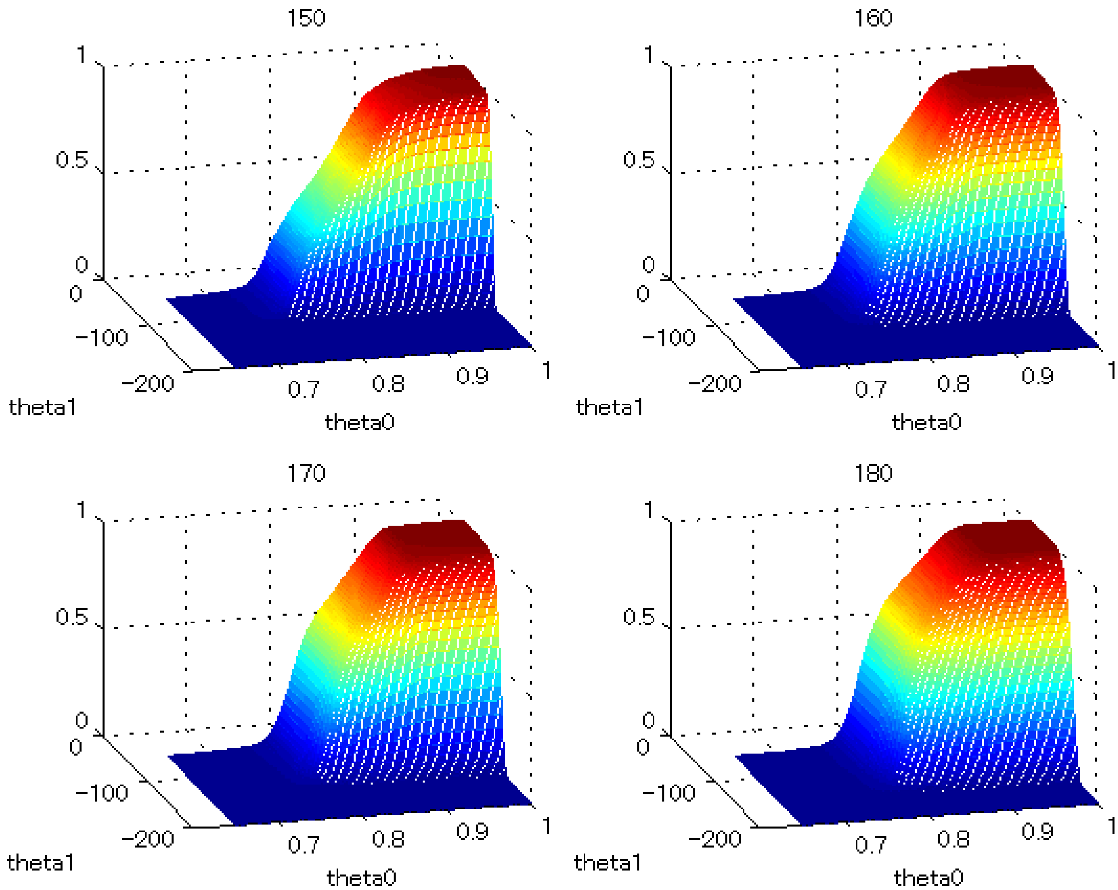

Figure 3: Estimated cdf of $\boldsymbol{\theta}$. Each panel corresponds to the scoring auctions with the quality upper bound equal to 150 (top left), 160 (top right), 170 (bottom left), and 180 (bottom right). The Gaussian kernel is used. The bandwidths for θ^0 and θ^1 for the quality upper bound: 160 are, e.g., .0022 and 0.7176, respectively.

Variable*[1]	Obs	Mean	SD	Min	Max
Number of Bidders	5,142	9.88	6.39	2	34
Engineers' Estimates*[2]	5,142	477.0	1,100.0	200.0	37,600
Win Price Bids*[2]	5,142	423.0	972.0	169.0	34,300
Win Quality-Bid Points	5,142	158.17	11.34	132.60	200.00
Win Scores	5,142	177.23	15.088	109.39	310.12
Price bids*[2]	36,688	531.0	984.0	160.0	37,100
Quality-Bid Points	36,688	153.19	11.11	101.50	200.00
Scores	36,688	180.19	15.315	109.39	310.12

*[1]The top five rows are the statistics for each auction; the bottom three rows are the statistics for each bid. *[2]Units are Yen million.

Table 1: Sample statistics

Quality-Bid Upper bound	Obs	Mean*1	SD*1	Min*1	Max*1
150	1,182	290.0	174.0	200.0	4,470.0
160	2,124	339.0	412.0	200.0	5,950.0
170	1,114	504.0	1,050.0	200.0	12,200
180	495	666.0	1,280.0	200.0	12,400
190	220	1,990.0	2,830.0	207.0	28,300
200	7	8,110.0	13,400	397.0	37,600
Total	5,142	477.0	1,100.0	200.0	37,600

*1Numbers represent the statistics regarding the engineers' estimates. Units are Yen million.

Table 2: Project sizes (by Quality-Bid Upper bound)

Form	$C(q, \boldsymbol{\theta})$	$\bar{q}$	Obs	Mean	SD	Min	Max	Change[*3]	
FS[*1]	-	-	5,142	177.2	15.09	109.4	310.1	-	
		130	5,142	163.7	14.94	98.21	580.4	-7.66%	(0.53%)
		140	5,142	171.3	16.58	95.53	589.0	-3.34%	(0.39%)
	Quadratic	150	5,142	175.3	18.58	91.86	631.1	-1.10%	(0.25%)
		160	5,142	175.5	20.65	87.89	640	-0.96%	(0.18%)
		170	5,142	172.5	22.22	83.83	576.5	-2.69%	(0.24%)
		130	5,142	163.2	15.15	96.21	572.7	-7.94%	(0.50%)
		140	5,142	170.9	16.71	90.26	605.9	-3.55%	(0.38%)
SP[*2]	Cubic	150	5,142	175.0	18.79	83.19	637.9	-1.24%	(0.24%)
		160	5,142	174.7	21.38	75.73	638	-1.44%	(0.37%)
		170	5,142	169.8	23.97	68.36	575.7	-4.20%	(0.53%)
		130	5,142	161.9	14.87	93.79	561.9	-8.68%	(0.51%)
		140	5,142	170.1	16.44	84.88	593.7	-4.00%	(0.40%)
	Quartic	150	5,142	174.5	18.74	74.47	626.1	-1.51%	(0.26%)
		160	5,142	173.7	22.07	63.90	636	-2.00%	(0.28%)
		170	5,142	166.9	25.89	54.08	574.8	-5.82%	(0.51%)

[*1]Observed FS auctions. [*2]Counterfactual second-price auctions. [*3]Change in mean from FS to SP auction; numbers in parentheses are standard deviations generated by bootstrapping samples. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 3: Buyer's Utilities (Price-only vs FS Auctions)

Form	$C(q, \boldsymbol{\theta})$	$\bar{q}$	Obs	Mean	SD	Min	Max	Change[*3]	
FS[*1]	-	-	5,142	0.059	0.075	0.000	0.762	-	
		130	5,142	0.040	0.067	0.000	0.674	-30.82%	(1.50%)
		140	5,142	0.046	0.071	0.000	0.716	-21.37%	(1.46%)
	Quadratic	150	5,142	0.053	0.076	0.000	0.745	-10.05%	(1.39%)
		160	5,142	0.060	0.083	0.000	0.774	2.58%	(1.38%)
		170	5,142	0.068	0.090	0.000	0.852	15.83%	(1.48%)
		130	5,142	0.041	0.067	0.000	0.685	-29.09%	(1.37%)
		140	5,142	0.046	0.072	0.000	0.722	-20.69%	(1.45%)
SP[*2]	Cubic	150	5,142	0.053	0.079	0.000	0.816	-8.82%	(1.55%)
		160	5,142	0.063	0.089	0.000	1.032	6.92%	(1.71%)
		170	5,142	0.074	0.103	0.000	1.275	26.57%	(2.01%)
		130	5,142	0.041	0.067	0.000	0.684	-29.25%	(1.37%)
		140	5,142	0.046	0.073	0.000	0.724	-21.15%	(1.54%)
	Quartic	150	5,142	0.054	0.082	0.000	1.027	-8.13%	(1.81%)
		160	5,142	0.065	0.098	0.000	1.413	11.52%	(2.23%)
		170	5,142	0.082	0.122	0.000	1.893	39.60%	(2.89%)

[*1]Observed FS auctions. [*2]Counterfactual second-price auctions. [*3]Change in mean from FS to SP auction; numbers in parentheses are standard deviations generated by bootstrapping samples. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 4: Bidders' Payoffs (Price-only vs FS auctions)

Form	$C(q, \boldsymbol{\theta})$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	177.23	15.088	109.39	310.12	-	
	Quadratic	5,142	178.51	21.292	98.26	646.5	0.72%	(0.12%)
SS[*2]	Cubic	5,142	178.50	21.232	98.26	644.9	0.71%	(0.12%)
	Quartic	5,142	178.48	21.135	98.26	636.8	0.71%	(0.12%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the PQR rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 5: Buyer's utilities (FS vs SS auctions)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS*1	-	5,142	158.17	11.338	132.60	200.00	-	
	Quadratic	5,142	158.09	11.510	130.18	201.81	-0.05%	(0.02%)
SS*2	Cubic	5,142	158.09	11.457	131.93	201.56	-0.06%	(0.01%)
	Quartic	5,142	158.10	11.424	132.09	201.29	-0.05%	(0.01%)

*1Observed FS auctions (PQR). *2Hypothetical SS auctions with the PQR rule. *Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 6: Contracted Quality Levels (FS vs SS auctions)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	0.0585	0.0752	0.0000	0.7616	-	
	Quadratic	5,142	0.0572	0.0776	0.0000	0.8251	-2.32%	(1.09%)
SS[*2]	Cubic	5,142	0.0571	0.0775	0.0000	0.8248	-2.34%	(1.09%)
	Quartic	5,142	0.0571	0.0775	0.0000	0.8246	-2.34%	(1.09%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the PQR rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 7: Bidder's Payoffs (FS vs SS auctions)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	177.23	15.088	109.39	310.12	-	
	Quadratic	5,142	178.46	20.041	103.25	547.73	0.69%	(0.11%)
SS[*2]	Cubic	5,142	178.46	20.094	102.79	549.31	0.69%	(0.11%)
	Quartic	5,142	178.47	20.132	102.67	550.68	0.70%	(0.11%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the QL rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 8: Buyer's Utilities (QL vs PQR Scoring Rules)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS*1	-	5,142	0.0585	0.0752	0.0000	0.7616	-	
	Quadratic	5,142	0.0562	0.0752	0.0000	0.8157	-4.03%	(1.05%)
QL*2	Cubic	5,142	0.0562	0.0753	0.0000	0.8160	-3.89%	(1.05%)
	Quartic	5,142	0.0563	0.0754	0.0000	0.8161	-3.81%	(1.05%)

*1Observed FS auctions (PQR). *2Hypothetical SS auctions with the QL rule. *Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 9: Bidder Payoffs (QL vs PQR Scoring Rules)

Form	$C(q,\theta)$	Obs	Mean	SD	Min	Max	Change	
FS[*1]	-	5,142	158.17	11.338	132.60	200.00	-	
	Quadratic	5,142	158.27	11.484	126.33	200.81	0.06%	(0.04%)
SS[*2]	Cubic	5,142	158.25	11.428	129.24	200.71	0.05%	(0.03%)
	Quartic	5,142	158.23	11.402	129.99	200.59	0.04%	(0.02%)

[*1]Observed FS auctions (PQR). [*2]Hypothetical SS auctions with the QL rule. [*]Sample auctions with the number of bidders equal to or greater than 2; in FS auctions, profits are less than 1, and normalized bids are less than 150% of reservation prices; in simulated SP auctions, profits are less than 1, and price bids are less than 200% of reservation prices. Numbers in parentheses are standard deviations generated by bootstrapping samples.

Table 10: Contracted Quality Levels (QL vs PQR Scoring Rules)

Online Appendix (Not for Publication)

Online Appendix I Proof of the Existence, Uniqueness, and Strict Monotonicity of the Solution to (6)

Proof. We first show that a solution to the maximization problem (6) exists for all bidder types.

Define $\boldsymbol{\theta}^+ := \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} z(\boldsymbol{\theta})$ to denote the least efficient bidder among $\boldsymbol{\Theta}$. To the bidder type, choosing $s < \bar{s}$ brings a strictly negative payoff when winning, because $u(s, \boldsymbol{\theta}^+) < 0$ for all $s < \bar{s}$. On the other hand, choosing $s > \bar{s}$ is weakly dominated by choosing $s = \bar{s}$, because the probability of winning is zero for all $s > \bar{s}$. Therefore, choosing $\bar{s}$ is optimal.

For any other bidder type in $\boldsymbol{\Theta}$, i.e., $\boldsymbol{\theta} \in \boldsymbol{\Theta}\backslash\{\boldsymbol{\theta}^+\}$, we have $z(\boldsymbol{\theta}) < \bar{s}$. Then, the derivative of its objective function, (6), is given by

$$u_s(s, \boldsymbol{\theta})(1 - G(s)) - (n-1)u(s, \boldsymbol{\theta})g(s). \tag{OA-1}$$

We have $\lim_{s\to\bar{s}} 1 - G(s) = 0$. In addition, $\lim_{s\to\bar{s}} u(s, \boldsymbol{\theta}) > 0$, and $\lim_{s\to\bar{s}} g(\bar{s}) > 0$ given (Guess). Therefore, (OA-1) is strictly negative as s approaches $\bar{s}$. On the other hand, if $s = z(\boldsymbol{\theta})$, then $1 - G(s) > 0$, $u(s, \boldsymbol{\theta}) = 0$, and $g(s)$ being bounded given (Guess). Therefore, (OA-1) is strictly positive. Because the bidder objective function is smooth in s, there exists s in $(z(\boldsymbol{\theta}), \bar{s})$ with which (OA-1) vanishes. Therefore, for any bidder type $\boldsymbol{\theta} \in \boldsymbol{\Theta}\backslash\{\boldsymbol{\theta}^+\}$, the solution exists in $(z(\boldsymbol{\theta}), \bar{s})$.

Next, we show that the solution is unique and strictly increasing in $\boldsymbol{\theta}$. As shown above, $\bar{s}$ is the optimal for the least efficient type, $\boldsymbol{\theta}^+$. Hence, the solution is unique if $\boldsymbol{\theta} = \boldsymbol{\theta}^+$.

For bidder type $\boldsymbol{\theta} \in \boldsymbol{\Theta}\backslash\{\boldsymbol{\theta}^+\}$, let s^* denote a solution to the maximization problem. Then, the log-supermodularity of $u(s, \boldsymbol{\theta})$ implies that

$$u_s(s^*, \hat{\boldsymbol{\theta}})(1 - G(s^*)) - (n-1)u(s^*, \hat{\boldsymbol{\theta}})g(s^*) \gtreqless 0 \tag{OA-2}$$

for any $\hat{\boldsymbol{\theta}} \in \boldsymbol{\Theta}$ if and only if $\hat{\boldsymbol{\theta}} \gtreqless \boldsymbol{\theta}$. This suggests that s^* is suboptimal – i.e., too low for all $\hat{\boldsymbol{\theta}} \geq \boldsymbol{\theta}$ and too high for all $\hat{\boldsymbol{\theta}} \leq \boldsymbol{\theta}$.[44] This, in turn, implies that, for all $\hat{\boldsymbol{\theta}} \geq \boldsymbol{\theta}$, the solution is strictly greater than s^* and that, for all $\hat{\boldsymbol{\theta}} \leq \boldsymbol{\theta}$, the solution is strictly smaller than s^*. Because this is true for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}\backslash\{\boldsymbol{\theta}^+\}$, we conclude that the optimal solution to

[44] Here, "$\geq$" and "$\leq$" denote vector inequalities.

(6) is unique and strictly increasing in $\boldsymbol{\theta}$ under (Guess). $\square$

Online Appendix II Proof of $\sum_{\ell=1}^{L-1} C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) q^\ell_{ss}(s,\boldsymbol{\theta}) \geqq 0$

Proof. Establish the following Lagrangian:

$$\mathcal{L}(\mathbf{q},\lambda) = \bar{p} - C(\mathbf{q},\boldsymbol{\theta}) + \lambda(V(\mathbf{q}) - \bar{p} + s).$$

In the interior of $\{(s,\boldsymbol{\theta})\}$ in which the constraint is binding, the Kuhn-Tucker condition gives

$$\nabla_{\mathbf{q}} C(\mathbf{q}(s,\boldsymbol{\theta}),\boldsymbol{\theta}) = \lambda(s,\boldsymbol{\theta}) \nabla_{\mathbf{q}} V(\mathbf{q}(s,\boldsymbol{\theta})), \tag{OA-3}$$

$$V(\mathbf{q}(s,\boldsymbol{\theta})) = \bar{p} - s, \tag{OA-4}$$

$$\lambda(s,\boldsymbol{\theta}) > 0. \tag{OA-5}$$

Equation (OA-4) implies that $V(\mathbf{q}(s,\boldsymbol{\theta}))$ is linear in s. By differentiating $V(\mathbf{q}(s,\boldsymbol{\theta}))$ twice with respect to s, we obtain:

$$\sum_{\ell=1}^{L-1} V_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta})) q^\ell_{ss}(s,\boldsymbol{\theta}) = -\sum_{\ell=1}^{L-1}\sum_{m=1}^{L-1} V_{q^\ell q^m}(\mathbf{q}(s,\boldsymbol{\theta})) q^\ell_s(s,\boldsymbol{\theta}) q^m_s(s,\boldsymbol{\theta}).$$

Because the Hessian of $V(\mathbf{q})$ is negative semidefinite, the right-hand side is nonnegative, and so is the left-hand side for all $\boldsymbol{\theta}$ and s. Applying this and (OA-5) to (OA-3), we have:

$$\sum_{\ell=1}^{L-1} C_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta})) q^\ell_{ss}(s,\boldsymbol{\theta}) = \lambda(s,\boldsymbol{\theta}) \sum_{\ell=1}^{L-1} V_{q^\ell}(\mathbf{q}(s,\boldsymbol{\theta})) q^\ell_{ss}(s,\boldsymbol{\theta}) \geqq 0.$$

$\square$

Online Appendix III Generating counterfactual SS auction samples from the estimated parameters

With $L = 1$, define

$$q^z(\boldsymbol{\theta}) = \mathbf{q}(z(\boldsymbol{\theta}),\boldsymbol{\theta}) \in \mathbb{R}. \tag{OA-6}$$

Under the PQR scoring rule, $q^z(\boldsymbol{\theta}) = \{q | C_q(q, \boldsymbol{\theta})q = C(q, \boldsymbol{\theta})\}$. Hence, solving the following polynomial:

$$\left(q^z + \hat{\theta}^1_{i,t}\right)^\beta - q^z \beta \left(q^z + \hat{\theta}^1_{i,t}\right)^{\beta-1} + \hat{\theta}^0_{i,t} = 0,$$

gives us the estimate of q^z under the PQR scoring rule. Using $\hat{q}^z_{i,t}$, the break-even score, $z(\boldsymbol{\theta}) = C(\hat{q}^z, \boldsymbol{\theta})/\hat{q}^z$, is estimated as

$$\hat{z}(\hat{\boldsymbol{\theta}}_{i,t}) = \frac{1}{\hat{q}^z_{i,t}} \left[\left(\hat{q}^z_{i,t} + \hat{\theta}^1_{i,t}\right)^\beta + \hat{\theta}^0_{i,t}\right].$$

Given that the contract quality (as well as price) matches the second-lowest score $s_{(2)} = z(\boldsymbol{\theta}_{(2)})$, it is obtained as

$$\hat{q}^{post}_{\mathrm{II,t}} = \left(\frac{z(\hat{\boldsymbol{\theta}}_{(2),t})}{\beta}\right)^{\frac{1}{\beta-1}} - \hat{\theta}^1_{(1),t}.$$

The winner's payoff is, thus, given by

$$u(s_{(2),t}, \hat{\boldsymbol{\theta}}_{(1),t}) = \hat{q}^{post}_{\mathrm{II,t}} \cdot s_{(2),t} - \left(\hat{q}^{post}_{\mathrm{II,t}} + \hat{\theta}^1_{i,t}\right)^\beta - \hat{\theta}^0_{i,t}.$$

Online Appendix IV Generating the counterfactual SS auctions with the QL scoring rule

Under the QL rule, the bidder's *pseudotype* is given by $z(\boldsymbol{\theta}) = \min_q C(q, \boldsymbol{\theta}) - \phi(\beta)q$. The minimizer is $q^z(\boldsymbol{\theta}) \equiv q(z(\boldsymbol{\theta}), \boldsymbol{\theta})$ as defined in (OA-6). Because $P_q(s, q) = \phi(\beta)$ for all s and q, q^z is given by

$$\hat{q}^z_{\mathrm{QL},i,t} = \left(\frac{\phi(\beta)}{\beta}\right)^{\frac{1}{\beta-1}} - \hat{\theta}^1_{i,t}. \tag{OA-7}$$

Hence, the bidder's pseudotype is estimated by

$$\hat{z}_{i,t} = \left(\hat{q}^z_{\mathrm{QL},i,t} + \hat{\theta}^1_{i,t}\right)^\beta + \hat{\theta}^0_{i,t} - \phi(\beta)\hat{q}^z_{\mathrm{QL},i,t}. \tag{OA-8}$$

The lowest pseudotype bidder wins and receives the payment $p_{\mathrm{QL}} = z(\boldsymbol{\theta}_{(2)}) + q^z(\boldsymbol{\theta}_{(1)})$ in the SS auction with the QL scoring rule. Thus, both are estimated from observations. The buyer's utility from the contract is then estimated by

$$w_{\mathrm{QL},t} = \hat{p}_{\mathrm{QL},t} / q^{z}_{\mathrm{QL}}(\hat{\boldsymbol{\theta}}_{(1),t}).$$