

Multi-Agent Search with Deadline*

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Abstract

This paper studies a finite-horizon search problem in which two or more players are involved. Players can agree upon a proposed object by a unanimous decision. Otherwise, search continues until the deadline is reached, at which players receive predetermined fixed payoffs. If players can benefit from the object of search only at the deadline, the limit payoffs as the realizations of objects become frequent are efficient but sensitive to the distribution of possible payoff profiles. In this case the limit expected duration of search relative to the length of time before the deadline is more than a half, and approximates one in the limit as the number of involved players goes to infinity. If the benefits are received as soon as they agree, the payoff approximates the Nash bargaining solution in the limit, and an agreement is reached almost immediately.

1 Introduction

This paper studies search problems which involve two features that arise in many real-life situations: The decision to “stop” is made by *multiple individuals*, and there is a *predetermined deadline* at which a decision has to be made, whatever it is. The goal of this paper is to provide a tractable framework to analyze such situations, and investigate the implications of the two assumptions, especially in terms of welfare of the involved players, and the duration of search.

To fix the idea of the kind of situations that we would like to analyze, let us provide a concrete example: A husband and a wife are searching for an apartment in a city that

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they are moving in. The contract with their current apartment terminates at the end of August, they need to find a place by September 1. Since they are not familiar with the city, they ask a broker to pass information of available apartments to them. The availability of new apartments depends on many specific factors, and so the news do not come every single day. Whenever the broker passes an information to the couple, each person (the husband and the wife) expresses whether they are willing to rent it or not. The couple's preferences are not perfectly aligned with respect to the type of apartments they would like to reside, and both players need to agree to rent an apartment. The search ends once an offer is accepted. The market is seller's market, so the couple cannot "hold" an offer and search a better one. If the couple cannot agree on anything by September 1, they will end up homeless.

To analyze this type of situations, we consider an n -player search problem with a deadline. Time is continuous, and according to a Poisson process an "opportunity" arrives. At each opportunity, a payoff profile for all involved players realizes from some set according to an iid distribution, and each player says "accept" or "reject." The search ends if all players accept, otherwise it continues. If no agreement is reached by the deadline, players obtain an a priori specified fixed payoff. Since there is a trivial subgame perfect equilibrium in which all players always reject, we analyze an (appropriately defined) trembling-hand equilibrium of this game in the limit as the Poisson arrivals become frequent. Our main results are the following: If the benefits are received only at the deadline (which corresponds to the situation in which the couple can only rent an apartment only in September), the limit payoffs are efficient but sensitive to the probability distribution of possible payoff profiles. In this case the limit expected duration of search relative to the length of time before the deadline is more than a half, and approximates one in the limit as the number of involved players goes to infinity. If players can benefit from the object of search as soon as they agree (in the example this means the couple can rent the apartment as soon as they sign the contract), the payoff approximates a point in the Nash set (Maschler et al. (1988), Herrero (1989)) which generalizes the Nash bargaining solution (Nash (1950)) to nonconvex domains. They reach an agreement almost immediately in the limit. We further investigate the structure of equilibrium and relate the forms of equilibria in these two situations.

As would be clear at this point, there are two key features in our model: *deadline* and *multiple agents*. These two factors give rise to new challenges for model-making and analysis of search problems, and as a consequence they lead us to obtain new insights to search problems. In particular, these two things *interact* with each other. First, the existence of deadline implies that the problem is *nonstationary*. That is, the problems faced by the agents at different moment of time are different. Nonstationarity often results in intractability, but we overcome this by using a continuous-time specification and Poisson arrivals as described in the preceding paragraph, which turns out to be useful

in recent developments of finite horizon games.¹ Second, one may argue that since each player’s decision at any given opportunity is essentially conditional on the situation where all other agents say “yes,” the problem essentially boils down to a single-player search problem. This argument misses an important key property of our model: It is indeed true that at each given opportunity the decisions by the opponents do not affect a player’s decision. However, the player’s expectation about the opponents’ future decisions affect her decision today, and such future decisions by opponents are in turn affected by the decisions at the further future by involved agents. These two features interact because the two “futures” discussed in the previous sentence are different exactly due to the nonstationarity.

1.1 Literature Review

Let us relate our work with the literature. First, although there is a large body of literature on search problems with a single agent over infinite horizon, there are very few works that diverge from these two assumptions.² Some recent papers in economics discuss infinite-horizon search models in which a group of multiple decision-makers determine when to stop. Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011) consider a search model in which a unanimous agreement is required to accept an alternative, and show that the equilibrium outcome is close to the Nash bargaining solution when players are patient. Despite absence of a deadline, these convergence results to the Nash bargaining solution have a similar flavor to ours when payoffs realize as soon as an agreement is reached. In Section 6.3, we will discuss a common logic behind these convergence. Compte and Jehiel (2010) also analyze the general majority rule to discuss the power of each individual to affect outcomes of search, and the size of the set of limit equilibrium outcomes. Albrecht et al. (2010) consider the general majority rule, and show that cutoffs in their strategies are lower for the decision-makers than for a player in the corresponding single-person search model, and the expected duration of search is shorter if they are more patient. Alpern and Gal (2009), and Alpern et al. (2010) analyze a search model in which a realized object is chosen when one of two decision-makers accepts it, unless one of them cast a veto which can be exercised only finite times in the entire search process.³ Note that all of the above works consider an infinite-horizon framework.⁴

Second, there is an emerging new field on “revision games,” which concerns players’

¹See Ambrus and Lu (2010a), Kamada and Kandori (2009), Kamada and Sugaya (2010a,b), and Calcagno and Lovo (2010).

²See Rogerson et al. (2005) for a survey.

³Recent papers by Moldovanu and Shi (2010), and Bergemann and Välimäki (2011) analyze infinite-horizon search problems where each player receives a private signal in every period.

⁴There is large literature of search models in Operations Research. Fewer works, however, consider multi-person decision problems (See Abdelaziz and Krichen (2007) for a survey). Sakaguchi (1978) proposed a two-player continuous-time infinite-horizon stopping game in which opportunities arrive according to a Poisson process as in our model.

interactions over continuous-time with finite horizon, where opportunities to “revise” actions arise according to a Poisson process (Kamada and Kandori (2009), Kamada and Sugaya (2010a,b), Calcagno and Lovo (2010)). An insight from these works is that when action space is finite (as in our case) the set of equilibria is typically small and the solution can be obtained by (appropriately implemented) backwards induction. Another insight is that a differential equation is useful when characterizing the equilibrium. In our paper we follow and extend these methods to characterize equilibrium.

Third, the multi-agent search problem is similar to the bargaining problem in that both predict what outcome in a prespecified domain is chosen as a consequence of strategic interaction between agents. On the other hand, the search model is distinguished from bargaining in which players have full control of the proposals, as discussed by Compte and Jehiel (2004, 2010). As opposed to the well-known bargaining models in which a player is chosen as a “proposer” and makes an offer to other players, we assume that there is no “proposer” but rather all players are passive. This assumption captures the feature of situations that we would like to analyze. For example neither the husband nor the wife designs and builds their house for themselves, but looks for an apartment which is already built. The distinction between these “active” and “passive” players is also important when we consider the difference between our work and the standard bargaining literature.⁵

There are papers discussing continuous-time bargaining models with finite horizon, in which players have full control of proposals. Ma and Manove (1993) argue continuous-time bargaining with deadline where two players propose alternately, having options to wait with retaining the right of proposal. They show that players reach an agreement near the deadline as the delay of transmission of the proposal shrinks. Ambrus and Lu (2010a) consider a model of coalitional bargaining in a similar context with ours.⁶ They show general uniqueness of the Markov perfect equilibrium, and characterize the core as the limit equilibrium outcomes in convex games.

Perhaps the most related are the papers by Gomes et al. (1999) and Imai and Salonen (2009), who analyze models that have common features with ours, although their models are different in several important aspects. As a consequence, the set of analyses implemented are different, the proof methods are different, and importantly the results are different even with respect to what seem common in these models. Detailed comparison is made in Remark 2 of Section 6.2. Here we describe their works very briefly and point out several important similarity and differences. Gomes et al. (1999) consider a model of n -player discrete-time bargaining with a general coalitional form, in which a player

⁵Cho and Matsui (2011) present another view: A drawn payoff profile in the search process can be considered as an outcome in a (unique) equilibrium in a bargaining game which is not explicitly described in the model. From this viewpoint, every player is “active” although the “activeness” is hidden in the model.

⁶See Ambrus and Lu (2010b) for an application of their model to legislative processes.

is selected as a proposer with equal probability. They introduce a vector field on the boundary of the set of feasible payoff profiles, and identify the limit of the subgame perfect equilibrium payoffs as breakdown probability vanishes and the horizon length tends to infinity (while the duration of a single period is kept fixed). In addition to the modeling assumption and the difference with respect to the limit payoff, their models and ours have different implication about the duration of search: Search ends immediately for any discount rate (including the case with no discounting) in their model, while the duration depends crucially on discount rate in our model. Imai and Salonen (2009) consider a two-player discrete-time bargaining. They axiomatize their bargaining solution defined as the limit equilibrium payoff as the number of periods in a fixed length of horizon tends to infinity first and then the horizon length shrinks to zero or diverges to infinity. The limit as the horizon length shrinks is the Raiffa bargaining solution, while the other limit is the Nash bargaining solution. In contrast to their results, in our search model, the limit payoff profile as realization of allocations becomes frequent is the Nash bargaining solution irrespective of the length of time horizon for any positive discount rate. This is because there is no “last opportunity” in our model.

The paper is organized as follows. Section 2 provides a model. In Section 3 we provide basic results. In particular, we show that trembling-hand equilibrium takes the form of cutoff strategies, by which we mean each player at each moment of time has a “cutoff” of payoffs below which they reject offers and otherwise accept. In Section 4 we consider the case in which discounting is not so much important relative to the frequency of search (the case corresponding to the situation where the couple can rent an apartment only in September), and in Section 5 we consider the opposite case, that is, the case in which discounting is important relative to the frequency of search (the couples can rent an apartment as soon as they sign the contract). Section 6 discusses further topics and Section 7 concludes. All proofs are provided in Appendix.

2 Model

The Basic Setup

There are n players who face a search problem (X, x^d) where $X \subset \mathbb{R}^n$ is the set of possible payoff profiles (which we call “allocations”), and $x^d \in \mathbb{R}^n$ is the disagreement point assumed to be $x^d = (0, \dots, 0) \in \mathbb{R}^n$. Let $N = \{1, \dots, n\}$ be the set of players. A typical player is denoted by i , and the other players are denoted by $-i$.

Players with a common discount rate $\rho > 0$ search within a finite time interval $[-T, 0]$ with $T > 0$, on which opportunities of agreement arrive according to the Poisson process with arrival rate $\lambda > 0$. At each opportunity, the nature draws an object which is characterized by an allocation $x = (x_1, \dots, x_n) \in X$ following an identical and independent

probability measure μ defined on the Borel subsets of X . After allocation x is realized, each player simultaneously responds by either accepting or rejecting x without a lapse of time. Let $B = \{\text{accept}, \text{reject}\}$ be the set of responses in this search process. If all players accept, the search ends, and players receive the corresponding payoffs. We will consider two specifications with respect to the timing that the payoffs realizes. The first is when the payoffs realize only at the deadline, irrespective of the timing of agreement. In this case the payoff profile from agreeing to allocation x is $e^{-\rho T}x$. The second is when the payoffs realize upon agreement. In this case the corresponding payoff profile is $e^{-\rho(T-t)}x$. Notice that the coefficient $e^{-\rho T}x$ in the first case is a strictly positive constant, so the first case is essentially equivalent to the case with $\rho = 0$ in the second case. If at least one of the players rejects the offer, then they continue search. If players reach no agreement before the deadline at time 0, they obtain the disagreement payoffs $x^d = (0, \dots, 0)$.

Assumptions

We make the following mild assumptions about X and μ throughout the paper.

Assumption 1.

- (a) X is a compact subset of \mathbb{R}^n .
- (b) There exists an allocation $(x_1, \dots, x_n) \in X$ with $x_i > 0$ for all $i \in N$.
- (c) For all $x \in X$, any neighborhood of x in X has a positive volume with respect to the Lebesgue measure in \mathbb{R}^n .
- (d) μ is absolutely continuous with respect to the Lebesgue measure in X , and admits a continuous probability density function f whose support is X .
- (e) f is bounded away from zero, i.e., $\min_{x \in X} f(x) > 0$.⁷

Condition (a) is standard and assumed in all the papers we cited in the introduction. Note however that X may not be convex.⁸ Condition (b) is an assumption not to make the problem meaningless because a player should reject any allocation that gives her a nonpositive payoff at any time before the deadline. Condition (c) rules out degenerate domains such as points and line segments. This condition follows from the next condition if X is convex. Conditions (d) and (e) are standard regularity conditions of the probability measure.

⁷The minimum exists because of Assumption 1 (d).

⁸Convexity is assumed in Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011), although Cho and Matsui (2011) discuss an application of their analysis to nonconvex cases.

Histories and Strategies

Let us define strategies in this game. A history at $-t \in [-T, 0]$ consists of

1. a series of time (t^1, \dots, t^k) when there was an opportunity of Poisson arrival before $-t$, where $k \geq 0$ and $-T \leq -t^1 < -t^2 < \dots < -t^k < -t$,
2. allocations x^1, \dots, x^k drawn at opportunities t^1, \dots, t^k respectively,
3. acceptance/rejection decision profiles (b^1, \dots, b^k) , where each decision profile b^l ($l = 1, \dots, k$) is contained in $B^n \setminus \{(\text{accept}, \dots, \text{accept})\}$,
4. allocation $x \in X \cup \{\emptyset\}$ at $-t$ ($x = \emptyset$ if no Poisson opportunity arrives at $-t$).

We denote a history at time $-t$ by $((t^1, x^1, b^1), \dots, (t^k, x^k, b^k), (t, x))$. Let $\tilde{\mathcal{H}}_t$ be the set of all such histories at time $-t$, and $\tilde{\mathcal{H}} = \bigcup_{-t \in [-T, 0]} \tilde{\mathcal{H}}_t$. Let

$$\mathcal{H}_t = \{((t^1, x^1, b^1), \dots, (t^k, x^k, b^k), (t, x)) \in \tilde{\mathcal{H}}_t \mid x \neq \emptyset\}$$

be the history at time $-t$ when players have an opportunity, and $\mathcal{H} = \bigcup_{-t \in [-T, 0]} \mathcal{H}_t$. A (behavioral) *strategy* σ_i of player i is a function from \mathcal{H} to a probability distribution over the set of responses B .⁹ Let Σ_i be the set of all strategies of i , and $\Sigma = \prod_{i \in N} \Sigma_i$. For $\sigma \in \Sigma$, let $u_i(\sigma)$ be the expected payoff when players play σ . A strategy profile $\sigma \in \Sigma$ is a *Nash equilibrium* if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$ for all $\sigma'_i \in \Sigma_i$ and all $i \in N$. Let $u_i(\sigma \mid h)$ be the expected payoff of player i given that a history $h \in \tilde{\mathcal{H}}$ is realized and strategies taken after h is given by σ . A strategy profile $\sigma \in \Sigma$ is a *subgame perfect equilibrium* if $u_i(\sigma_i, \sigma_{-i} \mid h) \geq u_i(\sigma'_i, \sigma_{-i} \mid h)$ for all $\sigma'_i \in \Sigma_i$, $h \in \mathcal{H}$, and all $i \in N$.

Equilibrium Notions

A strategy $\sigma_i \in \Sigma_i$ of player i is a *Markov strategy* if for history $h \in \mathcal{H}_t$ at $-t$, $\sigma_i(h)$ depends only on the time $-t$, and the present allocation x_i^k for player i herself. A strategy profile $\sigma \in \Sigma$ is a *Markov perfect equilibrium* if σ is a subgame perfect equilibrium, and σ_i is a Markov strategy for all $i \in N$. We will later show that players play a Markov perfect equilibrium (except for histories in a zero-measure set) if they follow a trembling-hand equilibrium defined below. For $\varepsilon \in (0, 1/2)$, let Σ^ε be the set of strategy profiles which prescribe probability at least ε for both responses in $\{\text{accept}, \text{reject}\}$ after all histories in \mathcal{H} . A strategy profile $\sigma \in \Sigma$ is a *trembling-hand equilibrium* if there exists a sequence $(\varepsilon^k)_{k=1,2,\dots}$ and a sequence of strategy profiles $(\sigma^k)_{k=1,2,\dots}$ such that $\varepsilon^k > 0$ for all k , $\lim_{k \rightarrow \infty} \varepsilon^k = 0$, $\sigma^k \in \Sigma^{\varepsilon^k}$, σ^k is a Nash equilibrium in the ε^k -constrained game with a

⁹We assume that the set of histories has a measure naturally induced by the Cartesian product of the Lebesgue measures for t 's and x 's, and the counting measures for b 's. A strategy has to be a measurable function with respect to this measure on \mathcal{H} .

restricted set of strategies Σ^{ε^k} for all k , and $\lim_{k \rightarrow \infty} \sigma^k(h) = \sigma(h)$ for all $h \in \tilde{\mathcal{H}}$, i.e. the convergence is pointwise with respect to histories.¹⁰

Some More Terminologies

An allocation $x = (x_1, \dots, x_n) \in X$ is (strictly) *Pareto efficient* in X if there is no allocation $y = (y_1, \dots, y_n) \in X$ such that $y_i \geq x_i$ for all $i \in N$ and $y_j > x_j$ for some $j \in N$. An allocation $x \in X$ is *weakly Pareto efficient* in X if there is no allocation $y \in X$ such that $y_i > x_i$ for all $i \in N$. Let $\hat{X} = \{v \in \mathbb{R}_+^n \mid x \geq v \text{ for some } x \in X\}$ be the nonnegative region of the comprehensive extension of X .

3 Preliminary Results

In this section, we present preliminary results which will become useful in the subsequent sections. We will show that there exists essentially a unique trembling-hand equilibrium, in which every player plays a strategy with a cutoff manner. We will derive an ordinary differential equation that characterizes the cutoff profile in the equilibrium. In addition, we will show a basic invariance: The equilibrium continuation payoff when enlarging the arrival rate is the same as that when lowering the discount rate and stretching the duration from the deadline at the same rate. We hereafter consider the case with discount rate $\rho \geq 0$, because the case when the payoffs realize only at the deadline is equivalent to the case with $\rho = 0$, as discussed in the previous section.

We show that any trembling-hand equilibria yield the same continuation payoff profile after almost all histories at time $-t \in [-T, 0]$. Therefore the trembling-hand equilibrium is essentially unique and Markov.

Proposition 1. *Suppose that σ, σ' are two trembling-hand equilibria. Then $u_i(\sigma \mid h) = u_i(\sigma' \mid h')$ for almost all histories $h, h' \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ and all $i \in N$.*

We make two remarks about Proposition 1. First, we ruled out histories in \mathcal{H}_t , because different realization of payoffs that players accept clearly give rise to different continuation payoffs on the equilibrium path. Second, there exist subgame perfect equilibria in which all players reject any allocations, since they move simultaneously.¹¹ We introduced trembling-hand equilibrium to rule out such trivial equilibria. In an ε -constrained game, a player will optimally accept a favorable allocation for herself, expecting the others to accept it with a small probability.

¹⁰This equilibrium concept is an analogue of extensive-form trembling-hand equilibrium, as opposed to its normal-form counterpart. Although our extensive-form game involves uncountably many nodes and hence the standard definitions of trembling-hand equilibria are not directly applicable, it is for this reason that we call this notion a trembling-hand equilibrium.

¹¹If players respond sequentially, we can show that any subgame perfect equilibrium consists of cutoff strategies. Therefore our results are essentially independent of the timing of responses of players.

A Markov strategy σ_i of player $i \in N$ is a *cutoff strategy* with cutoff $v_i(t) \geq 0$ if player i who is to respond at time $-t$ accepts allocation $x \in X$ whenever $x_i \geq v_i(t)$, and rejects it otherwise. For a cutoff profile $v = (v_1, \dots, v_n) \in \mathbb{R}_+^n$, we denote the “acceptance set” by $A(v) = \{x \in X \mid x_i \geq v_i \text{ for all } i \in N\}$. The following argument will show that there exists a trembling-hand equilibrium that consists of cutoff strategies. To show this, we construct a Markov perfect equilibrium.

Suppose that all players play Markov strategies σ , and there is no Poisson arrival at time $-t \in [-T, 0]$. Then player i has an expected payoff $u_i(\sigma \mid h)$ at $-t$ independent of history $h \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ played before time $-t$. We denote the continuation payoff at time $-t$ by $v_i(t, \sigma) = e^{-\rho(T-t)} u_i(\sigma \mid h)$.

We hereafter fix a cutoff strategy profile σ , and simply denote by $v_i(t)$ the continuation payoff of player i at time $-t$. Let $A(v(t)) \subset X$ be the set of allocations accepted by the cutoff strategies with cutoff profile $v(t) = (v_1(t), \dots, v_n(t))$. We often denote this set by $A(t)$ with a slight abuse of notation. If σ is a Markov perfect equilibrium, $v_i(t)$ must be characterized by the following recursive expression: For $i \in N$,

$$\begin{aligned} v_i(t) &= \int_0^t \left(\int_{X \setminus A(\tau)} v_i(\tau) d\mu + \int_{A(\tau)} x_i d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau \\ &= \int_0^t \left(v_i(\tau) + \int_{A(\tau)} (x_i - v_i(\tau)) d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau. \end{aligned} \quad (1)$$

After time $-t$, players find the first Poisson opportunity at time $-\tau$ with probability density $\lambda e^{-\lambda(t-\tau)}$. If the drawn allocation x falls in $A(\tau)$, they reach agreement with x , or otherwise, they continue search with continuation payoffs $v(\tau)$.

Bellman equality (1) implies that $v_i(t)$ is differentiable in t . Multiplying both sides of (1) by $e^{(\lambda+\rho)t}$ and differentiating both sides yield

$$v_i'(t) = -\rho v_i(t) + \lambda \int_{A(t)} (x_i - v_i(t)) d\mu$$

for $i \in N$. Therefore we obtain the following ordinary differential equation (ODE) of the continuation payoff profile $v(t) = (v_1(t), \dots, v_n(t))$ defined in \hat{X} :

$$v'(t) = -\rho v(t) + \lambda \int_{A(t)} (x - v(t)) d\mu \quad (2)$$

with an initial condition $v(0) = (0, \dots, 0) \in \mathbb{R}^n$. Let us make a couple of observations about ODE (2). Figure 1 describes a typical illustration of the vectors that appear in this ODE for $n = 2$. The shaded area shows the acceptance set $A(t)$, whose barycenter with respect to the probability measure μ is $\int_{A(t)} x d\mu / \mu(A(t))$. Therefore the term $\lambda \int_{A(t)} (x - v(t)) d\mu$ is parallel to the vector from $v(t)$ to the barycenter of $A(t)$, which represents the gain upon agreement. On the other hand, $-\rho v(t)$ captures the loss from discounting.

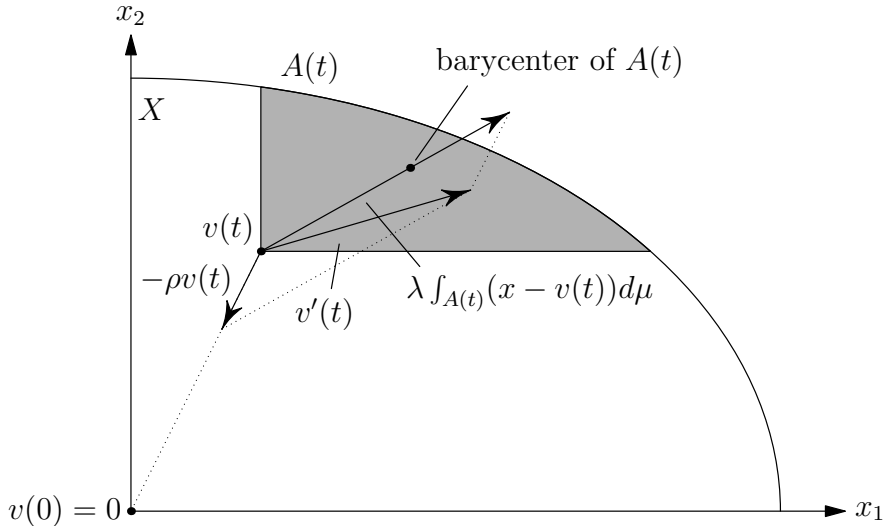


Figure 1: Vectors in ODE (2)

Velocity vector $v'(t)$ is the sum of these two vectors. The absolute value of the integral on the right hand side of (2) is proportional to the weight $\mu(A(t))$. If $\rho = 0$, (2) immediately implies $v'_i(t) \geq 0$ for all t and $i \in N$, and $v'_i(t) = 0$ if and only if $\mu(A(t)) = 0$.

Now we see that a standard argument of ordinary differential equations shows that ODE (2) has a solution whenever Assumption 1 holds.¹² The cutoff strategy profile with a cutoff profile given by this solution of ODE (2) turns out to be a trembling-hand equilibrium, as shown in the following proposition.

Proposition 2. *There exists a trembling-hand equilibrium that consists of (Markov) cut-off strategies.*

By Proposition 1, the solution of ODE (2) is unique. Therefore the game has essentially a unique trembling-hand equilibrium for any given X and μ . Let us denote the unique solution of (2) by $v^*(t; \rho, \lambda)$, the continuation payoff profile in the trembling-hand equilibrium. We simply denote this by $v^*(t)$ as long as there is no room for confusion. To prepare for the main sections in which we investigate the asymptotic behavior of $v^*(t)$ as the arrival rate λ becomes large, we show a useful lemma which is directly derived from the form of ODE (2).

Lemma 3. *For any $\alpha > 0$, $v^*(t; \rho, \alpha\lambda) = v^*(\alpha t; \rho/\alpha, \lambda)$ if $-t, -\alpha t \in [-T, 0]$.*

Next we note that, under certain assumptions, $v^*(t; \rho, \lambda)$ converges to an allocation v^* independent of t in the limit of $\lambda \rightarrow \infty$. This convergence is obvious if $\rho = 0$. Since $v'(t)$ is always nonnegative and \hat{X} is compact, $\lim_{T \rightarrow \infty} v^*(T; 0, \lambda)$ clearly exists. Since $v^*(T; 0, \lambda) = v^*(\lambda T; 0, 1)$ by Lemma 3, $v^* = \lim_{\lambda \rightarrow \infty} v^*(T; 0, \lambda)$ also exists, and is

¹²This is because the right hand side of ODE (2) is continuous in v , and \hat{X} is compact. See Coddington and Levinson (1955, Chapter 1) for a general discussion about ODE.

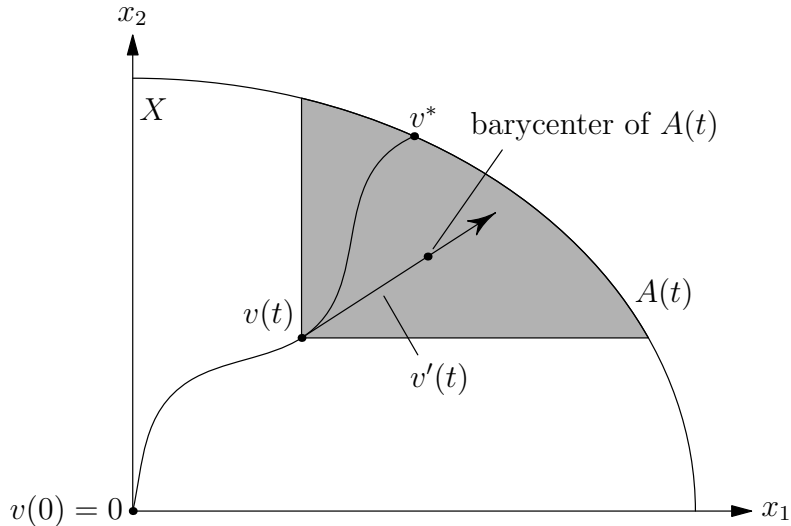


Figure 2: The path and the velocity vector of ODE (3)

independent of T . If $\rho > 0$, the existence of $v^* = \lim_{\lambda \rightarrow \infty} v^*(t)$ is not straightforward since $v'_i(t)$ may be negative. We postpone a proof of the existence of this limit in the case of positive ρ until Proposition 12 in Section 5.

In the following two sections, we analyze the limit of the continuation payoffs in the equilibrium and the expected duration that the search process continues, as the frequency of Poisson arrival goes to infinity. This limit is considered in two cases: the case when the payoffs realize at the deadline (Section 4), and the case when the payoffs realize as soon as the agreement is reached (Section 5).

4 Asymptotic Results when the Payoffs Realize at the Deadline

In this section, we consider the case in which players receive benefits of the agreement only at the deadline even when they stop searching earlier. Mathematically, this is equivalent to analyzing the limit of the continuation payoff profile $\lim_{\lambda \rightarrow \infty} v^*(t)$ in the equilibrium when $\rho = 0$. In this case, $v^*(t)$ is characterized as the unique solution of the following ordinary differential equation given by letting $\rho = 0$ in ODE (2):

$$v'(t) = \lambda \int_{A(t)} (x - v(t)) d\mu \quad (3)$$

with an initial condition $v(0) = (0, \dots, 0) \in \mathbb{R}^n$. Figure 2 describes a path of the solution of ODE (3) for $n = 2$. At time $-t \in [-T, 0]$, the velocity vector $v'(t)$ equals $\lambda \int_{A(t)} (x - v(t)) d\mu$, which is parallel to the vector from $v(t)$ to the barycenter of $A(t)$ weighted by μ .

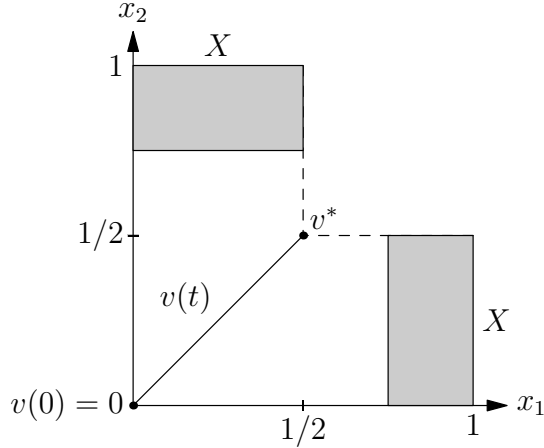


Figure 3: A path that converges to a weakly Pareto efficient allocation.

We present asymptotic results of the continuation payoffs $v^*(t)$ as the arrival rate λ tends to infinity. If λ is very large, one may conjecture that players have so many opportunities that they can find a good allocation. We discuss efficiency of the limit $v^* = \lim_{\lambda \rightarrow \infty} v^*(t; 0, \lambda)$ in X to show that this intuition is basically correct. Let us note that we sometimes consider the limit $\lim_{t \rightarrow \infty} v^*(t; 0, \lambda)$ with enlarging the time interval $[-T, 0]$. By Lemma 3, $v^* = \lim_{t \rightarrow \infty} v^*(t; 0, \lambda)$ for all λ . This implies that we can consider the two limits interchangeably; the limit of $v^*(t)$ as $t \rightarrow \infty$, and the limit as $\lambda \rightarrow \infty$ for fixed t .

In general, v^* is not necessarily Pareto efficient in X . There is an example of a probability density function f satisfying Assumption 1 in which $v^*(t)$ converges to an allocation that is not strictly Pareto efficient.

Example 1. Let $n = 2$, $X = ([0, 1/2] \times [3/4, 1]) \cup ([3/4, 1] \times [0, 1/2])$, and f be the uniform density function on X , which is shown in Figure 3. By the symmetry with respect to the 45 degree line, we must have $v_1^*(t) = v_2^*(t)$ for all t . Therefore $v^* = (1/2, 1/2)$, which is not Pareto efficient in X . \square

Note that v^* is weakly Pareto efficient, and that X is a non-convex set in this example. In fact, we will show that v^* is always weakly efficient for general X , and strictly Pareto efficient if X is convex. Furthermore, even if X is not convex, we can show v^* is “generically” Pareto efficient, that is, v^* is Pareto efficient in X for any generic f that satisfies Assumption 1.

Remember $\hat{X} = \{v \in \mathbb{R}_+^n \mid x \geq v \text{ for some } x \in X\}$ is the positive region of the comprehensive extension of X .

Lemma 4. *The solution $v^*(t)$ of equation (3) converges to a weakly Pareto efficient allocation in \hat{X} as $\lambda \rightarrow \infty$.*

Let \mathcal{F} be the set of density functions that satisfy Assumption 1. We consider a topology on \mathcal{F} defined by the following distance in \mathcal{F} : For $f, \tilde{f} \in \mathcal{F}$,

$$|f - \tilde{f}| = \sup_{x \in X} |f(x) - \tilde{f}(x)|.$$

Proposition 5. *The set*

$$\{f \in \mathcal{F} \mid v^* \text{ is Pareto efficient in } X\}$$

is open and dense in \mathcal{F} .

This proposition shows that v^* is efficient only for generic f . However, if X is convex, then v^* is efficient for all f .

Proposition 6. *Suppose that X is a convex set. Then v^* is Pareto efficient in X .*

Weak Pareto efficiency leads to an observation that players reach an agreement almost surely if t is very large. Let $p(t)$ be the probability that players reach an agreement in the equilibrium before the deadline given no agreement at time $-t$. Then the continuation payoffs $v^*(t)$ must fall in the set $\{p(t)v \mid v \in \hat{X}\}$, which implies $v^*(t)/p(t) \in \hat{X}$. We have $v_i^*(t) > 0$ for all $t > 0$ and $i \in N$ since $v_i^*(t)$ is nondecreasing and $v_i^{*'}(0) > 0$ by equation 3. Since there is a positive probability that no opportunity arrives before the deadline, $p(t)$ is smaller than one. Therefore $v^*(t)/p(t) \in \hat{X}$ strictly Pareto dominates $v^*(t)$. This implies $\lim_{t \rightarrow \infty} p(t) = 1$ since by Lemma 4 v^* is weakly Pareto efficient in \hat{X} . Now we show the following proposition:

Proposition 7. *The probability of agreement before the deadline converges to one as the time interval becomes large.*

In Propositions 5 and 6, we showed that $v^*(t)$ almost always converges to the Pareto frontier of X . We consider the inverse problem. For any Pareto efficient allocation w in X which is not at the edge of the Pareto frontier,¹³ we show that one can find density f which satisfies Assumption 1 such that the limit of the solution $v^*(t)$ of equation (3) is w .

Proposition 8. *Suppose that w is a Pareto efficient allocation in X such that $w_i > 0$ for all $i \in N$, and w is not located at the edge of the Pareto frontier. Then there exists a probability measure μ satisfying Assumption 1 such that the equilibrium continuation payoff profile $v^*(t)$ converges to w as λ tends to infinity.*

In the proof, we construct a probability density function f to have a large weight near $w \in X$, and show that the limit continuation payoffs is w if there is a sufficiently large

¹³We formally define this property in the proof given in Appendix A.8.

weight near w . Note that this claim is not so obvious as it seems. Indeed, we will see in Section 5 that the limit is independent of density f if there is a positive discount rate $\rho > 0$, as long as Assumption 1 holds.

In the rest of this section, we make assumptions on regularity of X around v^* in addition to Assumption 1.

Assumption 2. (a) The limit v^* is Pareto efficient in X .

(b) The Pareto frontier of X is smooth in a neighborhood of v^* .

(c) For the unit normal vector $\alpha \in \mathbb{R}_+^n$ at v^* , $\alpha_i > 0$ for all $i \in N$.

(d) For all $\eta > 0$, there exists $\varepsilon > 0$ such that $\{x \in \mathbb{R}_+^n \mid |v^* - x| \leq \varepsilon, \alpha \cdot (x - v^*) \leq -\eta\}$ is contained in X , where “ \cdot ” denotes the inner product in \mathbb{R}^n .¹⁴

The next lemma shows that $v^*(t)$ converges to v^* with a speed of order $(\lambda t)^{-1/n}$.

Lemma 9. *Suppose that Assumption 2 holds. As either $t \rightarrow \infty$ or $\lambda \rightarrow \infty$, $(v_i^* - v_i^*(t))(\lambda t)^{1/n}$ converges to a positive and finite value which is written as*

$$\lim_{t \rightarrow \infty} (v_i^* - v_i^*(t))(\lambda t)^{\frac{1}{n}} = \left(\frac{n+1}{f(v^*)n^{n+1}} \prod_{j \neq i} \frac{\alpha_j}{\alpha_i} \right)^{\frac{1}{n}}$$

for all $i \in N$, where $\alpha \in \mathbb{R}_{++}^n$ be the unit normal vector of X at v^* .

In the present model with finite λ , it always takes positive time for players to reach an agreement. The next proposition shows that if the Pareto frontier of X is smooth and $v^*(t)$ converges to the Pareto frontier, then the expected duration of the search process in the time interval $[-T, 0]$ is $(n^2 T)/(n^2 + n + 1)$.

Proposition 10. *Suppose that Assumption 2 holds. Then the expected duration of search in the equilibrium is $\frac{n^2}{n^2 + n + 1} T$.*

The proposition implies that a positive fraction of time is spent on search, but players do not spend all the time they have. This is a result of a tradeoff between two effects: On one hand, players do not want to wait too much, as doing so would result in disagreement, or the agreement in low payoffs which they would receive if it takes place close to the deadline. On the other hand, players do not want to stop their search immediately, as they are very picky when the deadline is very far away. Being picky is optimal for the players, as there is no discounting. In the next section, we will see that if there is a significant effect of discounting we expect the search to end immediately. Putting it another way, there are two effects of making the arrival rates large. One is that there exist many realizations of payoffs in a given time interval, which makes the possibility of

¹⁴Only condition (b) is necessary if X is convex.

Number of agents	1	2	3	5	10	100
Limit expected duration	.33	.57	.69	.81	.90	.99

Table 1: Limit expected duration of search as opportunities arrive more and more frequently.

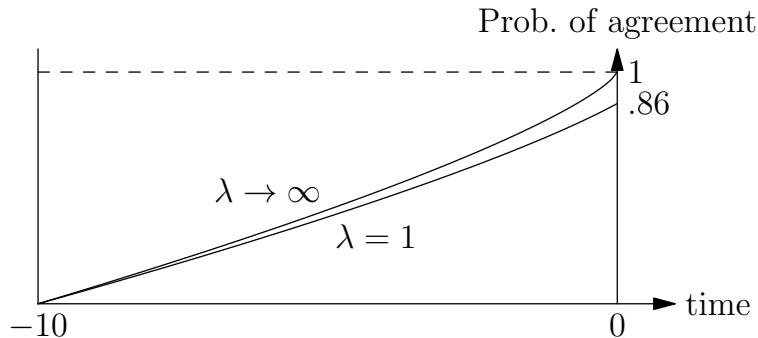


Figure 4: A numerical example of the cumulative probability of agreement.

agreement more likely. The other is that as the result of the increase of opportunities, players expect more opportunities in the future, which makes them pickier.

The solution of the expected duration provided in the theorem implies that, if only two players are involved in search, the expected duration is $\frac{4}{7}T$, it monotonically increases to approach T as n gets larger. Table 1 shows the limit expected duration for several n 's when $T = 1$.¹⁵ We do not think the reason for this is a simple one, which would say that if there are many people it is difficult to all agree on something. The result is rather the consequence of two distinct effects explained above.

The limit duration of Proposition 9 is not far away from those with finite λ in many cases. In the apartment search example, let us suppose that the couple has ten weeks before the deadline, and a broker provides information of an apartment once in every week on average. Figure 4 shows a graph of the cumulative probability of agreement for $\lambda = 1$ and $\lambda \rightarrow \infty$ when $T = 10$, $N = \{1, 2\}$, $X = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 1\}$, and μ is the uniform distribution on X . The probability of agreement before any time $-t$ does not change more than 15% if one considers quite a frequent search instead of searching once a week. A straightforward computation shows that the expected duration of search is decreasing in λ , and it is 6.08 for $\lambda = 1$, which is only 6.5% longer than the limit expected duration $(4/7) \cdot 10 = 5.71$.

¹⁵Although we assumed $n \geq 2$, one can easily show that the same analysis applies to the search problem with a single agent.

5 Asymptotic Results when the Payoffs Realize upon Agreement

In this section, we consider the limit of the equilibrium continuation payoffs $v^*(t)$ as $\lambda \rightarrow \infty$ when the payoffs realize as soon as an agreement is reached. First, we note that if $\rho = 0$, exactly the same analysis as in the previous section applies, as we have already discussed. Hereafter we consider the case where $\rho > 0$. Let us revisit the ordinary differential equation (2) that characterizes the equilibrium continuation payoff profile $v^*(t)$ as its unique solution:

$$v'(t) = -\rho v(t) + \lambda \int_{A(t)} (x - v(t)) d\mu \quad (2)$$

with an initial condition $v(0) = (0, \dots, 0)$.

If λ is large, the right hand side of equation (2) is approximated by the right hand side of equation (3). Therefore, $v^*(t)$ is close to the solution of equation (3) in the case of $\rho = 0$, for λ large relative to ρ . This resemblance of trajectories holds until $v^*(t)$ approaches the boundary of \hat{X} . In particular, we can show that $v^*(t)$ approaches v^0 arbitrarily closely as $\lambda \rightarrow \infty$, where v^0 is defined as the limit of the solution of equation (3). Note that v^0 is weakly Pareto efficient by Lemma 4.

Proposition 11. *For all $\varepsilon > 0$, there exists $\bar{\lambda} > 0$ such that for all $\lambda \geq \bar{\lambda}$,*

$$|v^0 - v^*(t)| \leq \varepsilon \quad \text{for some } t.$$

Remark 1. Before analyzing $v^* = \lim_{\lambda \rightarrow \infty} v^*(t; \rho, \lambda)$, let us consider another limit $v^*(\infty; \rho, \lambda) = \lim_{t \rightarrow \infty} v^*(t; \rho, \lambda)$. Since the right hand side of equation (2) is not proportional to λ , these two limits do not coincide for positive $\rho > 0$. If the limit $v^*(\infty)$ exists, this must satisfy

$$\rho v^*(\infty) = \lambda \int_A (x - v^*(\infty)) d\mu \quad (4)$$

where $A = \{x \in X \mid x \geq v^*(\infty)\}$. For $\rho > 0$, equality (4) shows $\mu(A) > 0$, which follows that $v^*(\infty)$ is Pareto inefficient in X . This will contrast efficiency of $v^* = \lim_{\lambda \rightarrow \infty} v^*(t; \rho, \lambda)$ shown in Proposition 12.

Equality (4) also implies that $v^*(\infty)$ is parallel to the vector from $v^*(\infty)$ to the barycenter of A , as shown in Figure 5 in the two-dimensional case. \square

We assume the following simplifying conditions until the end of this section.¹⁶

Assumption 3. The Pareto frontier of X is smooth, and every component of the normal vector at any Pareto efficient allocation in X is strictly positive.

¹⁶We can show basically the same results without this assumption. We avoid complications derived from the indeterminacy of a normal vector on the boundary of X .

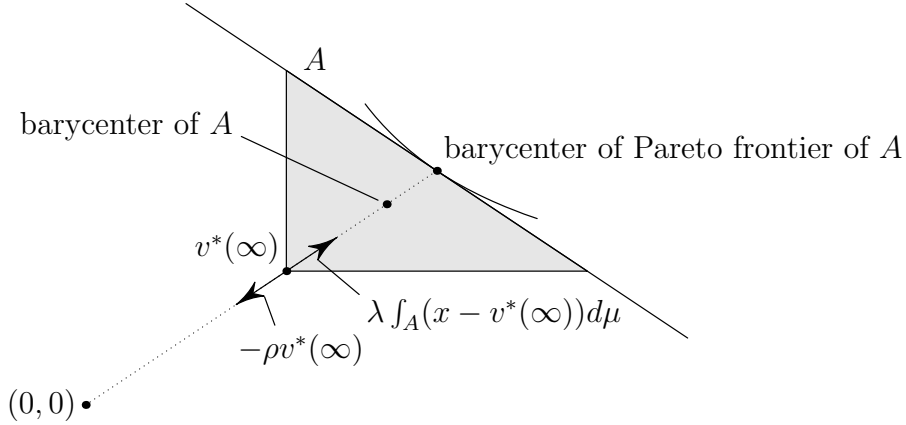


Figure 5: Vectors when $t \rightarrow \infty$.

Now suppose that λ is very large. Then $\mu(A)$ must be very small, which means that $v^*(\infty)$ is very close to the Pareto frontier of X . By Assumption 1, the density f is approximately uniform in A if A is a set with a very small area. To obtain an intuition, suppose that A is a small n -dimensional pyramid. The vector in the right hand side of equality (4) is parallel to the vector from $v^*(\infty)$ to the barycenter of the Pareto frontier of A .

If x is the barycenter of the Pareto frontier of A , it turns out that the boundary of A at x is tangent to the hypersurface defined by $\prod_{i \in N} x_i = a$ for some constant a . We refer to such a Pareto efficient allocation as a *Nash point*, and the set of all Nash points as the *Nash set* of $(X, 0)$ (Maschler et al. (1988), Herrero (1989)). The Nash set contains all local maximizers and all local minimizers of the Nash product. If X is convex, there exists a unique Nash point, which is called the Nash bargaining solution.

The above observation leads to the next proposition.

Proposition 12. *Suppose that any Nash point is isolated in X . Then the limit $v^* = \lim_{\lambda \rightarrow \infty} v^*(t)$ exists and belongs to the Nash set of the problem $(X, 0)$. In particular, if X is convex, this limit coincides with the Nash bargaining solution of $(X, 0)$.*

Therefore, the trajectory of $v^*(t)$ for very large λ starts at $v^*(t) = 0$, approaches v^0 , and moves along the Pareto frontier until reaching a point close to a Nash point.

Finally we consider the duration of search in the equilibrium. In contrast to Proposition 10 in the case of $\rho = 0$, we show that players reach an agreement almost immediately if λ is very large.

Proposition 13. *For all $-t \in (-T, 0]$ and all $\varepsilon > 0$, there exists $\bar{\lambda} > 0$ such that the probability that players reach an agreement before time $-t$ in the equilibrium is larger than $1 - \varepsilon$ for all $\lambda \geq \bar{\lambda}$.*

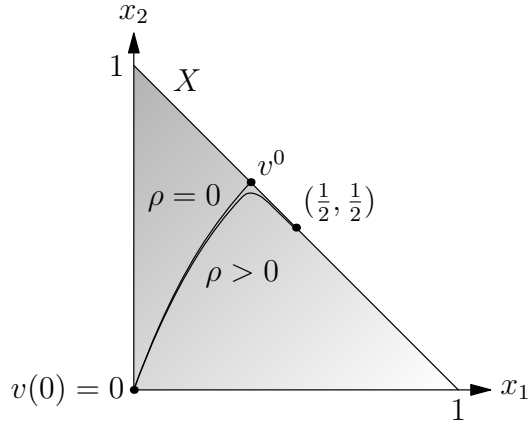


Figure 6: Paths of continuation payoffs. The probability density is low near $(1, 0)$, and high near $(0, 1)$.

6 Discussion

6.1 Dynamics of Bargaining Powers

Consider the case where $X = \{x \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 1\}$ and a density f such that $f(x) > f(x')$ if $x_2 - x_1 > x'_2 - x'_1$. Suppose $\rho > 0$ is very small. In this case, the limit of the solution of ODE (3) with $\rho = 0$, denoted v^0 , locates at the boundary of X by Proposition 6, and it is to the north-west of $(\frac{1}{2}, \frac{1}{2})$, which is the Nash bargaining solution and is the limit of the solution of ODE (2). Hence, by Proposition 11, the continuation payoff when the players receive payoffs upon the agreement starts at a point close to $(\frac{1}{2}, \frac{1}{2})$, and goes up along the boundary of X and reaches a point close to v^0 , and then goes down to $(0, 0)$. On this path of play, player 1's expected payoff is monotonically decreasing over time. On the other hand, player 2's expected payoff changes non-monotonically. Specifically, it rises up until it reaches close to v_2^0 , and then decreases over time. Figure 6 illustrates this path.

Underlying this non-monotonicity is the change in bargaining power between the players. When the deadline is far away, there will be a lot of opportunities left until the deadline, so it is unlikely that players will accept allocations that are far from the Pareto efficient allocations, so the probability distribution over such allocations does not matter so much. Since X is convex and symmetric, two players expect roughly the same payoffs. However, as the time passes, the deadline comes closer, so players expect more possibility that Pareto-inefficient allocations will be accepted. Since player 2 expects more realizations favorable to her than player 1 does, player 2's expected payoff rises while player 1's goes down. Finally, as the deadline comes even closer, player 2 starts fearing the possibility of reaching no agreement, so she becomes less pickier and the cutoff goes down accordingly.

6.2 Simultaneous Convergence

In Section 4, we have shown that if $\rho = 0$, the limit equilibrium payoffs as $\lambda \rightarrow \infty$ are efficient but dependent on the distribution μ . In Section 5, in contrast, the limit is the Nash bargaining solution if $\rho > 0$ is fixed. In this section, we show that the limit equilibrium payoffs as $\lambda \rightarrow \infty$ and $\rho \rightarrow 0$ simultaneously depends on where $\lambda\rho^n$ converges to.

Proposition 14. *Suppose that Assumptions 2 and 3 hold. The limit $v^* = \lim_{\lambda \rightarrow \infty, \rho \rightarrow 0} v^*(t; \rho, \lambda)$ satisfies the following claims: (i) If $\lambda\rho^n \rightarrow 0$, then $v^* = \lim_{\lambda \rightarrow \infty} v^*(t; 0, \lambda)$, which is the limit analyzed in Section 4, and (ii) if $\lambda\rho^n \rightarrow \infty$, then $v^* = \lim_{\lambda \rightarrow \infty} v^*(t; \rho, \lambda)$ for $\rho > 0$, which is the limit shown in Section 5.*

An insight behind this result is as follows: The limit depends on whether the first term in ODE (2) is negligible or not when compared to the second term. Let $z(t; \rho, \lambda)$ be the Hausdorff distance from $v^*(t; \rho, \lambda)$ to the Pareto frontier of X . If ρ is very small and λ is not very large, Proposition 11 shows that $v^*(t; \rho, \lambda)$ is close to $\lim_{\lambda \rightarrow \infty} v^*(t; 0, \lambda)$ which is on the Pareto frontier. Then we can apply Lemma 9 to show that $z(t; \rho, \lambda)$ is approximately proportional to $\lambda^{-1/n}$. Since $\mu(A(t))$ approximates $z(t; \rho, \lambda)^n$ (times some constant), and the length of the vector from $v^*(t; \rho, \lambda)$ to the barycenter of $A(t)$ is linear in $z(t; \rho, \lambda)$, the second term is of order $\lambda \cdot \lambda^{-1/n} \cdot \lambda^{-1} = \lambda^{-1/n}$. Therefore if $\lambda\rho^n \rightarrow 0$ the first term, which approximates ρv^* , is negligible because ρ vanishes more rapidly than $\lambda^{-1/n}$. If $\lambda\rho^n \rightarrow \infty$, the first term is significant because ρ does not vanish rapidly compared to $\lambda^{-1/n}$.

Remark 2. Gomes et al. (1999) consider a discrete-time finite-horizon n -player bargaining with a general coalitional form. A proposer is selected with equal probability in every period, and if her proposal is rejected, she drops out of bargaining with a small probability. They investigate the limit subgame perfect equilibrium payoff profile as the probability of breakdown vanishes and the horizon length tends to infinity. The limit is shown to be the Raiffa solution (defined appropriately in the setting of a general coalitional form) if the speed that the probability of breakdown vanishes is sufficiently faster than the speed that the horizon length goes to infinity, and it is the Nash solution (or more general concept of the NTU consistent value) if the latter is sufficiently faster than the former. Imai and Salonen (2009) consider a related two-player model. They analyze a limit solution as the number of periods in the fixed length of horizon becomes large. This solution turns out to be between the Raiffa and the Nash solution, which are borderline cases of Gomes et al. (1999).

Despite common features such as presence of deadline, and analysis of limits in multiple cases, the present paper differs from these works in the following three aspects. First, proposers in the above bargaining model have a full control of their proposals. In

a subgame perfect equilibrium, players reach an agreement immediately because a proposer always offers an allocation accepted by the others given continuation payoffs in subsequent subgames. On the other hand, realization of an allocation at each moment is stochastic in our model, hence it is not ex ante obvious whether the duration of search shrinks as we let the arrival rate go to infinity. Indeed, we have shown in Proposition 10 that when the payoffs realize at the deadline, players spend a significant amount of time for search.

Second, the conditions that determines whether the limit equilibrium payoffs is Nash solution or not are different. In the bargaining models, the condition is expressed by the limit of the probability that there is no breakdown in the entire bargaining conditional on no agreement at all (Gomes et al. (1999, Theorem 4.4)). Since imposing a breakdown with probability π at an opportunity is mathematically equivalent to discounting payoffs with discount factor $1 - \pi$, and there are λ opportunities in average in a unit time interval in the Poisson process, the discount rate ρ is given by $e^{-\rho} = (1 - \pi)^\lambda$. Therefore the probability of no breakdown corresponds to $(1 - \pi)^{\lambda T} = e^{-\rho T}$ which does not depend on λ . This contrasts our Proposition 14, in which the term in the convergence condition depends on λ . This distinction derives from the difference in the probabilities of agreement at an opportunity on the equilibrium path. In our search model, the probability of realization falling into the acceptance set is the n th order of the distance to the limit payoff profile, while players reach an agreement with probability one in the bargaining model.

Third, the processes by which opportunities arrive differ in two models. In Gomes et al. (1999) and Imai and Salonen (2009), players have an opportunity for sure in every fixed time interval. This implies that there is the “last period” (unless a breakdown occurs). In contrast, our model have a positive probability that there comes no opportunity after any time $-t$. This distinction affects the limit as $\lambda \rightarrow \infty$. Let the discount rate be positive. Then, the limit payoffs in Imai and Salonen (2009) comes between the Raiffa and the Nash bargaining solutions as the number of periods in the fixed length of horizon becomes large, whereas the limit in our search model is exactly the Nash bargaining solution independent of the level of discount rate $\rho > 0$. If there is the last period, players expect to obtain positive payoffs at the last period for sure. This effect does not disappear in backward induction unless the time horizon goes to infinity. \square

6.3 Infinite-Horizon and Static Games

Although we consider a finite-horizon model, our convergence result in Proposition 12 is suggestive of that in infinite-horizon models such as Wilson (2001), Compte and Jehiel (2010), and Cho and Matsui (2011), all of whom consider the limit as the discount factor goes to one in discrete-time infinite-horizon models. This is because the threatening power of disagreement at the deadline is quite weak if the horizon is very far away, and

thus the infinite-horizon model is similar to a finite-horizon model with $T \rightarrow \infty$ if $\rho > 0$. In fact, we can show that the iterated limit as $T \rightarrow \infty$ and then $\rho \rightarrow 0$ is the Nash bargaining solution in our model. By Proposition 14, $\lim_{\rho \rightarrow 0} v^*(T; \alpha\rho, \alpha\lambda)$ with $\alpha = \rho^{-a}$ is the Nash bargaining solution for all $a > n/(n+1)$. As $a \rightarrow \infty$, we see that the iterated limit $\lim_{\rho \rightarrow 0} \lim_{\alpha \rightarrow \infty} v^*(T; \alpha\rho, \alpha\lambda)$ is also the Nash bargaining solution. Since Lemma 3 shows that enlarging T is equivalent to rising λ and ρ in the same ratio, the iterated limit as $T \rightarrow \infty$ first and then $\rho \rightarrow 0$ must be the Nash bargaining solution.

Propositions 1 and 2 imply that the limit continuation payoff is essentially equal to the cutoff, which is expressed by a single variable. In this sense, there is some connection between our model and a static game considered by Nash (1953) himself, who provided a characterization of the Nash bargaining solution by introducing a static demand game with perturbation described as follows.¹⁷ Suppose that X is convex. The basic demand game is a one-shot strategic-form game in which each player i calls a demand $x_i \in \mathbb{R}_+$. Players obtain $x = (x_1, \dots, x_n)$ if $x \in X$, or 0 otherwise. In the perturbed demand game, players fail to obtain $x \in X$ with a positive probability if x is close to the Pareto frontier. Under certain conditions, he showed that the Nash equilibrium of the perturbed demand game converges to the Nash bargaining solution as the perturbation vanishes.

Let us compare the perturbed Nash demand game with our multi-agent search model with a positive discount rate. Let $p(x) = \mu(A(x))$ be the probability that players come across an allocation which Pareto dominates or equals $x \in X$ at an opportunity. If T is very large and t is close to T , players at time $-t$ choose almost the same cutoff profile, say x , contained in the interior of X . The average duration that players wait for an allocation falling into $A(x)$ is almost $1/\lambda p(x)$. During this time interval, payoffs are discounted at rate ρ . Since x_i must be equal to her continuation payoff in an equilibrium, i would lose nearly $(1 - e^{-\rho/\lambda p(x)})y_i$ on average by insisting cutoff x_i where y is the expected allocation conditional on $y \in A(x)$. Note that this loss vanishes as $\rho \rightarrow 0$ for every x in the interior of X . Let probability $P(y)$ satisfy $P(y) = e^{-\rho/\lambda p(x)}$. Player i loses the same expected payoff when $y \in X$ is demanded in the perturbed demand game where the probability of successful agreement is $P(y)$.

The key tradeoff in this game, the attraction to larger demands or the fear of failure of agreement, is parallel to that in the multi-agent search, to be pickier or to avoid loss from discounting.

7 Conclusion

We investigated an n -person search problem with deadline in which agreement opportunities arrive according to the Poisson process, and the drawn object is adopted by a

¹⁷We here follow a slightly modified game considered by Osborne and Rubinstein (1990, Section 4.3). Despite the difference, the model conveys the same insight as the original.

unanimous acceptance. If players cannot reach any agreement before the deadline, they obtain a predetermined payoff profile. We analyzed the limit of the equilibrium continuation payoffs as objects are drawn more and more frequently. If players receive payoffs at the deadline, the continuation payoffs are efficient but sensitive to the distribution of the payoffs of the object in search. The limit expected duration of search is longer than a half of the length of the given time interval, increases in n , and converges to one as n goes to infinity. On the other hand, if players receive payoffs immediately after they agree, the continuation payoff profile converges to a Nash point, and the duration of search is zero in the limit.

We are currently working on several extensions and further investigation of our model. First, we have assumed that the disagreement payoff is 0. This assumption is not entirely innocuous in the presence of discounting, but we expect none of our results change substantially even if we relax this assumption. Second, we have assumed that the discount rate is homogeneous across all the players. Allowing for heterogeneous discount rates will change the Nash bargaining solution to weighted Nash bargaining solution when the payoffs realize upon agreement, but the duration of search will not change. Third, our results pertain to the limit case as the arrival rates go to infinity. We are exploring the case of finite arrival rates to conduct comparative statics exercises. We will address these and other points in our future research.

Appendix

A.1 More Definitions

First we introduce definitions and notations used in proofs.

Let $\bar{x}_i = \max\{x_i \mid (x_i, x_{-i}) \in X \text{ for some } x_{-i}\}$ be the maximum payoff attainable for player i in X , $f_H = \max_{x \in X} f(x) < \infty$ be the upper bound, and $f_L = \min_{x \in X} f(x) > 0$ be the lower bound of f in X . Note that Assumption 1 ensures existence of these values.

A.2 Proof of Proposition 1

Suppose that there exists at least one trembling-hand equilibrium. We show that the continuation payoff of player i at time $-t$ is unique in any trembling-hand equilibrium.

Let $\bar{v}_i^\varepsilon(t)$ and $\underline{v}_i^\varepsilon(t)$ be the supremum and the infimum of the set of continuation payoffs $u_i(\sigma \mid h)$ of player i after all histories $h \in \tilde{\mathcal{H}}_t \setminus \mathcal{H}_t$ at time $-t$ in all Nash equilibria σ in the ε -constrained game. Let $w_i^\varepsilon(t) = \bar{v}_i^\varepsilon(t) - \underline{v}_i^\varepsilon(t)$, $\bar{w}^\varepsilon(t) = \max_{i \in N} w_i^\varepsilon(t)$. We will show that $\bar{w}^\varepsilon(t) = 0$ for all $\varepsilon > 0$ for any time $-t \in [-T, 0]$. Note that $\bar{w}^\varepsilon(0) = 0$ for all ε .

Let us consider the ε -constrained game. If player i accepts an allocation $x \in X$ at time $-t$, she will obtain x_i with probability at least ε^{n-1} . Accepting x is a dominant

action of player i if the following inequality holds:

$$\varepsilon^{n-1}x_i + (1 - \varepsilon^{n-1})\underline{v}_i^\varepsilon(t) > \bar{v}_i^\varepsilon(t),$$

which implies,

$$x_i > \bar{v}_i^\varepsilon(t) + \frac{1 - \varepsilon^{n-1}}{\varepsilon^{n-1}}w_i^\varepsilon(t).$$

Let $\tilde{v}_i^\varepsilon(t) = \bar{v}_i^\varepsilon(t) + \frac{1 - \varepsilon^{n-1}}{\varepsilon^{n-1}}w_i^\varepsilon(t)$. Then $\tilde{v}_i^\varepsilon(t) - \underline{v}_i^\varepsilon(t) = \frac{1}{\varepsilon^{n-1}}w_i^\varepsilon(t)$.

Let $X_i^1(t) = \{x \in X \mid x_i \geq \tilde{v}_i^\varepsilon(t)\}$, $X_i^m(t) = \{x \in X \mid \underline{v}_i^\varepsilon(t) \leq x_i \leq \tilde{v}_i^\varepsilon(t)\}$, and $X_i^0(t) = \{x \in X \mid x_i \leq \underline{v}_i^\varepsilon(t)\}$. Any player i accepts $x \in X_i^1(t)$ and rejects $x \in X_i^0(t)$ with probability $1 - \varepsilon$ after almost all histories at time $-t$. Note that $X = (\bigcup_{j \in N} X_j^m(t)) \cup (\bigcup_{(s_1, \dots, s_n) \in \{0,1\}^n} \bigcap_{j \in N} X_j^{s_j}(t))$ (although not disjoint). Then

$$\begin{aligned} \bar{v}_i^\varepsilon(t) &\leq \int_0^t \left(\sum_{j \in N} \int_{X_j^m(\tau)} \bar{x}_i d\mu \right. \\ &\quad + \sum_{(s_1, \dots, s_n) \in \{0,1\}^n} \int_{\bigcap_{j \in N} X_j^{s_j}(\tau)} \left((1 - \varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N} (1-s_j)} x_i \right. \\ &\quad \left. \left. + (1 - (1 - \varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N} (1-s_j)}) \bar{v}_i^\varepsilon(\tau) \right) d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau, \end{aligned}$$

and

$$\begin{aligned} \underline{v}_i^\varepsilon(t) &\geq \int_0^t \left(\sum_{(s_1, \dots, s_n) \in \{0,1\}^n} \int_{\bigcap_{j \in N} X_j^{s_j}(\tau)} \left((1 - \varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N} (1-s_j)} x_i \right. \right. \\ &\quad \left. \left. + (1 - (1 - \varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N} (1-s_j)}) \underline{v}_i^\varepsilon(\tau) \right) d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau. \end{aligned}$$

Therefore $w_i^\varepsilon(t) = \bar{v}_i(t) - v_i(t)$ is estimated as follows:

$$\begin{aligned}
w_i^\varepsilon(t) &\leq \int_0^t \left(\sum_{j \in N} \int_{X_j^m(\tau)} \bar{x}_i d\mu \right. \\
&\quad \left. + \sum_{(s_1, \dots, s_n) \in \{0,1\}^n} \int_{\cap_{j \in N} X_j^{s_j}(\tau)} (1 - (1 - \varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N} (1-s_j)}) w_i^\varepsilon(\tau) d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau \\
&\leq \int_0^t \left(\sum_{j \in N} f_H \bar{x}_i \frac{1}{\varepsilon^{n-1}} w_j^\varepsilon(\tau) \prod_{k \neq j} \bar{x}_k \right. \\
&\quad \left. + \sum_{(s_1, \dots, s_n) \in \{0,1\}^n} \int_X (1 - (1 - \varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N} (1-s_j)}) w_i^\varepsilon(\tau) d\mu \right) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau \\
&\leq \int_0^t \left(\sum_{j \in N} f_H \max_{k \in N} \{\bar{x}_k\} \frac{1}{\varepsilon^{n-1}} \prod_{k \neq j} \bar{x}_k \right. \\
&\quad \left. + \sum_{(s_1, \dots, s_n) \in \{0,1\}^n} (1 - (1 - \varepsilon)^{\sum_{j \in N} s_j} \varepsilon^{\sum_{j \in N} (1-s_j)}) \right) \bar{w}^\varepsilon(\tau) \lambda e^{-(\lambda+\rho)(t-\tau)} d\tau.
\end{aligned}$$

Since the above inequality holds for all $i \in N$, there exists a constant $L > 0$ such that the following inequality holds:

$$\bar{w}^\varepsilon(t) \leq \int_0^t L \bar{w}^\varepsilon(\tau) e^{-(\lambda+\rho)(t-\tau)} d\tau.$$

Let $W^\varepsilon(t) = \int_0^t \bar{w}^\varepsilon(\tau) e^{(\lambda+\rho)\tau} d\tau$. Then

$$\begin{aligned}
W^{\varepsilon'}(t) &= \bar{w}^\varepsilon(t) e^{(\lambda+\rho)t} \\
&\leq W^\varepsilon(t).
\end{aligned}$$

Therefore we have $\frac{d}{dt}(W^\varepsilon(t)e^{-t}) \leq 0$, which implies $\bar{w}^\varepsilon(t) \leq W^\varepsilon(t) \leq 0$ since $\bar{w}^\varepsilon(0) = 0$. Hence, $\bar{w}^\varepsilon(t) = 0$ for all t and all $\varepsilon > 0$. Any trembling-hand equilibria yield the same continuation payoffs after almost all histories at time $-t \in [-T, 0]$.

A.3 Proof of Proposition 2

We show that the unique solution $v^*(t)$ of ODE (2) characterizes a trembling-hand equilibrium. For $s_i \in \{+, -\}$, and $v_i \in [0, \bar{x}_i]$ let

$$I_i^{s_i}(v_i) = \begin{cases} [0, v_i] & \text{if } s_i = +, \\ [v_i, \bar{x}_i] & \text{if } s_i = -, \end{cases}$$

and $p^+ = 1 - \varepsilon$, $p^- = \varepsilon$. For $\varepsilon > 0$, let us write down a Bellman equation similar to (1) with respect to a continuation payoff profile $v^\varepsilon(t)$ in the ε -constrained game:

$$v_i^\varepsilon(t) = \int_0^t \left(\sum_{s \in \{+, -\}^n} \int_{(I_1^{s_1}(v_i^\varepsilon(\tau)) \times \dots \times I_n^{s_n}(v_i^\varepsilon(\tau))) \cap X} (p^{s_1} \dots p^{s_n} \cdot x_i + (1 - p^{s_1} \dots p^{s_n}) v_i^\varepsilon(\tau)) d\mu \right) \cdot \lambda e^{-(\lambda + \rho)(t - \tau)} d\tau$$

This implies that

$$v_i^{\varepsilon'}(t) = -(\lambda + \rho)v_i^\varepsilon(t) + \lambda \sum_{s \in \{+, -\}^n} \int_{(I_1^{s_1}(v_i^\varepsilon(t)) \times \dots \times I_n^{s_n}(v_i^\varepsilon(t))) \cap X} (p^{s_1} \dots p^{s_n} \cdot x_i + (1 - p^{s_1} \dots p^{s_n}) v_i^\varepsilon(t)) d\mu.$$

This ODE has a unique solution because the right hand side is uniformly Lipschitz continuous in v_i^ε . Let $v^\varepsilon(t)$ be this solution, which is a cutoff profile of a Nash equilibrium in the ε -constrained game by construction. Since $A(t) = (I_1^+(v_i^\varepsilon(\tau)) \times \dots \times I_n^+(v_i^\varepsilon(\tau))) \cap X$, and $p^+ \rightarrow 1$, $p^- \rightarrow 0$ as $\varepsilon \rightarrow 0$, ODE 2 is obtained by letting $\varepsilon \rightarrow 0$. Therefore $v^\varepsilon(t)$ converges to $v^*(t)$ as $\varepsilon \rightarrow 0$ because the above ODE is continuous in ε .¹⁸ Hence the cutoff strategy profile with cutoffs $v^*(t)$ is a trembling-hand equilibrium.

A.4 Proof of Lemma 3

Let $w^*(t; \rho, \lambda) = v^*(\alpha t; \rho/\alpha, \lambda)$. By equation (3), $w^*(t; \rho, \lambda)$ is the solution of

$$w'(t/\alpha) = -\frac{\rho}{\alpha} w(t/\alpha) + \lambda \int_{A(w(t/\alpha))} (x - w(t/\alpha)) d\mu,$$

which is equivalent to

$$\frac{d}{d\tau} w(\tau) = -\rho w(\tau) + \alpha \lambda \int_{A(w(\tau))} (x - w(\tau)) d\mu$$

where $\tau = t/\alpha$, $w(t) = v(\alpha t)$. The solution of the second equation is $w^*(t; \lambda) = v^*(t; \alpha \lambda)$. Therefore we have $v^*(t; \rho, \alpha \lambda) = v^*(\alpha t; \rho/\alpha, \lambda)$ as desired.

A.5 Proof of Lemma 4

By Lemma 3, it suffices to show that $v^* = \lim_{t \rightarrow \infty} v^*(t)$ is weak Pareto efficient.

Let $A = \{x \in X \mid x \geq v^*\}$. Suppose that there exists $x = (x_1, \dots, x_n) \in A$ such that $x_1 > v_1^*, \dots, x_n > v_n^*$. By Assumption 1, there exists a closed subset $Y \subset A$ such

¹⁸See, e.g., Coddington and Levinson (1955, Theorem 7.4 in Chapter 1).

$\mu(Y) > 0$, and $y_i = \inf\{x_i \mid (x_1, \dots, x_n) \in Y\} > x_i$. By ODE (3), we have

$$\begin{aligned} v_i^{*'}(t) &= \lambda \int_{A(t)} (x_i - v_i^*(t)) d\mu \\ &\geq \lambda \int_W (y_i - v_i^*) d\mu \\ &= \lambda(y_i - v_i^*)\mu(Y) > 0. \end{aligned}$$

This inequality implies that $v_i^*(t) \geq \lambda(y_i - v_i^*)\mu(Y)t + v_i^*(0)$, which tends to infinity as $t \rightarrow \infty$. This contradicts the fact that $v_i^*(t)$ is convergent. Hence x is weakly Pareto efficient in \hat{X} .

A.6 Proof of Proposition 5

Let $v^*(t; f)$ be the solution of ODE (3) for density $f \in \mathcal{F}$, and $v^*(f) = \lim_{\lambda \rightarrow \infty} v^*(t; f) = \lim_{t \rightarrow \infty} v^*(t; f)$.

First we show that the set is open, i.e., for all $f \in \mathcal{F}$ with $v^*(f)$ Pareto efficient, $\varepsilon > 0$, and a sequence $f_k \in \mathcal{F}$ ($k = 1, 2, \dots$) with $|f_k - f| \rightarrow 0$ ($k \rightarrow \infty$), there exist $\delta > 0$ and \bar{k} such that

$$|v^*(f_k) - v^*(f)| \leq \varepsilon$$

for all $k \geq \bar{k}$.

Since $\lim_{t \rightarrow \infty} v^*(t; f) = v^*(f)$, for all $\delta > 0$ there exists $\bar{t} > 0$ such that $|v^*(f) - v^*(t; f)| \leq \delta$ for all $t \geq \bar{t}$. By Pareto efficiency of $v^*(f)$, let $\delta > 0$ be sufficiently small so that $A(v^*(\bar{t}; f) - (\delta, \delta, \dots, \delta))$ is contained in the ε -ball centered at $v^*(f)$. Since the right hand side of ODE (3) is continuous in v , the unique solution of (3) is continuous with respect to parameters in (3). Therefore, for a finite time interval $[0, T]$ including \bar{t} , there exists \bar{k} such that $|v^*(t; f_k) - v^*(t; f)| \leq \delta$ for all $t \in [0, T]$ and all $k \geq \bar{k}$. This implies that $v^*(t; f_k) \in A(v^*(\bar{t}; f) - (\delta, \delta, \dots, \delta))$, thereby $v^*(f_k) \in A(v^*(\bar{t}; f) - (\delta, \delta, \dots, \delta))$. Hence we have $|v^*(f_k) - v^*(f)| \leq \varepsilon$.

Second we show that the set is dense, i.e., for all $f \in \mathcal{F}$ with $v^*(f)$ not strictly Pareto efficient in X and all $\varepsilon > 0$, there exists $\tilde{f} \in \mathcal{F}$ such that $|f - \tilde{f}| \leq \varepsilon$ and $v^*(\tilde{f})$ is Pareto efficient. Since $v^*(f)$ is only weakly Pareto efficient in \hat{X} , there exists Pareto efficient $y \in X$ which Pareto dominates $v^*(f)$. Let $I = \{i \in N \mid y_i = v_i^*(f)\}$ and $J = N \setminus I$. Since y is Pareto efficient, there is $\delta > 0$ such that if $x \in X$ is weakly Pareto efficient, satisfies $|y - x| \leq \delta$, and $y_i = x_i$ for some $i \in N$, then there is no $\tilde{x} \in X$ such that $\tilde{x}_i > y_i$ and $|y - \tilde{x}| \leq \delta$.

By Assumption 1, for any small $\delta/2 > \eta > 0$, there is a small ball contained in X centered at \tilde{y} with $|y - \tilde{y}| \leq \eta$. Let g be a continuous density function whose support is

the above small ball, takes zero on the boundary of the ball, and the expectation of g is exactly \tilde{y} . Let $\tilde{f} = (1 - \frac{\varepsilon}{|f|+|g|})f + \frac{\varepsilon}{|f|+|g|}g \in \mathcal{F}$. Since f and g are bounded from above, $|f - \tilde{f}| \leq \varepsilon$.

Since $v^*(f)$ is weakly Pareto efficient, if $v^*(f) \in A(v)$, then $A(v) \subset \bigcup_{i \in N} ([v_i, v_i^*(f)] \times \prod_{j \neq i} [0, \bar{x}_j])$. If $|v^*(f) - v| \leq \xi$ where $\xi > 0$ is very small,

$$\begin{aligned} \int_{A(v)} (x_i - v_i) f(x) dx &\leq f_H \sum_{j \in N} (v_j^*(f) - v_j) \prod_{k \in N} \bar{x}_k \\ &\leq \xi n f_H \prod_{k \in N} \bar{x}_k \end{aligned}$$

If $v^*(f) \in A(v)$, $\min_{j \in N} (y_j - v_j) \geq 2\eta$ and $|v^*(f) - v| \leq \xi$, we have

$$\begin{aligned} \int_{A(v)} (x_i - v_i) \tilde{f}(x) dx - \int_{A(v)} (x_i - v_i) f(x) dx &= \int_{A(v)} (x_i - v_i) (\tilde{f}(x) - f(x)) dx \\ &= \frac{\varepsilon}{|f| + |g|} \int_{A(v)} (x_i - v_i) (g(x) - f(x)) dx \\ &\geq \frac{\varepsilon}{|f| + |g|} \left((\tilde{y}_i - v_i) - \left(\xi n f_H \prod_{k \in N} \bar{x}_k \right) \right). \end{aligned}$$

If $j \in J$ and $|v^*(f) - v| \leq \xi$ where $\xi > 0$ is very small, then

$$\int_{A(v)} (x_j - v_j) \tilde{f}(x) dx - \int_{A(v)} (x_j - v_j) f(x) dx \geq \frac{\varepsilon}{2(|f| + |g|)} (\tilde{y}_j - v_j^*(f)).$$

Let $w(t) = v^*(t; \tilde{f}) - v^*(t; f)$. Since ODE (3) is continuous in the parameters, for all $\zeta > 0$, there exists $\varepsilon > 0$ such that $|w(t)| \leq \zeta$ for all $t \in [0, T]$. Suppose that T and t are very large so that $|v^*(f) - v^*(t; f)| \leq \xi$. For $j \in J$, $w'_j(t)$ is estimated as follows:

$$\begin{aligned} w'_j(t) &= \lambda \int_{A(v^*(t; \tilde{f}))} (x_j - v_j^*(t; \tilde{f})) \tilde{f}(x) dx - \lambda \int_{A(v^*(t; f))} (x_j - v_j^*(t; f)) f(x) dx \\ &= \lambda \int_{A(v^*(t; \tilde{f}))} (x_j - v_j^*(t; \tilde{f})) \tilde{f}(x) dx - \lambda \int_{A(v_j^*(t; \tilde{f}))} (x_j - v_j^*(t; \tilde{f})) f(x) dx \\ &\quad + \lambda \int_{A(v^*(t; \tilde{f}))} (x_j - v_j^*(t; \tilde{f})) f(x) dx - \lambda \int_{A(v^*(t; f))} (x_j - v_j^*(t; f)) f(x) dx \\ &\geq \frac{\lambda \varepsilon}{2(|f| + |g|)} (\tilde{y}_j - v_j^*(f)) - \lambda \int_{A(v^*(t; f)) \cap A(v^*(t; \tilde{f}))} w_j(t) f(x) dx \\ &\quad - \lambda \int_{A(v^*(t; f)) \setminus (A(v^*(t; f)) \cap A(v^*(t; \tilde{f}))} (x_j - v_j^*(t; f)) f(x) dx \\ &\geq \frac{\lambda \varepsilon}{2(|f| + |g|)} (\tilde{y}_j - v_j^*(f) - \zeta) - \lambda \zeta \xi \sum_{k \in N} \prod_{l \neq k} \bar{x}_l - \lambda \xi n f_H \prod_{k \in N} \bar{x}_k. \end{aligned}$$

Therefore when $\xi > 0$ is sufficiently small, $w'_j(t)$ is bounded away from zero:

$$w'_j(t) \geq \frac{\lambda \varepsilon}{4(|f| + |g|)} (\tilde{y}_j - v_j^*(f) - \zeta).$$

This implies that for small $\varepsilon > 0$ and large t , $v_j^*(t; \tilde{f}) > v^*(f)$ for all $j \in J$. Then the similar method to Step 3 in the proof of Proposition 6 shows that $v^*(t; \tilde{f})$ converges to a Pareto efficient allocation in X .

A.7 Proof of Proposition 6

Let $A = \{x \in X \mid x \geq v^*\}$. Let $I = \{i \in N \mid x_i = v_i \text{ for all } x \in A\} \subset N$, and $J = N \setminus I$. Suppose that there exists $x \in X$ which Pareto dominates v^* , thereby $J \neq \emptyset$.

Step 1: We show that I is nonempty. If there is no such player, there exist $y(1), \dots, y(n)$ such that $y(j) \in A$ and $y_j(j) > v_j^*$ for all $j \in N$. This implies that $y = \frac{1}{n} \sum_{j \in N} y(j)$ strictly Pareto dominates v^* . Since X is convex, y also belongs to A . This contradicts the weak Pareto efficiency of v^* shown in Lemma 4.

Step 2: Next we show that if v^* is not Pareto efficient in X , and $i \in I$, then $x_i \leq v_i^*$ for all $x \in X$.

Let i be the player in I . Suppose that there exists $y \in X$ with $y_i > v_i^*$. Since X is convex, $\alpha y + (1 - \alpha)x \in X$ for all $0 \leq \alpha \leq 1$ and $x \in X$. Since we assumed that there exists $x \in X$ which Pareto dominates v^* , $x_j > v_j^*$ for $j \in J$. Then there exists $\alpha > 0$ such that $\alpha y + (1 - \alpha)x \geq v^*$, and $\alpha y_j + (1 - \alpha)x_j > v_j^*$ for some j . By Step 1, we must have $x_i = v_i^*$. Therefore, $\alpha y_i + (1 - \alpha)x_i > v_i^*$, which contradicts the fact that $i \in I$.

Step 3: Finally we show that $v^*(t)$ converges to a Pareto efficient allocation in X as $t \rightarrow \infty$.

By convexity of X , we may find y_j, \bar{y}_j ($j \in J$) such that $v_j^* < y_j < \bar{y}_j$, and $\prod_{i \in I} [v_i^* - \varepsilon, v_i^*] \times \prod_{j \in J} [y_j, \bar{y}_j]$ is contained in X for small $\varepsilon > 0$. Let $\varepsilon \in (0, 1/2)$ be sufficiently small such that $\varepsilon \leq \frac{2f_L \prod_{j \in J} (\bar{y}_j - y_j)}{f_H \prod_{j \in J} \bar{x}_j}$. Since $v^*(t)$ converges to v^* as $t \rightarrow \infty$, there exists \bar{t} such that $\max_{i \in N} \{v_i^* - v_i^*(t)\} \leq \varepsilon$ whenever $t \geq \bar{t}$. Let $Y(t) = \prod_{i \in I} [v_i^*(t), v_i^*] \times \prod_{j \in J} [y_j, \bar{y}_j] \subset A(t)$.

We have $A(t) \subset \prod_{i \in I} [v_i^*(t), v_i^*] \times \prod_{j \in J} [0, \bar{x}_j]$ since there is no $x \in A(t)$ with $x_i > v_i^*$. By equation (3), for $i \in I$,

$$\begin{aligned} v_i^{*'}(t) &= \lambda \int_{A(t)} (x_i - v_i^*(t)) d\mu \\ &\leq \lambda \int_{\prod_{i' \in I} [v_{i'}^*(t), v_{i'}^*]} (x_i - v_i^*(t)) \int_{\prod_{j \in J} [0, \bar{x}_j]} f_H \prod_{j \in J} dv_j \prod_{i' \in I} dv_{i'} \\ &\leq \frac{1}{2} \lambda f_H (v_i^* - v_i^*(\bar{t})) \prod_{i' \in I} (v_{i'}^* - v_{i'}^*(t)) \prod_{j \in J} \bar{x}_j \end{aligned}$$

for all $t \geq \bar{t}$. On the other hand, for $j \in J$,

$$\begin{aligned}
v_j^{*'}(t) &= \lambda \int_{A(t)} (x_j - v_j^*(t)) d\mu \\
&\geq \lambda \int_{Y(t)} (y_j - v_j^*) d\mu \\
&= \lambda (y_j - v_j^*) \mu(Y(t)) \\
&\geq \lambda f_L(y_j - v_j^*) \prod_{i \in I} (v_i^* - v_i^*(t)) \prod_{j' \in J} (\bar{y}_{j'} - y_{j'}).
\end{aligned}$$

Then for $i \in I$ and $j \in J$,

$$\begin{aligned}
\frac{v_i^{*'}(t)}{v_j^{*'}(t)} \cdot \frac{v_j^* - v_j^*(\bar{t})}{v_i^* - v_i^*(\bar{t})} &\leq \frac{f_H(v_i^* - v_i^*(\bar{t}))(v_j^* - v_j^*(\bar{t})) \prod_{j \in J} \bar{x}_j}{2f_L \prod_{j' \in J} (\bar{y}_{j'} - y_{j'})} \\
&\leq \frac{(v_i^* - v_i^*(\bar{t}))(v_j^* - v_j^*(\bar{t}))}{\varepsilon} \\
&\leq \varepsilon \leq \frac{1}{2}
\end{aligned}$$

for all $t \geq \bar{t}$. Therefore,

$$\frac{v_i^{*'}(\bar{t})}{v_j^{*'}(\bar{t})} \leq \frac{v_i^* - v_i^*(\bar{t})}{2(v_j^* - v_j^*(\bar{t}))}$$

holds for all $t \geq \bar{t}$. This inequality implies

$$v_i^*(t) - v_i^*(\bar{t}) \leq \frac{v_i^* - v_i^*(\bar{t})}{2(v_j^* - v_j^*(\bar{t}))} (v_j^*(t) - v_j^*(\bar{t}))$$

for all $t \geq \bar{t}$. By letting $t \rightarrow \infty$ in the above inequality, we have $0 < v_i^* - v_i^*(\bar{t}) \leq (v_i^* - v_i^*(\bar{t}))/2$, a contradiction. Hence v^* is strictly Pareto efficient in X .

A.8 Proof of Proposition 8

First, we define the notion of the edge of the Pareto frontier. Suppose that w is Pareto efficient in X , and $w_i > 0$ for all $i \in X$. Let us denote an $(n-1)$ -dimensional subspace orthogonal to w by $D = \{z \in \mathbb{R}^n \mid w \cdot z = 0\}$. For $\xi > 0$, let D_ξ be an $(n-1)$ -dimensional disk defined as

$$D_\xi = \{z \in D \mid |z| \leq \xi\},$$

and let S_ξ be its boundary. We say that a Pareto efficient allocation w in X is *not* located at the edge of the Pareto frontier of X if there is $\xi > 0$ such that for all vector $z \in D_\xi$ there is a scalar $\alpha > 0$ such that $\alpha(w + z)$ is Pareto efficient in X . We denote this Pareto efficient allocation by $w_z \in X$.

Let $B_\varepsilon(y) = \{x \in X \mid |w - x| \leq \varepsilon\}$ for $y \in X$ and $\varepsilon > 0$. We denote the volume

of $B_\varepsilon(y)$ by $V_\varepsilon(y)$, and the volume of the n -dimensional ball with radius ε by V_ε . Note that $\min_{y \in X} V_\varepsilon(y) > 0$ by Assumption 1. Let g be a continuous density function on an n -dimensional ball centered at $0 \in \mathbb{R}^n$ with radius ε , assumed to take zero on the boundary of the ball. Let \tilde{f} be the uniform density function on X . For a Pareto efficient allocation y , we define a probability density function f_y on X by

$$f_y(x) = \eta \tilde{f}(x) + (1 - \eta)g(y - x) \frac{V_\varepsilon}{V_\varepsilon(y)}$$

where $\eta > 0$ is small. Note that $f_y(x)$ is uniformly bounded above and away from zero in x and y .

For $z \in D_\xi$, let $\tilde{\varphi}(z)$ be the limit of the solution of ODE (3) with density f_{w_z} , and define a function φ from D_ξ to D by $\varphi(z) = \tilde{\varphi}(z) + \delta w \in D$ for some $\delta \in \mathbb{R}$. By the form of ODE (3), the solution of (3) with density f_{w_z} is continuously deformed if z changes continuously. Since w is not at the edge of the Pareto frontier, $\tilde{\varphi}(z)$ is also Pareto efficient in X and comes close to w if ξ , ε , and η are small. Therefore $\varphi(z)$ is a continuous function. The rest of the proof consists of two steps.

Step 1: We show that for any $\xi > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that $|\varphi(z) - z| \leq \xi$ for all $z \in D_\xi$. If a density function has a positive value only in $B_\varepsilon(y)$ for some y in the Pareto frontier of X , then the barycenter of $A(t)$ is always contained in $B_\varepsilon(y)$. In such a case, the limit allocation with density f_y belongs to $B_\varepsilon(y)$. As $\eta \rightarrow 0$, f_y approaches the above situation. Therefore, for sufficiently small $\eta > 0$, the distance between the limit allocation and y is smaller than 2ε . For $y = w_z$ and letting ε very small, we have $|\varphi(z) - z| \leq \xi$. Since D_ξ is compact, such we can take such small $\varepsilon > 0$ and $\eta \rightarrow 0$ uniformly.

Step 2: We show that there is $z \in D_\xi$ such that $\varphi(z) = 0$. Let $\psi(z) = z - \varphi(z)$. By Step 1, $\psi(z)$ belongs to D_ξ for all $z \in D_\xi$. By Brouwer's fixed point theorem, there exists $z \in D_\xi$ such that $\psi(z) = z$. Therefore there exists $z \in D_\xi$ such that $\varphi(z) = 0$.

Hence for $z \in D_\xi$ such that $\varphi(z) = 0$, the limit allocation with density f_{w_z} coincides with w .

A.9 Proof of Lemma 9

Let $f_H(t) = \max_{x \in A(t)} f(x)$, and $f_L(t) = \min_{x \in A(t)} f(x)$. Since f is continuous, both $f_H(t)$ and $f_L(t)$ are continuous and converge to $f(v^*)$ as $t \rightarrow \infty$. For $\varepsilon > 0$, there is \bar{t} such that $|v^* - v^*(t)| \leq \varepsilon$ for all $t \geq \bar{t}$. For $\eta > 0$, let

$$\begin{aligned} \underline{A}(t) &= \{x \in \mathbb{R}_+^n \mid x \geq v(t), \alpha \cdot (x - v^*) \leq -\eta\}, \text{ and} \\ \overline{A}(t) &= \{x \in \mathbb{R}_+^n \mid x \geq v(t), \alpha \cdot (x - v^*) \leq \eta\}. \end{aligned}$$

The volume of $\underline{A}(t)$ (with respect to the Lebesgue measure on \mathbb{R}^n) is

$$V(\underline{A}(t)) = \frac{1}{n} \prod_{j \in N} \left(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_j} - \eta \right),$$

and the volume of $\overline{A}(t)$ is

$$V(\overline{A}(t)) = \frac{1}{n} \prod_{j \in N} \left(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_j} + \eta \right).$$

Suppose that $\varepsilon > 0$ is small and \bar{t} is large. Then by Assumption 2, $\underline{A}(t) \subset A(t) \subset \overline{A}(t)$ holds for all $\eta > 0$ and all $t \geq \bar{t}$. The rest of the proof consists of two steps.

Step 1: We show that for any two players $i, j \in N$, $\lim_{t \rightarrow \infty} v_j^{*'}(t)/v_i^{*'}(t) = \alpha_i/\alpha_j$. The i th coordinate of the right hand side of equation (3) is estimated as

$$\begin{aligned} f_L(t) \int_{\underline{A}(t)} (x_i - v_i^*(t)) dx \\ \leq \int_{A(t)} (x_i - v_i^*(t)) f(x) dx \leq f_H(t) \int_{\overline{A}(t)} (x_i - v_i^*(t)) dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\lambda f_L(t) V(\underline{A}(t))}{n+1} \left(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_i} - \eta \right) \\ \leq v_i^{*'}(t) \leq \frac{\lambda f_H(t) V(\overline{A}(t))}{n+1} \left(\frac{\alpha \cdot (v^* - v^*(t))}{\alpha_i} + \eta \right) \end{aligned} \quad (\text{A.1})$$

for all $t \geq \bar{t}$ and $i \in N$. By letting $\eta \rightarrow 0$, $\varepsilon \rightarrow 0$, and $t \rightarrow \infty$, we have $\lim_{t \rightarrow \infty} v_j^{*'}(t)/v_i^{*'}(t) = \alpha_i/\alpha_j$.

Step 2: By Step 1, for i and small $\delta > 0$, there exist \tilde{t} such that such that

$$(1 - \delta) \frac{\alpha_i}{\alpha_j} \leq \frac{v_j^* - v_j^*(t)}{v_i^* - v_i^*(t)} \leq (1 + \delta) \frac{\alpha_i}{\alpha_j}$$

for all $t \geq \tilde{t}$ and $j \in N$. Therefore,

$$n(1 - \delta)(v_i^* - v_i^*(t)) \leq \frac{\alpha \cdot (v^* - v^*(t))}{\alpha_i} \leq n(1 + \delta)(v_i^* - v_i^*(t)).$$

By inequality (A.1), we have

$$\begin{aligned} & \frac{\lambda f_L(t)}{n(n+1)} (n(1-\delta)(v_i^* - v_i^*(t)) - \eta)^2 \prod_{j \neq i} \left(n(1-\delta) \frac{\alpha_i}{\alpha_j} (v_i^* - v_i^*(t)) - \eta \right) \\ & \leq v_i^{*'}(t) \leq \frac{\lambda f_H(t)}{n(n+1)} (n(1+\delta)(v_i^* - v_i^*(t)) + \eta)^2 \prod_{j \neq i} \left(n(1+\delta) \frac{\alpha_i}{\alpha_j} (v_i^* - v_i^*(t)) + \eta \right) \end{aligned}$$

for all $t \geq \tilde{t}$ and $j \in N$. Thus for large t , $v_i^*(t)$ is approximated by the solution of the following ordinary differential equation:

$$v_i'(t) = \frac{\lambda f(v^*) n^n}{n+1} (n(v_i^* - v_i^*(t)))^2 \prod_{j \neq i} \frac{\alpha_i}{\alpha_j} (v_i^* - v_i(t)).$$

By solving this, we have an approximation

$$v^* - v_i^*(t) = \left(C + \frac{\lambda f(v^*) n^{n+1} t}{n+1} \prod_{j \neq i} \frac{\alpha_i}{\alpha_j} \right)^{-\frac{1}{n}}$$

where C is a constant. Hence,

$$\lim_{t \rightarrow \infty} (v_i^* - v_i^*(t)) (\lambda t)^{\frac{1}{n}} = \left(\frac{n+1}{f(v^*) n^{n+1}} \prod_{j \neq i} \frac{\alpha_j}{\alpha_i} \right)^{\frac{1}{n}},$$

which is a positive constant.

A.10 Proof of Proposition 10

By Assumption 2, $A(t)$ is approximated as

$$\{x \in \mathbb{R}_+^n \mid x \geq v^*(t), \alpha \cdot (x - v^*) \leq 0\}.$$

By Lemma 9, $\mu(A(t))$ is approximated as

$$\begin{aligned} \mu(A(t)) &= f(v^*) n^{n-1} \prod_{i \in N} (v_i^* - v_i^*(t)) \\ &= f(v^*) n^{n-1} \prod_{i \in N} \left(\left(\frac{n+1}{f(v^*) n^{n+1}} \prod_{j \neq i} \frac{\alpha_j}{\alpha_i} \right)^{\frac{1}{n}} (\lambda t)^{-\frac{1}{n}} \right) \\ &= \frac{n+1}{n^2 \lambda t} \end{aligned}$$

if t is large. For $s \in [0, T]$, the probability that players reach an agreement before time $-(T-s)$ is

$$1 - e^{\int_{T-s}^T \mu(A(t)) \lambda dt} = 1 - \left(\frac{\lambda n^2 (T-s) + n+1}{\lambda n^2 T + n+1} \right)^{\frac{n+1}{n^2}}.$$

This probability is approximated by $1 - \left(\frac{T-s}{T}\right)^{\frac{n+1}{n^2}}$. Therefore the expected duration of the search process is

$$\int_0^T s \frac{d}{ds} \left[1 - \left(\frac{T-s}{T}\right)^{\frac{n+1}{n^2}} \right] ds = \frac{n^2}{n^2 + n + 1} T.$$

A.11 Proof of Proposition 11

Let $v^0(t; \lambda)$ be the solution of (3) for $\rho = 0$. Fix any $t \in [0, T]$. Recall that $v^0(t; \alpha\lambda) = v^0(\alpha t; \lambda)$ for all $\alpha > 0$. Since $\lim_{\lambda \rightarrow \infty} v^0(t; \lambda) = v^0$, there exists $\bar{\lambda}^1 > 0$ such that

$$\begin{aligned} |v^0 - v^0(t; \lambda)| &= |v^0 - v^0(\lambda t; 1)| \\ &\leq \varepsilon/2 \end{aligned} \tag{A.2}$$

for all $\lambda \geq \bar{\lambda}^1$.

Since the right hand side of ODE (2) is continuous in ρ, λ , and uniformly Lipschitz continuous in v , the unique solution $v^*(t; \rho, \lambda)$ is continuous in ρ, λ for all $t \in [0, T]$. Recall that $v^*(t; \rho, \alpha\lambda) = v^*(\alpha t; \rho/\alpha, \lambda)$ for all $\alpha > 0$. Therefore by continuity in ρ , there exists $\bar{\lambda}^2 > 0$ such that

$$\begin{aligned} |v^*(t; \rho, \lambda) - v^0(t; \lambda)| &= |v^*(\lambda t; \rho/\lambda, 1) - v^0(\lambda t; 1)| \\ &\leq \varepsilon/2 \end{aligned} \tag{A.3}$$

for all $\lambda \geq \bar{\lambda}^2$. By adding (A.2) and (A.3), we obtain the desired inequality for $\bar{\lambda} = \max\{\bar{\lambda}^1, \bar{\lambda}^2\}$.

A.12 Proof of Proposition 12

Let $v(t)$ be the solution of ODE (2). The proof consists of five steps.

Step 1: We show that for any $t > 0$, $\mu(A(t)) \rightarrow 0$ as $\lambda \rightarrow \infty$. If not, there exist a positive value $\varepsilon > 0$ and an increasing sequence $(\bar{\lambda}_k)_{k=1,2,\dots}$ such that $\mu(A(t)) \geq \varepsilon$ for all $\bar{\lambda}_k$. Since X is compact and f is bounded from above, there exists $\eta > 0$ such that $\mu(A(v(t) + (\eta, \dots, \eta))) \geq \varepsilon/2$. In fact, since

$$\begin{aligned} \mu(A(v(t)) \setminus A(v(t) + (\eta, \dots, \eta))) &\leq \sum_{i \in N} \mu\left([v_i(t), v_i(t) + \eta] \times \prod_{j \neq i} [0, \bar{x}_j]\right) \\ &\leq f_H \sum_{i \in N} \eta \prod_{j \neq i} \bar{x}_j, \end{aligned}$$

we have $\mu(A(v(t) + (\eta, \dots, \eta))) \geq \varepsilon/2$ for $\eta = \frac{\varepsilon}{2f_H \sum_{i \in N} \prod_{j \neq i} \bar{x}_j}$. For this η , the integral

in ODE (2) is estimated as

$$\begin{aligned} \int_{A(t)} (x_i - v_i(t)) d\mu &\geq \int_{A(v(t) + (\eta, \dots, \eta))} (x_i - v_i(t)) d\mu \\ &\geq \int_{A(v(t) + (\eta, \dots, \eta))} \eta d\mu \\ &\geq \eta\varepsilon/2. \end{aligned}$$

By ODE (2),

$$v'_i(t) \geq -\rho\bar{x}_i + \bar{\lambda}_k \eta \varepsilon / 2,$$

which obviously grows infinitely as $\bar{\lambda}_k$ becomes large. This contradicts compactness of X .

Step 2: We compute the direction of $\int_{A(t)} (x_i - v_i(t)) d\mu$ in the limit as $\lambda \rightarrow \infty$. By Step 1, the boundary of X contains all accumulation points of $\{v_i(t) \mid \lambda > 0\}$ for fixed $t > 0$. Fix an accumulation point $v^*(t)$. There exists an increasing sequence $(\lambda_k)_{k=1,2,\dots}$ with $v^*(t) = \lim_{k \rightarrow \infty} v(t)$. By Assumption 3, there exists a unit normal vector of X at $v^*(t)$, which we denote by $\alpha \in \mathbb{R}_{++}$.

Step 1 implies that $v(t)$ is very close to the boundary of X when λ_k is very large. By smoothness of the boundary of X , $A(t)$ looks like a polyhedron defined by convex hull of $\{v(t), v(t) + (z_1(t), 0, \dots, 0), v(t) + (0, z_2(t), 0, \dots, 0), \dots, v(t) + (0, \dots, 0, z_n(t))\}$ where $z_i(t)$'s are positive length of edges such that the last n vertices are on the boundary of X . This vector $z(t)$ is parallel to $(1/\alpha_1, \dots, 1/\alpha_n)$. Let $r(t)$ be the ratio between the length of $z(t)$ and $(1/\alpha_1, \dots, 1/\alpha_n)$, i.e., $r(t) = z_1(t)\alpha_1 = \dots = z_n(t)\alpha_n$.

Since density f is bounded from above and away from zero, distribution μ looks almost uniform on $A(t)$ if λ_k is large. Then the integral $\int_{A(t)} (x_i - v_i(t)) d\mu$ is almost parallel to the vector from $v(t)$ to the barycenter of the polyhedron, namely, $z(t)/(n+1)$. Therefore $\int_{A(t)} (x_i - v_i(t)) d\mu$ is approximately parallel to $(1/\alpha_1, \dots, 1/\alpha_n)$ when λ_k is large.

Step 3: We show that $\sum_{i \in N} \alpha_i v'_i(t) \geq 0$ for large λ . Let $(\lambda_k)_{k=1,2,\dots}$ be the sequence defined in Step 2. For large λ_k , $A(t)$ again looks like a polyhedron with the uniform distribution. By Step 2, the ODE near $v_i(t)$ is written as

$$v'_i(t) = -\rho v_i(t) + \lambda_k \frac{z_i(t)}{n+1} \cdot \mu(A(t)). \quad (\text{A.4})$$

Note that $v_i(t)$ is close to $v_i^*(t)$ and $\mu(A(t))$ is order n of the length of $z(t)$. By replacing the above equation by $r(t)$, ODE (A.4) approximates

$$r'(t) = \rho a - \lambda_k b r(t)^{n+1} \quad (\text{A.5})$$

for some constants $a, b > 0$. Since $r(t)$ is large when t is small, the above ODE shows that $r(t)$ is decreasing in t . Therefore $\mu(A(t))$ is also decreasing in t . For large λ_k , this implies that

$$\alpha \cdot v'(t) = \sum_{i \in N} \alpha_i v'_i(t) \geq 0.$$

Step 4: We show that the Nash product is nondecreasing if λ is large. By ODE (A.4), we have

$$\alpha_i v'_i(t) = -\rho \alpha_i v_i(t) + \beta \tag{A.6}$$

where $\beta = \lambda_k \mu(A(t))/(n+1)$ independent of i . Let us assume without loss of generality that $\alpha_1 v'_1(t) \geq \dots \geq \alpha_n v'_n(t)$. Then we must have $1/\alpha_1 v_1(t) \geq \dots \geq 1/\alpha_n v_n(t)$.

Let $L(t) = \sum_{i \in N} \log v_i(t)$ be log of the Nash product. Then $L'(t) = \sum_{i \in N} v'_i(t)/v_i(t)$. By Chebyshev's sum inequality,

$$\begin{aligned} L'(t) &= \sum_{i \in N} \frac{v'_i(t)}{v_i(t)} \\ &\geq \frac{1}{n} \left(\sum_{i \in N} \alpha_i v'_i(t) \right) \left(\sum_{i \in N} \frac{1}{\alpha_i v_i(t)} \right) \geq 0. \end{aligned}$$

Hence, $L(t)$ is nondecreasing if λ_k is large. Moreover, equality holds if and only if $\alpha_1 v'_1(t) = \dots = \alpha_n v'_n(t)$ or $\alpha_1 v_1(t) = \dots = \alpha_n v_n(t)$.

Step 5: We show that $v(t)$ converges to a point in the Nash set as $\lambda \rightarrow \infty$. Step 4 shows that $L'(t)$ converges to zero as $\lambda \rightarrow \infty$. Then $\alpha_1 v'_1(t) = \dots = \alpha_n v'_n(t)$ or $\alpha_1 v_1(t) = \dots = \alpha_n v_n(t)$ in the limit of $\lambda \rightarrow \infty$. The former case implies $v'_i(t) = 0$ for all $i \in N$ by Step 3. Then ODE (A.6) shows that the latter case holds. Therefore the latter case always holds in the limit of $\lambda \rightarrow \infty$. This implies that the boundary of X at $v^*(t)$ is tangent to the hypersurface defined by "Nash product = $\prod_{i \in N} v_i^*(t)$." Hence any accumulation point $v^*(t)$ belongs to the Nash set.

Since we assumed that the Nash set consists of isolated points, $v^*(t)$ is isolated. If $v(t)$ does not converge to $v^*(t)$, there is $\delta > 0$ such that for any $\bar{\lambda}$ there exists $v(t)$ with $\lambda \geq \bar{\lambda}$. Let $\delta > 0$ be small such that there is no point in the Nash set in $\{x \in X \mid |v^*(t) - x| \leq \delta\}$. Since $v(t)$ is continuous with respect to λ , for any $\bar{\lambda}$, there exists $\lambda > \bar{\lambda}$ such that $\delta/2 \leq |v^*(t) - v(t)| \leq \delta$. Since $\{x \in X \mid \delta/2 \leq |v^*(t) - x| \leq \delta\}$ is compact, $v(t)$ must have an accumulation point in this set. This contradicts the fact that any accumulation point is contained in the Nash set. Furthermore, $v^*(t)$ does not depend on t since $v^*(t)$ is continuous in t .

A.13 Proof of Proposition 13

(Sketch of proof): The ODE (A.5) is approximated by a linear ODE, which has a solution converging to v^* with an exponential speed. Therefore for large λ , $r(t)$ is approximated by $r(t) = \left(\frac{\rho a}{\lambda b}\right)^{\frac{1}{n+1}}$. Since $\mu(A(t))$ is proportional to $r(t)^n$, $\mu(A(t)) = c\lambda^{-\frac{n}{n+1}}$ for a constant $c > 0$. the probability that players reach an agreement before time $-(T - s)$ is

$$1 - e^{-\int_{T-s}^T \mu(A(t))\lambda dt} = 1 - e^{-sc\lambda^{\frac{1}{n+1}}},$$

which converges to one as $\lambda \rightarrow \infty$.

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