Generic Impossibility of Partial Ex Post Implementation with General Utility Functions

Tadashi Hashimoto*

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Abstract

This paper examines possibility of partial ex post implementation under general utility functions which is not necessarily differentiable or quasi-linear with respect to money. We deal with an interdependent-value model in which there are two agents, two alternatives and each agent receives more than two dimensional private signal. The main result of this paper is that under generic utility functions, a public decision rule must be almost constant if it can be ex post incentive compatible with some transfer rule.

Keywords: robust mechanism design, ex post equilibrium, interdependent values, multi-dimensional signals.

1 Introduction

Harsanyi doctrine, which was introduced by Harsanyi (1967-68), is now a standardized assumption in modeling situations with asymmetric information. It asserts that the prior distribution over the state of nature is shared by all players and common knowledge. Although Harsanyi doctrine and its central concepts, Bayesian game and Bayesian equilibrium, have been contributing to deepen our understanding of asymmetric information situations, they have few convincing foundations[1]. There are plenty of situations to which Mertens and Zamir (1985)'s universal type space models are well suited rather than naïve Bayesian games with Bayesian equilibria.

In considering mechanism design, using Bayesian equilibrium as a solution concept, is inappropriate for situations in the realm of a universal type space model. If Harsanyi doctrine is satisfied among agents and the mechanism designer know this fact and the prior distribution, then mechanisms based on Bayesian incentive compatibility can generate desired outcomes.

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[1]See, for example, Morris (1995) and Gul (1998).

However, it is possible that a slight deviation from Harsanyi doctrine may cause a hazardous outcome.

On the other hand, *ex post equilibrium*[2] does not suffer from the above problem. it is a belief-free concept in which each agent have no incentive to deviate from the equilibrium strategy whichever beliefs agents may form. As formally argued in Bergemann and Morris (2005), ex post incentive compatibility is a natural concept of mechanism design in situations where Harsanyi doctrine is not guaranteed.

Unfortunately, a negative result about ex post equilibrium was found. Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2006, hereafter JMMZ) considered a model with multi-dimensional signals and interdependent values, and proved that under generic utility functions, an ex post incentive compatible mechanism must generate almost constant social outcomes. Their argument depends on the assumption that utility functions are quasilinear and differentiable.

This paper studies partial ex post implementation in the case that utility functions are not necessarily quasilinear nor differentiable. We considers a model with two agents and two alternatives. Each agent receives a multi-dimensional signal, and has a utility function depending on the other player's information. The mechanism designer can use monetary transfer, and as already mentioned agents may exhibit nonlinear preference with respect to money.

The main result of this paper is that even in the space of such general utility functions, an ex post incentive compatible mechanism returns almost constant social outcomes for generic utility functions. In a model with multi-dimensional signals, the functional form of each agent i's utility function puts a severe restriction on functional forms of incentive compatible mechanisms, and these two restrictions generically contradict each other. In fact, deriving such a contradiction for a general utility function is technically challenging. Instead, we focus on densely existing treatable functions and prove that this kind of contradiction occurs under them (Section 4). Also, we can prove that each of them has some neighborhood in which partial ex post implementation is impossible for almost all functions (Section 5).

Although quasilinear utility functions are predominant due to their tractability, they provide quite rough approximation of agents' behavior. As JMMZ and Milgrom (2004) mentioned, quasi-linear utility functions nicely fit when amount of monetary transfer is sufficiently small or agents have abundant liquidity. There are however several interesting situations in which large amounts of money are transfered and agents face liquidity problems. For example, Salant (1997) reported as a participant of spectrum actions that bidders had budget problems, and they were strategically important and unknown to the other bidders. It justifies us in considering that in general bidding-cost functions are non-linear, and depend on signals. There have been papers studying budget constrained agents[3] (Pitchik and Schotter (1988), Laffont and Robert (1996), Maskin (2000), Che and Gale (1998, 2000), Fang and Parreiras (2001), and

[2] It is introduced by Holmstrom and Myerson (1983) as *uniform incentive compatibility*, and termed by Cremer and Mclean (1985).

[3] In this paper only continuous utility functions are considered while budget constrained utility functions (hereafter BCUs) are discontinuous, so we cannot directly say BCUs are directly treated in this paper. However it is difficult to justify for payoff functions to take such a extreme form. BCUs are approximation of plausible continuous functions, and in fact BCUs can be seen as "limits" of continuous utility functions. For example,

Benoît and Krishna (2001)). Also, non-linear bidding-cost functions are extensively studied mainly in the context of all-pay auction. Examples of such studies are Moldovanu and Sela (2001, 2007) and Gavious, Moldovanu and Sela (2002).

Several papers found positive results about ex post incentive compatibility. See, for example, Dasgupta and Maskin (2000), Perry and Reny (2002), Chung and Ely (2006) and Bikhchandani (2005).

We make two remarks on the relation between JMMZ and this paper. First their result does not imply that of this paper. It is because their set of preferences is quite small, or nowhere dense mathematically speaking, in the space of general utility functions. Second, their techniques are not applicable to our environments, even if we restricted utility functions to be differentiable. The key part of their proof is that the first order derivative of agent i's utility function depends only on the state of nature. Obviously, this is true for quasilinear utility functions but in general false for non-quasilinear utility functions.

It is worth mentioning the difference between the result of this paper and Gibbard-Satterthwaite theorems, originated by Gibbard (1973) and Satterthwaite (1975). The most similar work is Barbera (1983), in which each agent has a continuous preference. The main difference is that, in this paper, agents state only finite-dimensional signals, while in Barbera (1983) they state their whole preferences, which contains infinite-dimensional information. In other words, this paper puts a strong restriction on the domain of social choice functions: the domain, or the set of possible preferences, can be embedded to a finite-dimensional space.

This paper is organized as follows.In section 2, we present the model and show the statement of the main theorem. In section 3, we derive a geometric necessary condition for ex post incentive comaptibility. In section 4, we define $\mathcal{L}[i]$ and see it is dense set and an impossibility result holds in it. In section 5, we finish the proof of the main theorem. In section 6, we discuss about extensions of the main theorem and related issues. Section 7 is the conclusion.

2 The Model

First we introduce mathematical notations. Let X be a topological space and S a subset of X. We denote by S° the interior of S, by $\overline{S}$ the closure of S, and by ∂S the boundary of S. We usually use topologies of Euclidean spaces rather than relative topologies. Otherwise, the underlying topology is mentioned.

We consider an environment with two agents $\mathcal{N} = \{1, 2\}$, and two alternatives $\mathcal{A} = \{a_1, a_2\}$. We denote agents by i and j, and $i \neq j$ unless otherwise mentioned. Each agent $i \in \mathcal{N}$ receives private signal $\theta^i \in \Theta^i$, where the signal space Θ^i is a compact convex subset of $\mathbb{R}^{d^i}$ ($d^i \in \{2, 3, \ldots\}$) whose interior is nonempty. Let $\Theta = \Theta^1 \times \Theta^2$. We allow the mechanism

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$$U(x,t) = \begin{cases} V(x) + t & \text{if } t \geq T, \\ -\infty & \text{otherwise,} \end{cases}$$

is the pointwise limit of

$$U_n(x,t) = \begin{cases} V(x) + t & \text{if } t \geq T, \\ V(x) + t - n(T-t) & \text{otherwise.} \end{cases}$$

designer to use monetary transfer, but the amount of money m is restricted to a nonempty compact interval M of $\mathbb{R}$.

Each agent $i \in \mathcal{N}$ has a utility function $u^i(a, m; \theta)$, where $a \in \mathcal{A}$ is an alternative, $m \in M$ is an amount of money and $\theta \in \Theta$ is a state of nature. For each such u^i, we define $\mu^i[u^i] : \Theta \times M^2 \to \mathbb{R}$ by

$$\mu^i[u^i](\theta; m_1, m_2) = u^i(a_1, m_1; \theta) - u^i(a_2, m_2; \theta). \tag{1}$$

We omit the argument u^i when it is apparent. $\mu^i[u^i]$ stands for the relative attractiveness of a_1 compared to a_2, when each a_k is accompanied by m_k amount of money.

Take a fixed non-zero $\eta^i \in \mathbb{R}^{d^i} \setminus \{0\}$ for each $i \in \mathcal{N}$. We let $\mathcal{U}^i$ be the set of $u^i : \mathcal{A} \times M \times \Theta \to \mathbb{R}$ satisfying the following properties:

1. u^i is continuous;
2. $u^i(a, m; \theta)$ is strictly increasing in m;
3. $\mu^i[u^i](\theta^i + \alpha\eta^i, \theta^j; m_1^i, m_2^i)$ is strictly increasing in α.

We refer to the third property as *monotonicity* of μ^i. We endow $\mathcal{U}^i$ with the uniform metric[4]. Let $\mathcal{U} = \mathcal{U}^1 \times \mathcal{U}^2$.

A *public decision rule*[5] is defined to be a mapping $f : \Theta \to \mathcal{A}$.

Definition 1. A public decision rule f is *almost constant*[6] if f is constant within Θ°.

We restrict mechanisms to direct mechanisms and equilibria to the truth-telling equilibrium. It is justified because the revelation principle is also applicable to ex post incentive compatibility. A *transfer rule* is defined as a pair of mappings $t = (t^1, t^2) : \Theta \to M^2$. A *direct mechanism* is a pair (f, t) of public decision rule f and transfer rule t.

Definition 2. A direct mechanism (f, t) is *ex post incentive compatible under* $u \in \mathcal{U}$ if for all $i \in \mathcal{N}$, $\theta \in \Theta$ and $\hat{\theta}^i \in \Theta^i$,

$$u^i(f(\theta), t^i(\theta); \theta) \geq u^i(f(\hat{\theta}^i, \theta^j), t^i(\hat{\theta}^i, \theta^j); \theta). \tag{2}$$

A public decision rule f is *ex post incentive compatible under* $u \in \mathcal{U}$ if there exists a transfer rule t such that (f, t) is ex post incentive compatible under u.

Using sections 3-5, we prove the following theorem. In section 6, this theorem will be extended in various directions.

Theorem 1. *There exists a residual set $\mathcal{R}$ of $\mathcal{U}$ such that, for all $u \in \mathcal{R}$ and public decision rule f, if f is ex post incentive compatible under u then f is almost constant.*

[4] In this case, the uniform metric d_∞ is defined by

$$d_\infty(u^i, v^i) = \max_{(a,m;\theta) \in \mathcal{A} \times M \times \Theta} |u^i(a, m; \theta) - v^i(a, m; \theta)|.$$

[5] JMMZ calls it a *social choice function.*

[6] In the terminology of JMMZ, an almost constant public decision rule is a *trivial* social choice function.

A residual set is topologically large set. It contains as a subset the countable intersection of open dense sets, or equivalently, its complement is at most the countable union of nowhere dense sets. Thus, we can say that a residual set contains as elements almost all or *generic* elements of the universal set.

The above theorem tells us that in generic situations, the mechanism designer is unable to construct any meaningful mechanism satisfying ex post incentive compatibility.

3 Geographical Properties

First, we generalize the *ex post taxation principle*,[7] which was proved by Chung and Ely (2006), to our non-quasilinear model. For convenience, we allow monetary transfer m to be $-\infty$ and define $u^i(a, -\infty; \theta) = -\infty$. We denote $M \cup \{-\infty\}$ by $\overline{M}$, and define T^i by

$$T^i = \{t^i : \mathcal{A} \times \Theta^j \to \overline{M} \mid \forall \theta^j \in \Theta^j,\ \exists a \in \mathcal{A},\ t^i(a, \theta^j) \neq -\infty\}. \tag{3}$$

Let $T = T^1 \times T^2$.

Proposition 1 (Chung and Ely 2006)**.** *Assume that a direct mechanism (f, t_*) is ex post incentive compatible under $u \in \mathcal{U}$. Then there exists $t \in T$ such that, for all $i \in \mathcal{N}$ and $\theta \in \Theta$,*

$$f(\theta) \in \operatorname*{argmax}_{a \in \mathcal{A}} u^i(a, t^i(a, \theta^j); \theta). \tag{4}$$

Proof. Since $u^i(a, m; \theta)$ is strictly increasing with respect to m, ex post incentive compatibility implies that $t^i_*(\theta) = t^i_*(\theta')$ if $f(\theta) = f(\theta')$ and $\theta^j = \theta'^j$. Thus

$$t^i(a, \theta^j) = \begin{cases} t^i_*(\hat{\theta}^i, \theta^j) & \text{if there exists } (\hat{\theta}^i, \theta^j) \in f^{-1}(a) \\ -\infty & \text{otherwise,} \end{cases} \tag{5}$$

is well-defined, and satisfies the above property. □

For notational simplicity, we denote $t^i(a_k, \theta^j)$ by $t^i_k(\theta^j)$ $(k = 1, 2)$, and $(t^i_1(\theta^j), t^i_2(\theta^j))$ by $t^i(\theta^j)$.

The taxation principle reduces agent i's decision problem to the following situation. Agent i chooses a from $\mathcal{A}$ after observing both θ^i and θ^j. If agent i's choice is a_k, then agent i receives $t^i_k(\theta^j)$ amount of money. When $t^i_\ell(\theta^j) = -\infty$, the alternative a_ℓ is "not for sale," so agent cannot buy a_ℓ and thus must choose $a_m \neq a_\ell$.

The ex post taxation principle tells us useful properties of ex post incentive compatible public choice rule f. Choose $t \in T$ satisfying (4). Then, for fixed $\bar{\theta}^j \in \Theta^j$, $\mu^i(\theta^i, \bar{\theta}^j; t^i(\bar{\theta}^j))$ is a continuous function of θ^i. Since μ^i increases as θ^i goes in η^i-direction, the sign of μ^i is as depicted in Figure 1. Due to (4), we know that $f(\theta^i, \bar{\theta}^j) = a_1$ for θ^i at which $\mu^i > 0$, and that $f(\theta^i, \bar{\theta}^j) = a_2$ for θ^i at which $\mu^i < 0$. As in Figure 1, two areas $\{\theta^i \mid f(\theta^i, \bar{\theta}^j) = a_1\}$ and $\{\theta^i \mid f(\theta^i, \bar{\theta}^j) = a_2\}$ are separated by

$$\big\{\theta^i \in \Theta^i \bigm| \mu^i(\theta^i, \bar{\theta}^j; t^i(\bar{\theta}^j)) = 0\big\}. \tag{6}$$

[7] This name is due to JMMZ.

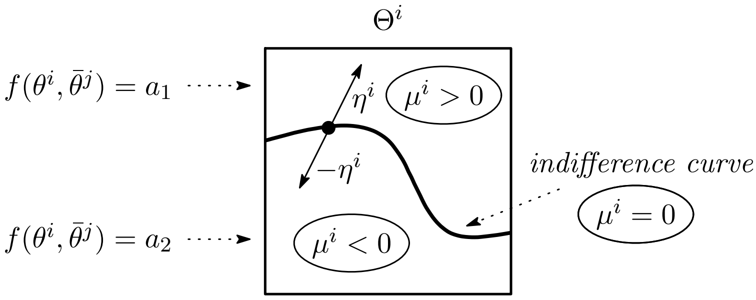

Figure 1: The relationship between the indifference curve and f.

We call the set (6) agent i's *indifference curve*[8] at $\bar{\theta}^j$, on which two alternatives are indifferent for agent i. Thus we can conclude that agent i's indifference curve at $\hat{\theta}^j$ determines the value of $f(\theta^i, \bar{\theta}^j)$ for almost all θ^i.

In the rest of the paper, we fully exploit good properties of closed sets. Although $f^{-1}(a_1)$ and $f^{-1}(a_2)$ are not necessarily closed sets, we can interpret f as a pair of closed set by taking the closures of $f^{-1}(a_1)$ and $f^{-1}(a_2)$. Define

$$A_k(f) = \overline{f^{-1}(a_k)} \cup \partial\Theta, \tag{7}$$
$$B(f) = A_1(f) \cap A_2(f) \tag{8}$$

for each public decision rule f and $k = 1, 2$. $B(f)$ is the *border* between $A_1(f)$ and $A_2(f)$. The reason these sets include $\partial\Theta$ is to avoid various singularities on it.

Given a metric space X, we denote $\mathcal{C}(X)$ to the set of all non-empty closed subsets of X, and endow $\mathcal{C}(X)$ with the Hausdorff metric. Let $\mathcal{S}$ be the set of $(A_1, A_2) \in \mathcal{C}(\Theta)^2$ such that $\partial\Theta \subseteq A_k$ for each k. Obviously $(A_1(f), A_2(f)) \in \mathcal{S}$ for all public decision rule f.

Using $(A_1(f), A_2(f))$, we obtain another necessary condition of ex post incentive compatibility.

Definition 3. $(u, t) \in \mathcal{U} \times T$ *implements* $(A_1, A_2) \in \mathcal{S}$ if

$$\left\{\theta \in \Theta \;\middle|\; \mu^i[u^i](\theta; t^i(\theta^j)) \geq 0\right\} \subseteq A_1 \quad \text{and} \tag{9}$$
$$\left\{\theta \in \Theta \;\middle|\; \mu^i[u^i](\theta; t^i(\theta^j)) \leq 0\right\} \subseteq A_2 \tag{10}$$

for each $i \in \mathcal{N}$. Also, $u \in \mathcal{U}$ *implements* $(A_1, A_2) \in \mathcal{S}$ if there exists $t \in T$ such that (u, t) implements (A_1, A_2).

Lemma 1. *Assume that a public decision rule f is ex post incentive compatible under $u \in \mathcal{U}$. Then u implements $(A_1(f), A_2(f))$.*

Proof. Take $t \in T$ satisfying equation (4). What we need to show is that if $\tilde{\mu}^i(\theta) = 0$ then $\theta \in B$. It is apparent when $\theta \in \partial\Theta$, so we assume $\theta \in \Theta^\circ$. In this case, by the monotonicity of μ^i, $f(\theta^i + \varepsilon\eta^i, \theta^j) = a_1$ and $f(\theta^i - \varepsilon\eta^i, \theta^j) = a_2$ for sufficiently small $\varepsilon > 0$. This implies $\theta \in B$. □

[8] Of course, this definition is different from the usual definition of indifference curves.

However, this condition is too weak. For example, any $u \in \mathcal{U}$ can implement $(A_1, A_2) = (\Theta, \Theta)$. This motivates us to put some restriction on (A_1, A_2).

Definition 4. $(A_1, A_2) \in \mathcal{S}$ is *monotonic* if, for all $\theta, \theta' \in \Theta^\circ$ satisfying $\theta' = \theta + (\alpha^1\eta^1, \alpha^2\eta^2)$ with some $\alpha^1, \alpha^2 > 0$,

1. if $\theta \in A_1$ then $\theta' \in A_1 \setminus A_2$, and
2. if $\theta' \in A_2$ then $\theta \in A_2 \setminus A_1$.

Let $\mathcal{S}^*$ be the set of monotonic $(A_1, A_2) \in \mathcal{S}$.

Lemma 2. *Assume that a public decision rule f is ex post incentive compatible under $u \in \mathcal{U}$. Then $(A_1(f), A_2(f)) \in \mathcal{S}^*$.*

Proof. See Appendix. □

The monotonicity of $(A_1(f), A_2(f))$ is a natural consequence of the monotonicity of μ^1 and μ^2. For simplicity, assume $\bar{\theta} \in \Theta^\circ$ satisfies $f(\bar{\theta}) = a_1$. Take small positive real numbers α^1 and α^2. Because of the monotonicity of μ^1, $f(\theta^1, \bar{\theta}^2) = a_1$ in a neighborhood of $\bar{\theta}^1 + \alpha^1\eta^1$. Also by the monotonicity of μ^2, we can find a neighborhood of $\bar{\theta}^2 + \alpha^2\eta^2$ in which $f(\theta^1, \theta^2) = a_1$. There is no a_2 around $(\bar{\theta}^1 + \alpha^1\eta^1, \bar{\theta}^2 + \alpha^2\eta^2)$, so it is not in $A_2(f)$.

The next proposition summarizes this section's argument. Let $\mathcal{S}^*_{\text{nc}}$ be the set of monotonic $(A_1, A_2) \in \mathcal{S}$ which is neither $(\Theta, \partial\Theta)$ or $(\partial\Theta, \Theta)$. Given $\mathcal{S}' \subseteq \mathcal{S}$, we define $\mathcal{U}(\mathcal{S}')$ as the set of $u \in \mathcal{U}$ implementing some $(A_1, A_2) \in \mathcal{S}'$.

Proposition 2. *Assume that a public decision rule f which is not almost constant is ex post incentive compatible under $u \in \mathcal{U}$. Then $u \in \mathcal{U}(\mathcal{S}^*_{\text{nc}})$.*

Thus, in order to prove Theorem 1, it suffices to show that $\mathcal{U} \setminus \mathcal{U}(\mathcal{S}^*_{\text{nc}})$ is residual, or equivalently that $\mathcal{U}(\mathcal{S}^*_{\text{nc}})$ is meager. For a technical reason, for each $i \in \mathcal{N}$ we introduce a set $\mathcal{S}^*[i]$, which is the set of $(A_1, A_2) \in \mathcal{S}^*$ such that $A^j_k(\theta^i) \neq A^j_k(\theta^{i\prime})$ for some $\theta^i, \theta^{i\prime} \in \Theta^{i\circ}$ and $k \in \{1, 2\}$, where $A^j_k(\theta^i)$ is defined by $A^j_k(\theta^i) = \{\theta^j \mid (\theta^i, \theta^j) \in A_k\}$. Since $\mathcal{S}^*_{\text{nc}} = \mathcal{S}^*[1] \cup \mathcal{S}^*[2]$, what we should show is that each $\mathcal{S}^*[i]$ is meager.

4 Density of Impossible Sets

In this section, we introduce a dense set $\mathcal{L}[i]$ of $\mathcal{U}$, which is proved to be disjoint with $\mathcal{U}(\mathcal{S}^*[i])$. This implies that the complement of $\mathcal{U}(\mathcal{S}^*[i])$ is a dense subset of $\mathcal{U}$ (Proposition 3).

4.1 Piecewise Linear Functions

Here we define $\mathcal{L}[i]$ for each i and prove that it is dense in $\mathcal{U}$ (Lemma 4). Before doing these things, we define piecewise linear functions.

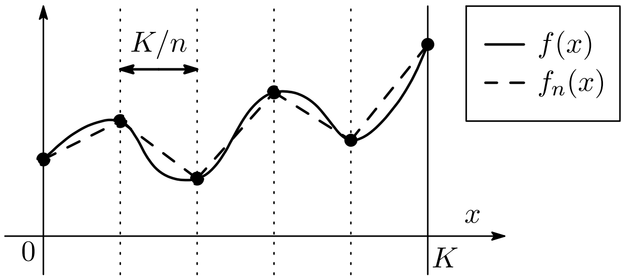

Figure 2: Piecewise linear approximation.

Definition 5. A function $f : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^d$) is *piecewise linear* if there exist non-empty open convex sets $U_1, \dots, U_n \subseteq D$ such that $D \subseteq \bigcup_{k=1}^{n} \overline{U_n}$ and $f|_{U_k}$ is affine for each k.[9] Also, $u^i_L \in \mathcal{U}^i$ is a *piecewise linear utility function* of agent i if $\mu^i[u^i_L]$ is piecewise linear.

We let $\mathcal{L}^i_0$ denote the set of piecewise linear utility function of agent i.

Lemma 3. *$\mathcal{L}^i_0$ is dense in $\mathcal{U}^i$.*

Proof. See Appendix. □

It is a mathematical fact that any continuous function whose domain is a compact subset of $\mathbb{R}^d$ is approximated by some sequence of piecewise linear functions. Figure 2 show an approximation procedure for the one-dimensional case. The original function f is uniformly continuous. Therefore, by taking $n \to \infty$, the sequence of approximating piecewise linear functions $\{f_n\}$ converges to the original function. By following a similar procedure, we obtain that $\mathcal{L}_0$ is a dense set of $\mathcal{U}$.

For each $i \in \mathcal{N}$, define $\mathcal{L}[i]$ as the set of $u \in \mathcal{U}$ such that there exists non-zero $\zeta^i \in \mathbb{R}^{d^i}$ orthogonal to η^i satisfying these properties:

1. There exists $u^i_L \in \mathcal{L}^i_0$ such that, for all $\theta \in \Theta$ and $m \in M^2$,
$$\mu^i[u^i](\theta; m) = \mu^i[u^i_L](\theta, m) + (\zeta^i \cdot \theta^i)(\eta^j \cdot \theta^j). \tag{11}$$
2. $u^j \in \mathcal{L}^j_0$.
3. For all open set $U \subseteq \Theta \times M^2$, if $\mu^j[u^j]|_U$ is affine then the coefficient of θ^j in $\mu^j[u^j]|_U$ is not equal to $\alpha\eta^j$ for any $\alpha \in \mathbb{R}$.

We call $\mu^i[u^i_L]$ the *linear part* of $\mu^i[u^i]$ and $(\zeta^i \cdot \theta^i)(\eta^j \cdot \theta^j)$ its *cross-term part.*

Lemma 4. *For all $i \in \mathcal{N}$, $\mathcal{L}[i]$ is dense in $\mathcal{U}$.*

Proof. By adding a small perturbation, any element of $\mathcal{L}_0$ can be an element of $\mathcal{L}[i]$. Since $\mathcal{L}_0$ is dense (Lemma 3), $\mathcal{L}[i]$ is also dense. □

[9] $f|_{U_k}$ is the restriction of f to U_k. An affine function is a function $f : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^d$) satisfying $f(x) = \alpha_0 + \sum_{n=1}^{d} \alpha_n x_n$ with some $\alpha_0, \dots, \alpha_d \in \mathbb{R}$.

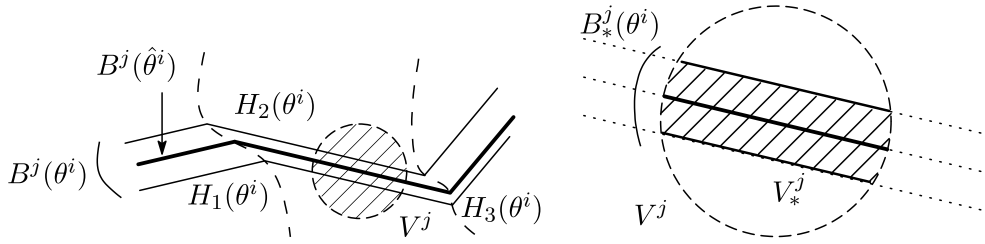

Figure 3: $B^j(\theta^i)$, $H_n(\theta^i)$ and V^j.

4.2 Impossibility

In this subsection, we prove that $\mathcal{L}[i]$ is disjoint with $\mathcal{U}(\mathcal{S}^*[i])$ (Lemma 6).

Given $(A_1, A_2) \in \mathcal{S}$, we denote $A_1 \cap A_2$ by B. We define $B^j(\bar{\theta}^i) = \{\theta^j \in \Theta \mid (\bar{\theta}^i, \theta^j) \in B\}$ for $\bar{\theta}^i \in \Theta^{i\circ}$.

Lemma 5. *Assume that $u \in \mathcal{L}[i]$ implements $(A_1, A_2) \in \mathcal{S}^*$. Then the interior of $B^j(\theta^i)$ is empty for all $\theta^i \in \Theta^{i\circ}$.*

Proof. The second statement is an corollary of the first. The proof of the first part is in Appendix. □

Assume that there are $\hat{\theta}^i \in \Theta^{i\circ}$ and open set $U^j \subseteq \Theta^{j\circ}$ such that $(\hat{\theta}^i, \theta^j) \in B$ for all $\theta^j \in U^j$. Then, for "almost all" $\theta^j \in U^j$, agent i's indifference curve $\{\theta^i \mid \mu^i(\theta^i, \theta^j; t^i(\theta^j)) = 0\}$ goes through $\hat{\theta}^i$. Let θ^j go η^j-direction. Then, μ^i's cross-term $(\zeta^i \cdot \theta^i)(\eta^j \cdot \theta^j)$ rotates i's indifference curve around $\hat{\theta}^i$. At the same time, the monotonicity of (A_1, A_2) pulls back i's indifference curve in $(-\eta^i)$-direction. These two effects contradict each other, and therefore $B^i(\theta^j)$ cannot have a nonempty interior for any $\theta^i \in \Theta^{i\circ}$.

As a corollary of this lemma, the following would be intuitively obvious: $B^j(\cdot) : \Theta^{i\circ} \to \mathcal{C}(\Theta^j)$ is continuous. Its formal proof is in Appendix.

The next lemma is the main result of this subsection:

Lemma 6. *If $u \in \mathcal{L}[i]$ implements $(A_1, A_2) \in \mathcal{S}^*$, then $(A_1, A_2) \notin \mathcal{S}^*[i]$.*

Sketch of Proof. Here we see a rough sketch of the proof. Details are in Appendix.

Assume that $u \in \mathcal{L}[i]$ implements $(A_1, A_2) \in \mathcal{S}^*[i]$. Since $(A_1, A_2) \in \mathcal{S}^*[i]$, $B^j(\theta^i)$ cannot be constant. Take $\hat{\theta}^i$ at which $B^j(\theta^i)$ is not local constant.

Remember that μ^j is a piecewise linear function, so each $B^j(\theta^i)$ consists of segments of hyperplains (and $\partial\Theta^j$)[10]. We can take $\hat{\theta}^i$ as an element of $\Theta^{i\circ}$ whose $B^j(\theta^i)$ consists of the largest number of segments of hyperplains among $\{B(\theta^i) \mid \theta^i \in \Theta^{i\circ}\}$. Denote $B^j(\hat{\theta}^i)$'s (closed) segments of hyperplains by $\hat{H}_1, \ldots, \hat{H}_N$. Then, because of the continuity of $B^j(\cdot)$, we obtain:

Fact: There exist a neighborhood U^i of $\hat{\theta}^i$ and continuous mappings $H_1, \ldots, H_N : U^i \to \mathcal{C}(\Theta^i)$ such that

[10] To avoid unnecessarily complicated arguments, here we skip the definitions of "a segment of a hyperplain." It is formally defined in Appendix.

1. $H_n(\hat{\theta}^i) = \hat{H}_n$,

2. $H_n(\theta^i)$ *is a segments of hyperplain parallel to* $\hat{H}_n$*, and*

3. $B^j(\theta^i) = \bigcup_{n=1}^{N} H_n(\theta^i) \cup \partial\Theta^j$,

for all n *and* $\theta^i \in U^i$.

There must be n such that $H_n(\theta^i)$ is not locally constant at $\hat{\theta}^i$. Take sufficiently small open sets $\hat{\theta}^i \in V^i \subseteq U^i$ and $V^j \subseteq \Theta^j$ such that

1. $B^j(\theta^i) \cap V^j = H_n(\theta^i) \cap V^j \neq \varnothing$, and
2. $B^j(\theta^i) \cap V^j = H \cap V^j$ for some hyperplain (not a segment) H,

for all $\theta^i \in V^i$. Define a mapping $B^j_* : V^i \to \mathcal{C}(V^j)$ by

$$B^j_*(\theta^i) = B^j(\theta^i) \cap V^j. \tag{12}$$

Let $\mathcal{H}$ be the image of B^j_*, i.e., $\{B^j_*(\theta^i) \mid \theta^i \in V^i\}$, and V^j_* be the union of $\mathcal{H}$. Also, define $B^i_* : V^j_* \to \mathcal{C}(V^i)$ by

$$B^i_*(\theta^j) = B^i(\theta^j) \cap V^i. \tag{13}$$

The elements of $\mathcal{H}$ satisfy an all-or-nothing property: for all $H, H' \in \mathcal{H}$, H and H' are equal or disjoint. This implies the following fact:

Fact: $B^i_*(\cdot)$ *is nonempty and constant within* $H \in \mathcal{H}$.

It is very difficult for $B^i_*(\theta^j)$ to be constant because μ^i's cross-term twists i's indifference curve $\{\theta^i \mid \mu^i(\theta; t^i(\theta^j)) = 0\}$, and obviously it is possible only when $B^i_*(\theta^j)$ has a non-empty interior. Now what we should do is to find $\theta^j \in V^j_*$ whose $B^i_*(\theta^j)$ has the empty interior. The following fact is useful to find such θ^j.

Fact: For an open set $U^j \in \Theta^j$*, there is* $\theta^j \in U^j$ *at which* $B^i(\theta^j)$ *has a non-empty interior.*

$V^{j\circ}_*$ is a non-empty open set since $B^j_*(\theta^i)$ is continuous and non-constant. Thus there must be $\theta^j \in V^j_*$ such that the interior of $B^j_*(\theta^j)$ is empty; this completes the proof. □

The proposition below is the summary of this section.

Proposition 3. $\mathcal{U} \setminus \mathcal{U}(\mathcal{S}^*[i])$ *is dense in* $\mathcal{U}$ *for all* $i \in \mathcal{N}$.

Proof. This is because $\mathcal{L}[i] \subseteq \mathcal{U} \setminus \mathcal{U}(\mathcal{S}^*[i])$ and Lemmata 4 and 6. □

5 The Proof of The Main Theorem

In order to prove $\mathcal{U}(\mathcal{S}[i])$ is meager, it suffices to find $\{\mathcal{K}_n\}_{n=1}^{\infty}$ ($\mathcal{K}_n \subseteq \mathcal{U}(\mathcal{S}[i])$) such that (i) the complement of each $\mathcal{K}_n$ is dense, (ii) each $\mathcal{K}_n$ is closed, and (iii) $\mathcal{U}(\mathcal{S}^*_{\mathrm{nc}}) = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. In the previous section we proved the complement of $U(\mathcal{S}^*[i])$ is dense, so (i) is automatically satisfied. Thus, what we should do is to find $\{\mathcal{K}_n\}_{n=1}^{\infty}$ satisfying (ii) and (iii). The next lemma is a useful tool to check (ii).

Lemma 7. *Let $\mathcal{T}$ be a closed subset of $\mathcal{S}$. Then $\mathcal{U}(\mathcal{S}^* \cap \mathcal{T})$ is closed in $\mathcal{U}$.*

Proof. See Appendix. □

To get an intuition, consider the following simple situation. Assume that (u_n, t_n) implements $(A_{1,n}, A_{2,n}) \in \mathcal{S}^* \cap \mathcal{T}$ and that limits of these sequences exist. First, we informally see that (u, t) implements (A_1, A_2). By the definition of implementation, we know that, for each i and n,

$$\left\{\theta \mid \mu^i[u^i_n](\theta; t^i_n(\theta^j)) \geq 0\right\} \subseteq A_{1,n} \quad \text{and} \tag{14}$$

$$\left\{\theta \mid \mu^i[u^i_n](\theta; t^i_n(\theta^j)) \leq 0\right\} \subseteq A_{2,n}. \tag{15}$$

These inclusion relationships are preserved even in the limit:

$$\left\{\theta \mid \mu^i[u^i](\theta; t^i(\theta^j)) \geq 0\right\} \subseteq A_1 \quad \text{and} \tag{16}$$

$$\left\{\theta \mid \mu^i[u^i](\theta; t^i(\theta^j)) \leq 0\right\} \subseteq A_2 \tag{17}$$

It is because $\{\theta^i \mid \mu^i[u^i](\theta^i, \bar{\theta}^j; m_1, m_2) \geq 0\} \cup \partial\Theta$ and $\{\theta^i \mid \mu^i[u^i](\theta^i, \bar{\theta}^j; m_1, m_2) \leq 0\} \cup \partial\Theta$ are continuous with respect to u^i, m_1 and m_2 (Lemma 18 in Appendix). Also, we obtain $(A_1, A_2) \in \mathcal{T}$ because $\mathcal{T}$ is closed, although proving $(A_1, A_2) \in \mathcal{S}^*$ requires a subtle argument.

We see an example of closed subsets of $\mathcal{S}$. We denote the Hausdorff metric by ρ.

Example 1. Let $\Theta^i(\varepsilon)$ be the set of $\theta^i \in \Theta^i$ satisfying $\|\theta^i - \hat{\theta}^i\| \geq \varepsilon$ for all $\hat{\theta}^i \in \partial\Theta^i$. Define

$$\delta[i, \varepsilon](A_1, A_2) = \sup_{\substack{k \in \{1,2\} \\ \theta^j, \theta^{j\prime} \in \Theta^j(\varepsilon)}} \rho(A^i_k(\theta^j), A^i_k(\theta^{j\prime})). \tag{18}$$

$\delta[i, \varepsilon](A_1, A_2)$ represents $(A^i_1(\theta^j), A^i_2(\theta^j))$'s *degree of dependence* on θ^j within $\Theta^j(\varepsilon)$. Define $\mathcal{T}[i](\varepsilon)$ by

$$\mathcal{T}[i](\varepsilon) = \left\{(A_1, A_2) \in \mathcal{S} \;\middle|\; \delta[i, \varepsilon](A_1, A_2) \geq \varepsilon\right\}. \tag{19}$$

Then $\mathcal{T}[i](\varepsilon)$ is a closed subset of $\mathcal{S}$ (the proof is routine and left to the reader). Notice that $\delta[i, \varepsilon](A_1, A_2) = 0$ for all $\varepsilon > 0$ if and only if (A_1, A_2) ignores agent i.

Proposition 4. *For each $i \in \mathcal{N}$, there are $\mathcal{K}_1, \mathcal{K}_2, \ldots \subseteq \mathcal{U}(\mathcal{S}^*[i])$ such that each $\mathcal{K}_n$ is closed in $\mathcal{U}$ and $\mathcal{U}(\mathcal{S}^*[i]) = \bigcup_{n=1}^{\infty} \mathcal{K}_n$.*

Proof. Let $\mathcal{K}_n = \mathcal{U}(\mathcal{T}[i](1/n) \cap \mathcal{S}^*)$. Due to Lemma 7 and Example 1, we know that each $\mathcal{K}_n$ is closed. $\mathcal{U}(\mathcal{S}^*[i]) = \bigcup_{n=1}^{\infty} \mathcal{K}_n$ is also true since $\mathcal{S}[i] = \bigcup_{n=1}^{\infty} \mathcal{T}[i](1/n)$ as explained in Example 1. □

Finally we complete the proof of Theorem 1.

Proof of Theorem 1. Each $\mathcal{U}(\mathcal{S}^*[i])$ is meager by Propositions 3 and 4. Since any finite union of meager sets is also meager, $\mathcal{U}(\mathcal{S}^*_{\mathrm{nc}}) = \mathcal{U}(\mathcal{S}^*[1]) \cup \mathcal{U}(\mathcal{S}^*[2])$ is also meager. Let $\mathcal{R} = \mathcal{U} \setminus \mathcal{U}(\mathcal{S}^*_{\mathrm{nc}})$. Then $\mathcal{R}$ is residual. Also, by Proposition 2 and the definition of $\mathcal{U}(\mathcal{S}^*_{\mathrm{nc}})$, we obtain that for all $u \in \mathcal{R}$, if f is ex post incentive compatible under u then f must be almost constant. □

6 Extensions and Discussion

In this section, we discuss about several extensions of Theorem 1 and related issues.

To make our model general, we introduce new notations. Let $M \subseteq \mathbb{R}$ and $\Theta^i \subseteq \mathbb{R}^{d^i}$ for each i. Otherwise mentioned, we assume M is a compact interval and each Θ^i is a compact convex set whose interior is nonempty. We define the set $\mathcal{U}_0^i$ to consist $u^i : \mathcal{A} \times M_0 \times \Theta_0 \to \mathbb{R}$ such that

1. u^i is continuous, and
2. $u^i(a, m; \theta)$ is strictly increasing in m.

We endow $\mathcal{U}_0^i$ with the topology of uniform convergence on compacta[11]. Note that in the case that M and each Θ^i are compact, the topology of uniform convergence on compacta and the topology induced by the uniform metric coincide.

Let η^i be a non-zero element of $\mathbb{R}^{d^i}$ and denote $(\eta^i)_{i\in\mathcal{N}}$ by η. Define $\mathcal{U}_0^i(\eta)$ be the set of $u^i \in \mathcal{U}_0^i$ such that

$$\mu^i[u^i](\theta^i + \alpha\eta^i, \theta^j; m_1, m_2)$$

is strictly increasing in α.

Let $\mathcal{U}_0 = \mathcal{U}_0^1 \times \mathcal{U}_0^2$ and $\mathcal{U}_0(\eta) = \mathcal{U}_0^1(\eta) \times \mathcal{U}_0^2(\eta)$. Notice what we have considered is $\mathcal{U}_0(\eta)$.

6.1 n-Agent Models

Theorem 1, and other results we will argue in this section, can be easily extended to n-agent cases. We can reduce a n-agent model to a two-agent model by fixing $(\theta^3, \ldots, \theta^n)$, which is justified since we are studying ex post equilibrium.

6.2 The Case $M = \mathbb{R}$

We have restricted M to be a compact interval. Here we consider the case $M = \mathbb{R}$. Given $A > 0$, we say that a transfer rule $t : \Theta \to M^2$ is *A-bounded* if $\sup_{i,\theta} |t^i(\theta)| < A$, and that t is *bounded* if t is A-bounded for some $A > 0$.

Theorem 2. *Assume $M = \mathbb{R}$. Then, there exists a residual set $\mathcal{R}$ of $\mathcal{U}_0(\eta)$ such that the following holds for all $u \in \mathcal{R}$: For all public decision rule f and bounded transfer rule t, if (f, t) is ex post incentive compatible under u then f is constant.*

Proof. Applying the technique used in the proof of Theorem 1,[12] we can take a residual set $\mathcal{R}_n$ for each $n \in \{1, 2, \ldots\}$ whose element u satisfy the following: For all public decision rule f and n-bounded transfer rule t, if (f, t) is ex post incentive compatible under u then f is constant. Let $\mathcal{R} = \bigcap_{n=1}^{\infty} \mathcal{R}_n$. Then $\mathcal{R}$ is also residual and satisfies the desired property. □

[11] $\{u_n^i\}_{n=0}^{\infty}$ uniformly converges on compacta to u^i if and only if, for all compact set $K \subseteq M_0 \times \Theta_0$, $\max_{a\in\mathcal{A},\, x\in K} |u_n^i(a, x) - u^i(a, x)|$ converges to 0 as n goes to infinity. The topology of uniform convergence on compacta is the topology induced by this convergence concept.

[12] A few modifications are needed in approximating utility functions by piecewise linear functions.

6.3 Quasilinear Utility Functions

For quasilinear utility functions, we can achieve the generic impossibility result by using the completely same procedure as non-quasilinear utility functions.

6.4 Differentiable Utility Functions

We can also prove the generic impossibility theorem for differentiable utility functions, although $\mathcal{L}[i]$ is not dense in the space of differentiable utility functions. We use the following mathematical fact: In a topological space X, if $S \subseteq X$ is dense and $R \subseteq X$ is residual, then $S \cap R$ is residual in S. It is an easy exercise of general topology.

Define $C^n(\eta)$ $(n = 1, 2, \ldots, \infty)$ to be the set of $u \in \mathcal{U}_0(\eta)$ such that each $u^i(a, m; \theta)$ is n-times continuously differentiable with respect to m and θ.

Theorem 3. *There exists a residual set $\mathcal{R}$ of $C^n(\eta)$ such that if the pair of a public decision rule f and a bounded transfer rule t is ex post incentive compatible under $u \in \mathcal{R}$, then f is almost constant.*

Proof. This is because $C^n(\eta)$ is dense in $\mathcal{U}_0(\eta)$. □

6.5 Without the Monotonicity of μ^i

We can drop the assumption of the monotonicity of μ^i. First we see the cases of differentiable quasilinear utility functions, which are analyzed in JMMZ (2006). Let C^n_{QL} be the set of $u \in \mathcal{U}_0$ satisfying the following for each i:

1. $u^i(a, m; \theta)$ is quasilinear with respect to m,
2. $u^i(a, m; \theta)$ is n-times continuously differentiable with respect to θ, and
3. $\nabla_{\theta^i} \mu^i[u^i](\theta; m_1, m_2) \neq 0$ for all $m_1, m_2 \in M$ and $\theta \in \Theta^\circ$.

Theorem 4. *There exists a residual set $\mathcal{R}$ of C^n_{QL} such that the following holds for all $u \in \mathcal{R}$: For all public decision rule f, if f is ex post incentive compatible under u then f is constant.*

Proof. Let $\mathbb{Q}_{++} = \mathbb{Q} \cap (0, \infty)$, $\Theta_* = \mathbb{Q}^{d^1+d^2} \cap \Theta^\circ$ and $B^i(r; \theta^i) = \{\hat{\theta}^i \in \Theta^{i\circ} \mid \|\hat{\theta}^i - \theta^i\| < r\}$.

Take $q^i : \Theta_* \to \mathbb{Q}_{++}$ and $\eta^i : \Theta_* \to \mathbb{Q}^{d^i} \setminus \{0\}$ for each i. We denote (q_1, q_2) by q and (η^1, η^2) by η. Let $\mathcal{V}(q, \eta)$ be the set of $u \in C^n_{\mathrm{QL}}$ satisfying the following for all $\theta_* \in \Theta_*$: For all $m_1, m_2 \in M$ and $\theta \in B^1(q_1(\theta_*), \theta^1_*) \times B^2(q_2(\theta^1_*), \theta^2_*)$,

$$\eta^i(\theta_*) \cdot \nabla_{\theta^i} \mu^i[u^i](\theta; m_1, m_2) > 0. \tag{20}$$

Note that any $u \in C^n_{\mathrm{QL}}$ is in $\mathcal{V}(q, \eta)$ for some (q, η).

Applying the technique used in Theorem 3 to $\mathcal{V}(q, \eta)$, for each $\theta_* \in \Theta_*$, there exists a residual set $\mathcal{R}(q, \eta; \theta_*)$ in which ex post incentive compatible public decision rules must be constant within $B^1(q_1(\theta_*), \theta^1_*) \times B^2(q_2(\theta_*), \theta^2_*)$. Let $\mathcal{R}(q, \eta) = \bigcap_{\theta_* \in \Theta_*} \mathcal{R}(q, \eta; \theta_*)$. Then $\mathcal{R}(q, \eta)$

is residual in $\mathcal{V}(q,\eta)$, and under $u \in \mathcal{R}(q,\eta)$ ex post incentive compatible public decision rules must be almost constant. Define $\mathcal{M}(q,\eta) = C^n_{\mathrm{QL}} \setminus \mathcal{R}(q,\eta)$. Then $\mathcal{M}(q,\eta)$ is meager in C^n_{QL}.

Let $\mathcal{M} = \bigcup_{q,\eta} \mathcal{M}(q,\eta)$. Since candidates of q and η are countable, $\mathcal{M}$ is also meager. Let $\mathcal{R}$ be the complement of $\mathcal{M}$, in which ex post incentive compatible public decision rules must be almost constant. □

This theorem can be extended to the case $M = \mathbb{R}$. Note that under quasilinear utility functions, transfer rules always can be bounded. That is, if (f,t) is ex post incentive compatible then there is bounded $\hat{t}$ such that $(f,\hat{t})$ is ex post incentive compatible.

The difference between this result and the main theorem of JMMZ (2006) is the used toplogies. In this paper, we are using the uniform metric, while JMMZ (2006) considered the topology of the C^n-uniform convergence[13].

In order to derive the impossibility result, differentiable but not necessarily quasilinear utility functions are required that their μ^i are locally monotonic in some direction, which is automatically satisfied for quasilinear utility functions. For $n = 1,2,\ldots,\infty$, let C^n be the set of $u \in \mathcal{U}_0$ satisfying the following for each i:

1. $u^i(a,m;\theta)$ is n-times continuously differentiable with respect to m and θ, and
2. for all $\theta \in \Theta^\circ$, there exists non-zero $\eta^i \in \mathbb{R}^{d^i}$ such that

$$\eta^i \cdot \nabla_{\theta^i} \mu^i[u^i](\theta; m_1, m_2) > 0 \tag{21}$$

   for all $m_1, m_2 \in M$.

For continuous utility functions, we need a little complicated condition. Take continuous functions $\eta^i_* : \Theta^\circ \to \mathbb{R}^{d^i}$ and $r^i, s^i : \Theta^\circ \to (0,\infty)$ for each i. Consider a utility function $u^i \in \mathcal{U}^i_0$ satisfying the following: For all $\theta \in \Theta^\circ$ and $\eta^i \in \{\eta \in \mathbb{R}^{d^i} \mid \|\eta^i - \eta^i_*(\theta)\| < r^i(\theta)\}$,

$$\mu^i[u^i]\Big|_S(\theta^i + \alpha\eta^i, \theta^j; m_1, m_2)$$

is strictly increasing in α, where $S = B^1(s^1(\theta),\theta) \times B^2(s^2(\theta),\theta) \times M^2$. Let C^0 be the set of such utility functions with some $(\eta^i, r^i, s^i)_{i=1,2}$.

Theorem 5. *Let $n \in \{0,1,2,\ldots,\infty\}$. Then, there exists a residual set $\mathcal{R}$ of C^n such that the following holds for all $u \in \mathcal{R}$: For all public decision rule f, if f is ex post incentive compatible under u then f is constant.*

6.6 A Cardinality-Free Topology

Debreu (1968) defined a metric on the space of the closed preferences, which is one of the most famous topologies on preference spaces. Let X be a metric space and $\mathcal{P}$ be the set of

[13] $u_k(a,m;\theta) = v_k(a;\theta) + m$ C^n-uniformly converges to $u(a,m;\theta) = v(a;\theta) + m$ if and only if v_k uniformly converges to v and all up to n-th order derivatives of v_k also uniformly converge to those of v.

continuous, transitive and complete preference relations. Each element $\succsim \in \mathcal{P}$ is a closed subset of X^2.[14] Debreu (1968) endowed the preference space $\mathcal{P}$ with the Hausdorff metric.

In our model, preferences are dependent on $\theta \in \Theta$ so we need to extend this topology. We take two approaches to the extension. The first approach is to consider a state-dependent preference $\succsim(\theta)$ as a bundle of preferences $(\succsim(\theta))_{\theta \in \Theta}$. For state-dependent preferences $\succsim(\cdot)$ and $\succsim'(\cdot)$, we define a "uniform metric":[15]

$$d_1(\succsim(\cdot), \succsim'(\cdot)) = \sup_{\theta \in \Theta} \rho(\succsim(\theta), \succsim'(\theta))m, \tag{22}$$

where ρ is the Hausdorff metric.

The other approach is regarding a state-dependent preference $\succsim(\cdot)$ as preferences $\dot{\succsim}$ on (x, θ). One possible definition of $\dot{\succsim}$ would be $\{(x, \theta; y, \theta) \mid x \succsim(\theta)\ y\}$, but θ is redundant. Instead we can define it by $\{(x, y, \theta) \mid x \succsim(\theta)\ y\}$, and we denote it by $S(\succsim(\cdot))$. We define the second metric by

$$d_2(\succsim(\cdot), \succsim'(\cdot)) = \rho(S(\succsim(\cdot)), S(\succsim'(\cdot))). \tag{23}$$

In fact, these two topologies are the same because of the continuity of utility functions. A proof is in Appendix.

By additionally assuming the following assumption, we can show that our arguments are valid even under the metrics defined above. A little more detailed explanations are in Appendix.

Assumption 1. For all $u \in \mathcal{U}$ and $\theta \in \Theta$,

$$\mu^i[u^i](\theta; \max M, \min M) > 0 > \mu^i[u^i](\theta; \min M, \max M) \tag{24}$$

for each $i \in \mathcal{N}$.

6.7 Weakening Ex Post Incentive Compatibility

We have found that in various utility function spaces meaningful public decision rules cannot be ex post incentive compatible under generic utility functions. A possible question is whether we can make incentive compatibility possible by weakening the concept of incentive compatibility.

One natural way to weaken ex post incentive compatibility is localizing this concept. Ex post incentive compatibility is a global concept in the sense that the inequality

$$u^i(f(\theta), t^i(\theta); \theta) \geq u^i(f(\hat{\theta}^i, \theta^j), t^i(\hat{\theta}^i, \theta^j); \theta). \tag{2}$$

is satisfied globally. We can weaken this concept by restricting the range of θ and $\hat{\theta}^i$.

[14]Here, as usual, a binary relation P on X is defined as a subset of X^2, and xPy denotes $(x, y) \in P$.

[15]Since $(\succsim(\theta))_{\theta \in \Theta}$ is an element of a product space $\prod_{\theta \in \Theta} \mathcal{P}$, conventionally it is natural to use the product topology instead of the topology induced by this metric. The product topology is however so fine that if $(\succsim_n(\cdot))_{n=0}^{\infty}$ converges to $\succsim(\cdot)$ then $\succsim_n(\theta)$ and $\succsim(\theta)$ must be coincide for almost all θ. In fact, $\succsim_n(\theta) = \succsim(\theta)$ for all $\theta \in \Theta$ because our utility functions are continuous with respect to θ. Under such a strong topology, our arguments are useless.

Definition 6. Let $U^1 \subseteq \Theta^{1\circ}$ and $U^2 \subseteq \Theta^{2\circ}$ be open sets. A public decision rule f is *locally ex post incentive compatible within* $U^1 \times U^2$ *under* $u \in \mathcal{U}_0$ if there exists a transfer rule t such that the inequality (2) holds for all $i \in \mathcal{N}$, $\theta \in U^1 \times U^2$ and $\hat{\theta}^i \in U^i$.

For example, this condition is satisfied in the situation that ex post agent i of type $\theta^i \in U^i$ has no incentive to deviate from the truth-telling, that is, in the situation that (2) is satisfied for all $i \in \mathcal{N}$, $\theta^i \in U^i$, $\theta^j \in \Theta^j$ and $\hat{\theta}^i \in \Theta^i$.

Can we avoid the generic impossibility under such an incentive compatibility concept? The next result gives us a negative answer.

Theorem 6. *There exists a residual set* $\mathcal{R}$ *of* $\mathcal{U}_0(\eta)$ *such that for all* $u \in \mathcal{R}$ *the following holds: For all public decision rule* f *and connected open sets* $U^1 \subseteq \Theta^{1\circ}$ *and* $U^2 \subseteq \Theta^{2\circ}$*, if* f *is locally ex post incentive compatible within* $U^1 \times U^2$ *under* u*, then* f *is constant within* $U^1 \times U^2$*.*

Proof. Define $B^i(r;\theta^i) = \{\hat{\theta}^i \in \Theta^{i\circ} \mid \|\hat{\theta}^i - \theta^i\| < r\}$. Let $\mathbb{Q}_{++} = \mathbb{Q} \cap (0,\infty)$, $\Theta^i_* = \mathbb{Q}^{d^i} \cap \Theta^{i\circ}$ and $\Theta_* = \Theta^1_* \times \Theta^2_*$.

Given $q = (q_1, q_2) \in \mathbb{Q}^2_{++}$ and $\theta_* \in \Theta_*$, applying the technique used in the proof of Theorem 1, we can take a residual set $\mathcal{R}(q,\theta_*)$ whose element u satisfies the following: For all public decision rule f, if f is ex post incentive compatible within $B^1(q_1,\theta^1_*) \times B^2(q_2,\theta^2_*)$ under u, then f is constant within $B^1(q_1,\theta^1_*) \times B^2(q_2,\theta^2_*)$.

Let $\mathcal{R} = \bigcap_{q,\theta_*} \mathcal{R}(q,\theta_*)$. Since $\mathbb{Q}_{++}$ and Θ_* are countable sets, $\mathcal{R}$ is also residual. Take arbitrary $u \in \mathcal{R}$ and connected open set $U^i \subseteq \Theta^{i\circ}$ for each i, and assume that f is ex post incentive compatible under u. We see that f is constant within $U = U^1 \times U^2$.

Let S be the set of $(q,\theta_*) \in \mathbb{Q}^2_{++} \times \Theta_*$ satisfying $B^1(q_1,\theta^1_*) \times B^2(q_2,\theta^2_*) \subseteq U$. Then

$$\bigcup_{(q,\theta_*)\in S} \Big[B^1(q_1,\theta^1_*) \times B^2(q_2,\theta^2_*)\Big] = U \tag{25}$$

since each U^i is open. For each $(q,\theta_*) \in S$, f is constant within $B^1(q,\theta^1_*) \times B^2(q,\theta^2_*)$ because $u \in \mathcal{R}(q,\theta_*)$. Since each U^i is connected, this implies that f is constant within $U^1 \times U^2$. $\square$

It would be worth mentioning about another weakened incentive compatibility. Consider a concept of incentive compatibility in which (2) holds for all $\theta \in \Theta$ and $\hat{\theta}^i \in \Theta$ satisfying $\|\theta^i - \hat{\theta}^i\| < \varepsilon$ ($\varepsilon > 0$). This concept is also impossible for sufficiently small ε, since we can derive the ex post taxation principle for this concept.

7 Conclusion

We proved impossibility theorems about ex post incentive compatibility. These results tell us that constructing meaningful ex post incentive compatible mechanisms is usually impossible.

One possible next step is to find other concepts which is robust against subjective belief ex post incentive compatibility. This paper partly answers it with a negative result: Locally ex post incentive compatible mechanisms are locally constant in generic situations. Thus, we need to search for concepts other than localized ex post incentive compatibility.

APPENDIX

A Piecewise Linear Approximation

Consider a d-dimensional cube $[0,1]^d$.We construct a simplicial complex $\mathscr{C}^d$ whose union is equal to $[0,1]^d$ and which consists of d-dimensional simplices smaller than $[0,1/n]^d$, where n is a positive integer. Set $\alpha = 1/n$.

For notational simplicity, we identify a convex set with its vertices. For example, we identify $[0,\alpha]^2$ with $\{(0,0),(0,\alpha),(\alpha,0),(\alpha,\alpha)\}$. Let $V^d = \{0,\alpha,\dots,1\}^d$ be the set of vertices of $\mathscr{C}^d$. Define a strict total order $\succ^d$ on V^d as a lexicographic order:

$$(a_1,\dots,a_d) \succ^d (b_1,\dots,b_d) \;\Leftrightarrow\; \exists k, \forall i<k,\, a_i = b_i \text{ and } a_k > b_k. \tag{26}$$

Denote by $\succcurlyeq^d$ the induced weak order.

When $d=1$, we define $\mathscr{C}^1 = \{\{0,\alpha\},\{\alpha,2\alpha\},\dots,\{(n-1)\alpha,1\}\}$, which is obviously a simplicial complex. Now assume that we constructed a simplicial complex $\mathscr{C}^{d-1}$. For each $S^{d-1} \in \mathscr{C}^{d-1}$, $z \in \{\alpha,\dots,1\}$ and $y \in S^{d-1}$, define $S^d(y,z,S^{d-1})$ as the union of $\{(y',z-\alpha) \mid y' \in S^{d-1},\, y' \succcurlyeq^{d-1} y\}$ and $\{(y',z) \mid y' \in S^{d-1},\, y \succcurlyeq^{d-1} y'\}$. Now we can define $\mathscr{C}^d$ as the set of $S^d(y,z,S^{d-1})$ such that $S^{d-1} \in \mathscr{C}^{d-1}$, $y \in S^{d-1}$ and $z \in \{\alpha,\dots,1\}$.

Lemma 8.

1. *$\mathscr{C}^d$ is a simplicial complex whose union is $[0,1]^d$.*
2. *For $S^d \in \mathscr{C}^d$,*
   a. *there exists $x \in V^d$ such that $S^d \subseteq \prod_{n=1}^d [x_n, x_n+\alpha]$, and*
   b. *for each $k = d-1, d$, there exist $x, x' \in S^d$ such that $x - x' = \alpha e_k$, where e_k's k-th element is 1 and the other elements are 0.*

The second statement is obvious. To prove the first statement, we use the following notion and lemma. We say $(\mathscr{C}^d, \succ^d)$ satisfies *symmetric property* if, for all $S_1, S_2 \in \mathscr{C}^d$,

$$\begin{aligned}&\forall a,b \in S_1 \cap S_2,\\ &[\exists x_1 \in S_1 \setminus S_2,\, a \succ x_1 \succ b] \Leftrightarrow [\exists x_2 \in S_2 \setminus S_1,\, a \succ x_2 \succ b].\end{aligned} \tag{27}$$

Lemma 9. *Let $d \in \{2,3,\dots\}$ and assume that $\mathscr{C}^{d-1}$ is a simplicial complex and that $(\mathscr{C}^{d-1}, \succ^{d-1})$ satisfies symmetric property. Then $\mathscr{C}^d$ is also a simplicial complex, and $(\mathscr{C}^d, \succ^d)$ also satisfies symmetric property.*

Proof. Take different $S_1, S_2 \in \mathscr{C}^d$ such that $\operatorname{co} S_1 \cap \operatorname{co} S_2 \neq \varnothing$, where $\operatorname{co} X$ is the convex hull of X. For each S_k, there are unique $T_k \in \mathscr{C}^{d-1}$, $y_k \in T_k$ and $z_k \in \{\alpha,\dots,1\}$ such that $S_k = S^d(y_k, z_k, S_k)$. Let $T = T_1 \cap T_2$ ($\neq \varnothing$), and denote elements of T by $y^1,\dots,y^n$ so as to satisfy $y^1 \succ^{d-1} \dots \succ^{d-1} y^n$. Without loss of generality, we assume $y_1 \succcurlyeq^{d-1} y_2$.

Here, we see a representation of $\operatorname{co} S_1 \cap \operatorname{co} S_2$ for each case. These representations obviously imply that $\operatorname{co} S_1 \cap \operatorname{co} S_2$ is a common face of S_1 and S_2 and the property (27) is satisfied.

Case 1: $T_1 = T_2$ and $z_1 = z_2$ ($= z$).

In this case, $\operatorname{co} S_1 \cap \operatorname{co} S_2$ is written as $\operatorname{co}\{(y^1, z-\alpha),\dots,(y_1,z-\alpha),(y_2,z),\dots,(y^n,z)\}$.

Case 2: $T_1 \neq T_2$ and $z_1 = z_2$ ($= z$).

Due to the assumption, $\operatorname{co} S_1 \cap \operatorname{co} S2$ has a representation as the convex hull of $\{(y^1, z-\alpha),\dots,(y_1,z-\alpha),(y_2,z),\dots,(y^d,z)\} \setminus X$, where $X = \{(y_1,z-\alpha),(y_2,z)\} \setminus (T \times \{z-\alpha, z\})$.

Case 3: $z_1 + \alpha = z_2$.

In this case, $\mathrm{co}\, S_1 \cap \mathrm{co}\, S_2$ is the convex hull of $\{(y, z_1) \mid y \in T,\ y_1 \succcurlyeq^{d-1} y \succcurlyeq^{d-1} y_2\}$.

Case 4: $z_1 = z_2 + \alpha$.

In this case we can use Case 3. □

Lemma 8 is an easy corollary of Lemma 9. Using Lemma 8, we can do approximation using piecewise linear functions as explained in section .

Appendix B Proof of Lemma 2

At first, we define functions χ^i which is useful in the proof of Lemma 2. They are also used to prove Lemma 7.

Given $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we define $U(\varepsilon; x) = \{y \in \mathbb{R}^d \mid ||x - y|| < \varepsilon\}$. For each i, we let X^i be the set of $(\theta, \tau^i, \lambda^i) \in \Theta \times M^2 \times \mathbb{R}$ satisfying $\theta^i + \lambda^i \eta^i \in \Theta^i$. We denote by X^i_+ the set of $(\theta, \tau^i, \lambda^i) \in X^i$ satisfying $\lambda^i > 0$, and by X^i_- the set of $(\theta, \tau^i, \lambda^i) \in X^i$ satisfying $\lambda^i < 0$. We define two mappings $\tilde{\chi}^i_+, \tilde{\chi}^i_- : X^i \times \mathcal{U}^i \to (0, \infty)$ by[16]

$$\begin{aligned}\tilde{\chi}^i_+(\theta, \tau^i, \lambda^i; u^i) = \max\{\gamma > 0 \mid\ & \mu^i[u^i](\hat{\theta}^i, \theta^j; \tau^i) > \mu^i[u^i](\theta, \tau^i) \\ & \text{for all } \hat{\theta}^i \in \mathbb{R}^{d^i} \text{ s.t. } ||(\theta^i + \lambda^i \eta^i) - \theta^i|| < \gamma\},\end{aligned} \tag{28}$$

$$\begin{aligned}\tilde{\chi}^i_-(\theta, \tau^i, \lambda^i; u^i) = \max\{\gamma > 0 \mid\ & \mu^i[u^i](\hat{\theta}^i, \theta^j; \tau^i) < \mu^i[u^i](\theta, \tau^i) \\ & \text{for all } \hat{\theta}^i \in \mathbb{R}^{d^i} \text{ s.t. } ||(\theta^i + \lambda^i \eta^i) - \theta^i|| < \gamma\}.\end{aligned} \tag{29}$$

Lemma 10. *$\tilde{\chi}^i_+$ and $\tilde{\chi}^i_-$ are continuous for each $i \in \mathcal{N}$.*

Proof. We prove the case of $\tilde{\chi}^i_+$ only. Assume there is $(\hat{\theta}, \hat{\tau}^i, \hat{\lambda}^i; \hat{u}^i) \in X^i \times \mathcal{U}^i$ at which χ^i_+ is discontinuous. Then, since ξ^i_+ is bounded, there is a sequence $(\theta_n, \tau^i_n, \lambda^i_n; u^i_n)_{n=0}^{\infty}$ converging to $(\hat{\theta}, \hat{\tau}^i, \hat{\lambda}^i; \hat{u}^i)$ satisfying $\lim_{n\to\infty} \gamma_n \neq \hat{\gamma}$, where $\gamma_n = \tilde{\chi}^i_+(\theta_n, \tau^i_n, \lambda^i_n; u^i_n)$ and $\hat{\gamma} = \tilde{\chi}^i_+(\hat{\theta}, \hat{\tau}^i, \hat{\lambda}^i; u^i)$. Let $V_n = \{\theta^i \in \Theta^i \mid ||(\theta^i_n + \lambda^i_n \eta^i) - \theta^i|| < \gamma_n\}$ and $\hat{V} = \{\theta^i \in \Theta^i \mid ||(\theta^i + \lambda^i \eta^i) - \theta^i|| < \hat{\gamma}\}$.

Case 1: $\lim_{n\to\infty} \gamma_n > \hat{\gamma}$.

Let $V^* = \{\theta^i \in \Theta^i \mid ||(\theta^i + \lambda^i \eta^i) - \theta^i|| \leq \hat{\gamma} + \varepsilon\}$, where $\varepsilon = (\lim_{n\to\infty} \gamma_n - \gamma)/2$. Then there is N such that, for all $n > N$, $V^* \subseteq V[n]$. Thus, for all $\theta^i \in V^*$,

$$\mu^i[u^i_n](\theta^i, \hat{\theta}^j; \hat{\tau}^i) \geq \mu^i[u^i_n](\hat{\theta}; \hat{\tau}^i) \tag{30}$$

by the continuity of $\mu^i[u^i_n]$, and

$$\mu^i[u^i](\theta^i, \hat{\theta}^j; \hat{\tau}^i) \geq \mu^i[u^i](\hat{\theta}; \hat{\tau}^i) \tag{31}$$

by the uniform convergence of $\mu^i[u^i_n]$. By the third property of $\mathcal{U}^i$, $\mu^i[u^i]$ must be positive in within V^*. This contradicts the definition of $\hat{\gamma}$, since the radius of V^* is strictly larger than $\hat{\gamma}$.

Case 2: $\lim_{n\to\infty} \gamma_n < \hat{\gamma}$.

[16] If $\mu^i(\hat{\theta}^i, \theta^j; \tau^i)$ is not defined, the proposition $\mu^i(\hat{\theta}^i, \theta^j; \tau^i) > \mu^i(\theta, \tau^i)$ is false. The maximum is well-defined because it coincides with distance between $\theta^i + \lambda^i \eta^i$ and the boundary of the closure of $\{\hat{\theta}^i \in \Theta^i \mid \mu^i(\hat{\theta}^i, \theta^j; \tau^i) > \mu^i(\theta, \tau^i)\}$.

Then for each n, there must be $\bar{\theta}^i_n \in \mathbb{R}^{d^i}$ satisfying $||(\hat{\theta}^i_n + \lambda^i_n \eta^i) - \bar{\theta}^i_n|| = \gamma_n$ and either $\bar{\theta}^i_n \in \partial\Theta^i$ or $\mu^i[u^i_n](\bar{\theta}^i_n, \theta^j_n; \hat{\tau}^i_n) \leq \mu^i[u^i_n](\theta_n; \tau^i_n)$. Without loss of generality, $(\bar{\theta}^i_n)^\infty_{n=0}$ can be a converging sequence since Θ^i is compact. In the limit, $\bar{\theta}^i = \lim_{n\to\infty} \bar{\theta}^i_n$ must satisfy either $\bar{\theta}^i \in \partial\Theta^i$ or $\mu^i[u^i](\bar{\theta}^i, \hat{\theta}^j; \hat{\tau}^i) \leq \mu^i[u^i](\hat{\theta}; \hat{\tau}^i)$. This is a contradiction since $||(\hat{\theta}^i + \hat{\lambda}^i\eta^i) - \bar{\theta}^i|| < \hat{\gamma}$. □

Now we can define a continuous mapping $\chi^i : (0, \infty) \times \mathcal{U}^i \to (0, \infty)$ by

$$\chi^i(\check{\lambda}^i; u^i) = \min_{\substack{(\theta,\tau^i,\lambda^i)\in X^i \\ \lambda^i \geq \check{\lambda}^i \\ k\in\{+,-\}}} \tilde{\chi}^i_k(\theta, \tau^i, \lambda^i; u^i). \tag{32}$$

We state this fact in the form of lemma.

Lemma 11. *χ^i is continuous.*

Now we prove Lemma 2.

Proof of Lemma 2. What we should prove is that $(A_1(f), A_2(f)) \in \mathcal{S}^*$. We prove the first part of the definition of $\mathcal{S}^*$ only. The second part is similarly proved. Simply denote $\Phi_{i\pm}(u^i, t^i)$ by $\Phi_{i\pm}$. Define $\Phi_{i++} = \Phi_{i+} \setminus \Phi_{i-}$.

Take arbitrary $\theta, \theta' \in \Theta^\circ$ satisfying $\theta' - \theta = (\alpha^1\eta^1, \alpha^2\eta^2)$ with some $\alpha^1, \alpha^2 > 0$. Since $\theta, \theta' \in \Theta^\circ$, there is $\gamma > 0$ such that $U(\gamma; \theta^i), U(\gamma; \theta^{i\prime}) \subseteq \Theta^{i\circ}$ for each $i \in \mathcal{N}$. Denote $\bigcup_{a\in[0,\alpha^i]} U(\gamma, \theta^i + a\eta^i)$ by S^i. By the convexity of Θ^i, $S_i \subseteq \Theta^{i\circ}$. Notice each of $\theta_\varepsilon, T_1, T_2$ defined below is an element or a subset of $S^1 \times S^2$.

Let $r = \{\min_i \chi^i(\alpha^i, u^i)\}/2$. Take sufficiently small $\varepsilon > 0$ satisfying $\varepsilon < \min\{\gamma, r\}$. We prove that $U(r, \theta^{1\prime}) \times U(r, \theta^{2\prime}) \subseteq \Phi_{2++}$ ($\subseteq A_1 \setminus A_2$). Notice that $\Phi_{i++} \subseteq f^{-1}(a_1) \subseteq \Phi_{i+}$.

Step 1: There is $\theta_\varepsilon \in \Phi_{1+}$ such that $||\theta - \theta_\varepsilon|| < \varepsilon$.

By the definition of $A_1(f)$, there is $\theta_\varepsilon \in f^{-1}(a_1) \subseteq \Phi_{1+}$.

Step 2: $T_1 = U(r; \theta^{1\prime}) \times \{\theta^2_\varepsilon\} \subseteq \Phi_{2+}$.

For each $\hat{\theta}^1 \in U(r; \theta^{1\prime})$,

$$||(\theta^1_\varepsilon + \alpha^1\eta^1) - \hat{\theta}^1|| \leq ||\theta^1_\varepsilon - \theta^1|| + ||\theta^1 + \alpha^1\eta^1 - \hat{\theta}^1|| \tag{33}$$

$$< \varepsilon + r < \chi^1(\alpha^1, u^1). \tag{34}$$

By the definition of χ^1, $(\hat{\theta}^1, \theta^2_\varepsilon) \in \Phi_{1++} \subseteq f^{-1}(a_1) \subseteq \Phi_{2+}$.

Step 3: $T_2 = U(r; \theta^{1\prime}) \times U(r; \theta^{2\prime}) \subseteq \Phi_{2++}$.

It is the same as Step 2. □

C Proof of Lemma 5

We use a proof by contradiction. Assume that $B^j(\hat{\theta}^i)^\circ \neq \varnothing$ for $\hat{\theta}^i \in \Theta^{i\circ}$. Take $\hat{\theta}^j \in B^j(\hat{\theta}^i)^\circ$ and let ℓ^j be the set of $\alpha \in \mathbb{R}$ satisfying $\hat{\theta}^j + \alpha\eta^j \in B^j(\hat{\theta}^i)^\circ$.

We decompose μ^i into the linear part μ^i_L and the cross-term part ψ^i. Put ζ^1 so as to satisfy $\psi^1(\theta) = (\zeta^1 \cdot \theta^1)(\eta^2 \cdot \theta^2)$. Note that

$$\sum_{n=1}^{d^1} \zeta^1_n \left(\frac{\partial\psi^1}{\partial\theta^1_n}\right) = ||\zeta^1||(\eta^2 \cdot \theta^2), \tag{35}$$

and that when we substitute $\theta^2 = \hat{\theta}^2 + \alpha\eta^2$ into this equation, the value becomes $\hat{p} + \gamma\alpha$ where $\gamma = ||\zeta^1||$ and $\hat{p} = ||\zeta^1||(\eta^2 \cdot \hat{\theta}^2)$. Let $\mathcal{D}$ be the set of all $(\sum_{n=1}^{d^1} \zeta_n^1(\partial\mu_L^1/\partial\theta_n^1), \sum_{n=1}^{d^1} \eta_n^1(\partial\mu_L^1/\partial\theta_n^1))$, which is finite by the definition of $\mathcal{V}^1$.

We reduce the problem to a two-dimensional case, by considering a space spanned by η^1 and ζ^1 at $\hat{\theta}^1$. Let $S^1 = \{\hat{\theta}^1 + x_1\eta^1 + x_2\zeta^1 \mid x_1 \in \mathbb{R},\ x_2 \leq 0\}$. For each $\alpha \in \ell^2$, we define $\tilde{B}^1(\alpha) = S^1 \cap B^1(\hat{\theta}^2 + \alpha\eta^2)$. We identify each element $\hat{\theta}^1 + x_1\eta^1 + x_2\zeta^1$ of S^1 with (x_1, x_2),[17] and then S^1 can be seen as $\tilde{S}^1 = \mathbb{R} \times (-\infty, 0]$. For $\varepsilon > 0$ and $y \in \mathbb{R}^2 \setminus \{0\}$, define $H(\varepsilon; y) = \{z \in \tilde{S}^1 \mid ||z|| \leq \varepsilon,\ y \cdot z = 0\}$.

For each $\alpha \in \ell^2$, we define a correspondence $\phi(\alpha)$ by

$$\phi(\alpha) = \left\{ d \in \mathcal{D} \;\middle|\; \exists \varepsilon > 0,\ H(\varepsilon; d_1, d_2 + \hat{p} + \gamma\alpha) \subseteq \tilde{B}^1(\alpha) \right\}. \tag{36}$$

By the definition of implementation, $\phi(\alpha)$ is not empty. The following fact leads a contradiction since an uncountable set $\bigcup_{\alpha \in \ell^2} \phi(\alpha)$ is a subset of a finite set $\mathcal{D}$.

Fact: $\phi(\alpha_1)$ and $\phi(\alpha_2)$ are disjoint for $\alpha_1 > \alpha_2$.

Suppose that there is $d \in \phi(\alpha_1) \cap \phi(\alpha_2)$. For each $i \in \{1, 2\}$, because $d \in \phi(\alpha_i)$, there is $\varepsilon_i > 0$ such that $H(\varepsilon_i; d) \subseteq \tilde{B}^1(\alpha_i)$. Since $d_1 > 0$, there is $\delta > 0$ such that $\{(x_1, -\delta) \mid x_1 \in \mathbb{R}\} \cap H(\varepsilon_i; d)$ is a singleton for both i; denote the unique element by $(x_1^i, -\delta)$. By the fact $(x_1^i, -\delta) \in H(\varepsilon; d)$, we obtain $x_1^i = \delta(d_2 + \hat{p} + \gamma\alpha_i)/d_1$, which is strictly increasing with respect to α_i. Thus we can say that $(x_1^1, -\delta) \in \tilde{B}^1(\alpha_1)$, $(x_1^2, -\delta) \in \tilde{B}^1(\alpha_2)$, and $x_1^1 > x_1^2$, which contradicts $(A_1, A_2) \in \mathcal{S}^*$. □

Appendix D Proof of Lemma 6

Lemma 12. *Assume $(u, t) \in \mathcal{U} \times T$ implements $(A_1, A_2) \in \mathcal{S}^*$, and denote $A_1 \cap A_2$ by B. For $i \in \mathcal{N}$ and $\bar{\theta}^j \in \Theta^{j\circ}$ $(j \neq i)$, if $B^i(\bar{\theta}^j)^\circ = \varnothing$ then*

1. *$A_1^i(\bar{\theta}^j) = \Phi_{i+}^i(\bar{\theta}^j | u^i, t^i)$ and $A_2^i(\bar{\theta}^j) = \Phi_{i-}^i(\bar{\theta}^j | u^i, t^i)$, and*
2. *$A_k^i(\cdot) : \Theta^j \to \mathcal{C}(\Theta^i)$ is continuous at $\bar{\theta}^j$ for each $k \in \{1, 2\}$.*

Proof. We start with a proof of the first statement. We prove $A_1^i(\bar{\theta}^j) = \Phi_{i+}^i(\bar{\theta}^j | u^i, t^i)$ only. The other equation can be proved similarly. Let $\theta_n^j = \bar{\theta}^j + n^{-1}\eta^j$. For notational simplicity, we denote $A_k^i(\bar{\theta}^j)$ by X_k, $A_k^i(\theta_n^j)$ by $X_{k,n}$, $\Phi_{i\ell}^i(\bar{\theta}^j | u^i, t^i)$ by Y_ℓ, and $\Phi_{i\ell}^i(\theta_n^j | u^i, t^i)$ by $Y_{\ell,n}$.

By the definition of implementation, we already know $X_1 \subseteq Y_+$. We use a proof by contradiction. Assume there is $x \in X_1 \setminus Y_+$. Since x is in an open set $\Theta \setminus Y_+$, which is a subset of X_2, there is $\alpha > 0$ and $\varepsilon_1 > 0$ such that $U(\varepsilon_1; x + \alpha\eta^i) \subseteq \Theta \setminus X_2$.

Step 1: There is $N > 0$ such that if $n > N$ then $x_n = x + n^{-1}\eta^i \in Y_{+,n}$.

Notice that $X_{1,n} \setminus X_{2,n} \subseteq Y_{+,n}$. By the monotonicity of (A_1, A_2), $x + n^{-1}$ is in $X_{1,n} \setminus X_{2,n}$ and thus in $Y_{+,n}$ if it is in $\Theta^{i\circ}$.

Step 2: There is a sequence $(\hat{x}_n)_{n=0}^\infty \in \prod_{n=0}^\infty Y_{-,n}$.

Otherwise, there is $\varepsilon_2 > 0$ and a subsequence $(Y_{+,n(m)})_{m=0}^\infty$ of $(Y_{+,n})_{n=0}^\infty$ such that $U(\varepsilon_2; \hat{x}) \subseteq Y_{+,n(m)}$ $(\subseteq X_{1,n(m)})$. By the closedness of A_1, $U(\varepsilon_2; \hat{x}) \subseteq X_1$, and thus $U(\min_i \varepsilon_i; \hat{x}) \subseteq X_1 \cap X_2$. This contradicts the assumption that the interior of $X_1 \cap X_2 = B^i(\bar{\theta}^j)$ is empty.

Step 3: Contradiction.

By Step 1, $\mu^i[u^i](x_n, \theta_n^j; t^i(\theta_n^j)) \geq 0$, and, by Step 2, $\mu^i[u^i](\hat{x}_n, \theta_n^j; t^i(\theta_n^j)) \leq 0$. Therefore we obtain

$$\mu^i[u^i](x_n, \theta_n^j; t^i(\theta_n^j)) - \mu^i[u^i](\hat{x}_n, \theta_n^j; t^i(\theta_n^j)) \geq 0. \tag{37}$$

[17]The function $(x_1, x_2) \mapsto \hat{\theta}^1 + x_1\eta^1 + x_2\zeta^1$ is a bijection since η^1 and ζ^1 are linearly independent.

Notice that $t^i(\theta^j_n)$ must be in M^2. Taking a converging subsequence of $(t^i(\theta^j_n))_{n=0}^{\infty}$, we obtain

$$\mu^i[u^i](x, \bar{\theta}^j; \tilde{\tau}^i) - \mu^i[u^i](x + \alpha\eta^i, \bar{\theta}^j; \tilde{\tau}^i) \geq 0 \tag{38}$$

by the continuity of $\mu^i[u^i]$, where $\tilde{\tau}^i$ is the limit of the subsequence. This contradicts the definition of $\mathcal{U}^i$.

Now we prove the second part. Take arbitrarily $\varepsilon > 0$ and a sequence $(\theta^j_n)_{n=0}^{\infty}$ of $\Theta^{j\circ}$ converging to $\bar{\theta}^j$. Again, for notational simplicity, we denote $A^i_h(\bar{\theta}^j)$ by S_h and $A^i_k(\theta^j_n)$ by $S_{h,n}$. Put $\ell \neq k$.

Step 1: There is no sequence $(x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} S_{k,n}$ such that $\min_{x \in S_k} ||x_n - x|| \geq \varepsilon$ for each n.

We use a proof by contradiction. Assume such a sequence exists. Since $(x_n)_{n=0}^{\infty}$ is a sequence of a compact space Θ^i, there is its converging subsequence $(x_{n(m)})_{m=0}^{\infty}$. Denote its limit by $\hat{x}$. Then $\hat{x}$ must not be in S_k by the construction of $(x_n)_{n=0}^{\infty}$, but we can obtain $\hat{x} \in S_k$ by the closedness of A_k. This is a contradiction.

Step 2: For all $x \in S^\circ$, it is impossible that, for all $N > 0$, there is $n > N$ such that $U(\varepsilon; x) \subseteq \Theta \setminus S_n$.

Again we use a proof by contradiction. Assume we can take a subsequence $(S_{n'(m)})_{m=0}^{\infty}$ of $(S_n)_{n=0}^{\infty}$ satisfying $U(\varepsilon; x) \subseteq \Theta \setminus S_{k,n'(m)}$ for each m. This implies that each $S_{\ell,n'(m)}$ contains $\overline{U}(\varepsilon; x)$ as its subset. Since A_ℓ is closed, $\overline{U}(\varepsilon; x)$ is also forced to be a subset of S_ℓ. Therefore a nonempty open set $U(\varepsilon; x) \cap S_k^\circ$ is a subset of $S_k \cap S_\ell = B^i(\bar{\theta}^j)$. This contradicts the assumption $B^i(\bar{\theta}^j)^\circ = \varnothing$.

Step 3: Convergence.

By Step 1, $\overline{\lim}_{n\to\infty} \max_{x' \in S_{k,n}} \min_{x \in S_k} ||x - x'|| \leq \varepsilon$.

We explain $\max_{x \in S_k} \min_{x' \in S_{k,n}} ||x - x'|| \leq \varepsilon$ for sufficiently large n. S_k is a union of three sets: $S_k \setminus S_\ell$, $(S_k \cap S_\ell) \setminus \partial\Theta^i$, and $\partial\Theta^i$. Since $S_k \setminus S_\ell$ is an open set, we can apply Step 2 to each $x \in S \setminus T$. As a corollary of the first statement of this lemma, each point of $(S_k \cap S_\ell) \setminus \partial\Theta^i$ can be approximated by some sequence of $S_k \setminus S_\ell$. Thus, for each $x \in S_k \setminus \partial\Theta^i$, $\min_{x' \in S_{k,n}} ||x - x'|| \leq \varepsilon$. For $x \in \partial\Theta^i$, $\min_{x' \in S_{k,n}} ||x - x'|| = 0$ since x is also in S_n. Therefore we obtain $\max_{x \in S_k} \min_{x' \in S_{k,n}} ||x - x'|| \leq \varepsilon$ for sufficiently large n.

Because ε is taken arbitrary, both $\max_{x' \in S_{k,n}} \min_{x \in S_k} ||x - x'||$ and $\max_{x \in S_k} \min_{x' \in S_{k,n}} ||x - x'||$ must converge to 0, as n goes to infinity. This is the definition of convergence of $S_{k,n}$ to S_k. □

D.1 Hyperplain Segments and Hyperplain Complexes

We define a (d-dimensional) *hyperplain segment* as a closed convex subset H of some d-dimensional hyperplain H' such that, under the relative topology of H', the interior of H is nonempty. We let $\mathcal{H}^n_{\text{seg}}$ denote the set of d-dimensional hyperplain segments. For each $d \in \mathbb{N} \setminus \{0\}$ and $H \in \mathcal{H}^d_{\text{seg}}$, we define $o(H)$ as the set of $x \in \mathbb{R}^{d+1}$ orthogonal to H, that is, x satisfying $x \cdot y = x \cdot y'$ for all $y, y' \in H$.

We call a finite set $\{H_1, \ldots, H_K\}$ of $\mathcal{H}^d_{\text{seg}}$ a (d-dimensional) *hyperplain complex* if there is no other $\{H'_1, \ldots, H'_L\}$ such that $\bigcup_{k=1}^K H_k = \bigcup_{\ell=1}^L H'_\ell$ and $L < K$. We denote by $\mathcal{H}^d_{\text{cx}}$ the set of hyperplain complexes. For each unit vector $\eta \in \mathbb{R}^{d+1}$ and subset $X \subseteq \mathbb{R}^{d+1}$ with $d \in \{1, 2, \ldots\}$, we define $\mathcal{H}^d_{\text{cx}}(\eta, X)$ as the set of $C \in \mathcal{H}^d_{\text{cx}}$ such that $(\bigcup C) \cap \{\alpha\eta + x \mid \alpha \in \mathbb{R}\}$ is the empty set or a singleton for all $x \in \mathbb{R}^{d+1}$. Let $\mathcal{H}^d_{\text{un}}(\eta, X) = \{\bigcup C \mid C \in \mathcal{H}^d_{\text{cx}}(\eta, X)\}$. For $S \in \mathcal{H}^d_{\text{un}}(\eta, X)$, define $c(S) = \{C \in \mathcal{H}^d_{\text{cx}}(\eta, X) \mid \bigcup C = S\}$ and $N(S) = |C|$, where $C \in S$.

Hereafter we consider a compact convex subset X of $\mathbb{R}^{d+1}$ with nonempty interior.

Lemma 13. *Assume that a sequence $(S_t)_{t=0}^{\infty}$ of $\mathcal{H}_{\text{un}}(\eta, X)$ converges to $S \in \mathcal{H}_{\text{un}}(\eta, X)$, that $N(S_t) \leq N(S)$ for all t, and that $\mathscr{O} = \{o(H) \mid \exists S' \in \{S, S_0, S_1, \ldots\},\ H \in C \in c(S')\}$ is a finite set. Let $C \in c(S)$. Then the following statements are satisfied:*

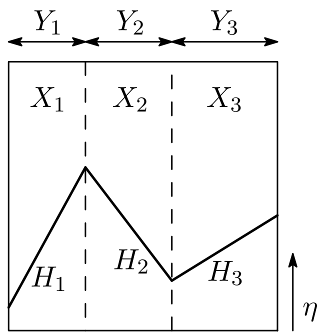

Figure 4:

1. *for each $H \in C$, there is a sequence $(C_t, H_t)_{t=0}^{\infty}$ such that*
   a. *$H_t \in C_t \in c(S_t)$ for each t,*
   b. *$(H_t)_{t=0}^{\infty}$ converges to H, and*
   c. *there is $t' > 0$ such that, for all $t > t'$, $o(H) = o(H_t)$;*
2. *there is $t' > 0$ such that, for all $t > t'$, $N(S) = N(S_t)$.*

The next lemma is used to prove Lemma 13. For $\varepsilon > 0$, we say that $S \subseteq \mathbb{R}^d$ is *ε-thick* if S contains some ε-open ball as a subset. For any $S \subseteq \mathbb{R}^d$, we say that S is *0-thick.* We define $r(S) = \sup\{r \geq 0 \mid S \text{ is } r\text{-thick}\}$ for each $S \subseteq \mathbb{R}^d$.

Lemma 14. *Let $S \subseteq \mathbb{R}^d$ be r-thick for $r > 0$. Assume that there is some disjoint open convex sets $A_1, \dots, A_K \subseteq S$ such that the union of their closure is S. Then A_k is $(2^{-(K+1)}r)$-thick for some $k \in \{1, \dots, K\}$.*

Proof. We use mathematical induction with respect to K. The proof is obvious when $K = 1$.

Now we give the proof for the case $K = \bar{N} > 1$ given that this lemma is true for $K = \bar{K} - 1$. By the separating hyperplain theorem, we can take a hyperplain $\{x \in \mathbb{R}^d \mid \alpha^\top x = \beta\}$ such that $A_1 \subseteq T_1$ and $A_2 \subseteq T_2$, where $T_1 = \{x \in \mathbb{R}^d \mid \alpha^\top x > \beta\}$ and $T_2 = \{x \in \mathbb{R}^d \mid \alpha^\top x < \beta\}$. We can immediately see that $S \cap T_1$ or $S \cap T_2$ is $(r/2)$-thick. If $S \cap T_i$ is $(r/2)$-thick, we can say that at least one of the elements of $\{A_i \cap T_i, A_3 \cap T_i, \dots, A_K \cap T_i\}$ is $(2^{-(K+1)}r)$-thick by the assumption of mathematical induction. When $A_\ell \cap T_i$ is $(2^{-(K+1)}r)$-thick, A_ℓ is also $(2^{-(K+1)}r)$-thick. □

Let ρ be the Hausdorff metric.

Proof of Lemma 13. The second statement is an easy corollary of the first. Thus we prove the first only.

We name elements of C by $C = \{H_1, H_2, \dots, H_K\}$. Let $Y = \eta^\perp$. We identify $\mathbb{R} \times Y$ and $\mathbb{R}^{d+1}$ using a function $\varphi : \mathbb{R} \times Y \to \mathbb{R}^{d+1}$ defined as $\varphi(\alpha, y) = \alpha\eta + y$. For each $k \in \{1, \dots, K\}$, we denote by Y_k the set of $y \in Y$ satisfying $(\alpha, y) \in H_k$ with some $\alpha \in \mathbb{R}$, and $(\mathbb{R} \times Y_k) \cap X$ by X_k. Then $X_k^\circ \cap X_\ell^\circ = \varnothing$ for $k \neq \ell$ by the definition of $\mathcal{H}_{\mathrm{cx}}(\eta, X)$. Denote by $Y_k(a)$ the set of $y \in Y_k$ such that $U(a; y) \subseteq Y_k$, and $X_k(a)$ be $(\mathbb{R} \times Y_k(a)) \cap X$. Let $Y(o) = \bigcup_{k:o(H_k)=o} Y_k$ and $Y(o; a) = \bigcup_{k:o(H_k)=o} Y_k(a)$.

Define a positive number $d = \min_{o_1, o_2 \in \mathscr{O},\, o_1 \neq o_2} \rho(o_1^\perp \cap K, o_2^\perp \cap K)$, where $K = \overline{U(1; 0)}$. For each t and $o \in \mathscr{O}$, define $H^t(o) = \bigcup\{H \mid o(H) = o,\, H \in C_t \in c(S_t)\}$. For each subset $X' \subseteq X$, we let $H^t(o|X') = H^t(o) \cap X'$. We let $Y^t(o|X') = \mathrm{proj}_Y\, H^t(o|X')$ and $r^t(o|X') = r(Y^t(o|X'))$.

Consider t satisfying $\rho(S_t, S) < \varepsilon$.

Step 1: $r^t(o|X_k(2\varepsilon)) < \varepsilon/d$ *for all* k *and* $o \neq o(H_k)$.

$r^t(o|X_k(2\varepsilon))$ must be strictly smaller than ε/d, since

$$\varepsilon > \max_{x \in H^t(o|X_k(2\varepsilon))} \min_{x' \in H_k} ||x - x'|| \geq dr^t(o|X_k(2\varepsilon)). \tag{39}$$

Step 2: $Y_k(2\varepsilon) \setminus Y^t(o(H_k)|Y_k(2\varepsilon))$ *is at most* $A\varepsilon$*-thick if* ε *is sufficiently close to 0, where* $A > 0$ *is some constant.*

Let $Z = Y_k(2\varepsilon) \setminus Y^t(o(H_k)|Y_k(2\varepsilon))$. Applying the separating hyperplain theorem, we can divide Z into $(Z_o)_{o \neq o(H_k)}$ such that each Z_o is a finite union of closed convex sets and $Y^t(o|Y_k(2\varepsilon)) \subseteq Z_o$ for each $o \neq o(H_k)$. By Lemma 14, there is $o^* \neq o(H_k)$ such that $r(Z_{o^*}) \geq 2^{-|\mathscr{O}|} r(Z)$. By Step 1, $Y^t(o^*|Y_k(2\varepsilon))$ is at most (ε/d)-thick, so $Z_{o^*} \setminus Y^t(o^*|Y_k(2\varepsilon))$ is at least $(\max\{(r(Z_{o^*})/2) - (\varepsilon/d), 0\})$-thick. Thus

$$\varepsilon > \max_{x \in (\mathbb{R} \times Z) \cap X} \min_{x' \in S_t} ||x - x'|| \geq \max\left\{\frac{r(Z)}{2^{|\mathscr{O}|+2}} - \frac{\varepsilon}{2d}, 0\right\}. \tag{40}$$

Thus we obtain $r(Z) < \{2^{|\mathscr{O}|+2}(1 + (1/2d))\}\varepsilon$, when ε is sufficiently small.

By Step 1 and 2, $Y^t(o)$ must converge to $Y(o)$ as ε goes to 0. $H^t(o)$ is also forced to converge to $H(o)$, since otherwise $\rho(S_t, S)$ does not converge to 0. Construction of $(C_t, H_t)_{t=0}^{\infty}$ is now easy and left to readers. ☐

D.2 Dense Existence of Thin $B^i(\theta^j)$

For each $(A_1, A_2) \in \S$ and $\varepsilon > 0$, we define $\Theta^i_\varepsilon(B)$ as the set of $\theta^i \in \Theta^{i\circ}$ such that $B^j(\theta^i)$ is at least ε-thick. We omit the argument B when it is obvious.

Lemma 15. $\Theta^i_\varepsilon(B)$ *is closed under the relative topology of* $\Theta^{i\circ}$ *for all* $(A_1, A_2) \in \S$.

Proof. Consider a sequence $\{\theta^i_n\}_{n=0}^{\infty}$ of Θ^i_ε converging to θ^i_∞. For each n, let K^j_n be an ε-closed ball of $B^j(\theta^i_n)$. Since Θ^j is compact, there is a subsequence of $\{K^j_n\}_{n=0}^{\infty}$ converging to some ε-closed ball K^j_∞. Because of the closedness of B, we obtain $K^j_\infty \subseteq B^j(\theta^i_\infty)$ and this implies $\theta^i_\infty \in \Theta^i_\varepsilon$. ☐

Lemma 16. *Let* $\bar{\theta}^i \in \Theta^{i\circ}$ *and* $\ell^i = \{\bar{\theta}^i + \alpha\eta^i \mid \alpha \in \mathbb{R}\}$ *for each* $i \in \{1, 2\}$. *Then for all* $(A_1, A_2) \in \mathcal{S}^*$,

1. *for all* $\varepsilon > 0$, $\#(\Theta^i_\varepsilon(B(f)) \cap \ell^i) < \infty$, *and*
2. $\Theta^i_{\text{thin}}(B(f)) \cap \ell^i$ *is dense in* ℓ^i.

Proof. By monotonicity of (A_1, A_2), $\mathcal{X}^j = \{B^j(\theta^i)^\circ \mid \theta^i \in \ell^i\}$ consists of disjoint sets. Since the Lebesgue measure of $\bigcup_{X^j \in \mathcal{X}^j} X^j \subseteq \Theta^j$ is finite, $\#(\Theta^i_\varepsilon \cap \ell^i)$ must be also finite. The second statement is because $\Theta^i_{\text{thin}} = \Theta^{i\circ} \setminus \bigcup_{n=1}^{\infty} \Theta^i_{1/n}$. ☐

D.3 Local Non-Constancy

For each set $S^2 \subseteq \Theta^2$ satisfying $\hat{S}^2 = \overline{S^2 \setminus \partial\Theta^2} \in \mathcal{H}_{\text{un}}(\eta^2, \Theta^2)$, we define $N'(S^2) = N(\hat{S}^2)$. We say a mapping f from topological space X to some set Y is *locally constant* at $x \in X$ if $f(U)$ is a singleton for some open neighborhood U of x.

Lemma 17. *Assume $(u,t) \in \mathcal{L}[i]$ implements $(A_1, A_2) \in \mathcal{S}^*$. Let $\bar{N} = \max_{\theta^1 \in \Theta^{1\circ}} N'(B^2(\theta^1))$. If $(A_1, A_2) \in \mathcal{S}^*[i]$ then there is $\hat{\theta}^1 \in \Theta^{1\circ}$ such that $N'(B^2(\theta^1)) = \bar{N}$ and B^2 is not locally constant at $\hat{\theta}^1$.*

Proof. We use the relative topology of $\Theta^{1\circ}$. Let S be the set of $\theta^1 \in \Theta^{1\circ}$ such that $N'(B^2(\theta^1)) = \bar{N}$, and T be the set of $\theta^1 \in \Theta^{1\circ}$ at which B^2 is locally constant. Applying Lemma 12, we also obtain the fact that S is closed.

Assume that $S \subseteq T$. Then S must be open, by the definition of local constancy. Since agent 2 is not a dictator and $\Theta^{i\circ}$ is a connected space, S must be the empty set. This is a contradiction. □

D.4 Proof of Lemma 6

Define $\hat{B}^2(\theta^1) = \overline{B^2(\theta^1) \setminus \partial\Theta^2}$. Then $\hat{B}^2(\theta^1) \in \mathcal{H}_{\mathrm{un}}(\eta^2, \Theta^2) \cup \{\varnothing\}$ for each $\theta^1 \in \Theta^{1\circ}$.

Step 1: Construction of a mapping F.

Take $\hat{\theta}^1$ of Lemma 6. Using Lemmata 12 and 13, there are U^1 which is an open neighborhood of $\hat{\theta}^1$, and mapping $H_k : U^1 \to \mathcal{H}_{\mathrm{seg}}$ for each $k \in \{1, \ldots, K\}$ such that

1. H_k is continuous,
2. $o(H_k(\theta^1)) = o(H_k(\theta^{1\prime}))$ for all $\theta^1, \theta^{1\prime} \in U^1$, and
3. $\hat{B}^2(\theta^1) = \{H_1(\theta^1), \ldots, H_K(\theta^1)\}$ for each $\theta^1 \in U^1$.

Since at $\hat{\theta}^1$ $\hat{B}^2$ is not locally constant, for some k, H_k is not locally constant at $\hat{\theta}^1$ either. Taking sufficiently small open ball $\hat{\theta}^1 \in V^1 \subseteq \Theta^1$ and closed ball $K^2 \subseteq \Theta^2$, we can construct a non-constant mapping $F : V^1 \to \mathcal{C}(K^2)$ such that

1. $F(\theta^1) = H_k(\theta^1) \cap K^2 = B^2(\theta^1) \cap K^2$, and
2. $F(\theta^1)$ is represented as an intersection of K_2 and some hyperplain (not hyperplain segment),

for all $\theta^1 \in V^1$.

Step 2: $B^1(\theta^2) \cap V^1 = B^1(\theta^{2\prime}) \cap V^1$ for all $\theta^1 \in V^1$ and $\theta^2, \theta^{2\prime} \in F(\theta^1)$.

Denote $F(\theta^1)$ by S^2, and take an arbitrary $\theta^1 \in B^1(\theta^2) \cap V^1$. Then $B^2(\theta^1)$ must contain θ^2, and thus $B^2(\theta^1)$ must be S^2. Since $\theta^{2\prime} \in S^2$, $\theta^{2\prime} \in B^2(\theta^1)$. This implies $(\theta^1, \theta^{2\prime}) \in B$ and $\theta^1 \in B^1(\theta^{2\prime})$. Therefore we obtain $B^1(\theta^2) \cap V^1 \subseteq B^1(\theta^{2\prime}) \cap V^1$, and the opposite inclusion is also proved.

Step 3: Contradiction.

By Lemma 16, there is $\bar{\theta}^2 \in \bigcup_{\theta^1 \in V^1} F(\theta^1)$ such that $B^1(\bar{\theta}^2)^\circ = \varnothing$. Let T^2 be the unique element of $\{F(\theta^1) \mid \theta^1 \in V^1\}$ which contains $\bar{\theta}^2$. Due to Step 2, $B^1(\cdot)$ is constant within T^2. Lemma 12 requires that $\Phi^1_{1\pm}(\cdot|u^1, t^1)$ also must be constant, but the cross-term μ^1_C makes it impossible. □

Appendix E Proof of Lemma 7

Let $M^* = \overline{M}^2 \setminus \{(-\infty, -\infty)\}$. Define $\Phi^i_+, \Phi^i_- : \mathcal{U}^i \times M^* \times \Theta^j \to \mathcal{C}(\Theta^i)$ by

$$\Phi^i_+(u^i, m; \theta) = \left\{\theta^i \in \Theta^i \;\middle|\; \mu^i[u^i](\theta; m) \geq 0\right\} \cup \partial\Theta^i \tag{41}$$
$$\Phi^i_-(u^i, m; \theta) = \left\{\theta^i \in \Theta^i \;\middle|\; \mu^i[u^i](\theta; m) \leq 0\right\} \cup \partial\Theta^i \tag{42}$$

Lemma 18. *Given $\theta^j \in \Theta^j$, $\Phi^i_+(\cdot; \theta^j)$ and $\Phi^i_-(\cdot; \theta^j)$ are continuous.*

Proof. Take arbitrary $(u^i, m) \in \mathcal{U}^i \times M^*$. Here we prove the continuity of $\Phi^i_+(\cdot;\theta^j)$ at (u^i, m). The continuity of $\Phi^i_-(\cdot;\theta^j)$ is similarly proved. If $m_k = -\infty$ for some $k \in \{1,2\}$ then the continuity is trivial. Assume $m \in M^2$.

Take arbitrary $\varepsilon > 0$ and let $S = \Phi^i_+(u^i, m; \theta^i)$. Since S is compact, there are a finite number of $(\varepsilon/2)$-balls $U_1, \dots, U_L \subseteq \Theta^i$ which cover S and each of which intersects with S. Since there is no local maxima or minima in every neighborhood of every internal points of Θ^i, there are $\alpha_\ell, \beta_\ell \in U_\ell$ such that

$$\mu^i(\alpha_\ell, \theta^j; m) > 0 > \mu^i(\beta_\ell, \theta^j; m). \tag{43}$$

Let $T = \Theta^i \setminus \bigcup_{\ell=1}^{L} U_\ell$ and $T' = T \cup \{\alpha_1, \beta_1, \dots, \alpha_L, \beta_L\}$. Since T' is compact, there is

$$\xi = \min_{\tilde{\theta}^i \in T'} \left|\mu^i(\tilde{\theta}^i, \theta^j; m)\right|. \tag{44}$$

Note $\xi > 0$ because $|\mu^i(\cdot, \theta^i; m)| > 0$ within T'.

μ^i is absolutely continuous since its domain $\Theta \times M^2$ is compact. Therefore, there is $\delta > 0$ such that, for all $x, y \in \Theta \times M^2$, if $\|x - y\| < \delta$ then $|\mu^i(x) - \mu^i(y)| < \xi/2$. Define an open set V by $V = \{(\tilde{u}^i, \tilde{m}) \in \mathcal{U}^i \times M^2 \mid d_\infty(u^i, \tilde{u}^i) < \xi/4,\ \|m - \tilde{m}\| < \delta\}$.

We prove that S and $\hat{S} = \Phi^i_+(\hat{u}^i, \hat{m}; \theta^j)$ are ε-close for all $(\hat{u}^i, \hat{m}) \in V$. First, we see that each $\theta^i \in S$ has ε-close $\hat{\theta}^i \in \hat{S}$. Choose ℓ so as to satisfy $\theta^i \in U_\ell$, and let $\hat{\mu}^i$ denote $\mu^i[\hat{u}^i]$. Then $\hat{\mu}^i(\alpha_\ell, \theta^j; \hat{m}) > 0$ because

$$\begin{aligned} &|\mu^i(\alpha_\ell, \theta^j; m) - \hat{\mu}^i(\alpha_\ell, \theta^j; \hat{m})| \\ \le\ &|\mu^i(\alpha_\ell, \theta^j; m) - \hat{\mu}^i(\alpha_\ell, \theta^j; m)| + |\hat{\mu}^i(\alpha_\ell, \theta^j; m) - \hat{\mu}^i(\alpha_\ell, \theta^j; \hat{m})| < \xi. \end{aligned} \tag{45}$$

Similarly we obtain $\hat{\mu}^i(\beta_\ell, \theta^j; \hat{m}) < 0$. Thus by the intermediate value theorem, there is some $\gamma_\ell \in U_\ell$ such that $\hat{\mu}^i(\gamma_\ell, \theta^j; \hat{m}) = 0$. Therefore $\gamma_\ell \in \hat{S}$, and γ_ℓ is ε-close to all elements of $S \cap U_\ell$.

Next, we see that each $\hat{\theta}^i \in \hat{S}$ has ε-close $\theta^i \in S$. Using the same technique as the previous paragraph, we obtain $\hat{\mu}^i(\theta^i_T, \hat{\theta}^j; \hat{m}) < 0$ for all $\theta^i_T \in T$. Thus $\hat{S}$ and T are disjoint, and hence $\hat{\theta}^i$ must be in some U_ℓ. Take θ^i from $U_\ell \cap S$. Then $\hat{\theta}^i$ and an element θ^i of S are ε-close. □

Lemma 19. *Let X be a metric space, and $(S_n)_{n=0}^\infty$ and $(T_n)_{n=0}^\infty$ be converging sequences of $\mathcal{C}(X)$. If $S_n \subseteq T_n$ for all n then $\lim_{n\to\infty} S_n \subseteq \lim_{n\to\infty} T_n$.*

Proof. The proof is routine, and left to the reader. □

Proof of Lemma 7. Take an arbitrary sequence $\{u_n\}_{n=0}^\infty$ of $\mathcal{U}(\mathcal{S}^* \cap \mathcal{T})$. By definition of $\mathcal{U}(\mathcal{S}^* \cap \mathcal{T})$, we can take $t_n \in T$ and $(A_{1,n}, A_{2,n}) \in \mathcal{S}^* \cap \mathcal{T}$ such that (u_n, t_n) implements (A_{1n}, A_{2n}) for each n. Without loss of generality, we can assume that $\{(A_{1n}, A_{2n})\}_{n=0}^\infty$ is a converging sequence, since $\mathcal{C}(\Theta)$ is a compact metric space. We define $t \in T$ by $t^i(a, \theta^j) = \liminf_{n\to\infty} t^i_n(a, \theta^j)$. We let $u = \lim_{n\to\infty} u_n$ and $A_k = \lim_{n\to\infty} A_{k,n}$ for each $k = 1, 2$. Since $\mathcal{T}$ is closed, $(A_1, A_2) \in \mathcal{T}$.

Step 1: (u, t) implements (A_1, A_2).

Take arbitrary $i \in \mathcal{N}$ and $\theta^j \in \Theta^j$ and fix them. We denote $t^i_n(\theta^j)$ by τ^i_n and $t^i(\theta^j)$ by τ^i. We can choose a subsequence $(\tau^i_{n(m)})_{m=0}^\infty$ of $(\tau^i_n)_{n=0}^\infty$ satisfying $\lim_{m\to\infty} \tau^i_{n(m)} = \tau^i$.[18] Also, a sequence $(A^i_{k,n(m)}(\theta^j))_{m=0}^\infty$ of $\mathcal{C}(\Theta^i)$ has a converging subsequence $(A^i_{k,n(m(\ell))}(\theta^j))_{\ell=0}^\infty$ for each $k = 1, 2$ because $\mathcal{C}(\Theta^i)$ is a compact space[19]. By definition of implementation, $\Phi^i_{i+}(\theta^j | u^i_n, \tau^i_n) \subseteq A^i_{1,n}(\theta^j)$ for each n.

[18] Of course, $n(\cdot)$ and $m(\cdot)$ are strictly increasing mappings from $\mathbb{N}$ to $\mathbb{N}$.

[19] $A^i_{k,n}(\theta)$ is non-empty since $\partial\Theta \subseteq A_{k,n}$.

Using Lemmata 18 and 19, we obtain

$$A_1^i(\theta^j) \supseteq \lim_{\ell\to\infty} A^i_{1,n(m(\ell))}(\theta^j) \tag{46}$$
$$\supseteq \lim_{\ell\to\infty} \Phi^i_{i+}(\theta^j | u^i_{n(m(\ell))}, \tau^i_{n(m(\ell))}) \tag{47}$$
$$= \Phi^i_{i+}(\theta^j | u^i, \tau^i). \tag{48}$$

Similarly we can show $\Phi^i_{i-}(\theta^j|u^i,\tau^i) \subseteq A_2^i(\theta^j)$. Since i and θ^j is taken arbitrary, we obtain $\Phi_{i+}(u^i,t^i) \subseteq A_1$ and $\Phi_{i-}(u^i,t^i) \subseteq A_2$ for each $i \in \mathcal{N}$.

Step 2: $(A_1, A_2) \in \mathcal{S}_{\text{mono}}$.

We use a proof by contradiction. Assume there are $\theta, \theta' \in \Theta^\circ$ and $\alpha^1, \alpha^2 > 0$ such that $\theta' = \theta + (2\alpha^1\eta^1, 2\alpha^2\eta^2)$ and $\theta, \theta' \in B$.

Since B is the limit of B_n, there are sequences $(\theta_n)_{n=0}^\infty$ and $(\theta'_n)_{n=0}^\infty$ such that $\lim_{n\to\infty}\theta_n = \theta$, $\lim_{n\to\infty}\theta'_n = \theta'$ and $\theta_n, \theta'_n \in B_n$. By $\theta, \theta' \in \Theta^\circ$, there is $\gamma > 0$ such that, for all $i \in \mathcal{N}$, $\{x^i \in \mathbb{R}^{d^i} \mid ||\theta^i - x^i|| < \gamma\}$ and $\{x^i \in \mathbb{R}^{d^i} \mid ||\theta^i - x^i|| < \gamma\}$ are subsets of $\Theta^{i\circ}$. Let $U^i = \{x^i \in \mathbb{R}^{d^i} \mid \exists \beta^i \in [0, 2\alpha^i], ||(\theta^i + \beta^i\eta^i) - x^i|| < \gamma\}$, and then S^i is a subset of $\Theta^{i\circ}$ by the convexity of Θ^i. Let $r^i = \phi^i(\alpha^i; u^i)/2$ for each i. By Lemma 11, there is $N \in \mathbb{N}$ such that, for all $n > N$, $\phi^i(\alpha^i; u^i_n) > r^i$. Take a sufficiently small $\varepsilon > 0$ so as to be $\varepsilon < \gamma/3$ and $\varepsilon < \min\{r^1, r^2\}/5$. Then there is $n > N$ such that $||\theta - \theta_n|| < \varepsilon$ and $||\theta' - \theta'_n|| < \varepsilon$. Fix such n.

We prove that $\theta'_n - (\alpha^1\eta^1, \alpha^2\eta^2) \in A_{1,n}$, which leads a contradiction to $\theta'_n \in A_{2,n}$ and $(A_{1,n}, A_{2,n}) \in \mathcal{S}_{\text{mono}}$. This is proved in the following steps. Notice that $\theta^{(1)}, \ldots, \theta^{(4)}$ defined below are all in $U^1 \times U^2$ and thus in Θ°.

Step 2.1: $\theta^{(1)} = \theta_n + (\varepsilon\eta^1, \varepsilon\eta^2)$ *is in* $\Phi_{1+}(u^1_n, t^1_n)$.

Notice that $A_{1,n} \setminus B_n \subseteq \Phi_{1+}(u^1_n, t^1_n)$. Since $(A_{1,n}, A_{2,n}) \in \mathcal{S}_{\text{mono}}$ and $\theta_n \in A_{1,n}$, we obtain $\theta^{(1)} \in A_{1,n} \setminus B_n \subseteq \Phi_{1+}(u^1_n, t^1_n)$.

Step 2.2: $\theta^{(2)} = (\theta^{1\prime}_n - (\alpha^1 + \varepsilon)\eta^1, \theta^2_n + \varepsilon\eta^2)$ *is in* $A_{1,n}$.

We obtain $\theta^{(2)} \in \Phi_{1+}(u^1_n, t^1_n) \subseteq A_{1,n}$ since

$$\left\| \left\{\theta^{1\prime}_n - (\alpha^1+\varepsilon)\eta^1\right\} - \left\{\theta^1_n + (\alpha^1+\varepsilon)\eta^1\right\} \right\|$$
$$\leq \left\| \theta^{1\prime}_n - \left\{\theta^1_n + 2\alpha^1\eta^1\right\} \right\| + 2\varepsilon \left\|\eta^1\right\| \tag{49}$$
$$< 4\varepsilon < \phi^1(\alpha^1, u^1_n). \tag{50}$$

Step 2.3: $\theta^{(3)} = (\theta^{1\prime}_n - \alpha^1\eta^1, \theta^2_n + 2\varepsilon\eta^2)$ *is in* $\Phi_{2+}(u^2_n, t^2_n)$.

This is proved by the same logic as Step 2.1.

Step 2.4: $\theta^{(4)} = \theta'_n - (\alpha^1\eta^1, \alpha^2\eta^2)$ *is in* $A_{1,n}$.

This is proved by the same logic as Step 2.2. □

Appendix F On the Extended Debreu (1968)'s Topology

Let $\mathcal{P}_i$ be the set of i's preferences represented by some $u^i \in \mathcal{U}^i_0$. We denote $\mathcal{P} = \mathcal{P}^1 \times \mathcal{P}^2$. For each $P \in \mathcal{P}^i$, we denote $\{(s,t;\theta) \in M^2 \mid (a_k, s)\ P(\theta)\ (a_\ell, t)\}$ by $P_{k\ell}$, and $\{(s,t) \in M^2 \mid (a_k, s)\ P(\theta)\ (a_\ell, t)\}$ by $P_{k\ell}(\theta)$, for each $k, \ell \in \{1,2\}$. Since P_{kk} and $P_{kk}(\theta)$ are common in $\mathcal{P}^i$, we ignore them.

Proposition 5. *In* $\mathcal{P}_i$*, metrics* d_1 *and* d_2 *generate the same topology.*

Proof. Since $d_2 \leq d_1$, d_1 generates a finer topology than d_2. We see that the converse is also true. Take arbitrary $P \in \mathcal{P}^i$ and $\varepsilon > 0$, and let $U = \{P' \in \mathcal{P}^i \mid d_1(P, P') < \varepsilon\}$. Assume that for all $\delta > 0$ there is $P_{(\delta)} \in \mathcal{P}^i$ such that $d_2(P, P_{(\delta)}) < \delta$ and $d_2(P, P_{(\delta)}) \geq \varepsilon$. Then we can take a sequence $(P_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} d_2(P, P_n) = 0$ but $d_1(P, P_n) \geq \varepsilon$ for all n.

Case 1: There are infinitely many n such that $\min_{(s_n,t_n)\in P_{12}(\theta_n)} ||(s_n, t_n) - (s, t)|| \geq \varepsilon$ for some $(s_n, t_n; \theta_n) \in P_{12,n}$.

In this case, by the compactness of M and Θ, we can take a converging subsequence $(s_{n(m)}, t_{n(m)}; \theta_{n(m)})_{m=0}^{\infty}$ satisfying the above property. Denote the limit by $(s_\infty, t_\infty; \theta_\infty)$. Since $\lim_{n\to\infty} d_2(P, P_n) = 0$, $(s_\infty, t_\infty; \theta_\infty)$ must be in $P_{12,n}$. This implies $\mu^i[u^i](\theta_\infty; s_\infty, t_\infty) \geq 0$, where u^i is a utility function representations of P respectively. By Assumption 1, there are (s', t') and $r > 0$ such that $||(s', t') - (s_\infty, t_\infty)|| < \varepsilon/2$ and $\{(s', t'; \theta) \mid ||\theta - \theta_\infty|| < r\} \subseteq \{(s, t; \theta) \mid \mu^i[u^i](\theta; s, t) > 0\} \subseteq P_{12}$. Thus for sufficiently large n, $(s', t') \in P_{12}(\theta_n)$ and $(s_n, t_n) \in P(\theta_n)$ and $||(s', t') - (s_n, t_n)|| < \varepsilon$. This is a contradiction.

Case 2: There are infinitely many n such that $\min_{(s,t)\in P_{12,n}(\theta_n)} ||(s_n, t_n) - (s, t)|| \geq \varepsilon$ for some $(s_n, t_n; \theta_n) \in P_{12}$.

As in Case 1, using Assumption 1 we obtain an contradiction. Details are left to readers. □

Due to the next lemma, we know that approximation in utility functions is also approximation in preferences. Let $P(u^i)$ be the preference induced by u^i.

Lemma 20. *Let $(u^i_n)_{n=0}^{\infty}$ be a sequence of $\mathcal{U}^i_0$. If $\lim_{n\to\infty} u^i_n = u^i$ then $\lim_{n\to\infty} P(u^i_n) = P(u^i)$.*

Proof. We can prove this lemma by techniques used in the proof of Lemma 18. Details are left to readers. □

Finally, all we need is the following lemma. The rest of the argument does not require any modifications. To prove this lemma, we can use quite similar techniques used in the original lemmata.

Lemma 21. *$\tilde{\chi}^i_{\pm}$ and $\Phi_{i\pm}$ are continuous.*

References

[1] Barbera, S. (1983): "Strategy-Proofness and Pivotal Voters: A Direct Proof of the Gibbard-Satterthwaite Theorem," *International Economic Review,* vol. 24(2), pages 413-17.

[2] Bergemann, D. and S. Morris (2005):"Robust Mechanism Design," *Econometrica,* 73(6), 1771-1813.

[3] Benoît, J.-P. and V. Krishna (2001): "Multiple-Object Auctions with Budget Constrained Bidders," *Review of Economic Studies,* vol. 68(1), pages 155-79.

[4] Bikhchandani, S. (2005): "The Limits of Ex Post Implementation Revisited," UCLA.

[5] Che, Y.-K., and I. Gale (1998): "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies,* 65, 1-21.

[6] Che, Y.-K., and I. Gale (2000): "The Optimal Mechanism for Selling to a Budget-Constrained Buyer," *Journal of Economic Theory,* 92, 198-233.

[7] Clarke, E. (1971): "Multipart Pricing of Public Goods," *Public Choice*, 8, 19-33.

[8] Cremer, J. and R. Mclean (1985): " Optimal Selling Strategies under Uncertainty for a Monopolist when Demands Are Interdependent, "*Econometrica,* 53, 345-361.

[9] Debreu, G. (1968): "Neighboring Economic Agents," *La Décision,* Colloques Internationaux du Centre National de la Recherche Scientifique no. 171, Paris, 85-90.

[10] Ely, J. C. and K.-S. Chung (2002): "Ex-Post Incentive Compatible Mechanism Design," Discussion Papers 1339, Northwestern University, Center for Mathematical Studies in Economics and Management Science.

[11] Fang, H. and S. O. Parreiras (2001): "Equilibrium of Affiliated Value Second-Price Auctions with Financially Constrained Bidders: The Two-Bidder Case," discussion paper, Yale University.

[12] Gavious, A., B. Moldovanu and A. Sela (2002): "Bid Costs and Endogenous Bid Caps," *The RAND Journal of Economics,* 33(4), 709-722.

[13] Gibbard, A. (1973): "Manipulation of Voting Schemes," *Econometrica*, 41,587-601.

[14] Groves, T. (1973): "Incentives in Teams," *Econometrica*, 41, 617-631.

[15] Gul, F. (1998): "A comment on Aumann's Bayesian view." *Econometrica*, 66(4), 923-927.

[16] Harsanyi, J. C. (1967-68): "Games with incomplete information played by Bayesian players," *Management Science* 14 , 159-182, 320-334, 486-502.

[17] Holmström, B., and R. Myerson (1983): "Efficient and Durable Decision Rules with Incomplete Information," *Econometrica*, 51, 1799-1820.

[18] Jehiel, P., M. Meyer-ter-Vehn, B. Moldovanu and W. R. Zame, (2006): "The Limits of ex post Implementation," *Econometrica*, 74(3), 585-610.

[19] Laffont, J.-J. and J. Robert (1996): "Optimal Auctions with Financially Constrained Bidders," *Economic Letters,* 52, 182-186.

[20] Maskin, E. (2000): "Auctions, Development, and Privatization: Efficient Auctions with Liquidity-Constrained Buyers," *European Economic Review,* 44, 667-681.

[21] Mertens, J.-F. and S. Zamir (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information." *International Journal of Game Theory,* 14, 1-29.

[22] Milgrom, P. (2004): *Putting Auction Theory to Work*, Cambridge University Press.

[23] Moldovanu, B. and A. Sela (2001): "The Optimal Allocation of Prizes in Contests," *The American Economic Review,* 91(3), 542-558.

[24] Moldovanu, B. and A. Sela (2006): "Contest Architecture," *Journal of Economic Theory,* 126(1), 70-97.

[25] Morris, S. (1995): "The Common Prior Assumption in Economic Theory," *Economics and Philosophy*, 11, 227-253.

[26] Perry, M. and P. J. Reny (2002): "An Efficient Auction," *Econometrica,* 70(3), 1199-1212.

[27] Pitchik, C. and A. Schotter (1988): "Perfect Equilibria in Budget-Constrained Sequential Auctions: An Experimental Study", *Rand Journal of Economics,* 19, 363-388.

[28] Salant D. (1997): "Up in the Air: GTE's Experience in the MTA Auction for Personal Communication Services Licenses," *Journal of Economics & Management Strategy,* 6(3), 549-572.

[29] Savage, L. (1954): *The Foundations of Statistics*, John Wiley and Sons.

[30] Satterthwaite, M. (1975): "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory*, 10, 187-217.

[31] Vikerey, W. (1961): "Counterspeculation, Auctions and Competitive Sealed Traders," *Journal of Finance*, 16, 8-37.