Middlemen: The Visible Market Makers

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Abstract

This paper presents a search-theoretic general equilibrium model where the size and number of intermediaries in the market are determined endogenously. I show how middlemen can emerge endogenously to intermediate between ex ante homogeneous buyers and sellers in the presence of coordination frictions. Middlemen compete against each other and the buyer-seller private market by setting price, and provide a high matching service by holding an inventory. Entry of middlemen, however, increases the amount of coordination frictions and reduces the amount of inventory each individual middleman holds. The resulting high bargaining power of middlemen lowers buyers' welfare. The frictionless Warlasian outcome emerges in the limit where a single middleman holds an excess inventory to wipe off buyer-seller direct trades. Remarkably, the asymptotic efficiency is attained, if and only if, the entry cost is set prohibitively high to induce a single middleman.

Keywords: Intermediation, Competitive Search Equilibrium, Matching Function, Market Microstructure, Inventory Adjustment

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1 Introduction

How do markets clear? Who creates the market and how is such a task accomplished? While the former question has been at the center of microeconomic thought, only recently has attention been given to the latter. Rubinstein and Wolinsky (1987) emphasize it is middlemen who fill a gap between production and consumption, and Spulber (1996) proposes they can clear markets. In addition to setting price, intermediaries often employ other market-clearing instruments. In particular, when there is a trade imbalance, they provide a buffer stock that offers their customers liquidity by holding inventories and allocate scarce resources by rationing.[1]

The aim of this paper is to present a unified framework that integrates these key functions of intermediaries. When buyers and sellers face trading risks arising from coordination frictions, middlemen emerge to (1) create markets to link producers and consumers, (2) set prices that allocate customers in such a way that markets clear, and (3) hold inventories to smooth trade imbalances and thereby improve the availability of goods for those who desire more immediate service.

Consider the following directed search framework where each period consists of two subperiods, *day* and *night*, and where a market is open for each sub-period.[2] Buyers wish to consume a unit of a homogeneous good, e.g. a potato, every day. Sellers produce one unit of potato every night. Buyers visit sellers in each period during the day but they cannot coordinate among each other over which seller to visit. Since more buyers may happen to arrive at a given location than the producer can accommodate, buyers face a risk of not purchasing. At the same time, because fewer buyers may show up at a location than the seller there can accommodate, sellers face the risk of not clearing out. To overcome these frictions, middlemen can invest in a technology to store multiple units of potatoes as an inventory in a warehouse. At midnight, when buyers are sleeping, they purchase potatoes from the farmers who have failed to make a sale during the day. In the next day market, they stand ready for buyers who prefer more sure service to a low price.

In the day-market equilibrium, it is shown that buyers are indifferent between visiting any seller who posts a low price and provides a low service rate, or any middleman who offers a high price

[1] Spulber (1999) suggests the importance of inventory holdings and customer-rationings in market-making (p.49). Recent microscopic empirical studies resonant this observation. Aguirregabiria (2003) finds intermediaries' inventory accumulation is crucial to explain their pricing patterns, when customers tradeoff the price against the service rate.

[2] Directed matching models are developed in Peters (1991), Montogomery (1991), Accemoglu and Shimer (1999a,b), and Burdett, Shi and Wright (2001). Likewise, the day-night trading-period structure is proposed in a mometary model of Lagos and Wright (2003) and Rocheteau and Wright (2003).

and a high service rate. Middlemen's inventory holdings generate two opposing effects. On the one hand, their capacity advantage creates a demand-stimulating effect to attract a large number of buyers, which gives them a monopoly power for *all units* of thier goods. On the other hand, the risk of holding unsold inventory weakends their bargaining position, when they divide the trading surplus of *each unit* of the goods with buyers.

After the day-market closes, middlemen restock their inventory at night. I assume that the steady-state level of inventory that each middleman holds is determined to clear the night-market, which is frictionless, each period.[3] As a result, middlemen's inventory holdings smooth out trade imbalances due to stochastic rationings in the day-market - all goods produced each period are distributed to buyers either at today's day-market or at tomorrow's day-market through middlemen. As entry occurs and more middlemen attempt to operate in the market, each middleman restocks a smaller amount of inventory at night. The resulting larger number of smaller-sized intermediaries imply that each middleman attracts only few buyers from whom it extracts a large amount of trading surplus per unit of inventory. Likewise, as the number of middlemen decreases, the middlemen' monopoly power gets weakened due to a reduction in their bargaining power.

In the limit, a single middleman provides a frictionless market to equate demand and supply, which could be termed as a *vertical centralization* phenomenon. The market-clearing process can manifest itself through intermediated trades. I will also show that a single middleman can achieve the Walrasian outcome where the buyer-seller direct trades are completely unraveled. The presence of the central trade coordinator, who can resolve matching frictions by itself, triggers the Bertrand competition to maximize buyers' welfare. The single middleman burdens itself with holding an excess-inventory, acting as a benevolent market-organizer, to part with all of its bargaining power.

The vertical centralization result also implies that entry of intermediaries does not benefit buyers although it stimulates competition in the day-market. An endogenous reduction in capacity, due to its congestion effect on the night-market, offsets the competitive fringe generated by entry. This contrasts sharply to Rust and Hall (2003), who argue that entry of intermediaries encourages competition and increases the matching efficiency and welfare, with assuming that inventory holdings

[3]This generalizes the similar assumption imposed in Rust and Hall (2003), in which the demand by buyers and the supply by sellers balance each period. In my model, the demand-supply balancing condition is stated in terms of middlemen's inventory restocking process at each night-market. The underlying assumption in my model is that sellers do not have access to storage technology and hence the goods are valueless for them if they remain unsold. The results presented here, however, do not depend on this assumption.

are not allowed and hence without considering the scale effect of inventories.[4]

The social planner's solution in my model is to utilize fully the single middleman's matching effectiveness. It implies that the entry of middlemen creates inefficiency by increasing the amount of coordination frictions and costly access to storage technology. Remarkably, I found that the efficient allocations can be decentralized at the single middleman equilibrium. The instruments that complement the market clearing role of the price can be mapped into tangible economic activities motivated by profit maximization - market makings, inventory holdings, and rationing.

Perhaps it would be interesting to relate my results to the 'market-makers story' prevailing in the directed-matching literature proposed by Moen (1997) and Mortensen and Wright (2002).[5] That is, matching efficiency can be achieved in the presence of fictitious market-makers that come to replace the Walrasian auctioneer. Given that the market-makers create submarkets and announce the price for each submarket, the market tighness adjusts itself to internalize externalities. The open question in the literature is: 'who can be the market-makers in this story?' (see Mortensen and Wright, p.19). My result suggests once middlemen replace this exogenous third party as profit maximizing market makers, the efficiency result breaks down endogenously. However, the asymptotic efficiency result implies the 'single' middleman can accomplish this market-making task efficiently. I propose intermediation is the optimal response to search externalities and matching frictions.

The paper is organized as follows. Section 2 studies the day-market equilibrium taking the size and the number of middlemen as given. It turns out that the Hosios' (1990) condition holds over the sellers and middlemen's day market prices, and hence the resultant allocation is efficient. This implies that the setup of intermediation fits quite naturally into the competitive search equilibrium. The size and number of middlemen are endogenized in section 3, where externalities arising from the inventory restocking process and entry are characterized. Section 4 solves the planner's problem and shows the asymptotic efficiency result. I will discuss my contributions to the related literature in Section 5, before concluding in Section 6. All the proofs are collected in the Appendix.

[4] One should note that although their model does not consider the inventory holdings, they point out its importance in intermediation (p.400-401). I will discuss the difference between their model and mine in detail in Section 5.

[5] The efficiency result is obtained in the literature based on other formulations, for example, in Accemoglu and Shimer (1999ab), Shi (2002ab), Shimer (2004) in labor economics, and Rochetau and Wright (2003) in monetary economics. Some inefficiency results are obtained in Faig and Jerez (2003) with incomplete information, and in Albrecht, Gautier, and Vroman (2003) with multiple applications.

2 Day-market equilibrium

2.1 Environment

Consider an economy where there are three types of agents: buyers, sellers and middlemen that are all continuum measured by one, S, and M respectively. s or m referrers to an agent whose name is seller or middleman. Time is discrete and continues forever. Each period is divided into two sub-periods - a day and a night. At the beginning of any day each seller has one unit of the good for sale and each middleman has I units of goods. All buyers wish to purchase a unit of the good that day. Before the day starts, each seller and middleman post a price they are willing to sell at. When the day starts buyers observe all prices and decide which trader to visit. A buyer can visit only one seller, or one middleman. Given the number of buyers who show up, the seller or middleman sell the good, or goods, at the price stated. If there are more buyers than goods available, any buyer purchases the good with equal probability. Buyers who purchase at price p, obtain period utility $1-p$, whereas those who do not purchase obtain period utility 0. Anyone who sells a k units of the good at price p obtains profit kp per period. The period is then repeated infinitely and all discount the future at rate r. How the sellers and middlemen obtain the goods, they supply in each day market will be discussed in the following section.

I will construct a symmetric steady-state equilibrium in the day market that has the following characteristics: (a) All buyers use identical mixed strategies. They visit some seller with probability Sx^s and visit some middleman with probability Mx^m, assigning an equal probability to each seller and to each middleman. They satisfy the adding-up restriction, $Mx^m + Sx^s = 1$; (b) All middlemen offer the identical price, p^m, expecting to attract the identical number of buyers x^m, and all sellers offer the identical price, p^s, expecting the identical number of buyers x^s; (c) Sellers' and middlemen's price postings are optimal given others' price offers and buyers' visiting strategies, and buyers' visiting strategies are optimal given the prices posted by sellers and middlemen and all other buyers' visiting strategies; (d) Agents' expectations are rational.

2.2 Buyers' directed search

The underlying matching environment is described as follows. Suppose a consumer visits a particular seller and n^s other consumers also visit it that period. In this case the buyer purchases the good with probability $1/(n^s+1)$. Suppose now the consumer visits a middleman and n^m others also visit it. If $n^m \geq I$, then the consumer purchases from this middleman with probability $I/(n^m+1)$. Otherwise, the buyer obtains the good with probability one.

Denote $x^{(i)}$ to represent an expected number of buyers who show up at a seller (if $i=s$) or at a middleman (if $i=m$), who posts a price $p^{(i)}$ (arbitrary). The following lemma shows the probability that a buyer is served when choosing $p^{(i)}$, denoted as $\eta^i = \eta^i(x^{(i)})$, can be expressed in terms of $x^{(i)}$ and the number of capacity.

Lemma 1 *Given $x^{(s)}, x^{(m)}$, the probabilities of obtaining a good from the respective seller and middleman are:*

$$
\begin{aligned}
\eta^m &= \frac{\Gamma\left(I, x^{(m)}\right)}{\Gamma\left(I\right)} + \frac{I}{x^{(m)}}\left(1 - \frac{\Gamma\left(I+1, x^{(m)}\right)}{\Gamma\left(I+1\right)}\right) \\
\eta^s &= \frac{1-e^{-x^{(s)}}}{x^{(s)}},
\end{aligned}
$$

where $\Gamma\left(I\right) = (I-1)! = \int_0^\infty t^{I-1}e^{-t}dt$ and $\Gamma\left(I,x\right) = \int_x^\infty t^{I-1}e^{-t}dt$. η^i is continuously twice differentiable and strictly decreasing in x^i with $\eta^i(0)=1$ and $\eta^i(\infty)=0$. $\eta^s=\eta^m$ when $I=1$.

The matching function η^m captures the stochastic rationing environment with multiple capacity. The key is that the service probability depends on the possibility of whether a stockout occuring or not.[6] If a smaller number of buyers than capacity visit the given middleman, any buyer gets a good with probability one from that middleman. This happens with probability $\frac{\Gamma(I,x^{(m)})}{\Gamma(I)}$, as appearing in the first term.[7] The probability that more than or equal to I number of buyers appear is $1-\frac{\Gamma(I+1,x^{(m)})}{\Gamma(I+1)}$, which will be referred to the *stockout probability.* In this case, I number of buyers are chosen randomly from among those at hand, and therefore any buyer would be served with probability $\frac{I}{x^{(m)}}$, as given in the second term. Clearly, the more an expected number of buyers appearing at a middleman, the smaller the probability that a buyer gets served by visiting the

[6] To the best of my knowledge, it is the first time that the urn-ball matching function is generalized to allow for any number of 'urns' (capacity). Burdett, Shi, and Wright (2001) and Shi (2002) study a case with $I=2$. An economic application of urn-ball models dates back to Butters (1977). Albrecht, Gautier, Tan and Vroman (2003) extends the standard urn-ball formula in the opposite direction to my generalization, by allowing for multiple 'balls' (applications) in a labor market context.

[7] In the standard urn-ball matching model with single urn, the probability of this event is binomial under finite agents and is approximated by a Poisson *point* process under continuum agents. In my situation with multiple capacities under contiuum agents, it follows from a Poisson *counting* process. For reference to the Poisson counting process, see Taylor and Karlin (1998) p.185-189, for example. $\Gamma(I,x)/\Gamma(I)$ is a continuously differentiable function with respect to $x \in [0,\infty] \subseteq \mathcal{R}_+$ and $I \in \mathcal{R}_+$ (See Erdelyl, Magnus, Oberhettinger, and Tricomi (1953)).

middleman. Hence, η^m is strictly decreasing in $x^{(m)}$. η^s can be derived by applying $I = 1$ to η^m, a standard urn-ball matching function.

In a day-market equilibrium, buyers adjust their visiting strategies so that they are indifferent between visiting any sellers and any middlemen, given the service probability stated above. Denote V^b to represent an discounted lifetime utility that buyers expect to obtain by choosing the best offer every period. A supplier i believes its price offer $p^{(i)}$ would attract on average $x^{(i)}$ number of buyers. As supplier i is infinitesimal, however, its price offer cannot affect this value. The equilibrium pines down V^b where all agents' expectations become self-fulfilling, as will be described later.

Given their belief and the service probability, $x^{(i)}$ is determined to reflect buyers' mixing strategies each period. If a supplier i attracts a positive number of buyers at some period, then it implies buyers must be indifferent between choosing its price $p^{(i)}$ and taking rV^b at that period. That is, $0 < x^{(i)} < \infty$ implies:

$$rV^b = \eta^i(x^{(i)})(1 - p^{(i)}).$$

If a buyer chooses $p^{(i)}$, it must reflect the buyer's expectation that the supplier i would attract $x^{(i)}$ number of buyers on average, and the good would be obtained with probability $0 < \eta^i(x^{(i)}) < 1$. At the same time, all other buyers have the same expectation when visiting i. Alternatively, if a supplier i attracts no consumers, then it implies its price is set so high that all buyers are better off by visiting the best alternative supplier, that is, $rV^b > \eta^i(0)(1 - p^i) = 1 - p^i$. In sum, the buyers' mixing strategies imply $x^{(i)}$ must satisfy the following each period for given values of V^b.[8]

$$x^{(i)} = \begin{cases} 0 & \text{if} \quad p^{(i)} > 1 - rV^b \\ x^{(i)}(p^{(i)}) \in (0, \infty) & \text{if} \quad p^{(i)} = 1 - \frac{rV^b}{\eta^i(\cdot)}. \end{cases} \tag{1}$$

I will refer to $x^{(i)}$ determined in equation (1) as an *expected demand*, since all anticipate that $x^{(i)}$ number of consumers are willing to purchase at a price $p^{(i)}$. It is downward sloping and reflects buyers' tradeoffs between the price and the probability of service. Note that *ex-ante* optimality describing an expect demand must be consistent with the *ex-post* gains to be realized in equilibrium.

[8]Given other buyers are taking the same strategy, $x^i = \infty$ cannot satisfy any buyer's optimality when $1 \leq I < \infty$, since it leads to the zero service probability and hence to the zero expect utility.

Such a consistency will be made clear when defining an equilibrium under the rational expectation. At this stage, however, we only need to keep in mind that $x^{(i)}$ is conditioned on given values of V^b.

2.3 Optimal pricing

Given the buyers' directed search embodied in an expected demand $x^{(i)}$, supplier i chooses optimally its price $p^{(i)}$.[9] At the start of each period, any middleman owns I units of goods, and any seller owns one unit of the good.

Denote $Q^i = Q^i(x^{(i)})$ to be an expected number of goods that a supplier i sells per period, when it posts a price $p^{(i)}$ and has an expect demand $x^{(i)}$. Supplier i's expected profit is the product of the expect number of sales times the price it posts:

$$Q^i p^{(i)} = x^{(i)} \eta^i p^{(i)} = \left\{ x^{(i)} \frac{\Gamma(k, x^{(i)})}{\Gamma(k)} + k \left(1 - \frac{\Gamma(k+1, x^{(i)})}{\Gamma(k+1)} \right) \right\} p^{(i)},$$

where $k = 1$ describes a seller's expect profit ($i = s$) and $k = I$ describes a middleman's ($i = m$). If fewer buyers show up at supplier i than its capacity, which occurs with probability $\frac{\Gamma(k,x^{(i)})}{\Gamma(k)}$, then it would sell $x^{(m)}$ units. Otherwise, a stockout occurs and hence all the goods or the good it has can be sold out without fail. Q^m/I represents the probability that any middleman sells out all the I units, while any seller expects to sell one unit of the good with probability $Q^S = 1 - e^{-x^{(s)}}$.

Taking V^b as given, each supplier chooses simultaneously a price to maximize its expect profit subject to the expected demand determined by (1). The downward sloping expect-demand imposes price competition among suppliers through buyers' directed search. Given its belief about others' price offers and buyers' search, a high price to induce zero demand is not optimal for any supplier. By using the demand constraint (1) to eliminate the price, the expect profit is expressed as follows:

$$Q^i p^{(i)} = x^{(i)} \eta^i (x^{(i)}) \left(1 - \frac{rV^b}{\eta^i(x^{(i)})} \right).$$

The problem is converted to a choice of $x^{(i)}$. A straightforward algebra shows this is strictly concave in $x^{(i)}$, hence the first order condition characterized the optimal price.

$$p^{(i)} = \varphi^i \equiv -\frac{d\eta^i/dx^{(i)}}{\eta^i/x^{(i)}} = \frac{1 - \frac{\Gamma\left(k+1, x^{(i)}\right)}{\Gamma(k+1)}}{Q^i/k} \tag{2}$$

[9] The price is assumed to be posted irrelevant of how many buyers show up. This assumption is, however, not restrictive since as shown by Coles and Eeckhout (2003), even if the pricing strategy were extended to be contingent on the ex post number of buyers, the equilibrium where the price does not depend on it still survives.

Each period, the price is set equal to the elasticity of the matching function, denoted as $\varphi^i = \varphi^i(x^{(i)})$.

The elasticity of the matching function plays a crucial role for the rest of the analysis. It is the ratio of the stockout probability to the probability of sold-out. When the stockout probability is high, this ratio is high and the price can be set high. Buyers accept a small fraction of trading surplus when there is a high risk of rationing. Naturally, more buyers are expected to yield a larger trading surplus to supplier i, as shown below.

Lemma 2 $\quad d\varphi^i/dx^i > 0; \quad \varphi^i(0) = 0; \quad \varphi^i(\infty) = 1$

Supplier i can extract all the trading surplus from buyers if a stockout or rationing occurs with probability one, while it is willing to give up all the surplus if fewer buyers appear than capacity with probability one.

2.4 Existence, uniqueness and characterization of the equilibrium

The optimality conditions for buyers, sellers and middlemen, described in the previous sections taking values of V^b as given, are now collected to construct a day-market equilibrium.

Definition 3 *A day-market equilibrium in this economy is a set of expected lifetime payoffs $\{V^j\}$ for $j = b, s, m$, and market outcomes $\{x^i, p^i\}$, for $i = s, m$, such that*

1. *The buyers' directed search satisfies equation (1);*
2. *The sellers' and middlemen's price-postings are optimal, satisfying the first order conditions (2) for $i = s, m$;*
3. *All agents employ the same strategy each period and the same type of agents take the same strategy and earn the same expected lifetime utility or profit, V^j for $j = b, s, m$.*
4. *Agents' expectations are rational, that is, the expected demands are consistent with their equilibrium realizations each period.*

We are now ready to close the price-search loop by finding a fixed point of V^b. The rational expectation requires that buyers' search should be directed towards the best offer, which yield the highest expected utility rV^b attainable over the equilibrium prices each period. Any buyers' expectation that a price $p^{(i)}$ would attract a demand $x^{(i)}$ is self-fulfilling at $x^i = x^{(i)}(p^i)$ in equilibrium, where no supplier i has an incentive to deviate from $p^{(i)} = p^i$. This implies:

$$rV^b = \eta^s(x^s)(1-p^s) = \eta^m(x^m)(1-p^m).$$

Each period, any buyers must be indifferent between visiting any seller and any middleman in equilibrium. The following theorem summarizes the main result of this section.

Theorem 4 *An equilibrium exists and is unique, where* $\{x^i, p^i, V^j\}$ *for* $i = s, m$ *and* $j = b, s, m$ *are the solution to the following equilibrium conditions:*

1. *Buyer Optimality:* V^b, x^s, x^m *satisfy*

$$\begin{aligned} rV^b &= \eta^s(x^s)\,(1-p^s) && (3)\\ &= \eta^m(x^m)\,(1-p^m) && (4)\\ Mx^m + Sx^s &= 1. && (5) \end{aligned}$$

2. *Seller Optimality:* V^s, p^s *satisfy*

$$\begin{aligned} rV^s &= Q^s(x^s)p^s && (6)\\ p^s &= \varphi^s(x^s). && (7) \end{aligned}$$

3. *Middleman Optimality:* V^m, p^m *satisfy*

$$\begin{aligned} rV^m &= Q^m(x^m)p^m && (8)\\ p^m &= \varphi^m(x^m). && (9) \end{aligned}$$

Further, the equilibrium implies $V^b, V^s, V^m > 0$.

Buyers are indifferent between visiting any seller who offers a low price and low service and any middleman who offers a high price and high service. Since the price are posted equal to the elasticity of matching function, the equilibrium implies:

$$\frac{\partial Q^m(x^m)}{\partial x^m} = \frac{\partial Q^s(x^s)}{\partial x^s}, \tag{10}$$

that is, the marginal contributions of an additional expect demand, or search intensity, on the matching rate is equated across all sellers and middlemen. As a result, the buyers' indifference condition can be simplified and stated in terms of the service possibility that sellers and middlemen offer. By using (7), (9), the prices in (3), (4) can be substituted out to yield:

$$\frac{\Gamma(I, x^m)}{\Gamma(I)} = e^{-x^s}. \tag{11}$$

It is clear from (11) that when $I = 1$ any seller and any middleman attracts the same number of buyers $x^s = x^m$, by offering the same price $p^s = p^m$, providing the same service $\eta^s = \eta^m$, and earning the same lifetime profit $V^s = V^m$.

Further, the equilibrium prices can be reinterpreted as a solution to the generalized Nash bargaining problem. That is, p^i satisfies:

$$p^i = \mathrm{argmax}(p^i)^{\phi^i} \left(1 - p^i\right)^{1-\phi^i}$$

where supplier i's bargaining power ϕ^i is equal to the elasticity of the matching function, $\phi^i = \varphi^i(x^i)$.[10] The risk arising from stockout gives suppliers a barganing power over the terms of trade. This implies with owning multiple units of inventory any middleman has a large barganing power when dividing a trading surplus with buyers, while a seller is in a weak position when bargaining.

The day-market equilibrium is fully characterized by the market tightness. As the number of middlemen increases, more goods are supplied into the day market. The intensified competition among sellers and middlemen lowers prices, and hence buyers enjoy a high welfare with a large number of succesful matches. See Figure 1.[11] An increase in inventory each middleman owns has a similar effect to increase buyers' welfare and trading volume.

However, the size of middlemen effects not uniformly on competition and search. On one hand, an increase in I creates a *demand stimulating effect* to give middlemen a monolpoy power, since buyers are benefited from visiting middlemen more intensively and visiting sellers less intensively. On the other hand, however, since a large amount of inventory has an effect of reducing the stockout probability, middlemen can extract only a small fraction of trading surplus from buyers per unit of inventory. An over-accumulation of inventory creates a risk of its remaining unsold which suppresses the barganing power of middlemen. In the limit as $I \to \infty$, the latter effect becomes dominant,

[10] This condition is called the Hosios condition in the matching literature. As will be described later, since the Hosios condition holds endogenously in my model, the matching externalities are internalized in the day-market allocation.

[11] These effects are evident from the equilibrium requirement, $Mx^m(V^b) + Sx^s(V^b) = 1$: As shown in Step 2 and 3 in the proof of Theorem 6, $x^s(\cdot)$ and $x^m(\cdot)$ are strictly decreasing in V^b. Therefore, an increase in M must lead to a decrease in V^b, and hence an increase in x^s and x^m.

hence the day market approaches to be perfectly competitive: both sellers and middlemen provide a full service $\eta^s, \eta^m \to 1$, with marginal cost pricing $p^s, p^m \to 0$. In this frictionless competitive equilibrium, buyers-sellers direct trades are completely unravelled, where buyers obtain maximam welfare $V^b \to 1/r$, while sellers and middlemen earn zero profits $V^s, V^m \to 0$.

3 Size and number of middlemen

3.1 Inventory restocking process at night market

This section endogenizes the steady state level of middlemen's inventory holding during the day. A market is open at night, where middlemen can restock an inventory from the sellers who do not make a sale during the day. Suppose now that any middlemen have a storage technology of the goods. Then, in order to hold I units at the beginning of each day, middlemen need to restock $Q^m(x^m)$ units at each night. Hence, there is a measure $MQ^m(x^m)$ of aggregate demand at every night market. Suppose further that any sellers can produce one unit of the good costlessly, but they do not have the technology to store it. Then, since the remaining goods are valueless for the sellers, there is a measure $S(1 - Q^s(x^s))$ of aggregate supply available to middlemen at every night market. How middlemen can obtain such a technology is discussed in the following section.

As already mentioned, matching frictions and competition are absent at night. I assume that the inventory level each middleman owns is determined to balance aggregate demand and aggregate supply at night market each period. To reflect the dependence of equilibria on I, let us change a notation slightly to $Q^m = Q^m(x^m, I)$. The steady state inventory level satisfies:

$$MQ^m(x^m, I) + SQ^s(x^s) = S, \tag{12}$$

where x^S, x^M are determined in Theorem 4. One could imagine this equation as the night-market 'clearing' condition, which determines the day-market inventory holding level. I will construct an equilibrium, which is referred to as a *middlemen equilibrium*, onto the day-market equilibrium.[12]

Theorem 5 *Assuming $M < S < 1$, a middlemen equilibrium exists and is unique, with the market equilibrium described in Theorem 4 and $1 \leq I < \infty$ satisfying (12).*

[12] The total number of ex post matches (i.e., the LHS of (12)) cannot exceed the measure of buyers. Hence, it is necessary to assume $S < 1$. $\overline{M}$ defines the upper bound of the measure of middlemen, satisfying $I = 1$ at $M = \overline{M}$ in equilibrium (see Step 2 in the proof of Theorem 5).

The middlemen equilibrium has the following properties. The goods produced each period are all distributed to buyers: a proportion, $SQ^s(x^s)$, at today's day market and the remaining proportion, $S(1-Q^s(x^s))$, at tomorrow's day market through middlemen. This implies the total welfare is fixed at the production level:

$$MV^m + SV^s + V^b = S/r.$$

This restocking process is repeated every period in the steady state. The essence of the middlemen equilibrium is that the middlemen's inventory restocking at night interacts with the day-market tightness through the night-market 'clearing' condition (12).

The day-market outcome reflects two different effects, namely, the *day-market effect* and the *night-market effect*. The former works within days while the latter works to feedback from nights to days. As described in the previous section, when the number of middlemen decreases, the day-market effect is to decrease buyers' welfare and total trading volumes. Since it implies the night-market gets less congested, each middleman can restock a large amount of inventory. As a result, each middleman provides a high matching service to buyers during the day, hence the night-market effect works to direct buyers' search more intensively towards middlemen and less intensively towards sellers. At the same time, however, a reduced risk of stockout gives buyers a bargaining power, which mitigates middlemen's monopoly power arising from the demand stimulating effect of the small number of the large-sized intermediation.

3.2 'Single' middleman equilibrium

As the measure of middlemen approaches to the minimum, namely, to a 'single' middleman, the single middleman collects all the goods remained during the day at night ($I \to \infty$). The resulting high matching service induces buyers to visit the middleman quite intensively ($x^m \to \infty$). However, since the quantity that the single middleman restocks cannot exceed the amount available at night, she cannot cover all buyers. This implies buyers are still benefited from visiting sellers with a positive probability, and hence sellers attract some buyers ($x^s \to x^{s'} > 0$).[13]

[13] $x^{s'}$ is a unique solution to $1/S = x^{s'} + e^{-x^{s'}}$. See the proof of Proposition 6.

Proposition 6 *At the limiting middleman equilibrium as $M \to 0$, the 'single' middleman provides a full service, owns balanced-inventory, and sells out all of her/his goods with probability one.*

The night-market effect is dominant when intermediated market is organized by a single middleman who provides frictionless trades ($\eta^m \to 1$) to equate demand and supply, namely, a *vertical centralization* phenomenon. Note that although the matching probability approaches to one, there exists a possibility of stockout. This allows the single middleman to extract a non-zero trading surplus from buyers per unit of inventory. As a result, she obtains the maximum expect profit $V^m \to \infty$ and sells out all inventory with probability one ($Q^m \to I$), by holding a balanced-inventory $Mx^m = MI$ and posting a price, $0 < p^s < p^m < 1$.

Since the short-side principle works in terms of the aggregate number of successful matches, a particular interest lies in the limiting case where the economy is endowed with the same amount of goods as population of buyers. The following proposition shows the Walrasian outcome emerges at the single middleman equilibrium in the limit.

Proposition 7 *In the limit as $S \to 1$, the single middleman equilibrium approaches to a competitive equilibrium. The complete unraveling of buyers-sellers trades occurs where the single middleman holds an excess-inventory and buyers maximize welfare.*

Although sellers are operating in the day-market with one unit of good for a sale, all buyers visit the single middleman with probability one ($Mx^m \to 1$), anticipating that the single middleman would serve them without fail ($\eta^m \to 1$). Their expectations are self-fulfilling in equilibrium only when the single middleman accumulates an excess-inventory ($Mx^m < MI$) at night to accommodate all buyers during the day ($S \to 1$). As a result, the competitive threat driven by the presence of the over-loaded supplier forces the market prices down to the competitive level ($p^S, p^M \to 0$), with zero expect profits. In the competitive equilibrium, all buyers buy a re-soled good from the single middleman, who burdens herself with holding an extra amount of remaining goods every period ($I - Q^m > 0$). This gives buyers all bargaining power to maximize welfare. In other words, for the single middleman to act as such a benevolent central trade-coordinator, she must come with an extra initial 'seed' to intermediate between sellers and buyers.

3.3 Entry

Intermediation requires an inventory-holding technology to store goods for future sales every period. Suppose now that the storage technology can be obtained by paying a fixed cost $c > 0$. Imagine a situation where middlemen buy a warehouse or a refrigerator before starting intermediation business.

The presence of rent induces entry of middlemen. Entry-exit occurs at night. After entering into the market, middlemen must pay c each night when they use the storage technology. Following the literature, I assume the number of middlemen active in the market is determined by free-entry condition,[14]

$$V^m = c/r, \tag{13}$$

Middlemen entering into the market earn zero profits. Assuming the bounds on the entry cost implies the existence of the equilibrium measure of middlemen.[15] The following proposition is immediate from the previous analysis, so the proof is omitted.

Theorem 8 *Assuing $\underline{c} \leq c < \infty$, a middlemen equilibrium under free-entry exists, with $0 < M \leq \overline{M}$ satisfying Theorem 5 and (13). Further, the single middleman equilibrium described in Proposition 6 emerges in the limit as $c \to \infty$, with $0 < Mc < MI$.*

The night-market effect implies that entry reduces the middlemen's inventory holdings. The externality generated from the large number of small-sized intermediation is eliminated only when the entry cost is prohibitively high to induce the 'single' middleman.

4 Constrained Efficiency

An allocation is constrained efficient if it maximizes the aggregate number of trades subject to the resources available in the economy and search restrictions - a standard definition in the competitive-search literature (see Moen (1997) and Shimer (1996)). This section first shows that middlemen are

[14] The free-entry condition is the commonly used assumption in the literature as in Accemoglu and Shimer (1999ab) and Moen (1997), in particular in Mortensen and Wright (2002) and Rocheteau and Wright (2003) in the context of the 'market makers' story. Another formulation can be examined but yields exactly the same result. See Watanabe (2004a).

[15] The lower bound, $0 < \underline{c}$, is defined to satisfy $\underline{c} = V^m$ at $M = \overline{M}$, yielding $I = 1$.

efficiency enhancing but not optimal in terms of their size and number. Next, I proceed to show that the 'single' middleman equilibrium characterized in Proposition 6, 8 is efficient, and that the efficient allocation can induce the underlying equilibrium conditions.

For the purpose of identifying exactly the source of inefficiency, consider the situation as in the day-market equilibrium where the size and the number of middlemen are exogenously given. The planner chooses an allocation $x^s, x^m > 0$ to maximize the total number of matches, subject to (5). The following proposition extends the constrained efficiency result obtained in the literature, which provides a useful benchmark to evaluate the matching efficiency of intermediation.

Proposition 9 *Given the size and the number of middlemen are exogenously fixed, the market-equilibrium allocation described in Theorem 4 is constrained efficient.*

The result is immediate from remembering that the Hosios' condition holds over the day-market prices.[16] Focusing our attention on the day-market allocation, our setup of intermediation naturally fits into the competitive search equilibrium. Search externalities arising from coordination frictions are internalized at the market allocation, given the size and the number of middlemen fixed.[17]

However, the optimal solution drastically changes when the capacity and the number of middlemen are endogenized to include the night-market constraint. The planner's problem is now formalized as follows:

$$\max_{\{x^s, x^m, I, M\}} MQ^M(x^m, I) + SQ^S(x^s) - Mc$$

subject to (5), the non-negativity constraints for all the choice variables, and the night-market constraint, (12). The last term in the planner's objective function, Mc, is the cost incurred when using middlemen or when accessing to the storage technology each period. Observe that the first two terms, the aggregate number of matches, are fixed at the measure of sellers by the night-market constraint. Although there are numerous possible combinations of $\{I, M\}$ that satisfy (12), the planner's objective is to minimize Mc. The following proposition shows the optimum is in the limit, where the planner collects the goods at the 'single' location intensively, to which buyers are assigned intensively.

[16]The proof is available in Watanabe (2004b).

[17]See also Albrecht, Gautier, and Vroman (2003), Faig and Jerez (2003), Mortensen and Wright (2002), Shi (2002b), Rocheteau and Wright (2003) which obtain the efficiency result under other circumstances.

Proposition 10 *The constrained efficient allocation under endogenous determination of the size and the number of middlemen satisfies:* $I_* \to \infty$*,* $M_* \to 0$*,* $x_*^m \to \infty$*, and* $0 \leq x_*^s \leq x^{s'}$*. The middlemen in equilibrium described in Theorem 5 and 8 are not efficient in terms of the size and the number, and the resulting allocations are not efficient.*

The 'single' middleman, who has the capacity advantage in serving goods, can minimize the coordination frictions by holding infinitely many goods. Indeed, the 'single' middleman result applies even when the night-market constraint is absent in the planner's problem.[18] The result here reflects the fact that the planner's objective of minimizing the storage cost is in line with the minimization of coordination frictions. The equilibrium allocations are obviously not efficient.

Note that since more buyers exist than the total amount of goods available under (12) and $S < 1$, the rationing problem is inevitable. However, the middleman's capacity can be allowed for an excess-demand ($Mx_*^m > MI_*$), an excess-supply ($Mx_*^m < MI_*$), or a balanced-stock ($Mx_*^m = MI_*$), because the buyers' search optimality does not matter to the planner's problem.

Proposition 11 *The 'single' middleman equilibrium given in Proposition 6 and Theorem 8 is efficient. Further, the underlying equilibrium conditions can be implemented at the efficient allocations.*

The set of the optimum allocations includes the limiting equilibrium allocations. Hence, the latter is mapped into the former. Further, the buyers' equilibrium search condition, (11), can be implemented at the social optimum, by assigning some buyers to sellers $x_*^s = x^{s'}$ to induce the balanced-stock. This leads to the other mapping from the optimum to the single middleman equilibrium allocations.

To conclude, the equilibrium cannot fully utilize their matching effectiveness due to the presence of too many middlemen. They hold an inefficiently small amount of inventory and attract only few buyers. The motive of entry is the presence of the rent, or the markups, that middlemen extract from buyers during the day and from sellers at night. However, the asymptotic matching efficiency is attained when the entry cost is set prohibitively high to induce the single middleman.

[18] Watanabe (2004b) proves even without the binding night-market constraint, the full utilization of intermediation maximizes the total number of matches by centralizing the resource allocation at a single location.

5 Discussion

Related literature - functions of middlemen The first of my contributions to the literature is to integrate the key functions of intermediation into a unified framework, each of which has been separately studied elsewhere - market makings, market clearing, inventory holdings. Second, middlemen in my paper emerge without resorting to any environment of randomness in matching technologies and ex ante heterogeneity, on which all the previous authors motivate the existence of middlemen.

The closely related areas to my paper can be classified into two.[19] The first strand studies the middleman's role of market making with frictions. Using a bilateral random matching and bargaining model, Rubinstein and Wolinsky (1987) show that middlemen are active in equilibrium by assuming that middlemen have a higher meeting rate than sellers and buyers.[20] The middlemen's advantage in the matching technology centers the subsequent extension, but to be derived endogenously as a result of their own investment decisions. Li (1998) considers an environment where the qualities of goods are private information, and claims that middlemen are persons who invest in identifying the quality of goods. Shevichenko (2004) provides an intriguing insight into the role of middlemen, by developing a model where middlemen collect various types of goods to meet various types of agents' tastes. His model and mine share the same mechanism where middlemen's inventory holdings are endogenously linked to their matching effectiveness. The demand stimulating effect, however, arises from the risk of trades, without resource to ex-ante heterogeneity in my model. Another feature in this strand is that since the terms of trades are determined by bilateral bargaining, competition among middlemen is absent.

The middlemen's role of clearing markets is explicitly considered with their price competition in Spulber (1996), but still within a random meeting framework. Because middlemen are assumed to be the only medium of exchange in his model, the subsequent extensions were made to incorporate an additional avenue of exchange to study the conditions under which middlemen survive in equilibrium - a monopolist market maker in Rust and Hall (2003) and buyer-seller direct trades in Hendershott

[19]There are many other important issues that I do not address in this paper. A non-exhaustive list includes Biglaiser (1993), Caillaud and Jullien (2003), Galeotti and Moraga-Gonzalez (2004), Gehrig (1993), Johri and Leach (2002), Masters (2004), Rochet and Tirole (2004), and Smith (2002).

[20]They also assume that only the middlemen are patient and live forever so that they can wait his own consumption. This non-compatibility problem of agents' time spans is also resolved in latter studies including my model.

and Zhang (2003). However, the two important issues are not considered in this strand. First, since the matching technology in this approach is exogenous, middlemen's advantage in the high meeting rate cannot be the issue. Second, since the demand-supply balance is imposed as a constraint each period, there is no room for inventories to play any economic roles - see Rust and Hall, p.400-01.

Finally, the middlemen's role of facilitating trades is similar to that of money. Both mitigate matching frictions. Corbae, Temzelides and Wright (2003) show that the essential role of money is robust to the agents' directed-matching environment assuming the double coincidence of wants and anonymity. My analysis provides a similar insight into a medium of exchange: middlemen survive even when agents are ex ante homogeneous and have ability to choose with whom to match.[21]

Welfare/efficiency property of market-makers under frictions The foregoing analysis shows the vertical centralization of resource allocation at a single location minimizes the risk of market trades arising from coordination frictions. It provides buyers the right incentive to visit middlemen intensively and generates a competitive fringe to limit their monopoly power. When the inventory restocking process at the night-market is explicitly considered, entry of middlemen mitigates the demand stimulating effect of inventory, as well as encourages price competition. The scale effect works through an endogenous reduction in the amount of inventory each individual middleman holds. In the competitive equilibrium, the scale effect dominates the competitive effect of entry.

Rust and Hall (2003) focus on the competition-stimulating effect of entry and show that entry of 'middlemen' or another intermediary called 'market maker' increases the amount of competition and total volume of trades. However, Rust and Hall's model differs from mine in the notion of search frictions and centralization of trades. In their formulation, buyers and sellers can choose between trading with market makers who run a centralized exchange with publicly observable prices, and searching for a price through a trade with middlemen whose prices are private information. As in the original model of Spulber (1996), trades with middlemen are subject to costly search. As

[21]Of course, money and middlemen are compliments as a medium of exchange. I presume the model could be extended to study the origin of money as a universal medium of exchange á la Kiyotaki and Wright (1993). Middlemen, who could be private banks or business enterprises, emerge to intermediate between sellers of money who want to consume goods and buyers who want money in exchange for their products. Because of the presence of the risk arising from coordination frictions, agents have an incentive to trade with middlemen who own an inventory of money and the goods. Agents' expectation that middlemen optimally collect money and goods to provide liquidity can be self-fulfilling in equilibria if they exist. Such a view might be related to the role of banks deposit for a safe keeping purpose proposed by He, Huang and Wright (2003), or the notion of fiat money emerging through organized exchange proposed by Howitt (2004).

search frictions vanish, free entry of market makers leads to an efficient outcome. An addition of the market maker induces the efficient types of traders to trade in the centralized market, which could be termed as a *horizontal centralization* phenomenon. Neeman and Vulkan (2003) construct a mechanism to induce complete unraveling of direct negotiation with middlemen. The issue of the market segmentation is studied sophisticatedly in Spulber (2002).

Shevichenko (2004) shows that large-scaled intermediaries increase buyers' welfare and that their size is too small in equilibrium. His result is, however, qualitatively different from mine in terms of the role played by inventory. First, since their inventory holdings do not reflect the effect of competition among middlemen, it is natural for his framework to obtain the matching inefficiency result. Second, his model is silent on where the inventory comes from. In contrast, my model considers the interaction between middlemen's inventory restocking process at night and matching frictions during the day. Indeed, it is exactly the mechanism through which the competitive equilibrium and the asymptotic efficiency result is obtained in my model.

6 Conclusion

This paper has proposed a unified framework to integrate three key functions of intermediation, each of which is separately studied in the literature: (1) market-makings by linking producers and consumers; (2) market-clearing by setting the price and by rationing customers; (3) providing liquidity or immediacy by holding an inventory. Middlemen arise endogenously to accomplish these triple tasks, and as a result, mitigate market frictions to enhance allocation efficiency. The model provides also a mechamism to determine the size and number of intermediaries. Middlemen's inventory holding creates a demand stimulating effect to attract many customers away from sellers, while at the same time it generates a risk of its remaining unsold. Buyer-seller direct trades are completely unravelled in the competitive equilibrium where a single middleman takes all the trading risk by holding an excess inventory. I found that the efficient allocations can be decentralized in equilibrium in which entry of middleman is prevented by a prohivitively high entry cost. The setup of intermediation fits naturally into the competitive search-equilibrium, and hence the result sends a strong message to the directed search literature - market makers matter.

Beyond the welfare and efficiency properties described above, the idea that middlemen can emerge to coordinate transactions by holding an inventory is quite simple and appealing. I presume that the model is applicable to various market exchanges and a number of extensions of the present model are feasible. First, an interesting extension would be to incorporate ex-ante heterogeneity in buyers' preferences or traveling costs with imperfect information. This would allow the model to describe the situations where a large number of small-sized middlemen fit well, and to study the related issues in the industrial organization literature. The 'single' middleman would be no longer optimal in this case. It is also interesting to endogenize the bid price middlemen offer to producers, and/or the middlemen's optimal inventory choice.[22] Then the equilibria would reflect a holdup problem in which middlemen incorporate uncertainty of consumer demands when investing in an inventory. The steady-state inventory condition assumed in this paper can also be relaxed. Finally, the model can be easily inserted into a standard neoclassical growth framework to endogenize the production process, which would provide a theoretical foundation to address a number of related issues in macroeconomics and labor economics.[23]

[22]The result presented in this paper goes through even if middlemen does not extract all the rent from sellers at night. See Watanabe (2004a) for such an extension, where an interesting multiplicity is obtained in the competitive search equilibria.

[23]Faig and Jerez (2004) construct this type of directed-search model with imperfect information, but without intermediation and inventory holdings.

7 Appendix

Throughout the proofs in this appendix, we simplify the notation to set $x = x^{(m)}$ and $\eta(x) = \eta^m(x)$, when there is no confusion.

7.1 Proof of Lemma 1

Suppose continuum of buyers are measured by an interval $(0, 1]$ and let $X \equiv X((0, 1])$ be a random variable counting the number of event that a buyer appears to a particular middleman or not. For $\forall i = 1, 2, ...$ and for any distinct points $b_0 = 0 < b_1 < b_2 < ... < b_i$, we define $X((a, b])$ to be the number of buyers b_i who visit a particular middleman, from the buyers located at an interval $(a, b]$, where $a < b_i \leq b$. Since each $X((b_{i-1}, b_i])$ is an independent random variable, the event that a single buyer shows up or not constitutes Bernoulli trials for any middleman.

Given the above postulations, the event that the buyer gets served by visiting the middleman follows a Poisson *counting* process and its mean is x (the expected number of buyers per middleman). When a buyer visits a particular middleman who has I units of inventory, she/he would get one of the goods with the following probability:

$$\begin{aligned}\eta(x) &= \Pr(X \leq I-1)\Pr(\text{served} \mid X \leq I-1) + \Pr(X \geq I)\Pr(\text{served} \mid X \geq I) \\ &= \sum_{j=0}^{I-1} \frac{x^j e^{-x}}{j!} + \sum_{j=I}^{\infty} \frac{x^j e^{-x}}{j!} \frac{I}{j+1}.\end{aligned}$$

If X counts less than the middleman's capacity, the buyer would get served with probability one by visiting the middleman. Otherwise, there might be a possibility of rationing and she/he gets served with probability $\frac{I}{j+1}$, where $j \in \{I, I+1, ..., \infty\}$ counts these events. By rearranging the terms, we get

$$\begin{aligned}\eta(x) &= \sum_{j=0}^{I-1} \frac{x^j e^{-x}}{j!} + I \sum_{j=0}^{\infty} \frac{x^j e^{-x}}{(j+1)!} - I \sum_{j=0}^{I-1} \frac{x^j e^{-x}}{(j+1)!} \\ &= \frac{\Gamma(I, x)}{\Gamma(I)} + \frac{I}{x}\left(1 - e^{-x}\right) - \frac{I}{x}\frac{\Gamma(I+1, x)}{\Gamma(I+1)} + \frac{I}{x}e^{-x} \\ &= \frac{\Gamma(I, x)}{\Gamma(I)} + \frac{I}{x}\left(1 - \frac{\Gamma(I+1, x)}{\Gamma(I+1)}\right).\end{aligned}$$

In the second equality we used $\sum_{j=0}^{I-1} \frac{x^j}{j!} = \frac{\Gamma(I,x)}{\Gamma(I)} e^x$ (by taking integration by part $I - 1$ times). $\Gamma(I, x) = \int_x^{\infty} t^{I-1} e^{-t} dt$ is the incomplete gamma function. It is immediately shown that η is strictly decreasing in x. The service probability from the sellers can be obtained by the standard urn-ball formula. QED

7.2 Proof of Lemma 2

The first order differentiation yields:

$$\frac{d\varphi(x)}{dx} = -\frac{d\eta(x)/dx}{\eta(x)^2}\left\{\eta(x)\left(1 + x\frac{d^2\eta(x)/dx^2}{d\eta(x)/dx}\right) - x\frac{d\eta(x)}{dx}\right\}.$$

It is sufficient to show that the first term in the bracket is positive. By noting that $\Gamma(k+1) = k\,\Gamma(k)$ and $\Gamma(k, x) = -\frac{1}{k}\left\{x^I e^{-x} - \Gamma(k+1, x)\right\}$, the term $\frac{d^2\eta(x)}{dx^2}$ can be written as follows:

$$\frac{d^2\eta(x)}{dx^2} = \frac{k}{x^3}\left\{\frac{\gamma(k+1, x)}{\Gamma(k+1)} - x\left(\frac{\Gamma(k+1, x)}{\Gamma(k+1)} - \frac{\Gamma(k, x)}{\Gamma(k)}\right)\right\},$$

where $\gamma(k+1,x) = \Gamma(k+1) - \Gamma(k+1,x)$. Hence, we obtain:

$$1 + x\frac{d^2\eta(x)/dx^2}{d\eta(x)/dx} = \frac{x\left\{\frac{\Gamma(k+1,x)}{\Gamma(k+1)} - \frac{\Gamma(k,x)}{\Gamma(k)}\right\}}{1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}} > 0.$$

This proves the first property. The second and third properties are immediate by applying the l'Hospital's rule once. QED

7.3 Proof of Theorem 4

The previous analysis has established that equations (3)-(9) describe necessary and sufficient conditions for an equilibrium. All that remains here is to establish a solution to those conditions, $\{x^i, p^i, V^j\}$, for $i = s, m$ and $j = b, s, m$, exists and is unique. The proof takes 4 steps. Step 1 establishes that equilibrium requires $V^b, V^s \in [0, \frac{1}{r}]$. For any $V^b \in [0, \frac{1}{r}]$, Step 2 establishes that equations (1),(4),(5) imply a unique solution for $x^s \geq 0$. With a slight abuse of notation, let $x^s(V^b)$ denote that solution. Similarly for any $V^b \in [0, \frac{1}{r}]$, Step 3 establishes that equations (2),(7) imply a unique solution for $x^m \geq 0$. We let $x^m(V^b)$ denote that solution. An equilibrium is then identified by noting equation (5) requires V^b satisfies the fixed point condition:

$$Mx^m(V^b) + Sx^s(V^b) = 1, \tag{14}$$

where M, S are positive constants. Using Steps 2 and 3, Step 4 establishes that a $V^b \in (0, \frac{1}{r})$ satisfying this condition exists and is unique. Hence Step 4 establishes an equilibrium exists and is unique: given V^b satisfying equation (14), x^s and x^m are uniquely determined in Step 2 and 3 respectively, $V^s > 0$ is uniquely determined in Step 1, and $V^m > 0$ is uniquely determined by (8). Finally, x^s and x^m in turn determine a unique solution $p^s \geq 0$ and $p^m \geq 0$ respectively by (7) and (9). By construction, this solution then satisfies equations (3)-(9) and so describes equilibrium.

Step 1 Equilibrium implies $V^b \in [0, \frac{1}{r}]$ and $V^s \in [0, \frac{1}{r}]$.

Proof of Step 1. Equations (3),(6),(7) imply V^b,V^s satisfies:

$$rV^b = e^{-x^s} \quad \text{and} \quad rV^s = 1 - e^{-x^s} - x^s e^{-x^s}.$$

Note that the RHS of the first equation is decreasing in x^s and the RHS of the second equation is increasing in x^s. As equilibrium implies $x^s \in [0, \infty]$, it follows that $V^b \in [0, \frac{1}{r}]$ and $V^s \in [0, \frac{1}{r}]$. This completes the proof of Step 1.

Step 2 For any $V^b \in [0, \frac{1}{r}]$, a solution for $x^s \geq 0$ defined by equations (3),(6),(7) exists, is unique and implies: $x^s(V^b)$ is continuous and strictly decreasing with $\lim x^s(V^b) = \infty$ as $V^b \to 0$ and $x^s(\frac{1}{r}) = 0$.

Proof of Step 2 The result is immediate by the inverse function theorem and by noting that for any $x^s \in [0, \infty]$, V^b is continuous, strictly decreasing in x^s, and hence invertible (by Step 1). This completes the proof of Step 2.

Step 3. For any $V^b \in [0, \frac{1}{r}]$, a solution for $x^m \geq 0$ defined by equations (4),(9) exists, is unique and implies: $x^m(V^b)$ is continuous and strictly decreasing in V^b with $\lim x^M(V^b) = \infty$ as $V^b \to 0$ and $x^m(\frac{1}{r}) = 0$.

Proof of Step 3. By substituting p^m in (9) into (4) and by using the result in Step 2, we obtain:

$$\frac{\Gamma(I, x^m)}{\Gamma(I)} = e^{-x^s(V^b)} \ .$$

By Step 2, the RHS of this equation is continuous, strictly increasing in V^b with taking 0 when $V^b = 0$ and taking 1 when $V^b = \frac{1}{r}$. Therefore since the LHS is strictly decreasing in x^m, we have a unique solution $x^m(V^b)$ which is continuous and strictly decreasing in V^b, with $\lim x^m(V^b) = \infty$ as $V^b \to 0$ and $x^m(\frac{1}{r}) = 0$. This completes the proof of Step 3.

Step 4 There exists a unique $V^b \in (0, \frac{1}{r})$ which satisfies (14).

Proof of Step 4 By Step 2 and Step 3, the LHS of (14) is continuous, strictly decreasing in V^b with approaching to ∞ as $V^b \to 0$ and taking zero when $V^b = \frac{1}{r}$. Because the RHS of (14) is a positive constant, we must have a unique $V^b \in (0, \frac{1}{r})$. This completes the proof of Step 4. QED

7.4 Proof of Theorem 5

The proof is similar to that of Theorem 4 but with modified according to the following steps. Step 1 solves for a unique x^s and x^m for any $I \in [1, \infty)$ using equations (3), (4), (5), (7), (9). With a slight abuse of notation, let $x^s(I)$ and $x^m(I)$ denote this solution. An equilibrium is then identified by noting equation (12) requires I satisfies the fixed point condition:

$$MQ^m(x^m(I), I) + SQ^s(x^s(I)) = S, \tag{15}$$

where M, S are positive constants. Using Step 1, Step 2 establishes that a $I \in [1, \infty)$ satisfying this condition exists and is unique under $M \in (0, \overline{M}]$, where $\overline{M}$ will be defined below. Hence, this implies an equilibrium exists and is unique: given I satisfying (15), Step 1 uniquely pins down x^s and x^m; given this solution, other equilibrium values are pinned down by the same procedure as in Theorem 4, using equations (3),(6),(7),(8). By construction, this solution then satisfies equations (3)-(9), (12) and so describes equilibrium.

Step 1 For any $I \in [1, \infty)$, a solution for x^s and x^m defined by equations (3), (4), (5), (7), (9) exists, is unique and implies: $x^m(I)$ is continuous and strictly increasing in I; $x^s(I)$ is continuous and strictly decreasing in I; $x^s(1) = x^m(1) = \frac{1}{S+M}$; $\lim x^m(I) = \frac{1}{M}$ and $\lim x^s(I) = 0$ as $I \to \infty$.

Proof of Step 1. As mentioned in the main text, equations (3), (4), (7), (9) can be reduced to equation (11). Substituting out x^s in (11) by using (5) yields:

$$\frac{\Gamma(I, x^m)}{\Gamma(I)} = e^{-\frac{1-Mx^m}{S}}.$$

The LHS of this equation is strictly decreasing in x^m and takes 1 when $x^m = 0$ and $\frac{\Gamma(I,1/M)}{\Gamma(I)} < 1$ when $x^m = \frac{1}{M}$. Similarly, the RHS is strictly increasing in x^m and takes $e^{-\frac{1}{S}} < 1$ when $x^m = 0$ and 1 when $x^m = \frac{1}{M}$. Hence, there exists a unique solution $x^m \in (0, \frac{1}{M}]$ for any $I \in [1, \infty)$.

Further, the LHS of the above equation is continuous, strictly increasing in I with taking e^{-x^m} when $I = 1$ and approaching to 1 when $I \to \infty$. This implies: $x^m = x^m(I)$ is continuous and strictly increasing in I; $x^m(1) = \frac{1}{S+M}$; $\lim x^m(I) = \frac{1}{M}$ as $I \to \infty$. Inserting this solution into (5) solves for x^s in terms of I, yielding the rest of the result. This completes the proof of Step 1.

Step 2 There exists a unique $I \in [1, \infty)$ which satisfies (15).

Proof of Step 2. By Step 1, the LHS of (15) is continuous and strictly increasing in I, with taking $(M+S)(1 - e^{-1/(S+M)})$ at $I = 1$ and taking 1 as $I \to \infty$. The RHS is a positive constant $S \in [\overline{M}, 1)$, where $\overline{M}$ defines a unique solution to $1 - e^{-1/(\overline{M}+S)} = \frac{S}{\overline{M}+S}$. Hence, this guarantees the existence of a unique solution, $I \in [1, \infty)$, to (15). This completes the proof of Step 2. QED

7.5 Proof of Proposition 6, 7

As shown in the proof of Theorem 5, x^s, x^m, I are fully characterized by equations (5), (11), (12). The limiting equilibrium as $M \to 0$ is constructed to satisfy these equilibrium requirements. First, note that the limit must satisfy: $x^m \to \infty$ and $I \to \infty$ as $M \to 0$. If either one of them is finite, $Q^m < \infty$ and hence $MQ^m \to 0$ as $M \to 0$. This contradicts to (12) given $x^s < \infty$.

Second, $x^s \to 0$ cannot be consistent with the equilibrium. If it is the case, $Mx^m \to 1$ by (5) and $\frac{\Gamma(I,x^m)}{\Gamma(I)} \to 1$ by (11). The latter implies:

$$Q^m = x^m \frac{\Gamma(I, x^m)}{\Gamma(I)} + I\left(1 - \frac{\Gamma(I+1, x^m)}{\Gamma(I+1)}\right) \to x^m,$$

hence the LHS of (12) exceeds the RHS, $Mx^m \to 1 < S$, a contradiction.

Third, x^m and I are required to evolve at the rate satisfying:

$$\text{(a) } 0 < Mx^m < \infty; \quad \text{(b) } 0 < MI < \infty; \quad \text{(c) } Mx^m = MI.$$

(a) and (b) follow from the fact that the aggregate number of matches must be bounded, as evident in (12), and from: $0 < MQ^m \leq \min\{Mx^m, MI\}$ given $x^s < \infty$. (c) is implied by (11) satisfying:

$$0 < \frac{\Gamma(I, x^m)}{\Gamma(I)} = e^{-x^s} < 1 \quad \text{as} \quad x^m \to \infty \quad \text{and} \quad I \to \infty$$

given $0 < x^s$. To prove it formally requires technique of a uniform expansion of the incomplete gamma function developed by Temme (1975) and given in the Additional Appendix.

Further, the item (c) implies:

$$\frac{\Gamma(I+1, x^m)}{\Gamma(I+1)} - \frac{\Gamma(I, x^m)}{\Gamma(I)} = \frac{(x^m)^{x^m} e^{-x^m}}{x^m!} \to 0$$

in the limit as $x^m \to \infty$. This results from the Stirling's approximation fomula, that is, when x is large, $\frac{x^x e^{-x}}{x!} \simeq \frac{1}{\sqrt{2\pi x}} \equiv \epsilon$. Hence, the limiting equilibrium implies:

$$\eta^m = \frac{\Gamma(I, x^m)}{\Gamma(I)} + \frac{I}{x^m}\left(1 - \frac{\Gamma(I+1, x^m)}{\Gamma(I+1)}\right) \simeq \frac{\Gamma(I, x^m)}{\Gamma(I)}\left(1 - \frac{I}{x^m}\right) + \frac{I}{x^m}(1 - \epsilon) \;\to\; 1.$$

The same argument implies in the limit that: $Q^m/I \to 1$ and $V^m \to \infty$.

Finally, (c) implies $0 < x^s < 1/S$ is a unique solution to $1 - Sx^s = Se^{-x^S}$ by (5) and (11). Hence, Proposition 7 is immediately shown by applying $S \to 1$ to this solution. QED

7.6 Proof of Proposition 10, 11

Let us simplify the notation to omit the subscript $*$ when there is no confusion. This proof will first characterize conditions for the optimal allocations. It will become clear that the limiting equilibrium obtained in Proposition 9 satisfies these optimality conditions. Then, I proceed to show the constrained efficient allocation can induce the equilibrium condition (11).

Notice first that if $M \to 0$ is the solution, then it must follow that: $x^m \to \infty$; $I \to \infty$; $0 < Mx^m < \infty$; $0 < MI < \infty$. There are three cases: (A) $x^m > I$; (B) $x^m = I$; (C) $x^m < I$. Additional Appendix shows the follows must be true in each case:

$$\text{(A')}\ \frac{\Gamma(I,x^m)}{\Gamma(I)} = 0; \quad \text{(B')}\ 0 < \frac{\Gamma(I,x^m)}{\Gamma(I)} < 1; \quad \text{(C')}\ \frac{\Gamma(I,x^m)}{\Gamma(I)} = 1.$$

Given this fact, the optimal allocations for each case are characterized as follows.

Case (A) $x^m > I$**:** If $\{x^m, I\}$ satisfies (A) and hence (A'), then (12) implies:

$$MQ^m = Mx^m \frac{\Gamma(I,x^m)}{\Gamma(I)} + MI\left(1 - \frac{\Gamma(I+1,x^m)}{\Gamma(I+1)}\right) = MI = Se^{-x^s}$$

as $I \to \infty$. Given x^s, this equation and (A') solve for Mx^m and MI. Given this solution, (5) determines some $x^s \in [0, x^{s'})$, satisfying $Mx^m - MI = 1 - Sx^s - Se^{-x^s} > 0$, where $x^{s'}$ is a unique solution to $\frac{1}{S} = x^{s'} + e^{-x^{s'}}$.

Case (B) $x^m = I$**:** (12) and (B') imply:

$$MQ^m = Mx^m = Se^{-x^s}$$

as $x^m \to \infty$, which solves for $x^s = x^{s'}$ by (5). This, in turn, determines $MI = Mx^m$.

Case (C) $x^m < I$**:** Similarly, (12) and (C') imply:

$$MQ^m = Mx^m = Se^{-x^s}$$

as $x^m \to \infty$. Given x^s, this equation and (C') solve for Mx^m and MI. Given this solution, $x^s = x^{s'}$ is determined by (5).

Now, it is clear that the limiting middlemen equilibrium characterized in Proposition 6, 7 as $c \to \infty$ corresponds to case (B) with $0 < Mc < \infty$, and obviously is asymptotically efficient. Because (5) and (12) are exactly the same as the corresponding equilibrium conditions, it remains to prove the existence of an optimal allocation that satisfies the other equilibrium condition, (11), which I briefly sketch below.

The main task is to show the existence of the limit, $q \equiv \frac{\Gamma(I,x^m)}{\Gamma(I)} \in (0,1)$ as $I \to \infty$. The problem can be solved by taking the asymptotic inversion of the incomplete gamma function, the procedure developed in Temme (1992): $x^m = I$ implies q can be approximated, inverted and arranged as follows (see Additional Appendix for details).

$$\frac{Mx^m - MI}{\sqrt{M}\sqrt{2MI}} = \text{inverfc}(2q)$$

$u(v) = \text{inverfc}(v)$ is the inversion of $v = \text{erfc(u)}$ (an error function), a well defined with $u(0) = +\infty$, $u(1) = 0$ and $u(2) = -\infty$. Given $0 < MI < \infty$, the LHS of this equality is bounded when $\frac{x^m}{I} = 1 + O(M^\delta)$ and $\delta \geq 1/2$ as $M \to 0$. Hence, given $x^s = x^{s'}$, an appropriate choice of δ guarantees the existence of a $q \in (0,1)$ that satisfies (11), and therefore: $Mx^m_* = MI_* = Mx^m = MI$ as $M \to 0$. QED

8 Additional Appendix (to referees)

8.1 Complementary Proof of Lemma 1

In this complementary proof, we show that the matching probability derived in Lemma 1 corresponds the limiting probability of its finite-agents setup counterpart. Suppose now B, S, M represent the number of buyers, sellers, and middlemen respectively that are positive integers. Any buyer assigns a probability α to individual middleman. Define $x \equiv \lim B\alpha$ to be the expect number of buyers at a middleman in the limit as B, S, M go to infinity with keeping the ratios B/M, S/M constant. When a buyer visits a middleman who has I units of inventory, he/she would get one of the goods with the following probability:

$$\eta^m = \sum_{h=0}^{I-1} C_{B-1}^h \alpha^h (1-\alpha)^{B-1-h} + \sum_{h=I}^{B-1} \frac{I}{h+1} C_{B-1}^h \alpha^h (1-\alpha)^{B-1-h},$$

where $C_{B-1}^h = \frac{(B-1)!}{h!(B-1-h)!}$. If the number of other buyers visiting the middleman is less than I, the buyer gets served with probability one (the first term), otherwise with probability $\frac{I}{h+1}$, $h \in \{I, I+1, ..., B-1\}$ (the second term). By rearranging the term to apply the binomial theorem, we get

$$\begin{aligned} \eta^m &= \frac{I}{B\alpha}\left\{1-(1-\alpha)^B\right\} + \sum_{h=0}^{I-1}\left(1-\frac{I}{h+1}\right) C_{B-1}^h (\alpha)^h (1-\alpha)^{B-1-h} \\ &\equiv \eta_1^m + \eta_2^m. \end{aligned}$$

It is obvious that $\lim \eta_1^m = \frac{I}{x}(1-e^{-x})$.

In order to calculate the limit of the second term, we view it as the summation of I number of terms as follows.

$$\begin{aligned} \eta_2^m &= (1-I)(1-\alpha)^{B-1} + \left(1-\frac{I}{2}\right)(B-1)\alpha(1-\alpha)^{B-2} + ... \\ &\quad ... + \left(1-\frac{I}{h+1}\right)\frac{(B-1)(B-2)\cdots(B-h)}{h!}\alpha^h(1-\alpha)^{B-1-h} + ... \\ &\quad ... + \left(1-\frac{I}{I}\right)\frac{(B-1)(B-2)\cdots(B-I+1)}{(I-1)!}\alpha^{I-1}(1-\alpha)^{B-I} \end{aligned}$$

The $h+1$-th term in the above expression, which we denote as $\eta_2^m[h]$, is the h-th order multi-nominal, and can be rearranged with a sequence of coefficient $\{c_i\}_{i=0}^h \subset \mathcal{R}$ as follows:

$$\begin{aligned} \eta_2^m[h] &= \frac{\left(1-\frac{I}{h+1}\right)}{h!}\alpha^h(1-\alpha)^{B-1-h}(B-1)(B-2)\cdots(B-h) \\ &= \frac{\left(1-\frac{I}{h+1}\right)}{h!}\alpha^h(1-\alpha)^{B-1-h}\left(c_0B^h + c_1B^{h-1} + ... + c_{h-1}B + c_h\right) \\ &= \frac{\left(1-\frac{I}{h+1}\right)}{h!}(1-\alpha)^{B-1-h}(B\alpha)^h\left(c_0 + c_1B^{-1} + ... + c_{h-1}B^{1-h} + c_hB^{-h}\right). \end{aligned}$$

By noticing that $c_0 = 1$, and $(1-\alpha)^{B-1-h} = (1-\frac{B\alpha}{B})^{B-1-h}$ converges to e^{-x}, it follows in the limit that

$$\lim \eta_2^m[h] = \frac{\left(1-\frac{I}{h+1}\right)}{h!}e^{-x}x^h.$$

This manipulation is allowed as I (over which h is summated) is finite. By summing up this term over $h \in \{0, 1, ..., I-1\}$, we obtain the limit expression as follows:

$$\begin{aligned}\lim \eta_2^m = \lim \sum_{h=0}^{I-1} \eta_2^m[h] &= e^{-x} \sum_{h=0}^{I-1} \frac{x^h}{h!} - e^{-x} \sum_{h=0}^{I-1} \frac{I}{h+1} \frac{x^h}{h!} \\ &= \frac{\Gamma(I,x)}{\Gamma(I)} - \frac{I}{x} \left(\frac{\Gamma(I+1,x)}{\Gamma(I+1)} - e^{-x} \right).\end{aligned}$$

In the first term of the third equality we have applied integration by part $I-1$ times:

$$\begin{aligned}e^{-x} \sum_{h=0}^{I-1} \frac{x^h}{h!} &= e^{-x} + xe^{-x} + ... + \frac{x^{I-2}e^{-x}}{(I-2)!} + \frac{x^{I-1}e^{-x}}{(I-1)!} \\ &= \frac{1}{(I-1)!} \left\{ \frac{(I-1)!}{0!} e^{-x} + \frac{(I-1)!}{1!} xe^{-x} + ... + \frac{(I-1)!}{(I-2)!} x^{I-2} e^{-x} + \frac{(I-1)!}{(I-1)!} x^{I-1} e^{-x} \right\} \\ &= \frac{1}{(I-1)!} \int_x^{\infty} t^{I-1} e^{-t} dt \quad \left(= \frac{\Gamma(I,x)}{\Gamma(I)} \right),\end{aligned}$$

where $\Gamma(I,x) = \int_x^{\infty} t^{I-1} e^{-t} dt$ is the incomplete gamma function, and $\Gamma(I) = \int_0^{\infty} t^{I-1} e^{-t} dt = (I-1)!$ is the gamma function. The second term has been similarly manipulated. These limiting terms are summated to yield the result:

$$\eta^m(x) = \lim \eta_1^m + \lim \eta_2^m = \frac{\Gamma(I,x)}{\Gamma(I)} + \frac{I}{x} \left(1 - \frac{\Gamma(I+1,x)}{\Gamma(I+1)} \right).$$

QED

8.2 Complementary Proof of Proposition 6, 7

Hereafter, simplify the notation to set $x = x^m$. In this complementary proof, I will show formally that the limiting equilibrium given in Proposition 9 must satisfy the item (c), that is, $0 < \frac{\Gamma(I,x)}{\Gamma(I)} < 1$ if and only if $Mx = MI$ as $M \to 0$. Given $x \to \infty$ and $I \to \infty$ as $M \to 0$, it is sufficient to prove in the limit that: $\frac{\Gamma(I,x)}{\Gamma(I)} \to D$ where $D = 1$ if and only if $x < I$ and $D = 0$ if and only if $x > I$.

For that purpose, I use a uniform asymptotic expansion for the incomplete gamma function obtained in Temme (1975, 1979): For any $x \geq 0$,

$$q \equiv \frac{\Gamma(I,x)}{\Gamma(I)} = \frac{1}{2} \text{erfc}(\xi \sqrt{I/2}) + R_I(\xi), \tag{16}$$

where

$$\xi = (\lambda - 1) \sqrt{\frac{2(\lambda - 1 - \ln \lambda)}{(\lambda - 1)^2}}; \quad \lambda = \frac{x}{I}; \quad \text{erfc}(v) = \frac{2}{\sqrt{\pi}} \int_v^{\infty} e^{-t^2} dt$$

with $\text{erfc}(0) = 1$, $\text{erfc}(+\infty) = 0$ and $\text{erfc}(-\infty) = 2$. See also Temme (1996) p.285. This expression is exact, and the asymptotic representation for the remainder, holding uniformly in $\lambda \geq 0$ for $I \to \infty$ and $x \to \infty$, is given as: $R_I(\xi) = O(I^{-1/2} e^{-I\xi^2/2})$ as $I \to \infty$ (see Temme (1982) p.245).

Clearly, $0 < Mx < \infty$ and $0 < MI < \infty$ guarantee that $0 < \lambda < \infty$ and $-\infty < \xi < +\infty$. Suppose now $x < I$. Then, we have $\lambda < 1$ and $\xi < 0$ as $I \to \infty$. This implies in the limit that:

$$\xi \sqrt{I/2} \to -\infty \quad \text{and} \quad \xi^2 I/2 \to +\infty \quad ,$$

therefore, $R_I(\cdot) \to 0$ and $q \to 1$. Since $q \to 1$ implies $R_I(\cdot) \to 0$ and $\xi < 0$ in the limit, this proves the first claim.

Similarly, suppose next that $x > I$. Then, $\lambda > 1$ and $\xi > 0$ as $I \to \infty$, implying:

$$\xi\sqrt{I/2} \to +\infty \text{ and } \xi^2 I/2 \to +\infty \quad ,$$

therefore, $R_I(\cdot) \to 0$ and $q \to 0$ as $I \to \infty$. Further, $q \to 0$ implies $R_I(\cdot) \to 0$ and $\xi > 0$ in the limit, proving the second claim. QED

8.3 Complementary Proof of Proposition 10, 11

By the complementary proof of Proposition 6, 7, it is now clear that the followings are true in the planner's solution described in Proposition 10, 11.

Case (A): $Mx > MI \iff \frac{\Gamma(I,x)}{\Gamma(I)} \to 0$ as $I \to \infty$ and $x \to \infty$.
Case (B): $Mx = MI \iff 0 < \frac{\Gamma(I,x)}{\Gamma(I)} < 1$ as $I \to \infty$ and $x \to \infty$.
Case (C): $Mx < MI \iff \frac{\Gamma(I,x)}{\Gamma(I)} \to 1$ as $I \to \infty$ and $x \to \infty$.

In this complementary proof, I will show that the limit exists for $0 < q \equiv \frac{\Gamma(I,x)}{\Gamma(I)} < 1$ satisfying the equilibrium condition (11). Specifically, I will show that the limit exists in case (B) if and only if $\frac{x}{I} = 1 + O(M^\delta)$ and $\delta \geq 1/2$ as $M \to 0$, that is, $x - I$ must disappear quickly enough in the limit. To prove it formally, I will follow the procedure of the asymptotic inversion of the incomplete gamma function developed in Temme (1992) as follows.

The proof is built on the complementary proof of Proposition 9. First, since $x = I$ implies ξ is small, an expansion of $\xi^2/2 = \lambda - 1 - \ln\lambda$ is allowed to yield:[24]

$$\lambda = 1 + \xi + O(\xi^2). \tag{17}$$

Second, the asymptotic expansion formula, equation (16), can be inverted as:

$$\xi_0 = \frac{\text{inverfc}(2q)}{\sqrt{I/2}}, \tag{18}$$

where $\xi = \xi_0 + \epsilon_1/I + \epsilon_2/I^2 + \ \dots$. $\xi_0 = \xi_0(q, I)$ has the following properties: $\xi_0(0, I) = +\infty$; $\xi_0(1/2, I) = 0$; $\xi_0(1, I) = -\infty$. The value of each ϵ_i is given in Temme (1992) p.760. $u(v) = \text{inverfc}(v)$ is the inversion of $v = \text{erfc(u)}$ with $u(0) = +\infty$, $u(1) = 0$ and $u(2) = -\infty$.

Finally, since $\xi = \xi_0 \to 0$ as $I \to \infty$ with $x = I$, equations (17) and (18) can be combined to yield:

$$x = I + \text{inverfc}(2q)\sqrt{2I},$$

which will further be arranged to:

$$\frac{Mx - MI}{\sqrt{M}\sqrt{2MI}} = \text{inverfc}(2q).$$

Since $0 < MI < \infty$ as $M \to 0$, the LHS of this equality is bounded when $\frac{x}{I} = 1 + O(M^\delta)$ and $\delta \geq 1/2$ as $M \to 0$. Hence, there exists $0 < q < 1$, which identifies $\{x, I\}$ by specifying the value of δ. QED

[24] $\xi^2/2 = \lambda - 1 - \ln\lambda = (\lambda-1)^2/2 - (\lambda-1)^3/3 + (\lambda-1)^4/4 + \dots$. This leads to: $\lambda = 1 + \xi + \xi^2/3 + \xi^3/36 - \xi^4/270 + \dots$

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Figure 1. Day-Market Equilibrium