Complementarities and Games: New Developments

Xavier Vives
INSEAD

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1 Introduction

Complementarities are pervasive in economics, ranging from coordination problems in macroeconomics and finance to pricing and product selection issues in industrial organization. At the heart of complementarity is the notion that the marginal value of an action or variable increases in the level of another action or variable. This is the notion of "Edgeworth complementarity". Complementarities have been a recurrent and somewhat contentious topic of study for economic analysis. Indeed, while Samuelson (1947) in his Foundations stated that "In my opinion, the problem of complementarity has received more attention than is merited by its intrinsic importance" (at the start of the section on complementarity, p.183, 1979 edition), he corrected himself later on in this very Journal in 1974, on the occasion of the 40th anniversary of the Hicks-Allen revolution in demand theory, when he stated at the very beginning of his paper that "The time is ripe for a fresh, modern look at the concept of complementarity. Whatever the intrinsic merits of the concept, forty years ago it helped motivate Hicks and Allen to perform their classical "reconsideration" of ordinal demand theory. And, as I hope to show, the last word has not yet been said on this ancient preoccupation of literary and mathematical economists. The simplest things are often the most complicated to understand fully".

The theory of supermodular games and monotone comparative statics, based on lattice-theoretic methods, has provided a powerful toolbox to analyze the consequences of complementarities in economics. Monotone comparative statics analysis provides conditions under which optimal solutions to optimization problems move monotonically with a parameter. In this paper I will provide an introduction to this methodology and I will apply it to study strategic interaction in the presence of complementarities. The approach exploits order and monotonicity properties in contrast to classical convex analysis. The central piece of attention will be games of strategic complementarities where the best response of a player to the actions of rivals is increasing in their level. The purpose of the article is to bring forward some recent applications of the lattice-theoretic methodology, mostly in industrial organization and finance, and at the same time provide an introduction to the toolbox. The paper will show the usefulness of the approach in:

- providing a common analytical frame to study complementarities,
- derive new results, and
- cast new light on old ones (by getting rid of unnecessary assumptions).

Modeling strategic interaction presents formidable problems. Nash equilibrium may not exist (at least in pure strategies) and even if it exists there may be multiple equilibria. Multiple equilibria present problems of interpretation (how do players coordinate on one of them?) and of policy analysis (how can the policy maker be sure that a change of a parameter will impinge in the desired direction?). Classical comparative statics analysis provides ambiguous results in the presence of multiple equilibria. Indeed, classical convex analysis imposes very strong regularity conditions and leaves the analyst orphan when those stringent conditions are not met. The regularity conditions become particularly strong when applied to games with complex functional strategy spaces like dynamic or Bayesian games. We will see how complementarities are intimately linked to multiple equilibria and how supermodular methods provide a natural tool to characterize them.

The approach has several advantages:

- Allows very general strategy spaces including indivisibilities and functional spaces such as those arising in dynamic or Bayesian games.
- Ensures the existence of equilibrium in pure strategies (without requiring quasiconcavity of payoffs, smoothness assumptions or interior solutions).
- Allows a global analysis of the equilibrium set, which has an order structure with largest and smallest elements.
- Equilibria have nice stability properties and there is an algorithm to compute extremal equilibria.
- Monotone comparative statics results are obtained with minimal assumptions.
- Results can be extended beyond games of strategic complemenatrities.

The above considerations are not only of theoretical interest. In many situations we would like to know how a change of a parameter affects the market equilibrium. Let me provide here two examples, developed in the text, of how the lattice-theoretic approach either obtains new results, hard or impossible to get with the classical approach, or improves upon the results obtained by getting rid of unnecessary assumptions.

- Consider an R&D race where each firm invests continuously to obtain a breakthrough and where we want to know what is the effect of increasing the number of participants n in the race (Lee and Wilde (1980)).

Under very weak assumptions this game is one of strategic complementarities and it will have multiple equilibria naturally. The problem of using the classical approach is that increasing n may make disappear some equilibria and some other may appear. Classical analysis will not help here but with the lattice approach we obtain an unambiguous comparative statics result: increasing n will necessarily increase R&D effort, provided that out of equilibrium adjustment dynamics are of a general adaptive form.

- Fudenberg and Tirole (1984) derived an "animal" taxonomy of strategic behavior in two-stage games under the stringent assumptions of concave payoffs yielding a unique an stable equilibrium at the second stage. What if those assumptions are not fulfilled? We show that in fact none of the strong regularity assumptions imposed are needed and that only the type of competition (strategic complements or substitutes) and whether investment makes the strategic incumbent soft or tough matters.

Instances of coordination failure with multiple equilibria abound: bank runs, debt runs on a country, low employment/activity equilibria, revolutions, and development traps provide some examples. A key issue is how to build coherent models of those situations that are useful for policy analysis. A challenging aspect of any crisis situation is to disentangle self-fulfilling from fundamentals-driven explanations that help answer questions such as: What is the effect of an increase in the amount of central bank reserves in the probability of a run on the currency? What is the impact of an increase in the solvency ratio on the probability of failure of a bank? What is the effect of a change in foreign short-term debt exposure on the probability of default of a small open economy?

Global games (Carlsson and van Damne (1993), Morris and Shin (2002)) are proving to be a popular methodology for equilibrium selection with applications to currency and banking crises and macroeconomics. Global games are Bayesian games and the lattice approach is particularly suited to analyze them. For example, recent major advances in the difficult problem of showing existence of Bayesian equilibrium in pure strategies have been done using the lattice-theoretic methodology. Furthermore, by realizing that in many situations of interest global games are games of strategic complementarities we understand immediately why and how iterated elimination of dominated strategies works and why and under what conditions equilibrium selection is successful. Indeed, we will see how equilibrium is unique precisely when strategic complementarities are weakened. The approach helps by providing a theory of crisis which is linked to the fundamentals of the economy building a bridge between self-fulfilling and fundamentals-driven theories. The implication is that the effect of policy instruments can be understood (the applications will provide an answer to the policy questions above).

The methodology of supermodular games provides tools and an appropriate framework to confront satisfactorily multiple equilibria and comparative statics. However, we should be aware also that the lattice-theoretic approach is not a panacea and cannot be applied to everything as some examples will make clear.

The paper starts in section 2 by introducing a simple class of games where many of the important issues are highlighted. Section 3 provides an introduction to the theory and basic results. Section 4 provides applications to oligopoly and comparative statics in the context of Cournot, Bertrand and R&D games and includes multimarket oligopoly competition. Section 5 deals with dynamic games starting with two-stage games, reviewing the taxonomy of strategic behavior by Fudenberg and Tirole (1984), studying when increasing or decreasing dominance will obtain, and ending with a characterization of strategic incentives in Markov games. Section 6 studies Bayesian games, characterizing equilibria in pure strategies, comparative statics and games of voluntary disclosure, and discussing global games with applications to currency and banking crisis. The Appendix provides a brief recollection of the most important definitions and results of the lattice-theoretic method.

2 A simple framework

Games of strategic complementarities are those in which players respond to an increase in the strategies of the rivals with an increase in their own strategy. I present in this section an example that suggests the flavor of many of the results that can be obtained with the approach.

Consider a game with a continuum of players in which the payoff to a player is $\pi(a_i, \widetilde{a}; \theta_i)$, where a_i is the action of the player, lying in a (normalized) compact interval $[0, 1]$, $\widetilde{a}$ the average or aggregate action and θ_i a, possibly idiosyncratic, payoff-relevant parameter.[1] I consider first the case with homogeneous players and then with heterogeneous players.

2.1 Homogeneous players

Consider the symmetric case wherethe payoff to a player is given by $\pi(a_i, \widetilde{a}; \theta)$, suppose that π is smooth in all arguments and strictly concave in a_i, and let

[1] The analysis with n players is similar.

$r(\cdot)$ be the best response of an individual player to aggregate action $\widetilde{a}$. In this framework equilibria will be symmetric because given any aggregate action $\widetilde{a}$ there is a unique best response $r(\widetilde{a};\theta)$. For interior solutions we will have that $\frac{\partial \pi}{\partial a_i}(r(\widetilde{a}),\widetilde{a};\theta)=0$. If $\frac{\partial^2 \pi}{(\partial a_i)^2}<0$ then r is continuously differentiable and $r'(\widetilde{a})=\left(\frac{\partial^2 \pi}{\partial a_i \partial \widetilde{a}}\right)/\left(\frac{\partial^2 \pi}{(\partial a_i)^2}\right)$. Therefore, $sign\ r'(\widetilde{a})=sign\frac{\partial^2 \pi}{\partial a_i \partial \widetilde{a}}$ and best replies are increasing if $\frac{\partial^2 \pi}{\partial a_i \partial \widetilde{a}}\geq 0$. A symmetric equilibrium is characterized by $r(a;\theta)=a$. Suppose also that $\frac{\partial^2 \pi}{\partial a_i \partial \theta}\geq 0$ so that an increase in θ increases the marginal profit of the action of a player and his best response $r(\cdot)$.

Two examples of the game are monopolistic competition and search. In monopolistic competition (section 6.6 in Vives (1999)) the action would be the price of a firm with $\widetilde{a}$ the average price in the market and θ a demand or cost parameter. For example, $\pi(a_i,\widetilde{a};\theta)=(a_i-\theta)D(a_i,\widetilde{a})$ with $D(\cdot)$ the demand function and θ the (common) marginal cost. For many demand systems $\frac{\partial^2 \log D}{\partial a_i \partial \widetilde{a}}>0$ (meaning that the elasticity of demand is increasing in the average price) and therefore $\frac{\partial^2 \log \pi}{\partial a_i \partial \widetilde{a}}=\frac{\partial^2 \log D}{\partial a_i \partial \widetilde{a}}>0$. Under this condition we will have that $r'(\widetilde{a})>0$ because best replies are invariant to an increasing transformation of the payoffs, such as the logarithm. In the search model (Diamond (1981)) the action a_i is the effort of trader i in looking for a partner. The benefit (probability of finding a partner) is proportional to own effort and increasing in the aggregate effort of others $\widetilde{a}$: $\pi(a_i,\widetilde{a};\theta)=\theta a_i f(\widetilde{a})-C(a_i)$, with θ the efficiency of the search technology, and $f(\cdot)$ and the cost of effort $C(\cdot)$ being increasing functions. In this case $\frac{\partial^2 \pi}{\partial a_i \partial \widetilde{a}}=a_i f'(\widetilde{a})\geq 0$. Models of aggregate demand externalities, and models of Keynesian effects have a similar flavor (see Cooper and John (1988)).

In those examples it is easy to generate multiple equilibria. For example, in the search model let $f(\widetilde{a})\equiv\widetilde{a}$ and C be increasing with $C(0)=0$, then $a_i=0$ for all i is always an equilibrium. If C is smooth and strictly convex with $C'(0)=0$, then there are two equilibria $a_i=0$ and $a_i=\widehat{a}>0$, with $\theta\widehat{a}=C'(\widehat{a})$, for all i. The latter equilibrium increases strictly with θ and is Pareto superior to the no effort equilibrium. Another possibility is when f has an S-shaped function and $C'(a)\equiv a$, then there will be three equilibria. They will be the solutions to $\theta f(a)=a:\underline{a},\widehat{a}$,and $\overline{a}$ as depicted in Figure 1 (lower branch). In this example $r(\widetilde{a})=\theta f(\widetilde{a})$. Obviously, equilibria are the solution to $r(a)=a$ and $r'(\widetilde{a})=\theta f'(\widetilde{a})$.

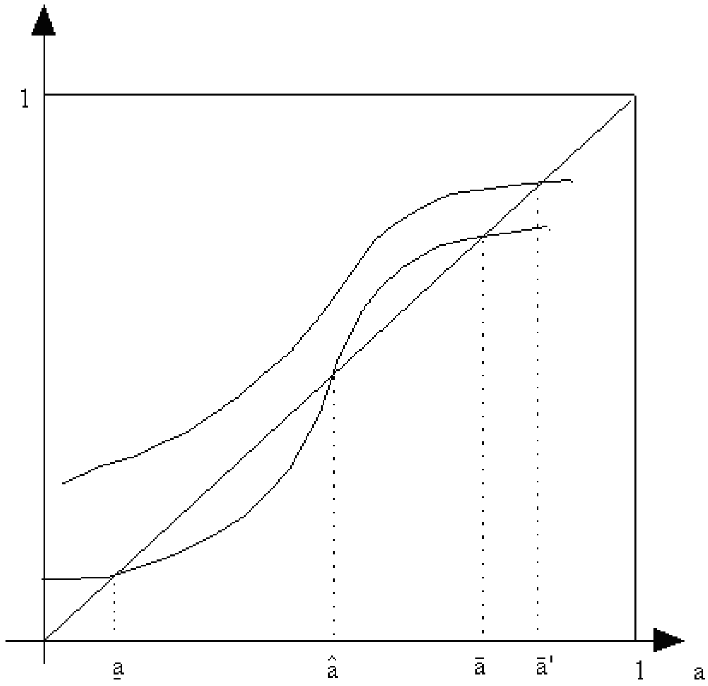


Figure 1

Several properties of the equilibria are worth noticing.

1. A sufficient condition to have multiple equilibria is that strategic complementarities be sufficiently strong. Namely, that $r'(a) > 1$ for some candidate equilibrium $r(a;\theta) = a$ (such as point $\widehat{a}$ in Figure 1).

2. The symmetric equilibria are ordered and there exits largest ($\overline{a}$) and smallest ($\underline{a}$) equilibria (this follows trivially here given that actions are one-dimensional) and equilibria can be Pareto ranked. This is a general property whenever π is increasing in $\widetilde{a}$ (positive externalities).

3. Extremal equilibria, $\underline{a}$ and $\overline{a}$, are stable with respect to the usual best reply dynamics. Indeed, it is immediate that best response dynamics starting at $a = 0$ (at $a = 1$) will converge to $\underline{a}$ (to $\overline{a}$). (See Figure 1.)

4. Iterated elimination of strictly dominated strategies defines two sequences that converge, respectively, to $\underline{a}$ and $\overline{a}$. For example, let $\underline{a}^0 = 0$.

Players will never use a strategy $a < r(0)$ because it is strictly dominated by $\underline{a}^1 = r(0)$. Now, knowing that no one will use a strategy in $[0, r(0))$ the region $[0, r(r(0)))$ will also be strictly dominated. Let $\underline{a}^2 = r(\underline{a}^1)$ and define $\underline{a}^k$ recursively. The sequence $\underline{a}^k$ is increasing and converges to $\underline{a}$ (indeed, it coincides with best reply dynamics starting at $a = 0$). (See Figure 1.) This means that rationalizable strategies will lie in the interval $[\underline{a}, \overline{a}]$ and if the equilibrium is unique the game will be dominance solvable. That is, the final outcome of the process of iterated elimination of strictly dominated strategies is unique, and is an equilibrium.

5. An increase in the parameter θ will lead to an increased action in equilibrium and this increase will be over and above the direct effect of the increase in the parameter. Indeed, increasing θ will move $r(\cdot)$ upwards (as in Figure 1) and the equilibrium level of a will increase. Starting at $a = \overline{a}$ the direct effect will lead us to $r(\overline{a}) > \overline{a}$ and the full equilibrium impact to $\overline{a}' > \overline{a}$.[2] The consequences of a common shock (or for that matter idiosyncratic) are amplified. Because of strategic complementarities there is a multiplier effect. Indeed, the direct effect of an increase in θ in the action of an agent, taken as given the average action, is amplified by the increase in the average action. This happens either focusing at extremal (or stable) equilibria or considering best response dynamics after the perturbation. Even starting at an unstable equilibrium, or at an equilibrium that disappears once θ increases, an increase in θ will result in an increase in a over and above the direct effect. In Figure 1 the unstable equilibrium $\widehat{a}$ disappears with the increase in θ, moving $r(\cdot)$ upwards and best reply dynamics lead to the new equilibrium $\overline{a}'$.

With strategic substitutability among strategies, $\frac{\partial^2 \pi}{\partial a_i \partial \tilde{a}} < 0$, there cannot be multiple symmetric equilibria. In this case it is immediate that there is a unique symmetric equilibrium (because $\frac{\partial^2 \pi_i}{(\partial a_i)^2} + \frac{\partial^2 \pi_i}{\partial a_i \partial \tilde{a}} < 0$ and $\frac{\partial \pi_i}{\partial a_i}(a, a; \theta) = 0$ will have a unique solution). It is easy to see that when $0 > r' > -1$ (or $|r'| < 1$) the game is dominance solvable.[3] This corresponds to the case where the symmetric equilibrium is stable according to the usual cobweb dynamics. Equivalently, in terms of iterated elimination of strictly dominated strategies

[2]Indeed, at a stable equilibrium $r' < 1$ or $\frac{\partial^2 \pi_i}{(\partial a_i)^2} + \frac{\partial^2 \pi_i}{\partial a_i \partial \tilde{a}} < 0$. Then from the first order condition it is immediate that $\frac{da}{d\theta} = -\left(\frac{\partial^2 \pi_i}{\partial a_i}\right) / \left(\frac{\partial^2 \pi_i}{(\partial a_i)^2} + \frac{\partial^2 \pi}{\partial a_i \partial \tilde{a}}\right) > \frac{\partial r}{\partial \theta} = -\left(\frac{\partial^2 \pi_i}{\partial a_i \partial \theta}\right) / \left(\frac{\partial^2 \pi_i}{(\partial a_i)^2}\right) > 0$.

[3]Guesnerie (1992) has shown this in a version of the model.

agents recognize that no one will take an action larger than $r(0)$ and this starts the process of elimination of strategies, this time with alternating regions on both sides of the candidate equilibrium.

2.2 Heterogeneous players

A variation of the search example encompasses heterogeneous agents.[4] Suppose an agent has to decide whether to adopt or not a new technology (or whether to "invest", "act", or "participate"). His action is $a_i = 0$ if there is no adoption, and $a_i = 1$ if there is adoption. The cost of adoption θ is idiosyncratic and follows a distribution function F on the interval $\left[\underline{\theta}, \overline{\theta}\right]$. The benefit of adoption is just the total mass adopting $\widetilde{a}$ (which will be between 0 and 1) and therefore $\pi\left(a_i, \widetilde{a}; \theta_i\right) = a_i\widetilde{a} - a_i\theta_i$. The agent will adopt if $\widetilde{a} - \theta_i \geq 0$. An equilibrium will be given by an adoption threshold $\widehat{\theta}$ and an adopting mass $\widetilde{a}$ such that $\widetilde{a} = \widehat{\theta} = F\left(\widehat{\theta}\right)$ and an agent will adopt if $\theta \leq \widehat{\theta}$. Letting $\underline{\theta} < 0$ and $\overline{\theta} > 1$, for $\theta < 0$ to adopt is a dominant strategy (that is, even if no one adopts it pays to adopt) and for $\theta > 1$ not to adopt is a dominant strategy (that is, even if everyone adopts it does not pay to adopt). The equilibria can be depicted as in Figure 1 where in the horizontal axis we have $\widehat{\theta}$ and $\widetilde{a}$, and in the vertical axis $\widetilde{a}$ and $F(\cdot)$.

It is instructive to think of the case in which θ follows a normal distribution $F \sim N\left(\mu_\theta, \sigma_\theta^2\right)$. We will have multiple equilibria if the distribution function $F(\cdot)$ is S-shaped and this will be so if σ_θ^2 is small enough. This happens when each player is not very uncertain about the costs of other players. In contrast, when σ_θ^2 is large and each players faces a lot of uncertainty about the costs of other players, then $F(\cdot)$ has a mild curvature and the equilibrium is unique. Indeed, when σ_θ^2 tends to infinity $F(\cdot)$ tends to a flat line at 1/2 (see Figure 2 where the case $\mu_\theta = \frac{1}{2}$ is displayed and $\widehat{\theta} = \mu_\theta = \frac{1}{2}$ is the equilibrium threshold). When $\mu_\theta = \frac{1}{2}$ a sufficient condition to have multiple equilibria is that $f(\frac{1}{2}) > 1$, where f is the density of the normal distribution, and this is true if and only if $\frac{1}{2\pi} > \sigma_\theta^2$.[5] Indeed, then $\widehat{\theta} = 1/2$ is one equilibrium and if $\frac{1}{2\pi} > \sigma_\theta^2$ two more equilibria appear.[6]

[4] See Chaterjee and Copper (1989), Pagano (1989), Dybvig and Spatt (1983) for related examples.

[5] Recall that if $x \sim N\left(\mu, \sigma^2\right)$ then $f\left(\mu\right) = \left(\sigma\sqrt{2\pi}\right)^{-1}$ where f is the density of x.

[6] More formally, the equilibrium is determined by $\widehat{\theta} = F\left(\widehat{\theta}\right)$ or $\theta = \Psi\left(\theta\right) \equiv \Phi\left(\frac{\theta - \mu_\theta}{\sigma_\theta}\right)$, where Φ is the cumulative standard normal distribution. Since $\Psi'\left(\theta\right) = \frac{1}{\sigma_\theta}\phi\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)$, where ϕ is the density of the standard normal which attains its (strict) maximum of $\frac{1}{\sqrt{2\pi}}$

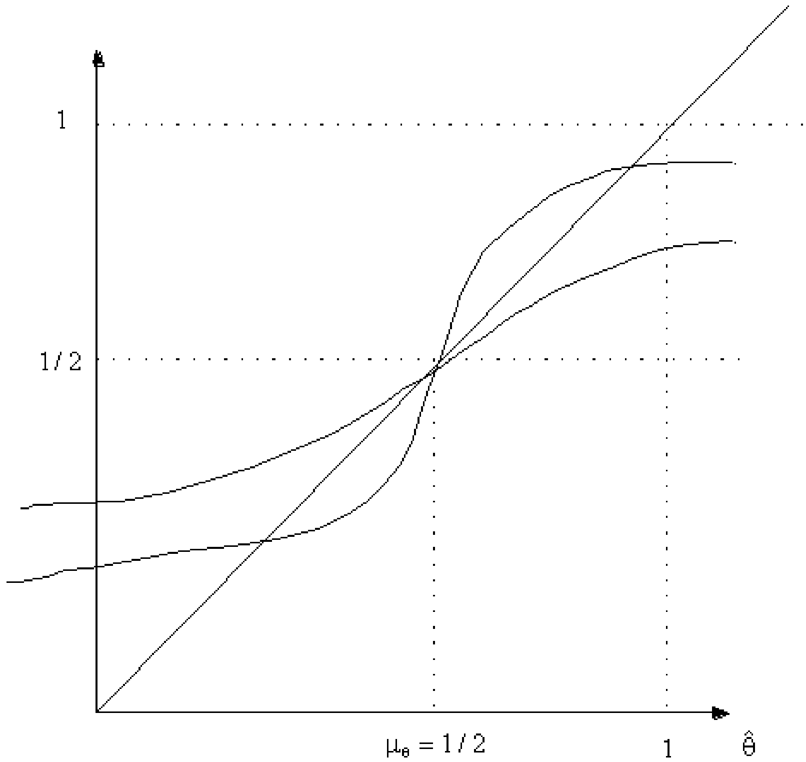


Figure 2

When each player faces a lot of uncertainty about the other players (or the heterogeneity of the population is very large) then strategic complementarities are weak and multiple equilibria cannot be supported. This is a very general principle that we will revisit when studying equilibrium selection in global games.

2.3 How general are the results?

The question that arises is how far those nice results about existence and characterization of equilibria and comparative static properties in our game extend to different specifications (what if payoffs are not concave and best responses non-unique?) or general games with strategic complementarities,

at $x = 0$, we have that $\Psi'(\theta) \leq 1$ if $\frac{1}{2\pi} \leq \sigma_\theta^2$ and $\Psi'(\theta) < 1$ except when $\theta - \mu_\theta = 0$. It follows then that equilibrium will be unique if $\frac{1}{2\pi} \leq \sigma_\theta^2$ (unique intersection of $\Phi\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)$ with the 45^o line). Conversely, if $\mu_\theta = \frac{1}{2}$ a sufficient condition to have multiple equilibria is that $\frac{1}{2\pi} > \sigma_\theta^2$.

or even beyond. As we will see in the next section most of the properties generalize to multidimensional strategy spaces, discrete or continuous, and even functional spaces, and non-smooth and non-concave payoffs. The basic insight of the next section will be that to obtain the desired results only the monotonicity properties of incremental payoffs and the order properties of strategies matter. Most of the regularity conditions typically assumed will not be crucial.

3 An introduction to games with strategic complementarities

In this section I provide first a brief introduction to the tools and main results and then comment on the scope of the theory.

3.1 Modeling complementarities and results

In this paper we will use the intuitive concept of game of strategic complementarities (GSC) whenever the best responses of the players in the game are increasing in the actions of rivals, and the technical concept of supermodular game that provides sufficient conditions for best responses to be increasing.

The analysis of supermodular games is based on lattice-theoretic results that exploit order and monotonicity properties of strategy sets and payoffs. The basis of the approach are monotone comparative statics results developed by Topkis (1978) and the application of Tarski's fixed point theorem for increasing functions (Tarski (1955)). In a nutshell, minimal assumptions are put on strategy sets and payoffs so that best responses are increasing and move monotonically with the parameters of interest. Then Tarski's fixed point theorem delivers existence of equilibria as well as order properties of the equilibrium set, and comparative statics results follow naturally.

This approach provides a powerful analytical tool that confronts the usual obstacles when analyzing a game: existence of pure-strategy equilibria, comparison of equilibria, and comparative statics. In particular, in games of strategic complementarities the presence of multiple equilibria need not be an obstacle to perform comparative statics analysis.

The emphasis of the exposition will be on the intuition and not the technical details. The section will provide the minimal background necessary for a reader to follow the rest of the article and the Appendix contains the technical definitions and intermediate results. Some examples will be developed to illustrate the methodology.[7]

I will provide a definition of a supermodular game in a smooth context. This is done in order to keep to a minimum the mathematical apparatus but it is by no means the most general way to define it. Consider the game $(A_i, \pi_i; i \in N)$ where N is the set of players, $i = 1, ..., n$, A_i is the strategy set, a compact cube in Euclidean space, and π_i the payoff of player $i \in N$ (defined on the cross product of the strategy spaces of the players A). Let $a_i \in A_i$ and $a_{-i} \in \Pi_{j \neq i} A_j$ (that is, we denote by a_{-i} the strategy profile $(a_1, ..., a_n)$ except the ith element). Let a_{ih} denote the hth component of the strategy a_i of player i.

We will say that the game $(A_i, \pi_i; i \in N)$ is s*mooth supermodular* if for all i

- A_i is a compact cube in Euclidean space;
- $\pi_i (a_i, a_{-i})$ is twice continuously differentiable,

    1. supermodular in a_i for fixed a_{-i} or $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$, and
    2. has increasing differences in (a_i, a_{-i}) or $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $j \neq i$ and for all h and k.

The game is smooth strictly supermodular if the inequality in (2) is strict.

Condition (1) is the complementarity property (supermodularity) in own strategies. It means that the marginal payoff to any strategy of player i is increasing in the other strategies of the player. Condition (2) is the strategic complementarity property in rivals' strategies a_{-i}. It means that the marginal payoff to any strategy of player i is increasing in any strategy of any rival player. This property of π_i is termed increasing differences in (a_i, a_{-i}). In the general formulation of a supermodular game strategy spaces need to be only "complete lattices", differentiability of payoffs is not needed, only continuity, and properties (1) and (2) are stated in terms of increments. See the Appendix for the general definitions of lattices, supermodularity, increasing differences, and supermodular game.

In a supermodular game very general strategy spaces can be allowed, including indivisibilities, as well as functional strategy spaces such as those

[7]The reader is referred to Chapter 2 Vives (1999) for a more thorough and general treatment of the theory, as well as proofs, and further references and applications.

arising in dynamic or Bayesian games (as we will see in Sections 5.3 and 6 below). Regularity conditions such as concavity and interior solutions can be dispensed with. The complementarity properties are robust in the sense that they are preserved under addition or integration, pointwise limits, and maximization (with respect to a subset of variables, preserving supermodularity for the remaining variables).[8]

Two leading oligopoly models fit, in many specifications, the complementarity assumptions made. A first example is a Cournot oligopoly with complementary products. In this case the strategy sets are compact intervals of quantities and the complementarity assumptions are natural (what Spence (1976) called "strong complements"). A second example is a Bertrand oligopoly with differentiated substitutable products with each firm producing a different variety. The demand for variety i is given by $D_i(p_i, p_{-i})$ where p_i is the price of firm i and p_{-i} denotes the vector of the prices charged by the other firms. A linear demand system will satisfy the complementarity assumptions.

The application of the theory can be extended by considering increasing transformations of the payoff (which does not change the equilibrium set of the game). We say that the game is log-supermodular if π_i is nonnegative and $\log \pi_i$ fulfils conditions (1) and (2). In the Bertrand oligopoly example, the profit function of firm i, $\pi_i = (p_i - c_i) D_i(p_i, p_{-i})$, where c_i is the constant marginal cost, is log-supermodular in (p_i, p_{-i}) whenever $\frac{\partial^2 \log D_i}{\partial p_i \partial p_j} \geq 0$. This holds whenever for firm i the own-price elasticity of demand η_i is increasing in p_{-i}, as with constant elasticity, logit, or constant expenditure demand systems.[9]

The key results of the theory are obtained by a combination of monotone comparative statics results due to Topkis and Tarski's fixed point theorem. The results by Topkis (1978) deliver monotone increasing best responses, even when π_i is not quasiconcave in a_i. The basic monotone comparative statics result states that the set of optimizers of a function $u(x,t)$ parameterized by t, supermodular in x and with increasing differences in x and t, has a largest and a smallest element and that both are increasing in t.[10] In a supermodular

[8] Supermodularity and increasing differences can even be weakened to define an "ordinal supermodular" game relaxing supermodularity to the weaker concept of quasisupermodularity, and increasing differences to a single crossing property (see Milgrom and Shanon (1994)). However, those properties, as opposed to supermodularity and increasing differences, have no differential characterization and need not be preserved under addition or partial maximization operations.

[9] See Chapter 6 in Vives (1999). However, this is not a universal result as we will see in Section 3.2.

[10] See Lemma 1 in the Appendix for a precise statement of the result.

game this means that each player i has a largest, $\overline{\Psi}_i(a_i) = \sup \Psi_i(a_{-i})$, and a smallest, $\underline{\Psi}_i(a_{-i}) = \inf \Psi_i(a_{-i})$, best reply and that they are increasing in the strategies of the other players. If the game is strictly supermodular then any selection from the best reply correspondence is increasing.

The monotone comparative static result is of wide application. Some intuition can be gained from the one dimensional "classic" case where best responses are continuous functions. Let A_i be a compact interval. Suppose that the ith player best reply to a_{-i} is unique, interior, and equal to $r_i(a_{-i})$. We have then that $\frac{\partial \pi_i}{\partial a_i}(r_i(a_{-i}), a_{-i}) = 0$. Furthermore, if $\frac{\partial^2 \pi_i}{(\partial a_i)^2} < 0$, then r_i is continuously differentiable and $\frac{\partial r_i}{\partial a_j} = -\left(\frac{\partial^2 \pi_i}{\partial a_i \partial a_j}\right) / \frac{\partial^2 \pi_i}{(\partial a_i)^2}$, $j \neq i$. Therefore $sign \frac{\partial r_i}{\partial a_j} = sign \frac{\partial^2 \pi_i}{\partial a_i \partial a_j}$. Now, even when π_i is not quasiconcave the monotone comparative statics result implies that if $\frac{\partial^2 \pi_i}{\partial a_i \partial a_j} > 0$, $j \neq i$, then any selection from the best-reply correspondence of player i (which may have jumps) is increasing in the actions of the rivals. In summary, the positive cross-partial derivative of profits ensures that any best response of the firm is increasing even though it may have jumps. If it has jumps the jumps will be up and not down.

We could define also the (weaker) concept of game of strategic complementarities (GSC), under our maintained assumptions, as a game in which strategy sets are compact cubes (or "complete lattices"), the best reply of any player has extremal (largest and smallest) elements, and those are increasing. Similarly, we could define a game of strict strategic complementarities if furthermore any selection from the best reply of any player is increasing in the strategies of the rivals.[11] All the results stated below will hold then replacing (strictly) supermodular game by GSC (game of strict SC).

The following results hold in a supermodular game. Let $\overline{\Psi} = \left(\overline{\Psi}_1, ..., \overline{\Psi}_n\right)$ and $\underline{\Psi} = (\underline{\Psi}_1, ..., \underline{\Psi}_n)$ denote the extremal best reply maps.

Result 1. Existence and order structure (Topkis (1979)). In a supermodular game there always exist extremal equilibria. That is, there is a largest $\overline{a} = \sup\left\{a \in A : \overline{\Psi}(a) \geq a\right\}$ and a smallest element $\underline{a} = \sup\left\{a \in A : \underline{\Psi}(a) \geq a\right\}$ of the equilibrium set.

The result is shown using Tarski's fixed point theorem (which implies that an increasing function from a compact cube into itself has a fixed point) on the extremal selections of the best-reply map $\overline{\Psi}$ and $\underline{\Psi}$, which are monotone

[11] This definition was used in Vives (1985b) who concentrated attention on monotone increasing best responses as the defining characteristic of games with strategic complementarities. See the Appendix for a more formal definition along those lines.

because of the strategic complementarity assumptions. There is no reliance on quasiconcave payoffs and convex strategy sets to deliver convex-valued best replies as required when showing existence using Kakutani's fixed point theorem.[12]

In the Bertrand oligopoly, for example, when the payoff is supermodular or log-supermodular we have that extremal price equilibria do exist. The results can be extended to convex costs and multiproduct firms and provide thus a large class of Bertrand oligopoly cases where the classical non-existence of equilibrium problem encountered by Roberts and Sonnenschein (1977) does not arise.

Result 2. Symmetric games. In a symmetric supermodular game (exchangeable against permutations of the players):

- Symmetric equilibria exist since the extremal equilibria $\overline{a}$ and $\underline{a}$ are symmetric.[13] Therefore, if there is a unique symmetric equilibrium then the equilibrium is unique (since $\overline{a} = \underline{a}$).

- If the strategy spaces of the players are one dimensional (or completely ordered more in general) then a symmetric strictly supermodular game only has symmetric equilibria.[14] This result proves to be a very useful tool to show uniqueness in symmetric supermodular games. For example, in a symmetric version of the constant elasticity demand system Bertrand oligopoly with constant marginal costs it is easy to see that there is a unique symmetric equilibrium. Since the game is strictly log-supermodular we can conclude that the equilibrium is unique.

- For one-dimensional strategy spaces existence of symmetric equilibria can be obtained relaxing the monotonicity requirement of best responses. It is enough then that all the jumps in the best reply of a player be up. Existence follows from Tarski's intersection point theorem.[15] The result is easy to grasp considering a function $f : [0,1] \rightarrow [0,1]$ which when discontinuous jumps up but not down. The function must cross then the 45^o line at some point. Indeed, suppose that it starts above the 45^o line (otherwise, 0 is a fixed point), then it either stays

[12] The equilibrium set has additional order properties (see Vives (1985), Vives (1990), problem 2.5 in Vives (1999)), and Zhou (1994).

[13] Indeed, if $\overline{a}_1 \neq \overline{a}_2$ then, because the game is symmetric, $(\overline{a}_2, \overline{a}_1, \overline{a}_3, ..., \overline{a}_n)$ will also be an equilibrium and therefore, because $(\overline{a}_1, \overline{a}_2, \overline{a}_3, ..., \overline{a}_n)$ is the largest equilibrium $\overline{a}_1 \geq \overline{a}_2 \geq \overline{a}_1$ and $\overline{a}_1 = \overline{a}_2$.

[14] See footnote 23 in Vives (1999) for a proof of the statement.

[15] See Section 2.3.1 in Vives (1999).

above it (and then 1 is a fixed point) or it crosses the 45^o line. Versions of this fixed point theorem have been derived by McManus (1962, 1964) and Roberts and Sonnenschein (1976) to show existence of equilibria in symmetric Cournot games with convex costs.[16]

– The argument is very simple. Consider a symmetric game with compact intervals as strategy spaces and $\pi_i(a_i, a_{-i}) = \pi\left(a_i, \sum_{j\neq i} a_j\right)$, as in a Cournot game with homogeneous product and identical cost functions (or as in the game with a continuum of players in Section 2). Existence of symmetric equilibria follows then from the stated result if the best-reply Ψ_i of a player (identical for all i due to symmetry) has no jumps down. This is in fact true if costs are convex in the Cournot game. Symmetric equilibria are given by the intersection of the graph of $a_i = \Psi_i\left(\sum_{j\neq i} a_j\right)$ with the line $a_i = \left(\sum_{j\neq i} a_j\right)/(n-1)$. We have thus that in a symmetric game where the strategy space of each player is a compact interval and the payoff to a player depends only on its own strategy and the aggregate strategy of the rivals, if the best reply of a player has no jumps down, then symmetric equilibria exist. This implies in particular that in the game in Section 2.1 under very weak assumptions a symmetric equilibrium will always exist.

Result 3. Welfare (Milgrom and Roberts (1990a), Vives (1990)). In a supermodular game if the payoff to a player is increasing in the strategies of the other players (positive externalities) then the largest (smallest) equilibrium point is the Pareto best (worst) equilibrium. This is a very simple result which is at the base of the finding of equilibria which can be Pareto ranked in many games with strategic complementarities. For example, in the Bertrand oligopoly example the profits associated with the largest price equilibrium are also the highest for every firm.

Result 4. Stability and rationalizability. In a supermodular game with continuous payoffs, best-reply dynamics:

(i) Approach the "box" $[\underline{a}, \overline{a}]$ defined by the smallest and the largest equilibrium points of the game. Therefore, if the equilibrium is unique it is

[16] Milgrom and Roberts (1994) also state and prove the theorem with $S = [0,1]$.

globally stable. (Vives (1990).)

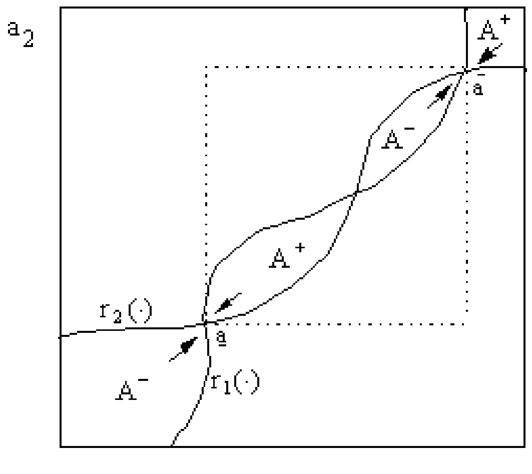
(ii) Converge monotonically downwards to an equilibrium starting at any point in the intersection of the upper contour sets of the largest best replies of the players (A^+ in Figure 3). Similarly, starting at any point in the intersection of the lower contour sets of the smallest best replies of the players (A^- in Figure 3) converge monotonically upwards to an equilibrium. (Vives (1990).)

(iii) The extremal equilibria $\underline{a}$ and $\overline{a}$ correspond to the largest and smallest serially undominated strategies. Therefore, if the equilibrium is unique the game is dominance solvable. (Milgrom and Roberts (1990a).)

This result implies that all relevant strategic action is happening in the box $[\underline{a}, \overline{a}]$ defined by the smallest and largest equilibrium points. For example, rationalizable outcomes (Bernheim (1984), Pearce (1984)) and supports of mixed strategy and correlated equilibria must lie in the box $[\underline{a}, \overline{a}]$. The argument for result (ii) is quite simple because, for example, starting at any point in A^+ (see Figure 3) best reply dynamics define a monotone decreasing sequence that converges to a point that, by continuity of payoffs, must be an equilibrium. In fact, starting at the largest (smallest) point of the strategy space A -recall it is a cube- best reply dynamics with the largest (smallest) best response map $\underline{\Psi}$ $\left(\overline{\Psi}\right)$ will lead to the largest (smallest) equilibrium $\underline{a}$,$(\overline{a})$ (Topkis (1979)). For example, starting at $\inf A$ (see Figure 3) best reply dynamics with the smallest best reply map $\underline{\Psi}$ define a monotone increasing sequence that converges to a point y that, by continuity of payoffs, must be an equilibrium. Furthermore, this must be the smallest equilibrium, $y = \underline{a}$. For any other equilibrium $x, x \geq \inf A$, and iterating the best reply map $\underline{\Psi}$, on both sides of the inequality, we obtain $x \geq \underline{a}$ because $\underline{\Psi}$ is increasing. Results (1) and (2) extend to a large class of adaptive dynamics.

Starting at an arbitrary point we cannot ensure convergence because, for example, a cycle is possible. For example, in Figure 3 starting at $a^0 = (\underline{a}_1, (\overline{a}_2))$ best reply dynamics cycle along the edges of the box $[\underline{a}, \overline{a}]$.

In the Bertrand oligopoly example with linear, constant elasticity, or logit demands the equilibrium is unique and therefore the game is dominance solvable and globally stable.



Cournot tatônnement in a supermodular game

(with best reply functions $r_1(\cdot)$, $r_2(\cdot)$)

Figure 3

Another interesting result is that properly mixed equilibria (i.e., Nash equilibria for which at least two players' strategies are not pure strategies) in strictly supermodular games are unstable with respect to best reply or more general adaptive dynamics (Echenique and Edlin (2001)). An example is the mixed strategy equilibrium in the battle of the sexes.

Result 5. Comparative Statics. Consider a n-player supermodular game with payoff for firm i, $\pi_i(a_i, a_{-i}; t)$, parameterized by a vector $t = (t_1, ..., t_n)$. If π_i has increasing differences in (a_i, t) ($\partial^2 \pi_i / \partial a_{ih} \partial t_j \geq 0$ for all h and j) then with an increase in t:

(i) the largest and smallest equilibrium points increase, and

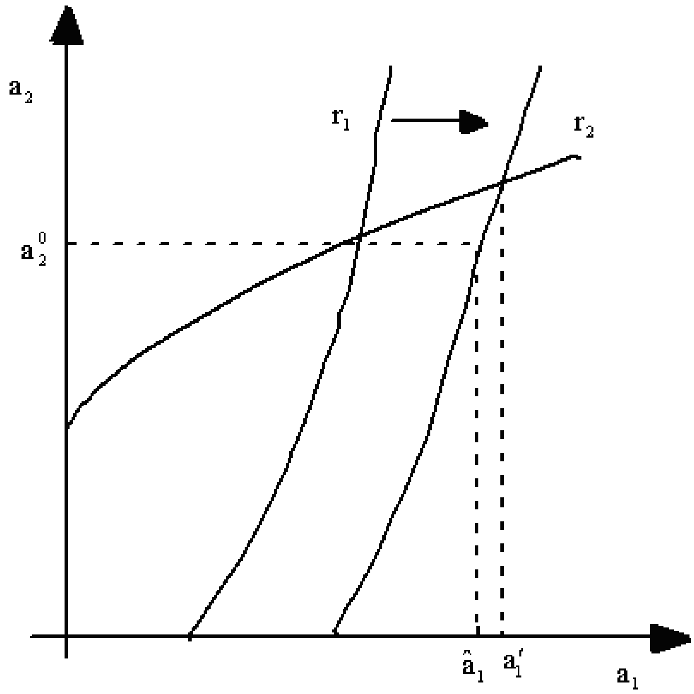
(ii) starting from any equilibrium, best reply dynamics lead to a (weakly) larger equilibrium following the parameter change.

Furthermore, the latter result can be extended to a class of adaptive dynamics including fictitious play and gradient dynamics; and continuous equilibrium selections that do not increase monotonically with t predict unstable equilibria (Echenique (2002)).

An heuristic argument for the result is as follows. The largest best reply of player i is increasing in t and from this it follows that the largest equilibrium point, determined by the largest best replies, also increases with t. Indeed, $\overline{a} = \sup \left\{ a \in A : \overline{\Psi}(a; t) \geq a \right\}$ and $\overline{\Psi}(a; t)$ is increasing in t. Obviously, for an increase in the equilibrium to take place it is only needed, for example, that the payoff to firm i is affected by t_i and not by any other t_j. An increase in t leaves the old equilibrium in A^- (see Figure 3) and therefore sets in motion, via best reply (or more in general via adaptive dynamics) a monotone increasing sequence that converges to a larger equilibrium. Increasing actions by one player reinforce the desire of all other players to increase their actions and the increases are mutually reinforcing (i.e., exhibit positive feedback).

Another way to look at the feedback loop is to think in terms of multiplier effects. As stated in Section 2 a multiplier effect in the parameter t_j obtains if the equilibrium reaction of each player to a change in the parameter is strictly larger than the reaction of the player keeping the strategies of the other players constant. This will happen, for example, in a smooth strictly supermodular game with one dimensional strategy spaces for which $\partial^2 \pi_i / \partial a_i \partial t_j \geq 0$ with strict inequality for at least one firm if we consider extremal equilibria or following best reply adjustment dynamics after a parameter change. In Figure 4, where there is a unique equilibrium, the effect of an increase in t_1 is to move outwards the best reply of player 1. If player 2 were to stay put at a_2^0 then the best response of player 1 would be $\widehat{a}_1$ but in equilibrium $a_1' > \widehat{a}_1$.[17]

[17] See Peitz (2000) for sufficient conditions for a price game to display multiplier effects.



Effect of an increase in t_1

Figure 4

In games with strategic complementarities, unambiguous monotone comparative statics obtain if we concentrate on stable equilibria. This is a multi-dimensional global version of Samuelson's (1979) Correspondence Principle, which links unambiguous comparative statics with stable equilibria and is obtained with standard calculus methods applied to interior and stable one-dimensional models.

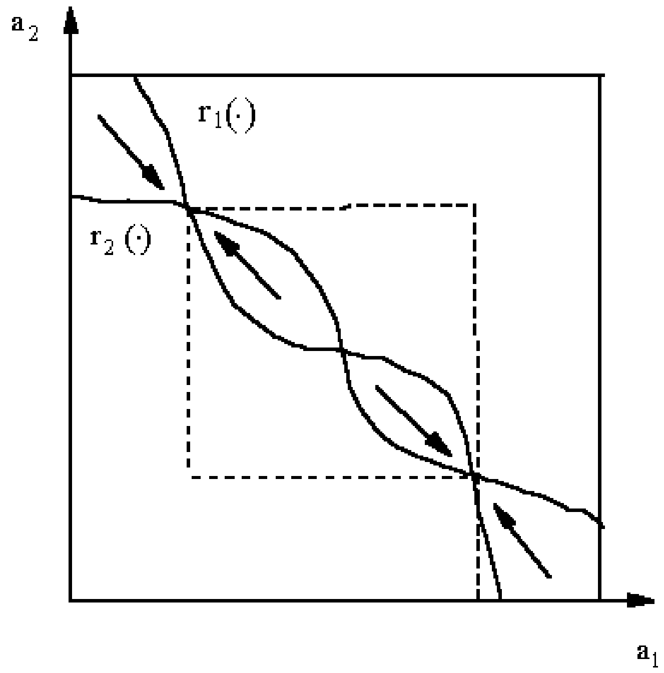
As an example consider the (supermodular or log-supermodular) Bertrand oligopoly. Then extremal equilibrium price vectors are increasing in an excise tax t. Indeed, we have that $\pi_i = (p_i - t - c_i) D_i(p)$ and $\frac{\partial^2 \pi_i}{\partial p_i \partial t} = -\frac{\partial D_i}{\partial p_i} > 0$.

One-dimensional strategy spaces and symmetric games With one-dimensional strategy spaces and symmetric games we have seen how symmetric equilibria exist if best replies have no jumps down. If equilibria are fixed points of a function $f : [0,1] \times \mathbb{R} \to [0,1]$ with no jumps down which is (strictly) increasing in a parameter $t \in \mathbb{R}$, then both the largest $\overline{x}(t) = \sup\{x \in S : f(x;t) \geq x\}$ and the smallest $\underline{x}(t) = \inf\{x \in S : f(x;t) \leq x\}$

equilibrium are (strictly) increasing in t. As t varies, the number of equilibria may change, but still the largest and the smallest equilibrium will be increasing in t. The argument for the result is immediate by looking, say, at Figure 1 where as the parameter increases, the number of equilibria goes from three to one and by realizing that upward jumps in the function can be allowed. This result generalizes the comparative statics property 5 of symmetric equilibria of the game considered in Section 2.1.

Result 6. Duopoly with strategic substitutability (Vives (1990)). If $n = 2$ and there is strategic complementarity in own strategies, with π_i supermodular in a_i or $\partial^2 \pi_i / \partial a_{ih} \partial a_{ik} \geq 0$ for all $k \neq h$, and strategic substitutability in rivals' strategies, with π_i with increasing differences in (a_i, a_j) or $\partial^2 \pi_i / \partial a_{ih} \partial a_{jk} \leq 0$ for all $j \neq i$ and for all h and k, then the transformed game with new strategies $s_1 = a_1$ and $s_2 = -a_2$ is smooth supermodular. (See Figure 5 and note that this is the mirror image of Figure 3 with respect to the ordinate axis.) Therefore, all the stated results above apply to this duopoly game. Unfortunately, the trick does not work for $n > 2$ and the extension to the strategic substitutability case for n players does not hold.

A typical example of duopoly with strategic substitutability is a Cournot market, where usually best replies are decreasing. In this case the welfare result is as follows. If for some players payoffs are increasing in the strategies of rivals, and for some others they are decreasing, then the largest equilibrium is best for the former and worst for the latter. This is the case in the Cournot duopoly with the strategy transformation yielding a supermodular game. The preferred equilibrium for a firm is the one in which its output is largest and the output of the rival lowest.



A duopoly game with decreasing best replies
Figure 5

3.2 The scope of the theory: Is anything a GSC?

If not everything is a game of strategic complementarities, where are the bounds of the theory?

First of all, if we take the view that the order of the strategy spaces are part of the description of the game or that there is a "natural" order in the strategy spaces, then there are many games that are not of strategic complementarities. For example, not all Bertrand games with product differentiation are supermodular games. Roberts and Sonnenschein (1977), Friedman (1983), and Section 6.2 in Vives (1999) provide examples, including games with avoidable fixed costs and the classical Hotelling model when firms are located close to each other. In those cases at some point best replies may jump down and a price equilibrium (in pure strategies) may fail to exist.[18] Indeed, with gross substitutes goods prices may be strategic substitutes because the

[18] See, however, the modified Hotelling game in Thisse and Vives (1992) where best responses may be discontinuous but are increasing.

own-price elasticity of demand need not increase in the prices charged by rivals. A price increase by rival j may lead to a *decrease* in the own-price elasticity of demand for firm i because it makes consumers of that brand who do not have strong preference for any product, i.e. who are more price sensitive, more willing to switch brands. Then it may be that for firm i it pays to cut the price to gain these consumers. Berry, Levinshon and Pakes (1999) find some empirical support for this in certain markets. Another instance of strategic price substitutability among prices may come from the presence of strong network externalities. For example, in the logit model with network externalities (Anderson, de Palma and Thisse (Ch. 7, 1992)) increasing the price set by a rival raises the value for consumers of the network of firm i and it may pay this firm to cut prices to enlarge this lead (although this will not happen for small network externalities). Put it another way, in many games best responses are just non-monotone. For example, they are increasing in some portion of the strategy space and decreasing in another.

However, we could take also the view that the order of the strategy sets of the players is a modeling choice at the convenience of the researcher. This is what we have done to extend the reach of the theory to duopolies with strategic substitutes. Then, if we allow also to construct this order ex post, with knowledge of the equilibria of the game, the answer to the question of the bounds of the theory is that most games are of strategic complementarities. This means that complementarities alone, in the weak sense stated, do not have much predictive power unless coupled with additional structure (Echenique (2004a)). Indeed, define a game with strategic complementarities (GSC) as one in which there is a partial order on strategies (that can be chosen by the modeler) so that best responses are monotone increasing (and with strategy sets having a lattice structure). Then (i) a game with a unique pure strategy equilibrium is a GSC if and only if Cournot best response dynamics (with unique or finite-valued best replies) have no cycles except for the equilibrium; (ii) a game with multiple pure strategy equilibria is always a GSC. As a corollary (iii) 2×2 games, generically, are either GSC or have no pure strategy equilibria (like matching pennies). Result (i) in particular means that a game with a unique and globally stable equilibrium is a GSC, according to the definition given. An example is the strategic substitutes case in the continuum of agents model of Section 2 when $r' > -1$. Note that in this case the game is dominance solvable.

Result (ii) is shown by taking one equilibrium to be the largest and another the smallest strategy profiles in a way that best responses are increasing.[19] Indeed, a game with multiple equilibria always involves a coordination

[19] If a correspondence ϕ from X to X has two fixed points a and b, then define an order

problem (i.e., coordinating on one equilibrium). We can find then an order on strategies that makes the game one of strategic complementarities. However, note that this is done with a priori knowledge of the equilibria and the defined order, indeed, may not be "natural" at all.[20]

The bottom line is that complementarities cannot be applied in the void, they need to be coupled with the structure of the problem at hand to deliver (hopefully powerful) results. Indeed, the fact that with multiple equilibria we can always define an order that makes best responses increasing by taking two equilibria and letting those strategy profiles be the extremal ones in the strategy sets of players, does not mean that the interval prediction (the "box"), once we use a natural order in the game, does not provide a narrowing down of the possible outcomes of strategic interaction.

4 Oligopoly and comparative statics

This section reviews some of the basic applications to oligopoly, surveys very recent ones and provides some new ones. It develops comparative statics results in Cournot markets (including entry), patent races, and multidimensional competition.

The analysis illustrates several points: The potential pitfalls of classical analysis, the extension of the methods to games that need not display complementarities globally, and the isolation of the crucial assumptions driving the results. As an example of the first issue, when studying the effects of increasing the number of firms n into a Cournot market, classical analysis obviates the point that some equilibria may disappear (or appear) when changing n, making any "local" study meaningless (Amir and Lambson (2000)). An analysis of a multimarket oligopoly coming from two-sided competition will exemplify the second issue. Using lattice-theoretic methods conditions for "perverse" comparative statics will be derived in a context where the underlying game is not supermodular (Cabral and Villas-Boas (2003)). Finally, an examination of a patent race will isolate the crucial assumptions behind the comparative statics of R&D effort with respect to the number of participants in the race (Vives (1999)). We deal in turn with a comparison of Cournot and

on $X, \geq$, as follows. Let $y \geq x$ if and only if any of the following is true: $x = y, x = a$ or $y = b$. Then in fact ϕ is weakly increasing, in the sense that if $y \geq x$ then there is z in ϕ and z'in $\phi(y)$ with $z' \geq z$. (In fact $(X, \geq)$ is a complete lattice, see the Appendix.)

[20] In fact, given that the order is endogenous it may be that in a GSC, as defined in this section, there is no equilibrium in pure strategies because strategy sets may not be complete lattices.

Bertrand equilibria, comparative statics in Cournot markets, patent races, and multidimensional competition.

4.1 Comparison of Cournot and Bertrand equilibria

Consider the same market as in the n-firm Bertrand oligopoly example with firm i producing q_i of variety i at cost $C_i(q_i)$. As before the demand for variety i is given by $D_i(p_i, p_{-i})$. In the Bertrand game firms compete in prices and $\pi_i = p_i D_i(p_i, p_{-i}) - C_i(D_i(p_i, p_{-i}))$. In the Cournot game firms compete in quantities and profits for firm i are given by $P_i(q_i, q_{-i}) q_i - C_i(q_i)$, where $P_i(q_i, q_{-i})$ is the inverse demand for firm i. Bertrand equilibria are typically thought to be more competitive than Cournot equilibria. The lattice-theoretical approach makes precise in what sense this is true and what drives the result.

It can be shown that with gross substitute, or complementary, products if the price game is supermodular and quasiconcave (that is, π_i is quasiconcave in p_i for all i) then at any interior Cournot equilibrium prices are higher than the smallest Bertrand equilibrium price vector. A dual result holds also. With gross substitute, or complementary, products, if the quantity game is supermodular and quasiconcave, then at any interior Bertrand equilibrium outputs are higher than the smallest Cournot equilibrium quantity vector (Vives (1985b, 1990)).

To show the result first note that Cournot prices p^c must lie in region A^+(Figure 3), that is, the region in price space defined by the intersection of the upper contour sets of the best replies of the firms in the Bertrand game. This is so because the perceived elasticity of demand for a firm is larger in price than in quantity competition and, consequently at Cournot price levels firms would have an incentive to cut prices if they were to compete in prices. Indeed, with quantity competition no market can be stolen from your competitor given their strategies. Then apply Result 4(ii) to the price game to conclude that starting at any Cournot price vector p^c best reply dynamics will lead the system to a Bertrand equilibrium with lower prices. A corollary is that starting at any interior Cournot equilibrium if firms were to compete in prices they will cut prices until the market stabilizes at a Bertrand equilibrium. The result for quantities is analogous.

4.2 Comparative statics in Cournot markets

Consider a Cournot market in which the profit function of firm i is given by $\pi_i = P(Q)\, q_i - C_i(q_i)$, where $P(\cdot)$ is the inverse demand, Q total output, $C_i(\cdot)$ the cost function of the firm and q_i its output level.

The standard approach (Dixit (1986)) assumes quasiconcavity of payoffs, downward sloping best replies, and that the equilibrium analyzed is unique and stable to derive comparative statics results. Are all those strong assumptions needed? What can we say if payoffs are not quasiconcave and/or there are multiple equilibria?

Let us review first the standard approach. Let P and C_i be smooth with $P' < 0$, $P + q_i P'' \leq 0$ (implying that the game is of strategic substitutes), and $C_i^{''} - P' > 0$ for all i. Those conditions ensure uniqueness and local stability (with respect to continuous best reply dynamics). Parameterize the cost function of firm i by θ_i and let $C_i(q_i; \theta_i)$ be such that $\frac{\partial C_i}{\partial \theta_i} > 0$ and $\frac{\partial^2 C_i}{\partial \theta_i \partial q_i} > 0$. Then it can be shown, using the standard calculus apparatus with the implicit function theorem, that an increase in θ_i decreases q_i and π_i, and increases q_j and π_i, $j \neq i$.

The lattice-theoretical approach (Amir (1996), Vives (1999)) makes the minimal assumptions to obtain the results. Let us consider two cases: a general (potentially asymmetric) oligopoly and a symmetric case.

In the general case consider a Cournot duopoly in which strategies are strategic substitutes and potentially there are multiple equilibria. A sufficient condition for best replies to be decreasing is that the inverse demand be log-concave (and costs strictly increasing in output for both firms).[21] According to Result 6 the strategic substitutes game is transformed into a strategic complements game by changing the sign of the strategy space of one player. We know then that extremal equilibria exist (Result 6) and that an increase in the parameter θ_i decreases q_i and increases q_j and π_i, $j \neq i$. The latter results follows from comparative statics Result 5, where the parameter of interest only affects directly the payoff function of one player. This explains why the increase in θ_i decreases q_i and increases q_j. The result for profits follows immediately from $\frac{\partial C_i}{\partial \theta_i} > 0$ and $\frac{\partial \pi_i}{\partial q_j} < 0$, $j \neq i$. What if we are not at an extremal equilibrium? Similarly as in Result 5(ii), best reply dynamics lead to the comparative static result following the increase in θ_i starting at any equilibrium.

Restrict attention now to a symmetric Cournot oligopoly. In the standard approach (Seade (1980 a, b)) it is assumed that payoffs are quasi-

[21]Decreasing best replies in fact imply the existence of a Cournot equilibrium in a n-firm game (see Theorem 2.7 in Vives (1999)). Decreasing best replies are considered the normal case with Cournot competition but it is easy to generate examples with increasing or nonmonotone best replies (see Section 4.1 in Vives (1999) for a discussion of the topic).

concave and conditions are imposed $((n+1)\,P'(nq) + nP''(nq)\,q < 0$ and $C''(q) - P'(nq) > 0$) so that there is a unique and locally stable symmetric equilibrium q^*. Let $\frac{\partial^2 C}{\partial\theta\partial q_i} \leq 0$. Then standard calculus techniques show that an increase in θ increases q^*, and that total output increases and profits per firm decrease as n increases. The comparative statics of output per firm with respect to the number of firms are ambiguous. The classical approach has several problems. First of all, it is silent about the potential existence of asymmetric equilibria. Second, it is restrictive and may be misleading. For example, if the uniqueness condition for symmetric equilibria does not hold and there are multiple symmetric equilibria, changing n either may make disappear the equilibrium considered or introduce more (as in Figure 1).

In the lattice-theoretic approach (Amir and Lambson (2000), Vives (1999)) it is assumed only that $P' < 0$ and $C'' - P' > 0$. As will be shown below, a symmetric equilibrium (and no asymmetric equilibrium) exists. Under the assumption that $\frac{\partial^2 C}{\partial\theta\partial q_i} \leq 0$, then at extremal (symmetric) Cournot equilibria: Individual outputs are increasing in θ, total output is increasing in n and profits per firm decrease with n. Furthermore, individual outputs decrease (increase) with n if demand is log-concave (log-convex and costs are zero). This approach does away with all the unnecessary assumptions of the standard approach and derives new results.

To illustrate the approach let us sketch the proof that, under the assumptions $P' < 0$ and $C'' - P' > 0$, a symmetric equilibrium (and no asymmetric equilibrium) Cournot exists; that individual outputs are increasing in θ and that total output is increasing in n. Let Ψ_i be the best reply map of firm i (identical for all i because of symmetry). Define the correspondence φ by assigning $(q_i + Q_{-i})\,(n-1)/n$, where $q_i \in \Psi_i\left(Q_{-i)}\right)$, to Q_{-i}. Symmetric equilibria are given by fixed points of this correspondence. Under the assumptions it can be checked that Ψ_i has slopes larger than -1.[22] This implies that all selections from $\Psi_i\left(Q_{-i)}\right) + Q_{-i}$ are (strictly) increasing and that no asymmetric equilibria can exist.[23] Furthermore, all selections from the correspondence φ will be increasing. We can use then Tarski's fixed point theorem to show existence of extremal equilibria. Those extremal equilibria can be found using the extremal selections of φ (which are well-defined in our context). Similarly as in Result 5(i), individual outputs at those extremal equilibria will be increasing in θ because, from the assumption $\frac{\partial^2 C}{\partial\theta\partial q_i} \leq 0$, extremal selections of φ (and Ψ_i) are increasing in θ. Let us see now that

[22] That is, the segment joining any two points on the graph of the correspondence Ψ_i has a slope larger than -1.

[23] This is so because for any total output there is a unique output for every firm, identical for all firms because of symmetry, consistent with optimization behavior (see remark 17 in Section 2.3 in Vives (1999)).

total output is increasing in n at extremal equilibria. First of all, it is easy to see that extremal selections of φ are increasing in n. This means that the total output of $(n-1)$ firms is increasing in n at any extremal equilibrium. It follows then that total output at extremal equilibria must be increasing in n because all selections from $\Psi_i\left(Q_{-i)}\right)+Q_{-i}$ are (strictly) increasing in Q_{-i}. The results for profits and individual outputs in relation to n follow along similar lines.[24]

4.3 Patent races

Suppose that n firms are engaged in a memoryless patent race and have access to the same R&D technology (Lee and Wilde (1980)). An innovating firm obtains the prize V and losers obtain nothing. If a firm spends x continuously the (instantaneous) probability of innovating is given by $h(x)$ where h is a smooth concave function with $h(0)=0$, $h'>0$, $\lim_{x\to\infty} h'(x)=0, h'(0)=\infty$ (a region of increasing returns for small x may be allowed). Without innovating the normalized profit of firms is zero. Under these conditions the expected discounted profits (at rate r) of firm i investing x_i if rival j invests x_j is given by $\pi_i=\left(h(x_i)V-x_i\right)/\left(h(x_i)+\Sigma_{j\neq i}h(x_j)+r\right)$ (see Lee and Wilde (1980) and Reinganum (1989)). Denote the best response of a firm by $x_i=R\left(\Sigma_{j\neq i}h(x_j)+r\right)$, this is well defined under the assumptions. Lee and Wilde (1980) restrict attention to symmetric Nash equilibria of the game and show that under a stability condition at a symmetric equilibrium, x^*, $R'\left((n-1)h(x^*)\right)(n-1)h'(x^*)<1$, x^* increases with n.

This approach, however, suffers from the same problems as the comparative statics of entry in Cournot markets. It requires assumptions to ensure a unique and stable symmetric equilibrium and cannot rule out the existence of asymmetric equilibria. However the following mild assumptions ensure that the game is strictly log-supermodular: $h(0)=0$ and h strictly increasing in $[0,\overline{x}]$ with $h(x)V-x<0$ for $x\geq\overline{x}>0$. It follows then from Result 2 that equilibria exist and all are symmetric. Let $x_i=x$ and $x_j=y$ for $j\neq i$, then $\log\pi_i$ has (strictly) increasing differences in (x,n) for all $y\,(y>0)$ and at extremal equilibria the expenditure intensity x^* is increasing in n. Furthermore, if h is smooth with $h'>0$ and $h'(0)=\infty$ then $\partial\log\pi_i/\partial x_i$ is strictly increasing in n and at extremal equilibria x^* is strictly increasing in n. This follows because under the assumptions equilibria are interior and have to fulfill the first order conditions.

[24] See pp. 42-43, 93-96 and Section 4.3.1 in Vives (1999) for details.

Similarly as before, starting at any equilibrium an increase in n will raise the research intensity with out-of-equilibrium adjustment according to best reply dynamics. This will be so even if by increasing n some equilibria disappear or some new appear.

4.4 Multidimensional competition

Multidimensional competition provides another fertile ground for application of the approach because it can handle readily multidimensional strategy spaces. We will consider first an example with advertising and price as strategies and then multimarket oligopoly situations.

4.4.1 Advertising and prices

Consider our Bertrand oligopoly example where the demand of firm i, $D_i(p; t_i)$, depends on advertising effort t_i with $\frac{\partial D_i}{\partial t_i} > 0$. Suppose that goods are gross substitutes $\frac{\partial D_i}{\partial p_j} \geq 0$, $j \neq i$ and demand downward sloping $\frac{\partial D_i}{\partial p_i} < 0$. Let $\pi_i = (p_i - c_i) D_i(p; t_i) - F_i(t_i)$ where F_i is the cost of advertising with $F_i' > 0$. The action of firm is $a_i = (p_i, t_i)$, with natural upper bounds for p_i and t_i. Profits π_i are strictly supermodular in $a_i = (p_i, t)$ if $\frac{\partial^2 \pi_i}{\partial p_i \partial t_i} = (p_i - c_i) \frac{\partial^2 D_i}{\partial p_i \partial t_i} + \frac{\partial D_i}{\partial t_i} > 0$. A sufficient condition for the condition to hold is that $\frac{\partial^2 D_i}{\partial p_i \partial t_i} \geq 0$ or that advertising increases the willingness to pay. Furthermore, π_i has increasing differences in $((p_i, t_i), (p_{-i}, t_{-i}))$ if $\frac{\partial^2 D_i}{\partial p_i \partial p_j} \geq 0$, $j \neq i$ (given that $\frac{\partial D_i}{\partial p_i \partial t_j} = 0$, $j \neq i$). Under these assumptions the game is supermodular and the largest (smallest) equilibrium have the feature of having high (low) prices and advertising levels. Multiple equilibria obtain with a symmetric linear demand system where t_i increases the demand intercept if F' is concave enough. We have found thus conditions under which high prices are associated to high advertising levels.

4.4.2 Multimarket oligopoly

Consider now a multimarket version of the Bertrand oligopoly where firm $i = 1, ..., n$ produces H_i varieties, $h = 1, ..., H_i$ and has profits $\pi_i = \sum_{h=1}^{H} (p_{ih} - c_{ih}) D_{ih}(p_i, p_{-i}; \theta_{ih})$.

If π_i is (log-)supermodular in prices and has increasing differences (log-supermodular) in $(p_i, (c_i, \theta_h))$, then extremal equilibrium prices are increasing in demand and cost parameters of any firm. For example, if demand D_{ih} is linear with intercept θ_{ih}, goods are gross substitutes, and θ_i is advertising effort by firm i, then advertising raises (extremal) equilibrium prices.

The approach can be extended to pricing games not supermodular (neither log-supermodular). A nice example of a multimarket Bertrand oligopoly is provided by the multiproduct logit model of Spady (1984). This is a case where best responses are increasing and there is a unique Bertrand equilibrium despite the fact that payoffs are nor quasiconcave, but single peaked, neither supermodular (or log-supermodular) in own actions (prices), but strategic complementarity across prices of different firms holds.

A multimarket mixed oligopoly with products demand complements within the firm and substitutes across firms provides another example. This situation is typical of two-sided markets where two groups of market participants benefit from interaction via a platform or intermediary. Intermediaries compete for business from both groups and set prices. Examples are numerous and include readers/viewers and advertisers in media markets, cardholders/consumers and merchants/retailers in payment systems such as credit cards, consumers and shops in shopping malls, authors and readers in academic journals, borrowers and depositors in banking, "subscription to a network" and "number of calls made to a network" in telecom markets, and in general buyers and sellers put together with the help of intermediaries (in real state, financial products or auction markets). The interaction between the two sides gives rise to complementarities or externalities between groups that are not internalized by end users. For example, when a consumer uses a credit card does not internalize the benefit that it confers to the other side of the market (the merchants).

Consider a situation of two-sided exclusive intermediation with two groups of participants (say columnists and readers, dating bars, workers and firms in a single region, consumers and shops in a mall) where each participant joins one of the two existing intermediaries only and where the utility derived by a member of a group by joining a particular intermediary is increasing in the number of members of the other group joining the same intermediary.[25] With linear demands arising from Hotelling-type preferences for the intermediaries (Armstrong (2002)) the result is that products are strategic complements across firms but strategic substitutes within the firm. The multimarket oligopoly game is therefore not a supermodular game as defined in Section 3. However, best replies will be increasing as long as the demand complementarity among the products of the same firm/intermediary is not very strong. In the context of the linear demand game with small and symmetric network effects, best replies are increasing and there is a unique symmetric

[25] See Armstrong (2002) for a survey of two-sided competition.

equilibrium.

An interesting result in the Hotelling game, where total demand is inelastic, is that an increase in the cross-group network effect, or, in other words, an increase in the degree of demand complementarity of the "products" of the intermediary, reduces equilibrium profits. An increase in the impact of the benefits that one side of the market confers on the other when they go to an intermediary has in fact a detrimental equilibrium effect on profits. The reason is that the externality increase has no positive direct impact in demand at a symmetric equilibrium in which the whole market is covered and it incentivates each intermediary to cut prices. Since total demand stays constant because it is price-inelastic equilibrium profits decrease. This result can be generalized whenever the direct effect of the externality on demand at symmetric equilibria is small, so that profits for any intermediary have decreasing differences in the price charged to a group (and in consequence the externality parameter and best replies shift inwards as the externality parameter increases), and total demand is fixed (or it is quite price-inelastic), so that the equilibrium price decrease translates into a profit decrease. In those circumstances the strategic pricing effect dominates the direct effect (Cabral and Villas-Boas (2003)). Economies of scope have a similar effect than demand externalities.

Peitz (2002) uses supermodular methods to study the effects of asymmetric access price regulation in telecom markets. He finds that networks are strategic complements around any cost-based (regulated access) equilibrium and that raising the access price for the incumbent shifts in the best response of both the incumbent and the entrant. The result is that subscription fees are decreased for both operators (whenever networks compete with two-part tariffs with termination-based price discrimination).

5 Dynamic games

This section will take a look at dynamic issues building on comparative statics results for supermodular games (like Result 5) that predict movements of equilibrium variables when a parameter changes.

I examine three issues. I review first the taxonomy of strategic behavior due to Fudenberg and Tirole (1984) and provide the minimal assumptions so that the classification of strategies (in terms of fat cats and puppy dogs) holds (Vives (1999)). It will be shown that all that matters is the character of competition (strategic substitutes or complements) and whether investment makes the incumbent tough or soft, and nothing else (in particular the strong regularity conditions implying uniqueness of equilibria usually imposed). The second application is to examine the conditions under which increasing or decreasing dominance occurs in oligopoly (Athey and Schmutzler (2001)). That is, whether leaders or laggards have more incentives to invest. This is particularly relevant in situations where investment today, which could be a larger firm size if there are learning effects and/or adjustment costs, affects competitive conditions tomorrow. These applications will illustrate the power of the approach to isolate the drivers of results and extend them beyond GSC. Finally, full-blown dynamic games are considered restricting attention to Markov games and Markov perfect equilibria (MPE). First I tackle how static complementarities translate into dynamic complementarities and use the methodology to characterize MPE. Conditions are given so that contemporaneous (intra-period) strategic complementarity (SC) and intertemporal (inter-period) SC obtain. The relationships between static and dynamic strategic substitutability and complementarity will be studied in alternating move games (Maskin and Tirole (1987, 1988)) and in games with adjustment costs (Jun and Vives (2003)). Finally, the problem of existence and characterization of Markov perfect equilibria will be addressed (Curtat (1996), Sleet (2001)).

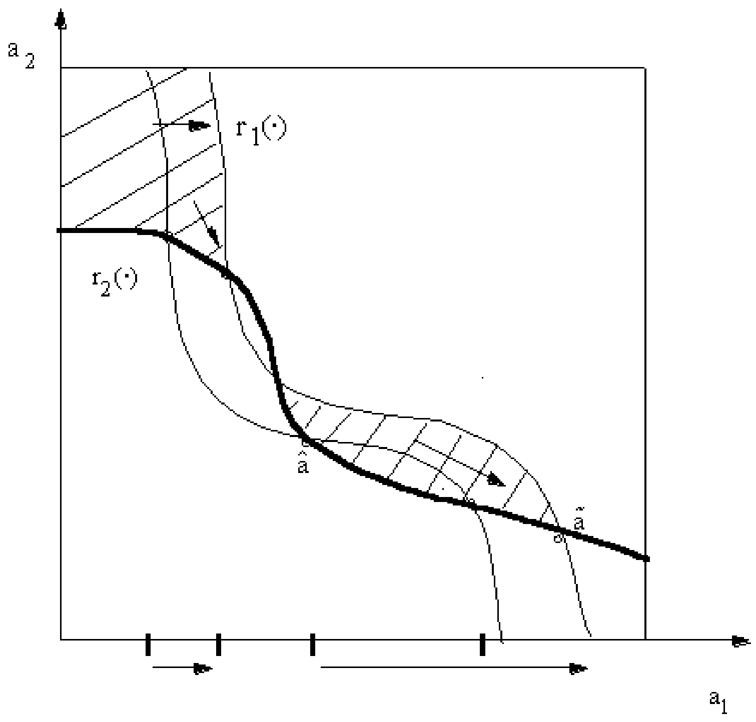
The outcome of the analysis are new results uncovered (characterization of dynamic strategic complementarity and linkage between static and dynamic complementarity concepts, existence of MPE) and isolation of crucial assumptions in know results (increasing dominance, monotonicity of dynamic reaction functions in alternating move games).

5.1 Taxonomy of strategic behavior

Fudenberg and Tirole (1984) provided a taxonomy of strategic behavior in the context of a simple two-stage game between an incumbent and an entrant. At the first stage the incumbent (firm 1) can make an observable investment t yielding at the market stage $\pi_1(a_1, a_2; t)$ where a_i is the market action of firm i. The payoff of the entrant is $\pi_2(a_1, a_2)$. The incumbent can influence the market outcome at the second stage by taking into account the effect of his investment on the equilibrium behavior of the rival at the market stage. The goal is to sign this strategic effect taking as benchmark "innocent" behavior where the incumbent when deciding about t only takes into account the direct effect of the investment on his payoff. Innocent behavior obtains in the open-loop equilibrium of the two-stage game, which is equivalent to the game with simultaneous choice by the incumbent of t and a_1.

The standard approach assumes that at the second stage there are well-defined best-response functions for both firms, and that there is a unique and (locally) stable Nash equilibrium that depends smoothly on t, $a^*(t)$. To obtain this it is assumed that $-\frac{\partial^2 \pi_i}{(\partial a_i)^2} > \left|\frac{\partial^2 \pi_i}{\partial a_i \partial a_j}\right|$, $i \neq j$, $i = 1, 2$. At a subgame-perfect equilibrium, we will have that $\frac{\partial \pi_i}{\partial t} + \frac{\partial \pi_1}{\partial a_2}\frac{\partial a_2^*}{\partial t} = 0$ where $S \equiv \frac{\partial \pi_1}{\partial a_2}\frac{\partial a_2^*}{\partial t}$ is the strategic effect. That is, the effect of the investment t on the equilibrium profits of the incumbent because of the modified market behavior of the entrant. Under the stated assumptions it follows, using standard calculus techniques, that $sign\frac{\partial a_2^*}{\partial t} = sign\left(\frac{\partial^2 \pi_2}{\partial a_1 \partial a_2}\frac{\partial^2 \pi_1}{\partial t \partial a_1}\right)$ and therefore $sign\ S = \frac{\partial \pi_1}{\partial a_2}\frac{\partial^2 \pi_1}{\partial t \partial a_1}\frac{\partial^2 \pi_2}{\partial a_1 \partial a_2}$. If $\frac{\partial \pi_1}{\partial a_2}\frac{\partial^2 \pi_1}{\partial t \partial a_1} < 0\,(< 0)$, we say that the investment makes firm 1 tough (soft). Indeed, suppose that $\frac{\partial \pi_i}{\partial a_j} < 0$, $j \neq i$ so that an increase in the market action of firm j hurts firm i. Then if $\frac{\partial^2 \pi_1}{\partial t \partial a_1} > 0$ an increase in t will shift the best response function of firm 1 out and this will be an aggressive move, making firm 1 tough.

A taxonomy of strategic behavior (see Table 1) can be provided then depending on whether competition is of the strategic substitutes $\left(\frac{\partial^2 \pi_2}{\partial a_1 \partial a_2} < 0\right)$ or complements $\left(\frac{\partial^2 \pi_2}{\partial a_1 \partial a_2} > 0\right)$ variety and on whether investment makes firm 1 soft $\left(\frac{\partial \pi_1}{\partial a_2}\frac{\partial^2 \pi_1}{\partial t \partial a_1} > 0\right)$ or tough $\left(\frac{\partial \pi_1}{\partial a_2} < 0\right)$. If competition is of the strategic substitutes type and investment makes firm 1 tough then the incumbent wants to overinvest $(S > 0)$ to push the entrant down his best response curve (see Figure 6). This is the top dog strategy. Cournot competition and investment in cost reduction are an example. If competition is of the strategic complements type and investment makes firm 1 tough then the incumbent wants to underinvest $(S > 0)$ to move the entrant up his best response curve. This is the puppy dog strategy. Price competition with differentiated products and investment in cost reduction are an example. Similarly, we can define the strategies "lean and hungry" and "fat cat".



Effect of t ↑ on unstable equilibrium: â→ã

Figure 6

Table 1

Taxonomy of strategic behavior

Investment makes player 1:

Strategic\	Tough	Soft
Substitutes	Overinvest (top dog)	Underinvest (lean and hungry)
Complements	Underinvest (puppy dog)	Overinvest (fat cat)

In the lattice theoretic version of the result (section 7.4.3, Vives (1999)) the taxonomy follows from minimal assumptions, the character of competition and investment, as applied to extremal equilibria. There is no need to impose the strong restrictions above to obtain a unique and stable equilibrium at the market stage. Indeed, comparative statics Result 5 holds, obviously, in a duopoly when only the parameter t_i changes and it affects only the payoff of one firm. Let us formulate the result in the duopoly case with actions of firms in a compact interval. Consider a supermodular duopoly game in which the payoff to player 1, parameterized by t, is $\pi_1(a_1, a_2; t)$, and to player 2 is $\pi_2(a_1, a_2)$. If $\partial^2\pi_1/\partial a_1\partial t \geq 0$ then extremal equilibria are increasing in t. Note then that if the game is of strategic substitutes then extremal duopoly equilibrium strategies for firm 1(2) are increasing (decreasing) in t if $\partial^2\pi_1/\partial a_1\partial t \geq 0$. The results are reversed if $\partial^2\pi_1/\partial a_1\partial t \leq 0$.

In our case this means that if the market game is supermodular $\left(\frac{\partial^2\pi_2}{\partial a_1\partial a_2} \geq 0\right)$ and $\frac{\partial^2\pi_1}{\partial t\partial a_1} \geq (\leq)\, 0$ then extremal equilibria are increasing (decreasing) in t. If the market game is of the strategic substitutes variety $\left(\frac{\partial^2\pi_2}{\partial a_1\partial a_2} \leq 0\right)$ then changing signs in the strategy space of one player the game becomes a supermodular game and extremal equilibrium strategies for player 1(2) are increasing (decreasing) in t if $\frac{\partial^2\pi_1}{\partial t\partial a_1} \geq 0$ and the result is reversed if $\frac{\partial^2\pi_1}{\partial t\partial a_1} \leq 0$. Therefore, $sign\frac{\partial a_2^*}{\partial t} = sign\left(\frac{\partial^2\pi_2}{\partial a_1\partial a_2}\frac{\partial^2\pi_1}{\partial t\partial a_1}\right)$ when a_2^* is an extremal equilibrium and the taxonomy follows for extremal equilibria.

What if at the market stage firms are sitting on a non-extremal equilibrium (for instance at the unstable equilibrium $\widehat{a}$ in Figure 6)? Then if out of equilibrium adjustment is governed by best reply dynamics the sign of the impact of a change in t is teh same as with an extremal equilibrium. In Figure 6, depicting teh case of strategic substitutes competition and investment that makes the incumbent tough, an increase in t will generate an adjustment process that will lead to the new equilibrium $\widetilde{a}$ with $\widehat{a}_2 > \widetilde{a}_2$.

In summary, the taxonomy of strategic behavior can be obtained with just the crucial assumptions on monotonicity of marginal payoffs without any need of quasiconcavity of payoffs and the requirement of a unique and stable market equilibrium.

5.2 Increasing or decreasing dominance?

The generalization of the taxonomy was based on a monotone comparative static result for a duopoly. The following comparative statics result deals with n-player competition in a strategic substitutes game and can be used to study in what situations leaders or laggards in an industry have more incentive to invest, in cost reduction or quality enhancement, and whether this leads to increasing or decreasing dominance (Athey and Schmutzler (2001)).

Suppose that the payoff to player i is given by $\pi_i(a_i, a_{-i}; t)$ with $t = (t_1, ..., t_n)$. The parameter t_i is to be interpreted as the state variable or initial conditions of player i in the game. Let both actions and state variables be one dimensional. We would like to find conditions under which if two firms only differ in their state variables and $t_i > t_j$ then at any equilibrium $a_i(t) \geq a_j(t)$, the interpretation being that an initial dominance is reinforced by the actions of the firms. For example, in the presence of a learning curve, the firm that has accumulated more output has incentive to produce more.

Suppose that $\pi_i(a_i, a_{-i}; t)$ has increasing differences in (a_i, a_j), $i \neq j$ (strategic complementarity) and increasing differences in $(a_i, (t_i, -t_{-i}))$. Assume also that all the players have the same strategy set and that the payoffs are exchangeable (any player does not care about the identity of the opponents, only about their action and payoff relevant parameters or state variables). This means that the payoffs of two players are the same if actions and state variables are exchanged among them. Suppose also that payoffs are strictly quasiconcave so that there is a unique best response function for any player, and that we have an equilibrium for which, without loss of generality, $a_1 < a_2$ with $t_1 > t_2$. Fix the actions of the players $n = 3,...,n$ at their equilibrium levels. Because of strict quasiconcavity and exchangeability we can write the best response of firm 1 as $r(a_2; t_1, t_2)$ and that of firm 2 as $r(a_1; t_2, t_1)$. Because of strategic complementarity and $a_1 < a_2$ we have that $a_1 = r(a_2; t_1, t_2) \geq r(a_1; t_2, t_1)$. Because $t_1 > t_2$ and the fact that $\pi_i(a_i, a_{-i}; t)$ has increasing differences in $(a_i, (t_i, -t_{-i}))$ we have that $r(a_1; t_1, t_2) \geq r(a_1; t_2, t_1) = a_2$, contradicting the supposition that $a_1 < a_2$. We conclude that $a_1 \geq a_2$, if $t_1 > t_2$.[26]

To help the intuition just think of the case $n = 2$ in Figure 7 starting from a symmetric equilibrium at $t_1 = t_2$ and increasing t_1. We see how the best reply of firm 1 shifts outwards while the best reply of player 2 shifts inwards and the equilibrium moves to a region with $a_1 \geq a_2$.

[26] Without requiring quasiconcavity we could make the same argument with the extremal best replies. The result would be true then for extremal equilibria.

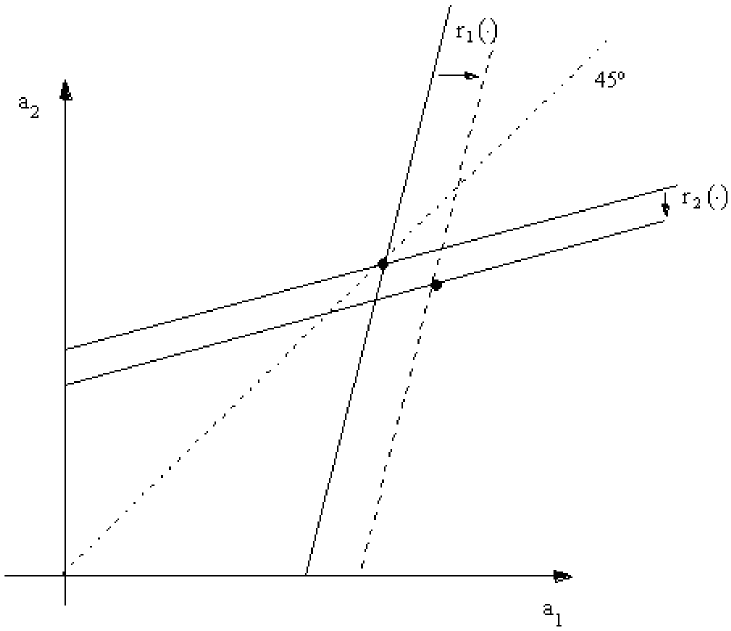


Figure 7

An example is provided by the Bertrand oligopoly model with product differentiation with learning by doing or, alternatively, with production adjustment costs, or still with switching costs. With learning by doing the profit function of firm i is $\pi_i = (p_i - (c - f(t_i)) D_i(p)$ where t_i is the accumulated output of the firm. Letting $a_i = -p_i$ we have that $\frac{\partial^2 \pi_i}{\partial a_i \partial t_i} > 0$ and $\frac{\partial^2 \pi_i}{\partial a_i \partial t_j} < 0$. With production adjustment costs we have that
$\pi_i = (p_i - c) D_i(p) - F(D_i(p) - D_i(t))$, where t_i is the price of the firm in the previous period and F is the increasing and convex production adjustment cost with $F(0) = 0$. Then $\frac{\partial^2 \pi_i}{\partial p_i \partial t_i} > 0$ and $\frac{\partial^2 \pi_i}{\partial p_i \partial t_j} < 0$. In both cases the firm starting with a higher output level (lower price) has an incentive to set lower prices in equilibrium. However, this does not mean that there is increasing dominance. Even tough in any period the larger firm sets a lower price it may well be that the price difference between the firms disappears overtime. In fact, this is exactly what happens at the MPE of an infinite horizon version of the model (see section 4.3 and Jun and Vives (2003)).

In the switching costs model (Beggs and Klemperer (1992)) firms compete in prices and t_i is the loyal customer base of firm i. In this case we have that $\frac{\partial^2 \pi_i}{\partial p_i \partial t_i} > 0$, because of a fat cat effect, lowering prices is more costly to a firm with a larger customer base, and $\frac{\partial^2 \pi_i}{\partial p_i \partial t_j} < 0$. It follows then that a firm with a larger customer base will be softer in pricing. This is to be interpreted as decreasing dominance (and indeed the authors show that at an MPE of the full-blown dynamic game initial asymmetries in market shares are eroded). The reader is warned however that in a dynamic game firms are forward looking and the continuation payoffs need not look like the static payoffs. Therefore the static dominance need not translate in dominance in the dynamic game. We will see in the next section the relationships between static and dynamic properties of payoffs.

The result can be extended to the strategic substitutes case ($\pi_i(a_i, a_{-i}; t)$ has decreasing differences in (a_i, a_j), $i \neq j$) with the restriction that $-\frac{\partial^2 \pi_i}{(\partial a_i)^2} > \left|\frac{\partial^2 \pi_i}{\partial a_i \partial a_j}\right|$, $i \neq j$ (this implies that the profit function of any player is concave and that a duopoly game would have a unique equilibrium).[27]

The conditions in the result are met typically when actions are investments in cost reduction and also in some models of quality enhancement.[28] Then profits at the market stage as a function of those investments display strategic substitutability both in Cournot and Bertrand models. The result can also be used to show that learning by doing in a Cournot market leads to increasing dominance. That is, the firm that is ahead of the learning curve remains ahead because it has incentives to produce more. Actions are current rates of output and state variables the inherited accumulated production of each firm. Let the profit function of firm i be given by $\pi_i = (P(Q) - (c - f(t_i)) q_i$ where $P(\cdot)$ is the inverse demand, Q total output, $f(\cdot)$ the learning curve, a differentiable and concave function of to-

[27] The argument is as follows. For $n = 2$ the result follows from the basic comparative statics result for equilibria in a supermodular game (Result 5) because the duopoly game with strategic substitutes can be seen as a game of strategic complementarities with increasing differences in the parameters considering the transformation $\widehat{a}_1 = a_1$, $\widehat{a}_2 = -a_2$, $\widehat{t}_1 = t_1$ and $\widehat{t}_2 = -t_2$. Result 5 implies directly that $a_1(t_1, t_2) \geq a_1(t_2, t_1)$ whenever $t_1 > t_2$. However, because of exchangeability, we have that and $a_1(t_2, t_1) = a_1(t_1, t_2)$, therefore $a_1(t_1, t_2) \geq a_2(t_1, t_2)$, whenever $t_1 > t_2$. Indeed, starting at $t_1 = t_2$ (with $a_1 = a_2$ because of exchangeability) and increasing t_1 will increase a_1 and $-a_2$. For $n > 2$, consider, without loss of generality, players 1 and 2. Let $t_1 > t_2$ and consider an equilibrium. Fix the actions of players $i = 3, ..., n$ at their equilibrium levels. Then, because of the assumption $-\frac{\partial^2 \pi_i}{(\partial a_i)^2} > \left|\frac{\partial^2 \pi_i}{\partial a_i \partial a_j}\right|$, $i \neq j$ there is a unique equilibrium of the duopoly game between players 1 and 2. We can reverse the roles of players 1 and 2 and the equilibrium actions of the rest of the players are not affected. Apply then the same reasoning as in the $n = 2$ case to conclude that $a_1(t) \geq a_2(t)$.

[28] This is so in the Shaked and Sutton (1982) model of vertical quality differentiation when the market is covered. However, in the classical linear Bertrand duopoly with product differentiation investments in quality that raise the intercept of demand for the own product (Vives (1985a)) or that increase the willingness to pay by lowering the absolute value of the slope of demand $\left|\frac{\partial D_i}{\partial p_i}\right|$(Vives (1990b)) are strategic complements.

tal accumulated output of the firm t_i with $f' > 0$, and q_i its current output level. If inverse demand is log-concave then best replies are downward sloping (strategic substitutes). Furthermore, we have that $\frac{\partial^2 \pi_i}{\partial q_i \partial t_i} = f' > 0$ and $\frac{\partial^2 \pi_i}{\partial q_i \partial t_j} = 0$. It follows then that $t_i > t_j$ implies that at the (unique) Cournot equilibrium $q_i(t) \geq q_j(t)$.[29]

5.3 Markov games

An important issue is how static complementarities translate or not into dynamic complementarities. In this section we will explore the issue in the context of discrete time Markov games.[30] A Markov strategy depends only on (state) variables, denoted y, that condense the direct effect of the past on the current payoff. Let the current payoff of player i be $\pi_i(x, y)$, where x is the current action profile vector, and y is the state evolving according to $y = f(x^-, y^-)$ where x^-, and y^- are, respectively, the lagged action profile vector and the lagged state. A Markov perfect equilibrium (MPE) is a subgame-perfect equilibrium in Markov strategies. That is, an MPE is a set of strategies optimal for any firm, and for any state of system, given the strategies of rivals.

What do we mean by dynamic strategic complementarity (SC) or dynamic strategic susbtitutability (SS)? We could think of "contemporaneous" SC when the value function at an MPE $V_i(y)$ displays SC (V_i has increasing differences in (y_i, y_{-i})). We could think also about "intertemporal" SC when dynamic best replies, or the policy function at an MPE, are monotone. There is intertemporal SC (SS) when a player raising his state variable today increases (decreases) the state variable of the rival tomorrow. I will investigate those properties for a class of simple dynamic Markov games which admits two-stage games, simultaneous move games with adjustment costs, and alternating moves games.

The class of simple dynamic Markov games is defined as follows. Consider the n-player game $(A_i, \pi_i; i \in N)$ where the actions of player i lie in A_i, a compact cube of Euclidean space, $\pi_i(x, y)$ is the current payoff for player i with $y \in A$ the action profile in the previous period (state variables), and $x \in A$ the current action profile. This simple class of games encompasses two-stage games and infinite horizon games of simultaneous moves with adjustment costs or of alternating moves. In a two-stage game $y \in A$ is the action profile

[29] A similar example with product differentiation and network demand externalities (Katz and Shapiro (1986)) would have $\pi_i = (P_i(q) - (c - f(t_i))) q_i$ where q is the vector of output levels of the firms and t_i the accumulated sales of product i.

[30] This section draws on Vives (2003).

in the first stage, $x \in A$ the action profile in the second stage, and $\pi_i(x, y)$ the payoff for player i. Consider now an infinite horizon, discrete time, game with discount factor δ. With simultaneous moves and adjustment costs the payoff to player i is given by $\pi_i(x, y) = u_i(x) + F_i(x, y)$, where $u_i(x)$ is the current profit in the period and $F_i(x, y)$ the adjustment cost in going from past (y) to current actions (x). Assume that $F_i(x, x) = 0$, $i = 1, 2$; that is, when actions are not changed, there is no adjustment cost. With alternating moves in a duopoly x is the action of the player moving now and y action of the player who moved last period.

We take in turn the issues of contemporaneous SC in two-stage games and intertemporal SC or SS in infinite horizon games. We end the section with some remarks on existence of MPE.

5.3.1 Contemporaneous SC in two-stage games

We will have the contemporaneous SC property when at the second stage, for any y, payoffs are SC, and when the SC property is preserved when payoffs are folded back at the first stage in a subgame-perfect equilibrium.

Suppose that $\pi_i(x, y)$ displays increasing differences (or is supermodular) in any pair of variables. Let $V_i(y) \equiv \pi_i(x^*(y), y)$, where $x^*(y)$ is an extremal equilibrium in the second stage. Extremal equilibria exist at the second stage for any y because the second stage game is supermodular. A particular case is when, contingent on y, a unique Nash equilibrium $x^*(y)$ obtains at the second stage. $V_i(y)$ is thus the first period reduced form payoff for player i. I claim that $V_i(y)$ is supermodular in y. The argument is very simple. We have that $V_i(y) \equiv \pi_i(x^*(y), y) = \max_{x_i} \pi_i\left(x_i, x^*_{-i}(y), y\right)$. Note that $x^*_j(y)$ increases in y because π_i has increasing differences in (x_i, y). It follows then that $V_i(y)$ is supermodular in y because (i) π_i is supermodular in all arguments; (ii) $x^*_j(y)$ is increasing in y; (iii) supermodularity is preserved by increasing transformations of the variables, and (iv) supermodularity is preserved under the maximization operation.

The result can be readily generalized to potentially multiple equilibria at the second stage, provided that only second stage equilibria resulting from best reply dynamics are considered after a first period strategy change, and to finite horizon games where the payoff to each player displays increasing differences in any two variables.[31]

[31] However, the result cannot be extended to the case where each payoff function $\pi_i(x, y)$ fulfils the ordinal complementarity conditions (or single crossing property SCP) in any pair of variables. Indeed, it is easy to construct examples where each payoff fulfills the SCP for all pairs of variables while the property is not preserved in the reduced form first period payoffs (Echenique (2004b)). Supermodularity/increasing differences cannot be weakened

An example of the result is provided by the Bertrand oligopoly with advertising. Under the assumptions made (Section 4.4.1) $\pi_i = (p_i - c_i) D_i (p; z_i) - F_i (z_i)$ is supermodular in any pair of arguments and the first stage value function at extremal equilibria is supermodular. That is, advertising expenditures are strategic complements. The assumptions are fulfilled in the classical linear differentiated product Bertrand competition model with constant marginal costs when either advertising or investment in product quality raises the demand intercept of the firm exerting the effort (Vives (1985a)) or increases the willingness to pay for the product of the firm by lowering the absolute value of the slope of demand $\left|\frac{\partial D_i}{\partial p_i}\right|$(Vives (1990b)). In this case for a given advertising effort there is a unique price equilibrium at the second stage.[32]

The result can be extended easily to a duopoly case in which for all i, $\pi_i (x, y)$ has increasing differences in $(x_i, -x_j), (y_i, -y_j)$ and $(x_i, (y_i, -y_j))$, $j \neq i$. An example is provided by a Cournot duopoly in which outputs are strategic substitutes and y_i is the cost reduction effort by firm i. Let $\pi_i = P(x_1, x_2)x_i - C_i (x_i, y_i)$ with $\frac{\partial^2 C_i}{\partial x_i \partial y_i} \leq 0$. Then the assumptions are fulfilled because $\frac{\partial^2 \pi_i}{\partial x_i \partial y_i} \geq 0$, and $\frac{\partial^2 \pi_i}{\partial x_i \partial y_j} = \frac{\partial^2 \pi_i}{\partial z_i \partial y_j} = 0$, $j \neq i$. We have then that cost reduction investments are strategic substitutes at the first stage. With linear demand there is a unique equilibrium at the second stage (see Vives (1990b) for a computed example where investment reduces the slope of marginal costs and a reinterpretation in terms of firms investing in expanding their own market).[33]

5.3.2 Intertemporal strategic complementarity

Consider a stationary MPE of an infinite horizon simultaneous move game with discount factor δ and let $V_i (y)$ be the value function associated to player i at the MPE. Player i solves $\max_{x_i} \{\pi_i (x, y) + \delta V_i (x)\}$. Let $x^* (y)$ be the (assumed unique) contemporaneous Nash equilibrium given y and the MPE

to the ordinal SCP. This happens even though the simultaneous move ("open loop") game would be an ordinal GSC and even though the second period equilibrium is monotone in first period choices.

[32] However, if firms invest in cost reduction the second stage SC is transformed into a first stage SS. The same happens with product enhancement investments in the Shaked and Sutton (1982) model of vertical quality differentiation when the market is covered.

[33] It is worth noting that with high enough spillovers firms' R&D cost reduction investment are SC in the two-stage game (d'Aspremont and Jacquemin (1988)). Ceccagnoli (2003) shows that adding fringe firms that do not invest in R&D and do not benefit from the spillover the degree of SC increases with the number of fringe firms.

policy functions for the players. From Result 5 we have that if for all i

1. $\pi_i(x,y) + \delta V_i(x)$ has increasing differences in (x_i, x_{-i}), and

2. π_i has increasing differences in (x_i, y),

then $x^*(y)$ is increasing in y (i.e., we have intertemporal SC: x_i^* increases with y_j , $j \neq i$).

For (1) to hold it is sufficient that both π_i and V_i have increasing differences in (x_i, x_{-i}).

Similarly as before we have the corresponding result for a duopoly with strategic substitutability. If for all i

1. $\pi_i(x,y) + \delta V_i(x)$ has increasing differences in $(x_i, -x_j)$, $j \neq i$, and

2. π_i has increasing differences in $(x_i, (y_i, -y_j))$,

then x_i^* increases in $(y_i, -y_j)$ (i.e., we have intertemporal SS: x_i^* decreases with y_j, $j \neq i$).

The question is when are the assumptions going to be fulfilled. We will consider in turn the adjustment cost model and the alternating move duopoly.

Simultaneous moves with adjustment costs Consider the adjustment cost model and interpret actions as either prices or quantities. Let production or price bear the convex adjustment cost F. Models with price adjustment costs, or "menu costs", are commonly used in macroeconomics. It is easy to see that with price competition (with static SC) and menu costs the marginal profit for firm i is increasing in the price y_i charged by the firm in the previous period and decreasing in the price y_j charged by the rival in the previous period. This case falls in the domain of the general result above provided that the value function V_i displays SC (this is true in the linear-quadratic specification). With quantity competition (static SS) and production adjustment costs the marginal profit for firm i is increasing in the production y_i of the firm in the previous period and decreasing in the production y_j of the rival in the previous period. This case falls in the domain of the duopoly result with SS above provided that the value function displays SS (as in the linear-quadratic specification).

In those two cases static SC or SS is transformed into intertemporal SC or SS. However, this need not be always so. Jun and Vives (2003) have fully characterized the linear and stable MPE in a symmetric differentiated duopoly model with quadratic payoffs and adjustment costs in a continuous time infinite horizon differential game building on the work of Reynolds

(1987) and Driskill and McCafferty (1989). Jun and Vives (2003) consider both SC (Bertrand) and SS (Cournot) competition and production or price (menu) adjustment costs. It is found that contemporaneous (dynamic) SC or SS are inherited from static SC or SS. Indeed, V_i displays increasing (decreasing) differences in (y_i, y_j) when there is static SC (SS). Intertemporal SC or SS obtains then depending on what variable bears the adjustment cost. We know already from the previous paragraph that with price competition with menu costs (quantity competition with production adjustment costs) static SC (SS) is transformed into intertemporal SC (SS). However, in the mixed case of price competition with production adjustment costs Jun and Vives show that the static SC is transformed into intertemporal SS. Then we have that the marginal profit for firm i is increasing in the price y_i of the firm in the previous period and decreasing in the price of the rival y_j in the previous period. The reason, much as in the learning curve model with price competition, is that a firm wants to make the rival small today in order to induce him to price softly tomorrow. Indeed, a smaller rival will face a stiff cost of increasing his output. A cut in price today therefore will bring a price increase by the rival tomorrow. The result is that if production is costly to adjust intertemporal SS obtains while if price is costly to adjust intertemporal SC obtains.

Having intertemporal SC or SS matters because it governs strategic incentives at the MPE with respect to innocent behavior at the open-loop equilibrium. Indeed, Jun and Vives show that with intertemporal SC (SS) steady state prices at the MPE will be above (below) the stationary open-loop equilibrium prices (which coincide with the static equilibrium prices with no adjustment costs). This in fact provides a generalization of the Fudenberg-Tirole taxonomy of strategic behavior in two stage games to the full-blown infinite horizon game.

Alternating move duopoly Consider a duopoly game in which the payoff to firm i, $i = 1, 2$, is $\pi_i(a_1, a_2)$ and the action set available to the firm is a compact interval. Two players in a duopoly interact repeatedly with player 1 moving in odd periods $t = 1, 3, ...$ and player 2 in even periods $t = 0, 2, ...$ (Cyert and DeGroot (1970), Maskin and Tirole (1987, 1988)). The action of player i, price or quantity for example, is fixed for one period. Denote by x the action of the player moving now and by y the action of the player who moved last period. The state variable for firm i is therefore the action taken in the previous period by firm j. A (pure) Markov strategy for firm i is a function $R_i(\cdot)$ that maps the past action of firm j into an action for firm i. This is truly a dynamic reaction function in contrast with the best-response functions derived in the static games considered in Section 3 (in which best-response functions are useful in finding equilibria and characterizing stability properties).

A Markov perfect equilibrium (MPE) is a pair of dynamic reaction functions $(R_1(\cdot), R_2(\cdot))$ such that for any state a firm maximizes its present discounted profits given the strategy of the rival. The pair $(R_1(\cdot), R_2(\cdot))$ is a MPE if and only if there exist value functions $(V_1(\cdot), V_2(\cdot))$ such that player 1 solves

$R_1(y) \in \arg\max_x \left\{\pi_1(x, y) + \delta \widetilde{V}_1(x)\right\}$where

$\widetilde{V}_1(x) = [\pi_1(x, R_2(x)) + \delta V_1(R_2(x))]$, and $V_1(y) = \max_x \left\{\pi_1(x, y) + \delta \widetilde{V}_1(x)\right\}$,

and similarly for player 2. That is, given the state variable y (the current action of firm 2) for firm 1, $V_1(y)$ gives the present discounted profits when it is the turn of firm 1 to move and both firms use the dynamic reaction functions $(R_1(\cdot), R_2(\cdot))$.

Suppose that MPE dynamic reaction functions exist. Then, according to our result, they will be monotone increasing (decreasing) if the underlying one-shot simultaneous move game is strictly supermodular (supermodular in $(x, -y)$) . That is, if $\pi_i(x, y)$ has strictly increasing differences in (x, y) (in $(x, -y)$). Then any selection $R_1(\cdot)$ from the set of maximizers of $\pi_1(x, y) + \delta \widetilde{V}_1(x)$ is increasing (decreasing). No other property is needed.

Existence of a MPE can be established easily with quadratic payoff functions: Cournot with homogeneous products (Maskin and Tirole (1987)) and Bertrand with differentiated products (exercise 9.12 in Vives (1999)). It can be shown that for any δ there is a unique linear MPE that is symmetric and (globally) stable and that the steady state action is increasing in δ and equals the static Nash equilibrium when $\delta = 0$.

In the Cournot case the strategic incentives for a firm are to increase its output in order to reduce the output of the rival, and in the Bertrand case (with product differentiation) to increase price to induce the rival to be softer in pricing. Thus, in the Cournot (Bertrand) case, the static strategic substitutability (complementarity) translates into intertemporal strategic substitutability (complementarity). In other words, in the Cournot (Bertrand) case, both static and dynamic reaction functions are downward (upward) sloping. An increase in the weight firms put into the future (a larger δ) increases the strategic incentives with the result of a higher output (price) in the Cournot (Bertrand) market. In any case the equilibrium action is larger than the static equilibrium when $\delta > 0$.

With homogenous products and price competition dynamic reaction functions are no longer monotone. This is so because with a homogeneous product the marginal profit of a firm is not monotone in the price charged by the rival.

For example, if the rival sets a (strictly) lower price than firm i, then firm i's marginal profit of changing its price is zero, and if the rival sets a (strictly) larger price, then firm i's marginal profit is positive, provided that its price is below the monopoly price. However, if the prices of both firms are equal, the marginal profit is negative. A consequence of this lack of monotonicity is the finding of multiple equilibria (including equilibria of the "kinked demand curve" type and price cycles, see Maskin and Tirole (1988b)).

5.3.3 Existence of MPE

Up to now we have not dealt with the existence problem for MPE, only with the characterization of equilibria. Lattice-theoretic methods can be used when there is enough monotonicity in the problem studied.

Curtat (1996) shows existence of MPE of stochastic games with complementarities with discrete time and infinite horizon. He considers multidimensional action spaces and a mutidimensional state evolving according to a transition probability as a function of the current state and action profile. Payoffs are smooth and display per period complementarities and positive externalities or spillovers (the payoff to a player is increasing in the actions of rivals and the state), the transition distribution function is smooth and displays complementarities and is stochastically increasing in actions and states, and the payoff to a player as well as the transition distribution function have a strict dominant diagonal condition. These strong assumptions allow to collapse the multiperiod problem to a reduced form static game (with continuation value functions increasing in the state variable) which is shown to be supermodular. An equilibrium can be found then with value functions increasing in the state. Examples of games fulfilling the assumptions are a dynamic version of the search game considered in Section 2 (where the parameter θ evolves stochastically in a monotone increasing way with the average search effort of the population: the higher the average effort $\widetilde{a}_t$ in period t the higher will θ_{t+1} be in expected terms); and a dynamic version of a Cournot oligopoly with complementary products and learning by doing where a high level of accumulated output by one firm yields stochastically higher levels of cumulated experience, and lower production costs, to the firm (learning by doing) and to the rivals (spillovers).

Another successful application of the techniques to proof existence of a MPE is provided by Sleet (2001). Sleet considers a version of the adjustment cost model of the previous section in an infinite horizon discrete game with a continuum of players and symmetric payoffs. Firms set prices and prices are costly to adjust. The payoff to a player in any period is given by $\pi(x, y, G, \theta) = R(x, G, \theta) - F(x, y)$ where x is the current price of the firm, y

the price in the previous period, G the distribution of prices chosen by other firms, θ a firm specific shock (which is iid across firms and dates), R is the net revenue function and F the adjustment cost. The payoff is increasing in G, has increasing differences in $(x, (G, \theta))$, and $-F(x, y)$ is supermodular. Under some further technical restrictions the existence of a symmetric monotone MPE is shown in which each firms uses the same increasing MPE policy function yielding the current period action contingent on last period's action, last period's distribution of actions, and the player's specific shock θ. This is done by showing the existence of a fixed point of an increasing function that maps (increasing) policy functions into themselves. The problem is simplified because with a continuum firms no firm can influence any aggregate and each firm faces a dynamic programming problem. Furthermore, as usual with the lattice-theoretic approach, an algorithm to compute the largest or the smallest equilibrium policy functions can be provided.

The model corresponds to a monopolistic competition model where each firm does not influence the market aggregates but retains some market power, the demand or the technology of a firm is subject to a period specific shock and prices are subject to continuous adjustment costs. For example, the demand for the product of a firm may depend on the average price charged in the market or on a price index. The assumptions are fulfilled with linear or constant elasticity demands and quadratic or constant elasticity production costs (subject to a multiplicatively shock) and with quadratic costs of price adjustment.[34]

6 Bayesian games

Bayesian games provide a fertile ground for applications of the lattice-theoretical approach. The reason is that it allows for general strategy spaces and payoff functions. In Section 6.1 I present the basic set up of the Bayesian game and basic approaches to the difficult issue of existence of equilibrium in pure strategies. Most recent advances are based on the lattice-theoretic approach, be it with supermodular games (Vives (1990)), single-crossing properties (Athey (2001)), or monotone supermodular games (Van Zandt and Vives (2003)). Section 6.2 presents a comparative statics application in monotone supermodular games that allows to extend substantially results by Okuno-Fujiwara et al (1990) on the existence of fully revealing equilibria in games of voluntary disclosure. Again, many of the regularity conditions assumed by these authors turn out not be necessary. Finally, Section 6.3 deals with

[34]See Section 8.1 in Blanchard and Fischer (1989) and Rotemberg (1982).

global games and applications to currency and banking crisis. It makes clear in one stroke that dominance solvability and uniqueness in the standard global game (Morris and Shin (2002)) obtain because the underlying game is one of strategic complementarities and the key to uniqueness is precisely that the strength of the strategic complementarities do not be too large. The lattice-theoretic approach does away with the need to go through the process of iterated elimination of dominated strategies to obtain a unique equilibrium. Indeed, we have seen that a supermodular game with a unique equilibrium is dominance solvable.

6.1 Bayesian Nash equilibrium: existence and characterization

In a Bayesian game the type of a player embodies all the decision-relevant private information. Let T_i be a subset of Euclidean space and the set of possible types of player i (t_i). The types of the players are drawn from a common prior distribution μ on $T = X_{i=0}^{n} T_i$ where T_0 is residual uncertainty not observed by any player. Let μ_i represent the marginal distribution on T_i. The action space of player i is a compact cube of Euclidean space A_i, and his payoff is given by the measurable and bounded function $\pi_i : A \times T \to \mathbb{R}$, where $A = X_{i=1}^{n} A_i$. The (ex post) payoff to player i when the vector of actions is $a = (a_1, ..., a_n)$ and the realized types $t = (t_1, ..., t_n)$ is thus $\pi_i(a; t)$. Endow strategy and type profiles with the usual component-wise or product order (that is, $a \geq a'$ if and only if $a_i \geq a'_i$ for all i, and $t \geq t'$ if and only if $t_i \geq t'_i$ for all i). Action spaces, payoff functions, type sets, and the prior distribution are common knowledge. The Bayesian game is then fully described by $(A_i, T_i, \pi_i; i \in N)$.

A (pure) strategy for player i is a (measurable) function $\sigma_i : T_i \to A_i$ which assigns an action to every possible type of the player. Let Σ_i denote the strategy space of player i and identify strategies σ_i and τ_i if they are equal with probability one (μ_i-almost surely (a.s.)). Let $\sigma = (\sigma_1, ..., \sigma_n)$. The expected payoff to player i, when agent j uses strategy σ_j, is given by $U_i(\sigma) = E\pi_i(\sigma_1(t_1), ..., \sigma_n(t_n); t)$.

A Bayesian Nash equilibrium (BNE) is a Nash equilibrium of the game where the strategy space of player i is Σ_i and his payoff function P_i. Given the strategies $\sigma_j(\cdot)$, $j \neq i$, denote by $\sigma_{-i}(t_{-i})$ the vector $(\sigma_1(t_1), ..., \sigma_n(t_n))$ except the i^{th} component. The expected payoff of player i conditional on t_i when the other players use σ_{-i} and player i uses a_i is $E\{\pi_i(a_i, \sigma_{-i}(t_{-i}); t \mid t_i)\}$. The profile of strategies σ is a BNE if and only if for every i the action $\sigma_i(t_i)$ maximizes over A_i the conditional payoff $E\{\pi_i(a_i, \sigma_{-i}(t_{-i}); t \mid t_i)\}$ (

μ_i-almost surely on T_i). Consider the game $(\Sigma_i, U_i, i \in N)$ and define a natural order in the strategy space $\Sigma_i : \sigma_i \leq \sigma_i'$ if $\sigma_i(t_i) \leq \sigma_i'(t_i)$ for μ_i-a.s. on T_i.

This formulation of a Bayesian game is general and encompasses common and private values as well as perfect or imperfect signals. In a "pure" private values, allowing for correlated types, we have $\pi_i(a; t_i)$. For example, types are private cost parameters of firms. A "common value" case is $\pi_i(a;t) = v_i(a; \Sigma_i t_i)$. For example, there is a common demand shock in an oligopoly and firm i observes component t_i only. As an example of imperfect signals, suppose firms observe with noise their cost parameters. Then t_0 could represent the n-vector of firms' cost parameters and t_i the private cost estimate of firm i. Both the cost parameters and the error terms in the private signals may be correlated.

6.1.1 Equilibrium existence in pure strategies

Existence of pure-strategy Bayesian equilibria in games with a continuum of types and/or actions has proved to be a difficult issue. Typical sufficient conditions for existence of pure-strategy Bayesian equilibria include conditionally independent types, finite action spaces, and atomless distributions for types (see Radner and Rosenthal (1982) and Milgrom and Weber (1985)).[35] Under these assumptions the authors show first the existence of mixed strategy equilibria and then obtain a purification result. For this approach to work independence, or at least, conditional independence, of the distribution of types is needed.

The lattice-theoretic method has provided three types of results:

1. for supermodular games with general action and type spaces (Vives (1990));

2. for games in which each player uses a strategy increasing in type in response to increasing strategies of rivals (Athey (2001)); and

3. for "monotone" supermodular games with general action and type spaces (Van Zandt and Vives (2003)).

Supermodular games In the first approach (Vives (1990) and Vives (section 2.7.3, 1999)), existence of pure strategy Bayesian equilibria follows from

[35] Khan and Sun (1995) show existence of pure-strategy equilibria when types are independent, payoffs continuous and action sets countable.

supermodularity of the underlying family of games defined with the ex post payoffs for given realizations of the types of the players. A key observation is that supermodularity of this underlying family of games is inherited by the Bayesian game. Let π_i be supermodular in a_i and have increasing differences in (a_i, a_{-i}). Then $U_i(\sigma)$ is supermodular in σ_i and has increasing differences in (σ_i, σ_{-i}) because supermodularity and increasing differences are preserved by integration. Furthermore, strategy spaces in the Bayesian game Σ_i can be shown to have the appropriate order structure (i.e., they are complete lattices). Then the game $(\Sigma_i, U_i, i \in N)$ is a GSC and for all $\sigma_{-i} \in \Sigma_{-i}$, $\beta_i(\sigma_{-i})$ contains extremal elements, $\bar{\beta}_i(\sigma_{-i})$ and $\underline{\beta}_i(\sigma_{-i})$. Existence of extremal pure strategy Bayesian equilibria follows then from the general versions of the results in Section 2 (Vives (1990, and Section 2.7.3, 1999)). This existence result holds for multidimensional action spaces and no distributional restrictions. The driving assumption is strategic complementarities.

Applications of this approach can be found in oligopoly games and team theory (as we will see below), Diamond's search model (1982), natural resource exploration games with private information (see Hendricks and Kovenock (1989) and Milgrom and Roberts (1990)), and global games (see Section 6.3).

Single-crossing properties In the second approach (Athey (2001)) conditions are imposed so that an equilibrium in monotone increasing strategies (in types) can be found. Suppose that both action A_i and types sets T_i for any player i are compact subsets of the real line, and that types have a joint density μ that is bounded, atomless and log-supermodular (i.e., types are affiliated). Suppose also that $\pi_i(a, t)$ is continuous and supermodular in a_i, has increasing differences in (a_i, a_{-i}) and (a_i, t) or, alternatively, $\pi_i(a, t)$ is nonnegative and log-supermodular in (a, t). Then the Bayesian game has a pure strategy equilibrium in increasing strategies. Note that in the first case the first approach outlined already delivers existence of a pure-strategy equilibrium.

The proof of those results relies on the standard Kakutani's fixed point theorem which relies on convex-valued correspondences. It turns out that with discrete action spaces, and under the assumptions, best-response correspondences are convex valued. A key step in the proof is to show that, under the assumptions, if the rivals of player i use increasing strategies, the payoff to player i is log-supermodular or has increasing differences (or, in general, fulfills an appropriate single crossing property) in action and type. This makes sure that a player uses a strategy increasing in its type as a best response to increasing strategies of rivals.[36] The existence result for discrete

[36] The result follows directly from the assumptions noting that if $f : \mathbb{R}^n \to \mathbb{R}$ is su-

action spaces can be used then to show existence with a continuum of actions using a purification argument.

An example of the result is our differentiated Bertrand oligopoly in which firm i has random marginal cost θ_i. Then it is immediate that $E(D_i(p_i, p_{-i}(\theta_{-i}) \mid \theta_i)$ is log-supermodular in (p_i, θ_i) if both $D_i(p_i, p_{-i})$ and the joint density of $(\theta_1, ..., \theta_n)$ are log-supermodular and if the strategies of rivals, $p_j(\cdot)$, $j \neq i$, are increasing in types. It follows then that $E(\pi_i \mid \theta_i) = (p_i - \theta_i)E(D_i(p_i, p_{-i}(\theta_{-i}) \mid \theta_i)$ is log-supermodular in (p_i, θ_i) and the best reply map of player i is increasing in θ_i.

The approach can be used also in games which are not of SC and with discontinuous payoffs. For example, in auctions the existence of monotone equilibria in pure strategies can be shown for

- first-price auctions with heterogeneous (weakly) risk averse bidders with private affiliated values or common value and conditionally independent signals (Athey (2001)), and for
- uniform price auctions with multiunit demand with non-private values and independent types (McAdams (2003)).[37]

Monotone supermodular games Combining both approaches Van Zandt and Vives (2003) show a stronger result for "monotone" supermodular games with multidimensional action spaces and type spaces. Let $\Delta(T_{-i})$ be the set of probability distributions on T_{-i} and let player i's posteriors be given by the (measurable) function $p_i : T_i \rightarrow \Delta(T_{-i})$, consistent with the prior μ, where $p_i(\cdot \mid t_i) \in \Delta(T_{-i})$ denotes i's posteriors on T_{-i} conditional on t_i. A monotone supermodular game is defined by

1. supermodularity and complementarity between action and type: π_i supermodular in a_i, and with increasing differences in (a_i, a_{-i}) and in (a_i, t); and by

permodular (or log-supermodular) so is the function $f(h_1(x_1), ..., h_n(x_n))$ if the h_i functions are increasing, and that if $g(x,t)$ is supermodular (log-supermodular) in (x,t) then $E\{g(x,t) \mid t_i\}$ is supermodular (log-supermodular) in (x, t_i) provided that (the random vector) t is affiliated. See Vives (1999, p.69 and p. 229-230) and Athey (2001) for details.

[37] The result is shown using an intermediate result, allowing multidimensional types and multidimensional (Euclidean) action spaces, that puts assumptions on non-primitives. The conditions are: atomless types and interim (conditional on type) expected payoff of each player (quasi)-supermodular in his action, and with single-crossing in own action and type given that other players use strategies increasing in types.

2. monotone posteriors: $p_i : T_i \to \Delta(T_{-i})$ increasing with respect to the partial order on $\Delta(T_{-i})$ of first order stochastic dominance (a sufficient but not necessary condition is that μ is affiliated).

Under these conditions there is a largest and a smallest Bayesian equilibrium and each one is in monotone strategies. The argument for the result is very powerful and simple.[38] First, the Bayesian game is of strategic complementarities. This follows as in the first approach to existence. This means that the extremal best reply maps are well defined for each player and are increasing. Second, the extremal best replies to monotone strategies are monotone. Let $V_i(a_i, t_i, P_{-i}) \equiv \int_{T_{-i}} \pi_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i})\, dP_{-i}(t_{-i})$. Then V_i has increasing differences in (a_i, t_i) and in (a_i, P_{-i}) because $\pi_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i})$ has increasing differences in (a_i, t) as π_i has increasing differences in $(a_i, (a_{-i}, t))$ and σ_{-i} is increasing, and because of the monotone posteriors condition. The latter implies that higher types believe that the other players are more likely to be of higher types as well (and this is implied by affiliation, for example). Furthermore, V_i is supermodular in a_i because π_i is supermodular in a_i. The result is that a higher type for i chooses a higher action both because a shift in beliefs (p_i is increasing and higher types believe that other players are more likely to be of higher types as well) and because the induced payoff has increasing differences in (a_i, t_i). Third, if the largest best reply map $\bar{\beta}_i(\sigma_{-i})$ is increasing, the largest best reply to monotone strategies is monotone, and payoffs are continuous then there is a largest equilibrium and it is in monotone strategies. This follows by starting a Cournot tâtonnement with strategies for each player i equal, for any type, to the largest element in the action set A_i (it exists because A_i is a cube in Euclidean space). Then the Cournot tâtonnement defines a decreasing sequence of monotone strategies and its limit must be an equilibrium because of the continuity of payoffs, and the limit is also in monotone strategies. Furthermore, it is easy to see that the limit must be the largest equilibrium. (The reader will appreciate that the argument is similar to the proof of Result 4 in Section 3.) There might be other equilibria which are in nonmonotone strategies but, if so, they will be "sandwiched" between the largest and the smallest one, which are monotone in type.

Two examples with muldimensional competition are the following. In the Bertrand multimarket oligopoly (Section 4.4.2) with $\pi_i = \sum_{h=1}^{H_i} (p_{ih} - c_{ih})\, D_{ih}(p_i, p_{-i}; \theta_{ih})$, let the type of firm i be $t_i = (c_i, s_i)$, where s_i is a signal about the random vector θ. If θ and $(c_i, s_i)_{i \in N}$ are affiliated, and D_{ih} linear and increasing in θ_{ih} with all goods gross substitutes, then extremal equilibrium prices p_i^* increase in (c_i, s_i). In the Bertrand oligopoly with advertising (Section 4.41)

[38] The result cannot be extended to log-supermodular payoffs.

with $\pi_i = (p_i - c_i)D_i(p_i, p_{-i}, z_i) - F_i(z_i, e_i)$, where e_i is the random efficiency of advertising effort z_i, let the type of firm i be $t_i = (c_i, e_i)$ and the action $a_i = (p_i, z_i)$. Under mild assumptions, with advertising increasing the willingness to pay and with affiliated types, the extremal equilibrium actions (p_i^*, z_i^*) increase in (c_i, e_i).

Another application is to a team problem (Radner (1962)) where the common function to be optimized is supermodular, there are increasing differences between actions and types, and the distribution of types yields monotone posteriors. Each member of the team chooses a decision rule, a strategy, contingent on his private information (type) in order to maximize the common objective. We know that the team optimum will be a Bayesian equilibrium of the game among team members (Radner (1962)). Suppose that there is a unique equilibrium. We conclude then that there is a team optimum and at the optimum players use decision rules monotone in type. For example, in a multidivisional firm in which the total profit of the firm has been internalized by the division's managers, a_j could be the vector of actions or "efforts" under the control of manager j and s_j his private information relating to cost and demand conditions for division j.

6.2 Comparative statics and strategic information revelation

In this section I provide an application of the monotone supermodular game framework. The starting point is to realize that in a monotone supermodular Bayesian game extremal equilibria are increasing in posteriors. This yields immediately an interesting comparative statics result for expected payoffs. Let $\mathcal{P}$ be the set of increasing posteriors (with respect to first order stochastic dominance FOSD) and $\{\Gamma(p) \mid p \in \mathcal{P}\}$ a parameterized family of monotone supermodular Bayesian games. Let $\bar{\sigma}(p)$ be the largest equilibrium in $\Gamma(p)$ and $W_i(p, t_i)$ player i's expected utility in $\bar{\sigma}(p)$ with type t_i. Then:

- If $p_i'(t_i) \geqslant_{FOSD} p_i(t_i)$ for (a.e.) $t_i \in T_i$ and i, then $\bar{\sigma}(p') \geq \bar{\sigma}(p)$ (because $\bar{\beta}_i'(\sigma_{-i}) \geqslant \bar{\beta}_i(\sigma_{-i})$), and
- if π_i is increasing in a_{-i} (positive externalities) then $W_i(p, t_i)$ is increasing in p_{-i} (because $\bar{\sigma}(p)$ is increasing in p_{-i}).

We have therefore that in a game with positive externalities the expected payoffs of each player in an extremal equilibrium is increasing in the posteriors of the other players (ordered by first order stochastic dominance).

This result can be strengthened when the family of games $\Gamma(p)$ is differentially strictly monotone (with $\partial u_j / \partial a_{jh}$ strictly increasing in t_j and t_i for all

h) and differentially strictly supermodular (with $\partial u_j / \partial a_{jh}$ strictly increasing in a_{ih} for all h) Bayesian game with strictly positive externalities. Then:

- If for a.e. $t_j \in T_j$ the marginal of $p'_j(t_j)$ on T_i strictly FOSD $p_j(t_j)$, then $\bar{\sigma}_{jh}(p') > \bar{\sigma}_{jh}(p)$.

- If π_i is strictly increasing in a_j, then $W_i(p, t_i)$ is strictly increasing in the marginal of $p_j(t_j)$ on T_i.

The comparative statics result has a ready application to games of voluntary disclosure. Let us see how a result by Okuno-Fujiwara et al (1990) can be improved upon. Consider a two-stage duopoly game in which at the first stage player i can send a message $m_i \in M_i$ about his type t_i which lies in a finite ordered set T_i. Types are independent. At the second stage a Bayesian equilibrium (assumed unique and interior) with updated beliefs conditional on the messages obtains. Types are verifiable and therefore players can conceal information but not lie. The mapping from message to lowest types is well-defined and covers all the types: For each $t_i \in T_i$, there is $m_i \in M_i$ such that $\min m_i = t_i$. It is assumed furthermore that A_i is a compact interval and π_i is concave and continuously differentiable in a_i. The cases of strategic complementarity and positive externalities (strategic substitutability and negative externalities) are allowed: with π_i strictly increasing (decreasing) in a_j, $\partial \pi_i / \partial a_i$ strictly increasing (decreasing) in a_j, and $\partial \pi_i / \partial a_i$ strictly increasing in t_i and increasing (decreasing) in t_j. Under these conditions any perfect Bayesian equilibrium (PBE) of the two-stage game is fully revealing. Player i with type t_i reports m_i such that $\min m_i = t_i$. The basic intuition of the result is that in equilibrium inferences are sceptical: if a player reports a set of types others believe the worst (that is, others believe that the player is of the most unfavorable type in the reported set). This unravels the information. For example, in a Cournot duopoly in which types are the (constant marginal) costs of firms, which can be high or low. Then a firm reporting nothing (the full set) will be assumed to have high costs because if teh firm had low costs it would have said so.

The comparative statics result presented generalizes the Okuno-Fujiwara et al (1990) result to a n-player GSC case, or to a duopoly with strategic susbtitutability, with multidimensional actions, affiliated types and allowing multiple non-interior second stage equilibria (provided they are extremal). For the generalized result to obtain there is no need to assume that payoffs are concave, and differentiability and interior solutions are needed only along one dimension of the strategy space. More precisely, consider a differentially strictly monotone, differentially strictly supermodular Bayesian game with strictly positive externalities. Then any PBE of the two-stage game in which a plausible refinement about out of equilibrium path beliefs holds (the "increasing posteriors" condition is satisfied in the second stage following nonequilibrium messages also) and for which extremal equilibria are selected in the second stage, is fully revealing.[39] Again beliefs are sceptical (on the equilibrium path): a player is believed to be of the lowest type in the reporting set. The argument is the following. Let $\bar{t}_i$ be highest type for i not fully revealed at an extremal PBE; hence $\bar{t}_i$ is pooled with lower types. Player i gains strictly by deviating and sending m_i such that $\min m_i = \bar{t}_i$ because then other players' posteriors about his type go up (according to strict FOSD). This is so because in a differentially strictly monotone, differentially strictly supermodular Bayesian game with strictly positive externalities the expected payoff of player i at extremal equilibria is strictly increasing in the posteriors of other players p_{-i} since larger posteriors (according to strict FOSD) increase strictly extremal equilibria. Therefore, the equilibrium is fully revealing. Suppose now that, for some i and type t_i, $t_i > \min m_i$. Since m_i is fully revealing, others believe that i is of type t_i. Then type $\min m_i$ could deviate from his message by sending instead m_i, causing a shift in all player's beliefs from his being of type $\min m_i$ to being of type t_i. Therefore, player i with type t_i reports m_i such that $\min m_i = t_i$.

6.3 Global games

Global games are games of incomplete information with type space determined by each player observing a noisy signal of the underlying state. The aim is equilibrium selection via perturbation of a complete information game. The basic idea is that when analyzing a complete information game with potentially multiple equilibria players have to entertain the "global picture" of slightly different possible games being played. Each player has a noisy estimate of the game being played and knows that the other players are also receiving noisy estimates.

Carlsson and van Damme (1993) show the following result. In 2×2 games if each player observes a noisy signal of the true payoffs and if ex ante feasible payoffs include payoffs that make each action strictly dominant then as noise becomes small, iterative strict dominance selects one equilibrium. The equilibrium selected is the Harsanyi-Selten (1988) risk dominant one if there are two equilibria in the complete information game. Carlsson and van Damme do not consider explicitly supermodular games but in the interesting

[39] In Okuno-Fujiwara et al (1990) the two refinements are satisfied automatically because types are independent (and therefore posteriors are trivially monotone) and at the second stage a unique equilibrium is assumed.

case in which there are two equilibria in the complete information game then the game is one of strategic complementarities.

I will analyze a standard symmetric binary action game of strategic complementarities with the tools of supermodular games, provide some applications to finance, and conclude with a robustness exercise.

6.3.1 A symmetric binary action game of SC

Consider a version of the game with a continuum of players of Section 2. The action set of player i is $A_i \equiv \{0, 1\}$ with $a_i = 1$ interpreted as "acting" and $a_i = 0$ "not acting" (and let $a_i = 1$ be "larger" than $a_i = 0$). To act may be to invest, adopt a technology or standard, revolt, attack a currency, or run on a bank. The fraction of people acting is $\widetilde{a}$ and the state of the world θ. There is a critical fraction of people $h(\theta)$ above which it pays to act with $h(\cdot)$ strictly increasing and crossing 0 at $\theta = \underline{\theta}$ and 1 at $\theta = \overline{\theta}$. (See Figure 8.)

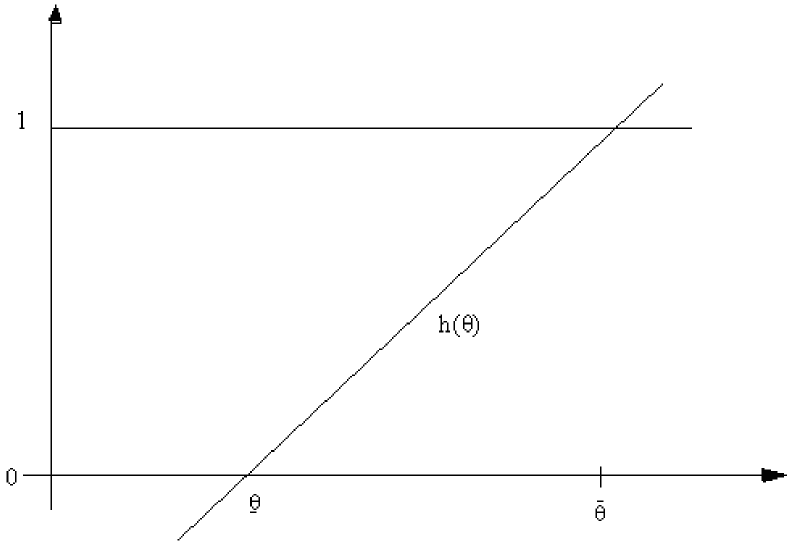


Figure 8

Let $\pi^1 = \pi(a_i = 1, \widetilde{a}; \theta)$ and $\pi^0 = \pi(a_i = 0, \widetilde{a}; \theta)$. The differential payoff to acting is given by:

	$\widetilde{a} \geq h(\theta)$	$\widetilde{a} < h(\theta)$
$\pi^1 - \pi^0$	$B - C > 0$	$-C < 0$

For any given state of the world θ this defines a supermodular game in which $\pi^1 - \pi^0$ is increasing in $\widetilde{a}$ and $-\theta$ or, equivalently, $\pi(a_i, \widetilde{a}; \theta)$ has increasing differences in $(a_i, (\widetilde{a}, -\theta))$. It is immediate that if $\theta \leq \underline{\theta}$ it is a dominant strategy to act; if $\theta \geq \overline{\theta}$ it is a dominant strategy not to act; and for $\theta \in (\underline{\theta}, \overline{\theta})$ there are multiple equilibria: Either everyone acting or no one acting are equilibria. We know also, according to Result 5 in Section 3, that extremal equilibrium strategies will be monotone (decreasing) in θ. Indeed, the largest equilibrium is $a_i = 1$ for all i if $\theta \leq \overline{\theta}$ and $a_i = 0$ for all i if $\theta > \overline{\theta}$ and it is (weakly) decreasing in θ.

Consider now the incomplete information game where players have a normal prior on the state of the world $\theta \sim N(\mu_\theta, \tau_\theta^{-1})$ and player i observes a private signal $s_i = \theta + \varepsilon_i$ with normally distributed noise $\varepsilon_i \sim N(0, \tau_\varepsilon^{-1})$, i.i.d. across players. Morris and Shin (2002) show then that iterated elimination of dominated strategies leads to a unique outcome provided that $\tau_\theta / \sqrt{\tau_\varepsilon}$ is small. We have then a unique Bayesian equilibrium. We will show here how using the tools of supermodular games we obtain the conclusion that the game is dominance solvable without actually having to go through the elaborate process of iterated elimination of dominated strategies. Furthermore, we will see how the approach brings in a very transparent way the intuition behind the uniqueness result.

Note first that the game is monotone supermodular because $\pi(a_i, \widetilde{a}; \theta)$ has increasing differences in $(a_i, (\widetilde{a}, -\theta))$ and signals are affiliated. This means that extremal equilibria exist, are symmetric (because the game is symmetric), and are in monotone (decreasing) strategies of the form $a_i = 1$ if and only if $s_i < t$ (according to the results in Section 6.1.1). Therefore, the extremal equilibrium thresholds $\overline{t}$ and $\underline{t}$ bound the set of rationalizable strategies.

Now, an equilibrium will be characterized by two thresholds (t^*, θ^*) with t^* yielding the acting signal threshold and θ^* the state of the world threshold below which the acting mass is successful and an acting player obtains the payoff $B - C > 0$ (the currency falls, the bank fails or the revolt succeeds). The critical thresholds must fulfil two equations:

1. $\widetilde{a}(\theta^*, t^*) = \Pr(s \leq t^* \mid \theta^*) = h(\theta^*)$, and
2. $E\{\pi(1, \widetilde{a}(\theta); \theta) - \pi(0, \widetilde{a}(\theta); \theta) \mid s\} = t^* = 0$, or $\Pr(\theta \leq \theta^* \mid t^*)(B - C) + \Pr(\theta > \theta^* \mid t^*)(-C) = 0$, or $\Pr(\theta \leq \theta^* \mid t^*) = \gamma$, where $\gamma \equiv C/B < 1$.

The first equation states that at the critical state of the world, in equilibrium, the fraction of acting players must equal the critical fraction above which it pays to act. The second equation states that at the critical signal threshold the expected payoff of acting and not acting is the same. Those equations may have multiple solutions. However, it can be shown that if (and only if) $\tau_\theta/\sqrt{\tau_\varepsilon}$ is small enough the solution is unique, in which case the equilibrium is unique and the game is dominance solvable because then $\overline{t} = \underline{t}$.

If $\tau_\theta/\sqrt{\tau_\varepsilon}$ is not small enough then typically there are three equilibria. The basic reason why with small noise in the signals the equilibrium is unique is that decreasing the amount of noise decreases the strength of the strategic complementarity among the actions of the players. Indeed, multiple equilibria come about when the strategic complementarity is strong enough.

It is instructive to sketch the proof of uniqueness to bring about the intuition. Suppose that $h(\cdot)$ is continuously differentiable with $h' > 0$. Let $\underline{h'} \equiv \min_{\theta \in [\underline{\theta}, \overline{\theta}]} h'(\theta) > 0$. Let $P(s,t)$ be the probability that the acting players succeed if they use a threshold t and the player receives a signal s. This is $P(s,t) \equiv \Pr[\theta < \widehat{\theta}(t)|s] = \Phi\left(\sqrt{\tau_\theta + \tau_\epsilon}\left(\widehat{\theta}(t) - \frac{\tau_\theta \mu_\theta + \tau_\epsilon s}{\tau_\theta + \tau_\epsilon}\right)\right)$ where $\widehat{\theta}(t)$ is the critical θ below which an attack succeeds when players use a strategy with threshold t, and Φ is the cumulative distribution of the standard normal random variable $N(0,1)$.

It is immediate that P is strictly decreasing in s, $\frac{\partial P}{\partial s} < 0$, and nondecreasing in t, $\frac{\partial P}{\partial t} \geq 0$. Given that other players use a strategy with threshold t the best response of a player is to use a strategy with threshold $\overline{s}$ where $P(\overline{s}, t) = \gamma$: act if and only if $P(s,t) > \gamma$ or, equivalently, if and only if $s < \overline{s}$ where $P(\overline{s}, t) = \gamma$. This defines a best response function $r(t) = \frac{\tau_\theta + \tau_\epsilon}{\tau_\epsilon}\widehat{\theta}(t) - \frac{\tau_\theta}{\tau_\epsilon}\mu_\theta - \frac{\sqrt{\tau_\theta + \tau_\epsilon}}{\tau_\epsilon}\Phi^{-1}(\gamma)$. The game is of strategic complementarities and we have that $r' = -\frac{\partial P/\partial t}{\partial P/\partial s} \geq 0:$ a higher threshold t by others induces a player to use also a higher threshold. Furthermore, we have that $r'(t) = \frac{\tau_\theta + \tau_\epsilon}{\tau_\epsilon}\widehat{\theta}'(t) \leq \frac{\tau_\theta + \tau_\epsilon}{\tau_\epsilon}\left[1 + \sqrt{\frac{2\pi}{\tau_\epsilon}}\underline{h'}\right]^{-1}$.[40] If $\frac{\tau_\theta}{\sqrt{\tau_\varepsilon}} \leq \sqrt{2\pi}\underline{h'}$ then $r'(t) \leq 1$ with equality only when $h(\theta) = 1/2$. This ensures that $r(t)$ crosses the 45^o line only once and the equilibrium is unique. When $h(\theta) = \theta$, $\underline{h'} = 1$ and if $\frac{\tau_\theta}{\sqrt{\tau_\varepsilon}} > \sqrt{2\pi}$ then $r'(t) > 1$ for $h(\theta) = \theta = 1/2$. Therefore, for example, for

[40] We have that $\widehat{\theta}(t)$ is the solution in θ of $\Pr(s \leq t \mid \theta) = \Phi\left(\sqrt{\tau_\epsilon}(t - \theta)\right) = h(\theta)$. From this equation we can solve for the inverse function and obtain $\widehat{t}(\theta) = \theta + \frac{1}{\sqrt{\tau_\epsilon}}\Phi^{-1}(h(\theta))$ with derivative $\widehat{t'} = 1 + \frac{1}{\sqrt{\tau_\epsilon}}h'(\theta)\left[\phi\left(\Phi^{-1}(h(\theta))\right)\right]^{-1}$, where ϕ is the density of the standard normal. Since ϕ is bounded above by $\frac{1}{\sqrt{2\pi}}$, $\widehat{t'}$ is bounded below: $\widehat{t'}(\theta) \geq 1 + \sqrt{\frac{2\pi}{\tau_\epsilon}}\underline{h'}$, where $\underline{h'} = \min_{\theta \in [\underline{\theta}, \overline{\theta}]} h'(\theta) > 0$. Hence, $\widehat{\theta}'(t) \leq \left[1 + \sqrt{\frac{2\pi}{\tau_\epsilon}}\underline{h'}\right]^{-1}$(with strict inequality except when $h(\theta) = 1/2$ because then $\Phi^{-1}(1/2) = 0$ and ϕ attains its maximum: $\phi(0) = \frac{1}{\sqrt{2\pi}}$).

γ such that $\theta^* = 1/2$ there are three equilibria.[41]

With small noise the strategic complementarity is lessened, and $r'(t) \leq 1$, because then a player faces a lot of uncertainty about the behavior of others. Indeed, consider the limit cases $\tau_\varepsilon \to +\infty$ (or equivalently a diffuse prior $\tau_\theta = 0$). Then it is not hard to see that the distribution of the proportion of acting players $\widetilde{a}(\theta, t^*)$ is uniformly distributed over $[0, 1]$ conditional on $s_i = t^*$. This means that players face maximal strategic uncertainty and cannot coordinate on different equilibria. In contrast, with complete information there are multiple equilibria when $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$. Indeed, at any of the equilibria players face no strategic uncertainty. For example, in the equilibrium in which everyone acts a player has a point belief that all other players will act.

It is worth to note that the uniqueness argument made is robust to general distributions for the uncertainty as long as noise is small. Indeed, with very precise signals all priors "look uniform" (Morris and Shin (2002)).

In summary, using the theory of supermodular games we bring forward the intuition for the uniqueness result, clarify the role of the assumptions and we need not solve the process of iterated eliminated of dominated strategies. Indeed, we start noting that the game is monotone supermodular. This means that extremal equilibria exist and are in monotone (threshold) strategies. Those extremal equilibria can be found starting at extremal points of the strategy sets of players ($\overline{t} = \infty$ and $\underline{t} = -\infty$) and iterating using best responses (Vives (1990)). The boundary assumptions on h guarantee that the process is not stuck at extremal points of strategy space, e.g., if $1 > h(\theta) > 0$ (or $\overline{\theta} = \infty$ and $\underline{\theta} = -\infty$) then both to act and not act coexist as equilibria no matter what signal realizations. The extremal equilibrium thresholds $\overline{t}$ and $\underline{t}$ bound set of rationalizable strategies and if the equilibrium is unique game is dominance solvable. The condition for equilibrium uniqueness is precisely that strategic complementarities do not be too strong and this holds when noise in the signals is small. It is in this situation when each player faces a lot of uncertainty about the aggregate action of the other players. This is exactly the same intuition as in the heterogenous population adoption externalities game in Section 2. A similar figure to Figure 2 would depict the situation here.

In the region where the equilibrium is unique (i.e., $\tau_\theta / \sqrt{\tau_\varepsilon}$ is small enough) we can obtain several useful results:

- When $\theta < \theta^*$ the acting mass of players succeeds. In the range $\left[\theta^*, \overline{\theta}\right)$ there is coordination failure from the point of view of players because if all them where to act then they would succeed.

[41] As γ ranges from 0 to 1, θ^* goes from $\overline{\theta} = 1$ to $\underline{\theta} = 0$.

- θ^* (and the probability that the acting mass succeeds) is decreasing in the relative cost of failure $\gamma \equiv C/B$ and in the expected value of the state of the world μ_θ.[42]

Comparative statics properties with respect to τ_θ and $\sqrt{\tau_\varepsilon}$ will depend on parameter configurations.

The approach is useful for policy analysis because it links the probability of occurrence of a "crisis" (successful mass action) at the unique equilibrium with the state of the world: $\Pr(\theta \leq \theta^*)$. This is in contrast with the complete information model where multiple self-fulfilling equilibria arise in the range $(\underline{\theta}, \overline{\theta})$. The theory builds a bridge therefore between the self-fulfilling theory of crisis (e.g., Diamond and Dybvig (1983)) and the theory that links crisis to the fundamentals (e.g, Gorton (1985, 1988)).[43] In the global game model there is both coordination failure and link between crisis and fundamentals.

Applications It is well known that multiple equilibria make comparative statics and policy analysis difficult. The uniqueness of equilibrium delivered by the global game approach comes to the rescue. We present here three applications that illustrate the power of the approach. Suppose in all of them that the uniqueness condition is fulfilled (i.e., $\tau_\theta/\sqrt{\tau_\varepsilon}$ is small enough).

The first application is a modified version of the currency attacks model of Morris and Shin (1998). This is an extremely streamlined model of currency attacks. Let θ be the reserves of the central bank (with $\theta \leq 0$ meaning that reserves are exhausted). There are a continuum of speculators and speculator i has one unit of resources to attack the currency ($a_i = 1$) at a cost C, and receives a signal about the level of resources of the central bank. Let $h(\theta) = \theta$ and the attack succeeds if $\widetilde{a} \geq \theta$. The (capital) gain if there is a depreciation is B (and it is fixed). The result is that the probability of a currency crisis is decreasing in C/B and in the expected value of the reserves of the central bank. In the region $[\theta^*, \overline{\theta})$ if speculators were to coordinate thier attack they would succeed but in fact the currency holds.

The second application is an instance of coordination failure in the interbank market proving a rationale for a Lender of Last Resort intervention (Rochet and Vives (2002)). Consider a market with three dates: $\tau = 0, 1, 2$. At date $\tau = 0$ the bank possesses own funds E and collects uninsured wholesale deposits (CDs for example) for some amount $D_0 \equiv 1$. These funds are used to finance some investment I in risky assets (loans), the rest being held

[42] This follows immediately because θ^* solves $\varphi(\theta) = \tau_\theta(\theta - \mu_\theta) - \sqrt{\tau_\epsilon}\Phi^{-1}(h(\theta)) - \sqrt{\tau_\theta + \tau_\epsilon}\Phi^{-1}(\gamma) = 0$, and $\varphi' < 0$ when $\tau_\theta/\sqrt{\tau_\epsilon}$ is small enough.

[43] An early model of incomplete information that bridges both approaches in the context of bank runs is Postlewaite and Vives (1987).

in cash reserves M. Under normal circumstances, the returns θI on these assets are collected at date $\tau = 2$, the CDs are repaid at their face value D, and the stockholders of the bank get the difference (when it is positive). However, early withdrawals may occur at an interim date $\tau = 1$, following the observation of private signals on the future realization of θ. If the proportion $\widetilde{a}$ of these withdrawals exceeds the cash reserves M of the bank, the bank is forced to sell some of its assets. A continuum of fund managers, make investment decisions in the interbank market. At $\tau = 1$ each fund manager, after having received a private signal about θ, decides whether to cancel ($a_i = 1$) or renew his CD ($a_i = 0$). Let $m \equiv M/D$ be the liquidity ratio, $\underline{\theta} \equiv \frac{D-M}{I}$ the solvency threshold of the bank, $\lambda > 0$ the fire sales premium of early sales of bank assets, and $\overline{\theta} \equiv (1+\lambda)\underline{\theta}$ the "supersolvency" point where a bank does not fail even if no fund manager renews his CDs. The bank fails if $\widetilde{a} \geq h(\theta)$, where $h(\theta) \equiv m + \frac{1-m}{\lambda}(\frac{\theta}{\underline{\theta}} - 1)$ for $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and $h(\theta) = 0$ for $\theta \leq \underline{\theta}$. A fund manager obtains $B > 0$ except if he renews the CDs and the bank fails, and the (reputation) cost of withdrawing is $C > 0$. The equilibrium failure threshold of the bank is $\theta^* \in \left[\underline{\theta}, \overline{\theta}\right]$ and in the range $[\underline{\theta}, \theta^*)$ the bank is solvent but illiquid. This provides a rationale for a LLR intervention with the discount window. Comparative statics results are also easily obtained. The critical θ^* (and probability of failure) is a decreasing function of the liquidity ratio m and the solvency (E/I) of the bank, of the critical withdrawal probability γ and of the expected return on the bank's assets μ_θ; and an increasing function of the fire sales premium λ and of the face value of debt D.

The third application is basically a reinterpretation of the second in terms of sovereign default and an international LLR. Suppose that θ is the unit return of the aggregate risky investment of a country which has foreign short-term debt with face value D and foreign reserves of M. At an interim period international fund managers may recall ($a_i = 1$) or roll over ($a_i = 0$) the foreign short-term debt of the country. The country defaults if $\widetilde{a} \geq h(\theta) = m + \frac{1-m}{\lambda}(\frac{\theta}{\theta_s} - 1)$, where $m \equiv M/D$ is the reserves ratio, $\underline{\theta} \equiv \frac{D-M}{I}$ the solvency threshold of the country, and λ the fire sales premium of early asset liquidation in the international market. The parameters B and C are as before. An interesting implication of the analysis, for example, is that the probability of default is decreasing in the reserves ratio.

Robustness Frankel, Morris and Pauzner (2003) obtain a generalization of the limit uniqueness result to games of strategic complementarities. They work in fact within the frame of monotone supermodular games (Section 6.1). The authors consider a Bayesian game (A_i, T_i, π_i) for $i \in N$, where A_i is a compact interval, $\pi_i(a_i, a_{-i}; \theta)$ is continuous and has increasing differences in $(a_i, (a_{-i}, \theta))$. The state θ is drawn from a continuous density with connected support and player i receives a private signal $s_i = \theta + \kappa\varepsilon_i$ with $\kappa > 0$ where ε_i is drawn from an atomless density with compact support (and the error terms are iid across players). The authors also assume that for extreme values of θ, extreme actions in A_i are strictly dominant (this is the equivalent of the assumption that $h(\cdot)$ crosses 0 and 1 at finite values) and the technical assumption that $\pi_i(a, \theta)$ has sensitivity to actions with a Lipschitz bound. The result is then that if θ is uniformly distributed or for κ tending to 0 there is a (essentially) unique Bayesian equilibrium in pure strategies (and it is increasing in type). However, the limit equilibrium may depend on the distribution of noise. Frankel, Morris and Pauzner give conditions for noise independent selection.

Extensions The framework can be extended:

- To include large players (see Corsetti, Dasgupta, Morris and Shin (2004) on currency attacks and Corsetti, Guimaraes and Roubini (2003) and Morris and Shin (2002) on the impact of the IMF as provider of "catalytic finance").
- To relax the strategic complementarity condition of actions to a single crossing condition and obtain a uniqueness result in switching strategies assuming that signals fulfill the monotone likelihood ratio property. However, then it cannot be guaranteed that there are no other equilibria in non-monotone strategies (see Athey (2001)); Goldstein and Pauzner (2003) apply a similar strategy to model bank runs when the depositor's game is not of strategic complementarities.
- To consider dynamic settings modeling, for example, contagion (Dasgupta (2003)) and dynamic speculative attacks (Chamley (2003)).

7 Concluding remarks

In the paper I have surveyed the theory and several applications of the lattice-theoretic approach in the study of complementarities in games. The survey has been by no means exhaustive. For example, no mention has been made of cooperative games (see Topkis (1998) for a survey). Furthermore, the method, as has been made clear in the text, can be applied fruitfully to comparative statics analysis and therefore is useful in basically all domains of economic theory: demand analysis, the theory of the firm and organizations, and dynamic optimization (see, for example, the applications in Milgrom and Roberts (1990b) and Milgrom and Shanon (1994)). The empirical analysis of complementarities based in the new methods is already taking off (see, e.g., Miravete and Pernías (2002) and Mohnen and Röller (2002)).

8 Appendix: Summary of lattice-theoretic methods

For the convenience of the reader I include a few definitions and results of lattice methods. More complete treatments can be found in Vives (1999, ch. 2) and Topkis (1998).

A binary relation $\geq$ on a nonempty set X is a *partial order* if $\geq$ is reflexive, transitive, and antisymmetric. An upper bound on a subset $A \subset X$ is $z \in X$ such that $z \geq x$ for all $x \in A$. A greatest element of A is an element of A that is also an upper bound on A. Lower bounds and least elements are defined analogously. The greatest and leat elments of A, when they exist, are denoted, repsectively, max A and min A. A supremum (resp., infimum) of A is a least upper bound (resp., greatest lower bound); it is denoted $\sup A$ (resp., $\inf A$).

A *lattice* is a partially ordered set $(X, \geq)$ in which any two elements have a supremum and an infimum. A lattice $(X, \geq)$ is *complete* if every non-empty subset has a supremum and an infimum. A subset L of the lattice X is a *sublattice* of X if the supremum and infimum of any two elements of L belong also to L.

Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. A function $f\colon X \to T$ is *increasing* if, for x, y in X, $x \geq y$ implies that $f(x) \geq f(y)$.

A function $g\colon X \to \mathbb{R}$ on a lattice X is *supermodular* if, all x, y in X, $g(\inf(x, y)) + g(\sup(x, y)) \geq g(x) + g(y)$. It is *strictly supermodular* if the inequality is strict for all pairs x, y in X that cannot be compared with respect to $\geq$ (i.e., neither $x \geq y$ nor $y \geq x$ holds). A function f is (*strictly*) *submodular* if $-f$ is (strictly) supermodular; a function f is (*strictly*) *log-supermodular* if $\log f$ is (strictly) supermodular.

Let X be a lattice and T a partially ordered set. The function $g\colon X \times T \to R$ has (*strictly*) *increasing differences* in (x, t) if $g(x', t) - g(x, t)$ is (strictly) increasing in t for $x' > x$ or, equivalently, if $g(x, t') - g(x, t)$ is (strictly) increasing in x for $t' > t$. Decreasing differences are defined analogously. If X is a convex subset of $\mathbb{R}^n$ and if $g\colon X \to R$ is twice-continuously differentiable, then g has increasing differences in (x_i, x_j) if and only if $\frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0$ for all x and $i \neq j$.

Supermodularity is a stronger property than increasing differences: If T is also a lattice and if g is (strictly) supermodular on $X \times T$, then g has (strictly) increasing differences in (x, t). The two concepts coincide on the product of linearly ordered sets: If X is such a lattice, then a function $g\colon X \to \mathbb{R}$ is supermodular if and only if it has increasing differences in any pair of variables.

The main comparative-statics tool for our purposes is the following.

Lemma 1 *Let X be a compact lattice and let T be a partially ordered set. Let $u\colon X \times T \to \mathbb{R}$ be a function that (a) is supermodular and continuous on the lattice X for each $t \in T$ and (b) has increasing differences in (x, t). Let $\varphi(t) = \arg\max_{x \in X} u(x, t)$. Then:*

1. *$\varphi(t)$ is a non-empty compact sublattice for all t;*
2. *φ is increasing in the sense that, for $t' > t$, for $x' \in \varphi(t')$ and $x \in \varphi(t)$, we have $\sup(x', x) \in \varphi(t')$ and $\inf(x', x) \in \varphi(t)$; and*
3. *$t \mapsto \max \phi(t)$ and $t \mapsto \min \phi(t)$ are well-defined increasing functions.*

Remark 2 *If u has strictly increasing differences in (x, t), then all selections of φ are increasing.*

Remark 3 *If $X \subset \mathbb{R}^m$, solutions are interior, and $\partial u / \partial x_i$ is strictly increasing in t for some i then all selections of φ are strictly increasing (Edlin and Shannon (1998)).*

The basic fixed point theorem in the lattice-theoretic approach is Tarski (1955).

Theorem 4 *(Tarski (1955)): Let A be a complete lattice (e.g., compact cube in $\mathbb{R}^m$). Then an increasing function $f : A \to A$ has a largest $\sup\{a \in A : f(a) \geq a\}$ and a smallest $\inf\{a \in A : a \geq f(a)\}$ fixed point.*

Supermodular game The game $(A_i, \pi_i; i \in N)$ is *supermodular* if for all i

- A_i is a compact lattice;
- $\pi_i(a_i, a_{-i})$ is continuous,
  1. supermodular in a_i, and
  2. has increasing differences in (a_i, a_{-i}).

Game of strategic complementarities Given a set of players N, strategy spaces A_i, and (non-empty) best reply maps Ψ_i, $i = 1, ..., n$, define a game of strategic complementarities (GSC) as one in which for each i, A_i is a complete lattice and Ψ_i is increasing and has well-defined extremal elements.

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