Monotone Equilibria in Bayesian Games of Strategic Complementarities*

Timothy Van Zandt and Xavier Vives
INSEAD

14 August 2003
Revised: 10 June 2004

For Bayesian games of strategic complementarities, we provide a constructive proof of the existence of a greatest and a least Bayes–Nash equilibrium, each one in strategies that are monotone in type. Our main assumptions, besides strategic complementarities, are that each player's payoff displays increasing differences in own action and the profile of types and that each player's interim beliefs are increasing in type with respect to first-order stochastic dominance (e.g., types are affiliated). The result holds for general action and type spaces (single-, multi-, or infinite-dimensional; continuous or discrete) and no prior is assumed. We also provide the following comparative statics result: the greatest and least equilibria are higher if there is a first-order stochastic dominant shift in the interim beliefs. We apply this result to strategic information revelation in games of voluntary disclosure.

KEYWORDS: Supermodular games, incomplete information, first-order stochastic dominance, Cournot tatônnement, monotone comparative statics, voluntary disclosure, local network effects.

AUTHORS' ADDRESSES:

Timothy Van Zandt
INSEAD
Boulevard de Constance
77305 Fontainebleau CEDEX
France

Email: tvz@econ.insead.edu
Web: faculty.insead.edu/vanzandt

Xavier Vives
INSEAD
Boulevard de Constance
77305 Fontainebleau CEDEX
France

Email: xavier.vives@insead.edu
Web: faculty.insead.edu/vives

*The authors thank Phil Reny for helpful input.

1. Introduction

This paper studies supermodular games of incomplete information in which (a) actions are strategic complements, (b) there is complementarity between actions and types, and (c) interim beliefs are increasing in type with respect to first-order stochastic dominance. We use lattice-theoretic methods to establish (i) existence of a greatest and a least pure-strategy Bayes–Nash equilibrium, each in strategies that are monotone in type, and (ii) that a first-order stochastic dominant shift in the interim beliefs causes these extremal equilibria to increase. These results hold for a general class of supermodular games, in which action and type spaces may be multidimensional and discrete or continuous, which we call "monotone supermodular".

Existence and characterization results of pure-strategy equilibria in Bayesian games include the following. Milgrom and Weber (1985) (see also Aumann et al. 1982; Radner and Rosenthal 1982) find stringent sufficient conditions for existence, such as conditionally independent and atomless distributions for types and finite action spaces; they use atomless distributions for purification. Vives (1990, Section 6) establishes existence with general action and type spaces when payoffs are supermodular in actions; he uses a lattice fixed-point theorem.

Athey (2001) shows existence of equilibria in monotone strategies when there are (a) either supermodular or log-supermodular cardinal utilities, (b) complementarity between actions and types, (c) one-dimensional action sets, and (d) one-dimensional atomless type spaces. McAdams (2003) presents an extension to multidimensional discrete action and atomless type spaces. Their proofs work for games in which players have monotone best responses to monotone strategies. The primary examples of such games are those with affiliated types and either supermodular or log-supermodular payoffs. By assuming atomless type spaces and finite action sets, they can represent monotone strategies by the cutoff values at which types switch from each action to the next highest action and then use a topological fixed-point theorem in this set of strategies. An extension to infinite action spaces is obtained as the limit of equilibria for finite approximations. The assumption of atomless types cannot be relaxed: the method of proof relies on it and there is an implicit purification without which pure-strategy equilibria may not exist when payoffs are log-supermodular.

Whereas supermodularity is preserved by taking expectations given incomplete information, log supermodularity and other ordinal single-crossing conditions are not. In our paper, by restricting attention to supermodular payoffs, we can exploit the full strength of strategic complementarities and thereby use a completely different and simpler proof to obtain stronger and otherwise more general results. Like Athey and McAdams, we need that players have monotone best responses to monotone strategies. Then Cournot tatônnement starting at the greatest or least strategy profile—as used in Vives (1990) to construct the greatest and smallest equilibria in Bayesian games with strategic complementarities—starts in monotone strategies and remains such, so that the limit equilibrium must also be in monotone strategies.

This method of proof has several advantages.

Simplicity. The simplicity makes it clearer why such monotone equilibria exist and facilitates further extensions.

Generalizations. A single proof works for multidimensional (even infinite dimensional) actions and types and for both discrete and continuous actions and types. Furthermore, we weaken the assumption that types are affiliated to the intuitive assumption that interim beliefs are increasing in type with respect to first-order stochastic dominance.[1] We do not need to assume that beliefs are derived from a prior; and our notion of Bayes–Nash equilibrium is interim, with everywhere best responses rather than almost-everywhere best responses.

Construction. The proof is constructive and can be used as an iterative numerical

1. We state in Appendix B some general comparative statics for first-order stochastic dominance.

method for computing the equilibria.

Bounds. Although neither method rules out the existence of nonmonotone equilibria, the two extremal monotone equilibria that we identify bound all equilibria and, as demonstrated in Milgrom and Roberts (1990), also bound the set of rationalizable strategy profiles and the set of strategy profiles that can be reached by a wide range of adaptive learning.

Comparative statics. We show that the extremal equilibria are increasing in the interim beliefs. That is, if we perturb the game such that, for each player and each type, there is a first-order stochastic dominant shift in the player's interim beliefs about the other players, then the greatest and least equilibrium strategies increase for each player and each type.[2]

The "monotone supermodular" class of games for which our results hold is broad. Besides the supermodular games mentioned in Vives (1990) and Milgrom and Roberts (1990), the following are all examples of games covered by our analysis and in which monotonicity of equilibria may be of interest: (a) many industrial organization games, such as firms with perhaps multiple differentiated products (for suitable restrictions on demand) who choose prices and advertising levels and have incomplete information about market conditions and each other's cost (as in Example 3 in Section 3); (b) various macroeconomic models in which investment and production decisions have complementarities (as in Cooper and John 1988; Murphy, Schleifer, and Vishny 1989); (c) most "global games", including multidimensional extensions of the games in Morris and Shin (2002) and Frankel, Morris, and Pauzner (2003); (d) many adoption games played by consumers when choosing among products with network externalities (see Example 1 in Section 3, which features local network effects and incomplete information about the network); (e) partnership games and multiagent principal–agent models when

2. The method of proof is related to, but not an application of, the comparative statics results of Milgrom and Roberts (1990).

the investments or effort levels are complements; and (f) many team-theory models as in Radner (1962) (such a model can be viewed as a game of incomplete information in which the optimal team solution is a Bayes–Nash equilibrium of the game).

One application of our comparative statics results is to the solution of multistage games through backward induction, where the beliefs in one stage are determined endogenously in an earlier stage. We can use the comparative statics to characterize the players' incentives in the earlier stage to influence beliefs. For example, suppose that the second-stage of a two-stage game satisfies the assumptions of this paper and that the actions have positive externalities, meaning that each player's payoff is increasing in the actions of the other players. An upward shift in second-stage beliefs shifts the equilibrium actions up and therefore benefits each player. Hence, each agent would like to shift the beliefs of other players upward. This conclusion might help us find separating monotone equilibria in generalized signaling games (e.g., with multidimensional actions and types and in which multiple players may choose actions in each stage). The application we develop is instead to strategic revelation of information in voluntary disclosure games, as a generalization of a result in Okuno-Fujiwara, Postlewaite, and Suzumura (1990) (which in turn generalized papers such as Grossman (1981) and Matthews and Postlewaite (1985)).

The plan of the paper is as follows. In Section 2 we set up the Bayesian game and state basic maintained assumptions. Section 3 summarizes the main results and outlines three applications. Section 4 shows how (under certain assumptions) Cournot tatônnement, starting at the greatest strategy profile and using the greatest best reply mappings, converges to the greatest Bayes–Nash equilibrium, which is in strategies that are monotone in type. Section 5 shows existence and monotonicity of the greatest best reply mapping. Section 6 builds on intermediate results about comparative statics under uncertainty and shows that the greatest best reply to monotone strategies is monotone. A strict version of this result is obtained in Section 7. The pieces are then in place to state the main existence result in Section 8. Section 9 provides an example illustrating that our approach cannot work for log-supermodular payoffs. Section 10 shows that the extremal equilibria are increasing in the interim beliefs; we give an application to games of voluntary disclosure in Section 11. We leave additional (more technical) discussion of related literature to Section 12, and then conclude in Section 13. Appendix A provides, for completeness, some basic lattice-theoretic definitions; Appendix B presents our results on comparative statics under uncertainty; and Appendix C compares affiliation and our weaker increasing beliefs condition.

2. The Bayesian game

We defer until Section 2.5 certain technical restrictions required for infinite actions sets or type spaces.

2.1. Interim formulation

We use an interim formulation of a Bayesian game and Bayes–Nash equilibrium, based on interim beliefs and interim best replies, rather than on a common prior and ex ante best replies. The interim formulation is stronger and, for the most part, more general. However, we eschew a common prior not for the sake of generality but rather because it would play no role in our analysis. When we state conditions on a common prior that would be sufficient for our assumptions, we denote the common prior by μ.

2.2. Components of a game

1. The set of players is $N = \{1, \ldots, n\}$, indexed by i.
2. The type space of player $i \in N$ is a measurable space $(T_i, \mathscr{F}_i)$. There is also a state space $(T_0, \mathscr{F}_0)$ capturing residual uncertainty not observed by any player.[3]

3. Allowing for such a state space does not add generality but it is convenient for certain applications.

(Let $T := T_0 \times T_1 \times \cdots \times T_n$; let $T_{-i} := \prod_{k \neq i} T_k$; let $\mathscr{F}$ be the overall product sigma-algebra; and let $\mathscr{F}_{-i}$ be the product sigma-algebra $\bigotimes_{k \neq i} \mathscr{F}_k$.)

3. Player i's interim beliefs are given by a function $p_i \colon T_i \to \mathscr{M}_{-i}$, where $\mathscr{M}_{-i}$ is the set of probability measures on $(T_{-i}, \mathscr{F}_{-i})$.

   (Because a probability measure is itself a function, we will denote the probability measure $p_i(t_i)$ by $p_i(\cdot \mid t_i)$; however, $p_i(\cdot \mid t_i)$ is not necessarily a conditional probability, since it is not necessarily derived from a prior on T.)

4. The action set of player i is A_i. The set of action profiles is $A = \prod_{i \in N} A_i$.

   (Let $A_{-i} = \prod_{j \neq i} A_j$.)

5. The payoff function of player i is $u_i \colon A \times T \to \mathbb{R}$.

Type spaces and action sets are nonempty.

Our formulation of a Bayesian game is general and encompasses common and private values as well as perfect or imperfect signals. We have pure private values if u_i does not depend on t_{-i} (but types may be correlated). For example, types are private cost parameters of the firms. In a common-values model, each u_i might depend on t through a common aggregate statistic such as $t_1 + \cdots + t_n$, as when there is a common demand shock in an oligopoly and each firm observes one component. As an example of imperfect signals, suppose firms imperfectly observe their cost parameters. Then t_0 could represent the n-vector of firms' cost parameters and t_i the private cost estimate of firm i. Not only the cost parameters may be correlated, so may the error terms in the private signals.[4]

2.3. Bayes–Nash equilibrium

A strategy for player i is a measurable function $\sigma_i \colon T_i \to A_i$. Let Σ_i denote the set of strategies for player i. Let $\Sigma = \prod_{i=1}^{n} \Sigma_i$ denote the set of strategy profiles and let $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$ denote the profiles of strategies for players other than i. For notational

4. See Vives (1999, Section 8.1.2) for parameterized examples of the cases discussed.

simplicity, a strategy profile is viewed as a map from T to A, even though it does not depend on t_0.

A Bayes–Nash equilibrium is a strategy profile σ such that each player and each type chooses a best response to the strategy profile of the other players. For future use, we disentangle how player i's payoff depends on her type and beliefs.

Given that player i's type is t_i, her interim beliefs are P_{-i}, and the strategy profile of the other players is σ_{-i}, her expected payoff from choosing action a_i is

$$V_i(a_i, t_i, P_{-i}; \sigma_{-i}) = \int_{T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i})\, dP_{-i}(t_{-i}). \tag{1}$$

Let $\varphi_i(t_i, P_{-i}; \sigma_{-i})$ be the set of actions for i that maximize this payoff:

$$\varphi_i(t_i, P_{-i}; \sigma_{-i}) = \arg\max_{a_i \in A_i} V_i(a_i, t_i, P_{-i}; \sigma_{-i}). \tag{2}$$

Then $\sigma \in \Sigma$ is a Bayes–Nash equilibrium if and only if, for $i \in N$ and $t_i \in T_i$, $\sigma_i(t_i) \in \varphi_i(t_i, p_i(t_i); \sigma_{-i})$.

Let $\beta_i \colon \Sigma_{-i} \to \Sigma_i$ denote player i's best-reply correspondence in terms of strategies:

$$\beta_i(\sigma_{-i}) = \{\sigma_i \in \Sigma_i \mid \sigma_i(t_i) \in \varphi_i(t_i, p_i(t_i); \sigma_{-i})\ \forall t_i \in T_i\}. \tag{3}$$

Then a Bayes–Nash equilibrium is a strategy profile σ such that $\sigma_i \in \beta_i(\sigma_{-i})$ for $i \in N$.

2.4. Order structure

Moving away from this canonical presentation of Bayesian games, we add some order structure to actions and types. We use the symbol $\geq$ for all partial orders. Expressions such as "greater than" and "increasing" mean "*weakly* greater than" and "*weakly* increasing". See Appendix A for lattice-theoretic definitions.

1. For each player i, A_i is a complete lattice.
2. For each $k = 0, 1, \ldots, n$, T_k is endowed with a partial order.

For each player, the set of strategies is also a lattice for the ordering "$\sigma_i \geq \sigma_i'$ if and only if $\sigma_i(t_i) \geq \sigma_i'(t_i)$ for all $t_i \in T_i$". We say that a strategy $\sigma_i \in \Sigma_i$ is *monotone* if, for all t_i, t_i' such that $t_i \geq t_i'$, we have $\sigma_i(t_i) \geq \sigma_i(t_i')$.

2.5. Continuity and measurability assumptions

We impose the following continuity and measurability restrictions, which are needed in case T or A is not finite. A simple case that satisfies these restrictions is where (a) T_i is a measurable subset of Euclidean space; (b) A_i is a compact sublattice of Euclidean space (e.g., the product of compact subsets of $\mathbb{R}$, in which case supermodularity in a_i is equivalent to increasing differences in any two components of i's action); (c) u_i is continuous in a, measurable in t, and bounded; and (d) interim beliefs for each player are derived from a prior.[5] However, T_i could be, for example, a universal type space as constructed in Mertens and Zamir (1985) (for the restriction to supermodular utilities) and A_i could be, for example, an order interval in an l^p space, for $1 \leq p < \infty$, with its norm topology.

Assumption 1.

1. For $i \in N$ and $F_{-i} \in \mathscr{F}_{-i}$, the function $t_i \mapsto p_i(F_{-i} \mid t_i)$ is measurable.
2. For $i \in N, A_i$ is a compact metric space. Furthermore: (a) any increasing or decreasing sequence in A_i converges to its supremum or infimum; (b) any order interval in A_i is closed; and (c) the lattice operations sup and inf are continuous.
3. For $i \in N$, player i's payoff function u_i has the following properties: (a) for all $a \in A$, $u_i(a, \cdot)\colon T \to \mathbb{R}$ is measurable; (b) for all $t \in T$, $u_i(\cdot, t)\colon A \to \mathbb{R}$ is continuous; and (c) u_i is bounded.

5. For the existence of such interim beliefs given a prior, see Dellacherie and Meyer (1978, III.70 and 71).

3. Summary of the main results and some examples

Our main results are Theorem 1 and Proposition 3, which we now summarize. There are also strict versions of these results, stated in Corollaries 2 and 3.

MAIN RESULTS. *Assume, for each player i, that*

1. *the utility function u_i is supermodular in a_i, has increasing differences in (a_i, a_{-i}), and has increasing differences in (a_i, t); and*
2. *the beliefs mapping $p_i \colon T_i \to \mathscr{M}_{-i}$ is increasing with respect to the partial order on $\mathscr{M}_{-i}$ of first-order stochastic dominance (e.g., μ is affiliated).*

Then there exist a greatest and a least Bayes–Nash equilibrium, and each one is in monotone strategies.

Furthermore, if we perturb the beliefs in the game such that each player's interim beliefs, for each type, shift up with respect to first-order stochastic dominance, then the greatest and least Bayes–Nash equilibria increase.

We now provide some examples that illustrate the applicability of our results as well as some differences with preceding ones. In particular, our first example has inherently discrete type spaces and hence is not covered by the work of Athey (2001) and McAdams (2003).

3.1. Example 1: An adoption game with local network effects

Sundarajan (2004) presents a model of network externalities on a graph. Players choose between buying a good ($a_i = 1$) or not ($a_i = 0$). (The extension to multiple complementary goods or to general quantities of demand is straightforward.) Consumption of the good has a network externality but only for neighbors in the graph. Players have only local knowledge of the network.

The details of the model are loosely as follows. We first describe the complete-information version of the game. Players are connected on an undirected graph, represented by the sets $(G_i)_{i \in N}$ of neighbors that the players have. Player i's payoff is $\pi_i(a_i, a_{-i}, G_i, \theta_i)$, where θ_i is a valuation parameter that will come into play in the incomplete-information game. This payoff is 0 if $a_i = 0$ and otherwise is the valuation of the good minus its price p. The player gets a network externality from each neighbor who also consumes the good, as follows. Her valuation of the good is $w_i((b_j)_{j \neq i}, \theta)$, where $b_j = a_j$ if $j \in G_i$ and $b_j = 0$ otherwise. The function w_i is increasing in b_j for each $j \neq i$; this is enough to guarantee that π has increasing differences in (a_i, a_j) for any $j \neq i$. (Otherwise the form of w_i can be general; for example, the network effects can vary across j and the marginal effect of one neighbor's consuming the good can diminish the more there are other neighbors who also consume the good.) By the standard theory of games with strategic complementarities, there is a greatest and least pure-strategy equilibrium in spite of the potential asymmetries in the game.

The incomplete-information version of the game captures the idea that players have only local knowledge about the structure of the network: (a) the graph is drawn randomly with a known distribution ρ on the set of possible graphs; and (b) each player observes only who her neighbors are. Her valuation parameter θ_i is also private information, so her type is $t_i = (G_i, \theta_i)$. We let $\Gamma_i = 2^{N \setminus \{i\}}$ be the set of possible neighborhoods for player i and let $\Gamma \subset \Gamma_1 \times \cdots \times \Gamma_n$ be the set of possible graphs. The partial order on Γ_i is that of set inclusion. Let Θ_i be the set of possible valuation parameters, which can be any measurable subset of Euclidean space. Then player i's type space is $\Gamma_i \times \Theta_i$.

Higher θ_i means higher valuation: w_i is increasing in θ_i for any $(b_j)_{j \neq i}$. Then π_i has increasing differences in (a_i, θ_i). Since w_i is increasing in b_j, π_i also has increasing differences in (a_i, G_i). The payoff π_i does not depend on θ_j or G_j for $j \neq i$.

To apply our results, we need only check that the increasing beliefs condition is satisfied. In particular, we need the distribution of the neighborhood sets to have the following property: if $G_i' \subset G_i''$ then, for any $\{G_j \in \Gamma_j \mid j \neq i\}$, the probabilities that all players $j \neq i$ have neighborhoods that include at least G_j should be weakly higher conditional on G_i'' compared to conditional on G_i'. This is a natural assumption; if one player has more neighbors, she would conclude that the network is probably more connected and hence that other players are also more likely to have more neighbors. It is satisfied, for example, for a Poisson random graph, in which the existence of an edge between any pair of agents is independent of the existence of other edges (for example, ρ is the uniform distribution on Γ). If the distribution over graphs is symmetric with respect to the players, as in Sundarajan (2004), then it is equivalent to assuming that the player's interim beliefs about other players' degrees (number of neighbors) is weakly increasing in her own degree with respect to first-order stochastic dominance.

Observe the following about the components Γ_i of the type spaces. First, types are inherently correlated because each player, by learning who his neighbors are, learns something about who the other players' neighbors are. Furthermore, these components are discrete and multidimensional (there is no natural linear order). Because of the discreteness, this game is not covered by Athey (2001) or McAdams (2003). Furthermore, the increasing beliefs condition is easier to check than affiliation.

From our main results, we may conclude as follows.

1. The game has a greatest and a least pure-strategy equilibrium, which are increasing in type: if player i purchases the good when of type (G_i, θ_i), then he would also purchase the good if he had additional neighbors or a higher valuation parameter.
2. If the network becomes more dense—in the sense that, for each pair of players, the probability that they are connected increases—then greatest and least equilibria of the altered game are higher than those of the original game. That is, the players and types who consume the good in the equilibrium before the shift also do so after the shift.
3. This game has positive externalities, meaning that each player's payoff is increas-

ing in the actions of the other players. Therefore, the greatest equilibrium Pareto dominates all other equilibria. Furthermore, if we have an equilibrium selection of either the greatest or the least equilibrium, then each player's interim payoff would increase as a consequence of the shift in the distribution of networks described in the previous item.

4. If the game is symmetric (requiring, for example, that the probability of any graph does not change if the names on the nodes of the graph are permuted) then, as is known for supermodular games, the greatest and least equilibria are symmetric. This implies, in this game, that each player's consumption decision depends on the number of neighbors and not on their identities. (We can restrict attention to such equilibria by assuming that the players observe the number of neighbors but not their identities.) In the greatest and least equilibria, the equilibrium strategies are increasing in the number of neighbors. That is, for each player and valuation parameter for that player, there is a cutoff number of neighbors above which the player adopts the product and below which she does not.

3.2. Example 2: Global games

Global games are games of incomplete information in which there is an underlying payoff-relevant state and each player observes a noisy signal of this state. The aim is equilibrium selection via perturbation of a complete-information game.

Carlsson and van Damme (1993) show the following result. In 2×2 games, if each player observes a noisy signal of the true payoffs and if ex ante feasible payoffs include payoffs that make each action strictly dominant then, as noise becomes small, iterative strict dominance selects one equilibrium. When there are two equilibria in the complete-information game—in which case the game is supermodular—the equilibrium selected is the Harsanyi and Selten (1988) risk-dominant one.

The extension by Frankel, Morris, and Pauzner (2003) to an arbitrary number of players and one-dimensional actions and types considers only games that satisfy our assumption that payoffs are continuous and have increasing differences in $(a_i, (a_{-i}, t))$. It is a common values model, in the sense that payoffs depend only on the one-dimensional component t_0 and each t_i is just a noisy signal $t_i = t_0 + \eta_i$ of t_0; the random variables t_0 and $\{\eta_i\}$ are independent and have continuous densities. Types satisfy our increasing beliefs condition because t_0 has a very diffuse distribution, nearly uniform on $\mathbb{R}$, and the support of each η is very small. (In the limit, the posterior on η_i is independent of the realization of t_i and the distribution of t_0 conditional on $t_i = \hat{t}_i$ is equal to the distribution of $\hat{t}_i - \eta_i$.) Therefore, the game is monotone supermodular.

A key step in the proof is to identify greatest and least strategy profiles that survive iterative deletion of strictly dominated strategies and to show that these are monotone in type. The other, more intricate, step is to show that, under additional assumptions and for a certain limit of the game, the two extremal equilibria are the same—and hence that the game is dominance solvable and has a unique equilibrium.

Since we have a game of incomplete information with supermodular payoffs, the results of this paper immediately yield the first key step. They also allow an extension of this step to games with multidimensional actions and types, with perhaps discreteness of some dimensions of the type spaces (as long as the increasing beliefs condition is still satisfied).

3.3. Example 3: Bertrand competition in prices and advertising

Consider the following Bertrand oligopoly with differentiated products in which firms compete both in price p_i and in advertising intensity z_i. The profit of firm i is given by $u_i = (p_i - c_i) D_i(p_i, p_{-i}, z_i) - F_i(z_i, e_i)$, where c_i is the per-unit cost, D_i yields the demand for the product of the firm, F_i yields the cost of advertising, and e_i measures the cost efficiency of advertising. The type of firm i is $t_i = (c_i, e_i)$ and its action is $a_i = (p_i, z_i)$.

We assume that D_i is decreasing in p_i, increasing in p_{-i}, and increasing in z_i; $F(z_i, e_i)$

is increasing in z_i and decreasing in e_i. Then u_i is supermodular in a_i if, for p_i^H and p_i^L such that $p_i^H > p_i^L$, $D_i(p_i^L, p_{-i}, z_i) - D_i(p_i^H, p_{-i}, z_i)$ is decreasing in z_i; this means, for example, that advertising increases demand by raising the valuations of existing consumers rather than by informing new consumers of the existence of the good. Observe that u_i has increasing differences in (a_i, a_{-i}) as long as, for $p_i^H > p_i^L$ and $z_i^H > z_i^L$, $D_i(p_i^L, p_{-i}, z_i) - D_i(p_i^H, p_{-i}, z_i)$ is decreasing in p_{-i} and $D_i(p_i, p_{-i}, z_i^H) - D_i(p_i, p_{-i}, z_i^L)$ is increasing in p_{-i}. All these conditions are satisfied, for example, when D_i is linear in all its terms. Also, u_i has increasing differences in a_i and t_i if F has decreasing differences in (z_i, e_i)—for instance, if higher e_i decreases the marginal cost of advertising. Interim beliefs are increasing in type if the joint distribution of types is affiliated.

Supposing also that there are natural upper bounds for p_i and z_i (e.g., that there is a choke-off price for demand as well as a point beyond which advertising has no further effect) and assuming continuity of D_i and F_i (but we still allow any variable to be discrete), we can apply our main results. It then follows that there exist extremal equilibria and that these are monotone in types; in other words, higher production cost or higher advertising efficiency induces higher prices and more advertising by firm i.

4. Cournot tatônnement and the greatest equilibrium

Our main results assume that the payoff of player i is supermodular in a_i and has increasing differences in (a_i, a_{-i}). It then follows immediately from Vives (1990, Theorem 6.1) or Milgrom and Roberts (1990, Theorem 5) that the game has extremal equilibria in pure strategies. However, we want to show that these extremal equilibria are in monotone strategies under additional assumptions.

The main idea is that Cournot tatônnement, starting at the greatest strategy profile and using the greatest best-reply (GBR) mappings, converges to the greatest Bayes–Nash equilibrium, which is in strategies that are monotone. We first state this result in terms of assumptions on the GBR mapping (in Lemma 1) and then derive the assumptions from more primitive ones (in Sections 5 and 6). An analogous result, which we do not bother stating, holds for the *least* best-reply mapping and the *least* Bayes–Nash equilibrium.

Definition 1. If $\beta_i(\sigma_{-i})$ has a greatest element, denote it by $\bar{\beta}_i(\sigma_{-i})$. If $\bar{\beta}_i(\sigma_{-i})$ is well-defined for all $\sigma_{-i} \in \Sigma_{-i}$, then we call $\bar{\beta}_i \colon \Sigma_{-i} \to \Sigma_i$ player i's *greatest best-reply* (GBR) mapping.

Lemma 1. *Assume the following for each player i.*

1. *The GBR mapping $\bar{\beta}_i$ is well-defined.*
2. *The GBR mapping is increasing: for $\sigma'_{-i}, \sigma_{-i} \in \Sigma_{-i}$ such that $\sigma'_{-i} \geq \sigma_{-i}$, $\bar{\beta}_i(\sigma'_{-i}) \geq \bar{\beta}_i(\sigma_{-i})$.*
3. *If the strategies σ_{-i} are monotone, then the strategy $\bar{\beta}(\sigma_{-i})$ is monotone.*

Then there is a greatest equilibrium and it is in monotone strategies.

Proof. The proof is constructive, using Cournot tatônnement. It is quite similar to the proof of Theorem 6.1 in Vives (1990), but with a few modifications because here we work with interim beliefs and have more general assumptions on types and actions. Also, we need to keep track of the monotonicity of strategies.

Define $\bar{\beta} \colon \Sigma \to \Sigma$ by $\bar{\beta}(\sigma) = (\bar{\beta}_1(\sigma_{-1}), \ldots, \bar{\beta}_n(\sigma_{-n}))$. Since each $\bar{\beta}_i$ is increasing, so is $\bar{\beta}$. By the third assumption, if σ is a profile of monotone strategies then so is $\bar{\beta}(\sigma)$.

For each player i, let $\bar{a}_i \in A_i$ be the greatest element of A_i (which exists because A_i is a complete lattice). Let $\sigma_i^0 \in \Sigma_i$ be the strategy that is equal to $\bar{a}_i$ for all t_i and let $\sigma^0 = (\sigma_1^0, \ldots, \sigma_n^0)$. Define recursively $\sigma^k = \bar{\beta}(\sigma^{k-1})$ for $k = 1, 2, \ldots$. Because σ^0 is the profile of greatest strategies, we have $\sigma^1 \leq \sigma^0$. Since $\bar{\beta}$ is increasing and since $\sigma^2 = \bar{\beta}(\sigma^1)$ and $\sigma^1 = \bar{\beta}(\sigma^0)$, we have $\sigma^2 \leq \sigma^1$. By induction, the sequence $\{\sigma^k\}$ is decreasing. Thus, for each player i and for all $t_i \in T_i$, $\{\sigma_i^k(t_i)\}$ is a decreasing sequence. Since every decreasing sequence in A_i converges to its infimum, it follows that σ_i^k converges pointwise (type-by-type). Denote the pointwise limit by σ_i^∞.

The pointwise limit of a sequence of measurable functions into a metric space is measurable. Hence, σ_i^∞ is in Σ_i. The limit must be an equilibrium (by a standard continuity argument) once we note that, for all $t_i \in T_i$,

$$\lim_{k\to\infty} \int_{T_{-i}} u_i(\sigma_i^k(t_i), \sigma_{-i}^{k-1}(t_{-i}), t_i, t_{-i})\, dp_i(t_{-i} \mid t_i) \\ = \int_{T_{-i}} u_i(\sigma_i^\infty(t_i), \sigma_{-i}^\infty(t_{-i}), t_i, t_{-i})\, dp_i(t_{-i} \mid t_i)$$

(and similarly when $\sigma_i^k(t_i)$ and $\sigma_i^\infty(t_i)$ are replaced by any $a_i \in A_i$) because u_i is continuous in a, σ_{-i}^k converges pointwise, and u_i is bounded (hence we are integrating a bounded function on T_{-i} that converges pointwise).

Furthermore, each term in the sequence $\{\sigma^k\}$ is in monotone strategies because (a) σ^0 is a profile of monotone strategies and (b) so is $\bar{\beta}(\sigma^k)$ if σ^k is such a profile. The pointwise limit of a decreasing sequence of monotone strategies is also monotone because, if $\{x^k\}$ and $\{y^k\}$ are decreasing sequences in a complete lattice and if $x^k \le y^k$ for all k, then $\inf(\{x^k\}) \le \inf(\{y^k\})$.

The limit σ^∞ must be the greatest equilibrium, as we now show. Any other equilibrium σ will be smaller than the greatest strategy profile σ^0, that is, $\sigma^0 \ge \sigma$. Since $\bar{\beta}$ is increasing, we have $\bar{\beta}(\sigma^0) \ge \bar{\beta}(\sigma)$. On the one hand, σ is a profile of best responses to σ because σ is an equilibrium; on the other, $\bar{\beta}(\sigma)$ is the greatest best response to σ. Therefore, $\bar{\beta}(\sigma) \ge \sigma$. Combining $\sigma^1 = \bar{\beta}(\sigma^0)$, $\bar{\beta}(\sigma^0) \ge \bar{\beta}(\sigma)$, and $\bar{\beta}(\sigma) \ge \sigma$ yields $\sigma^1 \ge \sigma$. Continuing by induction, $\sigma^k \ge \sigma$ for all k and hence $\sigma^\infty \ge \sigma$. □

5. Existence and complementarity of the greatest best reply

The assumptions needed for the existence of the greatest best reply for each type and the monotonicity of this best reply in the strategies of the other players follow from the standard theory of supermodular optimization, as summarized in Lemma A.1. We also need to show that this type-by-type greatest best reply is measurable.

PROPOSITION 1. *Assume for player i that, for all $t \in T$, $u_i(\cdot, t)$ is supermodular in a_i and has increasing differences in (a_i, a_{-i}). Then, for all $\sigma_{-i} \in \Sigma_{-i}$, $\beta_i(\sigma_{-i})$ contains a greatest element; that is, $\bar{\beta}_i(\sigma_{-i})$ is well-defined. Furthermore, $\bar{\beta}_i$ is an increasing function of σ_{-i}.*

Proof. Continuity, supermodularity, and increasing differences are preserved by integration. Therefore, for all $t_i \in T_i$, $V_i(\cdot, t_i, p_i(t_i); \sigma_{-i})$ is continuous and supermodular in a_i and has increasing differences in (a_i, σ_{-i}). It now follows from Lemma A.1 that $\varphi_i(t_i, p_i(t_i); \sigma_{-i})$ is a nonempty complete lattice and that $\sup \varphi_i(\cdot)$ belongs to $\varphi_i(\cdot)$ and is increasing in σ_{-i}.

The only remaining detail is that the mapping $t_i \mapsto \sup \varphi_i(t_i, p_i(t_i); \sigma_{-i})$ should be measurable so that it belongs to Σ_i. This is shown in Van Zandt (2004). □

6. The greatest best reply to monotone strategies is monotone

We now show that the greatest best reply to monotone strategies is monotone if, in addition to the assumptions of Proposition 1, u_i has increasing differences in (a_i, t) and a "monotone beliefs" condition is satisfied. We shall apply the results from Appendix B on monotone comparative statics under uncertainty.

We endow $\mathscr{M}_{-i}$ with the partial order of first-order stochastic dominance and assume that i's beliefs function $p_i : T_i \to \mathscr{M}_{-i}$ is increasing. That is, higher types of i believe that the other players are more likely to be of higher types as well. If there is a common prior μ, then this assumption is implied by—but is weaker than—the more familiar assumption that μ is affiliated (see Appendix C).

A higher type for i affects i's action through three interactions, all of which we must control for.

1. u_i depends on t_i. Hence, we assume that u_i has increasing differences in (a_i, t_i).
2. u_i depends on t_{-i} and t_i affects i's beliefs about t_{-i}. Hence, we assume that u_i has increasing differences in (a_i, t_{-i}) and that p_i is increasing.
3. u_i depends on a_{-i} and t_i affects i's beliefs about a_{-i}, since a_{-i} depends on t_{-i} through the strategies of the other players. Hence, we assume that u_i has increasing differences in (a_i, a_{-i}), we restrict attention to increasing strategies by players other than i, and we assume that p_i is increasing.

Proposition 2. *Let $i \in N$. Assume that:*

1. *u_i is supermodular in a_i, has increasing differences in (a_i, a_{-i}), and has increasing differences in (a_i, t); and*
2. *$p_i : T_i \to \mathcal{M}_{-i}$ is increasing with respect to the partial order on $\mathcal{M}_{-i}$ of first-order stochastic dominance (e.g., μ is affiliated).*

Then, for all monotone $\sigma_{-i} \in \Sigma_{-i}$, $\bar{\beta}_i(\sigma_{-i})$ is monotone.

Proof. Fix $\sigma_{-i} \in \Sigma_{-i}$. Recall from equation (3) that

$$\varphi_i(\cdot) = \arg\max_{a_i \in A_i} V_i(a_i, t_i, p_i(t_i); \sigma_{-i}).$$

Recall from the proof of Proposition 1 that V_i is supermodular in a_i. We now show that, if σ_{-i} is monotone, then V_i has increasing differences in (a_i, t_i) and (a_i, P_{-i}). Therefore, by Lemma A.1, $\sup \varphi_i(t_i, P_{-i}; \sigma_{-i})$ is increasing in t_i and in P_{-i}. Since $\bar{\beta}_i(\sigma_{-i})$ is equal to $t_i \mapsto \sup \varphi_i(t_i, p_i(t_i); \sigma_{-i})$ (see again the proof of Proposition 1) and since p_i is increasing, it follows that $\bar{\beta}_i(\sigma_{-i})$ is increasing in t_i.

Recall the definition of V_i from equation (1). If we view the payoff function that defines V_i solely as a function of a_i and t (because we are keeping σ_{-i} fixed and the actions of the other players are determined by t_{-i}), then this induced payoff function has increasing differences in (a_i, t) because u_i has increasing differences in $(a_i, (a_{-i}, t))$ and

σ_{-i} is increasing in t_{-i}. It follows from Lemma B.2 that V_i has increasing differences in (a_i, t_i) and in (a_i, P_{-i}). □

7. Strictly monotone best replies

We can strengthen the conclusion of Proposition 2 to "for all monotone $\sigma_{-i} \in \Sigma_{-i}$, $\bar{\beta}_i(\sigma_{-i})$ is *strictly* monotone" by adding some smoothness assumptions. We continue to rely on the lattice methods to obtain a weak inequality and then use differentiability to rule out equality—the inequality must then be strict.

For example, consider a choice problem $\max_{x \in X} u(x, y)$, where X is an interval of $\mathbb{R}$ and y is a parameter that belongs to a partially ordered set Y. Suppose x^H, x^L are interior solutions given $y^H, y^L \in Y$ such that $y^H > y^L$. Suppose we have determined (by using, e.g., monotone comparative statics) that $x^H \geq x^L$. Suppose also that u is differentiable in x and that $\partial u / \partial x$ is strictly increasing in y. The solutions x^H, x^L must satisfy the first-order condition; thus $\partial u(x^H, y^H)/\partial x = 0$ and $\partial u(x^L, y^L)/\partial x = 0$. Since $\partial u / \partial x$ is strictly increasing in y, we have $\partial u(x^L, y^H)/\partial x > 0$. Therefore, $x^H \neq x^L$ and instead $x^H > x^L$.

This kind of argument can be applied to a single dimension of a multidimensional choice set, thereby allowing for a mix of continuous and discrete choice variables. This is our approach. We refer to the smoothness conditions needed as the "smooth case".

Assumption 2. (*Smooth case for player i*) The following statements hold for player i:

1. $A_i = A_{i1} \times A_{i2}$, where A_{i1} is a compact interval of $\mathbb{R}$ and A_{i2} is a complete lattice;
2. u_i is continuously differentiable in a_{i1};
3. for all t_i, P_{-i}, and σ_{-i}, the elements of $\varphi_i(t_i, P_{-i}; \sigma_{-i})$ are such that a_{i1} is in the interior of A_{i1}.

In the smooth case for player i, a strategy σ_i is said to be *strictly monotone* if, for almost every $t_i^H, t_i^L \in T_i$ such that $t_i^H > t_i^L$, we have $\sigma_i(t_i^H) \geq \sigma_i(t_i^L)$ and $\sigma_{i1}(t_i^H) > \sigma_{i1}(t_i^L)$.

(Observe that the strict inequality is only for the dimension we have identified to satisfy the smoothness assumptions; if there are multiple such dimensions, we obtain a strict inequality for each one.)

We are now ready for our "strict" version of Proposition 2.

COROLLARY 1. *Given* (a) *the assumptions of Proposition 2,* (b) *the smooth case for player i, and* (c) *that* $\partial u_i/\partial a_{i1}$ *is strictly increasing in* t_i*, it follows for all monotone* $\sigma_{-i} \in \Sigma_{-i}$ *that* $\bar{\beta}_i(\sigma_{-i})$ *is strictly monotone.*

Proof. Let $\sigma_{-i} \in \Sigma_{-i}$ be monotone and let $\sigma_i = \bar{\beta}_i(\sigma_{-i})$. Let $t_i^H, t_i^L \in T_i$ be such that $t_i^H > t_i^L$. We know from Proposition 2 that $\sigma_i(t_i^H) \geq \sigma_i(t_i^L)$, so we only need to show that $\sigma_{i1}(t_i^H) \neq \sigma_{i1}(t_i^L)$.

Continuing from the proof of Proposition 2, $\sigma_i(t_i^H)$ and $\sigma_i(t_i^L)$ are solutions to (respectively) $\max_{a_i \in A_i} V_i(a_i, t_i^H, p_i(t_i^H))$ and $\max_{a_i \in A_i} V_i(a_i, t_i^L, p_i(t_i^L))$, where we have dropped the argument σ_{-i} from V_i for conciseness. Since u_i is continuously differentiable in a_{i1}, so is V_i. By assumption in the smooth case, $\sigma_{i1}(t_i^H)$ and $\sigma_{i1}(t_i^L)$ are interior. Therefore, we have the first-order conditions

$$\partial V_i(\sigma_i(t_i^H), t_i^H, p_i(t_i^H))/\partial a_{i1} = 0, \tag{4}$$

$$\partial V_i(\sigma_i(t_i^L), t_i^L, p_i(t_i^L))/\partial a_{i1} = 0. \tag{5}$$

The next step involves substituting $\sigma_{i2}(t_i^H)$, t_i^H, and $p_i(t_i^H)$ in the left side of equation (5) and showing that this causes the expression to increase, so that

$$\partial V_i\left((\sigma_{i1}(t_i^L), \sigma_{i2}(t_i^H)), t_i^H, p_i(t_i^H)\right)/\partial a_{i1} > 0. \tag{6}$$

On the one hand, we know that $\sigma_{i2}(t_i^H) \geq \sigma_{i2}(t_i^L)$ (a conclusion of Proposition 2), $t_i^H > t_i^L$ (by assumption), and $p_i(t_i^H) \geq p_i(t_i^L)$ (from the assumption that p_i is increasing). Since $\partial u_i/\partial a_{i1}$ is strictly increasing in t_i, so is $\partial V_i/\partial a_{i1}$. Furthermore, we established in the proofs of Propositions 1 and 2 that V_i is supermodular in a_i and has increasing differences in (a_i, P_{-i}); therefore, $\partial V_i/\partial a_{i1}$ is weakly increasing in a_{i2} and in P_{-i}. This establishes equation (6).

Comparing equations (4) and (6), we conclude that $\sigma_{i1}(t_i^H) \neq \sigma_{i1}(t_i^L)$. ☐

8. Monotone supermodular games have monotone extremal equilibria

Putting together Lemma 1 and Propositions 1 and 2 yields the first of our main results. We call games that satisfy the assumptions of Theorem 1 "monotone supermodular".

THEOREM 1. *Assume, for each player i, that*

1. *the utility function u_i is supermodular in a_i, has increasing differences in (a_i, a_{-i}), and has increasing differences in (a_i, t); and*
2. *the beliefs mapping $p_i\colon T_i \to \mathcal{M}_{-i}$ is increasing with respect to the partial order on $\mathcal{M}_{-i}$ of first-order stochastic dominance* (*e.g., there is a common prior μ that is affiliated*).

Then there exist a greatest and a least Bayes–Nash equilibrium, and each one is in monotone strategies.

Proof. According to Proposition 1, $\bar{\beta}_i$ is well-defined and increasing; according to Proposition 2, $\bar{\beta}_i(\sigma_{-i})$ is monotone if $\sigma_{-i} \in \Sigma_{-i}$ is monotone. Hence, the three assumptions of Lemma 1 are satisfied, so there exists a greatest equilibrium and it is in monotone strategies. (The same arguments apply to the least equilibrium.) ☐

COROLLARY 2. *Given* (a) *the assumptions of Theorem 1,* (b) *the smooth case for player i, and* (c) *that $\partial u_i/\partial a_{i1}$ is strictly increasing in t_i, it follows that the greatest and least Bayes–Nash equilibria are such that player i's strategies are strictly monotone.*

Proof. From Theorem 1, the greatest equilibrium is in monotone strategies. Player i is playing his greatest best response to a profile of monotone strategies of the other players, which according to Corollary 1 is strictly increasing in type. □

9. A counterexample for log-supermodular payoffs

Athey (2001) also obtains existence of a pure-strategy equilibrium for log-supermodular payoffs, affiliated types, and atomless type spaces (for single-dimensional types and actions, and extended to multidimensional types and actions by McAdams (2003)). We provide an example—with log-supermodular types and affiliated payoffs but finite types—that does not have a pure-strategy equilibrium. This shows (a) that our approach cannot work for log-supermodular payoffs and (b) that their results require the assumption of atomless type spaces. The problem, of course, is that log-supermodularity is not preserved by integration, and a Bayesian game with log-supermodular payoffs may not have strategic complementarities. Therefore, without purification via an atomless type space, the game may not have a pure strategy equilibrium.

There are two players, 1 and 2, with action sets $A_1 = \{1, 2\}$ and $A_2 = \{1, 2, 3\}$ and type spaces $T_1 = \{t_1\}$ and $T_2 = \{L, H\}$. Player 1 puts probability 1/2 on each of player 2's types. Since player 1's type space is degenerate and player 2's type space is one-dimensional, the distribution of types is trivially affiliated.

Player 1's utility depends only on the actions, with values shown in the following table.

	a_2		
	1	2	3
$u_1(1, a_2)$	2	8	2
$u_2(2, a_2)$	1/2	4	4
$\log_2 u(2, a_2) - \log_2 u(1, a_2)$	−2	−1	1

We first observe that this game does not have strategic complementarities.[6] Define strategies σ_2^L and σ_2^H for player 2 by $\sigma_2^L(L) = 1$, $\sigma_2^H(L) = 2$, and $\sigma_2^L(H) = \sigma_2^H(H) = 3$. From player 1's point of view, strategies σ_2^L and σ_2^H induce the probability distributions P^L and P^H over player 2's actions, as follows.

	a_2		
	1	2	3
$P^L(a_2)$	1/2	0	1/2
$P^H(a_2)$	0	1/2	1/2

Strategy σ_2^H is higher than σ_2^L, so P^H first-order stochastically dominates P^L; yet player 1's best response to σ_2^L is $a_1 = 2$ whereas her best response to σ_2^H is $a_1 = 1$.

To construct from this the nonexistence of a pure strategy equilibrium, we need only suppose that player 2 has a dominant action $a_2 = 3$ when observing $t_2 = H$ and that, when observing $t_2 = L$, player 2's best response to $a_1 = 1$ is 1 and his best response to $a_1 = 2$ is 2. (This is consistent with u_2 being either supermodular or log-supermodular.) Then player 2's best response to $a_1 = 1$ is σ_2^L whereas 1's best response to σ_2^L is $a_1 = 2$; likewise 2's best response to $a_1 = 2$ is σ_2^H whereas 1's best response to σ_2^H is $a_1 = 1$.

The more general message is that, whereas ordinal single-crossing properties are sufficient for existence of pure-strategy equilibria in games of complete information, we need the cardinal supermodularity and increasing differences properties in games of incomplete information because only these are preserved by integration. Hence, when relaxing these assumptions, it is likely we must resort to purification via atomless type spaces in order to obtain pure-strategy equilibria, even if we are not interested in the monotonicity of the equilibrium strategies.

6. This example amounts to a reinterpretation of an example in Appendix B in which a decision maker with log-supermodular utility over both action and state shifts his choice down in response to a first-order stochastic dominant shift of beliefs about the state.

10. The greatest equilibrium is increasing in the interim beliefs

Consider two monotone supermodular games that are identical except in the interim beliefs. Suppose the difference between the games is a shift in the information structure such that the interim beliefs *increase* from p_i to p_i', meaning that $p_i'(t_i) \geqslant p_i(t_i)$ for all $t_i \in T_i$. The logic in the proof of Proposition 2 tells us that the greatest and least best replies increase. We can then conclude that the greatest equilibrium increases.

To state the result, we fix all the parameters of the game except interim beliefs (players, actions, types, payoffs) as presented in Section 2. Assume that, for $i \in N$, u_i satisfies assumption 1 in Theorem 1. We denote interim beliefs $(p_i)_{i \in N}$ by p, we let $\mathscr{P}$ be the set of increasing interim beliefs, and we let $\Gamma(p)$ be the monotone supermodular game with interim beliefs p.

Proposition 3. *Consider two games $\Gamma(p)$ and $\Gamma(p')$ such that, for $i \in N$, $p_i' \geqslant p_i$. Then the greatest equilibrium of $\Gamma(p')$ is greater than the greatest equilibrium of $\Gamma(p)$.*

Proof. Let $\bar{\beta}_i$ and $\bar{\beta}_i'$ be player i's GBR mapping for the interim beliefs p_i and p_i', respectively. Fix an increasing strategy profile $\sigma_{-i} \in \Sigma_{-i}$ of the other players. Recall from the proof of Proposition 2 that $\max \varphi_i(t_i, P_{-i}; \sigma_{-i})$ exists and is increasing in P_{-i}. Since $\bar{\beta}_i(\sigma_{-i})$ is equal to $t_i \mapsto \max \varphi_i(t_i, p_i(t_i); \sigma_{-i})$ and $\bar{\beta}_i'(\sigma_{-i})$ is equal to $t_i \mapsto \max \varphi_i(t_i, p_i'(t_i); \sigma_{-i})$, and since $p_i'(t_i) \geqslant p_i(t_i)$, we have $\bar{\beta}_i'(\sigma_{-i}) \geqslant \bar{\beta}_i(\sigma_{-i})$.

Therefore, when we construct the greatest equilibria for the two information structures using Cournot tatônnement (as in the proof of Lemma 1), at each stage we have $\sigma'^k \geqslant \sigma^k$ and then—from $\bar{\beta}'(\sigma'^k) \geqslant \bar{\beta}'(\sigma^k)$ (because $\bar{\beta}'$ is increasing) and $\bar{\beta}'(\sigma^k) \geqslant \bar{\beta}(\sigma^k)$ (as shown above)—we obtain $\sigma'^{(k+1)} \geqslant \sigma^{k+1}$. Thus, in the limit, $\sigma'^{\infty} \geqslant \sigma^{\infty}$. □

Corollary 3 develops a strict version of Proposition 3, providing sufficient conditions for the equilibrium strategy of a particular player j to be strictly higher following a strict first-order stochastic dominant shift in j's beliefs about another player i (and a weak first-order stochastic dominant shift for all other beliefs of player j and of other players). One possibility is that j's action shifts up in direct response to a strict complementarity between a_{j1} and t_i. The other possibility is that player i's strategy is strictly monotone (because of strict complementarity between a_{i1} and t_i) and there is a strict complementarity between a_{j1} and a_{i1}.

Corollary 3. *Let $i, j \in \{1, \ldots, N\}$ with $i \neq j$. Given the assumptions of Proposition 3 and the smooth case for player j, assume also that either*

1. *$\partial u_j / \partial a_{j1}$ is strictly increasing in t_i or*
2. *$\partial u_j / \partial a_{j1}$ is strictly increasing in a_{i1} and the smooth case holds also for player i, with $\partial u_i / \partial a_{i1}$ strictly increasing in t_i.*

Then the greatest equilibria σ' and σ of $\Gamma(p')$ and $\Gamma(p)$, respectively, are such that, for all $t_j \in T_j$, if the marginal distribution of $p'_j(t_j)$ on T_i strictly first-order stochastically dominates that of $p_j(t_j)$, then $\sigma'_{j1}(t_j) > \sigma_{j1}(t_j)$.

Proof. Proposition 3 tells us that $\sigma'_{j1}(t_j) \geq \sigma_{j1}(t_j)$. We need to show that $\sigma'_{j1}(t_j) \neq \sigma_{j1}(t_j)$. The method of proof is the same as in Corollary 1.

Following first the proof of Proposition 2, we have that $\sigma'_j(t_j)$ and $\sigma_j(t_j)$ are solutions to (respectively) $\max_{a_j \in A_j} V'_j(a_j, t_j, p'_j(t_j); \sigma_{-j})$ and $\max_{a_j \in A_j} V_j(a_j, t_j, p_j(t_j); \sigma_{-j})$, where

$$\begin{aligned} V'_j(a_j, t_j, P_{-j}; \sigma_{-j}) &= \int_{T_{-j}} u_j(a_j, \sigma'_{-j}(t_{-j}), t_j, t_{-j})\, dP_{-j}(t_{-j}), \\ V_j(a_j, t_j, P_{-j}; \sigma_{-j}) &= \int_{T_{-j}} u_j(a_j, \sigma_{-j}(t_{-j}), t_j, t_{-j})\, dP_{-j}(t_{-j}). \end{aligned}$$

As in Corollary 1, we have the first-order conditions

$$\begin{aligned} \partial V'_j(\sigma'_j(t_j), t_j, p'_j(t_j)) / \partial a_{j1} &= 0, \\ \partial V_j(\sigma_j(t_j), t_j, p_j(t_j)) / \partial a_{j1} &= 0, \end{aligned}$$

and we need to show that

$$\partial V'\big((\sigma_{j1}(t_j), \sigma'_{j2}(t_j)), t_j, p'_j(t_j)\big)/\partial a_{j1} \\ > \partial V_j\big((\sigma_{j1}(t_j), \sigma_{j2}(t_j)), t_j, p_j(t_j)\big)/\partial a_{j1}, \tag{7}$$

implying that $\sigma'_{j1}(t_j) \neq \sigma_{j1}(t_j)$.

Inequality (7) involves three substitutions when comparing the right-hand side with the left-hand side, which we can make one at a time. First, we substitute $\sigma'_{j2}(t_j) \geq \sigma_{j2}(t_j)$, which raises the value weakly because $\partial u_j/\partial a_{j1}$ is increasing in a_{j2} (u_j is supermodular in a_j). Then we substitute $\sigma'_{-j} \geq \sigma_{-j}$, which raises the value weakly because $\partial u_j/\partial a_{j1}$ is increasing in a_{-j} (u_j has increasing differences in (a_j, a_{-j})). Finally we substitute $p'_j(t_j) > p_j(t_j)$, which causes a strict rise in the value because (a) $\partial u_j/\partial a_{j1}$ is increasing in t_{-j}; (b) $\partial u_j/\partial a_{j1}$ is increasing in a_{-j} and σ'_{-j} is increasing in t_{-j}; and (c) either $\partial u_j/\partial a_{j1}$ is strictly increasing in t_1 (assumption 1) or $\partial u_j/\partial a_{j1}$ is strictly increasing in a_{i1} and σ'_{i1} is strictly increasing in t_i (assumption 2 and Corollary 2). □

11. Games of voluntary disclosure

A leading application of the comparative statics result in Proposition 3 is to two-stage games in which information is revealed in the first stage. It is then important to know how the equilibria of the second stage—in particular, the players' second-stage payoffs—depend on the information structure that results from the first stage in order to understand the players' incentives to influence this information structure.

Consider the parameterized family $\{\Gamma(p) \mid p \in \mathscr{P}\}$ of monotone supermodular Bayesian games, as defined in Section 10. Each game has a greatest equilibrium, which we denote by $\bar{\sigma}(p)$. Let $W_i(p, t_i)$ be player i's expected utility in the equilibrium $\bar{\sigma}(p)$ of the game $\Gamma(p)$, conditional on i's type being t_i.

Assume that the Bayesian games have positive externalities, meaning that u_i is increasing in a_{-i} for all $i \in N$. According to Proposition 3, $\bar{\sigma}(p)$ is increasing in p_{-i}. It follows that $W_i(p, t_i)$ is increasing in p_{-i}. That is, higher beliefs by player $j \neq i$ lead to higher equilibrium actions, which lead to higher expected utility for player i. This is summarized in Proposition 4.

PROPOSITION 4. *Let $i \in \{1, \dots, N\}$ and assume that u_i is increasing in a_{-i}. For $p \in \mathscr{P}$ and for $t_i \in T_i$, let $W_i(p, t_i)$ be player i's expected utility in the greatest equilibrium of $\Gamma(p)$, conditional on being of type t_i. Then $W_i(p, t_i)$ is increasing in p_{-i}.*

Thus, if a unique equilibrium exists or if the equilibrium selection in the second stage is of the greatest or least equilibrium, then the players' incentives in the first stage are to induce the other players to increase their beliefs.

Corollary 4 states a strict version of this result. It follows immediately from Corollary 3.

COROLLARY 4. *Let $i, j \in \{1, \dots, N\}$ with $i \neq j$ be such that* (a) *the assumptions of Corollary 3 are satisfied and* (b) *u_i is strictly increasing in a_j. Then $W_i(p, t_i)$ is strictly increasing in the marginal probability measure of p_j on T_i. That is, if $p'_{-i} \geq p_{-i}$ and the marginal of $p'_j(t_j)$ on T_i strictly first-order stochastically dominates that of $p_j(t_j)$ for t_j in a $p_i(t_i)$-nonnull set of $t_j \in T_j$, then $W_i((p_i, p'_{-i}), t_i) > W_i((p_i, p_{-i}), t_i)$.*

Consider the setting in Okuno-Fujiwara, Postlewaite, and Suzumura (1990). In the first stage, there is only information revelation. Talk is cheap: it does not affect payoffs except through the play in the second stage. However, a player's message is a statement that her type belongs to a set of types, and she cannot lie because messages are verifiable. Stated another way, for each message there is a set of types who can send that message. Let M_i be the set of messages of player i; treat each $m_i \in M_i$ also as the set of i's types that can send message m_i. (We endow M_i with a σ-field for measurability restrictions.) Let $M = \prod_{i \in N} M_i$.

A first-stage strategy for player i is a measurable map $r_i \colon T_i \to M_i$ such that, for all

$t_i \in T_i$, we have $t_i \in r_i(t_i)$. A second-stage strategy is a measurable map $q_i \colon T_i \times M \to A_i$ and a second-stage belief function is a measurable map $\pi_i \colon T_i \times M \to \mathcal{M}_{-i}$ such that, for $t_i \in T_i$ and $m \in M$, $\pi_i(t_i, m)$ puts probability 1 on $\prod_{j \neq i} m_j$.

Observe that, given q_i and π_i, each realization $m \in M$ of the messages induces a beliefs mapping $\pi_i(\cdot, m) \colon T_i \to \mathcal{M}_{-i}$ and a strategy $q_i(\cdot, m) \colon T_i \to A_i$ in the second-stage game. Then $(r_i, q_i, \pi_i)_{i \in N}$ is a *perfect Bayesian equilibrium* (PBE) if the following statements hold.

1. (*Belief consistency*) π_i is a conditional beliefs mapping given the information $(t_i, (r_j(t_j))_{j \neq i})$.
2. (*Equilibrium in second stage*) For all $m \in M$, $(q_i(\cdot, m))_{i \in N}$ is a Bayes–Nash equilibrium of the game $\Gamma((\pi_i(\cdot; m))_{i \in N})$.
3. (*Equilibrium in first stage*) For all $t_i \in T_i$, $r_i(t_i)$ solves

$$\max_{\substack{m_i \in M_i: \\ t_i \in m_i}} \int_{T_{-i}} u_i\big(q_i(t_i, m_i, r_{-i}(t_{-i})), q_{-i}(t_{-i}, m_i, r_{-i}(t_{-i})), t_i, t_{-i}\big)\, dp_i(t_{-i} \mid t_i).$$

Proposition 5 states that there is a fully revealing equilibrium under the following conditions.

- There are strategic complementarities and positive externalities, and there are complementarities between actions and types (assumption 1 in Proposition 5).
- For each message, there is a lowest type who can send the message (assumption 2); for each type, there is message for which it is the lowest type (assumption 3).
- As a technicality, the following must be measurable: the "skeptical" second-stage beliefs, which conclude from each profile of messages that senders are of the lowest possible types (assumption 4); and a mapping that assigns to each type t_i a message such that t_i is the lowest type who can send the message (assumption 5).

Proposition 5. *Assume that, for each $i \in N$, the following statements hold:*

1. *u_i satisfies the assumptions of Theorem 1 and is increasing in a_{-i};*
2. *for each $m_i \in M_i$,* $\min m_i$ *exists;*

3. *for each* $t_i \in T_i$*, there exists an* $m_i \in M_i$ *such that* $\min m_i = t_i$*;*

4. *there is a measurable map* $\pi_i^* \colon T_i \times M \to \mathscr{M}_{-i}$ *such that, for* $t_i \in T_i$ *and* $m \in M$*,* $\pi_i^*(t_i, m)$ *puts probability* 1 *on* $(\min m_j)_{j \neq i}$*;*

5. *there is a measurable map* $r_i^* \colon T_i \to M_i$ *such that* $t_i = \min r_i^*(t_i)$ *for all* $t_i \in T_i$*.*

Let $q_i^* \colon T_i \times M \to A_i$ *be such that* $q_i^*(\cdot, m)$ *is the largest Bayes–Nash equilibrium in the game* $\Gamma((\pi_j^*(\cdot, m))_{j\in N})$ *for each* $m \in M$*. Then* $(r_i^*, q_i^*, \pi_i^*)_{i\in N}$ *is a perfect Bayesian equilibrium.*

Proof. The messages $(r_i^*)_{i\in N}$ are fully revealing. Since the second-stage beliefs $(\pi_i^*)_{i\in N}$ deduce (correctly, when on the equilibrium path) that a message m_j is sent by $\min m_j$, they satisfy belief consistency. Here q^* is defined so that $q^*(m)$ is an equilibrium in the second stage, given m. For each message m, the second-stage game is effectively one of complete information and satisfies the assumptions of Theorem 1 (in particular, the increasing beliefs condition is satisfied trivially because interim beliefs are type-independent). We can apply Proposition 4 to conclude that each player would like the other players to believe he is as high a type as possible. Given the skeptical beliefs, this is achieved for type t_i by reporting a message m_i such that $t_i = \min m_i$. Now $(r_i^*, q_i^*, \pi_i^*)_{i\in N}$ constitutes a perfect Bayesian equilibrium. □

Okuno-Fujiwara, Postlewaite, and Suzumura (1990) not only show existence of a fully revealing sequential equilibrium, they also provide conditions under which all sequential equilibria are fully revealing. We can do the same, with greater generality. They have unidimensional action spaces, strict concavity of payoffs (in own action), independent types, and unique interior equilibria in the second stage. All but one of their results concern two-player games.[7]

Our greater generality requires two equilibrium refinements that are automatically

7. The only case not covered by our results but covered in Okuno-Fujiwara et al. is an n-player strategic substitutes game with quadratic payoffs.

satisfied in Okuno-Fujiwara et al. First, to apply Proposition 4 and Corollary 4, the second-stage beliefs should be monotone in type, both on and off the equilibrium path. The independent-types assumption in Okuno-Fujiwara et al. guarantees that beliefs are type-independent (hence trivially monotone) on and off the equilibrium path in any sequential equilibrium. In our model, if types are one-dimensional and affiliated, then for any PBE the second-stage beliefs are increasing in type for any equilibrium messages: conditioning on an equilibrium message is like conditioning on a sublattice of types, given that type spaces are one-dimensional. We have not investigated whether the refinement of sequential equilibrium implies that this property holds for non-equilibrium messages; instead, we simply add this as an equilibrium refinement.

Second, whereas Okuno-Fujiwara et al. assume a unique equilibrium in any second-stage subgame, we instead require that the selection in the second stage be of the greatest (or least) equilibrium.

Proposition 6. *Assume that the prior distribution μ is affiliated and that, for each $i \in N$:*

1. *T_i is one-dimensional and finite;*
2. *$p_i(t_i)$ has full support for all $t_i \in T_i$;*
3. *u_i satisfies the assumptions of Theorem 1 and is increasing in a_{-i};*
4. *the smooth case holds for player i;*
5. *there is a player $j \neq i$ such that the assumptions of Corollary 3 hold and u_i is strictly increasing in a_{j1};*
6. *for each $m_i \in M_i$, $\min m_i$ exists; and*
7. *for each $t_i \in T_i$, there exists $m_i \in M_i$ such that $\min m_i = t_i$.*

Consider a perfect Bayesian equilibrium $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$ in which (a) *for $m \in M$ not in the range of r^*, $\pi_i^*(t_i, m)$ is increasing in t_i for $i \in N$, and* (b) *$(q_i^*(\cdot, m))_{i \in N}$ is the greatest (or least) Bayes–Nash equilibrium in the game $\Gamma((\pi_j^*(\cdot, m))_{i \in N})$ for each $m \in M$. Then,* *for each player $i \in N$, r_i^* is fully revealing—specifically, for each type t_i, $t_i = \min r_i^*(t_i)$.*

Note that beliefs are skeptical on the equilibrium path because, for any equilibrium message m, the player $j \neq i$ correctly deduces that player i is of type $\min m_i$.

Proof. Suppose $(r_i^*, q_i^*, \pi_i^*)_{i\in N}$ is a PBE that satisfies conditions (a) and (b) but is not fully revealing for player i. Let $\bar{t}_i$ be the highest type for i that is not fully revealed in the first round; hence $\bar{t}_i$ is being pooled with lower types. If she deviates and sends a message m_i such that $\bar{t}_i = \min m_i$, then the other players' interim beliefs about her type go up by strict first-order stochastic dominance (the assumption on full supports of interim beliefs rules out the case where, for example, types are perfectly correlated and hence messages have no effect on beliefs). Hence, according to Corollary 4, her second-stage payoff increases strictly. (Given the restriction on π_i^*, the second-stage game satisfies the assumptions in this paper.) This contradicts the assumption that $(r_i^*, q_i^*, \pi_i^*)_{i\in N}$ is a PBE.

Suppose that, for some player i and type t_i, $t_i > \min r_i^*(t_i)$. Because r_i^* is fully revealing, after receiving message $r_i^*(t_i)$ all other players believe with probability 1 that i is of type t_i. Then type $\min r_i^*(t_i)$ could deviate from his message by sending instead the message $r_i^*(t_i)$, causing a shift in all player's beliefs from his being of type $\min r_i^*(t_i)$ with probability 1 to his being of type t_i with probability 1. Again, according to Corollary 4, his second-stage payoff increases strictly; hence $(r_i^*, q_i^*, \pi_i^*)_{i\in N}$ is not a PBE. □

Results analogous to Propositions 5 and 6 can be obtained by replacing the assumption of positive externalities by negative externalities (each player's payoff is decreasing in the action of the other players) and replacing the "min" conditions on messages and beliefs by "max". Then each player would like to reduce the beliefs of other players, and there is a fully revealing equilibrium in which each type sends a message for which he is the highest possible type that can send the message (or, under the stricter assumptions of Proposition 6, every PBE satisfying the two refinements has this property).

12. Extensions and other related literature

All the existence proofs discussed in this paper circumvent a tension that arises whenever one tries to prove existence of equilibrium (in pure or mixed strategies, monotone or not) for games of incomplete information with infinite type spaces. The set of strategies is so large that—even when restricting attention to mixed strategies over finite action sets—a topology that is weak enough for compactness of the set of strategies (usually the weak or weak* topology), which is needed to apply a topological fixed-point theorem, is weaker than the topology needed for continuity of preferences (usually the norm or Mackey topology). Once Athey (2001) and McAdams (2003) establish that they can restrict attention to monotone strategies, they finesse this tension by representing the monotone strategies in a finite-dimensional set of cutoff values. An alternative method, employed by Fudenberg, Möbius, and Szeidl (2003), is to note that the weak and the strong topologies collapse on the set of monotone strategies, so that the tension between compactness and continuity disappears. A disadvantage of this approach is that one still needs convexity of best responses and hence action sets must be convex, whereas the methods of Athey (2001) and McAdams (2003) work for—and, in fact, are most direct for—finite action sets. Since our methods do not rely on a topological fixed-point theorem, this tension does not arise and we can deal simultaneously with finite and infinite action sets.

Though we do not take up any games with discontinuous payoffs, we note that one approach to such games (used, for example, in Lebrun 1996; Maskin and Riley 2000; Athey 2001) is to find equilibria for games with discretized action sets and then show that the equilibria converge to an equilibrium of the original game as the discretization of the action spaces becomes finer and finer (the difficult part is to show that the discontinuities of the payoffs do not disrupt the limiting argument). Any methods, such as ours, that yield monotone pure-strategy equilibria for finite action sets can be used as the first step in such arguments.

One method for obtaining uniqueness is to characterize the extremal equilibria and show that they are the same. As discussed in Example 2 on global games, we do not pursue such an exercise but the methods in this paper could constitute one step in such an argument. Another method is to show that the best-reply mapping is a contraction. This technique is employed by Mason and Valentinyi (2003) for games that in some directions are more general than ours but with assumptions that players be sufficiently heterogeneous, that types be sufficiently uncorrelated, and that types and actions be one-dimensional continua.

13. Concluding remarks

For games of incomplete information with supermodular payoffs (not merely payoffs with single-crossing properties), we are able to extend various results on existence of monotone pure-strategy equilibria by using quite different methods. For example, we are able to dispense with atomless type spaces, and we can easily handle multidimensional type and action spaces. Beyond such generalizations, the other value of this work is the simplicity with which the results can be obtained in comparison to games whose payoffs are not supermodular. Furthermore, we do not merely show existence; we also show that the greatest and least equilibria are in monotone strategies. We can thereby perform comparative statics on these equilibria.

We remind the reader that these results can be applied more generally by choosing the right direction of the orderings. For example, the main results can be applied to a submodular duopoly game—meaning that u_i is supermodular in a_i, has decreasing differences in (a_i, a_{-i}), has increasing differences in (a_i, t_i), and has decreasing differences in (a_i, t_{-i})—because changing the order of the strategy and type spaces of one player (via multiplying by -1) transforms the submodular game into a supermodular game (Vives 1990) with complementarity between actions and types. Similarly, if all payoffs have decreasing rather than increasing differences in actions and types yet the other assumptions of this paper hold, then we can reverse the ordering of types and apply the results of this paper. For example, under the assumptions of Theorem 1, there are greatest and least equilibria and these are *decreasing* in type (under the original ordering on types).

Appendix A: Summary of lattice and comparative statics methods

For the convenience of the reader and to fix some notation and terminology that may vary from author to author, we include a few definitions and results of lattice methods as used for monotone comparative statics. More complete treatments can be found in Topkis (1998) and Vives (1999, Chapter 2).

A binary relation $\geq$ on a nonempty set X is a *partial order* if $\geq$ is reflexive, transitive, and antisymmetric. An upper bound on a subset $A \subset X$ is $z \in X$ such that $z \geq x$ for all $x \in A$. A greatest element of A is an element of A that is also an upper bound on A. Lower bounds and least elements are defined analogously. The greatest and least elements of A, when they exist, are denoted $\max A$ and $\min A$, respectively. A supremum (resp., infimum) of A is a least upper bound (resp., greatest lower bound); it is denoted $\sup A$ (resp., $\inf A$).

A *lattice* is a partially ordered set $(X, \geq)$ in which any two elements have a supremum and an infimum. A lattice $(X, \geq)$ is *complete* if every nonempty subset has a supremum and an infimum. A subset L of the lattice X is a *sublattice* of X if the supremum and infimum of any two elements of L belong also to L.

Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. A function $f: X \to T$ is *increasing* if, for x, y in X, $x \geq y$ implies that $f(x) \geq f(y)$.

A function $g: X \to \mathbb{R}$ on a lattice X is *supermodular* if, for all x, y in X, $g(\inf(x, y)) + g(\sup(x, y)) \geq g(x) + g(y)$. It is *strictly supermodular* if the inequality is strict for all pairs x, y in X that cannot be compared with respect to $\geq$ (i.e., neither $x \geq y$ nor $y \geq x$ holds). A function f is (*strictly*) *submodular* if $-f$ is (strictly) supermodular; a function f is (*strictly*) *log-supermodular* if $\log f$ is (strictly) supermodular.

Let X be a lattice and T a partially ordered set. The function $g: X \times T \to R$ has (*strictly*) *increasing differences* in (x, t) if $g(x', t) - g(x, t)$ is (strictly) increasing in t for $x' > x$ or, equivalently, if $g(x, t') - g(x, t)$ is (strictly) increasing in x for $t' > t$. Decreasing differences are defined analogously. If X is a convex subset of $\mathbb{R}^n$ and if $g: X \to R$ is twice continuously differentiable, then g has increasing differences in (x_i, x_j) if and only if $\partial^2 g(x)/\partial x_i \partial x_j \geq 0$ for all x and $i \neq j$.

Supermodularity is a stronger property than increasing differences: If T is also a lattice and if g is (strictly) supermodular on $X \times T$, then g has (strictly) increasing differences in (x, t). The two concepts coincide on the product of linearly ordered sets: If X is such a lattice, then a function $g: X \to \mathbb{R}$ is supermodular if and only if it has increasing differences in any pair of variables.

The main comparative statics tool applied in this paper is the following. This version is a variant of that in Milgrom and Roberts (1990). A chain $C \subset X$ is a totally ordered subset of X. A function $f: X \to \mathbb{R}$ is *order upper semicontinuous* if $\lim_{x \in X, x \downarrow \inf(C)} \leq f(\inf(C))$ and $\lim_{x \in X, x \uparrow \sup(C)} \geq f(\sup(C))$ for any chain C.

Lemma A.1. *Let X be a complete lattice and let T be a partially ordered set. Let $u: X \times T \to \mathbb{R}$ be a function that* (a) *is supermodular and order upper semicontinuous on the lattice X for each $t \in T$ and* (b) *has increasing differences in (x, t). Let $\varphi(t) = \arg\max_{x \in X} u(x, t)$. Then:*

1. *$\varphi(t)$ is a nonempty complete sublattice for all t;*

2. *φ is increasing in the sense that, for $t' > t$ and for $x' \in \varphi(t')$ and $x \in \varphi(t)$, we have $\sup(x', x) \in \varphi(t')$ and $\inf(x', x) \in \varphi(t)$;*

3. *$t \mapsto \sup \varphi(t)$ and $t \mapsto \inf \varphi(t)$ are increasing selections of φ.*

Under the assumptions in Section 2.5, each u_i is order upper semicontinuous. The reason we need topological assumptions rather than "order continuity" assumptions in this paper is for the sake of measurability of various objects.

Appendix B: Extension of comparative statics under uncertainty

For monotonicity of best responses to monotone strategies, we extend the approach in Athey (2000, 2001) to our more general type and action spaces. The main idea is that we characterize when a first-order stochastic dominant shift in beliefs causes the solutions to a decision problem under uncertainty to increase. This is a straightforward generalization of classic results for univariate actions and states with differentiable and strictly concave utility (as presented, for example, by Hadar and Russell (1978)) and of the more recent results by Athey (2000, Example 2), which are also univariate but without the differentiability and strict concavity.

These comparative statics results are related to the one-dimensional results in Athey (2002) for utility functions that satisfy single-crossing properties. However, because we restrict attention to supermodular utility, we have weaker conditions on beliefs (first-order stochastic dominant shifts rather than log-supermodular densities) and the results are simpler and apply easily to discrete and multidimensional action and state spaces.

We first state and characterize a definition of first-order stochastic dominance for general partially ordered state spaces; it is the obvious extension of first-order stochastic dominance for probability measures on $\mathbb{R}$.

Let $(\Omega, \mathscr{F})$ be a measurable space and let $\geq$ be a partial order on Ω. A set $E \in \mathscr{F}$

is said to be *increasing* if $\omega \in E$, $\omega' \in \Omega$, and $\omega' \geq \omega$ imply $\omega' \in E$. Let P^H and P^L be two probability measures on $(\Omega, \mathscr{F})$. We say that P^H first-order stochastically dominates (f.o.s.d.) P^L if and only if $P^H(E) \geq P^L(E)$ for all increasing $E \in \mathscr{F}$.

LEMMA B.1. *The following statements are equivalent.*

1. P^H *f.o.s.d.* P^L.
2. *For all increasing functions* $f : \Omega \to \mathbb{R}$ *that are integrable with respect to* P^H *and* P^L,

$$\int_\Omega f(\omega)\, dP^H \geq \int_\Omega f(\omega)\, dP^L.$$

Proof. This is a simple "bootstrapping" of the result for the case where $\Omega = \mathbb{R}$.

(2) $\Rightarrow$ (1). A set $E \in \mathscr{F}$ is increasing if and only if its indicator $\mathbf{1}_E$ is an increasing function. Then $P^H(E) = \int \mathbf{1}_E \, dP^H \geq \int \mathbf{1}_E \, dP^L = P^L(E)$.

(1) $\Rightarrow$ (2). Consider the distributions π^H and π^L of the random variable f for the two probability measures P^H and P^L, respectively. We show that π^H f.o.s.d. π^L. The result then follows since, for example, $\int f(\omega)\, dP^H$ is the expected value for the distribution π^H.

Let $\alpha \in \mathbb{R}$. Then $f^{-1}([\alpha, \infty))$ and $f^{-1}((\alpha, \infty))$ are increasing measurable sets. (For instance, let $\omega \in f^{-1}([\alpha, \infty))$; then $f(\omega) \geq \alpha$. Let $\omega' \in \Omega$ be such that $\omega' \geq \omega$; then $f(\omega') \geq f(\omega)$ because f is increasing. Hence, $f(\omega') \geq \alpha$ and $\omega' \in f^{-1}([\alpha, \infty))$.) Therefore, $\pi^H([\alpha, \infty)) = P^H(f^{-1}([\alpha, \infty))) \geq P^L(f^{-1}([\alpha, \infty))) = \pi^L([\alpha, \infty))$. Similarly, $\pi^H((\alpha, \infty)) \geq \pi^L((\alpha, \infty))$. Therefore, π^H f.o.s.d. π^L. □

Let X be a partially ordered set and let $u : X \times \Omega \to \mathbb{R}$ be measurable in ω. Let $\mathscr{M}$ be the set of probability measures on $(\Omega, \mathscr{F})$, partially ordered by first-order stochastic dominance. Define $U : X \times \mathscr{M} \to \mathbb{R}$ by $U(x, P) = \int_\Omega u(x, \omega)\, dP(\omega)$, when well-defined.

LEMMA B.2. *Assume that* u *has increasing differences in* (x, ω). *Then, on the domain* *of U, U has increasing differences in* (x, P).

Proof. Let $x^H, x^L \in X$ be such that $x^H \geq x^L$. Define $h(\omega) = u(x^H, \omega) - u(x^L, \omega)$, which is increasing in ω because u has increasing differences in (x, ω). Then $U(x^H, P) - U(x^L, P) = \int h(\omega)\, dP$, which is increasing in P according to Lemma B.1. □

Suppose that X is a lattice. Since supermodularity is preserved by integration, U is supermodular in x if u is supermodular in x. Therefore, we have the following corollary.

Corollary B.1. *Assume that u is supermodular in x and has increasing differences in* (x, ω). *Then* $P \mapsto \arg\max_{x \in X} U(x, P)$ *is increasing in P.*

Athey (2002) presents the following comparative statics result for log-supermodular utility. Suppose u is log-supermodular and $f(\omega, \theta)$ is log-supermodular. Then

$$\theta \mapsto \arg\max_{x \in X} \int_u (x, \omega) f(\omega, \theta)\, d\theta$$

is increasing in θ. If we interpret f as a density, this provides conditions for an upward shift in beliefs, as parameterized by θ, to cause an upward shift in choices. Assuming that f is log-supermodular is stronger than assuming that higher θ implies a first-order stochastic dominant shift in beliefs. Athey has a converse to her result which implies that there are counterexamples to our Corollary B.1 when supermodular u is changed to log-supermodular u; put another way, there are counterexamples to her result when her assumption of supermodular density f is weakened to first-order stochastic dominant shifts in beliefs. We now provide one such counterexample.

Let the set of states be $\Omega = \{1, 2, 3\}$ and let the set of actions be $X = \{1, 2\}$. The payoff function $u \colon X \times \Omega \to \mathbb{R}$ is defined in the top of the following table.

	ω		
	1	2	3
$u(1,\omega)$	2	8	2
$u(2,\omega)$	$1/2$	4	4
$\log_2 u(2,\omega) - \log_2 u(1,\omega)$	-2	-1	1
$P^L(\omega)$	$1/2$	0	$1/2$
$P^H(\omega)$	0	$1/2$	$1/2$

We see that $\log u$ has increasing differences and hence u is log-supermodular. Consider the probability measures P^L and P^H defined at the bottom of the table. P^H first-order stochastically dominates P^L, yet the optimal action given P^L is $x = 2$ whereas the optimal action given P^H is $x = 1$.

Appendix C: Affiliation and increasing interim beliefs

A sufficient—but not necessary—condition for the "increasing interim beliefs" condition is affiliation. We follow the discussion of affiliation in Milgrom and Weber (1982, Appendix). Consider a probability space $(\Omega, \mathscr{F}, \pi)$ such that Ω is a lattice. If $\Omega = \mathbb{R}^k$ and π has a density f, then affiliation is equivalent to f being log-supermodular. The more general definition is that π is affiliated if and only if, for every measurable increasing set $A, B \subset \Omega$ and every measurable sublattice $S \subset \Omega$ (with positive measure), $P(A \cap B \mid S) \geq P(A \mid S)P(B \mid S)$.

Lemma C.1. *The measure μ is affiliated if and only if, for all increasing sets $A, B \subset \Omega$ and every sublattice $S \subset \Omega$, we have $P(A \mid B \cap S) \geq P(A \mid B^c \cap S)$.*

Proof. The inequality $P(A \cap B \mid S) \geq P(A \mid S)P(B \mid S)$ can be rewritten as

$$\frac{P(A \cap B \mid S)}{P(B \mid S)} \geq P(A \mid S)$$

or $P(A \mid B \cap S) \geq P(A \mid S)$. Since

$$P(A \mid S) = P(B \mid S)P(A \mid B \cap S) + P(B^c \mid S)P(A \mid B^c \cap S),$$

that is, since $P(A \mid S)$ is a weighted average of $P(A \mid B \cap S)$ and $P(A \mid B^c \cap S)$, it follows that $P(A \mid B \cap S) \geq P(A \mid S)$ is equivalent to $P(A \mid B \cap S) \geq P(A \mid B^c \cap S)$. □

Now suppose that $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 and Ω_2 are measurable sublattices of Euclidean space. Consider the probability measure $p(\omega_1)$ on Ω_2 conditional on the observation of ω_1.

Lemma C.2. *If π is affiliated then, for all a.e. $\omega_1^H, \omega_1^L \in \Omega_1$ such that $\omega_1^H > \omega_1^L$, it follows that $p(\omega_1^H)$ first-order stochastically dominates $p(\omega_1^L)$.*

Proof. Assume first that Ω is discrete. Let $\omega_1^H, \omega_1^L \in \Omega_1$ have positive measure and be such that $\omega_1^H > \omega_1^L$. Let $S = \{\omega_i^L, \omega_i^H\} \times \Omega_2$ and let $B = \{\omega \in \Omega \mid \omega_1 \geq \omega_1^H\}$. Clearly S is a sublattice and B is an increasing set. Furthermore, $B \cap S = \{\omega_1^H\} \times \Omega_2$ and $B^c \cap S = \{\omega_1^L\} \times \Omega_2$. Let $A_2 \subset \Omega_2$ be an increasing set and let $A = \Omega_1 \times A_2$ (which is also increasing). Since π is affiliated, $P(A|B \cap S) \geq P(A|B^c \cap S)$, or $P(\Omega_1 \times A_2 \mid \{\omega_1^H\} \times \Omega_2) \geq P(\Omega_1 \times A_2 \mid \{\omega_1^L\} \times \Omega_2)$. This can be restated as $P(A_2 \mid \omega_1^H) \geq P(A_2 \mid \omega^L)$, which is the first-order stochastic dominance conclusion we seek.

For arbitrary (nondiscrete) Ω, we first replace ω_1^H and ω_1^L in the previous argument by sublattices of Ω_1 with positive measure that are ordered (one lies entirely above the other). Then we use a standard limiting argument. □

The converse does not hold. Even if Ω_1 and Ω_2 are both subsets of $\mathbb{R}$ and are thus one-dimensional, $P(\cdot \mid \omega_1)$ and $P(\cdot \mid \omega_2)$ can still be increasing even if π is not affiliated. Consider the following symmetric distribution (provided to us by Phil Reny): $\Omega_1 = \Omega_2 = \{1, 2, 3\}$, and μ is defined in the following table:

		ω_2		
$\mu(\omega_1, \omega_2)$		1	2	3
	1	1/20	1/20	1/20
ω_1	2	1/20	4/20	3/20
	3	1/20	3/20	5/20

Here $P(\omega_2 \mid \omega_1)$ is increasing in ω_1 with respect to first-order stochastic dominance. However, the monotone likelihood ratio, a known implication of affiliation, does not hold. Specifically, $\mu(2,2)/\mu(1,2) > \mu(2,3)/\mu(1,3)$.

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