$P-$Best Response Set.

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Abstract

This paper introduces a notion of $p-$best response set ($p-$BR). We build on this notion in order to provide a new set-valued concept: the minimal $p-$best response set ($p-$MBR). After proving general existence results of the $p-$MBR, we show that it characterizes set-valued stability concepts in a dynamic with Poisson revision opportunities borrowed to Matsui and Matsuyama (1995). Then, we study equilibrium selection. In particular, using our notion of $p-$BR, we generalize Morris, Rob and Shin (1995) that aimed to provide sufficient conditions under which an equilibrium "spreads" through a state space.

1 Introduction

In the field of non-cooperative game theory, Nash equilibrium has played a central role as a solution concept. One reason for the widespread use of the Nash equilibrium (in mixed strategies) is that it has the advantage of existing in broad classes of games[1]. Many papers, some of which will be discussed

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[1]For instance, Nash (1950) has proved its existence in finite strategic-form games. Glicksberg (1952) has proved existence for strategic-form games when strategy spaces are

here, have stressed that in some games, none of the Nash equilibria could be seen as a "reasonable" prediction. Then, two ways can be followed. The first one consists in providing new equilibrium concepts avoiding some of Nash equilibrium drawbacks. The second one aims to find sufficient conditions under which a Nash equilibrium seems to be a good prediction. This paper contributes to both directions.

In bold strokes, two kind of criticisms can be addressed to the Nash equilibrium concept. The first is the *"epistemic criticism"*: it is now well-known that Nash equilibrium arises from restrictions on agents' expectations (see Bernheim (1984), Pearce (1984) and Aumann and Brandenburger (1995)). Nash hypothesis is far from being a consequence of rationality and it is a very stringent concept in terms of consistency of beliefs. The second is the *"evolutionist criticism"*. It has emerged when, being confronted with multiple equilibria, game theorists tried to find which equilibria, if any, are robust to some selection principle. While this research, some of which will be discussed here, has been extremely instructive, it remains inconclusive as far as the foundation of the Nash equilibrium. Indeed, this approach also made it clear that in many classes of games, selection processes could not lead to Nash equilibrium. For instance, Kajii and Morris (1997) showed how some games could have no Nash equilibrium that is robust to incomplete information. It is also well-known that a best reply dynamics can converge towards cycles[2] and never reach a Nash equilibrium. Results from the perfect foresight approach of Matsui and Matsuyama (1995) can be associated to the evolutionist criticism (see also Oyama (2002)). Roughly speaking, these papers show that in a model with a dynamics with Poisson revision opportunities, the belief that at each opportunity, the other players will choose the action of equilibrium, is not necessarily self-fulfilling.

Nevertheless, the evolutionist approach has provided conditions under which one can give a foundation to the Nash equilibrium concept. Indeed, many recent papers (see Morris, Rob and Shin (1995) and Kajii an Morris (1997) for the incomplete information approach, and Young (1993), Maruta (1997) and Ellison (2000) for the stochastic evolutionary dynamics approach, and Oyama (2002), for the perfect foresight approach) underlined the strong properties of the $p-$dominant equilibrium for $p < \frac{1}{2}$ (a notion introduced by Morris, Rob and Shin (1995)). Recall that an action profile $a \equiv (a_1, a_2)$ (we focus our attention on 2 player games), is a (strict) $p-$dominant equilibrium

nonempty and compact subset of a metric space and when payoff functions are continuous. More recently, Reny (1999), introducing the notion of better-reply secure, showed new results on the existence of mixed strategy Nash equilibria generalizing many existing conditions allowing for discontinuities in payoff functions.

[2]The result of Hurkens (1995) can be interpreted in that sense.

if for every player i, a_i is a (unique) best response to any conjecture putting probability at least p on other player choosing a_{-i}. This concept implicitly assumes that a player may have a lack of confidence about what the other player will play which contrasts with the Nash equilibrium concept which assumes that all players beliefs on opponents' plays are correct and thus it can be seen as a generalization of the pure Nash equilibrium (corresponding to $p = 1$). Nonetheless, while it can be shown that a $p-$dominant equilibrium with $p < \frac{1}{2}$ is unique whenever it exists, many generic games lack such equilibria.

Our **first purpose** in the present work consists in providing a set-valued concept that aims to avoid some criticism addressed to the Nash equilibrium concept. In order to do so, we introduce notions of $p-$best response set (hereafter $p-$BR) and minimal $p-$best response set (hereafter $p-$MBR). The concept of $p-$MBR as to be seen as an attempt to provide a set-valued concept that (1) generalizes the Nash consistency of beliefs on sets of action profiles and that (2) allows a lack of confidence on the play of the other players. Thus, this concept is less stringent in terms of consistency of beliefs than the Nash equilibrium concept. Its formal definition can be given as follows: a set profile $S \equiv (S_1, S_2)$ is a $p-$best response set if for every player i, for any conjectures putting probability at least p on other players choosing an action in S_{-i}, all best responses are in S_i. We will say that S is a minimal $p-$best response set if it is a $p-$best response set and if it does not contain any proper subset that is a $p-$best response set[3]. It is also interesting to note that our notion is a generalization of the $p-$dominance concept (which captures (2)) by passing from best reply related singletons of actions to best reply related sets of actions. Our first theorem shows many important properties of these sets in general strategic-form games where strategy spaces are nonempty and compact subset of a metric space and when payoff functions are continuous. For instance, for $p \in [0, 1]$, a $p-$MBR always exists and for $p \leq \frac{1}{2}$, the $p-$MBR set is unique.

As the $\frac{1}{2}-$dominant equilibrium has been proved to be stable in the dynamics with Poisson revision opportunities proposed by Matsui and Matsuyama (1995) (see for instance Oyama (2002)), we show that the $p-$MBR for $p = \frac{1}{2}$ characterizes set-valued stability concepts in this framework. In order to do so, we introduce two set-valued stability concepts : the *linearly stable set* (hereafter LSS) and the *absorbing set* that encompass respectively a global and a local stability property. These notions are the natural set-

[3]To the best of my knowledge, the only related concept can be found in Kalai and Samet (1984) and Basu and Weibull (1991). Notice that S is a $1-$best response set if and only if S is a Curb set in the terminology of Basu and Weibull (1991).

valued extensions of the one proposed by Matsui and Matsuyama (1995) and Oyama (2002). The model has a single large population of identical rational players who are repeatedly and randomly matched to play a symmetric 2 player I actions game. The dynamics process is characterized by frictions in the following sense. Each player must make a commitment to a particular action for a random time interval.

On the one hand, a LSS is a set of actions S such that, whatever the current action distribution is, if all players share a common belief that any player, given an opportunity, necessarily chooses an action in S, then they actually choose an action in S at every opportunity; moreover S does not contain any proper subset that satisfies this property. We show that our dynamic game has a unique LSS. Then whatever the current state (i.e. action distribution) of the society is, there must exist a "self-fulfilling" belief that leads us (linearly) to the LSS. Indeed, this allows to weaken the notion of self-fulfilling belief since instead of the "beliefs" of each player i that other players are playing a precise action, player i believes only that the other players will play in the LSS. One can see the LSS as the unique set that is globally stable in a strong sense.

On the other hand, an absorbing set S is a set of actions such that for any feasible path that starts from states involving only actions in S, the best replies to that path belong to S; moreover there does not exist any proper subset of S that satisfies this property. This set is absorbing in a very strong sense since there is no restriction on the path that an agent thinks possible, which contrasts with Matsui and Matsuyama (1995)'s approach (followed by Oyama (2002)) where agents are endowed with perfect foresight. Then, once the society reaches the absorbing set, it will never leave it. We prove that when the friction is sufficiently small, the LSS, the absorbing set, coincides with the $\frac{1}{2}-$MBR.

The **second purpose** of that work consists in providing sufficient conditions under which the event that some action profiles can be played with low *ex-ante* probability can affect the behavior at equilibrium. Using our notion of $p-$best response, we show how it allows for generalizations of existing results in the incomplete information framework of Morris, Rob and Shin (1995). First, we claim that although the game is not common knowledge, the fact that players will play in a non-trivial subset of the set of available action profiles can be common knowledge for a large class of games. This relies on a contagion[4] argument over sets of actions, and extends previous

[4]Roughtly speaking, contagion exists if the fact that a different behavior occurs in a state of the world (that can have a low probability *ex-ante*) implies that this behavior will be adopted in many other states of the world.

results that were based on contagion over actions. The logic behind is as follows. Suppose that players are known to play in a $p-$best response set S with low p at some information set (which may itself have a very small probability), then this knowledge implies best responses in S at information sets where the first information set is thought possible. This in turn determines how players respond to that knowledge, at a yet larger information set, and so on. Under some clear-cut conditions on the property of $p-$best response, this chain of reasoning results in playing in S at every state of the world[5].

Indeed, we show that we can go further in iterating this result. More precisely, there might exist a $p-$best response set with low p in the modified game where the set of available actions is restricted to S. Call this set S^1. Once we have established that it is common knowledge among players that they will play in S; and if players are assumed to play in S^1 at some information set, then by the same reasoning, we get that the set of rationalizable equilibria consists in playing in S^1 at each state. Now it is clear that in some case, this iteration will converge toward a Nash equilibrium. Since a $p-$dominant equilibrium is a $p-$best response set, our results admit Morris, Rob and Shin (1995)'s main theorem as a particular case. Therefore, our result shows that a more general condition applies to the selection of a unique equilibrium.

The organization of this paper is as follows. Section 2 introduces the notions of $p-$BR and of $p-$MBR, we state and prove the existence of a $p-$MBR for all $p \in [0,1]$ and its uniqueness for $p \leq \frac{1}{2}$, relates this notion to the concept of $p-$dominance and provides an extension to this definition. Finally, we present examples. Section 3 is divided into two parts. Part $3-1$ sets up our dynamic model and introduces the set-valued stability concepts of LSS and absorbing set. Part $3-2$ shows that the $\frac{1}{2}-$MBR coincides with the LSS and the absorbing set when frictions are sufficiently small. Section 4 shows how our set-valued concept allows for generalizations of the previous results on contagion under incomplete information. In section 5, we discuss some related works. We conclude in section 6.

[5]Notice that this allows to weaken the commonly used assumption in this literature of dominant regions (i.e. an underlying complete information game where there exists a strictly dominant equilibrium).

2 $p-$Best Response set and Minimal $p-$Best Response set

2.1 Definitions

We introduce here our main concepts. Attention in this section is focused on 2 players normal-form games $G = [N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}]$, where $N = \{1, 2\}$ is the set of players, A_i is player i's set of actions, and u_i is a mapping from A (the Cartesian product of A_i) into $\mathbb{R}$ such that $u_i(a)$ is player i's utility level when strategy profile a is played. We also assume that each strategy set A_i is a compact set in some Euclidean space and each payoff function $u_i : A \to \mathbb{R}$ is continuous. Let P be the collection of all products of non-empty and compact subsets of the players strategy sets, *i.e.* $X \in P$ if and only if X is the Cartesian product of nonempty compact sets $X_i \subseteq A_i$ $[i = 1, 2]$ (in particular $A \in P$). Note that P is closed under intersection provided that intersection is non-empty. (Notice that we will use $\subseteq$ for the weak inclusion and $\subsetneq$ for the strict one.)

In the sequel, for $S \in P$, we will note $G[S]$ as the game G where i's set of actions is restricted to S_i, i.e. $[N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}]$ (therefore, we implicitly note G for $G[A]$). Let $S_{-i} \subseteq A_{-i}$ and denote $\Delta[S_{-i}]$ be the set of Borel probability measures over S_{-i}. For a given $p \in [0, 1]$, we will note :

$$\Pi^p(S_{-i}) = \{\pi \in \Delta[A_{-i}] \mid \int_{a_{-i}\in S_{-i}} \pi(a_{-i})da_{-i} \geq p\}.$$

Let $\Lambda_i[S_{-i}, p]$ be the set of i's actions which are best responses with respect to some beliefs according to which the other player will play in S_{-i} with probability at least p. Formally:

$$\begin{aligned}\Lambda_i[S_{-i}, p] &= \{a_i \in A_i \mid \exists \pi \in \Pi^p(S_{-i}) \text{ such that} \\ \int_{a_{-i}\in A_{-i}} \pi(a_{-i})u_i(a_i, a_{-i})da_{-i} &\geq \int_{a_{-i}\in A_{-i}} \pi(a_{-i})u_i(a'_i, a_{-i})da_{-i} \ \forall a'_i \in A_i\}.\end{aligned}$$

In the sequel, we denote $\Lambda[S, p] = \times_{i\in N}\Lambda_i[S_{-i}, p]$ where $S = \times_{i\in N}S_i$.

Let us now introduce our concepts of $p-$best response set (hereafter $p-$BR) and of minimal $p-$best response set (hereafter $p-$MBR). $S = S_1 \times S_2$ will be said to be a $p-$best response set if when a player i believes with probability at least p that player $-i$ plays in S_{-i} all his best replies are in S_i. Formally :

Definition 1 *Let $S \subseteq A$. S is a $p-$best response set if $S \in P$ and $\Lambda[S,p] \subseteq S$. S is a minimal $p-$best response set if S does not contain any proper subset that is a $p-$best response set*[6].

Remark 1 *Since $A \in P$ and for every $p \in [0,1]$, $\Lambda[A,p] \subseteq A$, the set A is a (trivial) $p-$best response set for every $p \in [0,1]$. Hence there always exists a $p-$best response set.*

Remark 2 *Notice that $S'_{-i} \subseteq S_{-i} \Rightarrow \Lambda[S'_{-i},p] \subseteq \Lambda[S_{-i},p]$ and $p' \geq p \Rightarrow \Lambda[S_{-i},p'] \subseteq \Lambda[S_{-i},p]$.*

It follows from definition 1 that for all $\pi \in \Pi^p(S_{-i})$, there exists $a_i \in S_i$ such that

$$\int_{a_{-i}\in A_{-i}} \pi(a_{-i})u_i(a_i,a_{-i})da_{-i} > \int_{a_{-i}\in A_{-i}} \pi(a_{-i})u_i(a'_i,a_{-i})da_{-i} \ \forall a'_i \notin S_i.$$

Remark 3 *Note that Tercieux (2003) defines $p-$best response set in a weak sense in replacing the strict inequality by the weak one in the above equation. So we implicitely refer to the notion of strict $p-$best response set in Tercieux (2003). But since, for a generic choice of payoffs, these two defintions coincide, we will not distinguish them.*

Remark 4 *The notion of $p-MBR$ unifies a number of standard concepts: an action profile $a = (a_1, a_2)$ is a strict Nash equilibrium if and only if $(\{a_1\},\{a_2\})$ is a $1-MBR$ set. In Basu and Weibull (1991)'s terminology, for a set $S \subseteq A$; S is a curb set if $S \in P$ and $\Lambda[S,1] \subseteq S$. S is minimal Curb set if S does not contain any proper subset that is a curb set. Thus S is a minimal Curb set if and only if S is a minimal $1-$best response set. Also notice that the $p-MBR$ generalizes Harsanyi and Selten (1988) notion of risk-dominance, since it is easy to observe that in a symmetric 2×2 game, an action profile is a minimal $\frac{1}{2}-$best response set if and only if it is risk-dominant*[7].

For the sake of convenience, we will note in what follows $S = (S_1, S_2)$ instead of $S = S_1 \times S_2$.

[6] We could have defined a $\mathbf{p}-$BR (or $\mathbf{p}-$MBR) where $\mathbf{p} = (p_1, p_2)$ defining a p_i for each player $i \in \{1,2\}$. Our results would be unchanged.

[7] Recall that in a 2×2 game, (a_1,a_2) risk-dominates (b_1,b_2) equilibrium iff,
$(g_1(a_1,a_2) - g_1(a_2,a_2)) \times (g_2(a_1,a_2) - g_2(a_1,a_2)) >$
$(g_1(b_1,b_2) - g_1(b_2,b_2)) \times (g_2(b_1,b_2) - g_2(b_1,b_2))$. We say that (a_1,a_2) is the risk-dominant equilibrium.

2.2 Properties of the Minimal $p-$Best Response Set

Now we move to the main result of this section that proves several properties for our set-valued concept of $p-$MBR. Among others, we state that for any $p \in [0,1]$, a game always admits a $p-$MBR. And we obtain a uniqueness result for p less than $\frac{1}{2}$.

Theorem 1 *Fix $p \in [0,1]$.*

(1) *Every game has at least one $p-$MBR.*

(2) *Every minimal $p-$MBR contains the support of at least one (mixed) Nash equilibrium.*

(3) *Two distinct $p-$MBR are disjoint.*

(4) *If $p \leq \frac{1}{2}$, there exists a unique $p-$MBR.*

(5) *If (S_1, S_2) is the $p-$MBR with $p \leq \frac{1}{2}$ of a symmetric game, then $S_1 = S_2$.*

(6) *If S is a $p-$MBR, then $\Lambda[S,p] = S$.*

Proof. (1) Fix $p \in [0,1]$. Let Q be the (non-empty, see Remark 1) collection of $p-$best response set in A, partially ordered by (weak) set inclusion. By Hausdorff's Maximality Principle, Q contains a maximal nested sub-collection. Let $Q' \subseteq Q$ be such a sub-collection, and, for each $i \in N$, let $\tilde{S}_i$ be the intersection of all sets S'_i for which $S' \in Q'$ (*i.e.* $\tilde{S}_i = \cap_{S' \in Q'} S'_i$) Since S'_i is non-empty and compact, so is $\tilde{S}_i$, by the Cantor Intersection Theorem. Hence, $\tilde{S} \in P$. Suppose $s_i \in \Lambda_i[\tilde{S}_{-i}, p]$. Since $\Lambda_i[\tilde{S}_{-i}, p] \subseteq \Lambda_i[S'_{-i}, p]$ for all S'_{-i} such that $S' \in Q'$, we have $s_i \in \Lambda_i[S'_{-i}, p]$ for all S' in Q', and thus $s_i \in S'_i$ for all $S' \in Q'$ (since every S' is a $p-$best response set). Hence $s_i \in \tilde{S}_i$, so $\Lambda_i[\tilde{S}_{-i}, p] \subseteq \tilde{S}_i$ for each $i \in N$, *i.e.* $\tilde{S}$ is a $p-$best response set (necessarily minimal by construction).

(2) Since $\Lambda[S,p] \subseteq S$ implies $\Lambda[S,1] \subseteq \Lambda[S,p] \subseteq S$, every minimal $p-$best response set is a $1-$best response set. Now consider the game $G' = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$ obtained when players are restricted to the (non-empty and compact) strategy subset S_i. G' meets the conditions of Glicksberg (1952)'s Theorem concerning existence of Nash equilibrium in mixed strategies. Let us call such an equilibrium σ^*: it is clear that $Supp(\sigma^*) \subseteq S$. σ^* is also an equilibrium of the original game G, since by hypothesis each restricted action set S_i contains all the best replies to strategies in S_{-i}.

(3) Assume that S and S' are minimal $p-$best response set and $S \cap S' \neq \emptyset$. Note that $S \cap S' \in P$. By definition for each $i \in N$, for all $\pi \in \Pi^p(S_{-i} \cap S'_{-i})$, every best response must be simultaneously in S_i and S'_i (both are $p-$best response sets) and so in $S_i \cap S'_i$. Therefore $S \cap S'$ is a $p-$best response set contradicting the fact that S and S' are minimal $p-$best response sets.

(4) First note that since $p \leq \frac{1}{2}$, $\Lambda[S, \frac{1}{2}] \subseteq \Lambda[S, p] \subseteq S$. Now let us proceed by contradiction in assuming that S and S' are two minimal $p-$best response sets. By definition, for each $i \in N$, for $\pi \in \Delta(A_{-i})$ such that $\int_{a_{-i} \in S_{-i}} \pi(a_{-i}) da_{-i} = \frac{1}{2}$, and $\int_{a'_{-i} \in S'_{-i}} \pi(a'_{-i}) da'_{-i} = \frac{1}{2}$, (since $p \leq \frac{1}{2}$, $\pi \in \Pi^p(S_{-i})$ and $\pi \in \Pi^p(S'_{-i})$) every best responses are in S_i (since S is a $\frac{1}{2}-$best response set) and in S'_i (since S' is a $\frac{1}{2}-$best response set). But this contradicts the fact that $S \cap S' = \emptyset$ (by point (3)).

(5) It is clear that if (S_1, S_2) is a minimal $p-$best response set then by the symmetry of payoffs, (S_2, S_1) is a minimal $p-$best response set. Then together with our uniqueness result (point (4)), for $p \leq \frac{1}{2}$, we necessarily have $S_1 = S_2$.

(6) Suppose that it is not true. Then there exists some player $j \in N$ for which $\Lambda_j[S_{-j}, p] \subsetneq S_j$. Let $S'_j = \Lambda_j[S_{-j}, p]$ and $S'_i = S_i$ for $i \neq j$. Thus $S' \subsetneq S$, $\Lambda_j[S'_{-j}, p] = S'_j$ and $\Lambda_i[S'_{-i}, p] \subseteq \Lambda_i[S_{-i}, p] \subseteq S_i = S'_i$. Hence $\Lambda[S', p] \subseteq S'$. The payoff function u_j being continuous, the correspondence Λ_j is non-empty and upper hemi-continuous. Then $S'_j = \Lambda_j[S_{-j}, p]$ is non-empty. As S_{-j} is compact, it is well-known that $S'_j = \Lambda_j[S_{-j}, p]$ (by the upper semi-continuity of Λ_j) is compact, *i.e.* $S' \in P$. Thus, S' is a $p-$best response set, contradicting the minimality of S. ■

Let us briefly discuss how our set-valued concept varies with p. Note that for $p = 0$, the $p-$MBR eliminates actions if and only if these actions are never best replies[8] (in particular it eliminates all strictly dominated strategies). Moreover, for two sets S and S', one $p-$MBR and the other $p'-$MBR, where $p \leq p' \leq \frac{1}{2}$, we have that $S \supseteq S'$. Therefore our sets are nested when p is increasing and shrinks toward the $\frac{1}{2}-$MBR. When p is greater than $\frac{1}{2}$, our sets need not be nested any more when p increases. We will provide examples in section $2-4$ illustrating these points.

2.3 Relationship with $p-$dominance

We now relate our solution concept to the concept of $p-$dominance as stated by Morris, Rob and Shin (1995). As specified earlier, our concept can be seen as a generalization of the strict $p-$dominant equilibrium concept.

Definition 2 *Action profile (a_1^*, a_2^*) is a strict[9] $p-$dominant equilibrium of G if for all $i \in N$ and all $\pi \in \Pi^p(\{a_{-i}^*\})$,*

[8] Otherwise stated this corresponds to the first round in the rationalizability process of Bernheim (84) and Pearce (84).

[9] Without loss of generality, we will focus on strict $p-$dominant equilibrium since for generic choice of payoffs, $p-$dominant equilibria are strict.

$$\int_{a_{-i}\in A_{-i}} \pi(a_{-i})u_i(a_i^*, a_{-i})da_{-i} > \int_{a_{-i}\in A_{-i}} \pi(a_{-i})u_i(a_i', a_{-i})da_{-i} \ \forall a_i' \in A_i \backslash \{a_i^*\}$$

First, we note that it follows from our definition that (a_1^*, a_2^*) is a (strict) $p-$dominant equilibrium if and only if $(\{a_1^*\}, \{a_2^*\})$ is a $p-$MBR set.

2.4 Alternative Construction Generalizing $p-$dominance

In this part, we propose a construction that aims to generalize the idea of $p-$dominance using the notion of $p-$best response set. In order to present this new notion, assume that players coordinate on a given $p-$best response set S of G. Then in the modified game where the set of available actions is restricted to S ($G[S]$), we might find a smaller $p-$best response set S'. Again in $G[S']$, we could find a smaller $p-$best response set... Thus we can iterate this process. If a set is the issue of such a process, it is called an iterated $p-$best response set. Formally, this can be stated as follows :

Definition 3 *Let $S \subseteq A$. S is an iterated $p-$best response set if there exists a (decreasing) sequence $(S^0, S^1, ..., S^n)$ where S^h is a $p-$best response set in $G[S^{h-1}]$ for any $h \in \{1, ..., n\}$ where $S^0 = A$ and $S^n = S$. If $S = \{a^*\}$ then a^* is called an iterated $p-$dominant equilibrium.*

Remark 5 *Note that a $p-$best response set S, is an iterated $p-$best response set where $(S^0, S^1) = (A, S)$. Thus, a strict $p-$dominant equilibrium is an iterated $p-$dominant equilibrium. But as will be proved by the following example, the converse is not true.*

2.5 Examples

As the $p-$dominance concept seems to be particularly interesting for $p = \frac{1}{2}$ (see Morris, Rob and Shin (1995), Kajii and Morris (1997) and Maruta (1997)), we will focus in the following examples on its set-valued extension the $\frac{1}{2}-$MBR.

2.5.1 Example 1:

Unlike $p-$dominant equilibrium, minimal $p-$best response sets always exist. For instance, the following (finite) symmetric two person, three actions game borrowed from Young (1993) does not have any $p-$dominant equilibrium for $p \leq \frac{3}{5}$.

	Player 2		
	1	2	3
Player 1 1	6,6	0,5	0,0
2	5,0	7,7	5,5
3	0,0	5,5	8,8

More precisely, this game has three strict Nash equilibria : $(1,1)$, $(2,2)$ and $(3,3)$. $(1,1)$ is $p-$dominant for $p > \frac{7}{8}$; $(2,2)$ is $p-$dominant for $p > \frac{3}{5}$; $(3,3)$ is $p-$dominant for $p > \frac{5}{8}$. Thus, none of these equilibria is a $\frac{1}{2}-$dominant equilibrium. Nonetheless, let us explore what are the payoffs when a player believes with probability at least p that the other players will play in the set $\{2,3\}$. Subject to this (set of) beliefs, playing the action "1" implies a payoff of at most $6-6p$. Moreover, subject to these beliefs, playing the action "2" or "3" implies, respectively, a payoff of at least 5 and $5p$. Therefore, it is easy to show that for any belief of a player that assigns a probability strictly superior to $\frac{1}{6}$ to the other player playing in $\{2,3\}$, all his best replies are in $\{2,3\}$. Otherwise stated, for any $p > \frac{1}{6}$, $\Lambda[\{M,R\}\times\{M,R\},p] \subseteq \{2,3\}\times\{2,3\}$, and so, $\{2,3\}\times\{2,3\}$ is the $\frac{1}{2}-$MBR. To be more precise, one can show that for $p \in [0,\frac{1}{6}]$, the unique $p-$MBR is $\{1,2,3\}\times\{1,2,3\}$. And for $p \in (\frac{1}{6},\frac{3}{5}]$, the unique $p-$MBR is $\{2,3\}\times\{2,3\}$. For $p \in (\frac{3}{5},\frac{5}{8}]$, we obtain that $\{2\}\times\{2\}$ is the unique $p-$MBR. Then for $p \in (\frac{5}{8},\frac{7}{8}]$, $\{2\}\times\{2\}$ is still a $p-$MBR but $\{3\}\times\{3\}$ is also a $p-$MBR. Finally, for $p > \frac{7}{8}$ the three strict Nash equilibria coincide with the three $p-$MBR.

Remark 6 *Note also that in that game* $(3,3)$ *is an iterated* $p-$*dominant equilibrium for* $p > \frac{2}{5}$.

2.5.2 Example 2:

Consider the following (finite) symmetric two person 4×4 game.

	Player 2			
	1	2	3	4
Player 1 1	2,2	1,4	3,3	3,-9
2	4,1	2,2	1,3	3,0
3	3,3	3,1	0,0	3,0
4	-9,3	0,3	0,3	4,4

The unique strict Nash equilibrium is $(4,4)$; it is a $\frac{3}{4}-$dominant equilibrium. Interestingly, the unique strict Nash equilibrium is *not* included in $\{1,2,3\}\times\{1,2,3\}$ which is the $\frac{1}{2}-$MBR set. Note nonetheless that by point (2) of Theorem 1, the $\frac{1}{2}-$MBR set contains the support of at least one (mixed) Nash equilibrium.

Let us now move to a characterization of the $p-$MBR using the dynamic framework of Matsui and Matsuyama (1995).

3 Stability of the $p-$MBR set

We now prove that the $p-$MBR set for low p is stable in Matsui and Matsuyama (1995) framework in a sense that is closely related to Oyama (2002).

3.1 Framework

We study a dynamic model with Poisson revision opportunities originally proposed by Matsui and Matsuyama (1995). More precisely, we consider a symmetric two-player game with $I \geq 2$ actions. The set of actions and the payoff matrix, which are common to both players, are given by $A = \{1, ..., I\}$ and $u(i, j)$ $(i, j = 1, ..., I)$ (when no confusion arises and for the sake of lightness, we will note this function by u_{ij}): the payoff received by a player taking action i against an opponent playing action j. Note that in this section, since the game considered is assumed to be symmetric, we will omit the subscripts (for instance, A_i will be noted A).

We denote by $\mathbb{R}^I$ the $I-$dimensional real space with the sup norm $\mid . \mid$. The set of mixed strategies is identified with the $(I-1)-$dimensional simplex, denoted by Δ^{I-1}, which is a subset of the $I-$dimensional real space. For $F \subseteq \Delta^{I-1}$ and $\varepsilon > 0$, $B_\varepsilon(F)$ denotes the relative $\varepsilon-$neighborhoods of F in Δ^{I-1}, *i.e.* $B_\varepsilon(F) = \{y \in \Delta^{I-1}$ |there exists $x \in F$ such that $\mid y - x \mid < \varepsilon\}$. For $S \subseteq A$, we also note $\tilde{\Delta}(S) = \{\pi \in \Delta(A) \mid \pi(k) > 0 \Rightarrow k \in S\}$. We say that $x^* = (x_1^*, ..., x_I^*) \in \Delta^{I-1}$ is an equilibrium state if (x^*, x^*) is a Nash equilibrium, *i.e.* for all $x = (x_1, ..., x_I) \in \Delta^{I-1}$,

$$\sum_{ij} x_i^* u_{ij} x_j^* \geq \sum_{ij} x_i u_{ij} x_j^*.$$

We denote by $[i]$ the element of Δ^{I-1} that assigns probability one to the ith coordinate (and zero to the others).

The above game, called the *static game*, is played repeatedly in a society with a continuum of identical anonymous agents. At each point in time, agents are matched randomly to form pairs and play the static game. We assume that players cannot switch actions at every point in time. Instead, every player must make a commitment to a particular pure action for a random time interval. Time instants at which each player can switch actions follow a Poisson process with the mean arrival rate λ. The processes are independent across the agents. Thus, during a time interval $[t, t+h)$, approximately a fraction λh of the agents can switch action.

A path of action distribution, or simply a path, is a function $\phi : [0, \infty) \to \Delta^{I-1}$, where $\phi(t) = (\phi_1(t), ..., \phi_I(t))$ is the action distribution of the society (or the state of the society) at time t, with $\phi_i(t)$ denoting the fraction of the agents playing action i. Due to the assumption that the switching times follow independent Poisson processes with arrival rate λ, $\phi_i(t)$ is Lipschitz continuous with Lipschitz constant λ, which implies that it is differentiable at almost all $t \in [0, \infty)$.

Definition 4 *A path of action distribution $\phi : [0, \infty) \to \Delta^{I-1}$ is feasible if it is Lipschitz continuous with Lipschitz constant λ, and satisfies the condition that for almost all t, for all $i = 1, ..., I$,*

$$\dot{\phi}_i(t) \geq -\lambda \phi_i(t) \text{ a.e.} \tag{1}$$

Note that $\dot{\phi}_i(t) \geq -\lambda\phi_i(t)$ together with $\phi(t) \in \Delta^{I-1}$ implies $\dot{\phi}_i(t) \leq \lambda(1 - \phi_i(t))$. Since time instants at which it is possible to switch between actions form a Poisson process with arrival rate λ, the period of commitment to a fixed action has an exponential distribution. Denoting the common discount rate of the players by $\theta > 0$, it follows that the expected payoff of committing to action i at time t for a given anticipated path ϕ is calculated as:

$$V_i(\phi)(t) = (\lambda + \theta) \int_0^\infty \lambda e^{-\lambda z} \left(\int_0^z e^{-\theta s} \sum_{k=1}^{I} \phi_k(t+s) u_{ik} ds \right) dz$$

which can be simplified as:

$$V_i(\phi)(t) = (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} \sum_{k=1}^{I} \phi_k(t+s) u_{ik} ds$$

thus, payoffs can be written in the following way:

$$V_i(\phi)(t) = \sum_{k \in A} \pi(k) u_{ik} \text{ where } \pi(a_k) = (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} \phi_k(t+s) ds \tag{2}$$

where $\theta > 0$ is the common rate of time preference. Note that this expression is well-defined since $\phi_k(.)$ is bounded for each k. Given a feasible path ϕ, let $BR(\phi)(t) = \{i \in A \mid V_i(\phi)(t) \geq V_j(\phi)(t) \text{ for all } j\}$ be the set of best responses in pure strategies to ϕ at time t. Let Φ^0 be the set of all feasible paths, *i.e.*, the set of Lipschitz continuous paths satisfying (1).

Finally, following Matsui and Matsuyama (1995), we denote the degree of friction by $\delta = \frac{\theta}{\lambda} > 0$. In the sequel, the game described above will be called the *dynamic game*.

We propose a concept of linear path in the following way[10]. Let $S \subseteq A$. Denote $\Phi^{Li}(S) = \{\phi \in \Phi^0 \mid \phi(0) \in \Delta^{I-1}$ and $\phi_k(t) = e^{-\lambda t}\phi_k(0)$ for all $k \notin S\}$. $\Phi^{Li}(S)$ is the set of feasible paths that converge linearly towards S. The action distribution moves along a linear path when all players choose an action in S at revision opportunities. Formally :

Definition 5 *$\phi(.)$ is a linear path from $x \in \Delta^{I-1}$ toward $S \subseteq A$ if, $\phi(0) = x$ and $\phi \in \Phi^{Li}(S)$.*

Let $\Phi \subseteq \Phi^0$. Denote

$$\begin{aligned} \Psi[\Phi] &= \{\phi \in \Phi^0 \mid [\dot{\phi}_i(t) > -\lambda\phi_i(t) \Rightarrow \\ \text{there exists } \psi &\in \Phi \text{ such that } \psi(t) = \phi(t) \text{ and } i \in BR(\psi)(t)] \text{ a.e.}\}. \end{aligned}$$

$\Psi[\Phi]$ is the set of the paths in Φ^0 along which every agent takes a best response to some path in Φ.

Next, we provide a definition of linear stability. A linearly stable equilibrium (a^*, a^*) is such that, whatever the current action distribution is, if all players share a common belief that any player, given an opportunity, necessarily chooses action a^*, then their best response consists in choosing uniquely action a^* at every opportunity. This is slightly different from the notion in Oyama (2002) since we impose that playing action a^* at every opportunity is the *unique* best response. We will see later that this is without loss of generality since if there exists a linearly stable equilibrium then (generically) it verifies this latter property.

Definition 6 *$[i] \in \Delta^{I-1}$ is linearly stable if $\Phi^{Li}(\{i\}) = \Psi[\Phi^{Li}(\{i\})]$. A symmetric pure Nash equilibrium[11] of the static game (i, i) is linearly stable if the corresponding state $[i]$ is linearly stable.*

A natural extension of Oyama (2002)'s linear stability consists in the following set-valued stability concept called the linearly stable set (LSS) that can be viewed as a set-theoretic coarsening of linear stability. This LSS is such that whatever the current action distribution is, if all players share a common

[10] This concept is an extension of the concept of linear path in Oyama (2002).

[11] This restriction to symmetric pure Nash equilibrium is made without loss of generality. Indeed, Oyama (2002) does not do this restriction. But then he shows that a Nash equilibrium is linearly stable in his sense, only if it is pure and symmetric [Theorem 2, p.296].

belief that any player, given an opportunity, necessarily chooses an action in S, then they actually choose an action in S at every opportunity. Moreover for every proper subset S' of the LSS, there exists a current action distribution such that the common belief that any player given an opportunity, necessarily chooses an action in S' is not a self-fulfilling belief. Then whatever the current state of the society is, there must exists a "self-fulfilling" belief that leads us (linearly) to the LSS. Indeed, this allows to weaken the notion of self-fulfilling beliefs that is present in the concept of linear stability of Oyama (2002) since beliefs are self-fulfilling in a weak sense. Instead of the "beliefs" of each player i that other players are playing a precise action, our player i believes only that the other players will play in the LSS. We will show that this set always exists and is unique, therefore, one can see the LSS as the unique set that is globally stable in a strong sense.

Definition 7 *A non-empty subset S of A is a linearly stable set (LSS) if* (1) $\Psi[\Phi^{Li}(S)] \subseteq \Phi^{Li}(S)$*; and* (2) *there does not exist a proper subset of S that satisfies* (1).

We also state the following important property:

Proposition 1 *The dynamic game has a unique LSS.*

Proof. See Appendix A. ■

The following remark links linear stability with our concept of LSS.

Remark 7 $\{i\}$ *is a LSS if and only if* (i, i) *is linearly stable.*

We now move to the definition of the concept of absorbing set. $S \subseteq A$ is an absorbing set, if when the society has reached a distribution of action that only involves actions in S, the best responses to any feasible path must be in S. Moreover for every proper subset S' of the absorbing set, there exists a feasible path such that if a player believes that the society will follow this path, then best responses involves actions that are not in S', *i.e.* the society may leave S'. This set is absorbing in a very strong sense since there is no restriction on the path that an agent thinks possible. Then, once the society reaches the absorbing set, it will never leave it. We can also interpret this concept in the following way, an agent will play in the absorbing set if he plays actions that are best responses to at least one feasible path[12].

[12] This is less demanding than many other set-valued stability concepts such as the Globally Accessible Set of Oyama (2002) or Stable Sets under Rationalizable Foresight of Matsui and Oyama (2003).

Definition 8 *Let $S \subseteq A$. We say that S is an absorbing set if* (1) *for every $\phi \in \Phi^0$ with $\phi(0) \in \tilde{\Delta}(S)$, $BR(\phi)(0) \subseteq S$;* (2) *there does not exist a proper subset of S that satisfies* (1)*.*

In order to prove our next proposition, we need to restrict the attention to games that are generic in the $\frac{1}{2}-$MBR, a property that is stated as follows:

Definition 9 *G is said to be generic in the $\frac{1}{2}-MBR$ if S, the $\frac{1}{2}-MBR$ of G, is such that for all $S' \subsetneq S$, there exists $\bar{\rho} > 0$ such that for all $\rho < \bar{\rho}$, $\Lambda[S', \frac{1}{2} + \rho] \nsubseteq S'$.*

The following proposition claims that, assuming that the static game is generic in the $\frac{1}{2}-$MBR, and provided that friction is sufficiently low, the absorbing set must be unique.

Proposition 2 *Assume that the static game is generic in the $\frac{1}{2}-MBR$. There exists $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$, the dynamic game has a unique absorbing set.*

Proof. See Appendix B. ■

3.2 Main Results

In this part we state our main results. Since we consider a symmetric 2 person game, by a slight abuse of notation, we will note $\Lambda[S, p] \equiv \Lambda_1[S, p] = \Lambda_2[S, p]$.

Theorem 2 states that for a sufficiently small degree of friction, the unique $\frac{1}{2}-$MBR is the unique LSS. As discussed above, this implies that from any initial distribution of action, the $\frac{1}{2}-$MBR is the smallest set such that if all players share a common belief that any player, given an opportunity, necessarily chooses an action in S, then they actually choose an action in S at every opportunity. Formally :

Theorem 2 *There exists $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$ the LSS coincide with the $\frac{1}{2}-MBR$ set of the game.*

Proof. See Appendix A. ■

This result allows us to generalize Oyama (2002)'s main theorem in the following sense. One can check that generically, a $p-$dominant equilibrium for $p < \frac{1}{2}$ is a strict $\frac{1}{2}-$dominant equilibrium. Then restricting our attention to singletons, and together with Remark 7, we find Oyama (2002)'s main result[13] for generic games.

[13] This Corollary implies that our slightly modified definition of linear stability (compared with the one of Oyama (2002)) is without loss of generality.

Our next theorem relates our concept of $\frac{1}{2}-$MBR to the absorbing set. To be more precise, it states that for games that are generic in the $\frac{1}{2}-$MBR and for a sufficiently small degree of friction, the unique $\frac{1}{2}-$MBR set is exactly the unique absorbing set. Thus, the $\frac{1}{2}-$MBR is the smallest set such that once the society reaches that set, the best responses to any feasible path are in this set.

Theorem 3 *Assume that the static game is generic in the $\frac{1}{2}-$MBR. There exists $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$ the absorbing set coincides with the $\frac{1}{2}-$MBR set.*

Proof. See Appendix B. ■

Then the $\frac{1}{2}-$MBR is endowed with the strong stability properties (discussed above) of the LSS and of the absorbing set. The following example shows why we need the assumption of genericity in the $\frac{1}{2}-$MBR in our latter Theorem.

		Player 2	
		1	2
Player 1	1	1,1	0,0
	2	0,0	1,1

This game is non-generic in the $\frac{1}{2}-$MBR since $\{1,2\} \times \{1,2\}$ is the $\frac{1}{2}-$MBR and $\{1\} \times \{1\}$ and $\{2\} \times \{2\}$ are $p-$MBR for any $p > \frac{1}{2}$. One can easily show that $\{1\}$ and $\{2\}$ are two absorbing sets for all $\delta > 0$. Then in this non-generic game, the $\frac{1}{2}-$MBR does not coincide with any of the absorbing sets. (Note that Proposition 2 that establishes the uniqueness of the absorbing set does not hold when removing the assumption of genericity on G).

4 "Contagion" under Incomplete Information

A recent literature studies how a low probability event can dramatically affect the behavior at equilibrium (see Rubinstein (1989), Morris, Rob and Shin (1995) and Kajii and Morris (1997)). We propose to use our notion of $p-$BR in order to generalize these results of "contagion". In particular, we will focus on conditions under which an incomplete information structure can lead to the emergence of a unique equilibrium generalizing Morris, Rob and Shin (1995)'s main Theorem.

4.1 A leading example

Many recent papers (see Morris, Rob and Shin (1995), Kajii and Morris (1997)) have shown that by introducing a "slight" incomplete information, the $p-$dominant equilibrium for $p \leq \frac{1}{2}$ is selected when it exists. In this part, we prove through a simple example that this argument can be pushed one step further. In order to do so, we use a selection device in the flavour of Rubinstein (1989).

Consider the following incomplete information game. The state space is finite, with each state equally likely, we note $\Omega = \{1, 2, 3, ..., N\}$. There are two players: 1 observes information partition[14]: $Q_1 = (\{1, 2\}, \{3, 4\}, \{5, 6\}, ...\{N-2, N-1\}, \{N\})$, while 2 observes partition, $Q_2 = (\{1\}, \{2, 3\}, \{4, 5\}, ..., \{N-1, N\})$. Now suppose that at each information set, each player has a choice of three actions, M, D, and R. If the true state is $\omega \neq 1$, then the payoffs are given by :

Player 2

Player 1		M	D	R
	M	$5,5$	$1,4$	$1,0$
	D	$4,1$	$3,3$	$3,3.5$
	R	$0,1$	$3.5,3$	$4,4$

Notice that in this game, there exist two strict Nash equilibria. More precisely, (M, M) is $p-$dominant for $p > 2/3$ and (R, R) is $p-$dominant for $p > 4/5$, yet there is no $\frac{1}{2}-$dominant equilibrium. $(1, 1)$ is the Pareto-dominant equilibrium and has the “smallest” level of $p-$dominance. Nonetheless, the complete information game played at $\omega \neq 1$ possesses an interesting property, since $\{D, R\} \times \{D, R\}$ is a $p-$best response set for $p > \frac{1}{3}$. As our subsequent argument will show, this is a key feature in our contagion process.

In state 1, payoffs are different. Payoffs of the complete information game associated to world 1 are such that R is a dominant strategy for player 2 (whereas payoffs do not change for player 1). (Note that, although we are in the framework proposed by Morris, Rob and Shin (1995), their main result does not apply since it applies only if there exists a $p-$dominant equilibrium with $p \leq \frac{1}{2}$).

We first show that, the set of rationalizable equilibria in the sense of Bernheim (1984) and Pearce (1984) (*i.e.* action profiles that survive the iterative deletion of strategies that are not best response to some belief over the strategies of the other player) consists in playing in $\{D, R\} \times \{D, R\}$ at each

[14] We suppose that the state space has an odd number of states. But we could give the same example with an even number of states. With the partition, $Q_1 = (\{1, 2\}, \{3, 4\}, \{5, 6\}, ...\{N-2, N-1\}, \{N, N+1\})$, $Q_2 = (\{1\}, \{2, 3\}, \{4, 5\}, ..., \{N-1, N\}, \{N+1\})$, we would obtain the same results.

state of the world. At $\omega = 1$, player 2 plays R since it is a dominant strategy for him (note that each player knows his own payoffs). At information set $\{1, 2\}$, player 1 believes that with probability $\frac{1}{2}$ the real world is 1. Therefore player 1 believes with probability at least $\frac{1}{2}$ that player 2 plays R. Then, by definition of a $\frac{1}{2}-$best response set, player 1's best replies are in $\{D, R\}$. Then at information set $\{2, 3\}$, player 2 believes with probability at least $\frac{1}{2}$ that player 1 plays $\{D, R\}$ and so he plays in $\{D, R\}$. Thus, this chain of reasoning implies that the set of rationalizable equilibria consists in playing in $\{D, R\} \times \{D, R\}$ at each state of the world. Now the event "players play in $\{D, R\} \times \{D, R\}$" is common knowledge (in other words, we can restrict attention to the game $G[\{D, R\} \times \{D, R\}]$). Note that conditionally to this event and at each state of the world, both players have a dominant strategy to play R. Thus, the unique (rationalizable) equilibrium is (a Bayesian Nash Equilibrium) such that (R, R) is played at every state. We will show that the important concept to understand uniqueness in these frameworks, relies on the notion of $p-$best response set.

Remark 8 *As discussed above, in our leading example, at any state of the world,* $\{D, R\} \times \{D, R\}$ *is a* $p-$*best response set if* $p > \frac{1}{3}$ *(note that this includes state* 1 *where payoffs change). In addition, in* $G[\{D, R\} \times \{D, R\}]$, $\{R\} \times \{R\}$ *is a* $p-$*best response set for any* $p \in [0, 1]$ *(since* R *is strictly dominant in* $G[\{D, R\} \times \{D, R\}]$*). Therefore* (R, R) *is an iterated* $p-$*dominant equilibrium if* $p > \frac{1}{3}$*. More precisely, we have a decreasing sequence* (S^0, S^1, S^2) *where* $S^0 = A$, $S^1 = \{D, R\} \times \{D, R\}$ *and* $S^2 = \{R\} \times \{R\}$ *at any state of the world.*

In the next section we will state and prove our main results in an incomplete information framework.

4.2 Incomplete Information Framework

First, we introduce notation. An *information system* is a structure $I = \{\Omega, \{1, 2\}, \{Q_i\}_{i=1,2}, \Pi\}$ where Ω is a finite set of states of the world; $\{1, 2\}$ is the set of players, Q_i is the partition of states of the world representing the information of player i; and Π is a strictly positive prior probability distribution on Ω. Each player is assumed to share the same prior Π on Ω. We will write ω for a typical element of Ω. Then $\Pi(\omega)$ is the probability of state ω. We will also write $Q_i(\omega)$ for the element of i's partition, Q_i, containing state ω. Thus if the true state is ω, $Q_i(\omega)$ is the set of states which player i thinks possible. We write F_i for the field generated by i's partition, *i.e.* the set of unions and intersections of events in Q_i. We assume that there is some nontrivial information so that for some i, $Q_i \neq \{\Omega\}$.

In the following, we use the idea of *belief operators* (introduced by Monderer and Samet (1989)) on the state space. We write the conditional probability of event E, given event F, as $\Pi[E \mid F] = \frac{\sum_{\omega \in E \cap F} \Pi[\omega]}{\sum_{\omega \in F} \Pi[\omega]}$. Now we define player i's p-belief operator, $B_i^p : 2^\Omega \to 2^\Omega$, by $B_i^p E = \{w \in \Omega : \Pi[E \mid Q_i(w)] \geq p\}$ where $p \in [0,1]$. Thus, B_i^p is the set of states where i assigns a probability at least p to the event E.

Then, we introduce the concept of *belief potential* developed by Morris, Rob and Shin (1995). $B_i^p B_j^p E$ is the set of states where player i believes with probability at least p that player j believes with probability at least p that event E will occur. Then $B_i^p B_j^p E \cup E$ is the set of states where either player i believes with probability at least p that player j believes with probability at least p event E or E is true. Define the operator $H_i^p(.)$ as $H_i^p(E) \equiv B_i^p B_j^p E \cup E$. In the sequel, $[H_i^p]^1(E) = H_i^p(E)$ and, for $k \geq 2$, $[H_i^p]^k(E) = H_i^p([H_i^p]^{k-1}(E))$.

Definition 10 *The belief potential $\sigma_i(E)$ of an event E is the largest number p such that for some $k \geq 1$, $[H_i^p]^k(E) = \Omega$.*

Morris, Rob and Shin (1995) showed that every event has a well defined belief potential.

Definition 11 *The belief potential σ of an information system is the minimum belief potential of any nonempty measurable event in the system:*

$$\sigma \equiv \min_{i \in \{1,2\}, E \in F_i \backslash \emptyset} \sigma_i(E).$$

Remark 9 *In our leading example, it is easy to show that $\sigma = \frac{1}{2}$. Note that Morris, Rob and Shin (1995) have proved that for any information system $\sigma \leq \frac{1}{2}$. (See [Theorem $3-3$, p.151]).*

Having set-out properties of the information system, we now turn to payoffs. An *incomplete-information game* consists in $U = [I, \{A_i\}_{i=\{1,2\}}, \{u_i\}_{i=\{1,2\}}]$, where I is the information system as described previously; A_i is the set of actions available to player i; and $u_i : A \times \Omega \to \mathbb{R}$ is player i's payoff function, where $A = A_1 \times A_2$.

As in Morris, Rob and Shin (1995), we assume that players know their own payoff, *i.e.* $u_i(a;.)$ is measurable with respect to Q_i for every $a \in A$. A pure strategy for player i in the incomplete-information game is a function $s_i : \Omega \to A_i$, measurable with respect to his partition. Symmetrically, a (mixed) strategy for player i is a Q_i-measurable function $\mu_i : \Omega \to \Delta(A_i)$. A mixed-strategy profile is a function $\mu = (\mu_i)_{i \in \{1,2\}}$ where μ_i is a strategy for player i. We denote by $\mu_i(a_i \mid \omega)$ the probability that action a_i is chosen given ω under μ_i; $\mu(a \mid \omega)$ is the probability that action profile a is chosen given ω under μ; when no confusion arises, we extend the domain of each u_i to mixed strategies and thus write $u_i(\mu(\omega), \omega)$ for $\sum_{a \in A} u_i(a, \omega)\mu(a \mid \omega)$. We will also note $Supp(\mu_i(\omega)) = \{a_i \in A_i \mid \mu_i(a_i \mid \omega) > 0\}$ the support of $\mu_i(\omega)$.

Definition 12 *A mixed-strategy profile $\mu = (\mu_1, \mu_2)$ is a Bayesian Nash equilibrium of U if, for each $i \in \{1, 2\}$, $a_i \in A_i$, and $\omega \in \Omega$,*

$$\sum_{\omega' \in Q_i(\omega)} u_i(\mu(\omega'), \omega')\Pi[\omega' \mid Q_i(\omega)] \geq \sum_{\omega' \in Q_i(\omega)} u_i(\{a_i, \mu_{-i}(\omega')\}, \omega')\Pi[\omega' \mid Q_i(\omega)].$$

To each world $\omega \in \Omega$ is associated a game of complete information with payoff function given by $u_i(.; \omega) : A \to \mathbb{R}$ for each i and where those payoffs are common knowledge. We will refer to such a game as the *complete information game associated to* ω and denote $G[A, \omega]$ for $[\{1, 2\}, \{A_i\}_{i \in \{1,2\}}, \{u_i(.; \omega)\}_{i \in \{1,2\}}]$.

4.3 Results

In this section, we present our results that can be seen as two different ways of extending previous results on "contagion" arguments. Firstly they extend the "contagion" to sets of action profiles. Secondly, they provide a generalization of existing arguments on uniqueness. In addition, it proves that the class of games to which uniqueness arguments applies is much wider than the one proposed by Morris, Rob and Shin (1995). The two following theorems provide conditions under which it is possible to achieve common knowledge that a subset of the action profiles will be played.

Theorem 4 *Let $S = \times_{i \in \{1,2\}} S_i \subseteq A$ be such that :*
(1) *the information system has belief potential of σ,*
(2) *S is a $\sigma-$best response set at each $G[A, \omega]$,*
(3) *there exists $\omega^* \in \Omega$, there exists $i \in \{1, 2\}$ such that for player i, every action outside S_i is strictly dominated at the complete information game $G[A, \omega^*]$.*

Then rationalizable equilibria of the incomplete information game consist in playing in S at each ω.

Proof. See Appendix C. ■

In the same spirit, we can provide further conditions to achieve common knowledge that players play in the iterated $p-$best response set with low p. In order to do so, we will use the following important assumption.

Definition 13 *Let S be an iterated $p-$best response set at each state of the world. It is said to satisfy the Common Sequence Assumption (CSA) if there exists a (decreasing) sequence $(S^0, S^1, ..., S^n)$ such that* ***at every*** *$\omega \in \Omega$, S^h is a $p-$best response set in $G[S^{h-1}, \omega]$ for any $h \in \{1, ..., n\}$ where $S^0 = A$ and $S^n = S$. If S is a singleton set, the associated iterated $p-$dominant equilibrium will be said to satisfy the CSA.*

In our leading example, the iterated $p-$dominant equilibrium with $p > \frac{1}{3}$ (R, R) satisfies the CSA. This is due to the fact that we have the sequence $(S^0, S^1, S^2) = (A, \{D, R\} \times \{D, R\}, \{R\} \times \{R\})$ such that at any state $\omega \in \Omega$, S^h is a $p-$best response set with $p > \frac{1}{3}$ in $G[S^{h-1}, \omega]$ for any $h \in \{1, 2\}$. With this additional assumption, we are now able to use the notion of iterated $p-$best response set :

Theorem 5 *Let $S = \times_{i \in \{1,2\}} S_i \subseteq A$ be such that :*

(1) *the information system has belief potential of σ,*

(2) *S is an iterated $\sigma-$best response set at each $G[A, \omega]$ that satisfies the CSA,*

(3) *there exists $\omega^* \in \Omega$, there exists $i \in \{1, 2\}$ such that for player i, every action outside S_i is strictly dominated at the complete information game $G[A, \omega^*]$.*

Then rationalizable equilibria of the incomplete information game consist in playing in S at each ω.

Proof. See Appendix C. ■

At this point it is interesting to note that under (1), (2) and (3) in our two previous theorems, any Bayesian Nash equilibrium involves actions that are in S. Otherwise stated, any Nash equilibrium of an associated complete information game, that does not have its support included in S is "eliminated". Thus, by perturbing slightly payoffs (as in the leading example), we eliminate a lot of possible equilibria.

The following result shows how our argument can select a unique equilibrium at each state of the world by using our notion of iterated $p-$dominance. It generalizes Morris, Rob and Shin (1995)'s main result[15].

Corollary 1 *Suppose that (1) the information system has belief potential of σ, (2) (a_1^*, a_2^*) is an iterated σ-dominant equilibrium at each $G[A, \omega]$ that satisfies the CSA, (3) there exists $\omega^* \in \Omega$, there exists $i \in \{1, 2\}$ such that player i plays a_i^* at a complete information game $G[A, \omega^*]$.*

Then playing (a_1^*, a_2^*) at each ω is the unique rationalizable strategy profile of the incomplete information game.

[15] More precisely, this result extends Morris, Rob and Shin (1995) (Theorem $5-1$).

Proof. This is a straightforward implication of Theorem 5. ■

5 Discussion on Related Works

It is now clear that our notion of $p-$BR joint with iterated $p-$BR can generalized "contagion" results obtained with $p-$dominance. This paper joins a growing literature on equilibrium selection in noncooperative game theory. Indeed, some results using $p-$dominance have been generalized by the "potential maximizers" approach developed in the work of Morris and Ui (2002). One result of Morris and Ui (2002) links $p-$dominance with Generalized Potential Maximizer. For instance, in a 2 player finite game, if there exists a $p-$dominant equilibrium with $p < \frac{1}{2}$ then this equilibrium is a Generalized Potential Maximizer. A companion paper, Tercieux (2003) shows that the notion of $p-$best response set can be related to the notion of Generalized Potential Maximizer. In particular, in a 2 player finite game, when there exists a $p-$best response set with $p < \frac{1}{2}$, this set is a Generalized Potential Maximizer. Nonetheless, we note that the link between the iterated $p-$dominant equilibrium and the Potential Approach remains an interesting open question.

However, the potential maximizers approach is sometimes too strong. Firstly, our Theorem 3 and 4 together with Oyama (2002) main result shows that stability concepts using linear stability are fully characterized by the notion of $p-$MBR (at least in 2 players finite symmetric games). More importantly, in a stochastic evolutionary model *à la* Young (1993) one can show that the potential approach is not relevant. In the example 1 in our section 2-5-1, Young (1993) has shown that $(2, 2)$ is selected by his dynamics while Oyama, Takahashi, and Hofbauer (2003) have shown that $(3, 3)$ is a Generalized Potential Maximizer[16]. Therefore taking a (minimal) Generalized Potential Maximizer would lead to a prediction that is not in accordance with a stochastic evolutionary process (at least for Young's process). In addition, one can easily show that our Theorem 5 implies that any equilibrium excepted $(3, 3)$ is not robust to incomplete information in the sense of Kajii and Morris (2002) and thus is not a GP-maximizer. As will be discussed in the conclusion, this drawback is not present in our concept of $p-$MBR.

[16] Note that we can address the same criticism to the iterated $p-$dominant equilibrium since $(3, 3)$ in Example 1 is the iterated $p-$dominant equilibrium for some $p < \frac{1}{2}$.

6 Concluding Remarks

In this paper we proposed a set-valued concept that extends the notion of $p-$dominance. We defined minimal $p-$best response set that are shown to exist in broad classes of games. In addition, we show that this concept characterizes set-valued extension of stability concepts proposed by Matsui and Matsuyama (1995) and Oyama (2002) in 2 player finite symmetric games. Note that the extension to a many players framework of our existence and uniqueness results of the $p-$MBR does not involve any difficulties (and more generally the extension of Theorem 1). Nonetheless, our results of stability properties are still a first step. Enlarging our results (and also the one of Oyama (2002)) to many players asymmetric games is an interesting perspective[17].

Moreover, as specified earlier, a characterization using stochastic best-replies dynamics is an interesting way to follow. It is now well known that minimal Curb set can easily provide a characterization of best-replies dynamics[18]. As discussed in the paper, the $p-$MBR encompasses ideas of minimal Curb set. In particular, one can show that the $p-$MBR always contains at least one minimal Curb set. Thus the $p-$MBR seems to be an appropriate tool to characterize stochastic best-replies dynamics. Indeed, Maruta (1997) has shown that a $p-$dominant equilibrium with $p < \frac{1}{2}$ is selected by this dynamics when it exists. It is possible to show[19] that more generally, Young's process necessarily converges toward actions included in the $\frac{1}{2}-$MBR. This seems to be a fruitful approach also left for further research.

A Appendix A

In this part, we prove Theorem 2. Then, we show that Proposition 1 holds. First let us prove the following :

Lemma 1 *Fix $p > 0$. There exists $\bar{\eta} > 0$ such that for all $0 < \eta < \bar{\eta}$, if $S \subseteq A$ is a $p-$MBR, then it is also the $(p - \eta)-$MBR.*

Proof. Since there exists a finite number of $p-$MBR, showing the following statement is sufficient to prove our lemma: "Fix $p > 0$. If $S \subseteq A$ is a

[17] See for instance the recent work of Kojima (2003).

[18] See for instance Hurkens (1995).

[19] While focusing on minimal $\frac{1}{2}-$best response sets that are also Curb sets (or $1-$best response sets), our result in Durieu, Solal and Tercieux (2003) can easily be generalized in that sense.

$p-$MBR , then there exists $\bar{\eta} > 0$ such that for all $0 < \eta < \bar{\eta}$ it is also the $(p-\eta)-$MBR."
Let us prove that there exists $\bar{\eta} > 0$ such that for all $0 < \eta < \bar{\eta}$, $\Lambda[S, p-\eta] \subseteq S$.
For all $\pi \in \Pi^p(S)$, as noted previously, there exists $i \in S$ such that

$$\sum_{k \in A} \pi(k)u(i,k) > \sum_{k \in A} \pi(k)u(j,k) \ \forall j \notin S. \quad (3)$$

Because the inequality is strict in the above equation, one can prove that for all $\pi \in \Pi^p(S)$ there exists $\bar{\eta}_\pi > 0$, such that for all $\pi' \in B_{\bar{\eta}_\pi}(\pi)$, there exists $i \in S$, such that

$$\sum_{k \in A} \pi'(k)u(i,k) > \sum_{k \in A} \pi'(k)u(j,k) \ \forall j \notin S. \quad (4)$$

Let us prove that $\bar{\eta} = Inf\{\bar{\eta}_\pi \mid \pi \in \Pi^p(S)\} > 0$. Suppose that it is not the case *i.e.* $\bar{\eta} = 0$. Then there exist a sequence $\{\bar{\eta}_\pi^n\}_{n \geq 1}$ that converges to 0. Hence, we can build an associated sequence $\{\pi^n\}_{n \geq 1}$ in $\Pi^p(S)$. As $\Pi^p(S)$ is compact we must find a subsequence $\{\pi^{\alpha(n)}\}_{n \geq 1}$ (where $\alpha : \mathbb{N} \to \mathbb{N}$ is increasing) that converges toward $\pi^* \in \Pi^p(S)$. Then we have :

$$lim_{(n \to +\infty)} \mid \pi^{\alpha(n)} - \pi^* \mid = 0 \text{ and } lim_{(n \to +\infty)} \mid \bar{\eta}_\pi^{\alpha(n)} \mid = 0.$$

Thus, one can show that to π^* is associated a unique $\bar{\eta}_{\pi^*} = 0$. Thus $\pi^* \notin \Pi^p(S)$, which contradicts the fact that $\Pi^p(S)$ is compact. Then we can write that there exists $\bar{\eta} > 0$ such that for all $\eta < \bar{\eta}$, for all $\pi \in \Pi^{p-\eta}(S)$, there exists $i \in S$ such that

$$\sum_{k \in A} \pi(k)u(i,k) > \sum_{k \in A} \pi(k)u(j,k) \ \forall j \notin S. \quad (5)$$

This proves that there exists $\bar{\eta}$ such that for all $0 < \eta < \bar{\eta}$, $\Lambda[S, p-\eta] \subseteq S$. Now we need to show that S is minimal among the $(p-\eta)-$best response set. Since S is a $p-$MBR, we know that for any $S' \subsetneq S$, $\Lambda[S', p] \nsubseteq S'$. Then as $\Lambda[S', p] \subseteq \Lambda[S', p-\eta]$ (see remark 2), it is clear that for any $S' \subsetneq S$, $\Lambda[S', p-\eta] \nsubseteq S'$. Then S is a $(p-\eta)-$MBR set. ■

Now, the following lemma is sufficient to prove Theorem 2:

Lemma 2 *For* $S \subseteq A$. *S satisfies* $\Psi[\Phi^{Li}(S)] \subseteq \Phi^{Li}(S)$ *if and only if* $\Lambda[S, \frac{1}{2+\delta}] \subseteq S$.

Proof. (i) We prove the "if" part. For any initial condition $x^0 \in \Delta^{n-1}$, the expected discounted payoff to action j at t along the path $\phi(.)$ that is linear toward S is given by

$$V_j(\phi)(t) = \sum_{k \in A} \pi(k) u_{jk}$$

where $\pi \in \Delta(A)$ is such that for all $k \notin S$, (recall that $\phi_k(t) = x_k^0 e^{-\lambda t}$ for all $k \notin S$)

$$\begin{aligned} \pi(k) &= (\lambda+\theta) \int_0^{+\infty} e^{-(\lambda+\theta)s} x_k^0 e^{-\lambda(t+s)} ds \\ &= (\lambda+\theta) e^{-\lambda t} x_k^0 \int_0^{+\infty} e^{-(2\lambda+\theta)s} ds = x_k^0 e^{-\lambda t} \frac{1+\delta}{2+\delta}. \end{aligned}$$

Then for almost all $t \geq 0$,

$$\begin{aligned} \sum_{i \in S} \pi(i) &= 1 - \sum_{k \notin S} \pi(k) \\ &= 1 - \sum_{k \notin S} x_k^0 e^{-\lambda t} \frac{1+\delta}{2+\delta} \geq 1 - \frac{1+\delta}{2+\delta} \geq \frac{1}{2+\delta}, \end{aligned}$$

(since $e^{-\lambda t} \leq 1$ and $0 \leq \sum_{k \notin S} x_k^0 \leq 1$).
Since $\Lambda[S, \frac{1}{2+\delta}] \subseteq S$, we have that for all $t \geq 0$, there exists $l \in S$ such that $V_l(\phi)(t) > V_k(\phi)(t)$ for all $k \notin S$ *i.e.* for all $t \geq 0$, $BR(\phi)(t) \subseteq S$. So we go through a linear path toward S.

(ii) The "only if" part is proved by contradiction.

We know that $\Lambda[S, \frac{1}{2+\delta}] \nsubseteq S$. There then exists a probability distribution $\bar{\pi} \in \Delta(A)$ with $\sum_{k \in S} \bar{\pi}(k) = \frac{1}{2+\delta}$ such that for some $j \notin S$,

$$\sum_{k \in A} \bar{\pi}(k) u_{ik} \leq \sum_{k \in A} \bar{\pi}(k) u_{jk} \text{ for all } i.$$

Take the linear path $\phi \in \Phi^{Li}(S)$ such that $\phi(t) = x^1 - (x^1 - x^0)e^{-\lambda t}$ where $x_k^0 = 0$ and $x_k^1 = (2+\delta)\bar{\pi}(k)$ for all $k \in S$ and for all $k' \notin S$, $x_{k'}^0 = \frac{2+\delta}{1+\delta}\bar{\pi}(k)$ and $x_{k'}^1 = 0$. It follows that at $t = 0$:

$-$for $k \in S$, $\pi(k) = (\lambda+\theta) \int_0^{\infty} e^{-(\lambda+\theta)s} x_k^1 (1 - e^{-\lambda s}) ds = x_k^1 \frac{1}{2+\delta} = \bar{\pi}(k)$,

$-$for $k' \notin S$, $\pi(k') = (\lambda+\theta) \int_0^{\infty} e^{-(\lambda+\theta)s} x_{k'}^0 e^{-\lambda s} ds = x_{k'}^0 \frac{1+\delta}{2+\delta} = \bar{\pi}(k')$.

Then, the set of best responses to the linear path from x^0 to x^1 does not belong to S, *i.e.* for the linear path $\phi(t) = x^1 - (x^1 - x^0)e^{-\lambda t}$, $BR(\phi)(0) \nsubseteq S$. Then clearly, $\Psi[\Phi^{Li}(S)] \nsubseteq \Phi^{Li}(S)$. This proves lemma 2. ■

As noted previously, the proof of Theorem 2 can be completed very easily. Since by the previous lemma, for $\delta > 0$, the $\frac{1}{2+\delta}-$MBR coincide with the LSS. And since by lemma 1 there exists $\bar{\eta} > 0$ such that for all $0 < \eta < \bar{\eta}$ the $\frac{1}{2}-$MBR set is also the $(\frac{1}{2} - \eta)-$MBR set. Then one can check that for $\delta < \frac{4\bar{\eta}}{1-2\bar{\eta}}$, the $\frac{1}{2}-$MBR set coincide with the LSS.

Then we obtain Proposition 1 as a corollary of this latest result. To see why it is so, note that with the previous lemma, we know that for any degree of friction δ, the LSS coincides with the $\frac{1}{2+\delta}-$MBR. Since $\frac{1}{2+\delta} \leq \frac{1}{2}$, we know by Theorem 1 that the $\frac{1}{2+\delta}-$MBR exists and is unique. Therefore, this proves existence and the uniqueness of the LSS.

B Appendix B

In this part, we prove Theorem 3. Then, we note that Proposition 2 is a straightforward corollary of Theorem 3. Let us prove a useful lemma:

Lemma 3 *There exists $\bar{\gamma} > 0$ such that if S is the $\frac{1}{2}-$MBR, then for all $\gamma < \bar{\gamma}$, S is the unique $(\frac{1}{2} + \gamma)-$MBR.*

Proof. We know that for every $\gamma > 0$, $\Lambda[S, \frac{1}{2} + \gamma] \subseteq S$. Since we have assumed that the game is generic in the $\frac{1}{2}-$MBR, we have that for γ sufficiently small, S is still a $(\frac{1}{2} + \gamma)-$MBR. Now, let us prove that S is still the unique $(\frac{1}{2} + \gamma)-$MBR for γ sufficiently small. Assume that it is not true, then for every $\gamma > 0$, there exists $S' \neq S$ such that S' is a $(\frac{1}{2} + \gamma)-$MBR set. Then by Lemma 1, there must exist $S' \neq S$ that is a $\frac{1}{2}-$MBR set contradicting our uniqueness result of Theorem 1. ■

Now, in order to prove Theorem 3, it is sufficient to prove the following lemma.

Lemma 4 *For $S \subseteq A$. For every $\psi \in \Phi^0$ with $\psi(0) \in \tilde{\Delta}(S)$, $BR(\psi)(0) \subseteq S$ if and only if $\Lambda[S, \frac{1+\delta}{2+\delta}] \subseteq S$.*

Proof. (i) We first prove the "if part". Take any feasible path ψ with $\psi(0) \in \tilde{\Delta}(S)$. We want to show that $BR(\psi)(0) \subseteq S$. We know that the expected discounted payoff to action j at time 0 along a path ψ is written as

$$V_j(\psi)(0) = \sum_{k \in A} \pi(k) u_{jk},$$

where $\pi \in \Delta(A)$ is given by

$$\pi(k) = (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} \psi_k(s) ds.$$

Since $\psi(0) \in \tilde{\Delta}(S)$ holds, we have $\sum_{i \in S} \psi_i(s) \geq e^{-\lambda s}$, and therefore,

$$\sum_{i \in S} \pi(i) \geq (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} e^{-\lambda s} ds = \frac{1+\delta}{2+\delta}.$$

Since $\Lambda[S, \frac{1+\delta}{2+\delta}] \subseteq S$, it follows that there exists $j \in S$ such that $V_j(\psi)(0) > V_l(\psi)(0)$ for all $l \notin S$.

(ii) Let us prove by contradiction the "only if" part. Assume that $\Lambda[S, \frac{1+\delta}{2+\delta}] \nsubseteq S$. It is clear that there exists a probability distribution $\bar{\pi} \in \Delta(A)$ with $\sum_{i \in S} \bar{\pi}(i) = \frac{1+\delta}{2+\delta}$ such that for some $j \notin S$,

$$\sum_{k \in A} \bar{\pi}(k) u_{ik} \leq \sum_{k \in A} \bar{\pi}(k) u_{jk} \text{ for all } i.$$

Take the linear path $\psi(t) = x^1 - (x^1 - x^0)e^{-\lambda t}$ where $x_k^0 = \frac{2+\delta}{1+\delta}\bar{\pi}(k)$ and $x_k^1 = 0$ for all $k \in S$. $x_{k'}^0 = 0$ and $x_{k'}^1 = (2+\delta)\bar{\pi}(k')$ for all $k' \notin S$. Note that $x^0 = (x_1^0, ..., x_I^0) \in \tilde{\Delta}(S)$. We must prove that $BR(\psi)(0) \nsubseteq S$. We know that

$$V_j(\psi)(0) = \sum_{k \in A} \pi(k) u_{jk},$$

where,

$-$for $k \in S$, $\pi(k) = (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} x_k^0 e^{-\lambda s} ds = x_k^0 \frac{1+\delta}{2+\delta} = \bar{\pi}(k)$;

$-$for $k' \notin S$, $\pi(k') = (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} x_{k'}^1 (1 - e^{-\lambda s}) ds = x_{k'}^1 \frac{1}{2+\delta} = \bar{\pi}(k')$.

Then there exists $j \notin S$ such that $V_j(\psi)(0) \geq V_i(\psi)(0)$ for all $i \in A$. This completes the proof. ■

As noted previously, the proof of Theorem 3 can be completed very easily. Since, by the previous lemma, for $\delta > 0$, a $\frac{1+\delta}{2+\delta}-$MBR coincides with an absorbing set. And since by Lemma 3, there exists $\bar{\gamma} > 0$ such that if S is the $\frac{1}{2}-$MBR, then for all $\gamma < \bar{\gamma}$, S is the unique $(\frac{1}{2} + \gamma)-$MBR. It is clear that for $\delta < \frac{4\bar{\gamma}}{1-2\bar{\gamma}}$, the $\frac{1}{2}-$MBR coincides with the absorbing set.

Since the $\frac{1}{2}-$MBR set exists and is unique (Theorem 1), it is straightforward that we obtain Proposition 2 as a corollary of Theorem 3.

C Appendix C

In this part, we first prove Theorem 5. Then Theorem 4 is easily obtained.

Let us first recall that $(S^0, S^1, ..., S^n)$ is the sequence such that **at every** $\omega \in \Omega$, S^h is a $p-$best response set in $G[S^{h-1}, \omega]$ for any $h \in \{1, ..., n\}$ where $S^0 = A$ and $S^n = S$. In order to prove this part, we build a sequence of modified incomplete information game $\{U^h\}_{h=0}^{n-1}$ in the following way.

$U^h = [I, \{S_i^h\}_{i=\{1,2\}}, \{u_i^h\}_{i=\{1,2\}}]$, where I is $U's$ information system; and $u_i^h : S^h \times \Omega \to \mathbb{R}$ is player i's payoff function given by the restriction of u_i to $S^h = S_1^h \times S_2^h \subseteq A$. Therefore in this modified incomplete information game, only the set of available actions change. In the sequel, we note $R_i^h(\omega)$ for the set of rationalizable actions of i in the incomplete information game U^h at state ω. Now we provide two simple lemmas:

Lemma 5 *Let $h, h' \in \{0, ..., n-1\}$ where $h' \geq h$. If $R_i^h(\omega) \subseteq S_i^{h'}$ for all $i \in \{1,2\}, \omega \in \Omega$, then $R_i^h(\omega) \subseteq R_i^{h'}(\omega)$ for all $i \in \{1,2\}, \omega \in \Omega$.*

Proof. It is well-known[20] that for all $a_i \in R_i^h(\omega)$, there exists $\mu_{-i} : \Omega \to \Delta(S_{-i}^h)$ such that (1) for all $a_i' \in S_i^h$,

$$\sum_{\omega' \in Q_i(\omega)} \Pi[\omega' \mid Q_i(\omega)] u_i^h(\{a_i, \mu_{-i}(\omega')\}, \omega') \geq \sum_{\omega' \in Q_i(\omega)} \Pi[\omega' \mid Q_i(\omega)] u_i^h(\{a_i', \mu_{-i}(\omega')\}, \omega').$$

(2) $Supp(\mu_{-i}(\omega')) \subseteq R_{-i}^h(\omega')(\subseteq S_{-i}^{h'})$ for all $\omega' \in \Omega$. Then since from U^h to $U^{h'}$ only the set of available actions change, we have that for all $a_i \in R_i^h(\omega)$, there exists $\mu_{-i} : \Omega \to \Delta(S_{-i}^{h'})$ such that (1) for all $a_i' \in S_i^{h'}$,

$$\sum_{\omega' \in Q_i(\omega)} \Pi[\omega' \mid Q_i(\omega)] u_i^{h'}(\{a_i, \mu_{-i}(\omega')\}, \omega') \geq \sum_{\omega' \in Q_i(\omega)} \Pi[\omega' \mid Q_i(\omega)] u_i^{h'}(\{a_i', \mu_{-i}(\omega')\}, \omega').$$

(2) $Supp(\mu_{-i}(\omega')) \subseteq R_{-i}^h(\omega')(\subseteq S_{-i}^{h'})$ for all $\omega' \in \Omega$. Then this implies that $R_i^h(\omega) \subseteq R_i^{h'}(\omega)$ and completes the proof. ■

Lemma 6 *For any $h \in \{0, ..., n-1\}$, for each $i \in \{1,2\}$, for each $\omega \in \Omega$, $R_i^h(\omega) \subseteq S_i^{h+1}$.*

Proof. Fix $h \in \{0, ..., n-1\}$, and consider the incomplete information game U^h. Let us define $\Omega_i^{h+1} = \{\omega \in \Omega : R_i^h(\omega) \subseteq S_i^{h+1}\}$ and $\Omega_j^{h+1} = \{\omega \in \Omega : R_j^h(\omega) \subseteq S_j^{h+1}\}$, Ω_j^{h+1} is the event where player j plays in S_j^{h+1}. Let $E = Q_i(\omega^*)$, we know that, since the payoff function is measurable with respect to the partition, player i knows his payoffs at E and by (3), he will play in S_i, then $E \subseteq \Omega_i^{h+1}$. Moreover by (2), we know that S^{h+1} is a $\sigma-$best response set in $G[S^h, \omega]$ at any $\omega \in \Omega$. Then it must be that $B_j^\sigma \Omega_i^{h+1} \subseteq \Omega_j^{h+1}$ and $B_i^\sigma \Omega_j^{h+1} \subseteq \Omega_i^{h+1}$. Since $E \subseteq \Omega_i^{h+1}$, $B_j^\sigma(E) \subseteq B_j^\sigma(\Omega_i^{h+1}) \subseteq \Omega_j^{h+1}$. Then, $B_i^\sigma B_j^\sigma(E) \subseteq B_i^\sigma B_j^\sigma(\Omega_i^{h+1}) \subseteq B_i^\sigma(\Omega_j^{h+1}) \subseteq \Omega_i^{h+1}$. Thus, $B_i^\sigma B_j^\sigma(E) \cup E \subseteq \Omega_i^{h+1}$. Then by induction, $\forall k \geq 1$, $[H_i^\sigma]^k(E) \subseteq \Omega_i^{h+1}$. But by (1), $\exists k \geq 0$, such that $[H_i^\sigma]^k(E) = \Omega$. Therefore, $\Omega_1^{h+1} = \Omega_2^{h+1} = \Omega$. ■

[20] The proof uses standard arguments relying on rationalizability (see for details Pearce (1984) or Battigalli and Siniscalchi (2003)).

Now the proof of the Theorem can be completed. By Lemma 6, we know that for each $i \in \{1,2\}$, for each $\omega \in \Omega$, $R_i^0(\omega) \subseteq S_i^1$. Then, by Lemma 5, $R_i^0(\omega) \subseteq R_i^1(\omega)$ for all $i \in \{1,2\}, \omega \in \Omega$. Then again by Lemma 6, for each $i \in \{1,2\}$, for each $\omega \in \Omega$, $R_i^0(\omega) \subseteq (R_i^1(\omega) \subseteq) S_i^2$. Then again, by Lemma 5, $R_i^0(\omega) \subseteq R_i^2(\omega)$ for all $i \in \{1,2\}, \omega \in \Omega$ and so, for each $i \in \{1,2\}$, for each $\omega \in \Omega$, $R_i^0(\omega) \subseteq (R_i^2(\omega) \subseteq) S_i^3$. The proof of Theorem 5 is completed in iterating this reasoning.

Now note that if S is a $\sigma-$best response set at each $G[A,\omega]$, then by definition, S is an iterated $\sigma-$best response set that trivially satisfies the CSA (with the sequence $(S^0, S^1) = (A, S)$). Thus, Theorem 4 is obtained as a corollary of Theorem 5.

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