A dynamic two-country model of international trade

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Abstract

We present a dynamic two-country model of international trade with endogenous time preference. We show that if the two countries have similar preferences, production technologies, and labor endowments, there exists a unique and stable steady state such that both consumption and investment goods are produced in both countries. Unlike the case of constant time preferences, the steady state is independent of the initial international distribution of capital. We prove a dynamic Heckscher-Ohlin theorem such that the labor-abundant country exports the labor-intensive good.

- The paper is still very preliminary and incomplete. Please do not quote without the authors' permission.

1 Introduction

The determination of trade pattern is a central topic in international economics. This paper presents a dynamic Heckscher-Ohlin model that explains the long-run pattern of international trade.

The literature on dynamic Heckscher-Ohlin models originated in Oniki and Uzawa (1965). While Oniki and Uzawa assumed exogenous saving rate in each trading country, most subsequent contributions, including Stiglitz (1970), Baxter (1992), Chen (1992), Shimomura (1993), Ventura (1997), Atkeson and Kehoe (2000), and Nishimura and Shimomura (2002) assume that households maximize their discounted sum of utility, i.e., saving rates are endogenously determined.

Another common assumption shared in most of the literature is that the rate of time preference in each country is exogenously given and constant[1]. It is well known that in the standard two(-country, say Home and Foreign) by two(-factor, capital and labor) by two(-good, a pure consumption good and an investment good) dynamic Heckscher-Ohlin model with constatnt-returns-to-scale technologies this assumption and incomplete specialization in Home together imply that the rate of time preference is equal to the real interest rate in the steady state, i.e.,

$$\rho = r(p) - \delta, \tag{1}$$

where ρ, r, p, and δ denote the rate of time preference, the rental rate, the price of the pure consumption good in terms of the investment good, and the rate of capital depreciation. Note that it depends on the factor-intensity ranking whether $r(p)$ is increasing or decreasing in p in such a way that the Stolper-Samuelson theorem describes. If incomplete specialization holds in Foreign, we also have

$$\rho^* = r^*(p) - \delta^*, \tag{2}$$

where the variables attached an asterisk (*) are those belonging to Foreign.

Comparing (1) and (2), we see that it is purely a matter of chance that those equations hold together. That is, except for "measure zero" cases, production in at least either Home or Foreign is completely specialized. For example, suppose that production technologies are common in both countries, which implies the two functions $r(p)$ and $r^*(p)$ are the same with each other. Then, we can check that if $\rho + \delta >$ (resp. $<$)$\rho^* + \delta^*$, Home (resp. Foreign) is completely specialized to the production of the labor-intensive good and/or Foreign (resp. Home) is completely specialized to the production of the capital-intensive good.

As Baxter (1992) made clear, this property of the standard dynamic Heckscher-Ohlin model has an implication such that capital income taxation adopted by Home and Foreign governments drastically affects the long-run production/trade structure. Denote by τ and τ^* the Home and Foreign rates of tax on capital income. Then, under internationally identical production technologies (1) and (2) are replace by

$$\rho = (1 - \tau)r(p) - \delta$$

and

$$\rho^* = (1 - \tau^*)r(p) - \delta^*,$$

which means that the production/trade structure drastically change, according as the fiscal policies employed by Home and Foreign governments. One can find similar drastic properties under trade policies like tariff and subsidy.

The purpose of this paper is to construct a dynamic general equilibrium model of international trade in which a change in parameters and policy variables continuously influences on the steady-state production/trade structure, at least as long as the change is sufficiently small. Making use of such a model, one can derive fundamental propositions concerning the relationship between trade

[1] Shimomura (1993) realtes the pattern of international tarde to the international difference in the rate of tme preferences. Thus it is an exception.

pattern and international differences in preferences, technologies and factor endowments.

What makes this purpose achievable? That is one of Uzawa's ingenious invention, the discount rate function[2], $\rho(c)$, where c is the Home consumption and $\rho'(c)$ is assumed to be positive. Replacing it by the above constant rate of time preferences, we have

$$\rho(c) = (1-\tau)r(p) - \delta \tag{3}$$

and

$$\rho^*(c^*) = (1-\tau^*)r(p) - \delta^* \tag{4}$$

As we see later, these two equations are compatible with each other even if the rate of capital income tax, the rate of capital depreciation, discount rate functions, and the rental rate functions are internationally different. These two equations and the world market-clearing condition determines the steady-state c, c^* and the equilibrium price p of the dynamic general equilibrium model in this paper. Hence, under the conditions for the existence, uniqueness and (saddlepoint-)stability, which will be obtained in this paper, we can study the aforementioned relationship.

Section 2 sets the main assumptions and describes the model. Section 3 states the basic technical proposition concerning the existence, uniqueness and stability of the steady state in which the production of both Home and Foreign is incompletely specialized. Section 4 derives the trade-pattern propositions. Section 5 concludes. Appendix proves the basic technical proposition.

2 The model

Let us set up the two-country dynamic general equilibrium model.

2.1 Consumers

Each consumer maximizes the discounted sum of utility subject to her budget constraint.

$$\max \int_0^\infty u(c)X dt \tag{5}$$

subject to

$$\begin{aligned} \dot{a} &= ra + wl - pc && (6) \\ \dot{X} &= -\rho(c)X, && (7) \end{aligned}$$

where a is her net (physical and financial) asset, r is the real interest rate, w is the wage rate, l is her labor supply, and p is the price of the pure consumption good in terms of the investment good.

[2] See a now-classical article, Uzawa (196).

$u(c)$ is the felicity function of her consumption c with the properties $u'(c) > 0$ and $u''(c) < 0$ for any $c > 0$. We further assume that $\lim_{c \to 0} u(c) < 0$ and that there is $\bar{u}' > 0$ such that $\lim_{c \to \infty} u'(c) \equiv \bar{u}' > 0$ and for any $c > 0$ $c\bar{u}' - u(c) > 0$. Figure 1 illustrates an example of $u(c)$ function that satisfies those properties. It is clear that for any $c > 0$ the elasticity $|cu'(c)/u(c)| > 1$.

X is the discount factor with $X(0) = 1$.Following Uzawa (.), we assume that $\rho'(c) > 0$ for any $c > 0$. We also assume that $\rho(0) \equiv \rho_0 > 0$ and $\rho''(c) \geq 0$ and there is a positive value B such that $\lim_{c \to \infty} \rho'(c) = \lim_{c \to \infty} \frac{\rho(c)}{c} \equiv 1/B > 0$ and for any $c > 0$ $\rho(c) - c/B > 0$. Figure 2 illustrates an example of $\rho(c)$ function that satisfies those properties. It is clear that for any $c > 0$ the elasticity $c\rho'(c)/\rho(c) < 1$.

Associated with the above problem is the Hamiltonian

$$H = u(c)X + \lambda[ra + wl - pc] - \theta\rho(c)X \tag{8}$$

The necessary conditions for optimality are

$$\begin{aligned} 0 &= u'(c)X - \lambda p - \theta\rho'(c)X && (9)\\ \dot{\lambda} &= \lambda r && (10)\\ \dot{\theta} &= -u(c) + \theta\rho(c) && (11) \end{aligned}$$

Letting $y \equiv \lambda/X$. we can rewrite the conditions (9) and (10) as

$$\begin{aligned} yp &= u'(c) - \theta\rho'(c) && (12)\\ \dot{y} &= y[\rho(c) - r] && (13) \end{aligned}$$

2.2 Firms

There are two sectors in each country that produce a pure consumption good and an investment good by using physical capital and labor, respectively. Following the Oniki-Uzawa tradition, we assume that while the two goods are tradable, the existing capital is not internationally mobile and depreciates at a constant rate δ. We assume away an international credit market, while each country has a competitive domestic financial market. Thus, through arbitration, the real interest rate is equal to the rental rate, say R, minus the rate of depreciation and the net asset is equal to the capital stock at each point in time. That is,

$$r = R - \delta \text{ and } A \equiv \sum a = \sum k \equiv K^3$$

Both sectors are competitive and the production technology in each sector is described by a neoclassical CRS production function. If both sectors produce

[3] A and K are the aggregate national financial asset and the aggregate national physical capital stock, respectively.

respective goods, the price of each good is equal to its unit cost.

$$p = \Lambda^C(w, R) \quad (14)$$
$$1 = \Lambda^I(w, R), \quad (15)$$

where $\Lambda^C(w, R)$ is the unit cost of the pure consumption good and $\Lambda^I(w, R)$ is the unit cost of the investment good.

The unit cost functions have all the standard properties that are usually assumed. Moreover, we impose a couple of further conditions on them. First, for any positive p, w and R satisfying (14) and (15) are uniquely determined, $R(p)$ and $w(p)$[4]. Second, factor intensity reversal is assumed away, and either (I) or (II) below holds.

(I) The consumption good is more capital-intensive than the investment good in the sense that for any $p > 0$

$$\frac{\frac{\partial}{\partial R}\Lambda^C(w(p), R(p))}{\frac{\partial}{\partial w}\Lambda^C(w(p), R(p))} > \frac{\frac{\partial}{\partial R}\Lambda^I(w(p), R(p))}{\frac{\partial}{\partial w}\Lambda^I(w(p), R(p))} \quad (16)$$

(II) The consumption good is more labor-intensive than the investment good in the sense that for any $p > 0$

$$\frac{\frac{\partial}{\partial R}\Lambda^C(w(p), R(p))}{\frac{\partial}{\partial w}\Lambda^C(w(p), R(p))} < \frac{\frac{\partial}{\partial R}\Lambda^I(w(p), R(p))}{\frac{\partial}{\partial w}\Lambda^I(w(p), R(p))} \quad (17)$$

Third, the partial derivative of the national income

$$w(p)L + R(p)K,$$

where $L \equiv \sum l$, the aggregate labor supply, is equal to the aggregate national output of the consumption good.

$$Y_C(p) \equiv w\prime(p)L + R'(p)K$$

It is also well known in trade theory that the second derivative of the national income function is positive, $Y_C'(p) > 0$ when both goods are produced.

2.3 The dynamic system of a two-country world

We assume that there are two countries, Home and Foreign, which may have different production technologies, preferences and initial factor endowments. The population of each country is normalized to be one. Based on the foregoing

[4] If production technologies are of Cobb-Douglas, the unit-cost functions are also Cobb-Douglas,

$$\Lambda^i(w, R) = w^{\alpha_i} R^{1-\alpha_i}, \ 0 < \alpha_i < 1.$$

If $\alpha_1 \neq \alpha_2$, the system of equations, (14) and (15), has a unique soluition for any given $p > 0$.Denote it by a pair $(w(p), R(p))$.

argument, the dynamic general equilibrium two-coutry model can be described as follows.

$$
\begin{aligned}
yp &= u'(c) - \theta\rho'(c) && (18)\\
y^*p &= u^{*\prime}(c^*) - \theta^*\rho^{*\prime}(c^*) && (19)\\
\dot{K} &= (R(p) - \delta)K + w(p)L - pc && (20)\\
\dot{K}^* &= (R^*(p) - \delta^*)K^* + w^*(p)L^* - pc^* && (21)\\
\dot{y} &= y[\rho(c) + \delta - R(p)] && (22)\\
\dot{y}^* &= y^*[\rho^*(c^*) + \delta^* - R^*(p)] && (23)\\
\dot{\theta} &= -u(c) + \theta\rho(c) && (24)\\
\dot{\theta}^* &= -u^*(c^*) + \theta^*\rho^*(c^*) && (25)\\
0 &= w'(p)L + w^{*\prime}(p)L^* + R'(p)K + R^*(p)K^* - c - c^*, && (26)
\end{aligned}
$$

where an asterisk (*) is attached to foreign variables and functions. The state variables are K and K^*, and the jump variables are c, c^*, p, y, y^*, θ and θ^*.

3 The steady state

Let us define the steady state as (K^e, K^{*e}, c^e, c^{*e}, p^e, y^e, y^{*e}, θ^e, θ^{*e}) a time-invariant solution to the above dynamic system. Thus, the following equalities are established.

$$
\begin{aligned}
y^e p^e &= u'(c^e) - \theta^e\rho'(c^e) && (27)\\
y^{e*}p^e &= u^{*\prime}(c^{e*}) - \theta^{e*}\rho^{*\prime}(c^{e*}) && (28)\\
0 &= (R(p^e) - \delta)K^e + w(p^e)L - p^e c^e && (29)\\
0 &= (R^*(p^e) - \delta^*)K^{e*} + w^*(p^e)L^* - p^e c^{e*} && (30)\\
0 &= y^e[\rho(c^e) + \delta - R(p^e)] && (31)\\
0 &= y^{e*}[\rho^*(c^{e*}) + \delta^* - R^*(p^e)] && (32)\\
0 &= -u(c^e) + \theta^e\rho(c^e) && (33)\\
0 &= -u^*(c^{e*}) + \theta^{e*}\rho^*(c^{e*}) && (34)\\
0 &= w'(p^e)L + w^{*\prime}(p^e)L^* + R'(p^e)K^e + R^{*\prime}(p^e)K^{e*} - c^e - c^{e*}, && (35)
\end{aligned}
$$

Concerning the steady state, we can prove the following proposition.

THE BASIC TECHNICAL PROPOSITION ON THE EXISTENCE, UNIQUENESS AND STABILITY OF THE STEADY STATE: *Suppose that the differences in preferences, technologies and initial factor endowments between Home and Foreign are not very large. Then, there exists a unique steady state,* ($K^e > 0$, $K^{*e} > 0$, $c^e > 0$, $c^{*e} > 0$, $p^e > 0$, $y^e > 0$, $y^{*e} > 0$, $\theta^e > 0$, $\theta^{*e} > 0$), *which is saddlepoint-stable and does not exist a steady state such that complete specialization holds in at least one country.*

Proof: See Appendix.

REMARK: One may naturally wonder if there is a steady state such that at least one country is completely specialized to the production of either the pure consumption good or an investment good. The Appendix shows, however, that such complete specialization is impossible as long as the above differences are not very large.

4 Trade-Pattern Propositions

Let us focus on the Home excess demand for the pure consumption good in the steady state,

$$ED(p) \equiv c(R(p) - \delta) - [w'(p)L + R'(p)K(p)], \tag{36}$$

where $c(.)$ is the inverse function of the discount rate function $\rho(.)$ and

$$K(p) \equiv \frac{pc(R(p) - \delta) - w(p)L}{R(p) - \delta} \tag{37}$$

Denote by p^a the autarkic equilibrium price, $ED(p) = 0$. If the two countries are completely identical, the steady-state price p^e has to be equal to p^a.

Differentiating $ED(p)$ with respect to p, we obtain

$$\begin{aligned} \frac{d}{dp}ED(p) &= [1 - \frac{pR'(p)}{R(p)}\frac{R(p)}{(R(p)-\delta)}]R'(p)c'(R(p)-\delta) \\ &\quad -[w''(p)L + R''(p)K(p)] - \frac{R'(p)ED(p)}{R(p)-\delta} \end{aligned}$$

Since $[1 - \frac{pR^0(p)}{R(p)}\frac{R(p)}{(R(p)-\delta)}]R'(p) < 0$ due to the Stolper-Samuelson theorem and $[w''(p)L + R''(p)K(p)] > 0$[5], irrespective of the factor-intensity ranking between the two sectors, it follows that the slope of the excess demand curve is negative in a neighborhood of p^a[6]. Based on this fact, we obtain the following trade-pattern propositions.

4.1 Labor endowments

Substituting (37), let us rewrite (36) as

$$\begin{aligned} ED(p) &= [1 - \frac{pR'(p)}{R(p)}\frac{R(p)}{(R(p)-\delta)}]c(R(p)-\delta) \\ &\quad +[\frac{pR'(p)}{R(p)}\frac{R(p)}{(R(p)-\delta)} - \frac{pw'(p)}{w(p)}]\frac{w(p)L}{p} \end{aligned} \tag{38}$$

[5] This inequality holds under the standard neocassical tecnnologies and means that the supply function of the pure consumption good is positively sloped.

[6] Note that $ED(p^a) = 0$.

Suppose that Home and Foreign differ only in the labor endowments. If Foreign is labor abundant compared with Home ($L^* > L$) and the pure consumption good is more capital-intensive than the investment good ($\frac{pR^{\theta}(p)}{R(p)} > 1 > 0 > \frac{pw^{\theta}(p)}{w(p)}$). Then, we see from (38) that Foreign excess demand $ED^*(p)$ is larger than Home one for any given p. That is, the two excess demand curves can be depicted as in Figure 3A, which shows that the Home has the comparative advantage to the pure consumption good and exports it. If the pure consumption good is more labor intensive than the investment good ($\frac{pw^{\theta}(p)}{w(p)} > 1 > 0 > \frac{pR^{\theta}(p)}{R(p)}$), the two excess demand curves can be depicted as in Figure 3B. The trade pattern is reversed. Summarizing, we have the first main result.

PROPOSITION 1: *Suppose that Home and Foreign differ only in the labor endowments, L and L^*. In the steady state the labor abundant country exports (resp. imports) the labor- (resp. capital-)intensive good.*

4.2 Preferences

Next, let us assume that Foreign is more patient than Foreign in the sense that

$$\rho^*(c^*) \equiv \rho(c^*/\eta),$$

where $\eta > 1$ is a parameter. Then

$$c^* = \eta c(R(p) - \delta)$$

Thus, (38) can be written as

$$\begin{aligned} ED^*(p) &= [1 - \frac{pR'(p)}{R(p)} \frac{R(p)}{(R(p)-\delta)}]\eta c(R(p)-\delta) \\ &\quad +[\frac{pR'(p)}{R(p)} \frac{R(p)}{(R(p)-\delta)} - \frac{pw'(p)}{w(p)}]\frac{w(p)L}{p} \end{aligned} \tag{39}$$

Differentiating (39) with respect to η, we derive

$$\frac{\partial}{\partial \eta} ED^*(p) = [1 - \frac{pR'(p)}{R(p)} \frac{R(p)}{(R(p)-\delta)}]c(R(p)-\delta),$$

which is positive (resp. negative) if the pure consumption good is more labor-intensive than the investment good. Then, we can depict the excess demand curves of the impatient and patient countries in a way similar to Figure 3.

PROPOSITION 2: *Other things being equal, the patient country imports the labor-intensive good.*

4.3 Fiscal policy and trade pattern

Suppose that Home government imposes income tax and transfers the tax revenue to the Home households in a lump-sum manner. How the income taxation may affect the pattern of international trade?

First, let us consider the Home flow-budget constraint. Denoting by τ the rate of income tax, it is

$$\dot{K} = (1-\tau)[(R(p)-\delta)K + w(p)L] + T - pc,$$

where T is the lump-sum transfer from the government to the Home households, which is equal to $\tau[(R(p)-\delta)K + w(p)L]$. Solving for the utility-maximization problem for the Home household, we obtain, in the steady state

$$0 = \rho(c) - (1-\tau)(R(p)-\delta),$$

from which we have

$$c = c((1-\tau)(R(p)-\delta)) \tag{40}$$

Using this, and considering $T = \tau[(R(p)-\delta)K + w(p)L]$, we can rewrite the above flow-budget constraint in the steady state as

$$K(p) = \frac{pc((1-\tau)(R(p)-\delta)) - w(p)L}{R(p)-\delta} \tag{41}$$

From (40) and (41), we can express the steady-state excess demand as

$$\begin{aligned} ED(p) \quad = \quad & [1 - \frac{pR'(p)}{R(p)}\frac{R(p)}{(R(p)-\delta)}]c((1-\tau)(R(p)-\delta)) \\ & +[\frac{pR'(p)}{R(p)}\frac{R(p)}{(R(p)-\delta)} - \frac{pw'(p)}{w(p)}]\frac{w(p)L}{p} \end{aligned} \tag{42}$$

Differentiating (42) with respect to the income tax rate τ, we see that

$$\frac{\partial}{\partial\tau}ED(p) = -(R(p)-\delta)c' \cdot [1 - \frac{pR'(p)}{R(p)}\frac{R(p)}{(R(p)-\delta)}],$$

which implies that the imposition of the income tax shifts the excess demand curve to the right (resp. left) direction if the pure consumption good is more labor(resp. capital)-intensive.

PROPOSITION 3: *Other things being equal, a rise in the rate of income tax in Home makes the country imports the capital-intensive good*

4.4 Trade policy

Let us examine the effect of trade policy on the trade pattern. First, suppose that Home imports the pure consumption good under free trade and that the Home government imposes a small import tariff. Following the standard textbook of international trade, we assume that the Home government transfer the tariff revenue to the Home households in a lump-sum fashion. The flow-budget constraint of the Home household is

$$\dot{K} = (R(p+s) - \delta)K + w(p*s)L - pc + \Gamma$$

where Γ is the transfer of the tariff revenue

$$s \cdot [c - \{(R'(p+s)K + w(p+s)L\}]$$

In the steady state we have

$$c = c(R(p*s) - \delta) \tag{43}$$

Substituting it to the steady-state flow-budget constraint, we derive

$$K = K(p+s) \equiv \frac{pc(R(p*s) - \delta) - \{w(p+s) - sw'(p+s)\}}{R(p+s) - \delta - sR'(p+s)} \tag{44}$$

From (43) and (44), we can express the Home excess demand for the pure consumption good as

$$\begin{aligned} ED(p) &= c(R(p*s) - \delta) - \{R'(p+s)K(p+s) + w'(p+s)L\} \\ &\quad -\{\frac{R'(p+s)[pc(R(p*s) - \delta) - \{w(p+s) - sw'(p+s)\}]}{R(p+s) - \delta - sR'(p+s)} \\ &\quad +w'(p+s)L\} \end{aligned} \tag{45}$$

Differentiating (45) with respect to s and evaluating the derivative at $s = 0$, we have

$$\begin{aligned} \frac{\partial}{\partial s} ED(p)\bigg|_{s=0} &= c' \cdot R'(p)[1 - \frac{pR'(p)}{R(p)} \frac{R(p)}{(R(p) - \delta)}] \\ &\quad -[R''(p)K(p) + w''(p)L], \end{aligned}$$

which is negative, irrespective of the factor-intensity ranking.

Next, let us examine the case such that Home imports the investment good. Let $\bar{S}$, $P_1, P_2, \bar{W}$, and $\bar{R}$ be the tariff rate, the price of the pure consumption good, the price of the investment good, the wage rate and the rental rate in the nominal term. The flow-budget constraint is expressed as

$$\begin{aligned} (\bar{S} + P_2)\dot{K} &= \bar{R}K - \delta(\bar{S} + P_2)K + \bar{W}L - P_1 c \\ &\quad +\bar{S}[\delta K - \frac{1}{\bar{S} + P_2}\{\bar{R}K + \bar{W}L - P_1 Y_1)], \end{aligned}$$

where Y_1 is the Home output of the pure consumption good. Letting $\bar{s} \equiv \frac{\bar{S}}{S+P_2}, R \equiv \frac{\bar{R}}{S+P_2}, w \equiv \frac{\bar{W}}{S+P_2}, \bar{s} \equiv \frac{\bar{S}}{S+P_2}$ and $p \equiv \frac{P_1}{P_2}$, the above constraint can be rewritten as

$$\begin{aligned} \dot{K} &= (R-\delta)K + wL - pc \\ &\quad + \bar{s}[\delta K - \{RK + wL - (1-\bar{s})pY_1], \end{aligned}$$

where R and w are the functions of $(1-\bar{s})p$[7], and $Y_1 = R'((1-\bar{s})p)K + w'(1-\bar{s})p)L$. In the steady state we have

$$c = c(R((1-\bar{s})p) - \delta)$$

and

$$K = \frac{pc(R((1-\bar{s})p) - \delta) - \{w((1-\bar{s})p) + \bar{s}w'((1-\bar{s})p)\}}{R((1-\bar{s})p) - \delta + \bar{s}R'((1-\bar{s})p)}$$

Thus, in the steady state the excess demand for the pure consumption good is

$$\begin{aligned} ED(p) &= c((1-\bar{s})p) - \delta) - w'((1-\bar{s})p)L \\ &\quad - \frac{R'((1-\bar{s})p) - \delta)[pc(R((1-\bar{s})p) - \delta) - \{w((1-\bar{s})p) + \bar{s}w'((1-\bar{s})p)\}]}{R((1-\bar{s})p) - \delta + \bar{s}R'((1-\bar{s})p)} \end{aligned}$$

Differentiating the excess demand with respect to $\bar{s}$ and evaluating the derivative at $\bar{s} = 0$, we see that

$$\frac{\partial}{\partial \bar{s}} ED(p) = c' \cdot pR'[\frac{pR'}{R-\delta} - 1] + p(R''K + w''L) > 0$$

PROPOSITION 4: *The imposition of a small import tariff always makes the volume of import smaller.*

5 Concluding Remarks

We have presented a basic dynamic two-coutry model of international trade, which is regarded as an integration of Uzawa's two seminal contributions to economic theory. We show that there exists a unique and stable steady state with both countries being incompletely specialized. The steady state is independent of the initial international distribution of capital, which is different from the dynamic trade model with constant rate of time preference.

Since the model is nothing new and fundamental except that we introduce the Uzawa endogenous time preference, there seem to be many ways to discuss trade issues from the point of view of trade dynamics. One direction is to apply the model to the issues in the normative side of international trade like dynamic trade gains and international transfer.

[7] Note that $(1-\bar{s})p = \frac{P_2}{S+P_2}\frac{P_1}{P_2} = \frac{P_1}{S+P_2}$ is the dometic price of the pure consumption good.

6 References

[to be written]

7 Appendix

Here we shall prove the basic technical proposition.

7.1 the existence of the steady state with incomplete specialization

Assume that Home and Foreign are completely identical in preferences, technologies and endowments. Then, there exists a steady state if the following system of equations have a solution (K, p),

$$\begin{aligned} 0 &= (R(p)-\delta)K + w(p)L - pc(R(p)-\delta) && (46)\\ 0 &= R'(p)K + w'(p)L - c(R(p)-\delta) && (47) \end{aligned}$$

or, if the following system of equation

$$\begin{aligned} \Phi(p) &\equiv \frac{pc(R(p)-\delta)-w(p)L}{R(p)-\delta} - \frac{c(R(p)-\delta)-w'(p)L}{R'(p)} \\ &= [\frac{pR'(p)}{R(p)}\frac{R(p)}{R(p)-\delta} - 1]c(R(p)-\delta) \\ &\quad -\frac{w(p)}{R(p)}[\frac{R(p)}{R(p)-\delta} - \frac{pw'(p)/w(p)}{pR'(p)/R(p)}] \\ &= 0 && (48) \end{aligned}$$

has a solution.

It is well known from the Stolper-Samuelson Theorem that

- If the pure consumption good is more capital-intensive than the investment good, not only $R'(p) > 0$ and $w'(p) < 0$ but also $pR'(p)/R(p) > 1$.
- If the pure consumption good is more labor-intensive than the investment good, not only $R'(p) < 0$ and $w'(p) > 0$ but also $pw'(p)/w(p) > 1$.

Those properties are called magnification effects. In order to ensure the existence of the steady state, we specify production technologies a little more.

ASSUMPTION 1: (1) $R(0) = 0$, $\inf_{p>0} pR'(p)/R(p) > 1$ and $\infty > \sup_{p>0} |pw'(p)/w(p)| > 0$ if the pure consumption good is more capital-intensive good. (2) $w(0) = 0$, $\inf_{p>0} pw'(p)/w(p) > 1$ and $\infty > \sup_{p>0} |pR'(p)/R(p)| > 0$, if the pure consumption good is more labor-intensive good.

Cobb-Dopuglas technologies satisfy those properties.

LEMMA 1: *The system of equations, (46) and (47), has a unique and positive solution (p^e, K^e). The solution makes sense from the point view of economics in the sense that both goods are produced.*

Proof: First, assume that the pure consumption good is more capital-intensive than the investment good. Considering the properties of the function $\rho(.)$ stated in Section 2, we see that the inverse function $c(.)$ has to satisfy the following properties,

$$\begin{aligned} c(\rho_0) &= 0,\ c'(\rho) > 0 \text{ and } c''(\rho) \leq 0 \text{ for any } \rho > \rho_0, \\ \text{and } \lim_{\rho\to\infty} \frac{c(\rho)}{\rho} &= B > 0 \end{aligned} \tag{49}$$

Let p_0 be the solution to $R(p)-\delta = \rho_0$. Assumption 1(1) ensures the existence of p_0. It is clear from (48) that $\Phi(p_0) < 0$. Assumption 1(1) also implies that while the first term in (48) diverges to positive infinite as $p \to \infty$, the second term is bounded from the above. It follows that for a sufficiently large p, $\Phi(p_0) > 0$. Therefore, there exists a positive p^e such that $\Phi(p^e) = 0$.

From (46) and (47), we obtain

$$K^e/L = \frac{\frac{pc(R(p^e)-\delta)}{L} - w(p^e)}{R(p^e)-\delta} = \frac{\frac{c(R(p^e)-\delta)}{L} - w'(p^e)}{R'(p^e)} \tag{50}$$

Solving the second equality for $\frac{c(R(p^e)-\delta)}{L}$, we have

$$\frac{c(R(p^e)-\delta)}{L} = \frac{w(p)R'(p) - w'(p)(R(p)-\delta)}{pR'(p) - (R(p)-\delta)} \tag{51}$$

The substitution of (51) into (50) yields

$$K^e/L = \frac{w(p^e) - p^e w'(p^e)}{p^e R'(p^e) - (R(p^e)-\delta)} \tag{52}$$

It is well known that incomplete specialization, i.e., positive products in both production sectors, is guaranteed is K^e/L is in between the factor intensities of both sectors at $p = p^e$. The latter factor intensities are

$$K_1/L_1 = \frac{w(p^e) - p^e w'(p^e)}{p^e R'(p^e) - R(p^e)} \text{ (the pure consumption good)} \tag{53}$$

and

$$K_2/L_2 = -\frac{w'(p^e)}{R'(p^e)} \text{ (the investment good)} \tag{54}$$

Subtracting each of them from (52),

$$\begin{aligned} K^e/L - K_1/L_1 &= \frac{-\delta\{w(p^e) - p^e w'(p^e)\}}{\{p^e R'(p^e) - (R(p^e)-\delta)\}\{p^e R'(p^e) - R(p^e)\}} && (55) \\ K^e/L - K_2/L_2 &= \frac{R'(p^e)w(p^e) - w'(p^e)(R(p^e)-\delta)}{\{p^e R'(p^e) - (R(p^e)-\delta)\}R'(p^e)} && (56) \end{aligned}$$

It follows from (55) and (56) that

$$\min[K_1/L_1, K_2/L_2] < K^e/L < \max[K_1/L_1, K_2/L_2] \tag{57}$$

That is, incomplete specialization is established at (p^e, K^e).

Finally, let us prove the uniqueness. Differentiating $\Phi(p)$ with respect to p at $p = p^e$, we see that

$$\begin{aligned}\left.\frac{d\Phi(p)}{dp}\right|_{p=p^e} &= \frac{1}{R'(p^e)}[R'(p^e)\{\frac{p^e R'(p^e)}{R(p^e)}\frac{R(p^e)c'}{(R(p^e)-\delta)} \\ &\quad +(R''(p^e)K^e + w''(p^e)L)],\end{aligned}$$

which is *always* positive (resp. negative) if the pure consumption good is more capital(resp. labor)-intensive than the investment good[8]. Therefore, the pair $(p^e.K^e)$ is unique. (QED

Once the unique existence result is established for the case in which Home and Foreign are identical, we can assert that the result holds for the case in which the two countries are slightly different with each other, if the Jacobian is not zero at the symmetric case, which will be established at the subsequent subsection concerning stability.

7.2 the non-existence of the steady state with complete specialization

The foregoing result does not exclude the possibility that production is completely specialized at least in one of the two countries. Let us examine the possibility. Just for simplicity, we shall assume that $L = L^*$ and use the production functions $F^1(K, L)$ and $F^2(K, L)$ such that $F^i(0, L) = 0$, $F^i_K(K, L) > 0$ and $F^i_{KK}(K, L) < 0$, for any $K > 0$ and $i = 1, 2$.

Let us examine the case such that $R' > 0$. Suppose that Home is completely specialized to the production of the pure consumption good in the steady state $(\bar{p}^e, \bar{K}^e)$, while the production in Foreign is incompletely specialized at $(\bar{p}^e, \bar{K}^{*e})$. Let us define

$$\Delta_1(K) \equiv \bar{p}^e c(\bar{p}^e F^1_K(K, L)-\delta)-\{(\bar{p}^e F^1_K(K, L)-\delta)K+\bar{p}^e(F^1(K, L)-KF^1_K(K, L))\}$$

By definition, $\Delta_1(\bar{K}^e) = 0$. The differentiation of $\Delta_1(K)$ with respect to K yields

$$\Delta'_1(K) = (\bar{p}^e)^2 c' \cdot F^1_{KK}(K, L) - (\bar{p}^e F^1_K(K, L) - \delta),$$

which is negative, as long as $\bar{p}^e F^1_K(K, L) - \delta > 0$. Therefore, when K is equal to $(K_1/L_1)L$, (the factor intensity of the pure consumption good)$\times L$, Δ_1 has

[8] Note that $(R''(p^e)K^e + w''(p^e)L)$ is always positive as long as incomplete specialization holds.

to be positive.

$$\begin{aligned}
0 &< \Delta_1((K_1/L_1)L) \\
&= \bar{p}^e c(R(\bar{p}^e)-\delta) - [(R(\bar{p}^e)-\delta)(K_1/L_1) + w(\bar{p}^e)]L \\
&= \bar{p}^e c(R(\bar{p}^e)-\delta) - (R(\bar{p}^e)-\delta)(K_1/L_1)L - w(\bar{p}^e)L \\
&< \bar{p}^e c(R(\bar{p}^e)-\delta) - (R(\bar{p}^e)-\delta)\bar{K}^{*e} - w(\bar{p}^e)L \quad \because \bar{K}^{*e}/L < K_1/L_1 \\
&= 0,
\end{aligned}$$

a contradiction.

Next, suppose that Home is completely specialized to the production of the investment good in the steady state $(\bar{p}^e, \bar{K}^e)$, while the production in Foreign is incompletely specialized at $(\bar{p}^e, \bar{K}^{*e})$ Let us define

$$\Delta_2(K) \equiv \bar{p}^e c(F_K^2(K,L)-\delta) - \{(F_K^2(K,L)-\delta)K + (F^2(K,L) - KF_K^2(K,L))\}$$

By definition, $\Delta_2(\bar{K}^e) = 0$. The differentiation of $\Delta(K)$ with respect to K yields

$$\Delta_2'(K) = \bar{p}^e c' \cdot F_{KK}^2(K,L) - (F_K^2(K,K) - \delta),$$

which is negative, as long as $F_K^2(K,L) - \delta > 0$. Therefore, when K is equal to $(K_2/L_2)L$, (the factor intensity of the investment good)$\times L$, Δ_2 has to be negative.

$$\begin{aligned}
0 &> \Delta_2((K_2/L_2)L) \\
&= \bar{p}^e c(R(\bar{p}^e)-\delta) - (R(\bar{p}^e)-\delta)(K_2/L_2)L - w(\bar{p}^e)L \\
&> \bar{p}^e c(R(\bar{p}^e)-\delta) - (R(\bar{p}^e)-\delta)\bar{K}^{*e} - w(\bar{p}^e)L \quad \because \bar{K}^{*e}/L > K_2/L_2 \\
&= 0,
\end{aligned}$$

a contradiction. We can make a parallel argument for the factor-intensity ranking $R' < 0$.

What remains is the case such that each country is completely specialized to the production of a different good. Let us Without loss, let us assume that Home is specialized to the production of the pure consumption good, while Foreign to the production of the investment good. Let us focus on the factor-intensity ranking $R' > 0$.

First, let us note that $R' > 0$ means that

$$\begin{aligned}
0 &< \{\bar{p}^e F^1(\bar{K}^e, L) - \delta\bar{K}^e\} - (F^2(\bar{K}^{*e}, L) - \delta\bar{K}^{*e}) \\
&= [\bar{p}^e F_K^1(\bar{K}^e, L) - \delta)\bar{K}^e + \bar{p}^e(F^1(\bar{K}^e, L) - \bar{K}^e F_K^1(\bar{K}^e, L)] \\
&\quad -[(F_K^2(\bar{K}^{*e}, L) - \delta)\bar{K}^{*e} + (F^2(\bar{K}^{*e}, L) - \bar{K}^{*e} F_K^2(\bar{K}^{*e}, L))],
\end{aligned}$$

since the difference between OA and OD in Figure 4 is increasing in K. Second, $R' > 0$ also means that

$$\bar{p}^e F_K^1(\bar{K}^e, L) - \delta < F_K^2(\bar{K}^{*e}, L) - \delta$$

It follows that

$$
\begin{aligned}
0 &= \Delta_1(\bar{K}^e) \\
&= \bar{p}^e c(\bar{p}^e F_K^1(\bar{K}^e, L) - \delta) \\
&\quad -[\bar{p}^e F_K^1(\bar{K}^e, L) - \delta)\bar{K}^e + \bar{p}^e(F^1(\bar{K}^e, L) - \bar{K}^e F_K^1(\bar{K}^e, L)] \\
&< \bar{p}^e c(F_K^2(\bar{K}^{*e}, L) - \delta) \\
&\quad -[(F_K^2(\bar{K}^{*e}, L) - \delta)\bar{K}^{*e} + (F^2(\bar{K}^{*e}, L) - \bar{K}^{*e} F_K^2(\bar{K}^{*e}, L))] \\
&= \Delta_2(\bar{K}^e) \\
&= 0,
\end{aligned}
$$

a contradiction. Hence, it is impossible that two countries are completely specialized to the production of different goods when $R' > 0$. We can make a parallel argument when $R' < 0$. We arrive at the second lemma.

LEMMA 2: *When the two countries are sufficiently identical, it is impossible that there is a country whose production is completely specialized to one of the two goods in the steady state.*

7.3 local saddlepoint-stability

Let us assume that the two countries are identical. Our dynamic general equilibrium model is described as

$$
\begin{aligned}
\dot{K} &= (R(p) - \delta)K + w(p)L - pc \\
\dot{K}^* &= (R(p) - \delta)K^* + w(p)L - pc^* \\
\dot{y} &= y[\rho(c) + \delta - R(p)] \\
\dot{y}^* &= y^*[\rho(c^*) + \delta - R(p)] \\
\dot{\theta} &= -u(c) + \theta\rho(c) \\
\dot{\theta}^* &= -u^*(c^*) + \theta^*\rho(c^*) \\
0 &= yp - u'(c) + \theta\rho'(c) \\
0 &= y^*p - u'(c^*) + \theta^*\rho'(c^*) \\
0 &= w'(p)L + w'(p)L + R'(p)K + R'(p)K^* - c - c^*,
\end{aligned}
$$

Linearizing around the steady state, we obtain

$$
\begin{bmatrix} \dot{x}_K \\ \dot{x}_K^* \\ \dot{x}_y \\ \dot{x}_y^* \\ \dot{x}_\theta \\ \dot{x}_\theta^* \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix}
\rho & 0 & 0 & 0 & 0 & 0 & -p & 0 & 0 \\
0 & \rho & 0 & 0 & 0 & 0 & 0 & -p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho' y & 0 & -R'y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho' y & -R'y \\
0 & 0 & 0 & 0 & \rho & 0 & -yp & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho & 0 & -yp & 0 \\
0 & 0 & -p & 0 & -\rho' & 0 & u'' - \theta\rho'' & 0 & -y \\
0 & 0 & 0 & -p & 0 & -\rho' & 0 & u'' - \theta\rho'' & -y \\
R' & R' & 0 & 0 & 0 & 0 & -1 & -1 & 2\Xi
\end{bmatrix}
\begin{bmatrix} x_K \\ x_K^* \\ x_y \\ x_y^* \\ x_\theta \\ x_\theta^* \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

where $x_K \equiv K - K^e, x_K^* \equiv K^* - K^{*e}, x_y \equiv y - y^e, x_y^* \equiv y^* - y^{*e}, x_\theta \equiv \theta - \theta^e$, $x_\theta^* \equiv \theta^* - \theta^{*e}$, and $\Xi \equiv (R''K + w''L) > 0$.

Denote the above matrix by J, and the corresponding eigenvalue as z. Then z is determined by the characteristic equation

$$\Omega(z) \equiv |J - zI| = 0, \text{ where } I \equiv \begin{bmatrix} I_6 & 0 \\ 0 & O_3 \end{bmatrix}$$

If we subtract from the 9th row of $\Omega(z)$ by the first row multiplied by $R'/(\rho - z)$, and by the second row multiplied by $R'/(\rho - z)$, we obtain

$$\Omega(z) = (\rho - z) \begin{vmatrix} -z & 0 & 0 & 0 & \rho' y & 0 & R'y \\ 0 & -z & 0 & 0 & 0 & \rho' y & R'y \\ 0 & 0 & \rho - z & 0 & -yp & 0 & 0 \\ 0 & 0 & 0 & \rho - z & 0 & -yp & 0 \\ -p & 0 & -\rho' & 0 & u'' - \theta\rho'' & 0 & -y \\ 0 & -p & 0 & -\rho' & 0 & u'' - \theta\rho'' & -y \\ 0 & 0 & 0 & 0 & pR' - (\rho - z) & pR' - (\rho - z) & 2(\rho - z)\Xi \end{vmatrix}$$

Next, the 5th row minus the first row multiplied by p/z, and the 6th row minus the second row multiplied by p/z, to obtain

$$\Omega(z) = (\rho - z) \begin{vmatrix} \rho - z & 0 & -yp & 0 & 0 \\ 0 & \rho - z & 0 & -yp & 0 \\ -\rho' z & 0 & z(u'' - \theta\rho'') - p\rho' y & 0 & -zy + pR'y \\ 0 & -\rho' z & 0 & z(u'' - \theta\rho'') - p\rho' y & -zy + pR'y \\ 0 & 0 & pR' - (\rho - z) & pR' - (\rho - z) & 2(\rho - z)\Xi \end{vmatrix}$$

Finally, we add the first row multiplied by $\rho' z/(\rho - z)$ to the third row and the second row multiplied by $\rho' z/(\rho - z)$ to the fourth row, to get

$$\begin{aligned} \Omega(z) &= (\rho - z)^2 \begin{vmatrix} A(z) & 0 & F(z) \\ 0 & A(z) & F(z) \\ B(z) & B(z) & C \end{vmatrix} \\ &= (\rho - z)^2 A(AC - 2BF), \end{aligned}$$

where

$$\begin{aligned} A(z) &\equiv [z(u'' - \theta\rho'') - p\rho' y](\rho - z) - z\rho' yp \\ B(z) &\equiv pR' - (\rho - z) \\ C &\equiv 2\Xi > 0 \\ F(z9 &\equiv y(pR' - z) \end{aligned}$$

For $\Omega(z) = 0$, we have $z_1 = z_2 = \rho > 0$, $A(z) = 0$, and $A(z)C - 2B(z)F(z)$. We rewrite

$$A(z) = -(u'' - \theta\rho'')z^2 + \rho \cdot (u'' - \theta\rho'')z - p\rho' y\rho.$$

As $-(u'' - \theta\rho'') > 0$ (the second-order condition) and $-p\rho' y\rho < 0$, there exist two roots $z_3 > 0 > z_4$ such that $A(z_3) = A(z_4) = 0$.

Finally,

$$
\begin{aligned}
& H(z) \\
\equiv\ & A(z)C - 2B(z)F(z) \\
=\ & 2\Xi[-(u'' - \theta\rho'')z^2 + \rho \cdot (u'' - \theta\rho'')z - p\rho' y\rho] \\
& -2y(pR' - z)(pR' - \rho + z) \\
=\ & 2\{y - \Xi(u'' - \theta\rho'')\}z^2 - 2\rho\{y - \Xi(u'' - \theta\rho'')\}z \\
& -2py\{\Xi\rho'\rho + (pR' - \rho)R'\} \\
=\ & 0
\end{aligned}
$$

Since $2\{y - \Xi(u'' - \theta\rho'')\} > 0$ and $2py\{\Xi\rho'\rho + (pR' - \rho)R'\} > 0$, there exist $z_5 > 0 > z_6$ such that $H(z_5) = H(z_6) = 0$.

Therefore, there are four positive characteristic roots, i.e., z_1, z_2, z_3 and z_6, and two negative roots, i.e., z_4 and z_6. Since there are two state variables, K and K^*, it foll0ws that the steady state is a saddle point.

LEMMA 3: *When the two countries are sufficiently identical, the steady state with both countries being incompletely specialized is saddlepoint-stable.*

Lemma1-3 together imply the basic fundamental proposition.

Figure 1

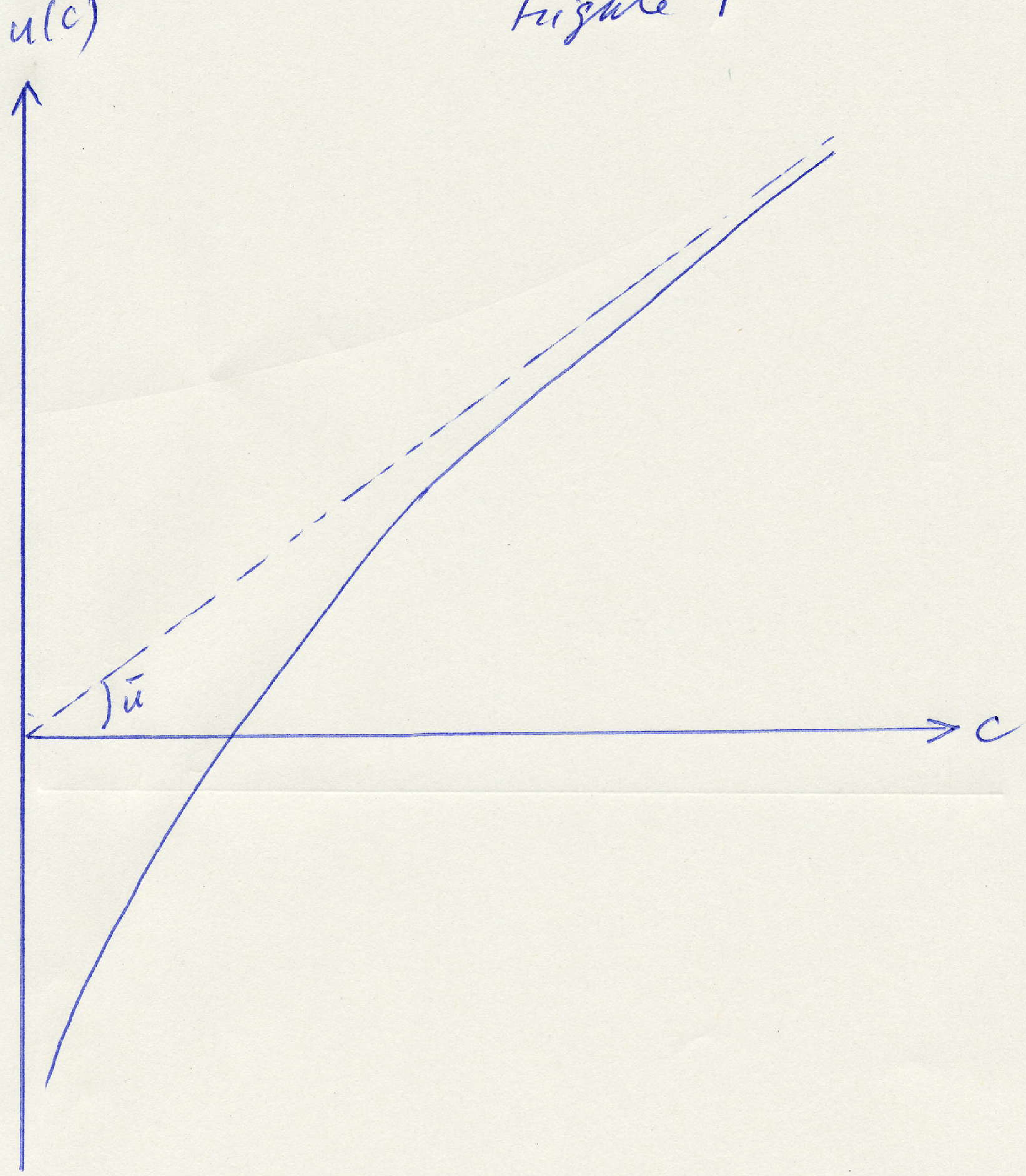
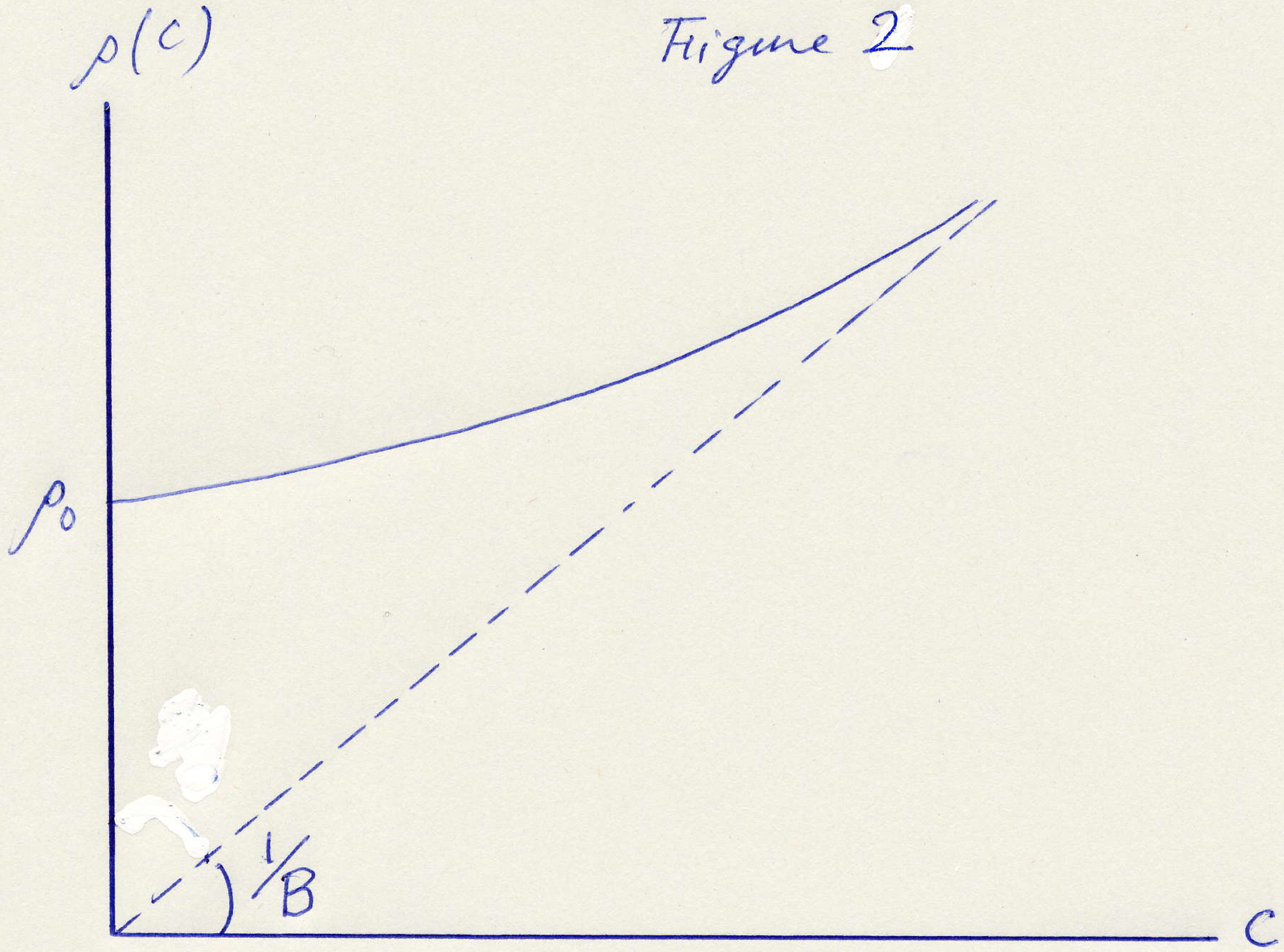


Figure 2



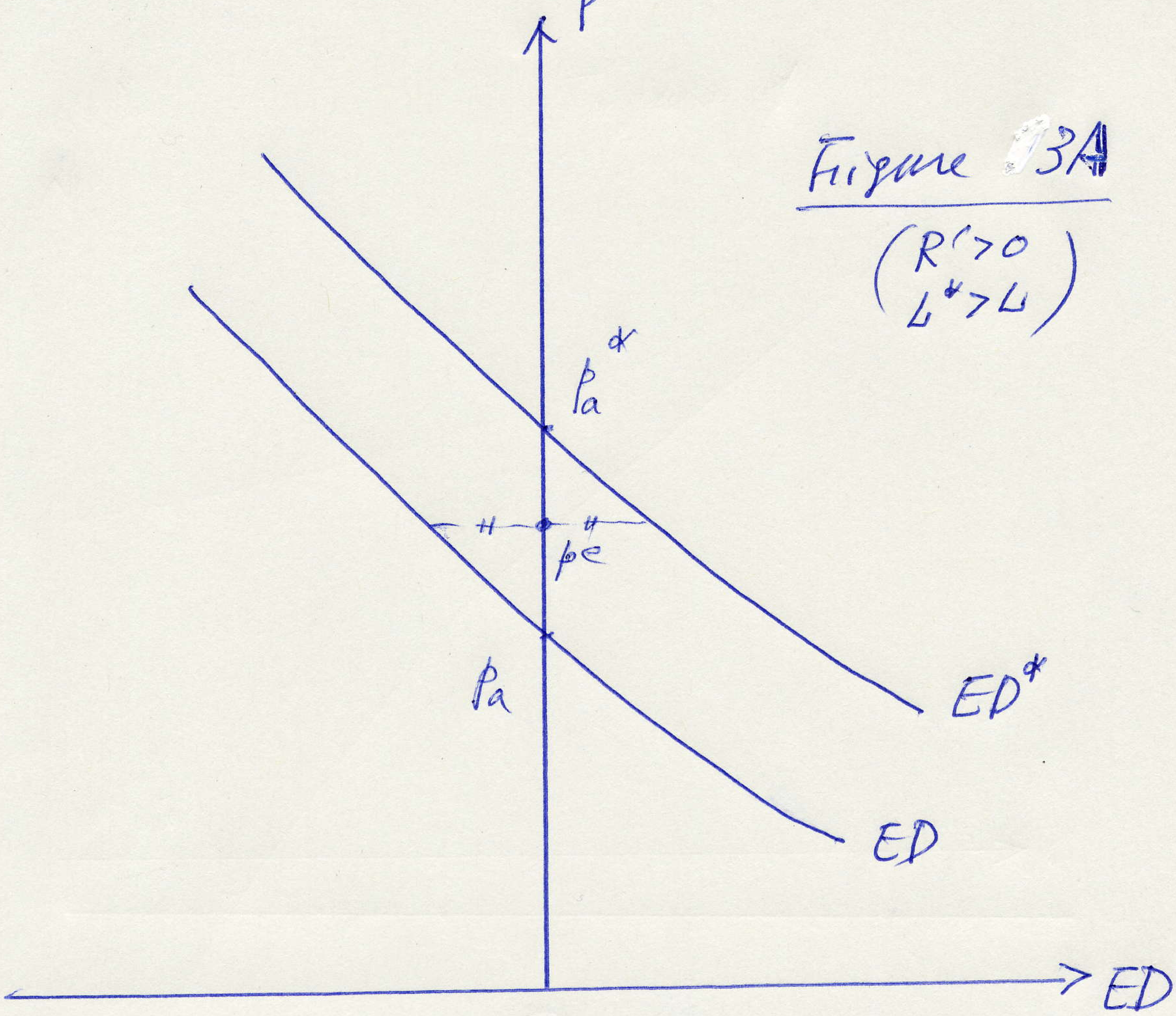


Figure 13A

($R' > 0$
$L^* > L$)

Figure 3B

$\begin{pmatrix} R' < 0 \\ L^* > L \end{pmatrix}$

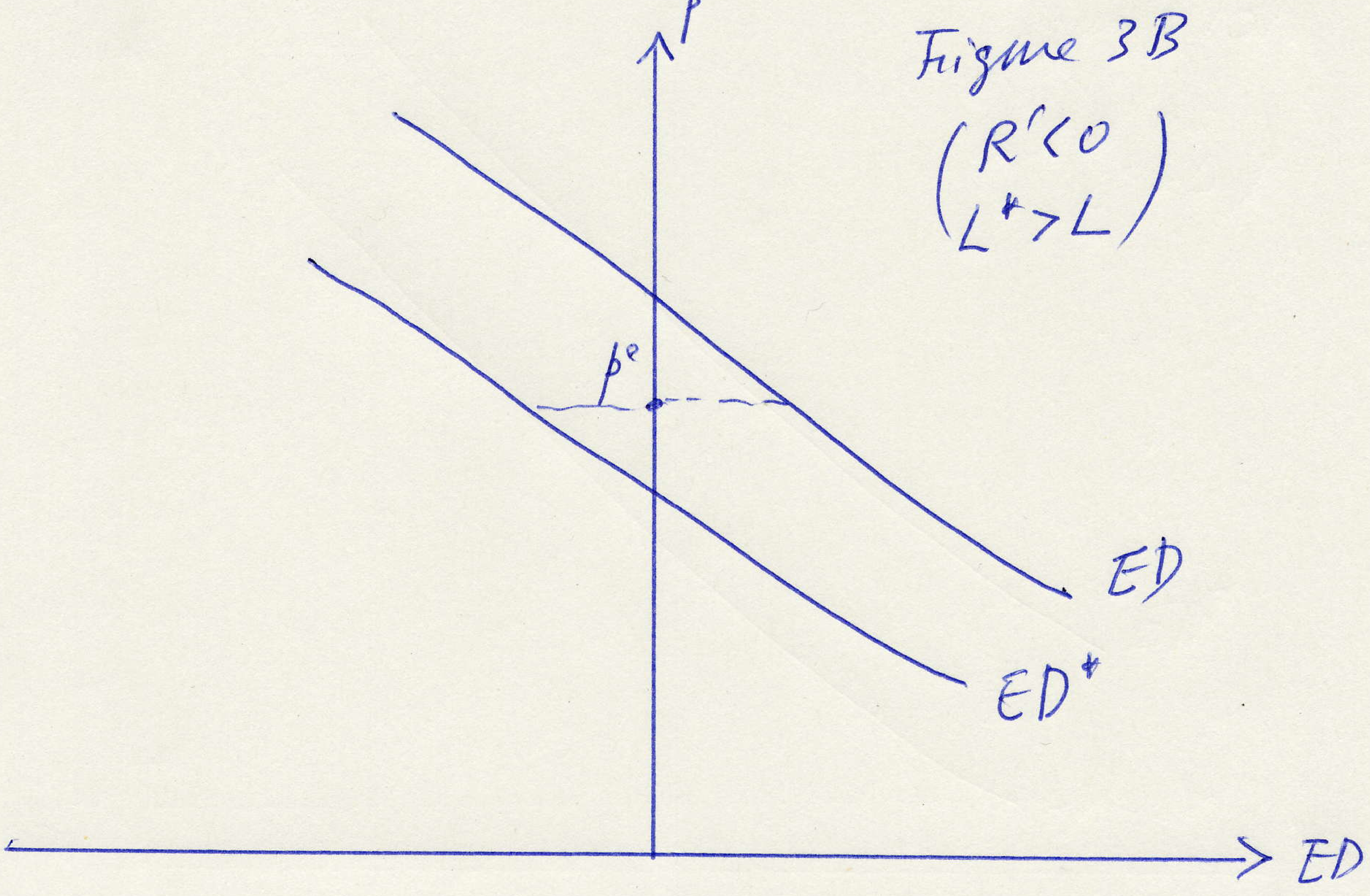
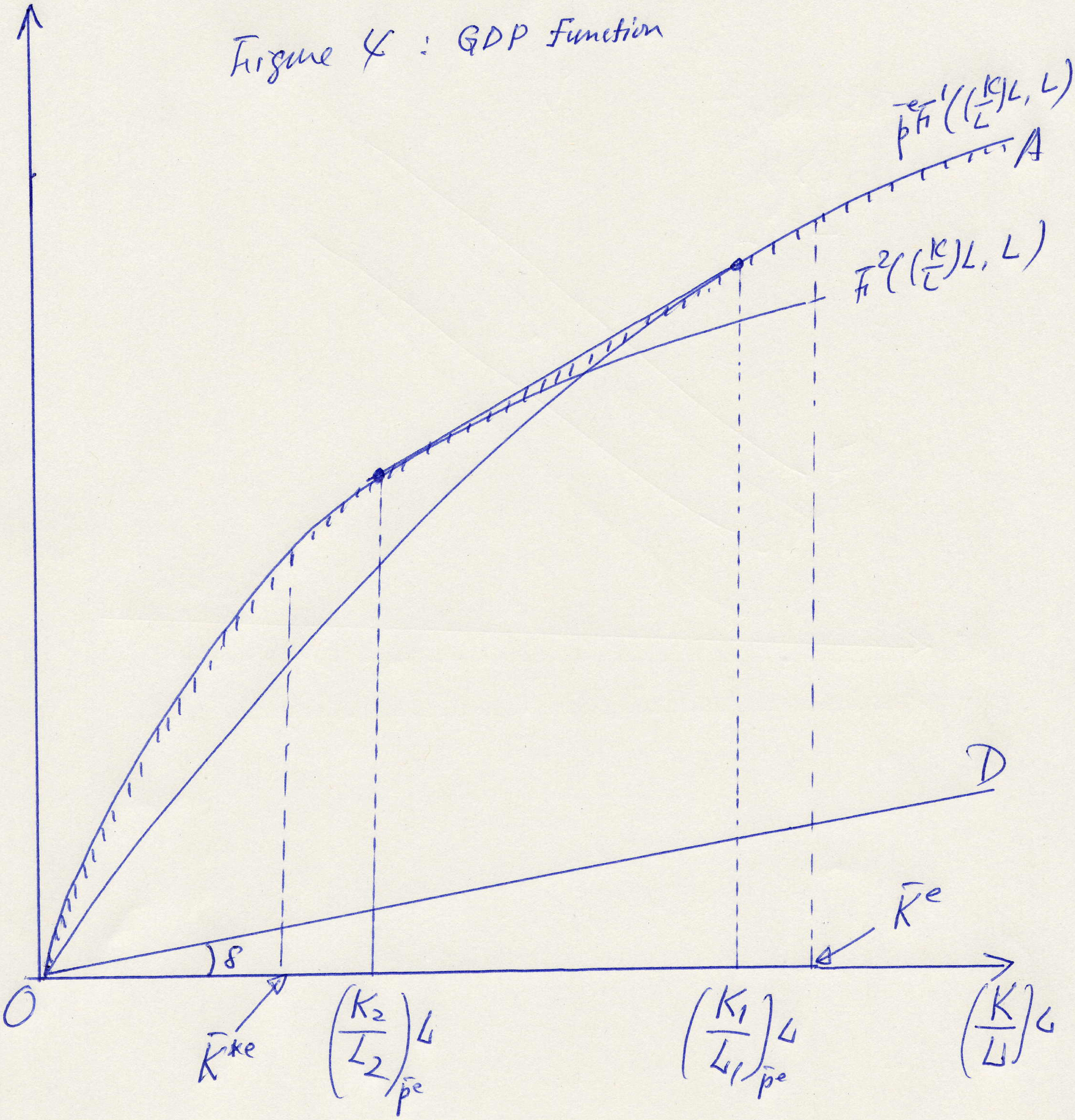


Figure 4 : GDP Function



[(OA) − (OD)] is increasing in $(\frac{K}{L})L = K$.