Tax-Subsidy Schemes in Money Search Models*

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Abstract

This paper studies the roles of tax-subsidy schemes in money search models with divisible money. Recently, real indeterminacy of stationary equilibria has been found both in specific and general search models with divisible money. In the literature, the welfare effect of monetary policy has often been discussed in search models with money. In most of these models, money is indivisible and the stationary equilibria are determinate. Thus the effects of the policies are determinate as well. However, if we assume the divisibility of money in these models, the stationary equilibria become indeterminate. Thus it is quite difficult to make accurate predictions of the effects of simple monetary policies in such models. Instead, in this paper we show that some tax-subsidy scheme selects a determinate efficient equilibrium. In other words, for a given efficient equilibrium, there exists a tax-subsidy scheme that makes the equilibrium locally determinate.

Keywords: Matching Model, Money, Tax-Subsidy Schemes, Real Indeterminacy
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1 Introduction

This paper studies the roles of tax-subsidy schemes in money search models. Recently, real indeterminacy of stationary equilibria has been found both in specific and general search models with divisible money. (See, for example, Green and Zhou [3] [4], Kamiya and Shimizu [6], Matsui and Shimizu [7], and Zhou [9].) In the literature, the welfare effect of monetary policy has often been discussed in search models with money. In most of these models, money is indivisible and the stationary equilibria are determinate. Thus the effects of the policies are determinate as well. However, if we assume the divisibility of money in these models, the stationary equilibria become indeterminate.

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Thus it is quite difficult to make accurate predictions of the effects of simple monetary policies in such models. Instead, we show that some tax-subsidy systems select a determinate efficient equilibrium. In other words, for a given efficient equilibrium, there exist a tax-subsidy scheme that make the equilibrium (locally) determinate.

Suppose money is perfectly divisible. We confine our attention to stationary equilibria in which, for some positive number p, all trades occur with its integer multiples of fiat money and the support of stationary distributions of money holdings has the form $\{0, p, \ldots, Np\}$. Kamiya and Shimizu [6] shows that there is a continuum of money holdings distributions which satisfies the stationary condition, which leads to the conclusion that under some regularity condition, there is also a continuum of stationary equilibria. These results are reviewed in Subsection 3.1.

Here we introduce government agents to this economy. Following Aiyagari et. al. [1], we describe them as those who are "programmed" in such a way that, following a rule, they collect tax from or give subsidy to the agents they are matched with. Then we show that, for any given stationary equilibrium, there exists tax-subsidy systems that almost lead the economy to the equilibrium and, moreover the government can select an efficient one.

The plan of this paper is as follows. In Section 2, we investigate a special model which can be considered as Zhou's model with government agents. In Section 3, we present general results by analyzing a wider class of models.

2 A Model with Government Agents

2.1 Model and Definitions

We first present a simple model with government agents. This model can be considered as Zhou [9]'s model with government agents in the sense of Aiyagari et. al. [1].

Time is continuous, and pairwise random matchings take place according to Poisson process with a parameter $\mu > 0$. Let the measure of private agents be normalized to one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Let κ be the reciprocal of k. Only one unit of good i can be produced and held by a type $i-1$ agent. The production cost is $c > 0$. A type i agent obtains utility $u > 0$ only when she consumes one unit of good i. We assume $u > c$. For every matched pair, the seller posts a take-it-or-leave-it price offer, ignorant of the buyer's money holdings. Let $M > 0$ be the nominal stock of fiat money, and $\gamma > 0$ be the discount rate.

We introduce government agents to this economy. They are "programmed" to follow a rule. That is, following the given rule, they collect tax from or give subsidy to the agents they are matched with. We assume that government agents can observe current money holdings of agents they are matched with. Let $G > 0$ be the measure of the government agents. Thus the total measure of agents is $1 + G$.

In what follows, we focus on a stationary distribution of money holdings of the private agents such that its support is the set $\{0, p, \dots\}$ for some $p > 0$. Thus the money holdings distribution can be expressed by $h_n, n = 0, 1, \dots,$ the measure of the set of private agents with money holding np. Of course, h satisfies $\sum_n h_n = 1$ and $h_n \geq 0$ for all n.

We describe government's policy by $(t_0, t_1, \dots)$. Each government agent gives subsidy p to the matched agent with n with probability $|t_n|$ when $t_n > 0$, while she collects tax p with probability $|t_n|$ when $t_n < 0$.

Let $\eta \in \mathbb{R}_+$ denote a private agent's money holding. A strategy of type i private agent is defined as a pair of an offer strategy $\omega(\eta) : \mathbb{R}_+ \to \mathbb{R}_+$ and a reservation price strategy $\rho(\eta) : \mathbb{R}_+ \to \mathbb{R}_+$. The former is a price that a type i agent with money holding η offers when she meets a potential buyer. A seller with money holding η offers $\omega(\eta)$. In case that a value function is continuous from the right, it will be shown that by the perfectness condition ρ gives the maximum price that a buyer is willing to defray for the consumption good, and so it becomes a function rather than a correspondence. Of course, since the reservation price cannot exceed the buyer's money holdings, ρ should satisfy the following feasibility condition:

$$\rho(\eta) \leq \eta. \tag{1}$$

From h, the stationary distribution of offer prices, Ω, and the stationary distribution of reservation prices, R, are defined as follows.

$$\Omega(x) = \sum_{n \in \{n' \in \mathbb{N} | \omega(n'p) \leq x\}} h_n, \tag{2}$$

$$R(x) = \sum_{n \in \{n' \in \mathbb{N} | \rho(n'p) < x\}} h_n, \tag{3}$$

where $\lfloor y \rfloor$ is the integer part of y. Let $\mathcal{V} : \mathbb{R}_+ \to \mathbb{R}_+$ be a value function. That is $\mathcal{V}(\eta)$ is the maximum value of discounted utility achievable by the agent's current money holding η. At every moment, a type i agent with money holding η meets a type $i - 1$

agent with probability $\mu\kappa/(1+G)$. Transaction does not occur and money holding does not change if the partner's offer x exceeds the type i's reservation price r. If partner's offer price x is not more than reservation price r, then transaction occurs and the type i agent derives utility u from consumption and enters in the next trading opportunity with money holding $\eta - x$. The probability that type i with money holding η meets a type $i+1$ agent is also $\mu\kappa/(1+G)$. Transaction does not occur if the type i's offer o is greater than the partner's reservation price. If type i's offer o does not exceed the partner's reservation price, then transaction occurs and faces the next matching opportunity with money holding $\eta + o$. The probability that an agent with money holding η meets a government agent is $\mu G/(1+G)$. If $\eta = np$ for some integer n, she obtain p from the government agent with probability $|t_n|$ if $t_n > 0$ and she pays p to the government agent with probability $|t_n|$ if $t_n < 0$. Then, using γ, μ, and h, the Bellman equation for $\mathcal{V}(\eta)$ is given by

$$\gamma\mathcal{V}(\eta) = \frac{\mu\kappa}{1+G}\max_{r\in[0,\eta]}\int_0^r [u + \mathcal{V}(\eta - x) - \mathcal{V}(\eta)]\, d\Omega(x) +$$

$$\frac{\mu\kappa}{1+G}\max_{o\in\mathbb{R}_+}[1 - R(o)][\mathcal{V}(\eta + o) - c - \mathcal{V}(\eta)] + \frac{\mu G t_{\lfloor \frac{\eta}{p}\rfloor}}{1+G}(\mathcal{V}(\eta + \mathrm{sign}(t_{\lfloor \frac{\eta}{p}\rfloor})p) - \mathcal{V}(\eta)), \quad (4)$$

where $t_{\lfloor \frac{\eta}{p}\rfloor} = 0$ if $\lfloor \frac{\eta}{p}\rfloor$ is not an integer.

In terms of $\mathcal{V}(\eta)$, it is optimal to accept offer o if $u + \mathcal{V}(\eta - o) \geq \mathcal{V}(\eta)$. The same condition in terms of reservation price ρ is $\rho(\eta) \geq o$. Then, in case that a value function is continuous from the right, the perfectness condition with respect to reservation price is as follows:

$$\rho(\eta) = \max\big\{r \in [0,\eta]\big|u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta)\big\}. \quad (5)$$

That is, type i's reservation price is her full value for good $i+1$, and thus it is a function of η. In order to assure that (5) is actually defined, we confine our attention to the case that a value function is continuous from the right hereafter.

The economy is stationary if h is an initial stationary distribution of the process induced by the optimal trading strategy (ω, ρ). Now we define the stationary equilibrium grounded on the above. We adopt stationary perfect Bayesian Nash equilibrium as our equilibrium concept.

Definition 1 $< h, \omega, \rho, \mathcal{V} >$, where $\mathcal{V}$ is a step function with step $p > 0$, is said to be a *stationary equilibrium* if

1. h is stationary under trading strategies ω and ρ, and the distribution of offer prices Ω and that of reservation prices R are derived from h by (2) and (3),

2. $\sum_{n=0}^{N} pnh_n = M$, and

3. given the distributions h, R and Ω, the reservation price strategy ρ and the offer strategy ω satisfy the feasibility condition (1) and the perfectness condition (5), respectively, and the value function $\mathcal{V}$, together with ρ and ω, solves the Bellman equation (4). Therefore,

$$\begin{aligned}\mathcal{V}(\eta) = \frac{1}{\phi+2}\Bigg[& \sum_{n=0}^{\lfloor \frac{\rho(\eta)}{p} \rfloor} \{u + \mathcal{V}(\eta - np)\} \left(\Omega(np) - \Omega((n-1)p)\right) \\ & + \{1 - \Omega(\rho(\eta))\}\, \mathcal{V}(\eta) + R(\omega(\eta))\mathcal{V}(\eta) + \{1 - R(\omega(\eta))\}\ \{\mathcal{V}(\eta + \omega(\eta)) - c\} \\ & + \frac{G}{\kappa} t_{\lfloor \frac{\eta}{p} \rfloor} \mathcal{V}(\eta + \mathrm{sign}(t_{\lfloor \frac{\eta}{p} \rfloor})p) + \frac{G}{\kappa}(1 - t_\eta)\mathcal{V}(\eta)\Bigg]\end{aligned}$$

holds, where $\phi = \frac{(1+G)\gamma}{\mu\kappa} + \frac{G}{\kappa}$.

We define the concept of a single-price equilibrium.

Definition 2 $< h, \omega, \rho, \mathcal{V} >$ is said to be a *single-price equilibrium (SPE)* with some price $p > 0$ if

1. it is a stationary equilibrium, and

2. with probability one, for a meeting between a buyer and a seller, either trade occurs with price p or trade does not occur.

We restrict our attention to single price equilibria (SPE) in which all trades occur with some price $p > 0$. In particular, we call our attention to SPE with the following features:

- the support of stationary distribution of money holdings is $\{0, p, 2p, \ldots, Np\}$ for some integer $N \geq 1$,

- on the equilibrium path, a seller with η, $0 \leq \eta < Np$, offers p,

- on the equilibrium path, a seller with η, $n \geq Np$, offers ∞ (i.e., she offer not to trade), and

- the reservation price of a buyer with η, $\eta \geq p$, is more than or equal to p.

Note that N is endogenously determined. Define the welfare as $W = \sum_{n=0}^{N} h_n V_n$.

2.2 SPE without Tax-Subsidy

In this subsection, we consider the case that $t_n = 0$ for all n. We first investigate SPEs with $N = 1$. The stationarity condition for $h = (h_0, h_1)$ is expressed as

$$0 = \dot{h}_0 = \frac{\mu\kappa}{1+G}\left[h_0(1-h_0) - h_1(1-h_1)\right], \tag{6}$$

$$0 = \dot{h}_1 = \frac{\mu\kappa}{1+G}\left[h_1(1-h_1) - h_0(1-h_0)\right]. \tag{7}$$

For example, in (6), $\dot{h}_0$ is the difference between the measure of agents whose money holdings change from 0 to other ones as results of trades and that of agents whose money holdings become 0 as results of trades at any moment. The former is the probabilities of an agent with p purchases her consumption good, $\frac{\mu\kappa}{1+G}h_1(1-h_1)$. A similar argument applies to $\dot{h}_1$. The latter is the probability of an agent with 0 sells her product at p, $\frac{\mu\kappa}{1+G}h_0(1-h_0)$. Substituting $h_1 = 1 - h_0$, it is clearly seen that both of the above equations are in fact identities. Thus any $h_0 \in [0,1]$, $h = (h_0, 1-h_0)$ satisfies the stationarity condition. p is determined by M and h as follows:

$$\frac{M}{p} = h_1.$$

First, we obtain the values at $\{0, p, \dots\}$. V_n, n = 0, 1, ... , the values at np, satisfy

$$V_0 = \frac{1}{\phi+2}[(1-h_0)(-c+V_1) + h_0V_0 + V_0], \tag{8}$$

$$V_n = \frac{1}{\phi+2}[(1-h_1)(u+V_{n-1}) + h_1V_n + V_n], \quad n \geq 1. \tag{9}$$

Note that $V_n, n \geq 2$, are the values in the off-equilibrium path. In case of (8), the strategy of an agent is to offer p. Suppose she meets a partner. If the partner is a buyer, she offers p and so the transaction results in a sale with probability $(1-h_0)$ and in no trade with probability h_0. Her money holding becomes p. Similar arguments apply to (9). Then we obtain

$$V_n = \frac{1}{\phi}\left[h_0u - \left(\frac{h_0}{\phi+h_0}\right)^n \frac{\phi+h_0}{\phi+1}\left\{h_0u + (1-h_0)c\right\}\right], n \geq 1, \tag{10}$$

and let

$$\mathcal{V}(\eta) = V(\lfloor \eta/p \rfloor). \tag{11}$$

By (10), it is seen that h and $\mathcal{V}$ satisfying (6), (7), (8), (9), and (11) are parametrized by h_0, since any h_0 satisfies (6) and (7). They are single-price equilibria if agents have incentive to take the specified strategy, and then there is a continuum of single-price equilibria.

The conditions for taking the strategy at $\eta = np$ for some integer n are

$$\begin{aligned} & -c + V_1 \geq V_0, \\ & V_n \geq -c + V_{n+1}, \quad n \geq 1, \\ & u + V_0 \geq V_1. \end{aligned}$$

The first condition is one for an agent with no money to try to sell her production good. The second condition is one for an agent with np not to sell her production good. The third condition is one for an agent with p to accept an offer price p. Note that the conditions at the other η follow from the above condition. (See Zhou [9].) Using (10), they can be expressed by h_0 as follows[1]:

$$1 + \frac{\phi}{h_0} \leq \frac{u}{c} \leq 1 + \frac{\phi(\phi + 1 + h_0)}{(h_0)^2}.$$

On the other hand, the welfare is expressed as

$$W = \frac{h_0(1 - h_0)}{\phi}(u - c).$$

It is easily seen that W has a single peak at $(h_0, h_1) = (1/2, 1/2)$ with $W = \frac{u-c}{4\phi}$.

Next, we investigate the case $N \geq 2$. We can apply almost the same argument as in the case of $N = 1$, though it is not very straightforward. As in the case of $N = 1$, the stationarity condition is expressed as:

$$0 = \dot{h}_n = \begin{cases} \frac{\mu\kappa}{1+G}\left[h_n(1 - h_0) - h_{n+1}(1 - h_N)\right], & n = 0, \\ \frac{\mu\kappa}{1+G}\left[h_n\left((1 - h_0) + (1 - h_N)\right) - \left(h_{n-1}(1 - h_0) + h_{n+1}(1 - h_N)\right)\right], & 1 \leq n \leq N - 1, \\ \frac{\mu\kappa}{1+G}\left[h_n(1 - h_N) - h_{n-1}(1 - h_0)\right], & n = N. \end{cases}$$

Let the nth equation in the above be $D_n(h) = 0$. It can be easily checked that $\sum_{n=0}^{N} D_n(h) = 0$ and $\sum_{n=0}^{N} nD_n(h) = 0$ always hold, i.e., they are identities. Thus if $D_n(h) = 0, n = 2, \ldots, N$, hold, then $D_0(h) = 0$ and $D_1(h) = 0$ automatically satisfied. In other words, two of the above equations are redundant. Thus the above system of equations has one degree of freedom.

Concretely, we obtain the following stationary distribution:

$$h_n = h_0 \left(\frac{1 - h_0}{1 - h_N}\right)^n, n = 1, \ldots, N.$$

[1] The following condition is slightly different from one obtained in Zhou [9], since she adopts a different equilibrium concept.

Note that

$$h_N(1-h_N)^N = h_0(1-h_0)^N$$

holds. It is verified that for any $h_0 \in (0,1)$ there are the corresponding stationary distribution h and real stock $m = M/p$.

As in the case of $N = 1$, the values at $np, n = 0, 1, \ldots$, are expressed as:

$$V_n = \begin{cases} \frac{1}{\phi+2}\left[(1-h_0)(-c+V_{n+1}) + h_0 V_n + V_n\right], & n = 0, \\ \frac{1}{\phi+2}\left[(1-h_0)(-c+V_{n+1}) + h_0 V_n + (1-h_N)(u+V_{n-1}) + h_N V_n\right], & 1 \le n \le N-1, \\ \frac{1}{\phi+2}\left[V_n + (1-h_N)(u+V_{n-1}) + h_N V_n\right], & n = N. \end{cases}$$

Then let $\mathcal{V}(\eta) = V_{\lfloor \frac{\eta}{p} \rfloor}$. Kamiya et al. [5] show that there exists a SPE with $N \ge 2$ in which h_0 is sufficiently close to 1 if

$$(\phi+1)^N < \frac{u}{c} < \frac{\phi(\phi+1)^{2N}}{(\phi+1)^N - 1}.$$

That is if the parameters satisfy this condition, there exists a continuum of single-price equilibria, where $h_0 \in (1, 1-\epsilon)$ for some $\epsilon > 0$. We also obtain

$$W = \frac{(1-h_0)(1-h_N)}{\phi}(u-c).$$

It is verified that W takes a value in $\left(0, \left(\frac{N}{N+1}\right)^2 \frac{u-c}{\phi}\right]$, and that the maximum value is attained at $h = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$.

2.3 SPE with Tax-Subsidy

Consider first SPEs with $N = 1$. Now consider the policy (t_0, t_1) for $t_1 < 0 < t_0$. Denote by $\tilde{h}$ a money holdings distribution with policy. Then the stationarity condition is

$$\begin{aligned} 0 = \dot{\tilde{h}}_0 &= \frac{\mu\kappa}{1+G}\left[\tilde{h}_0(\tilde{h}_1 + kGt_0) - \tilde{h}_1(\tilde{h}_0 - kGt_1)\right] \\ 0 = \dot{\tilde{h}}_1 &= \frac{\mu\kappa}{1+G}\left[\tilde{h}_1(\tilde{h}_0 - kGt_1) - \tilde{h}_0(\tilde{h}_1 + kGt_0)\right]. \end{aligned}$$

Besides trades between buyers and seller, an agent meets a government agent and pays (obtains) tax (subsidy). For example, with probability $\frac{\mu G t_1}{(1+G)}$, an agent with p meets a government agent and pays p. Let $\tilde{D}_0(\tilde{h}; G, t) = 0$ and $\tilde{D}_1(\tilde{h}; G, t) = 0$ be the first and the second equations, respectively. Then $\tilde{D}_0(\tilde{h}; G, t) + \tilde{D}_1(\tilde{h}; G, t) = 0$ always holds, i.e., it is an identity. Thus it is easily seen that one of the above equations is independent and, together with $\tilde{h}_0 + \tilde{h}_1 = 1$, the above equations determines h. In other words, the number of equations is equal to the number of variables. Indeed, we obtain the unique stationary distribution

$$(\tilde{h}_0, \tilde{h}_1) = \left(\frac{-t_1}{t_0 - t_1}, \frac{t_0}{t_0 - t_1} \right).$$

Note that, for almost all parameters, the incentive conditions are satisfied with strict inequalities in almost all the SPEs. (See Kamiya and Shimizu [6].) Therefore the introduction of government agents only slightly changes the incentive conditions as long as Gt_ns are sufficiently small. Thus if h^* constitute a stationary equilibrium without government agents, the policy satisfying $h_0^* = \frac{-t_1}{t_0 - t_1}$ makes h^* locally determinate almost always by setting Gt_ns sufficiently small. In particular, by setting $Gt_0 = \epsilon > 0$ and $Gt_1 = -\epsilon$ for small enough $\epsilon > 0$, we can make the most efficient SPE, $(\tilde{h}_0, \tilde{h}_1) = (\frac{1}{2}, \frac{1}{2})$, determinate.

Remark 1 The government uses t as a policy and, for a given $G > 0$, sets $Gt_0 = \epsilon$ and $Gt_1 = -\epsilon$. If possible, the government may choose G, though it is not necessary.

Consider next the SPE with $N \geq 2$. Now consider the policy $(t_0, 0, \ldots, 0, t_N)$, where $t_N < 0 < t_0$. Then the stationary condition is

$$0 = \dot{\tilde{h}}_n = \begin{cases} \frac{\mu\kappa}{1+G} \left[\tilde{h}_n(1 - \tilde{h}_0 + kGt_0) - \tilde{h}_{n+1}(1 - \tilde{h}_N) \right], & n = 0, \\ \frac{\mu\kappa}{1+G} \left[\tilde{h}_n(1 - \tilde{h}_0 + 1 - \tilde{h}_N) - \left(\tilde{h}_{n-1}(1 - \tilde{h}_0 + kGt_0) + \tilde{h}_{n+1}(1 - \tilde{h}_N) \right) \right], & n = 1, \\ \frac{\mu\kappa}{1+G} \left[\tilde{h}_n(1 - \tilde{h}_0 + 1 - \tilde{h}_N) - \left(\tilde{h}_{n-1}(1 - \tilde{h}_0) + \tilde{h}_{n+1}(1 - \tilde{h}_N) \right) \right], & 2 \leq n \leq N-2, \\ \frac{\mu\kappa}{1+G} \left[\tilde{h}_n(1 - \tilde{h}_0 + 1 - \tilde{h}_N) - \left(\tilde{h}_{n-1}(1 - \tilde{h}_0) + \tilde{h}_{n+1}(1 - \tilde{h}_N - kGt_N) \right) \right], & n = N-1, \\ \frac{\mu\kappa}{1+G} \left[\tilde{h}_n(1 - \tilde{h}_N - kGt_N) - \tilde{h}_{n-1}(1 - \tilde{h}_0) \right], & n = N. \end{cases}$$

Let the nth equation be denoted by $\tilde{D}_n(\tilde{h}; G, t) = 0$. As in the case of $N = 1$, by $\sum_{n=0}^{N} \tilde{D}_n(\tilde{h}; G, t) = 0$ only one of the above equations is redundant.

Combining the condition $\sum_{n=0}^{N} \tilde{h}_n = 1$, we obtain the following result. Suppose (t_0, t_N) satisfies $t_0 + t_N = 0$. Then

$$\tilde{h}_n = \begin{cases} \tilde{h}_0 & \text{if } n = 0 \text{ or } n = N, \\ \frac{\tilde{h}_0 \left(1 - \tilde{h}_0 + kGt_0\right)}{1 - \tilde{h}_0} & \text{if } 1 \leq n \leq N-1, \end{cases}$$

where $\tilde{h}_0$ is a solution of the equation

$$(N+1)(\tilde{h}_0)^2 - (3 + (N-1)(1 + kGt_0))\,\tilde{h}_0 + 1 = 0,$$

with $\tilde{h}_0 \in (0,1)$. It is verified that such $\tilde{h}_0$ is uniquely determined. In other words, the stationary distribution $\tilde{h}$ is uniquely determined. Moreover, as $Gt_0 \to 0$, $\tilde{h} \to \left(\frac{1}{N+1}, \dots, \frac{1}{N+1}\right)$.

When $t_0 + t_N \neq 0$,

$$\tilde{h}_n = \frac{1 - \tilde{h}_0 + kGt_0}{1 - \tilde{h}_N} \left(\frac{1 - \tilde{h}_0}{1 - \tilde{h}_N} \right)^{n-1} \tilde{h}_0 \quad \text{if } 1 \leq n \leq N-1,$$

where $\tilde{h}_0$, $\tilde{h}_N$ satisfy

$$\begin{aligned}
&\tilde{h}_N = b\tilde{h}_0, \\
&b(1 - b\tilde{h}_0 - kGt_N)(1 - b\tilde{h}_0)^{N-1} = (1 - \tilde{h}_0 + kGt_0)(1 - \tilde{h}_0)^{N-1},
\end{aligned}$$

and $b = -\frac{t_0}{t_N} > 0$. Then it is verified that the stationary distribution $\tilde{h}$ is uniquely determined. Moreover, as $Gt_0 \to 0$ and $Gt_N \to 0$, $\tilde{h}$ converges to

$$\tilde{h}_n = \frac{1 - b^{\frac{1}{N}}}{1 - b^{\frac{N+1}{N}}} b^{\frac{n}{N}}.$$

As in the case of $N = 1$, the incentive conditions are satisfied with strict inequalities in almost all the SPEs, see Kamiya and Shimizu [6], and therefore the policy approximates the target SPE and makes it locally determinate by setting Gt_0 and Gt_N sufficiently small.

To sum up, for any N, the policy above makes a stationary equilibrium locally determinate. Moreover, note that as $Gt_n, n = 0, \dots, N$ go to 0, the unique stationary distribution converges to the one perpendicular to t, i.e., $h^* \cdot t = 0$, since in equilibria $0 = \sum_{n=0}^{N} n\tilde{D}_n(h; G, t) = h^* \cdot t$ holds. Roughly speaking, if Gt is small enough, then the tax-subsidy scheme selects an equilibrium close to the single-price equilibrium without policy satisfying $h^* \cdot t = 0$. The reason is as follows. Introducing the policy, the stationarity condition without policy is slightly perturbed and the stationary distribution with the policy should be orthogonal to t. Note also that $h^* \cdot t$ can be considered as the budget deficit. Thus it is zero in equilibria on the stationary distribution while it can be nonzero out of stationary distributions.

2.4 Budget Balancing Rule

It is interesting to see that any policy with budget balancing cannot make equilibria determinate. For example, consider stationary equilibria with $N = 2$ and the policy $(\tilde{h}_2/\tilde{h}_0, 0, -1)$. Note that the budget balancing, $t_0\tilde{h}_0 + t_2\tilde{h}_2 = 0$, always holds even out of stationary equilibria. Then $\tilde{h}$ and t_0 must satisfy

$$
\begin{aligned}
&0 = \dot{\tilde{h}}_0 = \frac{\mu\kappa}{1+G}\left[\tilde{h}_0\left(1-\tilde{h}_0+t_0kG\right)-\tilde{h}_1(1-\tilde{h}_2)\right],\\
&0 = \dot{\tilde{h}}_1 = \frac{\mu\kappa}{1+G}\left[\tilde{h}_1\left(1-\tilde{h}_0+1-\tilde{h}_2\right)-\left(\tilde{h}_0\left(1-\tilde{h}_0+t_0kG\right)+\tilde{h}_2\left(1-\tilde{h}_2+kG\right)\right)\right],\\
&0 = \dot{\tilde{h}}_2 = \frac{\mu\kappa}{1+G}\left[\tilde{h}_2\left(1-\tilde{h}_2+kG\right)-\tilde{h}_1(1-\tilde{h}_0)\right],\\
&t_0 = \tilde{h}_2/\tilde{h}_0.
\end{aligned}
$$

Then for any $\tilde{h}_0 \in \left[\frac{4+kG-\sqrt{(4+kG)^2-12}}{6}, 1\right)$ and

$$
\begin{aligned}
\tilde{h}_1 &= \frac{-h_0-kG+\sqrt{(4-3h_0+kG)(h_0+kG)}}{2},\\
\tilde{h}_2 &= \frac{2-h_0+kG-\sqrt{(4-3h_0+kG)(h_0+kG)}}{2},
\end{aligned}
$$

the stationary condition is always satisfied, and thus stationary equilibria remain indeterminate.[2]

3 General Model

In this section, we investigate a general model which includes the above model as a special case. Note that the private sector in our model is a slightly special version of the general model presented by Kamiya and Shimizu [6] (hereafter, we call KS simply).

Time is continuous, and pairwise random matchings take place according to Poisson process with a parameter $\mu > 0$. Let the measure of private agents be normalized to one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Let κ be the reciprocal of k. A good i can be produced and held by a type $i-1$ agent. A type i agent obtains some positive utility only when she consumes a good i. We make no assumption on the divisibility of goods. We assume that fiat money is durable and perfectly divisible.

[2] The condition that $h_0 \geq \frac{4+kG-\sqrt{(4+kG)^2-12}}{6}$ is necessary for $t_0 \leq 1$.

We confine our attention to the case that, for some positive number p, all trades occur with its integer multiple amounts of money. In what follows, we focus on a stationary distribution of economy-wide money holdings on $\{0, \dots, N\}$ expressed by $h = (h_0, \dots, h_N)$, where h_n is a measure of the set of agents with np amount of money, and $N < \infty$ is the upper bound of the distribution. Our model includes the case of exogenously determined N as well as the case of endogenously determined N. Note that N is endogenously determined in the model in Section 2 and in Zhou [9]'s model. Of course, $h_n \geq 0$ and $\sum_{n=0}^{N} h_n = 1$ hold. Let $M > 0$ be a given nominal stock of money circulating in the private sector. Since p is uniquely determined by $\sum_{n=0}^{N} pnh_n = M$ for a given h, unless $h_0 = 1$, then, deleting p from $\{0, p, 2p, \dots, Np\}$, the set $\{0, \dots, N\}$ can be considered as the state space.

An agent with n chooses an action in $A_n = \{a_1, \dots, a_{k_n}\}$. For example, an action may include an offer price and/or a reservation price. Let $\beta_{nj} \geq 0$ be the proportion of the agents choosing an action a_j among the agents with n, and $\beta = (\beta_{01}, \dots, \beta_{nj}, \dots, \beta_{Nk_N})$. Thus $\sum_{j=1}^{k_n} \beta_{nj} = 1$ holds.

The monetary transition resulted from transaction among a matched pair is described as a function f. When an agent with (n, j) meets an agent with (n', j'), the former's and the latter's states, i.e., current money holdings will be $n + f((n, j), (n', j'))$ and $n' - f((n, j), (n', j'))$, respectively. That is f maps an ordered pair $((n, j), (n', j'))$ to a non-negative integer $f((n, j), (n', j'))$. Here "ordered" means, for example, that the former is a seller and the latter is a buyer. When N is exogenously determined, we assume

$$N \geq n + f((n, j), (n', j')) \quad \text{and} \quad n' - f((n, j), (n', j')) \geq 0.$$

When N is endogenously determined, we assume the latter condition while the former one should be satisfied on the equilibrium path.

Next, we introduce government agents. They are programmed to follow a rule which prescribes them how to collect tax from or give subsidy to the agents they are matched with. We assume that government agents can observe current money holdings of agents they are matched with. Let $G > 0$ be the measure of the government agents. Thus the total measure of agents is $1 + G$.

Then we describe government's policy by $(t_0, t_1, \dots, t_N)$, where $t_n \in [-1, 1]$. Each government agent gives subsidy p to the matched agent with n with probability $|t_n|$ when $t_n > 0$, while she collects tax p with probability $|t_n|$ when $t_n < 0$.

Let $\theta \in R^L$ be the parameter of the model and the policy. Of course, θ includes k,

μ, and G as well as t. We adopt a Bellman equation approach. Let V_n be the value of state n, $n = 0, \ldots, N$. The variables in the model are denoted by $x = (V, h, \beta)$. Let $W_{nj}(x;\theta)$ be the value of action j at state n. Thus, in equilibria, $W_{nj}(x;\theta) = V_n$ holds for j such that $\beta_{nj} > 0$. Note that $W_{nj}(x;\theta)$ includes the utility and/or the production cost of perishable goods.

3.1 Stationary Equilibria without Tax-Subsidy

First, we present the results in the case that $t_n = 0$ for all n, which have been proved in KS.

Let $h(n, j) = \beta_{nj}h(n)$. By the random matching assumption and the definition of f, the outflow O_n from state n and the inflow I_n into state n are defined as follows:

$$O_n(h, \beta; \theta) = \frac{\mu\kappa}{1+G}\left[\sum_{j,i',j'} h(n,j)h(i',j') + \sum_{i,j,j'} h(i,j)h(n,j')\right],$$

$$I_n(h, \beta; \theta) = \frac{\mu\kappa}{1+G}\left[\sum_{(i,j,i',j')\in B_n} h(i,j)h(i',j') + \sum_{(i,j,i',j')\in B'_n} h(i,j)h(i',j')\right],$$

where

$$B_n = \{(i,j,i',j') \mid i + f((i,j),(i',j')) = n\},$$
$$B'_n = \{(i,j,i',j') \mid i' - f((i,j),(i',j')) = n\}.$$

We denote $O_n - I_n$ by D_n. Then the condition for stationarity is $D_n = 0$, $n = 0, \ldots, N$ and $\sum_{n=0}^{N} h_n = 1$. Clearly, $\sum_{n=0}^{N} D_n = 0$ holds as an identity, and thus at least one equation is redundant. The following theorem shows that, moreover, one more equation is always redundant.

Theorem 1 Given β, then

$$\sum_{n=0}^{N} nD_n(h, \beta; \theta) = 0, \tag{12}$$

is an identity.

This implies that h is a stationary distribution if and only if $D_n = 0, n = 2, \ldots, N$, and $\sum_{n=0}^{N} h_n = 1$ hold. Namely, the condition for stationarity has at least one degree of freedom. This is the main cause of the indeterminacy.

Let B be the power set of $\{(n,j) \mid j = 1,\ldots,k_n, n = 0,\ldots,N\}$ and $\hat{B}$ be $\{b \in B \mid \forall n, \exists j, (n,j) \in b\}$. $b \in \hat{B}$ can be considered as a set of actions used in an equilibrium. For a given $b \in \hat{B}$, let

$$\Omega^b = \{(\beta_{nj})_{(n,j)\in b} \mid \beta_{nj} > 0 \quad \text{for} \quad (n,j) \in b\}. \tag{13}$$

Let $x^b = (V, h, \beta^b)$, where $\beta^b \in \Omega^b$. For a given $b \in \hat{B}$ and all $(n,j) \in b$, $W^b_{nj}(x^b;\theta)$ is defined from $W_{nj}(x;\theta)$ by setting $\beta_{n'j'} = 0$ for all $(n',j') \notin b$. In parallel with this, $D^b_n(h, \beta^b; \theta)$ is defined for $n = 2,\ldots,N$. Let $K = \sum_{n=0}^N k_n$.

Definition 3 For a given $b \in \hat{B}$, $x^b = (V, h, \beta^b) \in R^{N+1} \times R^{N+1}_+ \times R^{\sum_{n=0}^N k_n}_+$ is a *stationary equilibrium* with b if it satisfies the following:

$$\begin{aligned}
D^b_n(h, \beta^b; \theta) &= 0, & n &= 2,\ldots,N \\
\sum_{n=0}^N h_n - 1 &= 0, & & \\
V_n - W^b_{nj}(x^b;\theta) &= 0, & (n,j) &\in b \\
\sum_{j \in \{j' \mid (j',n) \in b\}} \beta_{nj} - 1 &= 0, & n &= 0,\ldots,N \\
V_n - W^b_{nj}(x^b;\theta) &\geq 0, & (n,j) &\notin b.
\end{aligned} \tag{14}$$

Let E^b_θ be the set of such an x^b, and $f^b : R^{N+1} \times R^{N+1}_{++} \times \Omega^b \times R^L \to R^{N-1} \times R \times R^{\#b} \times R^{N+1} \times R^{K-\#b}$ be the LHS of the above condition.

Remark 2 In addition to the above equilibrium conditions, the following conditions are typically required to be an "equilibrium" in most of matching models with money: (i) the existence of $p > 0$ satisfying $\sum_{n=0}^N pnh_n = M$, (ii) the incentive not to choose an action out of our action space,[3] and (iii) the incentive to take the equilibrium strategy at state $\eta \notin \{0, p, \ldots, Np\}$. However, they are not very restrictive. As for (i), it immediately follows from $h_0 \neq 1$. As for (ii) and (iii), KS presents a sufficient condition to assure that (ii) and (iii) hold, and it is satisfied in all of the matching models with divisible money known so far, such as Zhou [9]'s model, a divisible money version of Camera and Corbae [2]'s model, and a divisible money version of Trejos and Wright [8]'s model.

[3] For example in Section 2, a seller may offer a price which is not an integer multiple of p.

Let

$$C^b = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{R_{++} \times \cdots \times R_{++}}_{K-\#b},$$

and, for $(n,j) \notin b$,

$$C^{b(n,j)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{R_{++} \times \cdots \times R_{++} \times \{0\} \times R_{++} \times \cdots \times R_{++}}_{K-\#b},$$

where the last $\{0\}$ corresponds to $V_n - W^b_{nj}(x^b;\theta), (n,j) \notin b$. Moreover, for $(n,j),(n',j') \notin b$ such that $(n,j) \neq (n',j')$,

$$C^{b(n,j)(n',j')} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{R \times \cdots \times R \times \{0\} \times R \times \cdots \times R \times \{0\} \times R \times \cdots \times R}_{K-\#b},$$

where the last two $\{0\}$s correspond to $V_n - W^b_{nj}(x^b;\theta), (n,j) \notin b$, and $V_n - W^b_{n'j'}(x^b;\theta), (n',j') \notin b$, respectively. The it is verified that there is the indeterminacy of the stationary equilibrium under some regularity conditions.

Theorem 2 Let $\Theta \subset R^L$ be a C^2 manifold without boundary. For a given b^*, suppose that $E^{b^*}_\theta \neq \emptyset$ holds for all $\theta \in \Theta$, and that f^{b^*} is C^2 and is transversal to C^{b^*}, $C^{b^*(n,j)}$, and $C^{b^*(n,j)(n',j')}$ for all $(n,j),(n',j') \notin b^*$. Then, for almost every $\theta \in \Theta$, $E^{b^*}_\theta$ is a one-dimensional manifold with boundary. Moreover, at any endpoint of the manifold, only one $V^*(n) - W_{nj}(x^*;\theta) \geq 0$, $(n,j) \notin b^*$, can be binding, and at the other point of the manifold, no inequality is binding.[4]

KS also shows that this indeterminacy is indeed a real one; i.e., the welfare are typically not the same in a connected component of the equilibrium manifold.

3.2 Stationary Equilibria with Tax-Subsidy

In this section, we investigate the case that $t = (t_0, \ldots, t_N) \neq (0, \ldots, 0)$. In what follows, variables and functions with "tilde" denote the ones with nonzero t. The outflow at n, $\tilde{O}_n$, and the inflow at n $\tilde{I}_n$ are defined as follows:

$$\tilde{O}_n = O_n + \frac{\mu G}{1+G}|t_n|\tilde{h}_n,$$
$$\tilde{I}_n = I_n + \frac{\mu G}{1+G}\left(t^+_{n-1}\tilde{h}_{n-1} + t^-_{n+1}\tilde{h}_{n+1}\right),$$

[4] These assumptions imply that that $h(n) > 0, n = 0, \ldots, N$, hold and $D^b_n = 0, n = 2, \ldots, N$, can be independent. See KS for indeterminacy results of the other cases.

where $t_n^+ = \max\{0, t_n\}$, $t_n^- = -\min\{0, t_n\}$, and $t_{-1} = t_{N+1} = 0$.

Theorem 3 For a given β, consider the following system of the stationary condition:

$$(\tilde{D}_1, \ldots, \tilde{D}_N, \sum_{n=0}^{N} \tilde{h}_n - 1)^T = (0, \ldots, 0)^T,$$

where T denotes transpose. If the Jacobian matrix with respect to $\tilde{h}$ of the LHS of the above system is of full rank at a stationary distribution, then the stationary distribution is locally determinate. Moreover, the budget is balanced on this stationary distribution.

Proof:
The first statement is directly derived by the inverse function theorem. As for the second statement, it is verified that the budget deficit is equal to

$$\frac{\mu G}{1+G} \tilde{h} \cdot t = \sum_{n=0}^{N} n(\tilde{I}_n - \tilde{O}_n),$$

where the equation is derived from Theorem 1. Thus it is equal to 0 on the stationary equilibrium. ∎

For a sufficient condition for the assumption in the theorem, see the discussion following Assumption 1 below. For an example, see the discussions in Section 2.

Next, we show that, for a given stationary equilibrium, there exists a tax-subsidy scheme that almost leads the economy to the equilibrium. More precisely, for almost every stationary equilibrium without tax-subsidy, we can find a locally determinate stationary equilibrium with some tax-subsidy system in any neighborhood of the stationary equilibrium without tax-subsidy.

In what follows, we assume that the assumptions in Theorem 2 hold. We choose an arbitrary stationary equilibrium without tax-subsidy, denoted by x^*, which is in the relative interior of the equilibrium manifold. Thus (14) is satisfied with strict inequalities.

Lemma 1 Under the assumptions in Theorem 2, there exists an $(N+1)$-dimensional vector τ satisfying
(a) $\tau \neq (0, \ldots, 0)$,
(b) $\left(\frac{\partial D_n(h^*, \beta^*; \theta)}{\partial h_i}\right)_{i=0,\ldots,N} \cdot \tau = 0$ for $n = 2, \ldots, N$,
(c) $h^* \cdot \tau = 0$.

The above lemma clearly holds, since (b) and (c) have one degree of freedom.

We set $t = \epsilon\tau$. Here $\epsilon > 0$ is the "size" of the policy. In order for such a t to be a tax-subsidy scheme, we need to put the following assumption on tau:

Assumption 1 Under the assumptions in Theorem 2, there exists an $(N+1)$-dimensional vector τ satisfying
(d) $\tau_N \leq 0$, and
(e) $\tau_0 \geq 0$
in addition to (a), (b), and (c).

Recall that

$$\sum_{n=0}^{N} n(\tilde{I}_n - \tilde{O}_n) = \frac{\epsilon\mu G}{1+G}\tilde{h}\cdot\tau \tag{15}$$

holds. Since the LHS is equal to zero in the stationary distribution, (c) assures that $\tilde{h}$ approaches h^* as ϵ approaches zero if the stationary distribution is locally unique. (b) is a condition that τ is orthogonal to the gradient of $\tilde{D}_n$ w.r.t. $\tilde{h}$ for $n = 2, \ldots, N$. We will later show that under the assumptions in Theorem 2, this implies that a stationary distribution is locally unique for a sufficiently small ϵ.

Given b^*, the condition for a stationary equilibrium with a tax-subsidy scheme is as follows:

$$\begin{aligned}
\tilde{D}_n(\tilde{h}, \beta^*; \theta) &= 0, && n = 1, \ldots, N \\
\sum_{n=0}^{N} \tilde{h}_n - 1 &= 0, && \\
\tilde{V}_n - \tilde{W}_{nj}(\tilde{x}; \theta) &= 0, && (n,j) \in b^* \\
\sum_{j \in \{j' | (j',n) \in b^*\}} \beta^*_{nj} - 1 &= 0, && n = 0, \ldots, N \\
\tilde{V}_n - \tilde{W}_{nj}(\tilde{x}; \theta) &\geq 0, && (n,j) \notin b^*,
\end{aligned}$$

where $\tilde{x} = (\tilde{V}, \tilde{h}, \beta^*)$. Let $\tilde{f}_\epsilon^{b^*}(x)$ be the LHS of the above equations except the ones in the last line. By replacing $\tilde{D}_1$ by $\tilde{h}\cdot\tau$ in $\tilde{f}_\epsilon^{b^*}$, we define $\hat{f}_\epsilon^{b^*}$. We make the following assumption:

Assumption 2 $D_{x^{b*}} \hat{f}_0^{b^*}$ is of full rank at x^{*b^*}, where $x^{b^*} = (V, h, \beta^{b^*})$, $\beta^{b^*} \in \Omega^{b^*}$.

This is a weak condition, since it is veirfied that the first $N+1$ rows are independent for a sufficiently small ϵ. Recall that τ, which is the gredient of $\tilde{h}\cdot\tau$ w.r.t $\tilde{h}$, is orthogonal to the gradient of D_n w.r.t. h for $n = 2, \ldots, N$. Moreover, tau is independent of the gradient of $\sum_{n=0}^{N} \tilde{h} - 1$ w.r.t. h because of (a), (d), and (e). On the other hand, the assumptions in Theorem 2 implies that the gradient of $\tilde{D}_n$ for $n = 2, \ldots, N$, w.r.t. $\tilde{h}$, and the gradient of $\sum_{n=0}^{N} \tilde{h} - 1$ w.r.t. h are independent. Thus we conclude that the first $N + 1$ rows are independent for a sufficiently small ϵ.

Thus x^{*b^*} is a locally determinate solution to $\hat{f}_0^{b^*} = (0, \ldots, 0)^T$, since $h^* \cdot \tau = 0$. Note that, for any $\epsilon > 0$, the solution sets of $\hat{f}_\epsilon^{b^*} = (0, \ldots, 0)^T$ and of $\tilde{f}_\epsilon^{b^*} = (0, \ldots, 0)^T$ are clearly the same.[5]

Then applying the implicit function theorem to $\hat{f}_\epsilon^{b^*} = (0, \ldots, 0)^T$ at $(x^{b^*}, \epsilon) = (x^{*b^*}, 0)$, it can be clearly shown that, for all $\varepsilon > 0$, there exist $\epsilon > 0$ and $x_\epsilon^{b^*}$ such that $x_\epsilon^{b^*}$ is a solution to $\tilde{f}_\epsilon^{b^*} = (0, \ldots, 0)^T$ and is in the ε-neighborhood of x^{*b^*}. Finally, since x^{b^*} is in the relative interior of the equilibrium manifold, the agents have incentive to use the strategies for sufficiently small ϵ. Thus the following theorem holds.

Theorem 4 Suppose that Assumption 1, Assumption 2 and the assumptions in Theorem 2 hold. Then for any ε-neighborhood of $x^{*b^*} \in E_\theta^{b^*}$, there exists a tax-subsidy system such that a stationary equilibrium with the policy is in the neighborhood and is locally determinate.

In general, using some tax-subsidy systems, the government may obtain a more efficient equilibrium than any element in the set of stationary equilibrium without policy. Thus it will be the most important future research to seek for the best tax-subsidy systems.

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