A Model of Financial Markets with Endogenously Correlated Rational Beliefs*

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Abstract

This paper studies how communication amongst agents influences the equilibria of a financial economy. We set up a standard overlapping generations (OLG) model with assets, while allowing for heterogeneous beliefs. The paper explicitly describes how communication causes the beliefs of the agents to be correlated. In particular, it is shown that communication may generate large fluctuations even if the unconditional probability beliefs themselves are uncorrelated. Because of the complex nature of the problem, we use simulations to examine the characteristics of the equilibria.

1 Introduction

This paper studies how communication amongst agents influences the equilibria of a financial economy while allowing for heterogeneous beliefs. More specifically, we introduce communication in a standard overlapping generations (OLG) model with heterogeneous beliefs.

As in the companion paper (Nakata (2004)), we postulate that communication amongst different economic agents is a mechanism that causes the beliefs of the agents to be correlated with each other. Note that it is essential that heterogeneity of beliefs is present in order for communication to have an impact on the equilibrium (through correlation of beliefs), because the

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only role communication can play is to possibly remove (the impacts of initial) asymmetric information when *rational expectations* or a common prior is assumed.[1]

However, once correlation of beliefs is present in a general equilibrium model of a financial economy, it is obvious that the equilibria will be influenced by such correlation. Amongst the studies on heterogeneous beliefs, some studies of *rational beliefs* examine the effects of correlation of beliefs numerically — e.g. Kurz and Schneider (1996), Kurz and Beltratti (1997), and Kurz and Motolese (2001). In fact, their results indicate that correlation of beliefs amplifies the volatility of the economy. They, however, do not describe the mechanism that generates correlation of beliefs. The companion paper (Nakata (2004)) incorporates communication to these model, and provides an alternative interpretations to them, while restricting the class of communication.

This paper generalizes the companion paper (Nakata (2004)) so that communication may expand the state space of the economy, which was not the case there. With this generalization, we are able to study if communication alone may generate correlation of beliefs and consequently large fluctuations of the economy by restricting the class of beliefs to be uncorrelated *a priori* (in the absence of communication). To provide better insights, we construct a simulation model, in which the beliefs of the agents can be interpreted and classified in an intuitive way.

Such an intuitive classification of beliefs is helpful, because it is not clear in what ways communication generates correlation of beliefs, and thus, how it influences the equilibrium otherwise. Hence, it is hoped that such an intuitive classification of beliefs provide better answers to the following questions:

- *Does communication have a tendency to stabilize or destabilize the market?*
- *Under what conditions will the market be destabilized, versus being stabilized?*
- *How do the results depend on the nature of the market (with public information)?*
- *What are the implications of these results for public policy in financial markets?*

The rest of the paper proceeds as follows. In Section 2, the structure of the economy and the beliefs of the agents are explained first, and then the equilibrium of the economy is defined. Then, the definition of rational beliefs

[1] Blackwell and Dubins (1962) show that the conditional probabilities merge in the limit as agents accumulate the same information when mutual absolute continuity of measures is assumed. Also, Geanakoplos and Polemarchakis (1982) show that a merging of opinions occurs within finite steps when two agents communicate back and forth when the information structure is characterized by information partitions under the common prior assumption. However, a merging of opinions is a degenerate case when the mutual absolute continuity of measures and/or the common prior assumption is absent. See Freedman (1965) and Nakata (2003).

is given so as to define the rational belief equilibrium, which is central to our analysis. In Section 3, we examine a simulation model, and illustrate the effects of communication on the equilibrium to answer the above questions. Section 4 concludes the paper.

2 The Model

In this section, we introduce a standard OLG model with assets, albeit with communication. The main objective is to set up a model that elucidates the effects of communication on the equilibrium fluctuations of the economy. In particular, we regard communication as a mechanism that causes the beliefs of different agents to be correlated with each other.

In the remainder of this section, we first set up the model. To make the analyses tractable, we confine our attention to the Markovian economy. In so doing, we define the competitive equilibria of the economy, and also discuss the structure of the beliefs of the agents and that of the state space of the economy. Finally, we introduce the rationality conditions to define the rational belief equilibria (RBE) of the economy.

2.1 The Structure of the Model

2.1.1 The Structure of the Economy

The structure of the model is essentially the same as that of Kurz and Beltratti (1997), except that our model involves communication. Consider a standard OLG economy with H young agents in each generation which we denote by $h = 1, 2, ..., H$ (H is some finite positive integer). Also, there are H old agents in each period. There is a single perishable consumption good, whose price is normalized to unity in every period t. We assume that only young agents receive an endowment W_t^h ($t = 1, 2, ...$) of this consumption good, except that in the initial period ($t = 0$) old agents (in period 1; born in period 0) receive endowments of the stock specified below (θ_0^h with $\sum_{h=1}^{H} \theta_0^h = 1$). Furthermore, each young agent is a *replica* of the old agent who preceded him, where a *replica* refers to the preferences and beliefs (more precisely, the set of possible effective beliefs as will be explained below). This makes us interpret the streams of agents as 'dynasties' or 'types'. Also, there is a single infinitely lived firm owned by the agents. Let P_t denote the stock price of the firm in period t and θ_t^h the shareholding of young agent h purchased in period t. We assume without loss of generality that the aggregate supply of shares is fixed to unity in every period. The firm's technology generates an exogenous random stream of returns $\{D_t\}_{t=0}^{\infty}$, and we call it the dividend stream. We assume that $D_t > 0$ for all t. For the agents, shareholding yields income from the dividend. In addition, there is a market for a zero net supply, short term riskless debt instrument which we call a 'bill'.

To summarize, the economy has three markets: (a) a market for the consumption good with an aggregate supply equaling the total endowment and the total dividends, (b) a stock market with a total supply of unity, and

(c) a market for a zero net supply, short term riskless debt instrument which we call a 'bill'. We list our notation as follows: for each agent h,

C_t^{1h}: consumption of agent h when young in period t;
C_{t+1}^{2h}: consumption of h when old in $t+1$ (the agent was born in t);
$d_{t+1} := D_{t+1}/D_t$: the random growth rate of dividends;
θ_t^h: amount of stock purchases by young agent h in period t;
B_t^h: amount of one-period bill purchased by young agent h in period t;
W_t^h: endowment of young agent h in period t;
P_t: the price of the stock in period t;
$p_t := P_t/D_t$: the price/dividend ratio in period t;
q_t: the price of the one-period bill in period t. This is a discount price.

Next, we specify the structure of the dividend process. Our specification follows that of Mehra and Prescott (1985), which is standard in the literature. Namely,

$$D_{t+1} = d_{t+1} D_t, \tag{1}$$

where the stochastic process $\{d_t\}_{t=1}^{\infty}$ is a stationary and ergodic Markov process. Following Kurz and Beltratti (1997), the state space of the process is $\mathcal{D} := \{d^H, d^L\}$ with $d^H = 1.054$ and $d^L = 0.982$, and the stochastic process $\{d_t\}_{t=1}^{\infty}$ is driven by a transition probability matrix

$$\begin{bmatrix} .43 & .57 \\ .57 & .43 \end{bmatrix}. \tag{2}$$

With this specification, the dividends tend to rise over time; thus it is more convenient to focus on the growth rates of the economic variables. To this end, we define the following variables:

$w_t^h := W_t^h/D_t$:	the endowment/dividend ratio of young agent h;
$b_t^h := B_t^h/D_t$:	the bill/dividend ratio of young agent h in t;
$c_t^{1h} := C_t^{1h}/D_t$:	the consumption/dividend ratio when young;
$c_{t+1}^{2h} := C_{t+1}^{2h}/D_{t+1}$:	the consumption/dividend ratio when old.

In order to elucidate the sources of randomness of the economy, we assume that $w_t^h = w^h$ are constant for all h, t. Hence, the aggregate endowment of the consumption $\sum_{h=1}^{H} W_t^h$ is proportional to the total dividend D_t in each period t.

2.1.2 The Structure of Beliefs

Now we specify the structure of beliefs, which is the same as the one in the companion paper (Nakata (2004)). Instead of fixing a belief over generations within each dynasty h, we assume that the effective belief Q_t^h is random over time, and is governed by a probability measure μ^h. More specifically, the sequence of beliefs of dynasty h, i.e. $\{Q_t^h\}_{t=1}^{\infty}$ is an i.i.d. sequence, and μ^h is a probability measure on $(\mathcal{Q}^h, \mathcal{B}(\mathcal{Q}^h))$, where $\mathcal{Q}^h$ is the set of possible effective beliefs, which is assumed to be (at most) countable.[2] Hence, probability

[2]We assume $\mathcal{Q}^h$ to be finite to simplify the analysis later. However, our construction of beliefs is valid as long as $\mathcal{Q}^h$ is countable, and so is the analytical model.

$\mu^h(Q_t^h = k)$ is constant over time for all $k \in \mathcal{Q}^h$ since $\{Q_t^h\}_{t=1}^{\infty}$ is an i.i.d. sequence. While we can introduce a more complex structure here, for example $\{Q_t^h\}_{t=1}^{\infty}$ may be an AR(1) process, we do not complicate the matter here so as to keep the analysis simple.

Although there is a probability distribution μ^h over $\mathcal{Q}^h$ the set of possible effective beliefs, which also define probability distributions, we stress that μ^h is not part of young agent h's belief. Namely, young agent h does not form a belief such that $Q^h(\cdot) = \int_{k \in \mathcal{Q}^h} k(\cdot)\mu^h(dk)$.[3] Rather, $\{Q_t^h = k\}$ is a probability event with respect to measure μ^h, which is irrelevant for young agent h in period t, since his belief is completely represented by Q_t^h itself.

We note that the effective belief Q_t^h is really the theory with which young agent h in period t views the economy. Hence, when the set $\mathcal{Q}^h$ is not a singleton, there are multiple theories that might be adopted by young agent h. In fact, the randomness of Q_t^h means that the actual theory in use is chosen randomly in each period. This is possible when an agent is ambiguous about the choice of theory.

Now, we explain why it is reasonable to assume such an ambiguity. To begin with, it is reasonable to say that no agent actually knows the truth, and that, every 'intelligent' agent knows that he does not know the truth. Hence, with the knowledge that he does not know the truth, each agent relies on a theory, which always employs some assumption(s) by definition. Because each agent knows that it is impossible for an assumption to be always correct (otherwise it is the truth itself, which is not an assumption by definition), he is uncertain or ambiguous about the choice of theory as long as there are multiple theories available. Namely, each agent is ambiguous in the sense that he is not very sure which theory is the most relevant amongst others, yet he ultimately relies on a theory at the time when he is making decisions. This observation motivates us to randomize Q_t^h rather than to fix it as a particular measure over time, while assuming that the set of theories $\mathcal{Q}^h$ is inherited over generations within the dynasty h.

In addition, it is common that institutional investors including financial institutions adopt some sort of quantitative/statistical model to determine their portfolio choices in practice, and has become increasingly so recently. This means that they adopt a particular probabilistic model to make a portfolio choice, although they do alter the models from time to time. Alterations of models may simply be changes in the parameters of the models, or they may even involve changes in the structure of the model itself. Although such changes are common, there hardly exists a fixed rule/model for model selections. This observation is therefore clearly consistent with our construction of beliefs because each institution uses a probabilistic model out of several possible models in every period, whilst there remains ambiguity in the model selection process.

Moreover, we assume that the ambiguity concerning the choice of theories does not necessarily lead the agents to 'mix' different theories, e.g. put a weight of .3 on theory A and a weight of .7 on theory B, unlike a mixed

[3]The object Q^h is called the barycenter of μ, and there exists a unique barycenter if the underlying probability space is a standard space. See Gray (1988) for details.

strategy in game theory, because we do not allow for an agent to form a belief such that $Q^h(\cdot) = \int_{k \in \mathcal{Q}^h} k(\cdot)\mu^h(dk)$ as we noted above. Of course, it is possible that an agent forms a belief about beliefs. However, as long as the belief about beliefs changes over time, our construction remains valid. This is so, even if we consider beliefs about beliefs about beliefs ad infinitum, i.e. the infinite regress of hierarchical beliefs, unless the hierarchical structure of beliefs is time-invariant, in which case there is no ambiguity concerning the choice of theory or beliefs ultimately, because the hierarchy of beliefs as a whole defines a probability belief.[4] We come back to this point when we discuss the implications of the simulation results in the light of the *Expert Problem*, which is a literature in Bayesian theory (for example, Genest and Schervish (1985), Bayarri and DeGroot (1991), West (1992), and West and Crosse (1992)).

Our construction of beliefs is capable of describing a common situation in which the same investor sometimes becomes optimistic and sometimes becomes pessimistic even though the data at hand are the same. We stress that it is the belief of the agent that determines if he is optimistic or not, not the data. We do not adopt the view that a particular investor/institution always believes in a particular theory over the periods, and that a change in behaviour only occurs when the data changes. Rather, we allow for an agent to change his view (or mind) even though there is no change in data.

Note however that our construction of beliefs does not capture the concept of ambiguity found in the Ellsberg Paradox, because at any point of time, each agent forms a particular probability belief anyway, although we introduce a notion of 'set of probability beliefs'. Hence, we do not follow the literature that focuses on modelling this sort of ambiguity (or Knightian uncertainty), e.g. Gilboa and Schmeidler (1989), Epstein and Wang (1994, 1996).

Although as long as the set $\mathcal{Q}^h$ is countable, the analytical model remains the same, we assume for simplicity that for every agent h,

$$\mathcal{Q}^h := \{Q^h_H, Q^h_L\},$$

and for every h, t,

$$\mu^h\{Q^h_t = Q^h_H\} = \alpha^h. \tag{3}$$

Namely, the effective belief of young agent h in period t is Q^h_H with a frequency of α^h and Q^h_L with a frequency of $1 - \alpha^h$. Moreover, we may say *optimistic* when $Q^h_t = Q^h_H$, and *pessimistic* when $Q^h_t = Q^h_L$. Furthermore, we assume that Q^h_H and Q^h_L are stationary measures.

2.1.3 Announcements

We assume that in every period t each young agent makes an announcement concerning the price/dividend ratio in the next period p_{t+1}, with the announcements becoming public information thereafter. Let Y^h_t denote the

[4]See Brandenburger and Dekel (1993) for the discussion on hierarchical beliefs.

announcement of agent h, which is a measurable function of all available information. Agent h's information set (σ-field) then is defined as

$$\mathcal{G}_t^h \backslash \mathbf{Y}_t := \sigma((p_1, q_1, d_1, \mathbf{Y}_1), ..., (p_{t-1}, q_{t-1}, d_{t-1}, \mathbf{Y}_{t-1}), (p_t, q_t, d_t)),$$

with

$$\mathcal{G}_t^h := \sigma((p_1, q_1, d_1, \mathbf{Y}_1), ..., (p_t, q_t, d_t, \mathbf{Y}_{t-1})),$$

which is the σ-field generated by the state space of $((p_1, q_1, d_1, \mathbf{Y}_1), ..., (p_t, q_t, d_t, \mathbf{Y}_t))$, and the announcements of other agents (from agent h's perspective)

$$\mathbf{Y}_t := (Y_t^1, Y_t^2, ..., Y_t^H) \in \mathbf{R}^H.$$

Formally, we can define agent h's announcement as follows:

$$Y_t^h = v_t^h((p_1, q_1, d_1, \mathbf{Y}_1), ..., (p_t, q_t, d_t, \mathbf{Y}_{t-1})),$$

which depends on the effective belief Q_t^h implicitly.

One of the most natural examples of Y_t^h is agent h' s conditional expectation of p_{t+1}. However, to make computations tractable, we assume that each agent announces whether the price/dividend ratio is expected to go up or down in the next period. Namely, Y_t^h is defined as

$$Y_t^h = \begin{cases} 1 & \text{if} \quad E_{Q_t^h}\{p_{t+1} | \mathcal{G}_t^h \backslash \mathbf{Y}_t\} \geq p_t \\ 0 & \text{if} \quad E_{Q_t^h}\{p_{t+1} | \mathcal{G}_t^h \backslash \mathbf{Y}_t\} < p_t. \end{cases} \tag{4}$$

Hence, $Y_t^h = 1$ means that agent h believes that the price/dividend ratio will not drop in the next period on average, and $Y_t^h = 0$ means that he believes that it will fall on average.

To simplify the analyses, we make the following assumption on the announcements. Namely,

Assumption 1: *Each young agent h announces his opinion truthfully.*

This assumption will be maintained throughout the paper. By Assumption 1, we put strategic concerns about the announcement to one side, avoiding complications that would involve game theoretic considerations, which are not essential to our current focus. Because we are examining a general equilibrium model, we are interested in situations in which each single agent believes that he cannot affect the whole system as well as other agents' decisions, i.e. the competitive assumption. Hence, this assumption is compatible with the setting of a general equilibrium model.[5]

By computing the conditional expectation based upon the information set (sub σ-field) $\mathcal{G}_t^h$, each young agent determines his optimal portfolio — i.e. how many shares and bills to purchase. Then, the economy moves on to the next period, and the process will be iterated infinitely many times. We assume that trades occur at discrete times. The following activities occur within each period:

[5] Even without this assumption, i.e. even there are strategic concerns about announcements, the essential qualitative results of the paper do not change.

1. Each young agent forms an effective probability belief Q_t^h.
2. Each young agent observes the intrinsic data (p_t, q_t, d_t).
3. Each young agent makes an announcement Y_t^h publicly.
4. Transactions take place.

All activities but the first occur simultaneously. In what follows, we explicitly describe each young agent's problem, and then, describe the equilibrium of the economy.

2.1.4 Young Agent's Problem

The optimization problem of a young agent h in period t after observing the announcements of others is given by

$$\begin{array}{ll} \max_{(C_t^{1h}, \theta_t^h, C_{t+1}^{2h})} & E_{Q_t^h}\{u_h(C_t^{1h}, C_{t+1}^{2h}) \,|\mathcal{G}_t^h\} \\ \text{s.t.} & C_t^{1h} + P_t\theta_t^h + q_t B_t^h = W_t^h \\ & C_{t+1}^{2h} = \theta_t^h \cdot (P_{t+1} + D_{t+1}) + B_t^h, \end{array}$$

where C_t^{1h} denotes the consumption of h when young in period t, and C_{t+1}^{2h} denotes the consumption of h when old in period $t+1$ (bearing in mind h was born in period t).[6] To enable us to compute equilibria, we assume agent h's utility function to be of the CES form

$$u^h(C_t^{1h}, C_{t+1}^{2h}) = \frac{1}{1-\nu^h}(C_t^{1h})^{1-\nu^h} + \frac{\beta^h}{1-\nu^h}(C_{t+1}^{2h})^{1-\nu^h}, \quad \nu^h > 0,$$

where $\beta^h \in (0,1)$ is the discount factor and ν^h is the parameter that indicates the degree of relative risk aversion of agent h. Then, the first-order conditions (the Euler equations) for the optimization problem of a young agent h in period t will be

$$\begin{aligned} -P_t \cdot (C_t^{1h})^{-\nu^h} + \beta^h E_{Q_t^h}\{(C_{t+1}^{2h})^{-\nu^h} \cdot (P_{t+1} + D_{t+1}) \,|\mathcal{G}_t^h\} &= 0, \\ -q_t \cdot (C_t^{1h})^{-\nu^h} + \beta^h E_{Q_t^h}\{(C_{t+1}^{2h})^{-\nu^h} \,|\mathcal{G}_t^h\} &= 0. \end{aligned}$$

We can describe these conditions by using ratios $(p_t, d_t, c_t^{1h}, c_{t+1}^{2h}, b_t^h)$ instead of absolute values $(P_t, D_t, C_t^{1h}, C_{t+1}^{2h}, B_t^h)$ as follows:

$$\begin{aligned} p_t \cdot (c_t^{1h})^{-\nu^h} &= \beta^h E_{Q_t^h}\{(c_{t+1}^{2h} d_{t+1})^{-\nu^h}(p_{t+1}+1)d_{t+1} \,|\mathcal{G}_t^h\}, \\ q_t \cdot (c_t^{1h})^{-\nu^h} &= \beta^h E_{Q_t^h}\{(c_{t+1}^{2h} d_{t+1})^{-\nu^h} \,|\mathcal{G}_t^h\}, \\ c_t^{1h} &= -p_t\theta_t^h - q_t b_t^h + w^h, \\ c_{t+1}^{2h} &= \theta_t^h \cdot (p_{t+1}+1) + \frac{b_t^h}{d_{t+1}}. \end{aligned}$$

[6] We assume that each agent's preference is represented by a von Neumann-Morgenstern utility function.

From now on, *we assume that each agent believes that the economy is Markovian.* To be more specific, we assume that each agent believes that the joint process $(p_t, q_t, d_t, \mathbf{Y}_t, t = 1, 2, ...)$ is Markov. It follows that the conditions above will be rewritten as

$$\begin{aligned} p_t \cdot (c_t^{1h})^{-\nu^h} &= \beta^h E_{Q_t^h}\{(c_{t+1}^{2h} d_{t+1})^{-\nu^h} \cdot (p_{t+1}+1) d_{t+1} \,|\, p_t, q_t, d_t, \mathbf{Y}_t\}, \quad (5) \\ q_t \cdot (c_t^{1h})^{-\nu^h} &= \beta^h E_{Q_t^h}\{(c_{t+1}^{2h} d_{t+1})^{-\nu^h} \,|\, p_t, q_t, d_t, \mathbf{Y}_t\}, \quad (6) \\ c_t^{1h} &= -p_t \theta_t^h - q_t b_t^h + w^h, \\ c_{t+1}^{2h} &= \theta_t^h \cdot (p_{t+1}+1) + \frac{b_t^h}{d_{t+1}}. \end{aligned}$$

The conditional expectations in equations (5) and (6) are based only upon $(p_t, q_t, d_t, \mathbf{Y}_t)$, because of the Markov assumption. It follows that the demand correspondences of the young will be time-invariant: for every h, t,

$$\begin{aligned} \theta_t^h &= \theta^h(p_t, q_t, d_t, \mathbf{Y}_t; Q_t^h), \quad (7) \\ b_t^h &= b^h(p_t, q_t, d_t, \mathbf{Y}_t; Q_t^h). \quad (8) \end{aligned}$$

Observe however that the effective belief Q_t^h represents the non-stationarity in the sequence of effective beliefs of each dynasty h, and consequently, the demand is effectively non-stationary.

Moreover, the announcement can be defined on the true probability space as a time invariant mapping as follows:

$$Y_t^h = v^h(p_t, q_t, d_t; Q_t^h). \quad (9)$$

However, the map (9) is not understood by the young agent h himself in period t, because he does not form a belief about the effective beliefs. Instead, he only understands the following mapping:

$$Y_t^h = v_t^h(p_t, q_t, d_t), \quad (10)$$

which is not time invariant. Nevertheless, each young agent h in period t understands that the mapping (10) is determined randomly, although he does not form a belief about the determination of the mapping itself. In other words, he understands that the mapping (10) can be potentially different reflecting the ambiguity concerning the determination of the effective beliefs, and consequently, treats his own announcement Y_t^h as an intrinsic random variable. The companion paper (Nakata (2004)) examines a special case where (9) is a one-to-one mapping between Y_t^h and Q_t^h, and thus, (10) is not depending on (p_t, q_t, d_t).

2.2 The Equilibrium

In what follows, we provide the definition of the economy. In addition to the optimality conditions above, the equilibria of the economy are characterized by the market clearing conditions:

$$\sum_{h=1}^{H} \theta^h(p_t, q_t, d_t, \mathbf{Y}_t; Q_t^h) = 1, \quad (11)$$

$$\sum_{h=1}^{H} b^h(p_t, q_t, d_t, \mathbf{Y}_t; Q_t^h) = 0. \tag{12}$$

We by now are ready to state the definition of a stable Markov competitive equilibrium of our economy:

Definition: *Sequences of effective beliefs* $\{Q_t^h\}_{t=1}^{\infty}$ *(*$h = 1, 2, ..., H$*) and a joint stochastic process* $\{(p_t, q_t, d_t, \theta_t^h, b_t^h, Y_t^h), h = 1, 2, ..., H\}_{t=1}^{\infty}$ *with initial portfolios* $\{(\theta_0^1, b_0^1), h = 1, 2, ..., H\}$ *associated with the true probability measure* Π *constitute a* **stable Markov competitive equilibrium** *if*

1. $(p_t, q_t, d_t, \theta_t^h, b_t^h, Y_t^h; h = 1, 2, ..., H)$ *satisfy conditions (4) and (7)-(12) for all* t;

2. Π *is a stable measure, and every sequence of effective beliefs* $\{Q_t^h\}_{t=1}^{\infty}$ *constitutes a stable measure for all* h.

By construction, the equilibrium prices will be a sequence generated by a time-invariant map as follows:

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi(d_t, \mathbf{Y}_t, Q_t^1, Q_t^2, ..., Q_t^H) \quad \text{for all } t. \tag{13}$$

Note that the time-invariant map $\Phi(\cdot)$ is effectively time dependent due to the effective beliefs $(Q_t^1, Q_t^2, ..., Q_t^H)$, which are the very sources of non-stationarity of the economy.

To see that heterogeneity of beliefs is crucial for communication ever to have an impact on equilibria, we shall examine the special case of rational expectations equilibria (REE). In a stationary REE, $Q_t^h = \Pi$ for all h, t where the true probability measure Π is induced by (2) and by the equilibrium map (13).[7] By construction,

$$Y_t^h := v^h(p_t, q_t, d_t; \Pi) = Y_t,$$

Π-almost surely holds for all h, t. Hence, Y_t^h is a time-invariant function of (p_t, q_t, d_t). It follows that

$$\theta_t^h = \theta^h(p_t, q_t, d_t, \mathbf{Y}_t; \Pi) = \bar{\theta}^h(p_t, q_t, d_t; \Pi),$$

$$b_t^h = b^h(p_t, q_t, d_t, \mathbf{Y}_t; \Pi) = \bar{b}^h(p_t, q_t, d_t; \Pi).$$

Hence, given Π the equilibrium map will be reduced to

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi(d_t). \tag{14}$$

[7] In our set-up, rational expectations do not really make sense unless Π is a stationary measure. The basic principle behind rational expectations is compatibility between the empirical data and the probability law, which is the same as that of rational beliefs. However, rational expectations do not allow for heterogeneous beliefs. Hence, whenever the empirical data induces a stationary measure, each agent adopts it as its probability belief, and it is the true probability law as well.

This leads us to conclude the following:

Proposition 1: *Communication (or exchange of opinions) has no impact on a REE.*

As we can see from the above explanations, communication has no impact on a REE, because each agent can pinpoint the announcements of other agents correctly with probability one with respect to the true probability measure Π. Therefore, announcements are not really intrinsic random variables on the subjective probability spaces of agents in this case. In other words, announcements do not expand the state spaces of the agents. Note however that this particular result rests crucially on the assumption that $Q_t^h = \Pi$ for all h, t.

2.3 Rationality of Beliefs

Now, we allow for heterogeneous beliefs unlike the REE. Nevertheless, we require every sequence of effective beliefs $\{Q_t^h\}_{t=1}^{\infty}$ to constitute a rational belief. The generic condition/definition is the following:

Definition: *A sequence of effective beliefs $\{Q_t^h\}_{t=1}^{\infty}$ constitutes a **rational belief** if it induces a stationary measure that is equivalent to the one induced by the true probability measure Π.*

In what follows, we show that there exists a sequence of effective beliefs $\{Q_t^h\}_{t=1}^{\infty}$ that constitutes a rational belief. The argument here is essentially the same as in the companion paper (Nakata(2004)), although the state space is larger in the current paper than the one there. Let X denote the state space of $(p_t, q_t, d_t, \mathbf{Y}_t)$ for all t, and X^∞ the state space for the entire sequence. Let $\mathcal{B}(X^\infty)$ denote the Borel σ field generated by X^∞. Then, the true stochastic process of the economy is described by a stochastic dynamical system $(X^\infty, \mathcal{B}(X^\infty), T, \Pi)$. However, we can expand the probability space to incorporate the sequences of effective beliefs $\{Q_t^h\}_{t=1}^{\infty}$.

To do so, we $\hat{\Pi}^h$ denote the true probability measure on the space $((X \times \mathcal{Q}^h)^\infty, \mathcal{B}((X \times \mathcal{Q}^h)^\infty))$, whose 'marginal measure' for X^∞ is Π and that for $(\mathcal{Q}^h)^\infty$ is $\bar{\mu}^h$, while μ^h is the marginal measure for for $\mathcal{Q}^h$.[8] Then, the expanded true stochastic process is described as a dynamical system such that $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$, where $\Omega^h := (X \times \mathcal{Q}^h)^\infty$ and $\mathcal{B}^h := \mathcal{B}((X \times \mathcal{Q}^h)^\infty)$. This construction is similar to the rational belief structure in Nielsen (1996), although the set of beliefs consist of measures on $(X, \mathcal{B}(X))$ rather than on $(X^\infty, \mathcal{B}(X^\infty))$ there. Also, any interdependence between those measures and X is not allowed there, while it is allowed here. Hence, we follow Kurz and Schneider (1996) with respect to such an interdependence, whilst the structure that incorporates measures on measures follow Nielsen (1996).

[8]We can define a measure on measures as long as the underlying space is a standard space. See Gray (1988) for details.

With this in mind, we prove a theorem that is analogous to the conditional stability theorem (Theorem 2) in Kurz and Schneider (1996), although they introduce random variables that represent the effective beliefs as conditional measures, which they call the generating variables. Namely, while they study the stability properties of the joint system of X and the generating variables, we study the stability properties of the joint system of X and Q_t^h.

Before stating the theorem, we introduce some notation to be more precise concerning the construction of the probability space(s). Let $\hat{\Pi}_{\mathbf{k}}^h$ denote the conditional probability of $\hat{\Pi}^h$ given a particular sequence of effective beliefs $\mathbf{k} \in (\mathcal{Q}^h)^\infty$:

$$\hat{\Pi}_{\mathbf{k}}^h(\cdot) : (\mathcal{Q}^h)^\infty \times \mathcal{B}(X^\infty) \mapsto [0, 1].$$

For each $A \in \mathcal{B}(X^\infty)$, $\hat{\Pi}_{\mathbf{k}}^h$ is a measurable function of $\mathbf{k}$ and for each $\mathbf{k}$, $\hat{\Pi}_{\mathbf{k}}^h(\cdot)$ is a probability on $(X^\infty, \mathcal{B}(X^\infty))$. For $A \in \mathcal{B}(X^\infty)$ and $B \in \mathcal{B}((\mathcal{Q}^h)^\infty)$, we have

$$\hat{\Pi}^h(A \times B) = \int_{\mathbf{k}\in B} \hat{\Pi}_{\mathbf{k}}^h(A)\bar{\mu}^h(d\mathbf{k}),$$

where $\bar{\mu}^h$ is a probability measure on $((\mathcal{Q}^h)^\infty, \mathcal{B}((\mathcal{Q}^h)^\infty))$. Also, as we noted above,

$$\begin{aligned} \Pi(A) &= \hat{\Pi}^h(A \times (\mathcal{Q}^h)^\infty), \quad \forall A \in \mathcal{B}(X^\infty), \\ \bar{\mu}^h(B) &= \hat{\Pi}^h(X^\infty \times B), \quad \forall B \in \mathcal{B}((\mathcal{Q}^h)^\infty). \end{aligned}$$

When $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$ is a stable dynamical system with a stationary measure $m^{\hat{\Pi}^h}$, we define the two marginal measures of $m^{\hat{\Pi}^h}$ as follows:

$$\begin{aligned} m(A) &:= m^{\hat{\Pi}^h}(A \times (\mathcal{Q}^h)^\infty), \quad \forall A \in \mathcal{B}(X^\infty), \\ m_{Q^h}(B) &:= m^{\hat{\Pi}^h}(X^\infty \times B), \quad \forall B \in \mathcal{B}((\mathcal{Q}^h)^\infty). \end{aligned}$$

Also let $\hat{m}_{\mathbf{k}}$ denote the stationary measure of $\hat{\Pi}_{\mathbf{k}}^h$, which is a measure on $(X^\infty, \mathcal{B}(X^\infty))$.

When the dynamical system $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$ has the above construction, we have the following theorem:

Theorem 1: *Let $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$ be a stable and ergodic dynamical system. Then,*

(a) $(X^\infty, \mathcal{B}(X^\infty), T, \hat{\Pi}_{\mathbf{k}}^h)$ *is stable and ergodic for* $\hat{\Pi}^h$ *a.a.* $\mathbf{k}$.
(b) $\hat{m}_{\mathbf{k}}^h$ is independent of $\mathbf{k}$, and $\hat{m}_{\mathbf{k}}^h = m$.
(c) *If* $(X^\infty, \mathcal{B}(X^\infty), T, \hat{\Pi}_{\mathbf{k}}^h)$ *is stationary then the stationary measure of* $\hat{\Pi}_{\mathbf{k}}^h$ *is* Π. *That is*

$$\hat{m}_{\mathbf{k}}^h = m = \Pi.$$

(Proof) The proof essentially follows that of Theorem 2 in Kurz and Schneider (1996). First X is clearly a Polish space, since it is a set of countable isolated points.[9] Also, $\mathcal{Q}^h$ is a Polish space, because it is assumed that $\mathcal{Q}^h$ is at most a countable set, and thus, it is a set of countable isolated points. It

[9]It is a finite set in our model. However, we keep the theorem as generic as possible.

follows that $(X \times \mathcal{Q}^h)^\infty$ is also a set of countable isolated points; thus, it is a Polish space, too. Because $\mathcal{Q}^h$, whose role is essentially the same as that of the generating variable in Kurz and Schneider, is countable at most, it is straightforward that the dynamical system in this paper possesses the same properties as the one in Kurz and Schneider; thus, the proof of Theorem 2 of Kurz and Schneider applies. *Q.E.D.*

Suppose Q^h is a probability measure on $(\Omega^h, \mathcal{B}^h)$, and that, $(\Omega^h, \mathcal{B}^h, T, Q^h)$ is a stable and ergodic dynamical system. Then Theorem 1 states that any stable measure Q^h on $(\Omega^h, \mathcal{B}^h)$ implies a stationary measure $m^h_{\mathbf{k}} = m^h$ for all $\mathbf{k} \in (\mathcal{Q}^h)^\infty$, where m^h is the marginal measure of m^{Q^h} on $(X^\infty, \mathcal{B}(X^\infty))$, which is the stationary measure induced by Q^h.[10]

Recall that the definition of rational belief restricts the class of stable measures Q^h to satisfy the property such that $m^h = m$. Hence, *Theorem 1 ensures that every sequence of effective beliefs $\{Q^h_t\}_{t=1}^\infty$ constitutes a rational belief as long as the stable measure Q^h satisfy the condition (on the stationary measure m^h).*

In what follows, we explicitly show how to describe rational beliefs in our Markovian economy. Recall that the equilibrium map (13) is

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \hat{\Phi}(d_t, \mathbf{Y}_t, Q^1_t, Q^2_t, ..., Q^H_t), \quad \forall t.$$

Hence, the equilibrium is driven by a stable Markov process of $\{d_t, \mathbf{Y}_t, Q^1_t, Q^2_t, ..., Q^H_t\}_{t=1}^\infty$ as we noted above. Namely, we need to define a dynamical system on $(V^\infty, \mathcal{B}(V^\infty))$, where V is the state space of $(d_t, \mathbf{Y}_t, Q^1_t, Q^2_t, ..., Q^H_t)$, as a stable Markov process.

For the computation of the long-term frequencies or long-term averages of the economic variables, it is sufficient to specify a stationary transition matrix Γ that specifies the transition probabilities from $(d_t, \mathbf{Y}_t, Q^1_t, Q^2_t, ..., Q^H_t)$ to $(d_{t+1}, \mathbf{Y}_{t+1}, Q^1_{t+1}, Q^2_{t+1}, ..., Q^H_{t+1})$, i.e. Γ is on $V \times V$, which induces a stationary measure, and that, the stationary measure is the one that is induced by the true probability measure. Note that, the true process may not be stationary, but it is enough for us to specify its induced stationary measure that is fully characterized by the transition probability matrix Γ to compute the long-term frequencies.

On the other hand, we specified that the effective beliefs are determined randomly, either Q^h_H or Q^h_L. Hence, we define pairs of transition probability matrices that correspond to the pair of effective beliefs (Q^h_H, Q^h_L) as follows: young agent h in period t adopts a transition matrix $\bar{F}^h_t$ by the following rule:

$$\bar{F}^h_t = \begin{cases} \bar{F}^h_H & \text{if } Q^h_t = Q^h_H; \\ \bar{F}^h_L & \text{if } Q^h_t = Q^h_L. \end{cases} \tag{15}$$

With this specification, we require the sequence of effective beliefs to satisfy the rationality condition, which is analogous to the one found in papers

[10] We need the superscript h for m^h here while there is none in Theorem 1 (b), because the marginal measure of Q^h on $(X^\infty, \mathcal{B}(X^\infty))$ is subjective, while that of $\hat{\Pi}^h$ is the true probability.

on rational beliefs (e.g. Kurz and Schneider [1996], Kurz and Beltratti [1997], Kurz and Motolese [2001], etc.):

Rationality Condition: *The transition matrices* $\bar{F}_H^h$ *and* $\bar{F}_L^h$ *of each agent* h *must satisfy the following condition for the sequence of effective beliefs* $\{Q_t^h\}_{t=1}^{\infty}$ *to constitute a rational belief:*

$$\alpha^h \cdot \bar{F}_H^h + (1 - \alpha^h) \cdot \bar{F}_L^h = \Gamma, \quad \forall h. \tag{16}$$

Because the frequency of the event $\{Q_t^h = Q_H^h\}$ is α^h with respect to the true probability μ, agent h uses the transition probability matrix $\bar{F}_H^h$ with frequency α^h. Hence, the rationality condition (16) requires the sequence of beliefs $\{Q_t^h\}_{t=1}^{\infty}$ to be compatible with the data that is generated by the stationary transition probability matrix Γ. In other words, there is no way for the agents to reject the set of theories $\mathcal{Q}^h$ for being invalid by observing the data.

With this in mind, we define a Markov Rational Belief Equilibrium as follows:

Definition: *A* **Markov Rational Belief Equilibrium** *(RBE) is a stable Markov Competitive Equilibrium in which the sequences of effective beliefs* $\{Q_t^h\}_{t=1}^{\infty}$ $(h = 1, 2, ..., H)$ *satisfy (3), (15) and the rationality condition (16).*

The definition of a Markov Rational Belief Equilibrium allows for heterogeneous beliefs. However, it requires the sequence of beliefs to constitute a rational belief. Hence, it is required that both the true equilibrium process of the economic variables and the subjective process of them must be stable, but not necessarily stationary. However, it is not obvious at all how communication and/or the non-stationarity of beliefs impact the equilibrium of the economy. We therefore develop a simulation model to examine the impacts of communication and/or the non-stationarity of the beliefs. However, before examining the simulation model, we look into the structure of the state space of the economy.

2.4 Endogenous Expansion of the State Space

As we explained above, the state space of the observables $(p_t, q_t, d_t, \mathbf{Y}_t)$ in each period is X. Consequently, the subjective probability space of each young agent h in period t is $(X^{\infty}, \mathbf{B}(X^{\infty}), Q_t^h)$. On the other hand, effective beliefs Q_t^h are measurable with respect to the true probability measure Π. Hence, the state space with respect to the true probability measure in each period is $Z := X \times \mathcal{Q}^1 \times \cdots \times \mathcal{Q}^H$, and the true probability space is $(Z^{\infty}, \mathcal{B}(Z^{\infty}), \Pi)$.

It is important to recognize that the fundamentals (or the primitives) of the economy involve the effective beliefs $(Q_t^1, Q_t^2, ..., Q_t^H)$ as well as the announcements $\mathbf{Y}_t$. Furthermore, notice that the introduction of announcements expands the state space of the true probability space. Nevertheless, announcements are indeed endogenous, because they are functions defined by

(4). Therefore, we claim that *announcements create endogenous uncertainty.* In particular, we emphasize that announcements are by no means sunspot variables, because they are given endogenously, not given arbitrarily.

Recall that each effective belief Q_t^h represents the 'state of belief'. This is corresponding to an assessment variable, which is a random variable that represents the state of belief in the existing papers on rational belief models, e.g., Kurz and Beltratti (1997), Kurz and Motolese (2001). Moreover, in those papers, correlation between assessment variables are assumed *a priori*, and their simulation results exhibit sufficiently large volatilities of the financial economy with such correlation. They claim that such correlation is due to social interactions, which are not modelled explicitly in their papers.

Although one may claim that assessment variables are sunspot variables, and thus, rational belief equilibria and sunspot equilibria are equivalent, we can see a crucial difference between them in the light of our model. Recall that the conventional sunspot models assume rational expectations; thus, there is no room for endogenous expansion of the state space. Hence, the state space then is limited to a product of exogenous random variables (d_t in our model) and the sunspot variables.[11] Therefore, the RBE models are compatible with an endogenous expansion of the state space, while the conventional sunspot models are not. In fact, it is important to recognize that heterogeneity of beliefs is crucial to have some form of endogenous expansion of the state space.

Reverting our attention to the equilibrium map (13), the map implies that there is a range at most $2 \times (2 \times 2)^H$ distinct equilibrium price states (p_t, q_t). Observe that the state space of (p_t, q_t) is expanded by the introduction of communication and heterogeneity of beliefs. Now instead of describing the state space of the prices by that of $\{d_t, \mathbf{Y}_t, Q_t^1, ..., Q_t^H\}_{t=1}^{\infty}$, we may also consider a space S^{∞} where S is an index set such that $S := \{1, 2, ..., 2^{2H+1}\}$. In fact, we can define a one-to-one mapping ϕ_s between S and the state space of $(d_t, \mathbf{Y}_t, Q_t^1, ..., Q_t^H)$ such that

$$s = \phi_s(d_t, \mathbf{Y}_t, Q_t^1, ..., Q_t^H) \in S. \tag{17}$$

Because of the Markov assumption, the true equilibrium transition probabilities are defined as those from state s in period t to state s^+ in period $t+1$. Note that although the true process may not be stationary, it is sufficient to specify a stationary transition probability matrix Γ to compute the long-run frequencies of the economic variables.

3 Simulations

3.1 The Simulation Model

The simulation model here follows to some extent that of Kurz and Beltratti (1997) and/or Kurz and Motolese (2001), although our model involves communication. By following them, we assume $H = 2$ so as to make the

[11] In the conventional sunspot models, the private sunspots represent the Harsanyi types.

simulation tractable. To begin with, we specify the stationary transition probability matrix Γ. The matrix Γ must satisfy the following conditions:

(a) the marginal distribution for Q_t^h is i.i.d. with $\Pr\{Q_t^h = Q_H^h\} = \alpha^h$;
(b) the marginal distribution for d_t is Markov as specified by matrix (2).

The family of matrices that satisfy the above conditions is limited, yet there are quite a few of them. We choose the following matrix for simplicity and flexibility in parameterization:

$$\Gamma = \begin{bmatrix} 0.43A^H & 0.57B^H \\ 0.57A^L & 0.43B^L \end{bmatrix}, \tag{18}$$

where A^H, A^L, B^H and B^L are 16×16 matrices that have the following structure:

$$A^H = \begin{bmatrix} \eta^1 A^{H11} & (\alpha^1 - \eta^1)A^{H12} & (\alpha^2 - \eta^1)A^{H13} & (1 + \eta^1 - \alpha^1 - \alpha^2)A^{H14} \\ \eta^2 A^{H21} & (\alpha^1 - \eta^2)A^{H22} & (\alpha^2 - \eta^2)A^{H23} & (1 + \eta^2 - \alpha^1 - \alpha^2)A^{H24} \\ \eta^3 A^{H31} & (\alpha^1 - \eta^3)A^{H32} & (\alpha^2 - \eta^3)A^{H33} & (1 + \eta^3 - \alpha^1 - \alpha^2)A^{H34} \\ \eta^4 A^{H41} & (\alpha^1 - \eta^4)A^{H42} & (\alpha^2 - \eta^4)A^{H43} & (1 + \eta^4 - \alpha^1 - \alpha^2)A^{H44} \end{bmatrix}, \tag{19}$$

where A^{Hij} are 4×4 matrices for $i, j = 1, 2, 3, 4$ whose row coordinates sum up to one. Also, A^L, B^H, and B^L have the same structure.

Note that the upper half coordinates of Γ (the first 16 rows) are the transition probabilities from the current states being $d_t = d^H$, which we call the 'high dividend states', and the lower half coordinates are those from the 'low dividend states'. Also, the coordinates that contain A^{H1j} are the transition probabilities when the current state satisfies $(Q_t^1 = Q_H^1, Q_t^2 = Q_H^2, d_t = d^H)$, those containing A^{H2j} correspond to $(Q_t^1 = Q_H^1, Q_t^2 = Q_L^2, d_t = d^H)$, those containing A^{H3j} correspond to $(Q_t^1 = Q_L^1, Q_t^2 = Q_H^2, d_t = d^H)$ and those containing A^{H4j} correspond to $(Q_t^1 = Q_L^1, Q_t^2 = Q_L^2, d_t = d^H)$. Furthermore, the first row of A^{Hij} corresponds to $(Y_t^1 = 1, Y_t^2 = 1)$, the second corresponds to $(Y_t^1 = 1, Y_t^2 = 0)$, the third corresponds to $(Y_t^1 = 0, Y_t^2 = 1)$ and the fourth corresponds to $(Y_t^1 = 0, Y_t^2 = 0)$.

It is worthwhile to mention that the structure of Γ is slightly different from that of Kurz and Beltratti (1997) or Kurz and Motolese (2001). In their models, matrices A^H, A^L, B^H and B^L are assumed to be the same, but we eliminate this assumption. This is because the announcements of the agents depend upon p_t, the current price/dividend ratio, and that, p_t is positively correlated with d_t. In particular, we can expect that the price/dividend ratios in the high dividend states (i.e. $d_t = d^H$) are higher than those in the low dividend states (i.e. $d_t = d^L$) *ceteris paribus*, i.e. when other variables are the same. Hence, it is more likely for the agents to announce $Y_t^h = 1$ when the economy is in a low dividend state. Therefore, other things being equal, we can expect that d_t and (Y_t^1, Y_t^2) are correlated. This implies that A^H, A^L, B^H and B^L are different in general.

Next, we specify the pairs of transition probability matrices $(\bar{F}_H^h, \bar{F}_L^h)$ that satisfy the rationality condition (16) for the sake of computation. However, these transition probability matrices are 32×32 matrices, and that, it is not straightforward at all to interpret. To this end, we introduce some more structure to the transition probability matrices, instead of specifying $(\bar{F}_H^h, \bar{F}_L^h)$ directly.

We assume that each pair of transition probability matrices $(\bar{F}_H^h, \bar{F}_L^h)$ has the following structure for every h:

$$\begin{aligned}
\bar{F}_H^h &= \sum_{k=1}^{K^h} f^h(k,y) \cdot F_H^h(k), \\
\bar{F}_L^h &= \sum_{k=1}^{K^h} f^h(k,y) \cdot F_L^h(k),
\end{aligned}$$

where

$$f^h(k,y) = f^h(k) + \lambda_k^h \cdot (y - Z^h), \tag{20}$$

where $f^h(k)$ is the 'prior' weight (or 'prior probability') assigned to pair k of transition probability matrices $(F_H^h(k), F_L^h(k))$ by agent h, and thus, $\sum_{k=1}^{K^h} f^h(k) = 1$ (K^h is some integer) and $f^h(k) \geq 0$ for all h, k, while $f^h(k,y)$ is the 'posterior' weight (or 'posterior probability') assigned to pair k by agent h, and λ_k^h is a parameter, which is restricted so that $f^h(k,y)$ satisfies the law of probability. Also, y is the realization of the announcement of the other agent $Y_t^{(h)}$ and $Z^h = E_m Y_t^{(h)}$ is the expected value of the announcement of the other agent with respect to the stationary measure. Moreover, each pair of transition probability matrices $(F_H^h(k), F_L^h(k))$ satisfies the following condition for every h:

$$\alpha^h F_H^h(k) + (1-\alpha^h) F_L^h(k) = \Gamma, \quad \text{for } k = 1, 2, ..., K^h.$$

It follows that each pair of transition probability matrices $(\bar{F}_H^h, \bar{F}_L^h)$ always satisfies the rationality condition (16) under this construction.

Furthermore, we assume that $F_H^h(k)$ has the following structure:

$$F_H^h(k) := \begin{bmatrix} 0.43k^\gamma A^H & (1 - 0.43k^\gamma) \cdot B^H \\ 0.57k^\gamma A^L & (1 - 0.57k^\gamma) \cdot B^L \end{bmatrix}, \tag{21}$$

and $F_L^h(k) = (\Gamma - \alpha^h F_H^h(k))/(1-\alpha^h)$. Note that γ must satisfy the conditions such that $k^\gamma \leq 1/0.57$ and $k^\gamma \leq 1/0.43$ for all k. Amongst these conditions, we only need to check $(K^h)^\gamma \leq 1/0.57$. Moreover, by summing up both sides of equation (20) with respect to k, we obtain

$$1 = \sum_{k=1}^{K^h} f^h(k;y) = \sum_{k=1}^{K^h} f^h(k) + (\sum_{k=1}^{K^h} \lambda_k^h) \cdot (y - Z^h) = 1 + (\sum_{k=1}^{K^h} \lambda_k^h) \cdot (y - Z^h), \quad \forall y,$$

and it follows that

$$\sum_{k=1}^{K^h} \lambda_k^h = 0. \tag{22}$$

We adopt this construction, because it is very convenient to examine the difference between the cases with and without communication very easily. Namely, we can examine the cases without communication by selecting $\lambda_k^h =$

0 for all h, k, while there must exist communication otherwise. Note however that this does not necessarily mean that each agent's belief itself has this structure, although it is possible to introduce such an interpretation in a casual way as we discuss later.

Observe that the parameter k^γ is a proportional revision of the conditional probabilities of the 'high dividend states' ($s = 1, 2, ..., 16$) and the 'low dividend states' ($s = 17, 18, ..., 32$) relative to Γ. Because $k^\gamma > 1$ for $k \geq 2$, $F_H^h(k)$ for $k \geq 2$ involves higher probabilities of the 'high dividend states'. Hence, the effective beliefs Q_t^h have a simple interpretation: agent h is optimistic (relative to Γ) when $Q_t^h = Q_H^h$ at t about p_t at $t+1$.

Although there can be many possible set ups for λ_k^h, we specify λ_k^h as follows:

$$\lambda_k^h = \begin{cases} -\chi^h & \text{if} \quad k < \bar{k}^h, \\ 0 & \text{if} \quad k = \bar{k}^h, \\ \chi^h & \text{if} \quad k > \bar{k}^h, \end{cases} \tag{23}$$

where χ^h is some scalar and $\bar{k}^h := (1 + K^h)/2$. It is clear that this specification satisfies condition (22). Note however that there are restrictions on χ^h, because $f^h(k)$ and $f^h(k, y)$ must follow the law of probability, their ranges are $[0, 1]$:

$$|\chi^h| \leq \frac{f^h(k)}{Z^h}, \quad |\chi^h| \leq \frac{f^h(k)}{1 - Z^h}, \tag{24}$$

$$|\chi^h| \leq \frac{1 - f^h(k)}{Z^h}, \quad |\chi^h| \leq \frac{1 - f^h(k)}{1 - Z^h}. \tag{25}$$

With this specification of λ_k^h, we can classify the beliefs of the agents in a relatively simple way. When $\chi^h > 0$, agent h's optimism/pessimism will be enhanced by observing an announcement by the other agent saying that the price/dividend ratio is expected to rise in the next period (i.e. $Y_t^{(h)} = 1$), and his optimism/pessimism will be diminished when the other agent expects the price/dividend ratio to fall (i.e. $Y_t^{(h)} = 0$). Also, the converse holds when $\chi^h < 0$. Therefore, we can classify the beliefs as follows:

- Optimistic if $Q_t^h = Q_H^h$;
- Pessimistic if $Q_t^h = Q_L^h$;
- If $\chi^h > 0$, then conformist when optimistic and contrarian when pessimistic;
- If $\chi^h < 0$, then contrarian when optimistic and conformist when pessimistic.

Note that the larger $|\chi^h|$ is, the larger the impact of communication on the (conditional) beliefs $Q_t^h\{\cdot|Y_t^{(h)}\}$ becomes. In other words, the degree of conformism or contrarianism is larger when $|\chi^h|$ is larger.

However, it is *a priori* not trivial at all whether communication amplifies or mitigates the equilibrium fluctuations of the economy. Hence, we compute various equilibria by specifying various sets of parameters below to see the effects of communication on the equilibria explicitly. To evaluate the impact of communication on the equilibria, we compute the following statistics in the United States. The estimations are from Kurz and Motolese (2001) unless noted. They use the updated version of the data base for 1889 - 1998 compiled by Shiller (1981), whilst Mehra and Prescott (1985) use the same data base for 1889-1978.

- $\bar{p}$: the long term (average of the) price/dividend ratio. The estimated value is 22.84;
- σ_p: the standard deviation of the price/dividend ratio. The estimated value is 6.48;
- $\bar{R}$: the average risky rate of return on equities. The estimated value is 8%, whilst Mehra and Prescott (1985) report an estimated value of 6.98% for 1889 - 1978;
- σ_R: the standard deviation of the risky rate of return. The estimated value is 18.08%, whilst Mehra and Prescott (1985) report an estimated value of 16.67% for 1889 - 1978;
- r^F: the average riskless interest rate. Mehra and Prescott (1985) report an estimated value of .8% for 1889 - 1978 based on the 90 day T-bill rate for 1931 - 1978. For 1889 - 1931 one may use various alternative securities. We accept an estimate of around 1%;
- σ_{r^F}: the standard deviation of the riskless interest rate. Mehra and Prescott (1985) report an estimate of 5.67% for 1889 - 1978;
- ρ: the premium of equity return over the riskless rate. With the estimates of $\bar{R}$ and r^F above, it should be around 7%.

3.2 Computing the Rational Expectations Equilibria

First, we observe that we can compute a rational expectations equilibrium (REE) by selecting $\gamma = 0$ (and $\chi^1 = \chi^2 = 0$). This is because the prior $f^h(k)$ and/or the posterior $f^h(k; y)$ do not play any role in a REE. Namely, $\bar{F}^h_H = \bar{F}^h_L = \Gamma$ for all h must hold ($K^h = 1$ for all h). Also, because the announcements do not have any impact on the REE, the same results hold for any A^{Hij}, A^{Lij}, B^{Hij} and B^{Lij} ($i, j = 1, 2, 3, 4$). Moreover, the assessment variables do not play any role; thus, it does not matter what α^h's and η^i's are.

Table 1 reports a couple of REE with different choices of parameter values: the first column of Table 1 reports an REE with $\beta^1 = \beta^2 = 0.95$, $\nu^1 = \nu^2 = 3.0$, $w^1 = w^2 = 26.0$ (REE1), while the second column reports an REE with $\beta^1 = \beta^2 = 0.90$, $\nu^1 = \nu^2 = 3.25$, $w^1 = w^2 = 24.0$ (REE2). The results in Table 1 represent what Mehra and Prescott (1985) introduce as 'the equity premium puzzle'. It is clear that the computation results yield a rather small equity premium of less than 0.5%, while the empirical record shows that of about 7%. Also, the discrepancy in the riskless rate is huge:

Variable	REE1	REE2	Empirical Record
$\bar{p}$	25.31	23.13	23
σ_p	0.07	0.07	6.48
$\bar{R}$	5.84%	6.21%	8.00%
σ_R	4.08%	4.12%	18.08%
r^F	5.39%	5.72%	1.00%
σ_{r^F}	0.85%	0.88%	5.67%
ρ	0.45%	0.49%	7.00%

Table 1: REE Results

more than 5% versus 1%. It is however very important to recognize that all the volatility measures show that the REE predictions fail to reproduce the empirical record. For example, the standard deviation of price/dividend ratio is only about 0.07 according to the REE predictions, while the empirical record shows 6.48, which is more than 90 times larger than the REE predictions.

3.3 The Impact of Communication on RBE

To examine the effects of communication, we specify the unconditional joint distribution of the effective beliefs (Q_t^1, Q_t^2). In Kurz and Beltratti (1997), the assessment variables, which correspond to the effective beliefs in the current paper, are correlated with each other *a priori*. Although they do not model the mechanism that generates such correlation of beliefs, it is shown that correlation of beliefs makes the economy more volatile, and as a result, the equity premium puzzle is resolved. We, on the other hand, explicitly model the mechanism that causes the beliefs to be correlated, i.e. communication.

Moreover, to elucidate the effects of communication alone, we specify the unconditional joint distribution for the effective beliefs (Q_t^1, Q_t^2) to have the following property:

$$\Pi\{(Q_t^1, Q_t^2) \,|\, (Q_{t-1}^1, Q_{t-1}^2)\} = \Pi\{(Q_t^1, Q_t^2)\} = \Pi\{Q_t^1\} \cdot \Pi\{Q_t^2\}, \quad \forall t. \tag{26}$$

Namely, the unconditional joint distribution of the effective beliefs (Q_t^1, Q_t^2) has an i.i.d. as well as a mutual independence property. With this specification, there is no correlation between the effective beliefs of the two agents, and also, the effective beliefs are unaffected by the past effective beliefs *a priori*. Therefore, if there exists any kind of correlation of beliefs (either between agents or across time), it is caused by the observables, in particular (d_t, Y_t^1, Y_t^2). Note that the above properties are satisfied when

$$\eta^i = \alpha^1 \cdot \alpha^2 =: \eta \quad (i = 1, 2, 3, 4).^{12} \tag{27}$$

This is to highlight the effects of announcements rather than the correlation between the effective beliefs (Q_t^1, Q_t^2) unlike the companion paper (Nakata

[12] However, it is not necessarily the case vice versa.

(2004)). In other words, the current paper examines if communication generates correlation of beliefs that results in large fluctuations of the economic variables even when the effective beliefs (or the subjective probability measures) themselves are independent.

As pointed out above, it does not mean that the conditional joint distributions have a similar property, i.e.

$$\Pi\{(Q_t^1, Q_t^2) \,|\, d_t, Y_t^1, Y_t^2\} \neq \Pi\{Q_t^1 \,|\, d_t, Y_t^1, Y_t^2\} \cdot \Pi\{Q_t^2 \,|\, d_t, Y_t^1, Y_t^2\}.$$

On the contrary, it is the observables (d_t, Y_t^1, Y_t^2) that generate correlation between the effective beliefs (Q_t^1, Q_t^2). In particular, we shall show below that the announcements (Y_t^1, Y_t^2) play a dominant role in generating such correlation of beliefs.

To examine explicitly how large is the role of communication in generating correlation of beliefs, we specify two distinct sets of prior weights $(f^1(k), f^2(k))$.

<u>Set 1</u>
We assume that $K^h = K$. Also,
Agent 1:

$$f^1(k) = 1/K \quad \text{for all} \quad k;$$

Agent 2:

$$f^2(k) = \begin{cases} 1/K - 0.01 & \text{if} \quad k = 1; \\ 1/K + 0.01 & \text{if} \quad k = K; \\ 1/K & \text{otherwise.} \end{cases}$$

It is clear that both agents' priors are uniform or nearly uniform. In this case, the restrictions on χ^h are relatively loose. In fact, when $K = 5$,

$$|\chi^1| \leq \frac{0.2}{Z^1}, \qquad |\chi^1| \leq \frac{0.2}{1 - Z^1}, \quad |\chi^1| \leq \frac{0.8}{Z^1}, \quad |\chi^1| \leq \frac{0.8}{1 - Z^1}; \quad (28)$$

$$|\chi^2| \leq \frac{0.19}{Z^2}, \qquad |\chi^2| \leq \frac{0.19}{1 - Z^2}, \quad |\chi^2| \leq \frac{0.79}{Z^2}, \quad |\chi^2| \leq \frac{0.79}{1 - Z^2}. \quad (29)$$

Consequently, the posteriors may be quite different from the priors in this case. In other words, the effects of communication on the beliefs are large.

<u>Set 2</u>
We assume that $K^h = 5$ for $h = 1, 2$. Also,
Agent 1:

$$f^1(k) = 0.2 \quad \text{for all} \quad k;$$

Agent 2:

$$f^2(k) = k/15 \quad \text{for all} \quad k.$$

Agent 1's prior is again a uniform prior, while agent 2 assigns larger prior weights on larger k's. Recall that k^γ is a proportional revision of the conditional probabilities of the 'high dividend states' $(s = 1, 2, ..., 16)$ and the 'low dividend states' $(s = 17, 18, ..., 32)$ relative to Γ, a larger k corresponds to a more volatile stochastic process. Hence, agent 2 assigns larger prior weights on more volatile processes. However, the restrictions on χ^2 are severer than above:

$$|\chi^2| \leq \frac{1}{15Z^2}, \quad |\chi^2| \leq \frac{1}{15(1-Z^2)}, \quad |\chi^2| \leq \frac{2}{3Z^2}, \quad |\chi^2| \leq \frac{2}{3(1-Z^2)}.$$

Hence, the effects of communication on the beliefs are more limited under the proposed specification of λ_k^h above, i.e. equation (23).

3.3.1 RBE without Communication

First, we compute RBE without communication. To do so, we select $\chi^1 = \chi^2 = 0$. Because, there is no communication, the same results hold no matter what the matrices A^{Hij}, A^{Lij}, B^{Hij} and B^{Lij} are. For other parameters, we select $\beta^1 = \beta^2 = 0.90$, $\nu^1 = \nu^2 = 3.25$, $w^1 = w^2 = 24.0$, $\gamma = 0.346$ and $\alpha^1 = \alpha^2 = 0.57$. Table 2 reports the RBE for the two distinct sets of beliefs, where RBE2.1 corresponds to Set 1 with $K = 5$ and RBE2.2 corresponds to Set 2. It is clear from the table that Set 2 (RBE2.2) yields a slightly higher

Variable	RBE2.1	RBE2.2
$\bar{p}$	22.97	22.91
σ_p	0.34	0.42
$\bar{R}$	6.28%	6.32%
σ_R	5.00%	5.40%
r^F	5.70%	5.67%
σ_{r^F}	2.81%	3.36%
ρ	0.59%	0.65%

Table 2: RBE without Communication

volatility in general. As mentioned above, Agent 2's belief in Set 2 places larger weights on more volatile stochastic processes *a priori*, and that, the posterior has the exactly same property because there is no communication. Note that both Set 1 (RBE2.1) and Set 2 (RBE2.2) yield somewhat higher volatilities than the REE above. This is because the beliefs are correlated through the observations of d_t. However, the difference is quite limited. In other words, the difference in priors has a very little impact on the equilibrium fluctuations unless there is communication. Moreover, the results still do not show large fluctuations that are comparable with the empirical record, although they do show larger fluctuations than REE do.

It is worth noting that the computation results are again not actual sample statistics, but expected values that are based upon the invariant distribution derived from the stationary transition probability matrix Γ. Recall that although the agents do not know the true probability measure Π, their beliefs and the true measure have the same stationary measure, i.e. they agree on average in the long run. Hence, we can calculate the long run statistics by using the expected values based upon the stationary distribution. In fact, all agents know these long run statistics, and that sequence of their effective beliefs are compatible with these statistics as long as they satisfy the rationality conditions.

3.3.2 RBE with Communication

In what follows, we compute RBE with communication. Although we can expect that the results depend upon the prior weights of the agents, we use only Set 1 of prior weights. Moreover, if the equilibrium involves larger volatility than the previous results, that indicates communication plays a dominant role in amplifying fluctuations.

Because the announcements Y_t^h are determined endogenously by equation (4), we need to impose restrictions on the transition probability matrix Γ. Recall that $F_H^h(k)$ is a proportional revision relative to Γ. With the priors specified above, it is clear that

$$\sum_{k=1}^{K} f^1(k) \cdot k < \sum_{k=1}^{K} f^2(k) \cdot k$$

holds. Because p_t is supposed to be positively correlated with d_t, we should expect that when $(Q_t^1, Q_t^2) = (Q_H^1, Q_H^2)$,

$$E_{Q_t^1}\{p_{t+1} \,|\mathcal{G}_t^1 \backslash \mathbf{Y}_t\} < E_{Q_t^2}\{p_{t+1} \,|\mathcal{G}_t^2 \backslash \mathbf{Y}_t\}$$

holds, and consequently,

$$\Pr\{(Q_t^1, Q_t^2, Y_t^1, Y_t^2) = (Q_H^1, Q_H^2, 1, 0)\} = 0$$

must hold. Similarly,

$$\Pr\{(Q_t^1, Q_t^2, Y_t^1, Y_t^2) = (Q_L^1, Q_L^2, 0, 1)\} = 0$$

must hold. Note that $\Pr\{\cdot\}$ here refers to the long-term frequencies; thus, all agents observe and know these frequencies. Furthermore, when $(Q_t^1, Q_t^2) = (Q_H^1, Q_L^2)$, agent 1 is optimistic and agent 2 is pessimistic relative to the stationary process represented by Γ; thus,

$$E_{Q_t^1}\{p_{t+1} \,|\mathcal{G}_t^1 \backslash \mathbf{Y}_t\} > E_{Q_t^2}\{p_{t+1} \,|\mathcal{G}_t^2 \backslash \mathbf{Y}_t\}.$$

Hence,

$$\Pr\{(Q_t^1, Q_t^2, Y_t^1, Y_t^2) = (Q_H^1, Q_L^2, 0, 1)\} = 0$$

must hold. Similarly,

$$\Pr\{(Q_t^1, Q_t^2, Y_t^1, Y_t^2) = (Q_L^1, Q_H^2, 1, 0)\} = 0$$

must hold.

These indicate that there are restrictions on (A^H, A^L, B^H, B^L). However, it is not trivial at all to determine what/which matrices (A^H, A^L, B^H, B^L) indeed support an RBE. Note that by specifying some matrices (A^H, A^L, B^H, B^L)

and hence Γ exogenously, we can compute the 'equilibrium', but it is possible that condition (4) is not satisfied. Namely, an arbitrary choice of Γ is in fact an introduction of sunspot variables $(\hat{Y}_t^1, \hat{Y}_t^2)$, which may not satisfy condition (4). In other words, the random variables $(\hat{Y}_t^1, \hat{Y}_t^2)$ will be indeed the announcements (Y_t^1, Y_t^2) only when condition (4) is satisfied. With this observation, we propose the following algorithm to find the RBE.

An Algorithm to Find an RBE

- First, specify (A^H, A^L, B^H, B^L) arbitrarily.
- Compute the 'equilibrium' and also compute the endogenous announcements (Y_t^1, Y_t^2).
- Run a Monte Carlo simulation for $(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2)$.
- Compute the empirical distribution of $(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2)$ that corresponds to (A^H, A^L, B^H, B^L) by using the Monte Carlo samples.
- Use the empirical distribution as the entry of (A^H, A^L, B^H, B^L) in the next round.
- Iterate the above procedure until the empirical distribution converges.
- Such a limit distribution supports an RBE.

To begin with, we assume that $A^{Hij} = A^{Lij} = A^j$ and $B^{Hij} = B^{Lij} = B^j$ for all i, j. This means that given the dividend state, the transition probabilities are independent of the current state. We select $\beta^1 = \beta^2 = 0.90$, $\nu^1 = \nu^2 = 3.25$, $w^1 = w^2 = 24.0$, $\gamma = 0.346$ and $\alpha^1 = \alpha^2 = 0.57$. Note that by selecting $\gamma = 0.346$, $k^\gamma < 1/0.57 \approx 1.7544$ for all k is satisfied, and $5^\gamma \approx 1.7542$ in particular. By using the above algorithm, for a wide range of (χ^1, χ^2), only the following specifications of (A^H, A^L, B^H, B^L) support an RBE: for all $k \in \{1, 2, 3, 4\}$,

$$\begin{array}{rcl} a^1 & = & (0,0,0,1), \quad a^2 = (0,0,0,1), \quad a^3 = (0,0,0,1), \quad a^4 = (0,0,0,1) \\ b^1 & = & (1,0,0,0), \quad b^2 = (1,0,0,0), \quad b^3 = (1,0,0,0), \quad b^4 = (0,0,0,1), \end{array}$$

where a^j denotes coordinates of A^j. We omit the superscripts Hi and Li because the coordinates are independent of them. This implies that the long term frequencies are such that

$$\begin{array}{rcl} \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^H, Q_H^1, Q_H^2, 0, 0)\} = 0.16245; \\ \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^H, Q_H^1, Q_L^2, 0, 0)\} = 0.12255; \\ \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^H, Q_L^1, Q_H^2, 0, 0)\} = 0.12255; \\ \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^H, Q_L^1, Q_L^2, 0, 0)\} = 0.09245; \\ \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^L, Q_H^1, Q_H^2, 1, 1)\} = 0.16245; \\ \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^L, Q_H^1, Q_L^2, 1, 1)\} = 0.12255; \\ \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^L, Q_L^1, Q_H^2, 1, 1)\} = 0.12255; \\ \Pr\{(d_t, Q_t^1, Q_t^2, Y_t^1, Y_t^2) & = & (d^L, Q_L^1, Q_L^2, 0, 0)\} = 0.09245. \end{array}$$

Thus, the introduction of announcements does not expand the state space — i.e, there are only 8 price states. Namely, not only the announcements are perfectly correlated with each other, but also the state of announcements is a function of (d_t, Q_t^1, Q_t^2). However, it does not mean that the effect of communication is absent. Table 3 reports the computation results for various specifications of (χ^1, χ^2) when $K = 5$. The first column $(\chi^1, \chi^2) = (0, 0)$ is a

Variable	(χ^1, χ^2)					
	(0, 0)	(.337, .32)	(.05, .05)	(.05, .3)	(-.2, .3)	(-.3, .3)
$\bar{p}$	22.97	22.81	22.95	22.89	22.91	22.92
σ_p	0.34	0.83	0.38	0.56	0.42	0.37
$\bar{R}$	6.28%	6.52%	6.30%	6.38%	6.33%	6.31%
σ_R	5.00%	8.27%	5.33%	6.51%	5.61%	5.32%
r^F	5.70%	5.50%	5.69%	5.64%	5.69%	5.70%
σ_{r^F}	2.81%	5.38%	3.07%	4.11%	3.59%	3.44%
ρ	0.59%	1.02%	0.61%	0.74%	0.63%	0.61%

Table 3: RBE with Communication 1

reproduction of the first column of Table 2. We note that we cannot interpret this equilibrium as a situation where the agents do communicate but simply believe that the announcements of other agents are not informative, because condition (4) is not met.

It is clear from columns 2—4 of the table that the volatility is higher when both χ^1 and χ^2 are large. Recall that $\chi^h > 0$ indicates the following: when agents believe that the volatility is higher when the other agent expects the price/dividend ratio to rise, and vice versa. In particular, when χ^h is larger, the agent believes that such correlation is stronger. When both agents believe that such correlation is strong (i.e. both χ^1 and χ^2 are large), the volatility becomes larger, because both agents always react in the same direction, for the announcements are perfectly correlated with each other.

On the other hand, when the population is diverse in the sense that the reactions to the announcement of the other agent are opposite (i.e. the signs of χ^1 and χ^2 are different), the volatility is lower. Columns 1, 4—6 exhibit results where χ^2 is fixed to be 0.3 while χ^1 changes. From these results, when we fix χ^2, we can see some kind of monotonicity and continuity of volatility in the reaction parameter χ^1, i.e. the larger χ^1 is, the higher the volatility is, and there is no jump in the volatility changes.

However, it is clear that the fluctuations in Table 3 are small compared to the empirical record. Hence, we need to analyse further how a large fluctuation can be brought into place by examining several different settings of the simulation model.

Table 4 reports another two equilibria whose stationary transition probability matrices are more complex than the previous one. We again select $\beta^1 = \beta^2 = 0.90$, $\nu^1 = \nu^2 = 3.25$, $w^1 = w^2 = 26.0$, $\gamma = 0.33$ and $\alpha^1 = \alpha^2 = 0.57$. Also, we select $(\chi^1, \chi^2) = (0.33, 0.31)$ for RBE4.1 and $(\chi^1, \chi^2) = (0.30, 0.30)$ for RBE4.2. The transition probability matrices are reported in the appendix. While RBE4.1 has 10 price states, RBE4.2 has

Variable	RBE4.1	RBE4.2
$\bar{p}$	24.81	24.86
σ_p	0.74	0.57
$\bar{R}$	6.09%	6.00%
σ_R	7.29%	6.06%
r^F	5.20%	5.33%
σ_{r^F}	4.61%	3.73%
ρ	0.89%	0.67%

Table 4: RBE with Communication 2

16 price states. Namely, the announcements expand the state space more in RBE4.2. However, it is clear that RBE4.1 indicates a larger market volatility. This suggests that *an expansion of the state space does not necessarily imply a larger market volatility.*

To understand why this may be the case and also how a large fluctuation may arise, we refer to the results of Kurz and Motolese (2001). As they point out, a correlation of the assessment variables is the driving force of large market fluctuations. A close examination of their configurations suggests that asymmetry in the transition probabilities is the key in such correlation. We postulate that we should consider two kinds of asymmetry. First, we refer to a well known fact that large fluctuations can occur in a macro model only when the agents suffer severely in some states. In Kurz and Motolese (2001), the 'crash' occurs when the assessment variables disagree and the economy is in a recession $d_t = d^L$. In this state, the old agent who was an optimist in one of the two states of disagreement (i.e. $Q^1_{t-1} \neq Q^2_{t-1}$) suffers a lot, because he holds a large amount of shares whose price is much lower than what he expected. Another type of asymmetry is the following. According to the simulation results of Kurz and Motolese, a market experiences a large fluctuation when it could crash from any state while it takes several steps to reach the 'bull' market. Because the structure of the economy here is essentially the same as that of Kurz and Motolese, we should expect that asymmetry in the transition probabilities is the key to generating large fluctuations. However, all of the configurations we have employed so far lack these features.

To see indeed a large discrepancy between the expectations of the agents in a 'crash' state is crucial for the economy to experience a large volatility, we assume that $K = 2$. In this case, the priors are as follows:
Agent 1:

$$f^1(k) = 0.5 \quad (k = 1, 2);$$

Agent 2:

$$f^2(k) = \begin{cases} 0.49 & \text{if} \quad k = 1; \\ 0.51 & \text{if} \quad k = 2. \end{cases}$$

Moreover, equation (23) becomes:

$$\lambda_k^h = \begin{cases} -\chi^h & \text{if} \quad k = 1, \\ \chi^h & \text{if} \quad k = 2. \end{cases} \tag{30}$$

Table 5 reports equilibria with the same specifications of (A^H, A^L, B^H, B^L) as in Table 3. This is the simplest class of equilibria in the sense that the announcements do not expand the state space (there are only 8 price states). We select $\beta^1 = \beta^2 = 0.90$, $\nu^1 = \nu^2 = 3.25$, $w^1 = w^2 = 25.0$, $\gamma = 0.8108$ and $\alpha^1 = \alpha^2 = 0.57$. We choose $w^1 = w^2 = 25.0$ so that the average price/dividend ratio is close to 23 in the equilibrium with the largest fluctuation. Note also that $2^\gamma \approx 1.7542 < 1.7544 \approx 1/0.57$ and $1^\gamma = 1$ (recall that we need to satisfy $k^\gamma < 1/0.57$ for all k). Hence, the process is either stationary ($k = 1$) or extremely divergent from the stationary process ($k = 2$).

Variable	(χ^1, χ^2)					
	(0, 0)	(.45, .45)	(.80, .80)	(.8439, .8270)	(.8438, 0)	(.8438, -.3)
$\bar{p}$	23.97	23.84	23.55	23.42	23.73	23.74
σ_p	0.30	0.70	1.55	2.01	0.96	0.85
$\bar{R}$	6.09%	6.26%	6.85%	7.33%	6.41%	6.08%
σ_R	4.77%	7.29%	12.83%	16.15%	8.95%	8.20%
r^F	5.53%	5.44%	4.78%	4.46%	5.48%	5.58%
σ_{r^F}	2.49%	4.48%	8.88%	11.89%	7.38%	7.10%
ρ	0.55%	0.82%	2.07%	2.86%	0.93%	0.76%

Table 5: RBE with Communication 3

It is clear from the table that the economy is more volatile when the parameters (χ^1, χ^2) are larger, i.e. the degree of conformism is higher for both agents, as we saw in Table 3. However, the level of volatility is much higher now, and is reasonably comparable with the empirical record. Before explaining why such a difference arises, we make two additional observations about the table. As Kurz and Motolese (2001) point out, (a) the model prediction of σ_p is downward biased (both in the REE and RBE with or without communication) since the model assumes that dividends and GDP are proportional. Under the realistic assumption that profits are more volatile than GDP, the model predictions of σ_p would become larger. However, the extent cannot be large enough so that the REE predictions match the empirical record. Also, (b) the empirical record of σ_{r^F} is downward biased relative to the model assumptions since monetary policy during the second half of the 20th century tended to stabilize short term rates. Indeed, there is some evidence that σ_{r^F} was substantially higher than 5.67% before the Great Depression. Therefore, with these observations, we can claim that the model does have a prediction about volatility that closely matches the empirical record, although the prediction about the risk-free rate is not very close to the empirical record yet.

Now we explain why the modified version matches the empirical record far better than the previous one does. We observe that $f^1(2;1) \approx 1$ and $f^2(2;1) \approx 1$ when $(\chi^1, \chi^2) = (.8439, .8270)$. In other words, both agents assign weights that are almost 1 on the non-stationary process when the announcements are $(Y_t^1, Y_t^2) = (1, 1)$. Suppose the economy is in a recession ($d_t = d^L$) and both agents expect the share price to rise ($Y_t^1 = Y_t^2 = 1$). Then, the optimist is almost certain that the economy will be booming in the next period ($d_{t+1} = d^H$) while the pessimist is almost certain that the economy will be in a recession ($d_{t+1} = d^L$). We claim that this state is most likely to be a 'crash' state if such a state ever exists. This is because the price/dividend ratio p_t must be the lowest or nearly lowest in a crash state; thus, the announcements must be $(Y_t^1, Y_t^2) = (1, 1)$. Moreover, a crash state should be a low dividend state because p_t should be positively correlated with d_t. In particular, we claim that this state indeed becomes a crash state when the discrepancy between the beliefs of the agents is extremely large. This happens only when $f^1(2;1) \approx 1$ and $f^2(2;1) \approx 1$ with $Q_t^1 \neq Q_t^2$. Hence, these observations imply that all agents must be conformists when optimistic and contrarians when pessimistic (i.e. $\chi^h > 0$ for all h) for an equilibrium to involve a crash state.

We observe that the same argument does not hold when the economy is booming currently (i.e. $d_t = d^H$). More specifically, the optimist is *not* almost certain that the economy will continue to boom even if his posterior weight is such that $f^h(2;1) \approx 1$, while the pessimist is again almost certain that the economy will be in a recession. Therefore, there is an asymmetry in the transition probabilities, which is analogous to that in Kurz and Motolese (2001).

Note that although we selected $K = 2$ instead of $K = 5$ here, the number of K itself is not essential to obtain sharp results as in Table 5. If we modify the previous specification of λ_k^h, i.e. equation (23), so that the posterior weight for $\{k = K\}$ with $K^\gamma \approx 1.7542$ is close to 1 when $Y_t^{(h)} = 1$, a similar result should hold. In fact, so long as the prior weight for $\{k = K\}$ is not substantially less than .5 as well as Z^h is not substantially larger than .5, we can retain this feature.[13] This implies that the economy experiences a large fluctuation only when agents *a priori* put large weights on a process that is extremely divergent from the stationary one. Moreover, all agents must be conformists when optimistic and contrarians when pessimistic (i.e. $\chi^h > 0$ for all h) as we pointed out before. Thus, we can claim that the model has a good prediction about the class of beliefs.

Recall that so long as $\chi^h > 0$, an optimistic young agent becomes even more optimistic by listening to the announcement of the other agent which expects p_t to rise, while a pessimistic young agent becomes even more pessimistic. This means that when both agents expect p_t to rise, the optimism/pessimism of the agents will be amplified. In this sense, the agents believe that the economy is more volatile when both agents expect p_t to rise. As pointed out above, the discrepancy in the beliefs of the agents becomes

[13] If the prior weight is substantially less than .5, restrictions on λ_k^h prevent the posterior from being near 1.

largest in the crash states, where both agents expect p_t to rise. The simulation results indeed show that this is how communication makes the beliefs of the agents become correlated, and consequently, amplifies the fluctuations of the economy.

With such an observation concerning the beliefs of the agents in mind, we emphasize that the 'crash' states are created endogenously. It is the young agents who make 'mistakes' in terms of beliefs by themselves, and consequently hold 'wrong' portfolios so that they suffer a lot when they get old. Because the market as a whole cannot make an extreme mistake, it must be that some but not all agents make huge mistakes. This implies that a substantial discrepancy in the forecasts is the key to having a crash state. Indeed, an optimistic young agent in a crash state believes that the economy will be booming in the next period for sure, and thus, holds a portfolio whose position is extremely long in the common stock and extremely short in the 'bill'. However, when the economy remains in a crash state in the next period completely against his expectation, he suffers a lot (when he is old), because he needs to pay back for the shorting on the 'bill' whilst he fails to secure a capital gain from the common stock. This may well happen with the specification we have in Table 5.

Variable	Empirical Record	(ν^1, ν^2)		
		(3.00, 3.00)	(3.25, 3.25)	(3.50, 3.50)
$\bar{p}$	23	23.39	23.42	23.44
σ_p	6.48	1.93	2.01	2.07
$\bar{R}$	8.00%	7.25%	7.33%	7.39%
σ_R	18.08%	15.60%	16.15%	16.62%
r^F	1.00%	4.67%	4.46%	4.25%
σ_{r^F}	5.67%	11.11%	11.89%	12.60%
ρ	7.00%	2.58%	2.86%	3.15%

Table 6: RBE with Communication 4

To complete the discussion, we report the equilibria with various degrees of (relative) risk aversion in Table 6. Namely, we alter the values of ν^1 and ν^2, while fixing $(\chi^1, \chi^2) = (0.8439, 0.8270)$. Other than these parameters, the set-up of the equilibrium remains the same as that of Table 5, including the choice of Γ. Thus, there are only 8 price states.

It is clear that all simulation results in Table 6 are reasonably close to the empirical record, although the predicted average risk free rate is somewhat higher than the empirical record. Nevertheless, all volatility measures are fairly large. Therefore, we can claim that communication amplifies fluctuations in these equilibria.

3.4 Discussion

Now, we summarize the findings of the simulation results. By examining various equilibria, we have indeed seen that communication causes a larger volatility provided for a class of prior weights. We stress in particular that the prediction of the simulation model about volatility matches the empirical record well only by introducing communication. We can emphasize this point because we do not introduce any correlation of the beliefs *a priori*. To summarize, in a limited class of models and parameters we have shown the following:

- *Communication amplifies the fluctuations of the economy in general;*
- *Communication may play a dominant role in amplifying the fluctuations, and may be powerful enough to create large fluctuations that match the empirical record even if the effective beliefs themselves are independent a priori.*
- *The model is capable of generating large fluctuations without relying upon informational asymmetry.*
- The economy experiences large fluctuations if the agents are conformists when optimistic and contrarians when pessimistic.

The third point is particularly important when we are to analyse the behaviour of the market participants. For example, the press frequently makes statements such as 'this piece of information is already reflected in the market's expectations', or 'the stock price does not reflect the fundamentals' and so on. The idea behind such statements is that all participants interpret the newly arrived information in an identical way as long as they had the same information previously, which is exactly the idea of the rational expectations or the common prior. In short, the prices change only when information prevailing in the market changes according to this line of argument. This is partly the reason why the SEC is keen to alleviate informational asymmetry so as to eliminate unnecessary price fluctuations. However, the simulation results of our model reveal that the prices do change even if information other than the announcements remains unchanged.

Note that in our model, the dividend state corresponds to the market fundamentals in the usual sense, but in fact the effective beliefs as well as the announcements are also functioning as the market fundamentals. We point out that it is much more realistic to include them among the market fundamentals. For example, it is very common for the investors to refer to the opinions of financial analysts: even major market participants do refer to the opinions of the fellow market participants on a regular basis. Also, it is widely acknowledged that the 'atmosphere' of the market influences the market behaviour. We claim that the 'atmosphere' is represented by the effective beliefs and the announcements in our model. Therefore, although many statements seem to be based upon the idea of rational expectations, the actual behaviour of the market participants is not. Rather, the actual behaviour is more properly described by our model.

Moreover, we may interpret the results with respect to the public policy issue. Under rational expectations, it is more beneficial for the society as a whole to alleviate asymmetric information so that all the unnecessary instability of the market be eliminated. Our model, however, has shown that asymmetric information may not play any role in causing a large fluctuation. Hence, while we do not rule out the importance of eliminating insider dealing, we claim and have indeed shown that such a regulation is not key to stabilizing the market. Instead of focusing on informational asymmetry, we claim that the policy should be focused on the state of the beliefs. Recall that our model predicts that the economy experiences a large fluctuation only when the agents assign sufficiently large weights on extremely divergent processes *a priori*. Hence, we suspect that any public policy that encourages the agents to assign smaller weights on such extremely divergent processes would stabilize the market, although the formation of prior weights of the agents itself is not modelled in the current paper.

To conclude the discussion, we introduce an alternative interpretation of the structure of beliefs. When we interpret the prior weights $f^h(k)$ as prior beliefs about the possible processes, the posterior weights $f^h(k, y)$ can be interpreted as the posterior beliefs updated by listening to the announcements. In other words, each agent forms a belief about the combinations k, while he becomes optimistic of pessimistic randomly.[14] The formula (20) then is really the beliefs updating formula, and indeed, the formula can be justified by applying the result of Genest and Schervish (1985) in the literature of the *Expert Problem*. Their main result is the following. The reaction function of an agent to the announcements is linear in n moments of the announcements provided that the agent forms a belief up to the nth moment of the announcements, but does not specify the full joint distribution.[15] Of course, it is not necessarily the case that the agents form a belief only about he first moment of the announcements. Therefore, this interpretation is somewhat informal, yet provides a different insight.

4 Conclusion

We have examined a standard OLG model with financial assets while allowing for heterogeneous beliefs. The model is essentially the same as that of several existing studies on rational beliefs — e.g. Kurz and Beltratti (1997), and Kurz and Motolese (2001) except the introduction of communication. To introduce communication, we incorporated a result of Bayesian theory called the Expert Problem. To examine the impacts of communication on the equilibria, we set up a simulation model, which is a modification of model due to Kurz and Beltratti (1997) and/or Kurz and Motolese (2001). In general, the model involves an endogenous expansion of the state space through communication.

The simulation results reveal that communication alone is capable of amplifying the fluctuations of the economic variables. For a particular class of effective beliefs, it is shown that communication is powerful enough to cause

[14] This interpretation is in line with Barberis, Shleifer and Vishny (1998).

[15] See Theorem 2.1 of Genest and Schervish (1985).

strong correlation of beliefs so that the prediction of the model concerning the volatility measures matches the empirical record at a reasonable level even when there is no correlation of beliefs *a priori* (when communication is absent). This result is particularly important because it shows that informational asymmetry is not essential in causing large fluctuations when we allow for heterogeneous beliefs. This provides a sharp contrast with the fact that communication has no impact unless there is asymmetric information under rational expectations.

In the simulation model, the economy experiences a large fluctuation only when some young agents make a severe mistake in predicting the state of the economy in the next period (i.e. when they get old). Such a mistake arises when the economy is in a 'crash' state, where the price/dividend ratio slumps at a very low level. We emphasize that the agents are not forced to make mistakes arbitrarily, but they do so by themselves through communication. Hence, the crash states are not introduced exogenously, but are created endogenously.

Moreover, large fluctuations of the economy arise only when the agents are conformists when optimistic and contrarians when pessimistic. This is consistent to the results of the companion paper (Nakata (2004)). Of course, the simulation model is a very simple one, and thus, it is not too appropriate to claim that this is *the* structure of the beliefs and/or the economy. Yet, the model offers an intuitive way to classify the beliefs, i.e. optimism/pessimism and conformism/contrarianism, which should be useful when we reflect the empirical studies on the matter, including those of the psychology literature.

A Transition Probabilities for RBE4.1

The stationary transition probability matrix Γ in RBE4.1 is constructed from the transition probability matrices below. We choose $A^{Hij} = A^{H}$ for all i, j, and thus, we represent them by A^{H}.

$$A^{H} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}, \quad B^{H11} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{H12} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .6782 & .3218 & .0000 & .0000 \end{bmatrix}, \quad B^{H13} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7416 & .0000 & .2584 & .0000 \end{bmatrix},$$

$$B^{H14} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}, \quad B^{H21} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{H22} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{H23} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{H24} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}, \quad B^{H31} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{H32} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{H33} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{H34} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}, \quad B^{H41} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{H42} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{H43} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{H44} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}, \quad A^{L11} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$A^{L12} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad A^{L13} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$A^{L14} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad A^{L21} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$A^{L22} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad A^{L23} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$A^{L24} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad A^{L31} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$A^{L32} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad A^{L33} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$A^{L34} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad A^{L41} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{L42} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}, \quad A^{L43} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{L44} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}, \quad B^{L11} = \begin{bmatrix} 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$B^{L12} = \begin{bmatrix} .8409 & .1591 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad B^{L13} = \begin{bmatrix} .7363 & .0000 & .2637 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$B^{L14} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad B^{L21} = \begin{bmatrix} 1.0000 & .0000 & .0000 & .0000 \\ 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$B^{L22} = \begin{bmatrix} 1.0000 & .0000 & .0000 & .0000 \\ 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad B^{L23} = \begin{bmatrix} 1.0000 & .0000 & .0000 & .0000 \\ 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$B^{L24} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad B^{L31} = \begin{bmatrix} 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$B^{L32} = \begin{bmatrix} 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad B^{L33} = \begin{bmatrix} 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix},$$

$$B^{L34} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \end{bmatrix}, \quad B^{L41} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{L42} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{L43} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix},$$

$$B^{L44} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}.$$

B Transition Probabilities for RBE4.2

The stationary transition probability matrix Γ in RBE4.2 is constructed from the transition probability matrices below.

$$A^{H11} = \begin{bmatrix} .2521 & .0000 & .0000 & .7479 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7239 & .0000 & .0000 & .2761 \end{bmatrix}, \quad A^{H12} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .3146 & .0000 & .0000 & .6854 \end{bmatrix},$$

$$A^{H13} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2439 & .0000 & .0000 & .7561 \end{bmatrix}, \quad A^{H14} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{H21} = \begin{bmatrix} .1489 & .0000 & .0000 & .8511 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .5000 & .0000 & .0000 & .5000 \end{bmatrix}, \quad A^{H22} = \begin{bmatrix} .0357 & .0000 & .0000 & .9643 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2273 & .0000 & .0000 & .7727 \end{bmatrix},$$

$$A^{H23} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .1441 & .0000 & .0000 & .8559 \end{bmatrix}, \quad A^{H24} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{H31} = \begin{bmatrix} .1714 & .0000 & .0000 & .8286 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .5310 & .0000 & .0000 & .4690 \end{bmatrix}, \quad A^{H32} = \begin{bmatrix} .1000 & .0000 & .0000 & .9000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .1512 & .0000 & .0000 & .8488 \end{bmatrix},$$

$$A^{H33} = \begin{bmatrix} .0882 & .0000 & .0000 & .9118 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .1895 & .0000 & .0000 & .8105 \end{bmatrix}, \quad A^{H34} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{H41} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .8712 & .0000 & .0000 & .1288 \end{bmatrix}, \quad A^{H42} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .1683 & .0000 & .0000 & .8317 \end{bmatrix},$$

$$A^{H43} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2376 & .0000 & .0000 & .7624 \end{bmatrix}, \quad A^{H44} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{H11} = \begin{bmatrix} .5116 & .0000 & .0000 & .4884 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .9412 & .0000 & .0000 & .0588 \end{bmatrix}, \quad B^{H12} = \begin{bmatrix} .1712 & .0450 & .0000 & .7838 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .6923 & .2077 & .0000 & .1000 \end{bmatrix},$$

$$B^{H13} = \begin{bmatrix} .2240 & .0000 & .0960 & .6800 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .6500 & .0000 & .2167 & .1333 \end{bmatrix}, \quad B^{H14} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{H21} = \begin{bmatrix} .4918 & .0000 & .0000 & .5082 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .9286 & .0000 & .0000 & .0714 \end{bmatrix}, \quad B^{H22} = \begin{bmatrix} .2045 & .3636 & .0000 & .4318 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7931 & .1121 & .0000 & .0948 \end{bmatrix},$$

$$B^{H23} = \begin{bmatrix} .3030 & .0000 & .3030 & .3939 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7350 & .0000 & .1282 & .1368 \end{bmatrix}, \quad B^{H24} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{H31} = \begin{bmatrix} .6615 & .0000 & .0000 & .3385 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .9313 & .0000 & .0000 & .0688 \end{bmatrix}, \quad B^{H32} = \begin{bmatrix} .3077 & .3846 & .0000 & .3077 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .8136 & .1356 & .0000 & .0508 \end{bmatrix},$$

$$B^{H33} = \begin{bmatrix} .1667 & .0000 & .4333 & .4000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .8168 & .0000 & .1298 & .0534 \end{bmatrix}, \quad B^{H34} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{H41} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .9314 & .0000 & .0000 & .0686 \end{bmatrix}, \quad B^{H42} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7955 & .0606 & .0000 & .1439 \end{bmatrix},$$

$$B^{H43} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .8468 & .0000 & .0270 & .1261 \end{bmatrix}, \quad B^{H44} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{L11} = \begin{bmatrix} .0714 & .0000 & .0000 & .9286 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7500 & .0000 & .0000 & .2500 \end{bmatrix}, \quad A^{L12} = \begin{bmatrix} .0194 & .0000 & .0000 & .9806 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .9706 & .0000 & .0000 & .0294 \end{bmatrix},$$

$$A^{L13} = \begin{bmatrix} .0069 & .0000 & .0000 & .9931 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .9265 & .0000 & .0000 & .0735 \end{bmatrix}, \quad A^{L14} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{L21} = \begin{bmatrix} .1513 & .0000 & .0000 & .8487 \\ .7143 & .0000 & .0000 & .2857 \\ .2500 & .2500 & .2500 & .2500 \\ .9333 & .0000 & .0000 & .0667 \end{bmatrix}, \quad A^{L22} = \begin{bmatrix} .3790 & .0000 & .0000 & .6210 \\ .1250 & .0000 & .0000 & .8750 \\ .2500 & .2500 & .2500 & .2500 \\ .6667 & .0000 & .0000 & .3333 \end{bmatrix},$$

$$A^{L23} = \begin{bmatrix} .3084 & .0000 & .0000 & .6916 \\ .0385 & .0000 & .0000 & .9615 \\ .2500 & .2500 & .2500 & .2500 \\ .6786 & .0000 & .0000 & .3214 \end{bmatrix}, \quad A^{L24} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{L31} = \begin{bmatrix} .2102 & .0000 & .0000 & .7898 \\ .2500 & .2500 & .2500 & .2500 \\ .5172 & .0000 & .0000 & .4828 \\ .9792 & .0000 & .0000 & .0208 \end{bmatrix}, \quad A^{L32} = \begin{bmatrix} .2692 & .0000 & .0000 & .7308 \\ .2500 & .2500 & .2500 & .2500 \\ .1333 & .0000 & .0000 & .8667 \\ .5610 & .0000 & .0000 & .4390 \end{bmatrix},$$

$$A^{L33} = \begin{bmatrix} .3391 & .0000 & .0000 & .6609 \\ .2500 & .2500 & .2500 & .2500 \\ .0952 & .0000 & .0000 & .9048 \\ .6829 & .0000 & .0000 & .3171 \end{bmatrix}, \quad A^{L34} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$A^{L41} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .8855 & .0000 & .0000 & .1145 \end{bmatrix}, \quad A^{L42} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .3259 & .0000 & .0000 & .6741 \end{bmatrix},$$

$$A^{L43} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2881 & .0000 & .0000 & .7119 \end{bmatrix}, \quad A^{L44} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{L11} = \begin{bmatrix} .3750 & .0000 & .0000 & .6250 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{L12} = \begin{bmatrix} .5280 & .0880 & .0000 & .3840 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .5625 & .2083 & .0000 & .2292 \end{bmatrix},$$

$$B^{L13} = \begin{bmatrix} .4467 & .0000 & .1333 & .4200 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7069 & .0000 & .0690 & .2241 \end{bmatrix}, \quad B^{L14} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{L21} = \begin{bmatrix} .2672 & .0000 & .0000 & .7328 \\ .9048 & .0000 & .0000 & .0952 \\ .2500 & .2500 & .2500 & .2500 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{L22} = \begin{bmatrix} .8764 & .0000 & .0000 & .1236 \\ .8333 & .1667 & .0000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .9118 & .0882 & .0000 & .0000 \end{bmatrix},$$

$$B^{L23} = \begin{bmatrix} .9011 & .0000 & .0000 & .0989 \\ .9000 & .0000 & .1000 & .0000 \\ .2500 & .2500 & .2500 & .2500 \\ .9286 & .0000 & .0714 & .0000 \end{bmatrix}, \quad B^{L24} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{L31} = \begin{bmatrix} .3071 & .0000 & .0000 & .6929 \\ .2500 & .2500 & .2500 & .2500 \\ .9474 & .0000 & .0000 & .0526 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{L32} = \begin{bmatrix} .9072 & .0000 & .0000 & .0928 \\ .2500 & .2500 & .2500 & .2500 \\ .9000 & .1000 & .0000 & .0000 \\ .9333 & .0667 & .0000 & .0000 \end{bmatrix},$$

$$B^{L33} = \begin{bmatrix} .9205 & .0000 & .0000 & .0795 \\ .2500 & .2500 & .2500 & .2500 \\ .8500 & .0000 & .1500 & .0000 \\ 1.0000 & .0000 & .0000 & .0000 \end{bmatrix}, \quad B^{L34} = \begin{bmatrix} .0000 & .0000 & .0000 & 1.0000 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix},$$

$$B^{L41} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .9179 & .0000 & .0000 & .0821 \end{bmatrix}, \quad B^{L42} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7629 & .1753 & .0000 & .0619 \end{bmatrix},$$

$$B^{L43} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .7767 & .0000 & .1456 & .0777 \end{bmatrix}, \quad B^{L44} = \begin{bmatrix} .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .2500 & .2500 & .2500 & .2500 \\ .0000 & .0000 & .0000 & 1.0000 \end{bmatrix}.$$

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