Information Transmission in Repeated Auctions

T.Enkhbayar*

Abstract

In this paper we consider repeated first price auctions where two identical items are sold sequentially to two bidders who are interested in both items. In this framework information transmission is of great importance. In the course of an auction, information about the bidders' values become available as identity of the winner and/or bids are revealed. The aim of the paper is to study how different bid revelation policies affect equlibrium behavior.

Fevrier (2003) shows there exists a unique pure strategy equilibrium when no information is revealed after the first round. We show that if the bid revelation policy is changed so that only winning bid is revealed after the first round, no equilibrium with a pure non-decreasing bidding rule on first period exists. Furthermore if all bids are revealed after first round, no *weakly monotonic* mixed equilibrium exists.

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1 Introduction

In many real life auction markets, bidders often face the same competitors at several points in time. For example, a few large firms compete for government defense contracts, while local construction firms frequently compete for jobs in public and private sector. In all of these situations bidders desire multiple units and have serial persistence of bidder's attributes, e.g., valuations. However Ortega Reichert's pathbreaking work was as early as 1968, until recently only small amount of attention devoted to cases where bidders have multi-unit demand and values for all the items are correlated. The typical model of a multi-period auction has either a series of single item auctions in which participants receive new independent draws from the distribution of types at each period or each participant desires only one of the items (Weber 1983; Engelbrecht-Wiggans 1994; Jeitschko 1998). While some recent research addresses multi-unit demand and correlated value, very often attention is restricted to second price auctions (Katzman 1999; Monmartquette and Robert 1999; Menezes and Monteiro 2003). However correlated multi-unit demand is not rare and first price sealed bid auction framework are quite popular in practice as we have seen, therefore, certainly deserve more attention.

In this paper I study a model of two period repeated first-price repeated (sequential) auctions with two bidders. The model has 2 main properties.

1. Bidders would like to acquire both of the items.

*Graduate School of Economics, The University of Tokyo e-mail:ee36031@mail.ecc.u-tokyo.ac.jp

2. A bidder's values for the items are identical.

In repeated auctions, dynamic interaction results strategical information transmission and the equilibrium behavior significantly differs from that in one shot auction. Informational transmission and learning are of great importance in such environment. Each interaction provides information about privately known attributes of the participators, the form and precision of which depends on the auction framework used. This paper examines transmission of private information via outcome in auctions in which bidders desire multiple items and values of those items to a bidder are same, and shows how equilibrium behavior is affected by bid revelation rule (or market structure). Here 3 types of bid revelation policies are considered. The first one consists in revealing only the name of the winner but not the bids after first round. Such situations are realistic and could arise because of confidentiality. In many private sector auctions, for example, bids are considered proprietary information. The second one is the classical one where only winning bid is revealed. In the third one, all bids are revealed. For example, in municipal construction contracting, firms' bids are publicly revealed after the contract is awarded.

The primary obstacle faced when introducing multi-unit demands into a model of sequential auctions is assymmetry of bidder belief, even if beliefs are ex ante symmetric. Consider a situation in which bidders have symmetric belief prior to bidding in first period. The bidders update their beliefs prior to bidding in second period, thus, creating an assymmetry between the winner's and loser's belief. Fortunately some recent research such as Maskin and Riley(2000b and 2003), Lebrun(1999) and Landsberger et al.(2001) allow us to analyze asymmetric auctions.

Note if second-price auctions is used instead of first-price auctions in this model, the equilibrium of this new game is easy to find. It consists in playing his valuation in the two round. Indeed, bidding his valuation is dominant strategy in the second auction, so the revealed information has no impact on the second auction and the players bid their valuations in the first auction.

The rest of this paper is organized as follows. Section 2 introduces and analyzes general asymmetric auctions. Section 3 discusses equilibrium behavior in various revelations rules, while Section 4 briefly concludes.

2 Asymmetric One Shot Auction

An asymmetricity arises naturally in the second period, because some information about bidders' types is exchanged at first period. For that reason, it is necessary to examine asymmetric first frice auctions in order to analyze the model of repeated auctions. General asymmetric one shot auctions is modeled in the following way. Two risk neutral bidders,A and B, have private value for the item. Bidder i's valuation is drawn from a distribution function F^i with support[1] in X^i. Bidder i has utility $x_i - b$ if he wins with a bid of b and is of valuation of x_i. Ties are resolved at random, where each of the bidders has equal chance of winning. I will consider two cases: (i) X_i is a set of three points (the discrete value case) and (ii) X_i is a compact interval (the continious value case). A bidding function $\beta_i(\cdot)$ (possibly mixed rule) is a best response to $\beta_{-i}(\cdot)$ if, for all x_i and all $b_i \in$ real $\beta_i(x_i)$(the set

[1] The support of F is the smallest closed set of F-measure equal to one.

of realizations of $\beta_i(x_i)$) maximizes[2] buyer i's expected payoff given that the other players are using bid functions $\beta_{-i}(\cdot)$. An *equilibrium* is a vector $(\beta_A(\cdot), \beta_B(\cdot))$ such that, for all i, $\beta_i(\cdot)$ is a best response to $\beta_{-i}(\cdot)$. I am particularly interested in *monotonic equilibria* which is defined as follows.

Definition 1. *An equilibrium is monotonic if for all $i \in A, B$ and $x_i, x_i' \in X_i$, if $x_i > x_i'$, $b \in$ real $\beta_i(x_i)$ and $b \in$ real $\beta_i(x_i')$ then $b \geq b'$.*

Throughout this paper, we require following assumptions.

Assumption 1. *Bidder i never bids more than his valuation x_i in equilibrium.*[3] *Also if bidder i with valuation x never wins in equilibrium, then his bid is his valuation x in equilibrium.*

Assumption 2. *Bidder with valuation of $\min X_i$ do not submit bid in equilibrium.*

From Maskin and Riley (2000b) we have the following results.

Lemma 1. *Any equilibrium is monotonic. Furthermore in equilibrium, the distribution of winning bids of any bidder in equilibrium has a support consisting of an interval $[b_*, b^*]$, and is continuous on it.* (see Maskin and Riley, 2000b, Proposition 1 and Proposition 3, and Maskin and Riley, 2003, Lemma 4).

Note that this result is derived for any tie-breaking rule for which an equilibrium exists not just random tie-breaking rule. Moreover it holds for both the cases of discrete and continuous values.

2.1 The Continuous Value Case

Independent value

Firstly I consider a continuous value case, in which X^A and X^B are compact intervals. Bidder i's valuation is independently drawn from distribution $F^i(\cdot)$ on $[\underline{\alpha}_i, \bar{\alpha}_i]$, $0 \leq \underline{\alpha}_i < \bar{\alpha}_i$, that is twice continuosly differentiable and has no mass point in its domain. $F^A(\cdot)$ and $F^B(\cdot)$ are independent. I assume also density $f_i(\cdot)$ is positive on $(\underline{\alpha}_i, \bar{\alpha}_i]$. Assume, without loss of generality, that $\underline{\alpha}_A \leq \underline{\alpha}_B$. From Maskin and Riley (2003), we have following results.

Lemma 2. *Under Assumption 1, the auction has a pure strategy unique equilibrium where bidding functions are strictly increasing and twice differentiable. Also*

$$b_* = \max \arg \max_b F^A(b)(\underline{\alpha}_B - b)$$

where b_* is defined in Lemma 1.

Then I consider a case in which bidder A's valuation a is common knowledge and bidder B's valuation is drawn from a distribution $F(\cdot)$ on $[\alpha, \beta], 0 \leq \alpha < a$ and $\alpha < \beta$. $F(\cdot)$ is twice continuosly differentiable and has no mass point in its domain. This case is limit case of previous case as α_2 and β_2 converge to a. However there exist no pure strategy equilibrium. Instead the auction has a mixed strategy equilibrium. Vickrey (1961), a seminal paper, studied a special case in which $F(\cdot)$ is uniform distribution. Here I examine general case and characterize equilibrium behavior.

[2] Reference to measure, as well as to the fact that equalities hold almost surely, are suppressed in the text. A more formal approach can be found in Lebrun 1996

[3] Clearly it is weakly dominated strategy for a bidder to bid more than one's valuation at any period

Lemma 3. *There exists a unique equilibrium where bidder A's bid ditribution $H(b)$ has support in $[b_*, b^*]$ with positive density, and bidder B bids*

$$\beta(x) = \begin{cases} x & x \leq b_* \\ a - \frac{a-b_*}{F(x)} F(b_*) & b_* < x \leq 1 \end{cases} \tag{1}$$

where $b_* = \max \arg \max_b F(b)(a-b)$ and $b^* = \beta(1)$.

Proof: Suppose there exists an equilibrium. From Lemma 2, it is known that the distribution of winning bids of any bidder in equilibrium has a support consisting of an interval $[b_*, b^*]$ and continuous on it. Given Assumption 1, bidder B with valuation $b > b_*$ has positive payoff from bidding just above b_*. Let p_i be the probability that bidder i bids b_*. If, for all $i \in \{A, B\}$, $p_i > 0$, then bidding b_* results in tie with positive probability. Thus, bidders are strictly better off bidding slightly above b_*, since this increases his probability of winning discontinuously. Hence, for some i, $p_i = 0$. If $i \neq B$, then bidder B's probability of winning, hence his expected payoff, is zero from bidding b_*. But I have already argued that bidder B with valuation $b > b_*$ has positive equilibrium expected payoff, a contradiction. Hence, $p_B = 0$.

From Assumption 1, if bidder A bids $b \neq b_*$, his expected payoff is at least $F(b)(a-b)$ and if bidder A bids b_*, his expected payoff is $F(b_*)(a-b_*)$. It follows that for b_* to be an equilibrium bid for him

$$F(b)(a-b) \leq F(b_*)(a-b_*) \quad \text{for all } b.$$

Hence $b_* \in \arg\max F(b)(a-b)$. Suppose that b' solves this mazimization problem and $b_* < b'$. Since a distribution of winning bids of bidder B has no mass point, it follows bidder A's winning probability from bidding $\beta(b')$ is $F(b')$. Bidding b_* and $\beta(b')$ must be indifferent to bidder A, hence $\beta(b') = b'$. However bidder B with valuation b' can get positive payoff by deviating to b_*, a contradiction. Hence[4]

$$b_* = \max \arg \max_b F(b)(a-b)$$

Given Assumption 1, $\beta(x) = x$ for $x \leq b_*$. Further for x such that $b_* < x \leq 1$, the bidding function $\beta(x)$ should make bidder A indifferent about bidding any quantity in $[b_*, b^*]$. Therefore it must satisfy the following equation

$$F\big(\beta^{-1}(b)\big)(a-b) = F(b_*)(a-b_*) \quad \text{for all } b \in [b_*, b^*]$$

Then I get (1).

Now let me show there exists $H(b)$ such that $H(b)$ and $\beta(x)$ constitutes an equilibrium. Obviously bidder A has no incentive to bid outside of $[b_*, b^*]$. Now it is sufficient to show that there exists a distribution function $H(b)$ such that

$$\beta(x) = \arg\max_b H(b)(x-b) \quad \text{for all } x \in [b_*, 1]\ . \tag{2}$$

It can be easily seen that a function $H(b)$ defined by following differential equation satisfies (2)

$$H'(b) = \frac{H(b)}{\beta^{-1}(b) - b} \quad \text{given the initial condition } H(\beta(1)) = 1$$

[4] Existence of b_* can be proved by appealing to the Maximum Theorem.

From the fundamental theorem of ordinary differential equations this differential equation has unique solution. Q.E.D.

Ranking of valuations is common knowledge.

Further consider another case in which bidders do not only know their own valuation, but also know the ranking of valuations. Assume $F^A(\cdot) = F^B(\cdot)$ and $X^A = X^B = [0,1]$. Although the model is started as one where valuations are independent, and the distribution of valuations is common knowledge among bidders, after having incorporated the information of ranking, resulting conditional distributions are not common knowledge. Seemingly this poses a complication. Yet, the environment can be analyzed as a game with common knowledge of the distribution on types. One can assume that valuations are drawn from a commonly known joint distribution wiht triangular support and where the higher valuation is assigned to one particular bidder.

Landsberger et al. (2001) has studied this model and obtained following result.

Lemma 4. *The auction in which ranking of valuations is common knowledge has a unique pure strategy equilibrium.*

2.2 The Discrete Value Case

Consider a discrete value case in which $X_A = X_B = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$. For simplicity, assume $x_1 = 0$. I assume independent private value. Bidder A is type x_j with probability q_j, while bidder B is type x_j with probability r_j, $j \in \{1,2,3\}$. Denote $q = (q_1, q_2, q_3)$ and $r = (r_1, r_2, r_3)$. Assume $0 < q_1 < 1$, $q_1 < 1$, $r_2 > 0$ and $r_1 \leq q_1$. Let $F^i(b|x_j)$ be equilibrium bid distribution used by player i with type x_j. Let $\pi(x_j; q, r)$ denote expected payoff of bidder i.

Proposition 1. *An equilibrium in the asymmetric discrete value auction is in mixed strategies and has the following properties.*

1) *$F^i(b|x_2)$ and $F^i(b|x_3)$ have supports $[0, \hat{b}^i]$ and $[\hat{b}^i, b^*]$ respectively and continuous on its support.*

2) *$\hat{b}^A$, $\hat{b}^B$ and b^* are determined by following equaitons in equilibrium.*

   a) If $x_2(q_1 + q_2 - r_1 - r_2)r_1 \leq x_3 q_2 r_1 < x_2(q_1 + q_2 - r_1)r_1 + r_2 q_1(x_3 - x_2)$ *and* $q_1 + q_2 \geq r_1 + r_2$ *then*

$$\begin{aligned}\hat{b}^A &= x_3 - \frac{q_1 x_2(x_3 - \hat{b}^B)}{(q_1 + q_2)(x_2 - \hat{b}^B)} \\ \hat{b}^A &= x_2 - \frac{r_1 + r_2}{q_1 + q_2}(x_2 - \hat{b}^B) \\ b^* &= x_3 - (x_3 - \hat{b}^A)(q_1 + q_2)\end{aligned} \tag{3}$$

   b) If $q_1 + q_2 \geq r_1 + r_2$ *and* $x_3 q_2 r_1 \geq x_2(q_1 + q_2 - r_1)r_1 + r_2 q_1(x_3 - x_2)$ *then*

$$\begin{aligned}\hat{b}^B &= \frac{r_2 x_2}{(r_1 + r_2)} \\ \hat{b}^A &= x_2 - \frac{r_1 x_2}{q_1 + q_2} \\ b^* &= x_3 - (x_3 - \hat{b}^A)(q_1 + q_2)\end{aligned} \tag{4}$$

c) If $x_2(q_1+q_2-r_1-r_2) > x_3q_2$ *and* $q_1+q_2 \geq r_1+r_2$ *then*

$$\begin{aligned}\hat{b}^B &= 0\\ \hat{b}^A &= \frac{q_2x_3}{q_1+q_2}\\ b^* &= (1-q_1)x_3\end{aligned} \tag{5}$$

d) If $q_1+q_2 < r_1+r_2$ *then*

$$\begin{aligned}\hat{b}^A &= \frac{q_2x_2}{q_1+q_2}\\ \hat{b}^B &= x_3 - \frac{q_1+q_2}{r_1+r_2}(x_3-\hat{b}^A)\\ b^* &= x_3-(q_1+q_2)(x_3-\hat{b}^A)\end{aligned} \tag{6}$$

Proof 1) From Lemma 1, the winning bid distributions in equilibrium have common support of an interval $[b_*, b^*]$ and are continuous on it. Suppose $b_* = b^*$. Then $\beta^B(x_j) = b_*$ where $j = 2$ or 3. Suppose $x_j > b_*$. If bidder B deviates to bidding $b_* + \epsilon > 0$ from bidding b_*, then his probability of winning is discontinuously increases, while his payment when he wins goes up by ϵ. That is $\pi^B(b_*+\epsilon|x_j) > \pi^B(b_*|x_j)$ for small enough ϵ. Now suppose $x_j \leq b_*$. Since $q_1 > 0$ and bidder with valuation of x_1 stays out, it follows $\pi^B(x_j-\epsilon|x_j) > 0 \geq \pi^B(b_*|x_j)$ for small enough ϵ. Thus there is no pure strategy equilibrium.

Suppose both $F^A(b_*) > q_1$ and $F^B(b_*) > r_1$. Since $r_1 < 1$, there exists $j \in \{2,3\}$, such that $r_j > 0$. So it must be $b_* < x_j$ for such j. If a bidder deviates to bidding $b_*+\epsilon > 0$ from bidding b_*, then his probability of winning is discontinuously increases, while his payment when he wins goes up by ϵ. This is a profitable deviation for small enough ϵ. Thus $F^A(b_*) = q_1$ or $F^B(b_*) = r_1$.

Further suppose $b_* > 0$. If $F^A(b_*) = q_1$ ($F^B(b_*) = r_1$), then taking arbitriraly small $\epsilon > 0$ ensures that bidding $b_*-\epsilon$ yields a higher expected payoff than bidding b_* to bidder $B(A)$ with valuation x_2 and x_3. Therefore it must be $b_* = 0$. Then any bid realizes is winning bid, that is *ex ante* bid distribution $F(b)$ has a support consisting of $[b_*, b^*]$. And by Lemma 1, it is continuous. Also by Lemma 1, any equilibrium is monotonic. So $F^i(b|x_2)$ and $F^i(b|x_3)$ have at most one common point of support. Then it is proved.
2) I employ typical mixed strategy approach: Use indifference conditions. The indifference conditions are

$$\begin{aligned}F^A(b)(x_2-b) &= (x_2-\hat{b}^B)F^A(\hat{b}^B) \quad \text{if} \quad 0 \leq b \leq \hat{b}^B\\ F^B(b)(x_2-b) &= (x_2-\hat{b}^A)F^B(\hat{b}^A) \quad \text{if} \quad 0 \leq b \leq \hat{b}^A\end{aligned} \tag{7}$$

$$\begin{aligned}F^A(b)(x_3-b) &= x_3-b^* \quad \text{if} \quad \hat{b}^B \leq b \leq b^*\\ F^B(b)(x_3-b) &= x_3-b^* \quad \text{if} \quad \hat{b}^A \leq b \leq b^*\end{aligned} \tag{8}$$

Suppose $q_1+q_2 \geq r_1+r_2$ and $\hat{b}^A < \hat{b}^B$. Then $F^A(\hat{b}^A) < F^A(\hat{b}^B)$. Substituting $b = \hat{b}^B$ in (6), I get $F^A(\hat{b}^B) = r_1+r_2 \leq q_1+q_2 = F^A(\hat{b}^A)$. This is contradiction. So $q_1+q_2 \geq r_1+r_2$ is equivalent to $\hat{b}^A \geq \hat{b}^B$.

a) Suppose $\hat{b}^B > 0$ and $F^A(0) = q_1$. Since $\hat{b}^A \geq \hat{b}^B$, it follows $\hat{b}^A > 0$. Since bid distributions are continuous, $F^A(\hat{b}^A) = q_1 + q_2$ and $F^B(\hat{b}^B) = r_1 + r_2$. Substituting $b = 0$ and $\hat{b}^i, i = A, B$ in (7) and (8), I get

$$\hat{b}^A = x_3 - \frac{q_1 x_2 (x_3 - \hat{b}^B)}{(q_1 + q_2)(x_2 - \hat{b}^B)} \tag{9}$$

$$\hat{b}^A = x_2 - \frac{r_1 + r_2}{q_1 + q_2}(x_2 - \hat{b}^B) \tag{10}$$

$$F^B(0) = \frac{(q_1 + q_2)(x_2 - \hat{b}^A)}{x_2} \tag{11}$$

RHS of (9) is decreasing in $\hat{b}^B$, and RHS of (10) increasing in $\hat{b}^B$ and goes to negative infinity as $\hat{b}^B$ goes to x_2. So if RHS of (9) greater than RHS of (10) at $\hat{b}^B = 0$, then $0 < \hat{b}^i < x_2$. Then it follows if $x_2(q_1 + q_2 - r_1 - r_2) \leq x_3 q_2$, then $0 < \hat{b}^i < x_2$. Also it can be easily calculated that if $x_3 q_2 r_1 < x_2(q_1 + q_2 - r_1)r_1 + r_2 q_1 (x_3 - x_2)$, then $F^B(0) > r_1$. Therefore if and only if conditions of a) are satisfied, then $\hat{b}^i$ and b^* determined by (3) constitute an equilibrium. So we have proved part a). The remaining parts can be proved using same logic. Q.E.D.

3 Repeated Auctions

The main objective of this paper is to analyze following model of repeated auctions. Two identical items are auctioned sequentially through first-price sealed-bid auctions to two risk neutral bidders. Both bidders have symmetric, independent private value x_i, $i \in \{A, B\}$ for the items. Let $x_i, i = A, B$ be independently drawn from a common distribution function F with support in X. After having observed his valuation x_i, bidder i submits his bid $b \in R_+$ for first round. Once the first item is sold, the seller announces some information about outcome of the first auction. Assume both bidders is told if he wins the first round or not. Then after having learned outcome of first round, bidder i submits his bid $c \in R_+$ for second round. Assume time additive utility function with unit time discount ratio [5]. For example if a bidder with valuation x acquires both items bidding b and c respectively, then his utility equals to $2x - b - c$. Ties are resolved at random, where each of the bidders has equal chance of winning. Auctioneer's reserve price is zero. Further I assume that the "no bid" is revealed as a zero bid when revelation policy requires bid revelation.

I analyze this auction in the framework of dynamic game of incomplete information. The bidders update beliefs about their opponent's valuation for the object after learning the outcome of first round. I examine the symmetric perfect Bayesian equillibria (PBE). The fact that both bidders are ex-ante symmetric and restriction to symmetric equilibria allow me to analyze the auction from bidder A's point of view without loss of generality. Therefore I will not use subindex to identify players, unless it is necessary.

Outcome of first round for bidder i consists from publicly announced information and the bidder's own bid. Denote it by z_i. The bidding behavior of bidder of type x_i can be described by the pair of random variable $(\beta_i(x_i), \gamma_i(x_i, z_i))$ where each realization

[5] Introducing non-unitary time discocunt ratio doesn't singificantly change the results obtained in this paper.

is nonnegative number or the null bid corresponding to the choice not to participate. Formally a symmetric PBE consists of a first period bidding rule contingent on the bidder's valuation, denoted $\beta(x)$ and a second period bidding rule contingent on his valuation and the oucome of first period auction, denoted $\gamma(x,z)$, and belief system μ, such that

1) For each $x \in X$ and each possible outcome of first round z_i, if $c \in$ real $\gamma(x, z_i)$ (set of realizations of $\beta_i(x_i)$), then c maximizes his expected payoff under his belief system μ given that the outcome of first round for him is z_i and the other bidder bids at second period according to $\gamma(\cdot)$.

2) For each $x \in A$, if $b \in$ real $\beta(\cdot|x)$, then $\beta(x)$ maximizes his payoff given the other bidder bids according to $(\beta(\cdot), \gamma(\cdot))$, and he himself bids according to $\gamma(\cdot)$ at second period.

3) Belief μ is determined by Bayes' rule and equilbrium strategy $(\beta_i(x_i), \gamma_i(x_i, z_i))$.

Define *weakly monotonic PBE* as follows.

Definition 2. *A symmetric PBE* $\big(\beta(x), \gamma(x, z_i), \mu\big)$ *is weakly monotonic if following conditions are satisfied.*

1) $b \geq \inf$ *real* $\beta(x')$ *and* $\sup$ *real* $\beta(x) \geq \sup$ *real* $\beta(x')$ *for all* $x,\ x' \in X$*,* b *such that* $x > x'$ *and* $b \in$ *real* $\beta(x)$*.*

2) Equilibrium of any subgame on equilibrium path is monotonic.

3.1 The Continuous Values

Without loss of generality, let X be a compact interval $[0, 1]$. $F(\cdot)$ is twice continuously differentiable and has no mass point in its domain. I assume also density $f(\cdot)$ is positive on X.

Revealing No Bids

In this section I will examine the equilibria in the third rule in which no bid is revealed after first round, that is, players do not see other player's bid. The outcome of first period auction from bidder i's point of view, z, is pair of his bid b_i and the name of the winner $w \in \{A, B\}$. That is $z = (b, w)$. Let $\big(\beta(\cdot), \gamma(\cdot)\big)$ be an equilibrium strategy. Assume $\beta(\cdot)$ is strictly increasing pure action rule. First examine second period auction. Since strictly increasing pure bidding rule is assumed for first round, at the beginning of second period auction, players know ranking of their valuations. Then it is possible to analyze the second period auction using results obtained in Landsberger at al. (2001) which discussed in Section 2.

Fevrier (2001) has studied this model of repeated auctions and obtained following result. He characterizes an equilibrium in pure and monotonic strategies. Denote $\zeta(\cdot) = \beta_1^{-1}(\cdot)$, $\sigma(\cdot) = \gamma^{-1}(\cdot, \beta(x), A)$ and $\phi(\cdot) = \gamma^{-1}(\cdot, \beta(x), B)$.

Lemma 5. *Strategy* $(\beta(x), \gamma(x, z_i))$ *defined by*

$$\begin{aligned}
\zeta'(b) &= \frac{F(\zeta(b))}{f(\zeta(b))(\gamma(\zeta(b), b, B) - b)} \\
\sigma(b) &= \frac{F(\sigma(b)) - F(\phi(x))}{f(\sigma(b))(\phi(b) - b)} \\
\phi'(b) &= \frac{F(\phi(b))}{f(\phi(b))(\phi(b) - b)}
\end{aligned} \tag{12}$$

and the two limit conditions $\zeta(0) = \sigma(0) = 0$ *and such that* $sigma(k^{\sigma\phi}) = \psi(k^{\sigma\phi}) = 1$ *is a PBE of the game.* (see Fevrier, 2003 Proposition 1)

Revealing Winning Bid

If bid revelation rule is changed so that winning bid is revealed, mechanism of information transmission changes. The outcome of the first round from a bidder's point of view, z, is triple of his bid b, the winning bid $\bar{b}$, and the identity of the winner $w \in \{A, B\}$. That is $z = (b, \bar{b}, w)$. It is necessary to define PBE precisely for further analysis.

3′) First define

$$G(b) = \int_{\beta(x)<b} f(x)dx + \frac{1}{2}\int_{\beta(x)=b} f(x)dx$$

If $\bar{b} \in B$ and the bidder wins the first auction, then bidder A's belief $\mu(\cdot|\bar{b}, w)$ is defined by following equation

$$\mu(y|\bar{b}, w) = \begin{cases} f(y)/G(\bar{b}) & \text{if } w = A \ \& \ \beta(y) < \bar{b} \\ f(y)/\{2G(\bar{b})\} & \text{if } w = A \ \& \ \beta(y) = \bar{b} \\ f(y)/\int_{\beta(z)=\bar{b}} f(z)dz & \text{if } w = B \ \& \ \beta(y) = \bar{b} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Define $\tilde{F}(b)$ as probability of winning by bidding b in equilibrium. Let $(\beta(\cdot), \gamma(\cdot))$ be an equilibrium strategy. Assume $\beta(\cdot)$ is non-decreasing, pure bidding rule. When bidder B bids according to the equilibrium strategy, bidder A's maximized expected payoff function is

$$\begin{aligned}
\pi(b|x) = \tilde{F}(b)(x - b) + \sup_c \tilde{F}(b)\{Pr(c > c_-) + \frac{1}{2}Pr(c = c_-)\}(x - c) + \\
\sup_{c'}[1 - \tilde{F}(b)]\{Pr(c' > c'_-) + \frac{1}{2}Pr(c' = c'_-)\}(x - c')
\end{aligned} \tag{14}$$

where c_- and c'_- are bids of bidder b at second auction and distribute according to $\gamma(\cdot)$.

Lemma 6. *Assume only winning bid is revealed after first round. If* $(\beta(\cdot), \gamma(\cdot))$ *is a symmetric PBE with pure, non- decreasing strategy on first period, there exists no* $0 < x < 1$ *such that* $\beta(\cdot)$ *is strictly increasing to the right and continuous at* x*. Similarly, there exists no* $0 < x \leq 1$ *such that* $\beta(\cdot)$ *is striclty increasing to the left and continuous at* x*.*

Proof: I give proof of first part of the lemma here. Suppose to the contrary there exists such x. Denote $b = \beta(x)$. Take a strictly decreasing sequence $\{b_n\} \to b$ in $B(\equiv \beta(A)$. Since $\beta(\cdot)$ is strictly increasing to right and continuous at x, we can take such sequence. Define $x_n = \min \beta^{-1}(b_n)$. Obviously $x_n > x$ and $\{x_n\} \to x$. And take one more strictly decreasing sequence $\{b'_n\} \to 0$ in B, such that $b'_n < b_n$ and

$$\frac{b_n - b'_n}{b_n - b)} \to 1.$$

Also define $x'_n = \min \beta^{-1}(b'_n)$[6]. Then $x'_n < x$ and $\{x'_n\} \to x$. If a player with valuation of x_n bids b'_n, the other bidder believes that his valuation lies in $\beta^{-1}(b'_n)$. Since the attention is resticted to non-decreasing pure bidding rule for the first round, the support of bidder i's belief is an compact interval or a single point. Therefore second period auctions can be analyzed as asymmetric auctions discussed in Section 2. From Requirement 3′) of PBE, bidder i has belief with support of $[0, \sup \beta^{-1}(\bar{b})]$ in the second round if he loses the first one and has belief with support of $\beta^{-1}(\bar{b})$ in the second round if he wins the first one . Thus given Lemma 2 and Lemma 3, the winner bids in an interval in the second round . Define z_n and t_n as lower end point and upper end point of that interval conditioning on that the winner bids b_n in the first period in equilibrium. From Lemma 3, $z_n = \max_b F(b)(x_n - b)$. Then the winner's expected payoff at second round is

$$\frac{F(z_n)}{\tilde{F}(b_n)}(x_n - z_n)$$

Further define $\psi(x, y)$ as loser's expected payoff at second round when winner's valuation is y. Then I get

$$\begin{aligned} \pi(b_n|x_n) = {} & \tilde{F}(b_n)(x_n - b_n) + F(z_n)(x_n - z_n) + \\ & \frac{1}{2}\int_{\beta(y)=b_n} f(y)\psi(x_n, y)dy + \int_{\beta(y)>b_n} f(y)\psi(x_n, y)dy \end{aligned} \tag{15}$$

$$\begin{aligned} \pi(b'_n|x'_n) = {} & \tilde{F}(b'_n)(x'_n - b'_n) + F(z'_n)(x'_n - z'_n) + \\ & \frac{1}{2}\int_{\beta(y)=b'_n} f(y)\psi(x'_n, y)dy + \int_{\beta(y)>b'_n} f(y)\psi(x'_n, y)dy \end{aligned} \tag{16}$$

Suppose a bidder with valuation x_n deviate by bidding b'_n. It can be easily seen that, in case he wins at first round, it is optimal to bid t'_n. Define $t(y)$ as upper end point of the support of first round bid distribution of the winner who bids $\beta(y)$ at the first round in equilibrium. Then I get

$$\begin{aligned} \pi(b'_n|x_n) = {} & \tilde{F}(b'_n)(x_n - b'_n) + \tilde{F}(b'_n)(x_n - t'_n) + \frac{1}{2}\int_{\beta(y)=b'_n} f(y)(x_n - t'_n)dy \\ & + \int_{b_n \geq \beta(y) > b'_n} f(y)(x_n - t(y))dy + \int_{\beta(y)>b_n} f(y)\psi(x_n, y)dy \end{aligned} \tag{17}$$

[6] The existence of x'_n and x_n can be proved.

Suppose a bidder with valuation x_n' deviate by bidding b_n. Clearly in case he wins at first round, it is optimal to bid z_n'. Then I get

$$\pi(b_n|x_n') = \tilde{F}(b_n)(x_n' - b_n) + F(z_n')(x_n' - z_n') + \frac{1}{2}\int_{\beta(y)=b_n} f(y)\psi(x_n', y)dy + \int_{\beta(y)>b_n} f(y)\psi(x_n', y)dy \quad (18)$$

From the requirements of PBE, it follows

$$\pi(b_n'|x_n) \leq \pi(b_n|x_n) \quad (19)$$
$$\pi(b_n|x_n') \leq \pi(b_n'|x_n') \quad (20)$$

Adding equations (19) and (20), subsituting (15)-(18) into it and rearranging it, I obtain

$$[\tilde{F}(b_n) - \tilde{F}(b_n')](x_n - x_n') \geq \frac{1}{2}\int_{\beta(y)=b_n} f(y)[\psi(x_n', y) - \psi(x_n, y)]dy + \frac{1}{2}\int_{\beta(y)=b_n'} f(y)[(x_n - t_n') - \psi(x_n', y)]dy + \int_{b_n \geq \beta(y) > b_n'} f(y)\left[x_n - t(y) - \psi(x_n', y)\right]dy + \tilde{F}(b_n')(x_n' - t_n') - F(z_n)(x_n - z_n) + \tilde{F}(b_n')(x_n - x_n')$$

I divide both sides by $x_n - x_n'$ and take $n \to \infty$. Since $\psi(x_n', y)$ is maximized payoff, it follows $\psi(x_n', y) \leq x_n - t(y)$. Then second and third term of RHS become greater than zero. From Lemma 3, $\psi(x, y)$ is differentiable in x. So first term of RHS becomes zero. Fourth and fifth terms of RHS are

$$\lim_{n\to\infty} \frac{\tilde{F}(b_n')(x_n' - t_n') - F(z_n)(x_n - z_n)}{x_n - x_n'} = \lim_{n\to\infty} \frac{F(z_n')(x_n' - z_n') - F(z_n)(x_n - z_n)}{x_n - x_n'}$$

Then I get

$$0 \geq -\frac{d}{dx}[F(z(x))(x - z(x))] + F(x) \quad (21)$$

From Lemma 3, I have $z(x) \in \arg\max_b F(b)(x - b)$. Hence

$$\frac{d}{dz}F(z)(x - z) = 0$$

Substituting it in (21), I get

$$0 \geq -F(z) + F(x)$$

But it must be $F(z) < F(x)$ for $x > 0$, a contradiction. Q.E.D.

Proposition 2. *Assume only winning bid is revealed after first round. Then there exists no symmetric PBE with non-decreasing pure strategy on first period.*

Proof: From Lemma 2, it follows $\beta^{-1}(\beta(1))$ is not singleton. Denote $b_1 = \beta(1)$ and $a_1 = \min \beta^{-1}(b_1)$. Obviously $a_1 < 1$. Suppose a bidder with valuation of a_1 deviates by bidding $b_1 + \epsilon$. Since deviating in that way produces subgame off the equilibrium path, the opponent's belief at second period has not defined. But from Lemma 3, it can be easily seen that even if the opponent bids most aggresively, the payoff of the bidder who deviates does not change. (Note under Assumption 1, to bid most aggresively is to bid equally to his valuation.) Then bidder with valuation of a_1 does not reduce his payoff at second round by deviating to $b_1 + \epsilon$. Then I get

$$\pi(b_1 + \epsilon|a_1) - \pi(b_1|a_1) \geq \frac{F(1) - F(a_1)}{2}(a_1 - b_1 - \epsilon)$$

For small enough positive ϵ, RHS is positive. This means a player with valuation close enough to a_1 has incentive to deviate. Q.E.D.

Revealing All Bids

Ortega Reichert (1968) analyzes a model of sequential procurement auction, which differs from this model only in the values are not identical but stochastically correlated across time periods. In his model, each bidder's privetaly known cost $c_i, i = 1, 2$ is constant across periods and independently drawn an exponential distribution with unknown state parameter W, which is assumed to have a gamma distribution, at each period. Ortega Reichert shows that there exists a pure seperating equilibrium and each bidder has incentive to bid less aggressively at first periond than in a one-shot auction. Thus it seems both bidders are trying to deceive their opponent about their type. Though since the equilibrium is seperating, there really is no deception, paradoxically.

Here rather different result is obtained. In this model there exists no *weakly monotonic* PBE. Assume both bidders' first round bids are revealed after first round. The outcome of first period auction from bidder A's point of view, z, is pair of A's bid b and B's bid b_-. That is $z = (b, b_-)$. Let x and x_- be bidder A and B's valuation respectively. Let $\big(\beta(\cdot).\gamma(\cdot)\big)$ be an equilibrium strategy. When the other bidder bids according to the equilibrium strategy, bidder A's expected payoff function is

$$\begin{aligned} &\pi(b|x) = \pi^1(b|x) + \pi^2(b|x) \\ &\pi^1(b|x) = \tilde{F}(b)\big(x - b\big) \\ &\pi^2(b|x) = \sup_{c \in A}\big\{Pr(c > c_-) + \frac{1}{2}Pr(c = c_-)\big\}(x - c) \end{aligned} \tag{22}$$

(where c_- is bid of bidder B at second period and distributes according to $\gamma(\cdot)$)

Lemma 7. *Assume both bidders' first round bids are revealed after first round. Suppose there exists a weakly monotonic and symmetric PBE $(\beta(\cdot), \gamma(\cdot))$. Define*

$$p(b) = \{x \in X | b \in \mathrm{real}\beta(x)\}$$

Then $p(0)$ is neither a singleton nor empty set.

Proof: Since I assume bidders dont bid above their valuation, bidder with valuation 0 bids 0. Therefore $p(0)$ is not empty set. Suppose it is a singleton. Then if bidder A with valuation $x > 0$ bids 0 instead of $\beta(x)$, then from (22) his first period payoff $\pi^1(0|x)$ is 0. From the requirements of PBE, if bidder A bids 0, bidder B would believe bidder 1's valuation is 0. Given Assumption 2, any bidder with valuation 0 stay out. Then it is optimal to bid 0 for bidder B, under the belief. Using (22), I obtain

$$\pi(0|x) = \sup_c (x - c) = x$$

While $\pi(0|x)$ converges to 0 as x goes to 0, $\lim_{x\to 0} \frac{\pi(0|x)}{x} = 1$. On the other hand, if bidder A bids according to equilibrium strategy, his equilibrium payoff is expressed by

$$\pi(x) = \tilde{F}(b)(x - b) + \sup_c \{Pr(c > c_-) + \frac{1}{2} Pr(c = c_-)\}(x - c)$$

where $b \in \text{real}\beta(x)$ and c_- is bid of bidder B at second period and distributes according to $\gamma(x_-, \beta(x_-), \beta(x))$. It is easily seen that $\lim_{x\to 0} \frac{\pi^1(\beta(x)|x)}{x} = 0$. and $\lim_{x\to 0} \frac{\pi^2(\beta(x)|x)}{x} < 1$. Hence $\lim_{x\to 0} \frac{\pi(x)}{x} < 1$. As we have seen $\lim_{x\to 0} \frac{\pi(0|x)}{x} = 1$. That means, if x is small enough, it is strictly better to bid 0 than to bid $\beta(x)$. That is contradiction to that $\beta(\cdot)$ is an equilibrium strategy. Q.E.D.

Proposition 3. *Assume all bids are revealed after first round. Then there exists no weakly monotonic PBE.*

Proof: Suppose there exists a weakly monotonic PBE. Suppose $x \in p(0)$. By definition of weakly monotonic PBE, if $0 < x' < x$, then $x' \in p(0)$. This means p(0) is an interval or a point. Denote $a_0 = \sup p(0)$. From Lemma 8, $\beta^{-1}(0)$ must be an interval, that is, $a_0 > 0$. Therefore $\tilde{F}(0)$, probability of winning from bidding 0, is positive. In equilibrium, for all $x < a_0$

$$\pi(0|x) = \frac{1}{2}\tilde{F}(0)x + \pi^2(0|x)$$

From Section 2, clearly

$$\lim_{x\to 0} \frac{\pi^2(0|x)}{x} = 0$$

Hence,

$$\lim_{x\to 0} \frac{\pi(0|x)}{x} = \tilde{F}(0)$$

On the other hand, if that bidder deviates by bidding positive bid in the first period, his winning probability would increase to $2\tilde{F}(0)$ discountinuously. That is

$$\pi^1(b|x) = 2\tilde{F}(0)(x - b)$$

Thus

$$\lim_{x\to 0, b\to 0} \frac{\pi(b|x)}{x} \geq 2\tilde{F}(0) > \tilde{F}(0) = \lim_{x\to 0} \frac{\pi(0|x)}{x}$$

This means bidders with valuation near 0 have incentive to deviate. Q.E.D.
Remark: It can be shown that Proposition 1 holds in more general cases, e.g. in which players' valuations are not exactly same, but stochastically correlated across time periods and if the valuation is zero at first period then it is also zero at second period.

3.2 The Discrete Case

As we have seen in previous section, there exists no weakly monotonic equilibrium in a continuous value case where all bids are revealed after first auction. Now I introduce discrete value while preserving the revelation policy and examine existence of a weakly monotonic equilibrium. The general discrete value case is much complicated to solve. And an equilibrium exists trivially in the case of two values. Therefore I consider a case where bidders' valuation can take three discrete values, that is $X = \{x_1, x_2, x_3\}$. Bidder i is type x_j with probability $p_j \in (0, 1)$, $j \in \{1, 2, 3\}$. For simplicity, assume $x_1 = 0$ and $x_3 = 1$. Then a repeated auction of this kind is completely defined by x_2 and $p = (p_1, p_2, p_3)$.

I show there exists no seperating equilibrium in this repeated auction. Suppose to the contrary, there exists a seperating equilibrium. By Assumption 2, bidder with valuation x_1 do not submit bid. Remember that I assume "no bid" is revealed as zero bid. So if bidder with valuation x_2 bids 0, the opponent would believe he has valuation $x_1(= 0)$. Then bidder with valuation x_2 has expected payoff $p_1x_2+x_2$ from bidding 0. On the other hand, if he bids according to seperating equilibrium strategy, his payoff is $G(b)(x_2 - b) + p_1x_2$ where b belongs to set of realizations of his bid distribution in the equilibrium. Obviously $G(b)(x_2 - b) < x_2$. Then

$$G(b)(x_2 - b) + p_1x_2 < p_1x_2 + x_2$$

So bidder with valuation x_2 has incentive to deviate, a contradiction.

Then consider a concrete example of repeated auctions where $x_2 = 13/14$ and $p = (1/7, 3/7, 3/7)$. Then $G(b)$ defined by equations is a first period bid distribution of a (weakly monotonic) PBE.

$$\frac{(G(0) - p_1)^2}{2(G(0) + p_1)} x_2 = (x_2 - q_2x_3)(1 - G(0)) \tag{23}$$

$$G(b) = \begin{cases} \frac{G(0)x_2}{x_2 - b} & \text{if } b \in (0, \hat{b}] \\ \frac{G(0)x_3 + [1 - G(0 - g(0)(1 - G(0))]}{x_3 - b} & \text{if } b \in [\hat{b}, b^*] \end{cases} \tag{24}$$

$$g(b|x_3) = \frac{1 - G(0) - p_3 + g_0(1 - G(b))}{1 - G(b)} g(b) \quad \text{for } b \in (0, \hat{b}] \tag{25}$$

with initial condition $G(0|x_3) = 0$

$$G(\hat{b}) = \frac{1 - p_3 - G(0)}{1 - g_0} \tag{26}$$

where $b^* = \sup Supp\ G(b)$, $\hat{b} = \sup Supp\ G(b|x_2)$, $g_0 = 10/13$ and $g(b) = dG(b)/db$. I prove this claim below. Clearly $g(b|x_3)/g(b)$ is strictly increasing for $b \in [0, \hat{b}]$. Denote $q_1 = p_1/G(0)$ and $q_2 = 1 - q_1$. Define $q = (q_1, q_2, 0)$ and

$$h(b) = \left(0, \frac{g(b|x_2)}{g(b)}, \frac{g_3(b)}{g(b)}\right)$$

Using Proposition 1, I get that expected payoff to a bidder with valuation x_2 from bidding 0 is

$$\pi(0|x_2) = \frac{G(0) + p_1}{2} x_2 + \frac{p_1(G(0) - p_1)}{G(0) + p_1} x_2 + p_1x_2 + \int_0^{b^*} \pi^2(x_2; h(b), q)dG(b)$$

where $\pi^2(x_2; p, q)$ is expected payoff to bidder valuation x in one shot auction. Since

$$x_2 \frac{g_3(b)}{g(b)} > x_2 \frac{g_3(0)}{g(0)} > x_3 q_2$$

applying part c) of Proposition (1) I then obtain

$$\pi^2(x_2, \tilde{f}(b), q) = x_2 - q_2 x_3$$

Hence

$$\pi(0|x_2) = \frac{G(0) + p_1}{2} x_2 + \frac{2p_1(G(0) - p_1)}{G(0) + p_1} x_2 + p_1 x_2 + (1 - G(0))(x_2 - q_2 x_3) \qquad (27)$$

On the other hand

$$\pi(b|x_2) = G(b)(x_2 - b) + p_1 x_2 \quad \text{for } b \in (0, b^*] \qquad (28)$$

Subsituting the numerical values of x_2 and p into (27) and (28), I then get

$$\pi(b|x_2) = \pi(0|x_2) \quad \text{for } b \in (0, b^*]$$

That means bidder with valuation x_2 is indifferent between any $b \in [0, \hat{b}]$. From (25) and (26), $g_3(\hat{b})/g(\hat{b}) = 1$. Therefore deviating to b such that $b^* > b > \hat{b}$ from $\hat{b}$ does not increase his second period payoff since the opponent's belief concerning his valuation would not be changed. Also

$$G_2(b)(x_2 - b) = const \quad \text{for } b^* > b > \hat{b}$$

Thus deviating to $b > \hat{b}$ from $\hat{b}$ indeed decrease his first period payoff. Hence bidder with valuation x_2 has no incentive to deviate. Now I prove that bidder with valuation x_3 has no incentive to deviate. The expected payoff to bidder with valuation x_3 is

$$\pi(b|x_3) = G(b)(x_3 - b) + + \int_0^{b^*} \pi^2(x_2; h(b), h(\tilde{b})) dG(\tilde{b}) \qquad (29)$$

Applying Proposition 1, I then get

$$\pi^2(x_2; h(b), h(\tilde{b})) = \begin{cases} G(0) q_1 x_3 & \text{if } b = 0 \\ (1 - h(\tilde{b}))(x_3 - x_2) & \text{if } h(b) > h(\tilde{b}) \\ (1 - h(b))(x_3 - x_2) & \text{if } h(b) \le h(\tilde{b}) \end{cases} \qquad (30)$$

Substituting this expression into (29) and integrating it gives

$$\pi(b|x_3) = G(b)(x_3 - b) + G(0) x_3 q_1 + [1 - G_3(b) - G(0) - h(b)\big(1 - G(0)\big)](x_3 - x_2)$$

Substituting using (25), it then can be seen that bidder with valuation x_3 is indifferent between any $b \in [0, \hat{b}]$. On the other hand

$$\pi(0|x_3) = \frac{G(0) + p_1}{2} x_2 + G(0)(x_3 - \frac{G(0) - p_1}{G(0) + p_1} x_2) + (1 - G(0)) q_1 x_3$$

Numerical calculation yields $\pi(b|x_3) > \pi(0|x_3)$ for $b \in (0, b^*]$. Also learly bidder with valuation x_3 has no incentive to bid more than b^*. Hence is bidder with valuation x_3 has no incentive to deviate. Therefore the function G(b) defined by (23)-(26) is a first period bid distribution of a symmetric, weakly monotonic PBE.

4 Conclusion

The present paper studies first price repeated auctions where two identical items are sold sequentially to 2 bidders who are interested in both items. By assuming bidders desire multiple objects, and that the valuations of those items to each bidder are fully correlated, I address strategical problems that do not arise in traditional models of sequential auctions. In this framework, the information revealed between the two stages of the game is of great importance.

In Section 2, I summarize some results of recent research in general asymmetric auctions, which are necessary to examine repeated auctions. Also I analyze equilibrium behavior of some specific asymmetric auctions which is not covered in existing literature. In Section 3, I examine how bid revelation policies affect equilibrium behavior of repeated auctions. In cases of continuous value, Fevrier (2003) shows unique pure monotonic equilibrium exists when no information is revealed after first round. If the bid revelation policy is changed so as only winning bid is revealedi after the first period, no equilibrium with pure non-decreasing bidding rule on first period exists. However it turns out if all bids are revealed after first round, no *weakly monotonic* mixed equilibrium exists. Furthermore, I consider a case in which a bidder's valuation can be one of the three values and all bids are revealed after first round. A weakly monotonic mixed equilibrium is calculated in a concrete case.

References

[1] Engelbrecht-Wiggans, R. (1994). Sequential auctions of stochastically equivalent objects *Economic Letters* 44, 87-90.

[2] Fevrier, P. (2003) He who must not be named. *Review of Economic Design*

[3] Katzman, B. (1999) A Two stage sequential auction with multi-unit demands. *Journal of Economic Theory* 86, 77-99.

[4] Jeitschko, T. (1999). Learning in sequential auctions, *Southern Economic Journal* 65, 98-112

[5] Landsberger, M., Rubinstein, J., Wolfsetter, E. and Zamir, S. (2001). First price auctions when the ranking of valuations is common nnowledge. *Review of Economic Design* 3-4, 461-480

[6] Lebrun, B. (1996). Existence of an equilibrium in first-price auctions. *Economic Theory* 7, 421-23.

[7] Lebrun, B. (1999). First-price auction in the asymmetric n bidder case. *International Economic Review* 40, 125-42

[8] Maskin, E.S. and Riley, J.G. (2000a). Asymmetric auctions. *Review of Economic Studies* 67, 413-38.

[9] Maskin, E.S. and Riley, J.G. (2000b). Equilibrium in sealed high-bid auctions. *Review of Economic Studies* 67, 439-54.

[10] Maskin, E.S. and Riley, J.G. (2003). Uniquenss of equilibrium in sealed high-bid auctions. *Games and Economic Behavior* 67, 439-54.

[11] Monmartquette, C. and Robert, J. (1999). Sequential auctions with multi unit demand: theory and experiments. Working Paper, Universite de Montreal

[12] Ortega-Reichert, A. (1968). Model for competitive bidding under uncertainty. Ph.D thesis, Stanford University

[13] Thomas, C. (1996). Market structure and flow of information in repeated auction.

[14] Vickrey, W. (1961). Counterspeculation, auctions and competitive sealed tenders. *Journal of Finance* 16, 8-37.

[15] Weber, R. (1983). Multiple object auctions. In Engelbrecht-Wiggans, R., Shubik, M. and Stark, R.M.(eds) *Auctions, bidding and contracting: uses and theory.* New York, University Press