On the Robustness of Majority Rule †

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Abstract

We show that simple majority rule satisfies four standard and attractive properties—the Pareto property, anonymity, neutrality, and (generic) transitivity—on a bigger class of preference domains than (essentially) any other voting rule. Hence, in this sense, it is the most *robust* voting rule.

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1. Introduction

A *voting rule* is a method for choosing from a set of social alternatives on the basis of voters' stated preferences. Many different voting rules have been studied in theory and used in practice. But far and away the most popular method has been *simple majority rule*, the rule that chooses alternative x over alternative y if more people prefer x to y than vice versa.

There are, of course, good reasons for majority rule's[1] popularity. It not only is attractively straightforward to use in practice, but satisfies some compelling theoretical properties, among them the *Pareto property* (the principle that if all voters prefer x to y and x is available, then y should not be chosen), *anonymity* (the principle that choices should not depend on voters' labels), and *neutrality* (the principle that the choice between a pair if alternatives should depend only on the pattern of voters' preferences over that pair, not on the alternatives' labels). In fact, May (1952) establishes that majority rule is the *unique* voting rule satisfying the Pareto property, anonymity, and neutrality, and a fourth property called *positive responsiveness.*[2]. We shall come back to May's characterization below.

[1] For convenience, we will omit the modifier "simple" when it is clear that we are referring to *simple* majority rule rather to any of the many variants, such as the supermajority rules.

[2] A voting rule is positively responsive if whenever alternative x is chosen (perhaps not uniquely) for a given specification of voter's preferences and the only change that is then made to these preferences is to move x up in some voter's preference ordering, then x becomes *uniquely* chosen.

But majority rule has a well-known flaw, discovered by the Marquis de Condorcet (1785) and illustrated by the Paradox of Voting (or Condorcet Paradox): it can generate *intransitive* choices. Specifically, suppose that there are three voters 1, 2, 3, three alternatives x, y, z, and that the profile of voters' preferences is as follows:

$$\begin{array}{ccc} \underline{1} & \underline{2} & \underline{3} \\ x & y & z \\ y & z & x \\ z & x & y \end{array}$$

(i.e., voter 1 prefers x to y to z, voter 2 prefers y to z to x, and voter 3 prefers z to x to y). Then, as Condorcet noted, a two-thirds majority prefers x to y, y to z, and z to x, so that majority rule fails to select *any* alternative.

Despite the theoretical importance of the Condorcet Paradox, there are important cases in which majority rule avoids intransitivity. Most famously, when alternatives can be arranged linearly and each voter's preferences are *single-peaked* in the sense that his utility declines monotonically as one moves away from his favorite alternative, then, following Black (1948), majority rule is transitive for (almost) all[3] profiles of voters' preferences. Alternatively, suppose that, for every three alternatives, there is one that no voter ranks in the middle. This property, which is a special case of *value restriction* (see Sen 1966, Inada 1969, and Sen and Pattanaik 1969), seems

[3] We clarify what we mean by "almost all" in section 2.

to have held in recent French presidential elections, where the Gaullist and Socialist candidates have not engendered much passion, but the National Front candidate, Jean-Marie Le Pen, has inspired either revulsion or admiration, i.e., the vast majority of voters rank him either last or first, but not in between. Whether or not this pattern of preferences has been good for France is open to debate, but it is certainly "good" for majority rule: value restriction, like single-peakedness, ensures transitivity (almost always).

So, majority rule "works well"—in the sense of satisfying the Pareto property, anonymity, neutrality and generic transitivity—for some domains of voters' preferences but not for others. A natural question to ask is how its performance compares with that of other voting rules. Clearly, no voting rule can work well for *all* domains; this conclusion follows immediately from the Arrow impossibility theorem[4] (Arrow, 1951). But we might inquire whether there is a voting rule that works well for a bigger class of domains than does majority rule.[5]

[4] Our formulation of neutrality (see section 3)—which is, in fact, the *standard* formulation (see Sen, 1970 or Campbell and Kelly, 2002)—incorporates (i) Arrow's *independence of irrelevant alternatives*, the principle that the choice between two alternatives should depend only on voters' preferences for those two alternatives and not on their preferences for other alternatives and (ii) *symmetry with respect to alternatives*, the principle that permuting the alternatives in voters' preferences should permute social choices in the same way.

[5] It is easy to find voting rules that satisfy three out of our four properties on *all* domains of preferences. For example, majority rule and many of its variants, e.g., two-thirds majority rule (which deems two alternatives as socially indifferent unless one garners at least a two-thirds majority against the other), satisfy Pareto, anonymity, and neutrality on any domain. Similarly, rank-order voting (see below) satisfies Pareto, anonymity, and transitivity on any domain.

We show that the answer to this question is *no*. Specifically, we establish (see our Theorem) that if a given voting rule F works well on a domain of preferences, then majority rule works well on that domain too. Conversely, if F differs from majority rule[6], there exists some other domain on which majority rule works well but F does not.

Thus majority rule is essentially *uniquely* the voting rule that works well on the most domains; it is, in this sense, the most *robust* voting rule.[7] Indeed, this gives us a *characterization* of majority rule different from the one provided by May (1952) (see the corollary to the Theorem). Notice that, unlike May, we appeal to no monotonicity condition (May requires positive responsiveness), but instead invoke maximal robustness.

Our Theorem is closely related to a result obtained in Maskin (1995), but greatly improves on that earlier result. Maskin's proposition requires two strong assumptions, one of which is particularly unpalatable.

The first assumption is that the number of voters be *odd*. This is needed because Maskin (1995) demands transitivity for *all* preference profiles drawn from a given domain (oddness is also needed for much of the early work on majority rule, e.g., Inada, 1969). And, as we will see below,

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[6] More accurately, the hypothesis is that F differs from majority rule for a "regular" preference profile belonging to a domain on which majority rule works well.

[7] More precisely, any other maximally robust voting rule can differ from majority rule only for "irregular" profiles on any domain on which it works well (see the corollary to our Theorem).

even when preferences are single-peaked, intransitivity is possible if the population splits exactly 50-50 between two alternatives; an odd number of voters prevents this from happening. To capture the idea that such a split is unlikely, we will work with a large number of voters and ask only for *generic* transitivity. (Formally, we shall work with a *continuum* of voters, but it will become clear that we could alternatively deal with a finite number by defining generic transitivity to mean transitive with "sufficiently high probability." In that case, we would not need to impose "oddness" (a strong assumption, since it presumably holds only half the time).

Second, Maskin (1995) invokes the restrictive assumption that the voting rule F being compared with majority rule satisfies Pareto, anonymity, and neutrality on *any* domain. This is quite undesirable because, although it accommodates certain methods (such as the supermajority rules and the Pareto extension rules), it rules out such voting rules as the Borda count, plurality voting, approval voting, and runoff voting. These are the most common alternatives in practice to simple majority rule, yet fail to satisfy neutrality on the unrestricted domain. We show that this assumption can be eliminated altogether.

We proceed as follows. In section 2, we set up the model. In section 3, we define our five properties, Pareto, anonymity, neutrality, independence of irrelevant alternatives, and generic transitivity formally. We also characterize when rank-order voting—a major "competitor" of majority rule—satisfies all these properties. In section 4, we establish a lemma, closely related to a result of Inada (1969), that characterizes when majority rule is generically transitive. We use this lemma in section 5 to establish our main Theorem on majority rule. We obtain our alternative to May's (1950) characterization as a corollary. Finally, in section 6 we discuss a few extensions.

2. The Model

Our model is in most respects a standard social-choice framework. Let X be the set of social alternatives (including alternatives that may turn out to be infeasible). For technical convenience, we take X to be finite with cardinality $m(\geq 3)$. The possibility of individual indifference often makes technical arguments in the social-choice literature a great deal messier (see for example, Sen and Pattanaik, 1969). We shall simply rule it out by assuming that individual voters' preferences can be represented by *strict orderings* (of course, with only a finite number of alternatives, the assumption that a voter is not exactly indifferent between any two alternatives does not seem very strong). If R is a strict ordering, then, for any alternatives $x, y \in X$ with $x \neq y$, the notation "xRy" denotes "x is strictly preferred to y in ordering R."[8] Let $\Re_X$ be the set of all logically possible strict orderings of X. We shall typically suppose that voters' preferences are drawn from some subset $\Re \subseteq \Re_X$. For example, for some ordering $(x_1, x_2, \ldots, x_m)$ of the social alternatives, $\Re$ consists of *single-peaked* preferences (relative to this ordering) if, for all $R \in \Re$, whenever $x_i R x_{i+1}$ for some i, then $x_j R x_{j+1}$ for all $j > i$, and whenever $x_{i+1} R x_i$ for some i, then $x_{j+1} R x_j$ for all $j < i$.

For the reason mentioned in the Introduction (and elaborated on below), we shall suppose that there is a *continuum* of voters indexed by points in the unit interval $[0,1]$. A *profile* $\boldsymbol{R}$ on $\Re$ is a mapping

$$\boldsymbol{R} : [0,1] \to \Re,$$

where $\boldsymbol{R}(i)$ is voter i's preference ordering. Hence, profile $\boldsymbol{R}$ is a specification of the preferences of all voters.

We shall use Lebesgue measure m as our measure of the size of voting blocs.[9] Given alternatives x and y and profile $\boldsymbol{R}$, let

$$q_{\boldsymbol{R}}(x, y) = m\{i \mid x\boldsymbol{R}(i)\, y\}.$$

[8] Formally, a strict ordering is a binary relation that is reflexive, complete, transitive, and antisymmetric (antisymmetry means that if xRy and $x \neq y$, then it is not the case that yRx).

[9] Because Lebesgue measure is not defined for all subsets of $[0,1]$, we will restrict attention to profiles $\boldsymbol{R}$ such that, for all x and y, $\{i \mid x\boldsymbol{R}(i)\, y\}$ is a Borel set. Call these *Borel profiles*.

Then $q_{\boldsymbol{R}}(x, y)$ is the fraction of the population preferring x to y in profile $\boldsymbol{R}$.

Let $\mathscr{C}$ be the set of reflexive, complete, binary relations (not necessarily transitive or strict) on X. A *voting rule* F is a mapping that, for each profile $\boldsymbol{R}$ on $\Re_X$ (strictly speaking, we must limit attention to Borel profiles—see footnote 9—but henceforth we will not explicitly state this qualification), assigns a relation $F(\boldsymbol{R}) \in \mathscr{C}$. $F(\boldsymbol{R})$ can be interpreted as the "social preference relation" corresponding to $\boldsymbol{R}$ under F. More specifically, for any profile $\boldsymbol{R}$ and any alternatives $x, y \in X$, the notation "$xF(\boldsymbol{R})y$" denotes that x is socially weakly preferred to y under $F(\boldsymbol{R})$. If both $xF(\boldsymbol{R})y$ and $yF(\boldsymbol{R})x$, we shall say that x is socially *indifferent* to y and denote this by

$$\frac{F(\boldsymbol{R})}{x - y}.$$

Finally, the notation "$\sim xF(\boldsymbol{R})y$" denotes that x is not socially weakly preferred to y, given F and $\boldsymbol{R}$. Hence, if $xF(\boldsymbol{R})y$ and $\sim yF(\boldsymbol{R})x$, we shall say that x is socially *strictly* preferred to y under $F(\boldsymbol{R})$, which we will usually denote by

$$\frac{F(\boldsymbol{R})}{\substack{x \\ y}}.$$

For example, suppose that F^m is *simple majority rule*. Then, for all x, y, and $\boldsymbol{R}$

$$xF^{m}(\boldsymbol{R})\,y \quad \text{if and only if} \quad q_{\boldsymbol{R}}(x,y) \geq q_{\boldsymbol{R}}(y,x),$$

i.e., x is socially weakly preferred to y provided that the proportion of voters preferring x to y is no less than the proportion preferring y to x.

As another example, consider *rank-order voting*. Given $R \in \Re_X$, let $v_R(x)$ be m if x is the top-ranked alternative of R, $m-1$ if x is second-ranked, and so on. That is, a voter with preference ordering R assigns m points to her favorite alternative, $m-1$ points to her next favorite, etc. Thus, given profile $\boldsymbol{R}$, $\int_0^1 v_{\boldsymbol{R}(i)}(x)\,dm(i)$ is alternative x's rank-order score (the total number of points assigned to x) or *Borda count*. If F^{RO} is *rank-order voting*, then, for all x, y, and $\boldsymbol{R}$,

$$xF^{RO}(\boldsymbol{R})\,y \text{ if and only if } \int_0^1 v_{\boldsymbol{R}(i)}(x)\,dm(i) \geq \int_0^1 v_{\boldsymbol{R}(i)}(y)\,dm(i),$$

i.e., x is socially weakly preferred to y if x's Borda count exceeds that of y.

Speaking in terms of social *preferences* may seem somewhat indirect because the Introduction depicted a voting rule as a way of making social *choices*. Because, as noted at the beginning of the section, the set of feasible alternatives may not be known in advance, we cannot simply make a voting rule a mapping from profiles to outcomes; the designated outcome might turn out to be infeasible. However, we could define a voting rule as a mapping that to each profile $\boldsymbol{R}$ on $\Re_X$ assigns a choice *function* $C(\cdot)$, which, for each subset $Y \subseteq X$ (where Y is the "available" or "feasible" set), selects a subset $C(Y) \subseteq Y$ ($C(Y)$ consists of the "optimal" alternatives in Y).[10] However, partly because it is less cumbersome working with preference relations than choice functions, there is a tradition going back to Arrow (1951) of taking the former route. Furthermore, it is well known that there is a close connection between the two approaches.[11] In our setting, we shall take the statement "x and y are socially indifferent" to mean "if y is chosen and x is also available, then x must be chosen too." Similarly, "x is socially strictly preferred to y" should be interpreted as "if x is available, then y is not chosen."

3. The Properties

We are interested in four standard properties that one may wish a voting rule to satisfy.

Pareto Property on $\Re$: For all $\boldsymbol{R}$ on $\Re$ and all $x, y \in X$ with $x \neq y$, if, for all i, $x\boldsymbol{R}(i)y$, then $xF(\boldsymbol{R})y$ and $\sim yF(\boldsymbol{R})x$, i.e.,

$$\frac{F(\boldsymbol{R})}{\begin{matrix} x \\ y \end{matrix}}.$$

In words, the Pareto property requires that if all voters prefer x to y, then society should also (strictly) prefer x to y. Virtually all voting rules

[10] Indeed, we took this approach in an earlier version of the paper
[11] See, for example, Arrow (1959).

used in practice satisfy this property. In particular, majority rule and rank-order voting satisfy it on the unrestricted domain $\mathfrak{R}_X$.

Anonymity on $\mathfrak{R}$: Suppose that $p:[0,1]\to[0,1]$ is a measure-preserving permutation of $[0,1]$ (by "measure-preserving" we mean that, for all Borel sets $T\subset[0,1]$, $m(T)=m(p(T))$). If, for all $\boldsymbol{R}$ on $\mathfrak{R}$, $\boldsymbol{R}^p$ is the profile such that $\boldsymbol{R}^p(i)=\boldsymbol{R}(p(i))$ for all i, then $F(\boldsymbol{R}^p)=F(\boldsymbol{R})$.

In words, anonymity says that social preferences should depend only on the distribution of voters' preferences and not on who has those preferences. Thus if we permute the assignment of voters' preferences by p, social preferences should remain the same. The reason for requiring that p be measure-preserving is to ensure that the fraction of voters preferring x to y be the same for $\boldsymbol{R}^p$ as it is for $\boldsymbol{R}$.

Anonymity embodies the principle that everybody's vote should count equally.[12] It is obviously satisfied on $\mathfrak{R}_X$ by both majority rule and rank-order voting.

Neutrality on $\mathfrak{R}$: For all profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ on $\mathfrak{R}$ and all alternatives x, y, w, z, if

$$x\boldsymbol{R}(i)\,y \text{ if and only if } w\boldsymbol{R}'(i)\,z \text{ for all } i$$

[12] Indeed, it is sometimes called "voter equality" (see Dahl, 1989).

then

$$xF(\boldsymbol{R})\,y \text{ if and only if } wF(\boldsymbol{R}')\,z$$

and

$$yF(\boldsymbol{R})\,x \text{ if and only if } zF(\boldsymbol{R}')\,w.$$

In words, neutrality requires that the social preference between x and y should depend only on the set of voters preferring x and on that preferring y, and not on what the alternatives x and y actually are.

As noted in the Introduction, this (standard) version of neutrality embodies independence of irrelevant alternatives, the principle that the social preference between x and y should depend only on voters' preferences between x and y, and not on preferences entailing any other alternative:

Independence of Irrelevant Alternatives (IIA) *on* $\mathfrak{R}$: For all profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ on $\mathfrak{R}$ and all alternatives x and y, if

$$x\boldsymbol{R}(i)\,y \text{ if and only if } x\boldsymbol{R}'(i)\,y \text{ for all } i,$$

then

$$xF(\boldsymbol{R})\,y \text{ if and only if } xF(\boldsymbol{R}')\,y,$$

and

$$yF(\boldsymbol{R})\,x \text{ if and only if } yF(\boldsymbol{R}')\,x.$$

Clearly, majority rule satisfies neutrality on the unrestricted domain $\mathfrak{R}_X$. Rank-order voting violates neutrality on $\mathfrak{R}_X$ because, as is well known, it violates IIA on that domain. However, it satisfies neutrality on any domain $\Re$ on which "quasi-agreement" holds.

Quasi-agreement on $\Re$: Within each triple $\{x, y, z\} \subseteq X$, there exists an alternative, say x, such that either (a) for all $R \in \Re$, xRy and xRz; or (b) for all $R \in \Re$, yRx and zRx; or (c) for all $R \in \Re$, either $yRxRz$ or $zRxRy$.

In other words, quasi-agreement holds on domain $\Re$ if, for any triple of alternatives, all voters with preferences in $\Re$ agree on the relative ranking of *one* of these alternatives: either it is best within the triple, or it is worst, or it is in the middle.

Lemma 1: F^{RO} satisfies neutrality on $\Re$ if and only if quasi-agreement holds on $\Re$.

<u>Proof</u>**:** See appendix.

A binary relation $C \in \mathscr{C}$ is *transitive* if for all $x, y, z \in X$, xCy and yCz imply that xCz. Transitivity demands that if x is weakly preferred to y and y is weakly preferred to z, then x should be weakly preferred to z. We shall define transitivity of a voting rule F as follows:

Transitivity on $\Re$: $F(\boldsymbol{R})$ is transitive for all profiles $\boldsymbol{R}$ on $\Re$.

For our results on majority rule, we will, in fact, not require transitivity for *all* profiles in $\Re$ but only for *almost* all. To motivate this weaker requirement, let us first observe that, as mentioned in the

Introduction, single-peaked preferences do not guarantee that majority rule is transitive for all profiles. Specifically, suppose that $x < y < z$ and consider the profile

$$\begin{array}{cc} \underline{[0,\tfrac{1}{2})} & \underline{[\tfrac{1}{2},1]} \\ x & y \\ y & z \\ z & x \end{array}$$

That is, we are supposing that half the voters (those from 0 to $\frac{1}{2}$) prefer x to y to z and that the other half (those from $\frac{1}{2}$ to 1) prefer y to z to x. Note that these preferences are certainly single-peaked relative to the linear arrangement, $x < y < z$. However, the social preference relation under majority rule for this profile is not transitive: x is socially indifferent to y, y is socially strictly preferred to z, yet z is socially indifferent to x. We can denote the relation by:

$$\begin{array}{ll} x - y & \\ & z - x \end{array} .$$

Nevertheless, this intransitivity is a knife-edge phenomenon - - it requires that *exactly* as many voters prefer x to y as y to x, and *exactly* as many prefer x to z as prefer z to x. Thus, there is good reason for us to "overlook" it as pathological or *irregular*. And, because we are working with a continuum of voters, there is a formal way in which we can do so, as follows.

Let S be a subset of (0, 1). A profile $\boldsymbol{R}$ on $\Re$ is *regular* with respect to S (which we call an *exceptional set*) if, for all alternatives x and y,

$$q_{\boldsymbol{R}}(x, y) \notin S.$$

That is, a regular profile is one for which the proportions of voters preferring one alternative to another all fall outside the specified exceptional set.

Generic Transitivity on $\Re$: There exists a *finite* exceptional set S such that, for all profiles $\boldsymbol{R}$ on $\Re$ that are regular with respect to S, $F(\boldsymbol{R})$ is transitive.

In other words, generic transitivity requires that social preferences be transitive only for regular profiles, ones where the preference proportions do not fall into some finite exceptional set. For example, as Lemma 2 below implies, majority rule is generically transitive on a domain of single-peaked preferences because if the exceptional set consists of the single point $\frac{1}{2}$—i.e., $S = \{\frac{1}{2}\}$—social preferences are then transitive for all regular profiles.

In view of the Condorcet paradox, majority rule is not generically transitive on domain $\Re_X$. By contrast, rank-order voting is not only generically transitive on $\Re_X$ but *fully* transitive (i.e., generically transitive with exceptional set $S = f$).

We shall say that a voting rule *works well* on a domain $\Re$ if it satisfies the Pareto property, anonymity, neutrality, and generic transitivity on that domain. Thus, in view of our previous discussion, majority rule works well, for example, on a domain of single-peaked preferences, whereas rank-order voting works well on a domain with quasi-agreement.

Although we are considering only generic transitivity, we could easily accommodate generic versions of the other conditions too without changing any of the conclusions (indeed, we did exactly that in an earlier draft of this paper). The reason for concentrating only on transitivity is that, to our knowledge, no commonly-used voting rule has nongeneric failures except with respect that property.

4. Generic Transitivity and Majority Rule

We will show below (see the Theorem) that majority rule works well on more domains than (essentially) any other voting rule. To establish this result, it will be useful to have a characterization of precisely when majority rule works well, which amounts to asking when majority rule is generically transitive. We already suggested in the previous section that a single-peaked domain ensures generic transitivity. And we noted in the Introduction that the same is true when the domain satisfies the property that, for every triple of alternatives, there is one that is never "in the middle." But these are only sufficient conditions for generic transitivity; what we want is a condition that is both sufficient and necessary.

To obtain that condition, note that, for any three alternatives x, y, z, there are six logically possible strict orderings, which can be sorted into two Condorcet "cycles"[13]:

$$\begin{array}{ccc|ccc} x & y & z & x & z & y \\ y & z & x & z & y & x \\ z & x & y & y & x & z \end{array}$$

cycle 1 cycle 2

We shall say that a domain $\Re$ satisfies the *no-Condorcet-cycle* property [14] if it contains no Condorcet cycles. That is, for every triple of alternatives, at least one ordering is missing from each of cycles 1 and 2 (more precisely for each triple $\{x, y, z\}$, there do not exist orderings R, R', R'' in $\Re$ that, when restricted to $\{x, y, z\}$, generate cycle 1 or cycle 2).

Lemma 2: Majority rule is generically transitive on domain $\Re$ if and only if $\Re$ satisfies the no-Condorcet-cycle property.[15]

Proof**:** If there existed a Condorcet cycle in $\Re$, then we could reproduce the Condorcet paradox. Hence, the no-Condorcet-cycle property is clearly necessary.

[13] We call these *Condorcet* cycles because they constitute preferences that give rise to the Condorcet paradox

[14] Sen (1966) introduces this condition and calls it *value restriction.*

[15] For the case of an odd and finite number of voters, Inada (1969) establishes that the no-Condorcet-cycle property is necessary and sufficient for majority rule to be transitive.

To show that it is sufficient, we must demonstrate, in effect, that the Condorcet paradox is the *only* thing that can interfere with majority rule's generic transitivity. To do this, let us suppose that F^m is not generically transitive on domain $\mathfrak{R}$. Then, in particular, if we let $S=\{\frac{1}{2}\}$ there must exist a profile $\boldsymbol{R}$ on $\mathfrak{R}$ that is regular with respect to $\{\frac{1}{2}\}$ but for which $F^m(\boldsymbol{R})$ is *intransitive*. That is, there exist $x, y, z \in X$ such that $xF^m(\boldsymbol{R})\,yF^m(\boldsymbol{R})\,zF^m(\boldsymbol{R})\,x$, with at least one strict preference. But because $\boldsymbol{R}$ is regular with respect to $\{\frac{1}{2}\}$, $xF^m(\boldsymbol{R})\,y$ implies that

(1) $$q_{\boldsymbol{R}}(x,y) > \tfrac{1}{2},$$

that is, over half the voters prefer *x* to *y*. Similarly, $yF^m(\boldsymbol{R})\,z$ implies that

(2) $$q_{\boldsymbol{R}}(y,z) > \tfrac{1}{2},$$

meaning that over half the voters prefer *y* to *z*. Combining (1) and (2), we conclude that there must be some voters in $\boldsymbol{R}$ who prefer *x* to *y* to *z*, i.e.,

(3) $$\begin{matrix} x \\ y \\ z \end{matrix} \in \mathfrak{R}.$$ [16]

By similar argument, it follows that

$$\begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix} \in \mathfrak{R}.$$

Hence, $\mathfrak{R}$ contains a Condorcet cycle, as was to be shown.

[16] To be precise, formula (3) says that there exists an ordering in $\mathfrak{R}$ in which *x* is preferred to *y* and *y* is preferred to *z*. However, because F^m satisfies IIA we can ignore the alternatives other than x, y, z.

Q.E.D.

It is easy to check that a domain of single-peaked preferences satisfies the no-Condorcet-cycle property. Hence, Lemma 2 implies that majority rule is generically transitive on such a domain. The same is true of the domain we considered in the Introduction in connection with French elections.

5. The Robustness of Majority Rule

We can now state our main finding:

Theorem: Suppose that voting rule F works well on domain $\mathfrak{R}$. Then, majority rule F^m works well on $\mathfrak{R}$ too. Conversely, suppose that F^m works well on domain $\mathfrak{R}^m$. Then, if there exists profile R° on $\mathfrak{R}^m$, regular with respect to F's exceptional set, such that

(4) $$F\left(\boldsymbol{R}^\circ\right) \neq F^m\left(\boldsymbol{R}^\circ\right),$$

there exists a domain $\mathfrak{R}'$ on which F^m works well, but F does not.

Remark: Without the requirement that the profile $\boldsymbol{R}^\circ$ for which F and F^m differ belongs to a domain on which majority rule works well, the converse assertion above would be false. In particular, consider a voting rule that coincides with majority rule except for profiles that contain a Condorcet cycle. It is easy to see that such a rule works well on *any* domain for which majority rule does because it coincides with majority rule on such a domain.

Proof**:** Suppose first that F works well on $\Re$. If, contrary to the theorem, F^m does not work well on $\Re$, then, from Lemma 2, there exists a Condorcet cycle in $\Re$:

(5) $$\begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix} \in \Re \quad ,$$

for some $x, y, z \in X$. Let S be the exceptional set for F on $\Re$. Because S is finite (by definition of generic transitivity), we can find an integer n such that, if we divide the population into n equal groups, any profile for which all the voters in each particular group have the same ordering in $\Re$ must be regular with respect to S.

Let $\left[0, \frac{1}{n}\right]$ be group 1, $\left(\frac{1}{n}, \frac{2}{n}\right]$ be group 2, …, and $\left(\frac{n-1}{n}, 1\right]$ be group n. Consider a profile $\boldsymbol{R}_1$ on $\Re$ such that all voters in group 1 prefer y to x and all voters in the other groups prefer x to y. That is, the profile is

(7) $$\begin{matrix} \underline{1} \\ y \\ x \end{matrix} \quad \begin{matrix} \underline{2} \\ x \\ y \end{matrix} \quad \cdots \quad \begin{matrix} \underline{n} \\ x \\ y \end{matrix} \quad .$$

From (5), such a profile exists on $\Re$. From neutrality (implying IIA), the social preference between x and y under $F(\boldsymbol{R}_1)$ does not depend on voters' preferences over other alternatives.

There are three cases: either (i) x is socially strictly preferred to y under $F(\boldsymbol{R}_1)$; (ii) x is socially indifferent to y under $F(\boldsymbol{R}_1)$; or (iii) y is socially strictly preferred to x under $F(\boldsymbol{R}_1)$.

Case (*i*): $\dfrac{F(\boldsymbol{R}_1)}{\begin{matrix} x \\ y \end{matrix}}$

Consider a profile $\boldsymbol{R}_1^*$ on $\Re$ in which all voters in group 1 prefer x to y to z; all voters in group 2 prefer y to z to x; and all voters in the remaining groups prefer z to x to y. That is,[17]

(8) $$\boldsymbol{R}_1^* = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{n} \\ x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{array} .$$

Notice that, in profile $\boldsymbol{R}_1^*$, voters in group 1 prefer x to z and that all other voters prefer z to x. Hence, neutrality and the case (*i*) hypothesis imply that z must be socially strictly preferred to x under $F(\boldsymbol{R}_1^*)$, i.e.,

(9) $$\dfrac{F(\boldsymbol{R}_1^*)}{\begin{matrix} z \\ x \end{matrix}} .$$

Observe also that, in $\boldsymbol{R}_1^*$, voters in group 2 prefer y to x and all other voters prefer x to y. Hence from anonymity and neutrality and the case (*i*)

[17] This is not quite right because we are not specifying how voters rank alternatives other than *x*, *y*, and *z*. But from IIA, these other alternatives do not matter for the argument.

hypothesis, we conclude that x must be socially strictly preferred to y under $F(\boldsymbol{R}_1^*)$, i.e.,

(10)
$$\begin{array}{c} F(\boldsymbol{R}_1^*) \\ \hline x \\ y \end{array} \quad .$$

Now (9), (10), and generic transitivity imply that z is socially strictly preferred to y under $F(\boldsymbol{R}_1^*)$, i.e.,

(11)
$$\begin{array}{c} F(\boldsymbol{R}_1^*) \\ \hline z \\ y \end{array} \quad .$$

But (8), (11), and neutrality imply for any profile such that

$$\begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{n} \\ y & y & z & & z \\ z & z & y & & y \end{array},$$

z must be socially strictly preferred to y. Hence, from neutrality, for any profile $\boldsymbol{R}_2$ on $\Re$ such that

(12)
$$\begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{n} \\ y & y & x & & x \\ x & x & y & & y \end{array} \quad ,$$

x must be socially strictly preferred to y, i.e.,

(13)
$$\begin{array}{c} F(\boldsymbol{R}_2) \\ \hline x \\ y \end{array} \quad .$$

That is, we have shown that if x is socially strictly preferred to y when just *one* out of n groups prefers y to x (as in (7)), then x is again socially strictly preferred to y when *two* groups out of n prefer y to x (as in (12)).

Now choose $\boldsymbol{R}_2^*$ on $\mathfrak{R}$ so that

(14) $$\boldsymbol{R}_2^* = \begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & n \\ \hline x & y & y & z & & z \\ y & z & z & x & & x \\ z & x & x & y & & y \end{array} .$$

Arguing as above, we can use (12)– (14) to show that x is socially strictly preferred to y if *three* groups out of n prefer y to x. Continuing iteratively, we conclude that x is strictly socially preferred to y even if $n-1$ groups out of n prefer y to x, which, in view of neutrality, violates the case (i) hypothesis. Hence case (i) is impossible.

Case (ii): $$\begin{array}{c} F(\boldsymbol{R}_1) \\ \hline y \\ x \end{array}$$

But from the case (i) argument, case (ii) leads to the same contradiction as before. Hence we are left with

Case (iii): $$\begin{array}{c} F(\boldsymbol{R}_1) \\ \hline x - y \end{array}$$

Consider a profile $\hat{\boldsymbol{R}}$ on $\mathfrak{R}$ such that

$$\hat{\boldsymbol{R}} = \begin{array}{cccc} 1 & \cdots & n-1 & n \\ \hline x & & x & y \\ y & & y & z \\ z & & z & x \end{array} .$$

From anonymity, neutrality and the case (*iii*) hypothesis, we conclude that x is socially indifferent to y and x is socially indifferent to z under $F(\hat{\boldsymbol{R}})$, i.e.,

(15) $$\frac{F(\hat{\boldsymbol{R}})}{x - y} \quad .$$

and

(16) $$\frac{F(\hat{\boldsymbol{R}})}{x - z} \quad .$$

But the Pareto property implies that y is socially *strictly* preferred to z under $F(\hat{\boldsymbol{R}})$, which together with (15) and (16) contradicts generic transitivity. We conclude that case (*iii*) is impossible too, and so F^m must work well on $\Re$ after all, as claimed.

Turning to the converse, suppose that there exists domain $\Re^m$ on which F^m works well. If F does not work well on $\Re^m$ too, we can take $\Re' = \Re^m$ to complete the proof. Hence, assume that F works well on $\Re^m$ with exceptional set S and that there exists regular profile $\boldsymbol{R}^{\circ}$ on $\Re^m$ such that $F(\boldsymbol{R}^{\circ}) \neq F^m(\boldsymbol{R}^{\circ})$. Because $F(\boldsymbol{R}^{\circ})$ and $F^m(\boldsymbol{R}^{\circ})$ differ, there exist $a \in (0,1)$ with

(17) $$1 - a > a \quad ,$$

and alternatives $x, y \in X$ such that $q_{\boldsymbol{R}^{\circ}}(x, y) = 1 - a$ and $F(\boldsymbol{R}^{\circ})$ ranks x and y differently from $F^m(\boldsymbol{R}^{\circ})$. From (17), we have

$$\begin{array}{c} F^m\left(\boldsymbol{R}^{\circ}\right) \\ \hline x \\ y \end{array} \quad .$$

We thus infer that

(18) $$y\, F\left(\boldsymbol{R}^{\circ}\right) x \quad .$$

Because F is neutral on $\mathfrak{R}^m$, we can assume that $\boldsymbol{R}^{\circ}$ consists of just two orderings $R', R'' \in \mathfrak{R}$ such that

(19) $$y\, R' x \text{ and } x\, R'' y \quad .$$

Furthermore, because F is anonymous on $\mathfrak{R}^m$, we can write $\boldsymbol{R}^{\circ}$ as

(20) $$\boldsymbol{R}^{\circ} = \begin{array}{c} [0, a) \\ \hline R' \end{array} \quad \begin{array}{c} [a, 1] \\ \hline R'' \end{array} \quad ,$$

so that voters between 0 and a have preferences R', and those between a and 1 have R''.

Let us assume for the time being that F satisfies the Pareto property, anonymity, and neutrality on the *unrestricted* domain $\mathfrak{R}_X$. Consider $z \notin \{x, y\}$ and profile $\boldsymbol{R}^{\circ\circ}$ such that

(21) $$\boldsymbol{R}^{\circ\circ} = \begin{array}{c} [0, a) \\ \hline z \\ y \\ x \end{array} \quad \begin{array}{c} [a, 1-a) \\ \hline z \\ x \\ y \end{array} \quad \begin{array}{c} [1-a, 1] \\ \hline x \\ z \\ y \end{array} \quad .$$ [18]

Then from (18)-(21), anonymity, and neutrality, we have

[18] We have again left out the alternatives other than x, y, z, which we are entitled to do by IIA. To make matters simple, assume that the orderings of $\boldsymbol{R}^{\circ\circ}$ are all the *same* for these other alternatives. Suppose furthermore that, in these orderings, x, y, z are each preferred to any alternative not in $\{x, y, z\}$.

(22) $yF\left(\boldsymbol{R}^{\circ\circ}\right)x$ and $xF\left(\boldsymbol{R}^{\circ\circ}\right)z$.

From the Pareto property, we have

(23) $\dfrac{F\left(\boldsymbol{R}^{\circ\circ}\right)}{\begin{matrix} z \\ y \end{matrix}}$.

But, by construction, $\boldsymbol{R}^{\circ\circ}$ is regular with respect to F's exceptional set. Thus, (22) and (23) together imply that F violates generic transitivity on $\mathfrak{R}' = \left\{ \begin{matrix} z \\ y \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix}, \begin{matrix} x \\ z \\ y \end{matrix} \right\}$. Yet, from Lemma 2, F^m is generically transitive on $\mathfrak{R}'$, which implies that $\mathfrak{R}'$ is a domain on which F^m works well but F does not. Thus, we are done in the case in which F always satisfies the Pareto property, anonymity and neutrality.

However, if F does not always satisfy these properties, then we can no longer infer (22) from (18)-(21), and so must argue in a different way.

Consider R' and R'' of (19). Suppose first that there exists alternative $z \in X$ such that

(24) $zR'y$ and $zR''x$.

Let w be the alternative immediately below z in ordering R'. If $w \neq x$, let R'_* be the strict ordering that is identical to R' except that w and z are now *interchanged* (so that wR''_*z). By construction of R'_*, the domain $\{R', R'', R''_*\}$ does not contain a Condorcet cycle, and so, from Lemma 2, F^m works well on this domain. Hence, we can assume that F works well on this domain too (otherwise, we are done). Notice that neutrality of F and (18) then imply that if we replace R' by R''_* in profile $\boldsymbol{R}^\circ$ (to obtain profile $\boldsymbol{R}^\circ_*$) we must have

(25) $$yF\left(\boldsymbol{R}^\circ_*\right)x.$$

Now, if w_* is the alternative immediately below z in R'_* and $w_* \neq x$, we can perform the same sort of interchange as above to obtain R'_{**} and $\boldsymbol{R}^\circ_{**}$ and so conclude that F^m and F work well on $\{R', R''_*, R''_{**}\}$ and that

(26) $$yF\left(\boldsymbol{R}^\circ_{**}\right)x.$$

By such a succession of interchanges, we can, in effect, move z "downward" while still ensuring that F and F^m work well on the corresponding domains and that the counterparts to (18), (25) and (26) hold. The process comes to end, however, once the alternative immediately below z in R' (or R''_*, R''_{**}, etc.) is x. Furthermore, this must happen after finitely many interchanges (since X is finite). Hence, we can assume without loss of generality that $w = x$ (i.e., that x is immediately below z in R').

Let R''' be the strict ordering that is identical to R' except that x and z (which we are assuming are adjacent in R') are now interchanged. From Lemma 2, F^m works well on $\mathfrak{R}' = \{R', R'', R'''\}$, and we can suppose that F does too (otherwise, we are done). Hence, from the same argument we used for $\boldsymbol{R}^{\circ\circ}$ above, we can conclude that

(27) $$yF\left(\boldsymbol{R}_*^{\circ\circ}\right)x \text{ and } xF\left(\boldsymbol{R}_*^{\circ\circ}\right)z$$

and

(28) $$\frac{F\left(\boldsymbol{R}_*^{\circ\circ}\right)}{\begin{matrix} z \\ y \end{matrix}} \quad ,$$

where $\boldsymbol{R}_*^{\circ\circ}$ is the profile

$$\frac{[0,a)}{R'} \quad \frac{[a,1-a)}{R''} \quad \frac{[1-a,1]}{R'''} \quad ,$$

contradicting the generic transitivity of F on $\mathfrak{R}'$. Thus, we are done in the case where (24) holds.

Next, suppose that there exists $z \in X$ such that

(29) $$xR'z \quad \text{and} \quad yR''z\,.$$

But this case is the mirror image of the case where (24) holds. That is, just as in the previous case we generated R''' with

(30) $$xR'''zR'''y$$

through a finite succession of interchanges in which z moves *downwards* in R', so we can now generate R''' satisfying (30) through a finite succession of interchanges in which z moves *upwards* in R'. If we then take $\mathfrak{R}' = \{R', R', R'''\}$, we can furthermore conclude, as when (24) holds, that F^m

and F work well on $\mathfrak{R}'$. But, paralleling the argument for $\boldsymbol{R}_{*}^{\circ\circ}$, we can show that

$$yF\left(\boldsymbol{R}_{**}^{\circ\circ}\right)x \quad \text{and} \quad zF\left(\boldsymbol{R}_{**}^{\circ\circ}\right)y$$

and

$$\frac{F\left(\boldsymbol{R}_{**}^{\circ\circ}\right)}{\begin{matrix}x\\z\end{matrix}},$$

where $\boldsymbol{R}_{**}^{\circ\circ}$ is the profile

$$\frac{[0,a)}{R'} \quad \frac{[a,1-a)}{R''} \quad \frac{[1-a,1]}{R'''},$$

implying that $F\left(\boldsymbol{R}_{**}^{\circ\circ}\right)$ is intransitive. This contradicts the conclusion that F works well on $\mathfrak{R}'$, and so again we are done.

Finally, suppose that there exists $z \in X$ such that

(31) $$zR'y \quad \text{and} \quad xR''z\,R''y.$$

As in the preceding case, we can move z upwards in R' through a succession of interchanges. Only *this* time, the process ends when z and x are interchanged to generate $\hat{R}'$ such that

(32) $$z\hat{R}''x\hat{R}''y.$$

As in the previous cases, we can conclude that F and F^m work well on $\left\{R', R'', \hat{R}''\right\}$. Take $\hat{\boldsymbol{R}}^{\circ\circ}$ such that

$$\hat{\boldsymbol{R}}^{\circ\circ} = \frac{[0,a)}{R'} \quad \frac{[a,1-a)}{\hat{R}'} \quad \frac{[1-a,1]}{R''}.$$

Then, as in the arguments about $\boldsymbol{R}^{\circ\circ}_{*}$ and $\boldsymbol{R}^{\circ\circ}_{**}$, we infer that $F\left(\hat{\boldsymbol{R}}^{\circ\circ}\right)$ is intransitive, a contradiction of the conclusion that F works well on $\left\{R', R'', \hat{R}''\right\}$. This completes the proof when (31) holds. The remaining possible cases involving z are all repetitions or mirror images of one or another of the cases already treated.

Q.E.D.

As a simple illustration of Theorem 1, let us see how it applies to rank-order voting. If $X = \{x, y, z\}$, Lemma 1 implies that F^{RO} works well, for example, on the domain

$$\left\{\begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} z \\ y \\ x \end{matrix}\right\}.$$

And, as Theorem 1 guarantees, F^{m} also works well on this domain, since it obviously does not contain a Condorcet cycle. Conversely, on the domain

(*) $$\mathfrak{R}' = \left\{\begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ y \\ x \end{matrix}\right\},$$

$F^{m}(\boldsymbol{R}) \neq F^{RO}(\boldsymbol{R})$ for any profile $\boldsymbol{R}$ in which the proportion of voters with ordering $\begin{matrix} x \\ y \\ z \end{matrix}$ is a, the proportion with ordering $\begin{matrix} y \\ z \\ x \end{matrix}$ is b and

(**) $$1 < 2a < b + 1$$

(if (**) holds, then F^{RO} and F^{m} rank x and y differently). But, from Lemma 2, F^{m} works well on $\mathfrak{R}'$ given by (*). Hence, from Lemma 1, $\mathfrak{R}'$ constitutes a domain on which F^{m} works well but F^{RO} does not, as guaranteed by the Theorem.

We already mentioned May's (1952) characterization of majority rule in the Introduction. In view of our Theorem, we can provide an alternative characterization. Specifically, call two voting rules F and F' *generically the same* on domain $\mathfrak{R}$ if there exists a finite set $S \subset (0,1)$ such that $F(\boldsymbol{R}) = F'(\boldsymbol{R})$ for all $\boldsymbol{R}$ on $\mathfrak{R}$ for which $q_{\boldsymbol{R}}(x,y) \notin S$. Call F *maximally robust* if there exists no other voting rule that (*i*) works well on every domain on which F works well and (*ii*) works well on some domain on which F does not work well. The Theorem implies that majority rule is essentially uniquely the maximally robust voting rule:

Corollary: Majority rule is maximally robust, and any other maximally robust voting rule F is generically the same as majority rule on any domain on which F or majority rule works well.

6. Extensions

The symmetry inherent in neutrality is often a reasonable and desirable property—we would presumably want to treat all candidates in a presidential election the same. However, there are also circumstances in which it is natural to favor certain alternatives. The rules for changing the U.S. Constitution are a case in point. They have been deliberately devised so that, at any time, the current version of the Constitution—the status quo—is difficult to revise.

In related work (see Dasgupta and Maskin, 2004), we show that when neutrality is replaced by the weaker condition of IIA (and the requirement that ties be broken "consistently" is also imposed), then *unanimity rule with an order of precedence* [19] (the rule according to which x is chosen over y if it precedes y in the order of precedence, unless everybody prefers y to x) supplants majority rule as the most robust voting rule.

We have assumed throughout that voting rules must satisfy anonymity; this is part of the definition of "working well." But in practice there are many circumstances in which voters are, for good reason, not treated equally. Think, for instance, of the weighted voting system used by the council of the European Union, where more populous member nations have larger weights. Such examples suggest that it is worthwhile examining what becomes of our results when anonymity is relaxed.

Now, if we were to eliminate anonymity altogether as a requirement, nothing resembling our Theorem would continue to hold; instead, a

[19] For discussion of this voting rule in a political setting see Buchanan and Tullock (1962).

dictatorship (in which one particular voter's preferences determine social preferences) would now be the most robust voting rule, since it satisfies neutrality, the Pareto property, and transitivity on the unrestricted domain $\Re_X$. However, exploring what would happen if we replaced anonymity with weaker conditions seems useful. Consider, for example, the properties of voting-bloc responsiveness:

Voting-Bloc Responsiveness on $\Re$: For any $V \subseteq [0,1]$ with $m(V) > 0$, there exist profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ on $\Re$ such that $\boldsymbol{R}(i) = \boldsymbol{R}'(i)$ for all $i \notin V$ but $F(\boldsymbol{R}) \neq F(\boldsymbol{R}')$.

In words, voting-bloc responsiveness requires that every bloc of voters of positive size can sometimes affect the social ranking. The condition is clearly satisfied by any voting rule for which the Pareto property and anonymity hold. But it also holds for many non-anonymous voting rules, such weighted majority rule, defined as follows: Given a positive-valued, Lebesgue-measurable function w on $[0,1]$, F^w is *weighted majority rule with weight w*, if for all alternatives *x*, *y*, and profiles $\boldsymbol{R}$, $xF^w(\boldsymbol{R})y$ if and only if

$$\int_{i \in \{j | x\boldsymbol{R}(j)y\}} w(i)\,dm(i) \geq \int_{i \in \{j | y\boldsymbol{R}(j)x\}} w(i)\,dm(i) \quad .$$

Analogous to our Theorem, it can be shown (see Dasgupta and Maskin, 1998) that if a voting rule satisfies the Pareto property, neutrality, generic transitivity, and voting-bloc responsiveness on a domain $\Re$ then, for any *w*, F^w also satisfies those properties on $\Re$. We conjecture that the converse holds too. That is, if, for all *w*, $F(\boldsymbol{R}^\circ) \neq F^w(\boldsymbol{R}^\circ)$ for a regular profile $\boldsymbol{R}^\circ$ on domain $\Re^*$ where F^w satisfies these four properties, then there exists a domain $\Re'$ on which F^w satisfies all the properties, but F does not.

Another interesting extension to consider is strategic voting. It has long been known that there is a close connection between the problem of defining "reasonable" social preferences on a domain of preferences and that finding voting rules immune from strategic manipulation by voters (see Maskin 1979 and Kalai and Muller 1977). Because we have assumed a continuum of voters, sincere voting is automatically compatible with individual incentives for any voting rule in which a single voter's ordering makes no difference for social preferences. But the same is not true for coalitions (voting blocs). We conjecture that a counterpart to our Theorem can be derived when independence of irrelevant alternatives is replaced with the requirement that a voting rule be coalitionally strategy-proof.

Appendix

Lemma 1: For any domain $\Re$, F^{RO} satisfies neutrality on $\Re$ if and only if quasi-agreement holds on $\Re$.

Proof: Assume first that quasi-agreement holds on $\Re$. We must show that F^{RO} satisfies neutrality on $\Re$. Consider profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ on $\Re$ and alternatives *x*, *y*, *w*, and *z* such that

(A1) $$x\boldsymbol{R}(i)\,y \text{ if and only if } w\boldsymbol{R}'(i)\,z \text{ for all } i.$$

We must show that

(A2) $$xF^{RO}(\boldsymbol{R})\,y \text{ if and only if } wF^{RO}(\boldsymbol{R}')\,z$$

and

(A3) $$yF^{RO}(\boldsymbol{R}')\,x \text{ if and only if } zF^{RO}(\boldsymbol{R}')\,w.$$

If, for all i, $x\boldsymbol{R}(i)\,y$, then because F^{RO} satisfies the Pareto property, we have

$$\begin{array}{c} F^{RO}(\boldsymbol{R}) \\ \hline x \\ y \end{array} \qquad \text{and} \qquad \begin{array}{c} F^{RO}(\boldsymbol{R}') \\ \hline w \\ z \end{array} \quad ,$$

in accord with (A2) and (A3). Assume, therefore, that if we let

$$I_x = \{ i \mid x\boldsymbol{R}(i)\,y \} \quad \text{and} \quad I_y = \{ j \mid y\boldsymbol{R}(j)\,x \}$$

and

$$I'_w = \{ i \mid w\boldsymbol{R}'(i)\,z \} \quad \text{and} \quad I'_z = \{ j \mid y\boldsymbol{R}(j)\,w \} \quad ,$$

then I_x, I'_w, I_y, and I'_z are nonempty.

We claim that

(A4) $$v_{\boldsymbol{R}(i)}(x) - v_{\boldsymbol{R}(i)}(y) = v_{\boldsymbol{R}(j)}(y) - v_{\boldsymbol{R}(j)}(x) \quad \text{for all} \quad i \in I_x \quad \text{and} \quad j \in I_y.$$

Now, (A4) holds because, if there exist $i^* \in I_x$ and $z \in X$ such that

$$\begin{array}{c} \boldsymbol{R}(i^*) \\ \hline x \\ z \\ y \end{array} ,$$

then quasi-agreement implies

$$\begin{array}{c} \boldsymbol{R}(i) \\ \hline x \\ z \\ y \end{array} \quad \text{for all } i \in I_x \quad \text{and} \quad \begin{array}{c} \boldsymbol{R}(j) \\ \hline y \\ z \\ x \end{array} \quad \text{for all} \quad j \in I_y.$$

Similarly, we have

(A5) $$v_{\boldsymbol{R}'(i)}(w) - v_{\boldsymbol{R}'(i)}(z) = v_{\boldsymbol{R}'(j)}(z) - v_{\boldsymbol{R}'(j)}(w) \quad \text{for all} \quad i \in I'_w \quad \text{and} \quad j \in I'_z.$$

But from (A4) and (A5) and the definition of F^{RO}, we obtain (A2) and (A3), as required.

Next, suppose that quasi-agreement does not hold on domain $\mathfrak{R}$. Then there exist alternatives x, y, z and orderings $R, R' \in \mathfrak{R}$ such that

(A6) $$\begin{array}{c} R \\ \hline x \\ y \\ z \end{array}$$

and

(A7) $\dfrac{R'}{\begin{matrix} y \\ z \\ x \end{matrix}}$.

From (A6) and (A7) we have

(A8) $v_R(x) - v_R(y) < v_{R'}(y) - v_{R'}(x)$

(A9) $v_R(x) - v_R(z) > v_{R'}(z) - v_{R'}(x)$.

Choose

$$\boldsymbol{R} = \quad \frac{[0, \frac{1}{2})}{R} \quad \frac{[\frac{1}{2}, 1]}{R'} \quad .$$

Then from (A8) and (A9)

(A10) $\dfrac{F^{RO}(\boldsymbol{R})}{\begin{matrix} y \\ x \\ z \end{matrix}}$.

But, by construction, we have, for all i,

$$x\boldsymbol{R}(i)\,y \quad \text{if and only if} \quad x\boldsymbol{R}(i)\,z$$

and

$$y\boldsymbol{R}(i)\,x \quad \text{if and only if} \quad z\boldsymbol{R}(i)\,x\,.$$

Thus, if neutrality held we should have

$$yF^{RO}(\boldsymbol{R})\,x \quad \text{if and only if} \quad zF^{RO}(\boldsymbol{R})\,x\,,$$

which contradicts (A10).

Q.E.D.

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