Anti-Coordination Games and Dynamic Stability

Fuhito Kojima* Satoru Takahashi†

April 9, 2004

Abstract

We introduce the class of anti-coordination games, including the hawk-dove game as a special case. A symmetric two-player game is said to have the *anti-coordination property* if any worst response to a mixed strategy is in the support of that mixed strategy. Every anti-coordination game has a unique interior Nash equilibrium. We investigate stability of the static equilibrium under several dynamics with one-population setting. Specifically we focus on the best response dynamic (BRD), where agents in a large population take myopic best responses, and the perfect foresight dynamic (PFD), where agents take best responses to the time average of the action distributions from the present to the future. For any anti-coordination game we show (i) that, for any initial distribution, BRD has a unique solution, which reaches the static equilibrium in a finite time, (ii) that the same path is one of the solutions in PFD, and (iii) that no path escapes from the static equilibrium in PFD once the path reaches the equilibrium. Moreover, in some subclasses of anti-coordination games, we show that any solution from any initial state converges to the static equilibrium in PFD. All the results for PFD hold for any discount rate.

1 Introduction

Static and dynamic properties of the hawk-dove game have been a subject of much research in evolutionary game theory. It has been shown that the Nash equilibrium in the hawk-dove game is stable under most evolutionary dynamics in one-population setting.

*Harvard University. kojima@fas.harvard.edu

†Harvard University. stakahas@fas.harvard.edu

There are a number of possible extensions of these results to more general games. A natural direction is to investigate evolutionarily stable strategies (ESS). An interior ESS is known to have especially strong properties. If a game has an interior ESS, then it is a unique Nash equilibrium. Moreover, an interior ESS is globally stable for various dynamics including the replicator dynamic, the best response dynamic, smoothed best response dynamic.[1] Analyzing potential games is a second direction. Monderer and Shapley [13] show that the fictitious play converges in beliefs to some Nash equilibrium in any potential game.[2]

In this paper we propose a third direction. A symmetric game is said to have the *anti-coordination property* if any worst response to a mixed strategy lies in the support of that mixed strategy. In other words, a pure strategy is one of the worst responses against the action distribution in the society only if it is chosen by a positive fraction of agents in the society. This property is an abstraction of "strategic substitutability," but is different from submodularity of payoff functions. Congestion and product differentiation are economic examples of anti-coordination. We show that an anti-coordination game shares several properties with a game with an interior ESS. For example, an anti-coordination game has a unique Nash equilibrium, which is in the interior of mixed strategies. Yet we give an example to show that a game with an interior ESS may not have the anti-coordination property and that the unique Nash equilibrium of an anti-coordination game may not be an ESS.

We investigate stability of the static equilibrium under several dynamics with one-population setting. Specifically we focus on the best response dynamic (BRD) and the perfect foresight dynamic (PFD). BRD is a dynamic model of rational but myopic individuals, where agents in a large population take best responses to the current action distribution. PFD is a dynamic model of rational and forward-looking individuals, where agents take best responses to the time average of the action distributions from the present to the future.

For any anti-coordination game we show the following results. For BRD we show that there is a unique solution for each initial state, which reaches the static equilibrium in a finite time. For PFD we show two results. First, we show that the unique path in BRD is also a solution in PFD. Second, no path escapes from the static equilibrium in PFD once the path reaches the equilibrium. Moreover, in some subclasses of anti-coordination games,

[1] See Hofbauer [2, 3], Hofbauer and Sandholm [5], Hofbauer and Sigmund [6] and Shamma and Arslan [15].

[2] See the papers in footnote 1 and Sandholm [14].

we show that any solution from any initial state converges to the static equilibrium in PFD. All the results for PFD hold for any discount rate. We then discuss static properties of anti-coordination games, where we give an equivalent condition of anti-coordination property and also investigate its relation with interior ESS.

The rest of this paper is organized as follows. Section 2 introduces the anti-coordination property. Section 3 shows that the Nash equilibrium in each anti-coordination game is globally stable under the best-response dynamic. Section 4 introduces the perfect foresight dynamic, and investigates the stability of the equilibrium under the dynamic. Section 5 discusses static properties of anti-coordination games. Section 6 concludes.

2 Anti-Coordination Games

Consider a symmetric two-player game $G = (A, u)$, where A is the nonempty finite set of pure actions, $u = (u_{ij})$ is the payoff matrix, and u_{ij} is the payoff by choosing action $i \in A$ against action $j \in A$. The set of mixed actions is denoted by $\Delta = \{x \in \mathbb{R}^A \mid x_i \geq 0 \text{ for all } i \in A, \sum_{i \in A} x_i = 1\}$. For each $x \in \Delta$, $\operatorname{supp}(x) = \{i \in A \mid x_i > 0\}$ is the support of x, $\operatorname{br}(x) = \arg\max_i \sum_j u_{ij} x_j$ and $\operatorname{wr}(x) = \arg\min_i \sum_j u_{ij} x_j$ are the sets of best and worst responses to x in pure actions, respectively. x is a (symmetric) *Nash equilibrium* if $\operatorname{supp}(x) \subseteq \operatorname{br}(x)$.

Definition 1. G has the *anti-coordination property* if $\operatorname{wr}(x) \subseteq \operatorname{supp}(x)$ for any $x \in \Delta$.

$G = (A, u)$ has the anti-coordination property if and only if $(A, -u)$ has the total bandwagon property in the sense of Kandori and Rob [8]. The hawk-dove game is an example of an anti-coordination game.

Proposition 1. *Every anti-coordination game has a unique Nash equilibrium. The equilibrium is in the interior of* Δ*.*

Proof. The existence of a Nash equilibrium is clear. For any Nash equilibrium x, the anti-coordination property implies

$$\operatorname{wr}(x) \subseteq \operatorname{supp}(x) \subseteq \operatorname{br}(x).$$

Since $\operatorname{wr}(x) \subseteq \operatorname{br}(x)$ only if the two sets are equal to A, we have $\operatorname{supp}(x) = A$.

If there are two different interior Nash equilibria, then we can find one more Nash equilibrium on the boundary. This contradicts the fact that every Nash equilibrium is in the interior. □

Let x^* denote the unique Nash equilibrium of G. For any nonempty subset B of A, let $G(B)$ be the *restricted game* of G in which players choose actions only from B. If G has the anti-coordination property, then any restricted game of G also has the same property, and hence has a unique interior Nash equilibrium. The Nash equilibrium of $G(B)$ is denoted by $x^*(B)$.

3 The Best Response Dynamic

Consider the best response dynamic (BRD) over G, which is defined in Gilboa and Matsui [1] and Matsui [9]:

$$\phi\colon [0,\infty) \to \Delta, \tag{BRD-0}$$

$$\phi(0) = x, \tag{BRD-1}$$

$$\frac{d^+\phi}{dt}(t) = \alpha(t) - \phi(t), \tag{BRD-2}$$

$$\mathrm{supp}(\alpha(t)) \subseteq \mathrm{br}(\phi(t)). \tag{BRD-3}$$

A microfoundation of the dynamic is as follows. There is one large population of agents. The action distribution at time t is denoted by $\phi(t) \in \Delta$ (BRD-0). x is the initial action distribution (BRD-1). At each moment in time, an agent is matched randomly with another in the same population and play G. A fraction of the agents change their actions at each moment. The distribution of actions chosen at time t is proportional to $\alpha(t)$ (BRD-2), and every pure action chosen by a positive fraction of the agents has to be one of the best responses to the current action distribution (BRD-3).

Example 1. Consider the following payoff matrix on $A = \{1, 2, 3\}$,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

This game has the anti-coordination property. The unique Nash equilibrium is $x^* = (x_1^*, x_2^*, x_3^*) = (1/3, 1/3, 1/3)$. We explain that path ϕ from initial state x depicted in Figure 1 satisfies (BRD-0)–(BRD-3). The initial state x lies in the region where strategy 3 is a unique best response. Hence the path heads toward strategy 3 until the path reaches point P, where strategies 2 and 3 become indifferent. At this moment half the population begins to take strategy 2 and the rest begins to take strategy 3. At the aggregate level, the path kinks at P and moves toward Q. After a finite time the path reaches x^*, where all three strategies are indifferent. Each strategy is chosen by one third of the population and the path stays at rest afterwards.

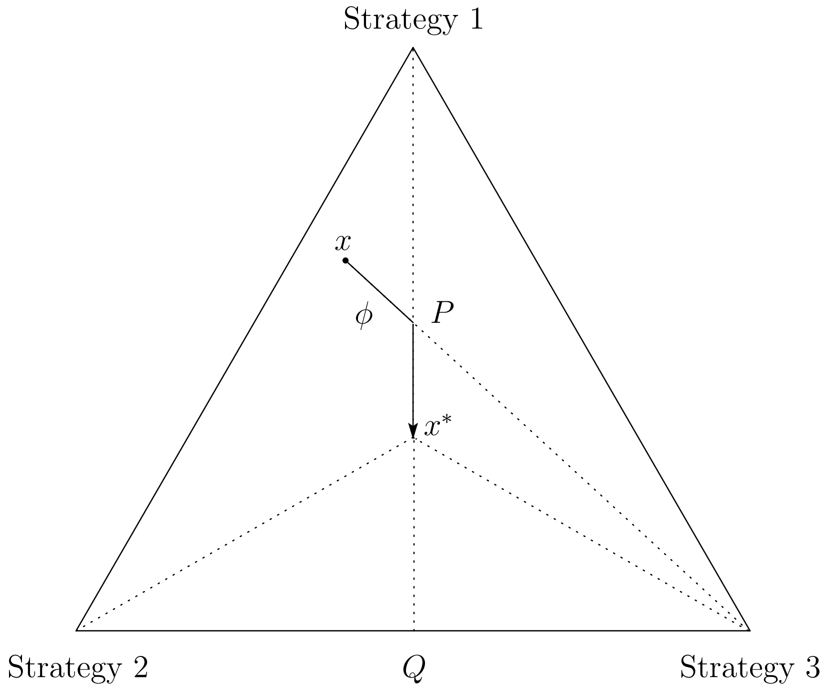

Figure 1: Example 1

Four features deserve comment here. First, x^* is globally stable. That is, from any initial state x, there exists a path ϕ satisfying (BRD-0)–(BRD-3) which converges to x^*. Moreover, the path ϕ is a unique solution for each x. Second, ϕ is not differentiable. It has a kink at P, where the fraction of agents choosing each action changes suddenly. Third, nevertheless, ϕ is right differentiable and piecewise linear. Fourth, $\alpha(t)$ in (BRD-2) may not be a pure strategy. That is, different agents may choose different actions at a point in time in general. After the path reaches P, strategies 2 and 3 are chosen simultaneously. Moreover, every strategy is taken by a positive fraction of the agents at x^*.

We restrict our analysis to piecewise linear solutions because of mathematical convenience. In this paper, a path is said to be *piecewise linear* if it has only finite kinked points in any bounded interval.[3]

The next proposition generalizes the first feature in Example 1, showing that the Nash equilibrium of any anti-coordination game is globally stable

[3]BRD may have non-piecewise linear solutions, and such solutions may change the stability of equilibria. For example, there exists a non-zero-sum rock-scissors-paper game which has no piecewise linear solution from the equilibrium except the constant path, but does have a more general solution which spirals out of the equilibrium. The solution is kinked infinitely often in a neighborhood of $t = 0$. See Hofbauer [2] for details.

under BRD. Global stability under BRD trivially implies local stability under BRD, which is called *social stability with respect to BRD* by Matsui [9]. This concept is known to be equivalent to *robustness against symmetric equilibrium entrants* by Swinkels [16], or *social stability against equilibrium entrants* by Matsui [9].

Proposition 2. *If G has the anti-coordination property, then, for any initial state $x \in \Delta$, there exists a unique piecewise linear path satisfying* (BRD-0)–(BRD-3). *The path arrives at x^* in a finite time and stays there afterwards.*

Proof. For every $t > 0$, there exists $t' > t$ such that $\alpha(s) = \alpha$ is constant for every $s \in [t, t')$. Then we have

$$\operatorname{supp}(\alpha) \subseteq \operatorname{br}(\phi(s)) = \operatorname{br}(c_\phi(s)\phi(t) + c_\alpha(s)\alpha)$$

for every $s \in [t, t')$, where $c_\phi(s) = e^{t-s}$ and $c_\alpha(s) = 1 - e^{t-s}$. α is a best response to $\phi(t)$ since the above inclusion holds for $s = t$, and given that, the above relation for $s > t$ implies that any pure strategy in $\operatorname{supp}(\alpha)$ is a best response to α within $\operatorname{br}(\phi(t))$. This means that $\alpha(t) = \alpha = x^*(\operatorname{br}(\phi(t)))$, the Nash equilibrium of $G(\operatorname{br}(\phi(t)))$.

Since any restricted game of G has a unique Nash equilibrium, $\alpha(\cdot)$ is uniquely determined by the above construction. Since $\operatorname{br}(\phi(t)) = \operatorname{wr}(x^*(\operatorname{br}(\phi(t)))) = \operatorname{wr}(\alpha(t))$, the payoff of α relative to the other strategies decreases in t. Therefore $\operatorname{br}(\phi(t))$ weakly increases in t in the set inclusion order, and strictly increases in a finite time until $\phi(t) = x^*$ is established. Therefore, ϕ arrives at x^* in a finite time and stays at x^* afterwards. □

Proposition 2 also follows from Hofbauer [2, Theorem 5.1.1]. He defines

$$V(x) = \max_i \sum_j u_{ij} x_j - w_{\mathbf{B}}(x),$$

where $\mathbf{B}$ is the set of mixed strategies b such that $\operatorname{supp}(b) \subsetneq A$ and every pure strategy in $\operatorname{supp}(b)$ is indifferent against b, and

$$w_{\mathbf{B}}(x) = \max\left\{ \sum_{b \in \mathbf{B}} \sum_{i,j} u_{ij} b_i b_j \lambda^b \;\middle|\; \lambda^b \geq 0,\ \sum_{b \in \mathbf{B}} \lambda^b = 1,\ \sum_{b \in \mathbf{B}} b \lambda^b = x \right\}.$$

He shows that if there exists $p \in \Delta$ with $\sum_{i,j} u_{ij} p_i b_j > \sum_{i,j} u_{ij} b_i b_j$ for all $b \in \mathbf{B}$, then V is a global Lyapunov function for BRD, and decreases except at x^*. These imply the global stability of x^*. It is easy to see that any anti-coordination game satisfies the above condition.[4] Note that Hofbauer's result applies to a broader class of games including games with interior evolutionarily stable strategies (ESS).

For anti-coordination games, however, Proposition 2 gives a sharper prediction than Hofbauer's theorem in two respects. First, the piecewise linear path satisfying (BRD-0)–(BRD-3) is shown to be unique. Second, the path is constructed explicitly. This construction turns out to be useful when we show in Proposition 3 that the same path is a solution to PFD as well.

4 The Perfect Foresight Dynamic

Consider the perfect foresight dynamic (PFD) over G, which is introduced by Matsui and Matsuyama [10]:

$$\phi\colon [0,\infty) \to \Delta, \tag{PFD-0}$$
$$\phi(0) = x, \tag{PFD-1}$$
$$\frac{d^+\phi}{dt}(t) = \alpha(t) - \phi(t), \tag{PFD-2}$$
$$\pi(t) = r\int_t^\infty e^{r(t-s)}\phi(s)\,ds, \tag{PFD-3}$$
$$\operatorname{supp}(\alpha(t)) \subseteq \operatorname{br}(\pi(t)). \tag{PFD-4}$$

The dynamic is similar to BRD, but different in one respect. Agents do not respond to the current action distribution. Rather, they form an expectation $\pi(t)$ by the discounted time average of the action distributions from the present to the future (PFD-3), and choose best responses to that expectation (PFD-4). $r > 0$ is called the effective discount rate.[5]

We again focus on piecewise linear solutions only.

Proposition 3 shows that, in anti-coordination games, solutions of BRD are preserved under PFD with any discount rate.

Proposition 3. *If G has the anti-coordination property, then, for any initial state $x \in \Delta$, there exists a piecewise linear path satisfying* (PFD-0)–(PFD-4) *which converges to x^*.*

[4]In an anti-coordination game, we have $\mathbf{B} = \{x^*(B) \mid B \subsetneq A\}$, that is, the set of Nash equilibria of strictly restricted games. Taking any totally mixed strategy as p, for instance, we can show $\sum_{i,j} u_{ij}p_ib_j > \sum_{i,j} u_{ij}b_ib_j$ for any $b = x^*(B)$ because any pure strategy outside B gives a higher payoff against $x^*(B)$ than any pure strategy inside B does.

[5]In the literature r is often written as $r = 1+\theta$, where θ is the rate of time preference relative to the arrival rate of action revision opportunities.

Proof. Let ϕ be the path constructed in Proposition 2. We will show that ϕ also satisfies (PFD-4). Since $\mathrm{br}(\phi(s)) \supseteq \mathrm{br}(\phi(t))$ for any $s \geq t$, we have $\mathrm{br}(\pi(t)) = \mathrm{br}(\phi(t))$ by (PFD-3). Therefore (BRD-3) implies (PFD-4). $\square$

By Proposition 3, we obtain the existence of a solution from any state to x^* under PFD, i.e., x^* is *globally accessible* in the terminology of Matsui and Matsuyama [10]. No matter how far from x^* the initial state is, it is possible that the action distribution in the society arrives at x^*.

The existence of such a solution, however, does not mean that the society *always* reaches x^*. For PFD typically entails serious multiplicity of solutions, the dynamic may have a solution which does not converge to x^*. Moreover, there may be a path which escapes even from x^*. See Matsui and Matsuyama [10] for the analysis of a 2×2 coordination game. In contrast, we will show that no path can escape from x^* in anti-coordination games.

We first show the following lemma, which claims that $\alpha(t)$ may not be a myopic best response to the current action distribution $\phi(t)$, but cannot be a myopic worst response to $\phi(t)$ unless $\phi(t)$ is equal to x^*. This lemma is powerful, for we obtain a restriction on $\alpha(t)$ without any reference to the future behavior $\phi(s)$ for $s > t$.

Lemma 1. *If G has the anti-coordination property and ϕ is a piecewise linear path satisfying* (PFD-0)–(PFD-4)*, then, for any $t \geq 0$, either $\phi(t) = x^*$ or* $\mathrm{supp}(\alpha(t)) \cap \mathrm{wr}(\phi(t)) = \emptyset$ *holds.*

Proof. Suppose $\phi(t) \neq x^*$ and $\mathrm{supp}(\alpha(t)) \cap \mathrm{wr}(\phi(t)) \neq \emptyset$ for some t. Let $\alpha = \alpha(t)$, and t' be the smallest kinked point greater than t. (If there is no kinked point after t, skip to the last paragraph of this proof.) Then we have

$$\mathrm{supp}(\alpha) \subseteq \mathrm{br}(\pi(s)) = \mathrm{br}\left(c_\pi(s)\pi(t) - c_\phi(s)\phi(t) - c_\alpha(s)\alpha\right) \qquad (1)$$

for every $s \in [t, t']$, where

$$c_\pi(s) = e^{r(s-t)}, \quad c_\phi(s) = \frac{r(e^{r(s-t)} - e^{t-s})}{1+r}, \quad c_\alpha(s) = \frac{e^{r(s-t)} + re^{t-s}}{1+r} - 1.$$

Since $c_\pi(\cdot)$, $c_\phi(\cdot)$ and $c_\alpha(\cdot)$ are linearly independent in the space of functions on $[t, t']$, (1) implies that every action in $\mathrm{supp}(\alpha)$ is indifferent against α. By the anti-coordination property, we have $\mathrm{supp}(\alpha) = \mathrm{wr}(\alpha)$.

Substitute $s = t'$ in (1). Since $c_\pi(t')$, $c_\phi(t')$ and $c_\alpha(t')$ are all positive, the set of best responses to $\pi(t')$ is given by the intersection of $\mathrm{br}(\pi(t))$, $\mathrm{wr}(\phi(t))$, and $\mathrm{wr}(\alpha)$ if the intersection is nonempty. Since we have $\mathrm{wr}(\alpha) = \mathrm{supp}(\alpha) \subseteq \mathrm{br}(\pi(t))$ and $\mathrm{supp}(\alpha) \cap \mathrm{wr}(\phi(t)) \neq \emptyset$, the intersection is actually nonempty and equal to $\mathrm{supp}(\alpha) \cap \mathrm{wr}(\phi(t))$. Therefore, $\mathrm{supp}(\alpha(t')) \subseteq$

$\mathrm{br}(\pi(t')) = \mathrm{supp}(\alpha) \cap \mathrm{wr}(\phi(t)) = \mathrm{wr}(\phi(t'))$. So t' also satisfies $\phi(t') \neq x^*$ and $\mathrm{supp}(\alpha(t')) \cap \mathrm{wr}(\phi(t')) \neq \emptyset$, and hence we can continue the same argument for the next kinked point. Since we assume that there are only finite kinked points in any bounded interval, we can show that $\mathrm{supp}(\alpha(s))$ is decreasing in the set inclusion order in $s \geq t$, and hence equal to some $\mathrm{supp}(\alpha(s_0))$ for sufficiently large s.

Finally, by (PFD-3), $\pi(t)$ is a convex combination of $\phi(t)$ and $\alpha(s)$ for $s \geq t$. Then we have $\mathrm{wr}(\pi(t)) = \mathrm{wr}(\phi(t)) \cap \bigcap_{s \geq t} \mathrm{wr}(\alpha(s)) = \mathrm{wr}(\phi(t)) \cap \mathrm{supp}(\alpha(s_0))$, which is nonempty but not equal to A because of $\phi(t) \neq x^*$. Therefore, no pure strategy in this set is a best response to $\pi(t)$. This contradicts the fact that any pure strategy in $\mathrm{supp}(\alpha)$ is a best response to $\pi(t)$ by (PFD-4). $\square$

As an immediate implication of Lemma 1, we obtain a local stability result under PFD.

Proposition 4. *If G has the anti-coordination property, then the constant path at x^* is a unique piecewise linear path satisfying* (PFD-0)–(PFD-4) *for $x = x^*$.*

Proof. Suppose $t_0 = \inf\{t \geq 0 \mid \phi(t) \neq x^*\} < \infty$. Then we have $\alpha(s) = \alpha \neq x^*$ for any $s \in [t_0, t]$ for some $t > t_0$. Therefore, $\alpha(t) = \alpha$ and $\phi(t)$ is a convex combination of x^* and α. This implies that $\phi(t) \neq x^*$ and $\mathrm{wr}(\phi(t)) = \mathrm{wr}(\alpha) \subseteq \mathrm{supp}(\alpha)$, which contradicts Lemma 1. $\square$

By Proposition 4, once the action distribution reaches the Nash equilibrium, it stays at rest. Takahashi [17] calls this property *absorption in the discrete topology* (*d-absorption*).

Proposition 4 and its proof are an extension of Matsui and Oyama [11, Lemma A.4]. They show the d-absorption of the unique Nash equilibrium in the hawk-dove game.[6]

Next, we turn to global stability under PFD. Although we cannot obtain a general result in the class of anti-coordination games, we can show the global stability in several "simple" games.

Example 2. Consider the hawk-dove game on $A = \{1, 2\}$,

$$\begin{pmatrix} 0 & a \\ 1-a & 0 \end{pmatrix}, \qquad 0 < a < 1.$$

[6]Actually their result in Lemma A.4 is not for the hawk-dove game, but for a 3×3 game to which the hawk-dove game is "embedded" as a restricted game. However, as they remark below Proposition 7.2, we can use the same technique as in Lemma A.4 to show the d-absorption in the hawk-dove game.

This game has the anti-coordination property. The unique Nash equilibrium is $x^* = (x_1^*, x_2^*) = (a, 1-a)$. For any given initial state $x = (x_1, x_2)$, the path constructed in Proposition 3 is a solution of PFD. Here we show that there is no other solution. Suppose that $\phi_1(t) > x_1^*$. Then $\mathrm{wr}(\phi(t)) = \{1\}$, which implies $\alpha(t) = (0, 1)$ by Lemma 1. That is, ϕ moves toward pure strategy 2. Similarly, if $\phi_1(t) < x_1^*$, then ϕ moves toward pure strategy 1. If $\phi_1(t) = x_1^*$, then, by Proposition 4, ϕ stays forever at x^*. In summary, the hawk-dove game has a unique solution of PFD from any initial state, which arrives at the Nash equilibrium in a finite time.

Example 3. Consider the following payoff matrix on $A = \{1, 2, 3\}$,

$$\begin{pmatrix} 0 & a & 1-a \\ 1-a & 0 & a \\ a & 1-a & 0 \end{pmatrix}, \qquad \frac{1}{3} \leq a \leq \frac{2}{3}.$$

Since $0 < a < 1$, this is an anti-coordination game with the unique Nash equilibrium $x^* = (1/3, 1/3, 1/3)$.

We will show that any solution of PFD converges to x^*, and that the solution reaches x^* in a finite time if $1/3 < a < 2/3$. We divide the state space Δ into three regions $\Delta_i = \{x \in \Delta \mid i \in \mathrm{wr}(x)\}$ for $i \in A$. Without loss of generality, we assume that $1/3 \leq a \leq 1/2$ and that the initial state x is in Δ_1. See Figure 2.

First, notice that no solution ϕ crosses the border from Δ_i to $\Delta_{i-1} \setminus \Delta_i$.[7] Otherwise, on the border $\Delta_i \cap \Delta_{i-1}$, the solution has to move toward pure strategy $i+1$ by Lemma 1. An increase in the proportion of $i+1$ makes the other strategies better off. However, since $a \leq 1/2$, the payoff of $i-1$ increases at least as much as that of i, which contradicts the direction in which the solution crosses the border.

Second, by the anti-coordination property, ϕ cannot stay forever in one region except at x^*. Therefore, ϕ goes from Δ_1 to Δ_2, Δ_3, Δ_1, Δ_2, and so on. In other words, ϕ moves counterclockwise around x^* in Figure 2.

Third, define $P_1, P_2, \ldots$ as follows. Let P_1 be the intersection of the border $\Delta_1 \cap \Delta_2$ and the segment connecting pure strategies 1 and 2, and let P_k be the intersection of the border $\Delta_k \cap \Delta_{k+1}$ and the segment connecting P_{k-1} and pure strategy $k+1$ for $k \geq 2$. See Figure 2 for P_1, P_2, and P_3. Observe that when the solution crosses the k-th border, the pass point has to be between x^* and P_k. This observation follows from Lemma 1.

[7] We take an element of A modulo 3. For example, $1 - 1 = 3$ and $3 + 1 = 1$.

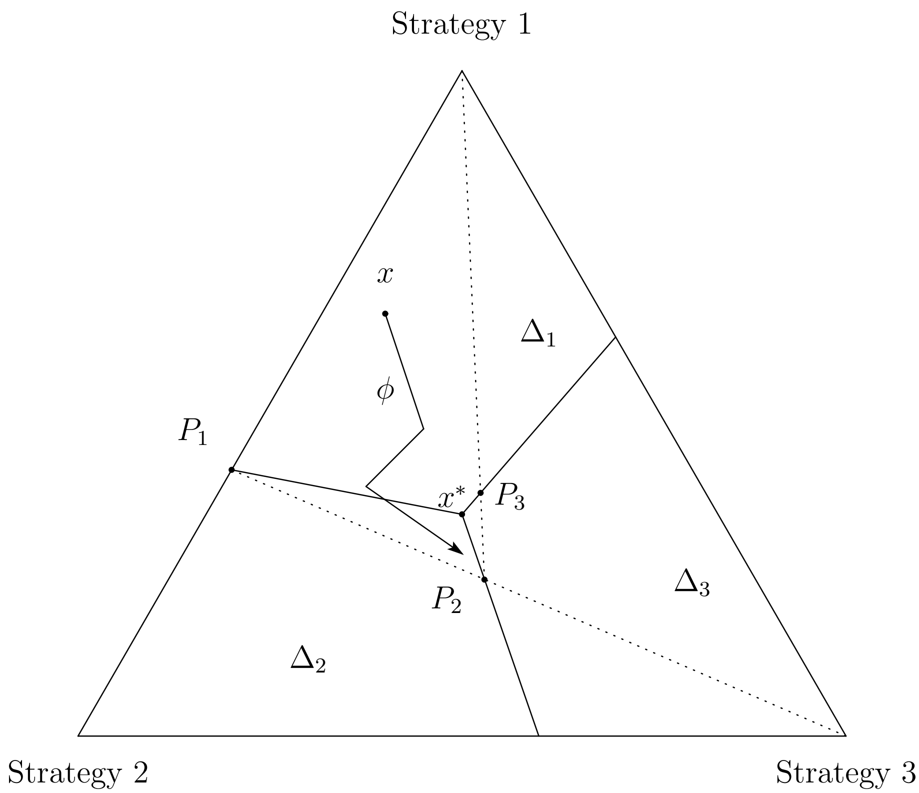

Figure 2: Example 3

Fourth, by a tedious computation, we have

$$d(P_k, x^*) = \begin{cases} \dfrac{\sqrt{2}}{3k} & \text{if } a = \dfrac{1}{3}, \\ \dfrac{(a-1/3)\sqrt{2(a^2-a+1/3)}}{a^{k+1}(1-2a)^{-k+1} - (a^2-a+1/3)} & \text{if } \dfrac{1}{3} < a \le \dfrac{1}{2}, \end{cases}$$

where $d(y,z) = \sqrt{\sum_i (y_i - z_i)^2}$ for y, $z \in \Delta$, and hence $P_k \to x^*$ as $k \to \infty$. This fact, combined with the third observation, implies that any solution converges to x^*.

Fifth, note that, in Δ_k, the fraction of strategy k decreases at a speed bounded away from zero. When a solution moves from a boundary $\Delta_{k-1} \cap \Delta_k$ to the next boundary $\Delta_k \cap \Delta_{k+1}$, the fraction of strategy k can change by at most $d(P_k, x^*) + d(P_{k+1}, x^*)$. Therefore, there exists a constant $C > 0$ such that it takes time at most $C(d(P_k, x^*) + d(P_{k+1}, x^*))$ for any solution to move from $\Delta_{k-1} \cap \Delta_k$ to $\Delta_k \cap \Delta_{k+1}$. Therefore, if $1/3 < a \le 1/2$, then $\sum_{k=1}^{\infty} d(P_k, x^*) < \infty$, and hence any solution reaches x^* in a finite time.

Example 4. Consider the following payoff matrix on $A = \{1, \dots, n\}$,

$$\begin{pmatrix} -a_1 & 0 & \cdots & 0 \\ 0 & -a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_n \end{pmatrix}, \qquad a_1 > 0, \dots, a_n > 0.$$

This game also has the anti-coordination property. The unique Nash equilibrium is $x^* = (\lambda a_1^{-1}, \lambda a_2^{-1}, \dots, \lambda a_n^{-1})$, where $\lambda = \left(\sum_i a_i^{-1}\right)^{-1}$.

We will show that any solution of PFD converges to x^* in a finite time.[8] As in Example 3, we divide Δ into n regions $\Delta_i = \{x \in \Delta \mid i \in \mathrm{wr}(x)\}$.

Similarly to Example 3, any solution ϕ has to cross the border from Δ_i to $\Delta_j \setminus \Delta_i$ for $i \neq j$ unless $\phi(t) = x^*$ for some t. Let t_0 be the moment of crossing the border. Then the ratio $\phi_i(t)/\phi_j(t)$ is equal to a_j/a_i for $t = t_0$, and is below a_j/a_i for t slightly greater than t_0. Therefore, $\phi_i(t)/\phi_j(t)$ has to be decreasing around $t = t_0$. This implies that $\alpha_j(t_0) > 0$, which contradicts Lemma 1.

Before concluding the section, we point out that all our results on PFD hold for any discount rate $r > 0$. In many games, in contrast, stability property typically depends on r. In a 2×2 coordination game, for example, the risk-dominant equilibrium is globally accessible only for r sufficiently close to 1. Moreover, this equilibrium is not d-absorbing for r less than 1 and close to 0. See Matsui and Matsuyama [10].

5 Static Properties of Anti-Coordination

This section investigates some static properties of anti-coordination games. First, we have the following result.

[8]The payoff matrix of Example 4 is symmetric, i.e., two players always get identical payoffs. For a symmetric payoff matrix, global accessibility in PFD (Proposition 3) is already obtained by Hofbauer and Sorger [7, Theorem 3] if the effective discount rate r is greater than but sufficiently close to 1. They also show in [7, Lemma 4] that any element of the ω-limit of each solution in PFD is a critical point of the potential function if $r > 1$. Since the ω-limit is connected and any connected component of critical points in Example 4 is a singleton, the ω-limit is a singleton, i.e., the solution converges to some limit. Because the limit has to be a Nash equilibrium, the solution converges to x^*. However, our results are stronger than Hofbauer and Sorger's in three respects. First, they need some assumptions on the discount rate. Second, they do not show d-absorption (Proposition 4 of this paper), i.e., they do not exclude the possibility that a solution escapes from x^* temporarily. Third, they do not show the finite-time convergence. According to their proof, the rate of convergence may become slower as r gets closer to 1, whereas the time needed for reaching x^* in our proof is bounded from above independently of r.

Proposition 5. *G has the anti-coordination property if and only if, for any $B \subsetneq A$, $G(B)$ has the anti-coordination property and* $\mathrm{wr}(x^*(B)) = B$.

Proof. See Appendix. □

We can use this proposition inductively on the size of restricted games to characterize the anti-coordination property. See below.

Example 5. Consider an arbitrary 3×3 payoff matrix on $A = \{1, 2, 3\}$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}.$$

We will give a necessary and sufficient condition for this payoff matrix to have the anti-coordination property.

First, we consider a restricted game $G(\{i\})$ for each $i \in A$. $G(\{i\})$ is obviously an anti-coordination game, and pure strategy i is a unique Nash equilibrium $x^*(\{i\})$. Then the condition that $\mathrm{wr}(x^*(\{i\})) = \{i\}$ for each $i \in A$ is written as

$$u_{ii} < u_{ji} \quad \text{for any } i \neq j. \tag{2}$$

The condition (2) means that each diagonal component is smaller than any other component in the same column. By Proposition 5, we know that each 2×2 restricted game $G(\{i, j\})$ is an anti-coordination game under the condition (2). Then, by Proposition 1, $G(\{i, j\})$ has a unique Nash equilibrium $x^*(\{i, j\})$, which is given by

$$x_i^*(\{i, j\}) = \frac{u_{ij} - u_{jj}}{u_{ij} + u_{ji} - u_{ii} - u_{jj}}, \quad x_j^*(\{i, j\}) = \frac{u_{ji} - u_{ii}}{u_{ij} + u_{ji} - u_{ii} - u_{jj}}.$$

Next, we consider each restricted game of the form $G(\{i, j\})$ with $i \neq j$. As we showed above, $G(\{i, j\})$ has the anti-coordination property if (2) is satisfied. Under this condition, the condition that $\mathrm{wr}(x^*(\{i, j\})) = \{i, j\}$ for each $i \neq j$ is equivalent to

$$u_{ij}u_{ji} - u_{ii}u_{jj} < u_{ki}(u_{ij} - u_{jj}) + u_{kj}(u_{ji} - u_{ii}) \quad \text{for any distinct } i,\ j,\ k. \tag{3}$$

Therefore, the system of inequalities (2) and (3) characterizes the anti-coordination property in the class of 3×3 games.

Second, we discuss relationship between anti-coordination games and other special classes of games. An interior ESS satisfies properties analogous to Propositions 1 and 2. There is no Nash equilibrium other than the

ESS, and the ESS is globally stable under BRD. See Hofbauer [3]. However, the two classes of games are not nested. That is, a game with an interior ESS may not have the anti-coordination property; the unique Nash equilibrium of an anti-coordination game may not be evolutionarily stable. For example, the following payoff matrix

$$\begin{pmatrix} 0 & a & b \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

has the anti-coordination property if and only if $a > 0$, $b > 0$, and $a + b > 1$, whereas it has an interior ESS if and only if $a+b > 1$ and $4(a+b+1) > (a-b)^2$.

¿From the above argument, we know that the existence of an interior ESS does not imply the anti-coordination property. Then, what additional condition is needed for a game with an interior ESS to have the anti-coordination property? Such a condition is given by the next proposition.

Proposition 6. *Suppose that G has an interior ESS. Then, G is an anti-coordination game if and only if $G(B)$ has an interior Nash equilibrium for any $B \subsetneq A$.*

Proof. See Appendix. □

The class of potential games is another one whose dynamic stability has been well investigated. It is not hard to see that, similarly to interior ESS, an anti-coordination game may not be a potential game and vice versa.

6 Conclusion

We investigated dynamic stability of the Nash equilibrium of anti-coordination games. For any initial state, there is a unique solution to the best response dynamic, which reaches the unique equilibrium in a finite time. Under the perfect foresight dynamic, the equilibrium is also stable in two senses. There exists a path from any initial state to the equilibrium, and once the path reaches the equilibrium, then the path stays there forever. For some subclasses of anti-coordination games, any solution to the perfect foresight dynamic converges to the equilibrium.

We should note that our results explicitly or implicitly depend on the following assumptions: one-population setting, exponential discounting, homogeneous action revision, the linearity of the payoff function in mixed strategies, the piecewise linearity of solutions and, above all, the anti-coordination property.[9] At the cost of these assumptions, we obtained rather strong predictions about perfect foresight behavior independently of the discount rate.[10]

This paper leaves an open question the stability of ESS under PFD, conjectured in Hofbauer and Sorger [7]. Although an interior ESS is known to be globally stable under BRD, stability under PFD is neither proved nor disproved.

Another open question is the stability of equilibria in anti-coordination games under other dynamics. We can show that a logit equilibrium à la McKelvey and Palfrey [12], which is a steady state of the smoothed best response dynamic with the logit choice function, is unique (Proposition 7 in Appendix).[11] Analysis of the stability in the corresponding dynamic is not completed, however. The stability in the replicator dynamic is also yet to be resolved.

A Appendix

A.1 Uniqueness of Logit Equilibria

Consider the following *logistic quantal response function* $\sigma^\lambda \colon \Delta \to \Delta$ with parameter $\lambda \geq 0$:

$$\sigma_i^\lambda(x) = \frac{\exp\left(\lambda \sum_j u_{ij} x_j\right)}{\sum_k \exp\left(\lambda \sum_j u_{kj} x_j\right)}.$$

$x \in \Delta$ is a (symmetric) *logit equilibrium* with parameter λ if $x = \sigma^\lambda(x)$. As $\lambda \to \infty$, σ^λ converges (in an appropriate sense) to the best response correspondence. See McKelvey and Palfrey [12].

---

[9] By homogeneous action revision we mean that who can change his action at each moment is independent of his name and any of the past history.

[10] The anti-coordination property in Propositions 2 and 3 can be relaxed. For instance, suppose that each restricted game $G(B)$ has at least one Nash equilibrium against which every pure strategy in B is indifferent. Then there is a path from any initial state to *some* equilibrium, which is a solution both in BRD and in PFD. (This may not be a unique solution even under BRD.) For example, this condition is satisfied in the following payoff matrix

$$\begin{pmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{pmatrix}$$

if and only if $ab \geq 0$, whereas the anti-coordination property is satisfied if and only if $a > 0$ and $b > 0$.

[11] A game with an interior ESS has an analogous property. See Hofbauer [3].

Proposition 7. *Every anti-coordination game has a unique logit equilibrium for any parameter $\lambda \geq 0$.*

Proof. The case for $\lambda = 0$ is obvious. Suppose that x and y with $x \neq y$ are both fixed points of σ^λ for some $\lambda > 0$. Let $B = \{i \in A \mid x_i > y_i\}$ and $C = A \setminus B = \{i \in A \mid x_i \leq y_i\}$. Since $x \neq y$, B and C are nonempty. For any $i \in B$ and $j \in C$, we have $x_i/y_i > 1 \geq x_j/y_j$, which implies $\sigma_i^\lambda(x)/\sigma_i^\lambda(y) > \sigma_j^\lambda(x)/\sigma_j^\lambda(y)$. This is equivalent to

$$\sum_k u_{ik}(x_k - y_k) > \sum_k u_{jk}(x_k - y_k). \tag{4}$$

Let $z_i = \max(x_i - y_i, 0)$ and $z = (z_i)_{i \in A}$. By the anti-coordination property, there exists a worst response $i^* \in B$ against z, and any $j \in C$ yields a higher payoff against z than i^* does.[12] Similarly, let $w_i = \max(y_i - x_i, 0)$ and $w = (w_i)_{i \in A}$. By the anti-coordination property, there exists a worst response $j^* \in C$ against w, and any $i \in B$ yields a higher payoff against w than j^* does. Therefore, we have the following inequalities:

$$\sum_k u_{i^*k} z_k < \sum_k u_{j^*k} z_k, \quad \sum_k u_{i^*k} w_k > \sum_k u_{j^*k} w_k.$$

Since $z - w = x - y$ by definitions of z and w, we have

$$\sum_k u_{i^*k}(x_k - y_k) = \sum_k u_{i^*k}(z_k - w_k) < \sum_k u_{j^*k}(z_k - w_k) = \sum_k u_{j^*k}(x_k - y_k),$$

which contradicts (4). □

Proposition 7 shows that the smoothed best response dynamic given by

$$\phi(0) = x, \quad \frac{d\phi}{dt}(t) = \sigma^\lambda(\phi(t)) - \phi(t) \quad \text{for } t \geq 0$$

has a unique steady state.

A.2 Proofs of Propositions 5 and 6

Proof of Proposition 5. The only if direction is shown as follows. For any $B \subsetneq A$, $G(B)$ is an anti-coordination game. Hence it has an interior Nash equilibrium $x^*(B)$ by Proposition 1. Then we have $\mathrm{wr}(x^*(B)) = B$ since

[12] We can extend the domain of the worst response to $\mathbb{R}_+^A \setminus \{\mathbf{0}\}$, where $\mathbb{R}_+^A = \{z \in \mathbb{R}^A \mid z_i \geq 0 \text{ for any } i \in A\}$ and $\mathbf{0} = (0, \ldots, 0)$. The anti-coordination property in Δ implies the same property in $\mathbb{R}_+^A \setminus \{\mathbf{0}\}$.

any strategy in $\mathrm{supp}(x^*(B))$ is indifferent against $x^*(B)$, and worse than any strategy outside $\mathrm{supp}(x^*(B))$ by the anti-coordination property of G.

For the if direction, we need to show $\mathrm{wr}(x) \subseteq \mathrm{supp}(x)$ for any $x \in \Delta$. If x is in the interior, then this relation is trivial. If not, we fix any such x. Then we construct $B^k \subseteq \mathrm{supp}(x) \subsetneq A$ and $c^k > 0$ for each $k = 0, 1, 2 \dots$ by

$$\begin{aligned} B^0 &= \mathrm{supp}(x), \\ c^k &= \min_{i \in B^k} \frac{x_i - \sum_{l<k} c^l x_i^*(B^l)}{x_i^*(B^k)}, \\ B^{k+1} &= B^k \setminus \arg\min_{i \in B^k} \frac{x_i - \sum_{l<k} c^l x_i^*(B^l)}{x_i^*(B^k)}. \end{aligned}$$

Since $B^0 \supsetneq B^1 \supsetneq B^2 \supsetneq \cdots$, we stop at the m-th step when $B^{m+1} = \emptyset$. Then we obtain

$$x = \sum_{k=0}^{m} c^k x^*(B^k).$$

Therefore, we have

$$\mathrm{wr}(x) = \bigcap_{k=0}^{m} \mathrm{wr}(x^*(B^k)) = B^m \subseteq \mathrm{supp}(x)$$

since $\mathrm{wr}(x^*(B^k)) = B^k \supseteq B^m$ for each k. $\square$

Proof of Proposition 6. The only if direction is obvious from Proposition 1.

For the if direction, for each $B \subseteq A$, let $\hat{x}(B)$ be an interior Nash equilibrium of $G(B)$. Since the original game G has an interior ESS, the payoff matrix $(u_{ij})_{i,j \in A}$ is negative definite with respect to $\mathbb{R}_0^A = \{x \in \mathbb{R}^A \mid \sum_i x_i = 0\}$.[13] (See Hofbauer and Sigmund [6, Exercise 6.4.3].) Since the submatrix $(u_{ij})_{i,j \in B}$ is also negative definite with respect to $\mathbb{R}_0^B$, it is not hard to verify that $\hat{x}(B)$ is a unique Nash equilibrium of $G(B)$.[14]

By Proposition 5, it is sufficient to show that, for any $B \subsetneq A$, any pure strategy outside B gets a higher payoff against $\hat{x}(B)$ than $\hat{x}(B)$ itself does. Suppose the contrary, i.e., there exist $B \subsetneq A$ and $i \in A \setminus B$ such that i is worse than or equal to $\hat{x}(B)$ against $\hat{x}(B)$. Then $\hat{x}(B)$ is a Nash equilibrium also in $G(B \cup \{i\})$, which contradicts the uniqueness of equilibrium in $G(B \cup \{i\})$. $\square$

[13] $(u_{ij})_{i,j \in A}$ is negative definite with respect to $\mathbb{R}_0^A$ if $\sum_{i,j} u_{ij}\xi_i\xi_j < 0$ for any $\xi \in \mathbb{R}_0^A$.

[14] Let x and y are Nash equilibria of $G(B)$ and $x \neq y$. Then $\sum_{i,j} u_{ij}x_i x_j \geq \sum_{i,j} u_{ij} y_i x_j$, $\sum_{i,j} u_{ij} y_i y_j \geq \sum_{i,j} u_{ij} x_i y_j$. Manipulating these inequalities, we obtain $\sum_{i,j} u_{ij}(x_i - y_i)(x_j - y_j) \geq 0$, which contradicts negative definiteness of the payoff matrix with respect to $\mathbb{R}_0^B$.

References

[1] Gilboa, I. and A. Matsui (1991) "Social Stability and Equilibrium". Econometrica, Vol. 59, pp. 859–867.

[2] Hofbauer, J. (1995) "Stability for the Best Response Dynamics". mimeo.

[3] Hofbauer, J. (2000) "From Nash and Brown to Maynard Smith: Equilibria, Dynamics, and ESS". Selection, Vol. 1, pp. 81–88.

[4] Hofbauer, J. and E. Hopkins (2004) "Learning in Perturbed Asymmetric Games". mimeo.

[5] Hofbauer, J. and W. H. Sandholm (2002) "On the Global Convergence of Stochastic Fictitious Play". Econometrica, Vol. 70, pp. 2265–2294.

[6] Hofbauer, J. and K. Sigmund (1998) *Evolutionary Games and Population Dynamics*. Cambridge University Press.

[7] Hofbauer, J. and G. Sorger (1999) "Perfect Foresight and Equilibrium Selection in Symmetric Potential Games". Journal of Economic Theory, Vol. 85, pp. 1–23.

[8] Kandori, M. and R. Rob (1998) "Bandwagon Effects and Long Run Technology Choice". Games and Economic Behavior Vol. 22, pp. 30–60.

[9] Matsui, A. (1992) "Best Response Dynamics and Socially Stable Strategies". Journal of Economic Theory, Vol. 57, pp. 343–362.

[10] Matsui, A. and K. Matsuyama (1995) "An Approach to Equilibrium Selection". Journal of Economic Theory, Vol. 65, pp. 415–434.

[11] Matsui, A. and D. Oyama (2003) "Rationalizable Foresight Dynamics: Evolution and Rationalizability". mimeo.

[12] McKelvey R. D. and T. R. Palfrey (1995) "Quantal Response Equilibria for Normal Form Games". Games and Economic Behavior, Vol. 10, pp. 6–38.

[13] Monderer, D. and L. S. Shapley (1996) "Fictitious Play Property for Games with Identical Interests". Journal of Economic Theory, Vol. 68, pp. 258–265.

[14] Sandholm, W. H. (2001) "Potential Games with Continuous Player Sets". Journal of Economic Theory, Vol. 97, pp. 81–108.

[15] Shamma, J. S. and G. Arslan (2003) "Unified Convergence Proofs of Continuous-time Fictitious Play". mimeo.

[16] Swinkels, J. M. (1992) "Evolutionary Stability with Equilibrium Entrants". Journal of Economic Theory, Vol. 57, pp. 306–332.

[17] Takahashi, S. (2004) "Perfect Foresight Dynamics in Two-Player Games and Time Symmetry". mimeo.