The Number of Public Goods: Are Two Public Goods Too Many?

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Abstract

This paper shows that there exists a basic difficulty for us to deal with multiple public goods in the theory of private provision of public goods. We can deal with two situations, i.e., either the situation where one individual contributes multiple public goods or that where many individuals contribute one kind of public good. This implies that the present state of theory does not describe a phenomenon where more than two number of individuals contribute multiple public goods, which is usually seen in real life.

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1 Introduction

In this paper, we will establish the fundamental fact that *the number of public goods contributed by many individuals must be unity* in almost all the models on the private provision of public goods. Here, the individual is said to *contribute a public good* when he supplies the public good by the positive amount.

It may appear easy for the economists in the theory of private provision of public goods to expand the model containing one public good to that containing multiple public goods. Kemp (1984) constructs a model where every individual contributes multiple public goods in the equilibrium. The situation is scrutinized by Bergstrom-Blume-Varian (1986) where multiple individuals contribute multiple public goods. Our result, on the other hand, shows that those economies are almost vacuous. That is, let H and D be a group consisting of two or more individuals and an arbitrary set of multiple public goods respectively. Then we can not deal with the economies where each individual in H contributes every public good in D. This result is established by a simple fact that the number of equations is strictly greater than the number of unknowns. We must note that

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the result does not imply that Nash equilibria do not exist but that the the number of contributors to public goods is limited in Nash equilibria. Warr (1983) suggests that the number of equations exceed that of unknowns when all the individuals were contributors to every public good. Our result is a generalization of his in two respects. One is that non-contributors to the public goods can exist in our setting. The other is that no operations concerning the differentiability of functions are exploited.

On the other hand, we daily encounter the situations where many persons contribute many public goods at the same time. This fact may be interpreted as the models on private provision of public goods still have a big problem to be solved.

2 The Number of Public Goods

Consider an economy which contains one private good and m kinds of public goods. Denote the index set of individuals and the set of public goods by $N \overset{\text{def}}{=} \{1, 2, \ldots, n\}$ and $M \overset{\text{def}}{=} \{1, 2, \ldots, m\}$ respectively[1]. An individual i's utility function is represented by $u_i : (x_i, G_1, \ldots, G_m) \in \mathbb{R}^{m+1}_{+} \mapsto u_i(x_i, G_1, \ldots, G_m) \in \mathbb{R}$, where x_i and $G_k, k \in M$ are the amount of private good and that of the k-th public good respectively[2]. Let $I \overset{\text{def}}{=} (I_1, \ldots, I_n) \in \mathbb{R}^n_{++}$ be a given income distribution. A list $(N, (u_i, I_i)_{i \in N})$ is an *economy*. Producing one unit of each public good requires one unit of private good. The price of the private good is unity.

We assume:

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[1]The symbol "$\overset{\text{def}}{=}$" implies that the left hand side is defined by the right hand side.

[2]The sets $\mathbb{R}$, $\mathbb{R}^{\ell}_{+}$ and $\mathbb{R}^{\ell}_{++}$ are the set of real numbers, ℓ dimensional non-negative vectors and ℓ dimensional strictly positive vectors, respectively.

Assumption 1 *The utility function $u_i(x_i, G_1, \ldots, G_m)$ is continuous, increasing, quasi-concave in $\mathbb{R}_+^{m+1}$, strictly increasing and strictly quasi-concave in $\mathbb{R}_{++}^{m+1}$, for $i \in N$.*

Denote the amount of contribution to the k-th public good by individual j as g_k^j. Suppose that vectors of all the individuals' contributions to public goods $g_k \stackrel{\text{def}}{=} (g_k^1, \ldots, g_k^n)$, $k \in M$ are given. We consider a maximization problem for each $i \in N$:

$$\left.\begin{array}{c} \max_{x_i, \tilde{g}_1^i, \ldots, \tilde{g}_m^i} u_i(x_i, \sum_{j \neq i} g_1^j + \tilde{g}_1^i, \ldots, \sum_{j \neq i} g_m^j + \tilde{g}_m^i) \\ \text{sub. to} \\ x_i + \tilde{g}_1^i + \cdots + \tilde{g}_m^i = I_i, \ \tilde{g}_k^i \geqq 0, k \in M \end{array}\right\}. \tag{1}$$

Denote the solution to (1) by $(x_i(g_1, \ldots, g_m), (\psi_k^i(g_1, \ldots, g_m))_{k \in M})$, $i \in N$. $x_i(g_1, \ldots, g_m)$ is the amount of private good of individual i and $\psi_k^i(g_1, \ldots, g_m)$ is that of k-th public good.

Let us consider the following maximization problem for each i:

$$\max_{x_i, G_1, \ldots, G_m} u_i(x_i, G_1, \ldots, G_m) \ \text{ sub. to } x_i + G_1 + \cdots + G_m = Y_i, \tag{2}$$

where Y_i is a positive real. The problem is auxiliary or artificial but turns out to be very useful later for us to know the characteristics of solution to (1). The solution to problem (2) is denoted by $(\xi^i(Y_i), \phi_k^i(Y_i))$, $k \in M, i \in N$, where the values of ξ^i and ϕ_k^i correspond to the amount of private good and the amount of k-th public good respectively.

Definition 1 [Nash Equilibrium]

An allocation $((x_i^*)_{i \in N}, (g_k^*)_{k=1}^m)$ is a Nash equilibrium in an economy $(N, (u_i, I_i)_{i \in N})$ when

$$\psi_k^i(g_1^*, \ldots, g_m^*) = g_k^{i*}, \ \forall i \in N, \ k \in M \tag{3}$$

$$x_i^* = x_i(g_1^*, \ldots, g_m^*), \ \forall i \in N. \tag{4}$$

2.1 Multiple Public Goods Economy

Consider simultaneous equations with respect to unknowns $Y_i, i \in N$:

$$\phi_k^i(Y_i) = \phi_k^j(Y_j), \ \forall i, j \in N, \ k \in M \tag{5}$$

$$\phi_1^1(Y_1) + \cdots + \phi_m^1(Y_1) = \sum_{i=1}^{n} \big(I_i - \xi^i(Y_i)\big) \tag{6}$$

Then we can establish the *equivalence theorem* which is stated as follows:

Theorem 1 *Suppose that Assumption* 1 *holds. A necessary and sufficient condition for a Nash equilibrium* $((x_i^*)_{i \in N}, (g_k^*)_{k \in M})$ *satisfying* $x_i^* > 0$ *and* $g_k^{i*} > 0, k \in M, i \in N$ *to exist, is that there exists a positive solution* $(Y_i^*)_{i \in N}$ *satisfying* $I_i - \xi^i(Y_i^*) > 0$ *and* $\xi^i(Y_i^*) > 0$, $i \in N$ *to the system* (5) *and* (6).

Proof. [**Necessity**] Let $((x_i^*)_{i \in N}, (g_k^*)_{k \in M}) \in \mathbb{R}_{++}^{nm+n}$ be the Nash equilibrium satisfying $x_i^* > 0$ and $g_k^{i*} > 0$, $k \in M$, $i \in N$. Define $Y_i^* \stackrel{\text{def}}{=} I_i + \sum_{k=1}^{m} \sum_{j \neq i} g_k^{j*}$, $i \in N$ and $G_k^* \stackrel{\text{def}}{=} \sum_{j \in N} g_k^{j*}$, $k \in M$. We have $Y_i^* = x_i^* + G_1^* + \cdots + G_m^*$, since $I_i = x_i^* + \sum_{j \in M} g_j^{i*}$. Let $(x_i^{\#}, G_1^{\#}, \ldots, G_m^{\#})$ be a solution to problem (2) when $Y_i = Y_i^*$. Then the inequality $u_i(x_i^*, G_1^*, \ldots, G_m^*) \leqq u_i(x_i^{\#}, G_1^{\#}, \ldots, G_m^{\#})$ holds. Suppose that $u_i(x_i^*, G_1^*, \ldots, G_m^*) < u_i(x_i^{\#}, G_1^{\#}, \ldots, G_m^{\#})$. Define $x_i(\lambda)$ and $G_k(\lambda)$ for λ satisfying $0 < \lambda < 1$ as

$$x_i(\lambda) \stackrel{\text{def}}{=} \lambda x_i^{\#} + (1 - \lambda) x_i^*,$$

$$G_k(\lambda) \stackrel{\text{def}}{=} \lambda G_k^{\#} + (1 - \lambda) G_k^*, \ k \in M.$$

We have $x_i(\hat{\lambda}) > 0$ and $G_k(\hat{\lambda}) > \sum_{j \neq i} g_k^{j*} > 0, k \in M$ for a sufficiently small $\hat{\lambda}$, since $x_i^* > 0$ and $G_k^* > \sum_{j \neq i} g_k^{j*} > 0, k \in M$. Therefore both points $(x_i^*, G_1^*, \ldots, G_m^*)$ and $(x_i(\hat{\lambda}), G_1(\hat{\lambda}), \ldots, G_m(\hat{\lambda}))$ are interior points of the domain of utility function. Due to strict quasi-concavity of utility functions, we have

$$u_i(x_i^*, G_1^*,\ \ldots, G_m^*) < u_i(x_i(\hat{\lambda}), G_1(\hat{\lambda}), \ldots, G_m(\hat{\lambda})).$$

Moreover, since $x_i(\hat{\lambda}) + G_1(\hat{\lambda}) + \cdots + G_m(\hat{\lambda}) = Y_i^*$, we have for sufficiently small $\hat{\lambda}$

$$\begin{aligned}
&\hat{g}_k^i \stackrel{\text{def}}{=} G_k(\hat{\lambda}) - \sum_{j\neq i} g_k^{j*} > 0, k \in M \\
&x_i(\hat{\lambda}) + \sum_{k=1}^{m} \left(G_k(\hat{\lambda}) - \sum_{j\neq i} g_k^{j*} \right) = I_i, \\
&u_i(x_i(\hat{\lambda}), G_1(\hat{\lambda}), \ldots, G_m(\hat{\lambda})) = u_i(x_i(\hat{\lambda}), \sum_{j\neq i} g_1^{j*} + \hat{g}_1^i, \ldots, \sum_{j\neq i} g_m^{j*} + \hat{g}_m^i).
\end{aligned}$$

This contradicts the fact that $((x_i^*)_{i=1}^n, (g_k^*)_{k\in M})$ is the Nash equilibrium. Therefore the $m+1$-tuple $(x_i^*, G_1^*, \ldots, G_m^*)$ is a solution to the problem (2) when $Y_i = Y_i^*$. This is true for any i. Therefore, $(Y_i^*)_{i\in N}$ is the solution to the simultaneous equations (5) and (6).

[**Sufficiency**] Let $(Y_i^*)_{i\in N}, i \in N$ be the positive solution to (5) and (6) satisfying $I_i - \xi^i(Y_i^*) > 0$, $i \in N$. Note that

$$\left(T \stackrel{\text{def}}{=}\right) \sum_{i=1}^{n} (I_i - \xi^i(Y_i^*)) = \sum_{k=1}^{m} \phi_k^1(Y_1^*) > 0.$$

Define for each $i \in N$ and $k \in M$

$$g_k^{i*} \stackrel{\text{def}}{=} \frac{1}{T} \left(I_i - \xi^i(Y_i^*)\right) \phi_k^1(Y_1^*) > 0, \quad x_i^* \stackrel{\text{def}}{=} \xi^i(Y_i^*) > 0.$$

By definition, we have

$$\begin{aligned}
x_i^* + g_1^{i*} + \cdots + g_m^{i*} &= \xi^i(Y_i^*) + \sum_{k=1}^{m} \frac{1}{T} \left(I_i - \xi^i(Y_i^*)\right) \phi_k^1(Y_1^*) \\
&= \xi^i(Y_i^*) + I_i - \xi^i(Y_i^*) = I_i,\ i \in N, \\
G_k^* \stackrel{\text{def}}{=} \sum_{j\in N} g_k^{j*} &= \frac{1}{T} \sum_{j=1}^{n} \left(I_j - \xi^j(Y_j^*)\right) \phi_k^1(Y_1^*) = \phi_k^i(Y_i^*),\ i \in N, k \in M.
\end{aligned}$$

This implies that $(x_i^*, g_1^{i*}, \dots, g_m^{i*})$ satisfies the budget constraint in (1). Let $(x_i^{\#}, g_1^{i\#}, \dots, g_m^{i\#})$ be a solution to the following problem:

$$\max u_i\left(x_i, G_1^{-i*} + g_1^i, \dots, G_m^{-i*} + g_m^i\right) \text{ sub. to } \; x_i + g_1^i + \cdots + g_m^i = I_i,$$

where $G_k^{-i*} \stackrel{\text{def}}{=} \sum_{j \neq i} g_k^{j*}$, $k \in M$. It is obvious that $u_i(x_i^{\#}, G_1^{-i*} + g_1^{i\#}, \dots, G_m^{-i*} + g_m^{i\#}) \geqq u_i(x_i^*, G_1^{-i*} + g_1^{i*}, \dots, G_m^{-i*} + g_m^{i*})$. Suppose that the strict inequality were true, i.e. $u_i(x_i^{\#}, G_1^{-i*} + g_1^{i\#}, \dots, G_m^{-i*} + g_m^{i\#}) > u_i(x_i^*, G_1^{-i*} + g_1^{i*}, \dots, G_m^{-i*} + g_m^{i*})$. Define $x_i(\lambda) \stackrel{\text{def}}{=} \lambda x_i^{\#} + (1-\lambda)x_i^*$, $G_k(\lambda) \stackrel{\text{def}}{=} G_k^{-i*} + \lambda g_k^{i\#} + (1-\lambda) g_k^{i*}$, $k \in M$ for any λ satisfying $0 < \lambda < 1$. It is obvious that $x_i(\lambda) > 0$ and $G_k(\lambda) > 0$ since $x_i^* > 0$ and $g_k^{i*} > 0$, $k \in M$. By the strict quasi-concavity in the interior of the domain of utility function, we have $u_i(x_i(\lambda), G_1(\lambda), \dots, G_m(\lambda)) > u_i(x_i^*, G_1^*, \dots, G_m^*)$. On the other hand, we obtain:

$$\begin{aligned}
x_i(\lambda) + \sum_{k=1}^m G_k(\lambda) &= \lambda x_i^{\#} + (1-\lambda)x_i^* + \sum_{k=1}^m \left(G_k^{-i*} + \lambda g_k^{i\#} + (1-\lambda) g_k^{i*}\right) \\
&= \lambda(x_i^{\#} + \sum_{k=1}^m g_k^{i\#}) + (1-\lambda)(x_i^* + \sum_{k=1}^m g_k^{i*}) + \sum_{k=1}^m G_k^{-i*} \\
&= I_i + \sum_{k=1}^m G_k^{-i*} = x_i^* + \sum_{k=1}^m \sum_{j=1}^n g_k^{j*} = Y_i^*.
\end{aligned}$$

This is a contradiction. And thus, $(x_i^*, g_1^{i*}, \dots, g_m^{i*})$ is the solution to (1) when a list $(g_1^*, \dots, g_m^*)$ is given. Therefore, $((x_i^*)_{i \in N}, (g_k^*)_{k=1}^m)$ is a Nash equilibrium. ∎

The above theorem implies that the Nash equilibrium where all the individuals contribute all the public goods can be fully described by the system (5) and (6). There are n kinds of unknowns: $Y_1, \dots, Y_n$, in the simultaneous equations (5) and (6). On the other hand, there are $m(n-1)+1$ equations in (5) and (6). The number of equations coincides with that of unknowns if and only if $m = 1$. In the general case containing multiple public goods, the number of unknowns is strictly less than that of equations. There are no solutions to the system (5) and (6) but exceptional cases.

We are to show the essence of the above theorem by way of an example. Let $n = m = 2$ and let utility functions of individuals be of Cobb-Douglas type:

$$u_i(x_i, G_1, G_2) = (x_i)^{\alpha^i}(G_1)^{\beta^i}(G_2)^{\gamma^i},$$
$$\alpha^i + \beta^i + \gamma^i = 1, \alpha^i > 0, \beta^i > 0, \gamma^i > 0, i = 1, 2.$$

The equations (5) and (6) are represented by:

$$\beta^1 Y_1 = \beta^2 Y_2$$
$$\gamma^1 Y_1 = \gamma^2 Y_2$$
$$\beta^1 Y_1 + \gamma^1 Y_1 = I_1 + I_2 - \alpha^1 Y_1 - \alpha^2 Y_2.$$

Clearly, there exist no solutions unless $\beta^1/\beta^2 = \gamma^1/\gamma^2$.

The above theorem leads us to the conclusion that the standard models in the theory of private provision of public goods cannot describe the situation that all the households contribute all the public goods. Next question we want to ask is whether the multiple households can contribute multiple public goods or not.

Let $((x_i^*)_{i\in N}, (g_k^*)_{k\in M})$ be a Nash equilibrium in the economy $(N, (u_i, I_i)_{i\in N})$. Let N' and M' be arbitrary subsets of N and M, respectively. We assume without loss of generality that $N' = \{1, \ldots, n'\}$, $2 \leqq n' < n$ and $M' = \{1, \ldots, m'\}$, $1 \leqq m' < m$. We can define a subeconomy containing n' persons and m' public goods of the economy

$(N, (u_i, I_i)_{i \in N})$. Define utility functions and incomes as:

$$v_i(x_i, G_1, \ldots, G_{m'}) \stackrel{\text{def}}{=} u_i\big(x_i, G_1 + \sum_{j \notin N'} g_1^{j*}, \ldots, G_{m'} + \sum_{j \notin N'} g_{m'}^{j*}, \sum_{j \in N} g_{m'+1}^{j*}, \ldots, \sum_{j \in N} g_m^{j*}\big),$$

$$y_i \stackrel{\text{def}}{=} I_i - \sum_{k \notin M'} g_k^{i*}, \qquad i \in N'.$$

The economy $(N', (v_i, y_i)_{i \in N'})$ thus derived is an economy containing n' number of individuals and m' number of public goods, the Nash equilibrium of which is $((x_i^*)_{i \in N'}, (g_k^*)_{k \in M'})$. Applying Theorem 1 to the economy $(N', (v_i, y_i)_{i \in N'})$, we have the equations corresponding to (5) and (6), which contain $m'(n'-1)+1$ unknown variables in the system consisting of n' number of equations. We must say that we cannot always solve the system when $m' > 1$. This discussion does not depend on the choice of the sets M' and N'.

Summarizing above, we can state:

Theorem 2 *Suppose that Assumption* 1 *holds. Any plural individuals cannot contribute any multiple public goods at Nash equilibria in almost all the economies.*

We can restate the above theorems in the most extreme form as:

> **We cannot necessarily depict an economy with n individuals and m public goods where two individuals contribute two public goods in Nash equilibrium.**

2.2 Two Public Goods Economy

In this section we scrutinize further the problem on the number of public goods. Our remaining problem is to answer whether one individual can contribute many public goods or not. We study the problem by introducing two public goods subeconomy of $(N, (u_i, I_i)_{i\in N})$ such as in the previous subsection.

Let $((x_i^*)_{i\in N}, (g_k^*)_{k\in M})$ be a Nash equilibrium in an economy $(N, (u_i, I_i)_{i\in N})$. We restrict the economy $(N, (u_i, I_i)_{i\in N})$ containing m kinds of public goods into the one containing two public goods. Define:

$$w_i(x_i, G_1, G_2) \stackrel{\text{def}}{=} u_i\left(x_i, G_1, G_2, \sum_{j=1}^{n} g_3^{j*}, \ldots, \sum_{j=1}^{n} g_m^{j*}\right),\ i \in N$$

$$y_i \stackrel{\text{def}}{=} I_i - \sum_{k\neq 1,2} g_k^{i*}$$

The economy $(N, (w_i, y_i)_{i\in N})$ thus derived is an economy with two public goods, where $((x_i^*)_{i\in N}, (g_k^*)_{k=1}^2)$ is a Nash equilibrium.

Define three types of the contributors to public goods as:

$$\begin{aligned}
J &\stackrel{\text{def}}{=} \{i \in N \mid g_1^{i*} > 0, g_2^{i*} > 0\}, \\
J_1 &\stackrel{\text{def}}{=} \{i \in N \mid g_1^{i*} > 0, g_2^{i*} = 0\}, \\
J_2 &\stackrel{\text{def}}{=} \{i \in N \mid g_1^{i*} = 0, g_2^{i*} > 0\}.
\end{aligned}$$

The key problem (2) can be rewritten according to the sets of contributors.

$$\max_{x_i, G_1, G_2} w_i(x_i, G_1, G_2) \ \text{ sub. to } x_i + G_1 + G_2 = Y_i,\ i \in J, \tag{7}$$

$$\max_{x_i, G_1} w_i(x_i, G_1, G_2^*) \ \text{ sub. to } x_i + G_1 = Y_i,\ i \in J_1, \tag{8}$$

$$\max_{x_i, G_2} w_i(x_i, G_1^*, G_2) \ \text{ sub. to } x_i + G_2 = Y_i,\ i \in J_2. \tag{9}$$

Denote the solutions to these problems by $\xi^i(Y_i)$, $\phi_1^i(Y_i)$ and $\phi_2^i(Y_i)$. Consider simultaneous equations with respect to unknowns $Y_i, i \in J \cup J_1 \cup J_2$ as follows:

$$\phi_1^i(Y_i) = \phi_1^j(Y_j), \ \forall i, j \in J \cup J_1 \tag{10}$$

$$\phi_2^i(Y_i) = \phi_2^j(Y_j), \ \forall i, j \in J \cup J_2 \tag{11}$$

$$\phi_1^i(Y_i) + \phi_2^i(Y_i) = \sum_{j \in J \cup J_1 \cup J_2} \left(y_j - \xi^j(Y_j)\right), \ \forall i \in J \tag{12}$$

$$\phi_1^i(Y_i) = \sum_{j \in J} \left(y_j - \xi^j(Y_j) - g_2^{j*}\right) + \sum_{j \in J_1} \left(y_j - \xi^j(Y_j)\right), \ \forall i \in J_1 \tag{13}$$

$$\phi_2^i(Y_i) = \sum_{j \in J} \left(y_j - \xi^j(Y_j) - g_1^{j*}\right) + \sum_{j \in J_2} \left(y_j - \xi^j(Y_j)\right), \ \forall i \in J_2. \tag{14}$$

We establish a theorem which is stated as follows:

Theorem 3 *Suppose that Assumption* 1 *holds. If there exist a Nash equilibrium* (x_i^*, g_k^{i*}), $k = 1, 2, \ i \in N$ *in the economy* $(N, (w_i, y_i)_{i \in N})$*, then there exists a solution* $Y_i^*, i \in J \cup J_1 \cup J_2$ *to the simultaneous equations* (10)*,* (11)*,* (12)*,* (13) *and* (14)*.*

Proof. Let $((x_i^*)_{i \in N}, (g_k^*)_{k=1}^2)$ be a Nash equilibrium. Define

$$Y_i^* \stackrel{\text{def}}{=} y_i + \sum_{j \neq i} (g_1^{j*} + g_2^{j*}), \ i \in J$$

$$Y_i^* \stackrel{\text{def}}{=} y_i + \sum_{j \neq i} g_k^{j*}, \ i \in J_k, k = 1, 2.$$

Since $G_k^* \stackrel{\text{def}}{=} \sum_{i \in J \cup J_k} g_k^{i*}, \ k = 1, 2$, it is clear that

$$\begin{aligned} Y_i^* &= x_i^* + G_1^* + G_2^*, && \text{if } i \in J \\ &= x_i^* + G_1^*, && \text{if } i \in J_1 \\ &= x_i^* + G_2^*, && \text{if } i \in J_2. \end{aligned}$$

We can repeat the same argument as in the proof of Theorem 1 to show that (x_i^*, G_1^*, G_2^*) is a solution to the maximization problem (7) for the individual $i \in J$ when $Y_i = Y_i^*$. By the similar procedure, for the individual $i \in J_1$, (x_i^*, G_1^*) is a solution to the maximization problem (8) when $Y_i = Y_i^*$. Similarly for the individuals $i \in J_2$, (x_i^*, G_2^*) is a solution to the problem (9) when $Y_i = Y_i^*$. The above discussion implies that $Y_i^*,\ i \in J \cup J_1 \cup J_2$ form a solution to the simultaneous equations (10), (11), (12), (13), and(14). ∎

By Theorem 3, a Nash equilibrium is characterized by the equations (10), (11), (12), (13) and (14). The equations depend on the sets J, J_1 and J_2 which is determined by the Nash equilibrium. Therefore, they may be impractical when we are to find a Nash equilibrium. However they are pertinent to our present objective, i.e. for us to make it clear whether one individual can contribute many public goods.

The numbers of equations in (10) and (11) are $\#J + \#J_1 - 1$ and $\#J + \#J_2 - 1$ respectively[3]. The number of equations in (12), (13) and (14) is three. In short, the number of unknowns $Y_i, i \in J \cup J_1 \cup J_2$ is $\#J + \#J_1 + \#J_2$ whereas the number of equations is $2 \times \#J + \#J_1 + \#J_2 + 1$.

Let us examine whether the numbers of these equations and unknowns are identical. We distinguish three cases (i) $\#J = 0$, (ii) $\#J = 1$ and (iii) $\#J \geqq 2$.

[Case 1: $\#J = 0$] In this case the equation (12) is null. Therefore the total number of the equations is equal to $\#J_1 + \#J_2$ which is the number of the unknowns. In other words, Nash equilibria can exist when all the individuals contribute to a single public good.

[Case 2: $\#J = 1$] If either $J_1 \neq \emptyset$ or $J_2 \neq \emptyset$ holds, the number of equations in (10), (11) and (12) is equal to $(1 + \#J_1 - 1) + (1 + \#J_2 - 1) + 1 = \#J + \#J_1 + \#J_2$. Therefore at least one of the equations (13) and (14) is redundant. In other words, there exist no Nash equilibria when $J_1 \neq \emptyset$ or $J_2 \neq \emptyset$. The remaining case is $J_1 = \emptyset$ and $J_2 = \emptyset$. In this case,

[3] Let A be a set. The symbol "$\#A$" represents the cardinality of the set A.

the equations (10) and (11) are null. In addition to this, the two equations (13) and (14) are equivalent to (12). Hence the number of unknowns coincides with that of equations.

[Case 3: $\#J \geqq 2$] The total number of equations in (10), (11) and (12) is $2 \times \#J + \#J_1 + \#J_2 - 1$ which is strictly greater than $\#J + \#J_1 + \#J_2$. Therefore the equations (13) and (14) are redundant. Thus there do not necessarily exist Nash equilibria.

In the above discussion, the choice of public goods 1 and 2 is arbitrary in defining the subeconomy $(N, (w_i, y_i)_{i \in N})$. Above inference applies to a subeconomy containing any pair of two public goods. Therefore, we can conclude as follows.

> **Three kinds of equilibria exist possibly in the models of private provision of public goods. One is the equilibrium where one individual contributes to multiple public goods. Another is that where many individuals contribute to one particular public good in equilibrium. The other is that where the former two situations can coexist.**

References

Bergstrom, T.C., Blume, L., Varian, H., 1986. On the private provision of public goods. Journal of Public Economics 29, 25-49.

Kemp, M.C., 1984. A note on the theory of international transfers. Economics Letters 14, 259-262.

Warr, P.G., 1983. The Private Provision of a Pure Public Good Is Independent of the Distribution of Income. Economics Letters 13, 207-211.