Is Bigger Better? Customer base expansion through word of mouth reputation*

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Abstract

A model of gradual reputation formation through a process of continuous investment is developed. We assume that the ability to produce high-quality products requires continuous investment and that as a consequence of informational frictions, such as search costs, information about firms' past performance diffuses only gradually in the market. This leads to a dual process of growth of a firm's customer base, and a steady increase in the firm's investment in quality. As long as a firm continues to deliver high quality products, its reputation as a high quality firm grows, new customers are attracted and the firm increases in size. However, if quality deteriorates, the firm's customer base shrinks and remains stagnant until it is able to resurrect its reputation through successful investment. Since a good reputation is costly to acquire and takes a long time to regain once it

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has been lost, it becomes increasingly valuable the longer a firm's tenure as a high quality producer. Therefore, the longer its tenure, the more a firm stands to lose from tarnishing its reputation and, hence, the more it invests to maintain it.

Key words: Reputation, Moral hazard, Investment in quality.

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1. Introduction

A firm's reputation is often its most valuable asset. For example, if a corporate giant like Coca Cola, McDonald's or Nike were stripped of its name - and the reputational resources associated with it - its value would be reduced to only a small fraction of what it is today. The importance of a firm's name and reputation for its balance sheet suggests that considerable managerial resources are devoted to establishing, maintaining, and enhancing the value of the firm's name and reputation. The goal of this paper is to develop a modeling framework in which a firm regards its reputation as a capital asset whose value is maintained through a process of active and continuous investment.

We consider a market for an experience good whose quality is unobserved at the time of purchase, but becomes known after purchase and consumption. It is natural for consumers of such goods to base their purchase decision on a firm's reputation, i.e., on its track record of delivered qualities. The model we develop here is based on this hypothesis and on two additional assumptions. First, we assume that a firm's ability to produce high quality products requires continuous investment in quality. Second, we assume that consumers are differentially informed about a firm's reputation. If a consumer buys from some firm (in some period), she knows the product quality that this firm delivered, but not the product qualities that other firms delivered to other consumers. Nonetheless, information about delivered product qualities spreads (gradually) in the market through word of mouth reputation. This enables a firm that consistently delivers high quality products to attract more consumers over time, increasing thereby the number of consumers who know the firm's reputation, and are likely to transact with it.

In this situation a firm is characterized not only by its technology (which is the same for all firms), but also by the size of its customer base, i.e., by the number of consumers who are aware of this firm's reputation. As a result, if we consider the cross section of firms in the industry at a given point in time, we see a distribution of firms according to their sizes. On the other hand, if we consider a single firm, we see that this firm's size fluctuates over time, depending on the realized sequence of product qualities it delivers.

Given this endogenous heterogeneity of firm sizes, our main objective is to determine the relationship between size and investment in quality. In particular, we address the question whether firms with large customer bases tend to invest much in quality, enhance their reputation, and thereby further increase the size of their customer base; or whether they tend to invest little in quality, milk their reputation, and lose some (or all) customers. Our main finding is that (at least in some equilibria) the former holds, i.e., that investment in quality is positively related to size. The reason for this is that reputation is costly to acquire and takes a long time to regain once it has been lost. Consequently, a good reputation is more valuable to a firm the larger its customer base is. But then, the longer its tenure as a high quality producer - and hence the larger its size - the more a firm stands to lose from tarnishing its reputation. As a result, the model predicts that the larger is a firm, the more it invests in quality, and the higher is the average quality it delivers.

In a related version of the model we consider entry and exit. As one might anticipate, exit is driven by an adverse quality shock that reduces the firm's future profits (as a result of losing customers), and the fact that each firm has to pay a fixed cost in each period that it stays in the industry. Entry, on the other hand, is driven by the fact that entering firms have a technology, which, on average, is better than the technology of exiting firms. What we show in this version of the model is that investment in quality, and hence average quality, are positively correlated with *age* (they continue to be correlated with size, since, as will be seen, age and size are themselves positively correlated). Hence, this version of the model generates the prediction that older firms deliver higher quality products.

The association between age and/or firm size and quality, predicted by our model, seems to fit the observation that producers of high quality products with a long history in the market tend to emphasize this characteristic in their advertising. For example, the New York Times heralds the year in which it was founded on its front page, and European quality beers vaunt the year in which the brand was established on their label. Similarly, advertising often seems to signal quality through market share. For example, the Hertz ad: "We're number one."

More systematic evidence supporting our results is found in empirical work on the experience rating scheme used by e-Bay. e-Bay gives buyers and sellers the opportunity to send feedbacks regarding the experience they have had with their trading partner. The feedback is in the form of a 'positive/negative/neutral' grade, and more extensive commentary (if desired) about the transaction. Various statistics of the results of these feedbacks are electronically posted by e-Bay, allowing future transactions to be informed by past transactions. If these statistics are interpreted as a seller's reputation, an empiricist is able to study the strategic response of buyers and sellers to reputation. One recent study that does that is Cabral and Hortacsu (2004). Analyzing panel data on sellers, Cabral and Hortacsu find that the growth rate of a seller's transactions drops significantly following the first negative feedback from buyers. They also find that the rate of arrival of negative feedbacks increases following the first negative feedback. These findings mirror the equilibrium behavior of firms in our model. Indeed, the equilibrium we construct is such that sullied reputation leads to a loss of market share, and to a reduction in the firm's investment in quality.

Brief Literature Review. This paper relates to several strands of literature. The first strand emanates from the (repeated) game theoretic idea that reputation (histories of realized actions) can be used to solve a moral hazard problem. Applications of this idea to quality provision problems similar to the one here is found in papers by Klein and Leffler (1981) and Shapiro (1983). A second strand of literature integrates elements of adverse selection into similar repeated game scenarios (we consider, however, a pure moral hazard scenario). Of the many applications in this strand of literature, the ones that seem related to the substantive issues we address here are Diamond (1989), Mailath and Samuelson (2001), Horner (2002), and Tadelis (2002). A third strand of relevant literature that deals with consumer dynamics (without reputation effects) is found in Radner (2003) and references therein. Finally, there is a strand of IO-theoretic literature (again without reputation effects) that considers size advantages that stem from other sources (e.g., economies of scale), a representative paper in this literature being Bagwell *et al.* (1998).

What differentiates our approach from all these papers is that reputation in our model spreads in the market through word of mouth, or referrals - consumers tell other consumers about their experience, causing some firms to grow and other firms to decline. As a consequence, a firm starts out small, grows gradually, and changes its investment as its reputation is established. These interrelated processes of firm growth, reputation formation, and the links between age, size, and investment in quality represent our main contribution to the literature.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 proves that a reputation equilibrium exists, and delineates its properties. Section 4 integrates entry and exit into the basic model, and discusses the relationship of the model to classical issues in industry dynamics. All proofs are found in a technical appendix.

2. The Model

Time is discrete and the horizon is infinite. There is a continuum of firms and consumers, both of measure 1.

Production and consumption. Firms are infinitely lived and produce either high or low quality products. All units that a firm produces within one period are of the same quality. The probability that a firm produces high quality products depends on how much it invests in quality in that period. Specifically, if a firm invests x at the beginning of a period, it produces high quality products with probability $q = f(x)$.

Each period's investment is restricted to $x \in [0, \overline{x}]$ with $\overline{x} < \infty$. f is strictly concave, strictly increasing, continuously differentiable, and $f(\overline{x}) < 1$. x is a fixed cost that affects the quality of all units produced, and has no effect on variable cost. Symmetrically, a firm's variable cost has no effect on quality. We normalize it to be zero.

A firm's *state* (sometimes called "tenure") t is the number of periods since it last produced a low quality product. If $t = 1$, the firm produced a low quality product last period; if $t = 2$, the firm produced a high quality product last period, but a low quality product in the period preceding it, etc. The investment of a firm depends (potentially) on its state, denoting it as x_t.

The investment of a firm x_t is its own private information. When a firm chooses some x_t, neither it nor consumers know the product quality which is about to be realized. After the product is bought and consumed, however, its quality becomes known to the firm and to consumers who *bought* it.

Consumers live one period, are price takers, and have identical downward sloping demand curves that come from expected utility maximization. Specifically, each consumer derives utility $u_H(z)$ from $z \geq 0$ units of the high quality, and utility $u_L(z)$ from z units of the low quality, product. Then, if a consumer faces probability q of getting the high quality product and a per unit price p for it, she chooses a z that maximizes

$$qu_H(z) + (1-q)u_L(z) - pz. \tag{2.1}$$

We assume that u_i's are strictly increasing, strictly concave, continuously differentiable, and that $u_H(0) \geq u_L(0) \geq 0$, $u'_H(z) > u'_L(z)$ for all $z > 0$, and $u'_H(0), u'_L(0) < \infty$. We let $D(p; q)$ be the maximizer of (2.1), which is a consumer's demand function. We let $S(p; q)$ be the maximized value of (2.1), which is a consumer's surplus.

Let

$$\Pi(p; q) \equiv pD(p; q)$$

be the per consumer period profit function (same as revenue since variable cost is assumed to be zero). We assume that $\Pi(\cdot; q)$ is single peaked for each q, and let $p(q)$ be the maximizer and $\pi(q)$ the maximized value of $\Pi(\cdot; q)$. We also let

$$s(q) \equiv S(p(q), q)$$

be a consumer's surplus under monopoly pricing.

If a consumer does not buy the product at all, she gets zero utility. Since u_L is strictly increasing in z, there is a $p > 0$ so that $D(p; 0) > 0$ and, consequently, $s(0)$ and $\pi(0)$ are positive. This means that even a low quality product generates positive sales, positive period profits, and a positive consumer's surplus. We also assume that $p(q)$, $\pi(q)$ and $s(q)$ are strictly increasing in q. Therefore, if a firm prices its product monopolistically, social surplus increases in quality,[1] and this increase is shared by firm and consumers.

Search and matching. At each period a new generation of consumers of measure 1 enters the market. Each consumer lives one period. Upon exiting the market an old consumer meets with probability δ, $0 < \delta < 1$, a new consumer and tells her which firm she bought from and which state this firm is in. This firm is called the (new) consumer's referral firm. The consumer can then become a customer of her referral firm, which means she does not transact with any other

[1] To abbreviate the language, the probability of getting the high quality product q is sometimes referred to as 'quality.'

firm. Alternatively, she can sample a randomly selected firm, learn its state, and become this firm's customer. This (other) firm is called the consumer's search firm. There is no going back to a referral firm if the consumer chooses to search. Search is costless, but only one search per period is feasible. If a new consumer does not meet an old consumer, the only way to locate a firm is by searching.

In this environment consumers search either if they have no referral firm, or, if they expect to get a higher consumer surplus (on average) from a search firm than from their referral firm. We assume searching consumers are divided uniformly across firms, i.e., each firm receives the same measure of searching consumers, which we denote by n (n is endogenously determined). By the assumption on the meeting technology, n is no less than $1-\delta$, which is positive.

Once search activity is over and consumers are matched to firms, a state t firm has a customer base, the size of which is determined by this firm's last period customer base (which determines the number of referrals this firm gets), by the decisions of consumers that are referred to it whether to transact or search, and by the way the population of searching consumers is divided between firms. We denote the measure of a state t firm's customer base by b_t, and specify momentarily a formula that links b_t to these three variables.

Within period profit maximization. Consider a state t firm. Then, all customers in this firm's customer base maintain the same belief, call it q_t, about the probability with which this firm delivers high quality products.[2] Given q_t and b_t, the overall period demand $b_t D(p; q_t)$ that a state t firm faces is determined. We assume that each firm enjoys monopoly power with respect to consumers in its customer base (these consumers are captive once search activity is over) and, hence, acts as a price setter. A state t firm chooses then the profit maximizing

[2] In equilibrium this belief equals the actual probability with which high quality products are delivered by a state t firm, $q_t = f(x_t)$. At this point however we have not defined an equilibrium yet, so we treat q_t as an independent variable.

price of $p(q_t)$, which is the maximizer of $b_t\Pi(p; q_t)$. Since marginal cost is constant, this profit maximizing price is independent of b_t.

Consumers' search problem. Consider a consumer that is referred to a state t firm, believing the quality of this firm's product to be q_t. The consumer can transact with this firm and realize a surplus of $s_t = s(q_t)$. Alternatively, the consumer can search. Let γ_τ be the proportion of state τ firms (γ_τ's are endogenous and yet to be determined), and let $\overline{s} \equiv \sum_{\tau=1}^{\infty} \gamma_\tau s_\tau$ be the expected surplus that a randomly selected firm delivers. Then the consumer chooses

$$\max \{s_t, \overline{s}\}. \tag{2.2}$$

If $s_t < \overline{s}$, the consumer searches and we denote this by $\sigma_t = 1$. If $s_t > \overline{s}$, the consumer buys from her referral firm and we denote this by $\sigma_t = 0$. If $s_t = \overline{s}$, the consumer is indifferent between buying and searching, so any value of σ_t, $0 \leq \sigma_t \leq 1$, is a best choice. We denote by $\sigma = (\sigma_t)_{t=1}^{\infty}$ the vector of consumers' choices.

Firms' Intertemporal maximization program. Given σ and n and assuming that a consumer searches if she is referred to a state 1 firm (the equilibrium we construct is such that if a consumer is referred to a state 1 firm, she optimally searches), the measure of a firm's customer base evolves as follows

$$\begin{aligned} b_1 &= n \\ b_{t+1} &= n + b_t\delta(1 - \sigma_{t+1}). \end{aligned} \tag{2.3}$$

Then, given $b = (b_t)_{t=1}^{\infty}$ and consumers' belief $q = (q_t)_{t=1}^{\infty}$, the net present value of a firm at state t, v_t, and the maximizing x_t are determined by the equations

$$v_t = b_t\pi(q_t) + \underset{x_t}{Max} \{-x_t + \beta[f(x_t)(v_{t+1} - v_1) + v_1]\}, \quad t = 1, 2, ..., \tag{2.4}$$

where $\beta \in (0, 1)$ is the discount factor.

Equilibrium. We seek a (symmetric, steady state) equilibrium, defined by the following objects:

- Investments x_t for firms at each possible state, $t = 1, 2, \ldots$.
- A decision σ_t whether to search or transact for a consumer who is referred to a state t firm.
- A belief q_t for consumers regarding the probability they get a high quality product if they transact with a state t firm.
- The measure γ_t of state t firms.
- The measure of search customers, n.

To ease the notation we refer to the vectors $(x_t)_{t=1}^{\infty}$, $(\sigma_t)_{t=1}^{\infty}$, $(q_t)_{t=1}^{\infty}$, and $(\gamma_t)_{t=1}^{\infty}$ as x, σ, q, and γ. $(x, \sigma, q, \gamma, n)$ is the tuple of endogenous variables, which is to be determined in equilibrium.

$(x, \sigma, q, \gamma, n)$ is an equilibrium if:

1. x is derived from firms' intertemporal maximization program (2.4).
2. σ is derived from consumers' maximization program (2.2).
3. Consumers' belief is correct,

$$q_t = f(x_t). \tag{2.5}$$

4. γ is in a steady state with respect to the transition probabilities induced by x, i.e.,

$$\begin{aligned} \gamma_1 &= \sum_{t=1}^{\infty} \gamma_t [1 - f(x_t)] \\ \text{and} \quad \gamma_{t+1} &= \gamma_t f(x_t), \text{ for } t \geq 1. \end{aligned} \tag{2.6}$$

5. The measure of search customers n is consistent with the transition probabilities induced by x, and with consumers' search rule:

$$n = 1 - \delta + \delta \sum_{t=1}^{\infty} \gamma_t b_t [f(x_t)\sigma_{t+1} + 1 - f(x_t)]. \tag{2.7}$$

3. Existence and properties of equilibria

3.1. Existence

We first note the existence of a trivial equilibrium, which replicates the static equilibrium under a one-shot interaction: Firms invest zero, all consumers search, and all firms have the same measure, 1, of consumers in their customer base. In this equilibrium consumers' self-fulfilling belief is that all firms, of whatever state, invest zero and hence deliver the same quality products (on average). Given this belief, the number of customers which a firm can access and the price it can charge do not depend on how much it invests and it, therefore, optimally invests zero.

As discussed in the introduction, we are interested in finding equilibria in which reputation matters, so that firms' investments increase in state as does their customer base and price. We call such equilibria 'reputation equilibria.' If consumers believe that firms that have had a long run of success (delivered high quality products for a long time) are more likely to deliver high quality products in the future, then consumers are more willing to transact with such firms and pay a higher price for their products. But, then, firms have an incentive to invest in quality so as to sustain this privileged status. And this may turn consumers beliefs, and firms' associated behavior, into an equilibrium, i.e., the belief becomes self-fulfilling. The existence of such equilibria is established in the Appendix under one extra assumption, and the net result is as follows.

Proposition 3.1. *Assume $f'(0) = \infty$. Then, there exists a reputation equilibrium in which firms' investments and the price they charge strictly increase in* *state, and consumers' optimal search rule is of the cutoff type, i.e., a consumer searches only if the state of her referral firm is sufficiently small. The cutoff state (below which consumers search), call it* $t^* > 1$, *is endogenously determined.*

We elaborate on properties of this equilibrium in the next subsection.

3.2. Properties of reputation equilibria

In a reputation equilibrium a firm invests more the longer its tenure as a high quality producer. Hence, the greater its tenure, the higher its expected quality and, therefore, the higher the price consumers are willing to pay for its product. The second feature of a reputation equilibrium is that firms' growth dynamics follow a cyclical pattern, which is determined by the endogenous cutoff tenure, t^*. t^* has the property that the surplus consumers expect to get from firms of tenure less than t^* is less than the average surplus in the industry, $\sum\limits_{\tau=1}^{\infty} \gamma_\tau s_\tau$, and the surplus consumers expect to get from firms with tenure greater than t^* is greater than the average surplus in the industry (at t^* consumers may exactly get the average surplus). Hence, consumers search if the tenure of their referral firm is below t^*, and buy from their referral firm otherwise. Thus all firms of tenure less than t^*, which sell only to randomly arriving search customers, are of the same size. Once a firm achieves tenure t^*, it starts to accumulate referral customers and its customer base begins to grow. It then continues to grow until it first produces a low quality product (which eventually it must). At that point it once again becomes a tenure 1 firm, loses all its accumulated referral customers, and only begins to grow again after regaining tenure t^*. These price and growth dynamics in turn account for why investment increases with tenure: Since it takes time and a fortunate sequence of successful investments for a high tenure firm to attain its large clientele and pricing power, it has more to lose from underinvestment than a low tenure firm, which has yet to attain these advantages. It therefore invests more to protect these assets. An alternative way to think of this is that a firm invests more in its reputation the better a reputation it already has. A low tenure firm, having recently been associated with a low quality product, does not have a good reputation and, hence, does not invest as much to attain one as a high tenure firm, with a good reputation, invests to keep it from being tarnished.

The preceding implies that success and failure tend to be reinforcing. High tenure firms, with a long history of successes, invest more and are, therefore, more likely to continue to be successful. Low tenure firms invest less and are, therefore, less likely to succeed in the future. As a result, quality is persistent over time, which is empirically supported by the findings of Cabral and Hortacsu (2004), in their study of e-Bay auctions, that the rate of negative feedback arrival increases twofold following the first negative feedback. In terms of our model, a seller becomes a tenure 1 firm when it receives its first negative feedback. Following this event such seller reduces its investment in quality and is, consequently, less likely to produce high quality products, which implies he is more likely to receive additional negative feedbacks.

These features of the reputation equilibrium also have an empirically meaningful implication about the size distribution of firms in the industry. Since a firm's customer base grows only as long as it continues to be of tenure greater than t^* and shrinks to the minimal size, n, as soon as it produces low quality, at any given point in time different firms will be of different sizes, depending on their tenure. The steady state tenure distribution, which is generated by these individual-firm fluctuations, has a specific structure. Namely, the relative proportion of firms of tenure t is decreasing in t. This comes from the fact that it takes a longer run of success to reach a larger tenure, which, since f is bounded away from 1, occurs with a smaller probability. This fact is then manifested (see (2.6)) in a downward sloping density over tenures, and since tenure and size are positively correlated, it is also manifested in a downward sloping density over sizes.[3]

[3] Most of the theoretical models of industry dynamics (see Jovanovic (1982) and Hopenhayn

4. Entry and Exit

In this section we integrate continual entry and exit into the model. To this end we add the feature that the provision of quality exhibits experience (or learning by doing) effects in the following sense. Consider two firms, one that delivered low quality products in the last period and one that delivered high quality. Then, we assume that the firm that delivered high quality products faces a superior technology in the sense that

$$\begin{aligned} f_H(0) &> f_L(0) \\ \text{and} \quad f_H'(x) &> f_L'(x), \;\; 0 < x < \overline{x}, \end{aligned}$$

where $f_H(x)$ is the probability of delivering a high quality product if the investment is x and if high quality products were delivered last period, and similarly for $f_L(x)$. Therefore, if the two firms make identical investments in quality, x, the firm that delivered high quality in the last period faces higher absolute and marginal probabilities of delivering high quality now.

The first result we prove under these circumstances is that a trivial equilibrium no longer exists.

Lemma 4.1. *In the experience-dependent model, an equilibrium in which all firms invest* 0 *does not exist.*

Proof. Assume there were such an equilibrium. Then the expected quality of state 1 firms, $f_L(0)$, is less than than the expected quality of state t firms, $f_H(0)$, for any $t > 1$. But then consumers who have a referral firm search if this firm is in state 1 and buy otherwise, i.e., $\sigma_1 = 1$ and $\sigma_t = 0$ for all $t > 1$. From this it follows that the customer base of a state t firm, for $t > 1$, is bigger than the

(1992)) do not generate concrete predictions about the size distribution of firms. An exception is Fishman and Rob (2003). On the empirical side, Cabral and Hortacsu (2004) report log-normal density of sales, which decreases beyond some point.

customer base of a state 1 firm and, in fact, that b_t is strictly increasing in t. But this implies that $v_t - v_1$ is strictly increasing and positive for all $t > 1$. Looking at a firm's maximization program, (2.4), we infer that optimal investments are strictly increasing in t, which contradicts the initial assumption that they are all zero. ■

Next we use this experience-dependent formulation to integrate continual entry and exit into the model. To do that we assume that each firm has to pay a fixed cost F, which is independent of state, in each period that it remains operative.[4] We also assume that there is an infinite pool of potential entrants that make zero profits in an alternative economic activity. At the beginning of each period, each potential entrant can pay F and enter as a firm that delivered a high quality product in the "last period," or as a firm that delivered a low quality product. The former occurs with probability $\alpha > 0$ and the latter with probability $1 - \alpha$. This means that a new entrant has the technology $f_e(x) \equiv \alpha f_H(x) + (1 - \alpha) f_L(x)$ in its initial period. A new entrant gets no referrals and, accordingly, serves n customers in its initial period, the same measure of customers that an existing state 1 firm serves. We consider entrants then as state 1 firms.

The timing of entry and exit is as follows. At the end of a period, existing firms know the realized quality of the product they just sold, and decide whether to pay F and stay, or not pay F and exit. Then, at the beginning of the next period a certain measure, which we denote by e and which is endogenously determined, of entrants enter. Since an existing state 1 firm has a lower value than an entrant (an entrant has the same customer base, but has access to a superior technology), all existing state 1 firms exit.[5] On the other hand, no state t firms with $t > 1$

[4] Another way to pin down the measure of operating firms is to introduce a *one time* cost of entry, K. This approach, however, does not give rise to *continual* entry and exit. One can, of course, combine a one time entry cost K with a fixed cost F that has to be paid each period. No new qualitative features arise from such combination, however.

[5] This is the reason we need an experience-based technology to study entry and exit. Other-

exit; if a state 2 firm exited, all entrants would not enter in the first place, which cannot be the case in equilibrium. And, a fortiori, state t firms with $t > 1$ do not exit. Thus, as long as it delivers a high quality product, a firm stays in the industry. The first time it delivers a low quality product, it exits. It follows, then, that the steady state measure of state 1 firms is e, while the steady state measure of state t firms, for $t > 1$, is determined by the scaled version of equation (2.6) (multiplying each equation by e).

We know from Proposition 3.1 that, corresponding to any value of e, there exists a reputation equilibrium of the type discussed above (without entry and exit), and that the steady state measure of all firms is proportional to e. For this to constitute a free entry equilibrium, the value of an entrant must be zero. Our last result, which we prove in an appendix, shows that a value of e can be found for which this is the case.

Proposition 4.2. *There exists an $e > 0$ which gives rise to a free entry equilibrium.*

In this version of the model, a firm remains operative only as long as it continues to deliver high quality products, and exits as soon as it first produces a low quality product. Hence, under this formulation, tenure is synonymous with age (the number of periods since entry) so that, in equilibrium, quality increases with age. Also, by the properties of the reputation equilibrium in the base model, it follows that all firms are of the same size up to the cutoff age, t^*, and increase in size after reaching this age. Hence, older firms are larger on average. And since investment increases with age, the probability of exit decreases with both size and age.

These features are consistent with an empirical literature (see Dunne, Roberts

wise there would be no difference between an entrant and an incumbent firm that exits, implying that there is no entry and exit.

and Samuelson (1988)), which has identified firm size and age among the characteristics most strongly associated with firm turnover and, specifically, has found that the probability of exit decreases with firm size and age. In our model this is due to the fact that older firms have more valuable reputations for quality, and invest more to maintain them. This reputational driven force contrasts with other models of industry dynamics, such as Jovanovic (1982), Hopenhayn (1992), and Ericson and Pakes (1995), in which the only characteristic of a firm is its cost and in which there is no such thing as reputation or a customer base.

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5. Appendix

A. Existence and Properties of Equilibrium in the Base Model

Let $\Gamma \equiv \left\{ \gamma \in [0,1]^\infty \middle| \sum_{t=1}^{\infty} \gamma_t = 1 \right\}$ be the set of distributions over firms' states. As stated in the text, an equilibrium is identified with a point $(x, \sigma, q, \gamma, n)$ that satisfies conditions 1-5, and that belongs to the domain $D_0 \equiv [0, \overline{x}]^\infty \times [0,1]^\infty \times [0, f(\overline{x})]^\infty \times \Gamma \times [0,1]$. To show that an equilibrium exists we set up a best-response correspondence, $\Phi : D_0 \rightrightarrows D_0$, and show that it satisfies the conditions of a fixed point theorem. Since Φ reflects maximizing behavior of each agent given the behavior of all other agents, a fixed point of Φ is an equilibrium.

To define Φ consider some point $(x, \sigma, q, \gamma, n) \in D_0$. Then, the set of values of Φ at this point, $\{(x', \sigma', q', \gamma', n')\} = \Phi(x, \sigma, q, \gamma, n)$, is found as follows. First, given q and b (where b is determined by (2.3), given σ and n), we have a functional equation (2.4), which, by the usual dynamic programming arguments (see [16]), has a unique solution, v', and a unique maximizer, x'. Second, given x', equations (2.5) and (2.6) uniquely define a q' and a γ'. Third, given q' and γ', equation (2.2) gives a set of σ''s. Finally for each σ' in this set, equations (2.3) and (2.7) are solved to give a b' and an n'.

To prove that Φ admits a fixed point we apply the Ky Fan fixed point theorem (see Berge, page 251). To this end, we have to show that D_0 is convex and compact, that Φ is non-empty and convex-valued, and that it is u.h.c. Convexity of D_0 is immediate. Compactness follows by defining the topology over D_0 to be the product topology, and applying Tychonoff theorem (see Berge, page 79). Convex-valuedness and non-emptiness of Φ come from the fact that the maximizer to a firm's program is unique, and the maximizer to a consumer's program is either unique or the whole $[0, 1]$ interval. Finally, Φ is u.h.c. by the theorem of the maximum (see Berge, page 116, or Dutta *et al.* (1994)).

As stated earlier, a trivial equilibrium with zero investments always exists. A potential problem with that is that the trivial equilibrium may correspond to a *unique* fixed point of Φ, so that there is no *reputation* (non-trivial) equilibrium. We now show that this is *not* the case and that, instead, there is a reputation equilibrium with an increasing investment profile. To this end, we confine Φ to a convex, compact sub-domain, D, that excludes the zero investment profile, and for which $\Phi(D) \subseteq D$. Then, we apply the Ky Fan theorem to D, giving us a non-trivial fixed point. The rest of this appendix shows how to choose a sub-domain D that has these properties.

To do that, we hypothesize that consumers do not search if their referral firm is in state T, for some $T > 1$, and, a fortiori, do not search if this firm is in state t for $t > T$. We analyze firms' best response program under this hypothesis, find properties of the solution to this program, and show that if T is "suitably chosen" this solution is consistent with the initial hypothesis (i.e., that if consumers' beliefs are derived from the solution, consumers do not search if their referral firm is in state T or higher). Because of this, the best response correspondence Φ maps a certain sub-domain of D_0, which does not include the zero investment profile, into itself, and from this it is a small step to show that Φ admits a fixed point over this sub-domain.

To execute this line of attack on the problem, fix an integer $T > 1$ and define D_T as follows

$$D_T \equiv \{(x, \sigma, q, \gamma, n) \in D_0 \mid x_{t+1} \geq x_t, \sigma_{t+1} \leq \sigma_t, \sigma_T = 0\},$$

The sub-domain D we are interested in is one of these D_T's for some T. To show how T is chosen we prove two lemmas. The first lemma analyzes firms' maximization program when this program is parameterized by some point in D_T. The lemma shows that the solution to this program is such that there is a minimal (and positive) "gap" between the investment of a state 1 firm and a state T firm.

Lemma 5.1. *Let the $T > 1$ in the definition of D_T be arbitrarily specified, and let $(x, \sigma, q, \gamma, n) \in D_T$. Let b and q be the values that x, σ, and n induce via (2.3), (2.5), and (2.7). Then, when firms' maximization program (2.4) is parameterized by b and q, it yields a solution x' that is strictly increasing, and satisfies $x'_T - x'_1 \geq \Delta$ for some $\Delta > 0$.*

Proof. Let $(x, \sigma, q, \gamma, n) \in D_T$ be as above, and consider a firm's maximization program (2.4) under the b and q that x, σ, and n induce. Since $\sigma_1 = 1$ and $\sigma_T = 0$, we know that $b_{t+1} > b_t$ for some t, $1 \leq t \leq T$. But this, together with the fact that q is (weakly) increasing, implies that the solution x' is strictly increasing (invoking the usual dynamic programming arguments; see [16]).

To show that $x'_T - x'_1 \geq \Delta$, for some $\Delta > 0$, observe first that

$$v_{T+1} - v_2 \geq b_{T+1}\pi(q_{T+1}) - b_2\pi(q_2) \geq (b_{T+1} - b_2)\pi(q_2) \geq \delta(1-\delta)\pi(0).$$

Second, since x'_1 is the solution to

$$-x + \beta f(x)(v_2 - v_1)$$

and since f is strictly concave and $f'(0) = \infty$, x'_1 is positive and satisfies

$$f'(x'_1) = \frac{1}{\beta(v_2 - v_1)} \equiv \frac{1}{A}.$$

Or,

$$x'_1 = \varphi(A) \equiv f'^{-1}(\frac{1}{A}),$$

where f'^{-1} is the inverse function of f'. Since $f'(0) = \infty$ and f' is continuous and strictly decreasing, φ is strictly increasing, bounded, continuous, and defined for all A, $0 \leq A \leq \overline{A}$, for some $0 < \overline{A} < \infty$. Likewise, x'_T is positive and satisfies

$$f'(x'_T) = \frac{1}{\beta(v_{T+1} - v_1)} \equiv \frac{1}{B}$$

or

$$x'_T = \varphi(B) \equiv f'^{-1}(\frac{1}{B}).$$

Now, since φ^{-1} is uniformly continuous (it is continuous because it is the inverse of a monotonic, continuous function; it is uniformly continuous because its domain is compact), we know that for every $\varepsilon > 0$ there exists an $\eta > 0$ so that if $B - A \geq \varepsilon$, then $\varphi(B) - \varphi(A) \geq \eta$. In particular, since, as shown above, $v_{T+1} - v_2 \geq \delta(1-\delta)\pi(0) > 0$, we know that a $\Delta > 0$ can be found for which

$$x'_T - x'_1 = \varphi(\beta(v_{T+1} - v_1)) - \varphi(\beta(v_2 - v_1)) \geq \Delta.$$

■

Lemma 5.1 shows, then, that if a firm's program is parameterized by an arbitrary $(x, \sigma, q, \gamma, n) \in D_T$ (for any $T > 1$), the solution to this program exhibits a positive gap between the investments (and hence the expected surplus) of state 1 firms and state t firms, for t sufficiently large. In addition, the steady state distribution of firms' states (again, for *any* x) is such that the relative frequency of state t firms is vanishingly small for sufficiently large t. But this suggests that, if we let $T \to \infty$, the upside potential of search is limited (because of the small probability) while the downside risk is not (because of the positive gap), implying that a consumer would not search if she has a referral firm that is in a sufficiently high state. The next lemma formalizes this intuition, showing that a T can be found so that consumers choose to patronize their referral firm if this firm is in state T or higher.

Lemma 5.2. *There exists a $T > 1$ so that any $(x, \sigma, q, \gamma, n) \in D_T$ yields (under Φ) a monotonic solution to consumers' programs for which $\sigma'_T = 0$.*

Proof. The previous Lemma tells us that - for any $T > 1$ - the solution to firms' program (under $(x, \sigma, q, \gamma, n) \in D_T$) is strictly monotonic and satisfies $x'_T - x'_1 \geq \Delta$. But, then, consumers' optimal search rule is monotonic, i.e., σ'_T is decreasing. We now show that if we start out with a specific T, then the solution, x', to firms' program is such that $\overline{s} = \sum_{\tau=1}^{\infty} \gamma_\tau s_\tau < s_T$, where γ and s are derived from x'. To see that, note that γ is stochastically dominated by the geometric distribution $\{[1-f(\overline{x})]f(\overline{x})^\tau\}_{\tau=0}^{\infty}$ (see equation (2.6)). Given this, we have

$$\begin{aligned}\sum_{\tau=1}^{\infty} \gamma_\tau s_\tau \;\; &< \;\; [1-f(\overline{x})]s(f(x'_1)) + [1-f(\overline{x})]f(\overline{x})^{T+1}s(f(\overline{x})) \\ &\quad + \{1-[1-f(\overline{x})]-[1-f(\overline{x})]f(\overline{x})^{T+1}\}s(f(x'_T)).\end{aligned}$$

The second term on the RHS goes to zero as T goes to infinity. Therefore, since $x'_T - \Delta \geq x'_1$, we can find a large enough T so that the RHS is indeed smaller than s_T. But this, in turn, implies that consumers optimally accept a state T firm, i.e., that $\sigma'_T = 0$ ■

Taken together, these two Lemmas show that $\Phi(D_T) \subseteq D_T$ for an appropriately chosen T. Given this, it remains to choose $D = D_T$ for one of these T's, and apply Ky Fan theorem to D. Since $x = 0$ cannot be the first component of any point in D, this shows that any fixed point of Φ over D is non-trivial. Furthermore, by Lemma 4, we know that the investment profile corresponding to this fixed point is strictly increasing.

B. Entry and Exit

Here we prove Proposition 4.2 in the text.

Proof. From the previous appendix we know that an equilibrium exists for any e, although this equilibrium is not unique. We also know, from fixed point theory, that the equilibrium correspondence is u.h.c. in e, and that the set of equilibria for any given e is compact. For every e, we choose an equilibrium that gives state 1 firms the largest value. This equilibrium is well defined because the set of equilibria is compact, and the value of a firm is a continuous function of $(x, \sigma, q, \gamma, n)$. Further, for sufficiently small e's, this largest equilibrium value is non-negative, and for sufficiently large e's, it is non-positive. We let e^* be the supremum of the set of e's for which this value is non-negative. Then, by u.h.c., the value at e^* is non-negative. On the other hand, if $e > e^*$, the value at *any* equilibrium corresponding to e is negative. Therefore, e^* is consistent with free entry: even if the largest equilibrium value at e^* is positive, the value of entry would be negative if the measure of entrants were any higher than e^*. ■