# Liquidity Crises and the Lender of Last Resort in a Monetary Economy[*]

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## Abstract

This paper studies the role of a lender of last resort (LLR) in a monetary model where a shortage of bank's monetary reserves (or a liquidity crisis) occurs endogenously. We show that while a discount window policy introduced by the LLR is welfare improving, it reduces the banks' ex ante incentive to hold reserves, which increases the probability of a crisis, and causes moral hazard in asset investments. We also examine the combined effect of other related policies such as a penalty in lending rate, liquidity requirements and constructive ambiguity.

**Keywords**: Monetary Equilibrium, Liquidity Crisis, Moral Hazard, Lender of Last Resort

**JEL Classification Number**: E40

# 1 Introduction

The 2007-09 financial crisis has renewed our interest in the role of the lender of last resort (LLR). After interbank markets stopped functioning properly, central banks in several developed countries conducted large-scale interventions beyond the traditional scope of open market operations. While the importance of the LLR has been stressed by many economists and policymakers, there is much less consensus on the nature of its role. As Fischer (1999, p.86) put it: "While there is considerable agreement on the need for a domestic lender of last resort, some disagreements persist about what the lender of last resort should do." The main concern about the LLR is a moral hazard problem. The conventional view is that the existence of a credible LLR will give financial institutions an incentive to take risks ex ante because they will expect ex post liquidity provision by the LLR in the event of a crisis, which will reduce the private cost of risk taking.

In this paper, we study the economic role and consequence of the LLR in a general equilibrium framework. We develop a monetary model in which individual agents face idiosyncratic uncertainty about the payment method, that is, whether money or credit can be used, and therefore banks are beneficial as liquidity insurance providers. Given aggregate uncertainty of money demand, banks are sometimes short of monetary reserves and fail to satisfy their depositors' needs who have monetary transactions. In such an illiquidity situation, which shall be referred to as a liquidity crisis, consumption inequality between money users and credit users become large, a money premium is high, and there is room for emergency liquidity assistance by the LLR.

In our model, banks allocate their deposits between monetary reserves and long-term assets. In the benchmark without the LLR, banks invest all their deposits in the reserves because non-monetary assets have no use as a medium of exchange. In the presence of the LLR, banks with a high liquidity demand can borrow money from the discount window whenever they hold a pledgeable long-term asset. That is, the discount window loans are fully collateralized in our model. This implies that with the LLR, long-term assets have a liquidity value during crises, and therefore banks invest a positive fraction of the deposits in it. This leads to a lower level

of monetary reserves and, thus, to a higher ex ante probability of liquidity crises relative to the benchmark case without the LLR. Nevertheless, the introduction of the LLR improves welfare because money and assets are substitutes during crises, and it mitigates losses associated with liquidity crises.

To examine asset choice, we consider two types of long-term assets: safe and risky assets. The safe asset is risk free. The risky asset yields a higher return if the gamble succeeds, but a lower return if not. We show that due to limited liability, which frees their payment responsibility in the case of default, banks can be induced to invest in the risky asset rather than the safe asset. Hence, the LLR can create moral hazard in asset investments, that is, banks take more financial risk in terms of long-term assets, resulting in a default on their discount window loans with a positive probability. We will refer to such an insolvent situation as a banking default.

To be clear, unlike in Diamond and Dybvig's (1983) model, depletion of bank's monetary reserves does not cause banking insolvency and bankruptcy in our model because banks can distinguish between two types of depositors and refuse to allow them to withdraw after their reserves are exhausted (suspensions of convertibility). That is, no self-fulfilling bank runs occur in our model. The distinction between bank's illiquidity (liquidity crisis) and insolvency (banking default) makes our model rich and possible to study the important problems associated with the LLR.

In addition, our model differs from the related banking models in the way we incorporate monetary factors. In fact, money matters in our model for several reasons. First, money serves as a medium of exchange and overcomes trading frictions such as a lack of commitment and imperfect monitoring in some decentralized transactions. Second, inflation has a positive impact on the likelihood and extent of a liquidity crisis because they depend a lot on the amount of monetary reserves banks have, and inflation increases the cost of holding the reserves. Third, inflation reduces welfare as in a standard monetary model. Finally, inflation affects the demand for the central bank loans during a crisis because it also depends on the amount of the bank's initial reserves. Considering all characteristics here is crucial to the analysis and to

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the contribution of this study.

Furthermore, we examine two additional banking policies, liquidity requirements and constructive ambiguity, which are often implemented in practice together with discount window lending. We find that both of these policies can increase bank's cash reserves, which eventually lowers the probability of liquidity crises and reduces the bank's risk-taking incentives, with accompanying higher welfare costs. This result may suggest the limitations of the central bank's credible commitment on preventing crises and moral hazard, due to its welfare deteriorating nature, which in turn causes the time consistency problem.

Our paper points to the tension in public debate among economists and policy makers between the classical doctrine (or the Bagehot principle) versus the moral hazard problem. The former suggests that the LLR should give liquid loans to illiquid but solvent banks at a high interest rate (or a "penalty" rate) against their good collateral (Thornton, 1802; Bagehot, 1873), while the latter concerns high financial risks taken by illiquid banks. This is one of the central issues of the LLR policy debates. The conventional view is that a high loan rate on the discount window prevents not only borrowing of unnecessary amounts of liquidity but also taking excessive risks. For example, Solow (1982) states that "the penalty rate is a way of reducing moral hazard (p.247)," and Fischer (1999) comments that "the lender of last resort should seek to limit moral hazard by imposing costs on those who have made mistakes. Lending at a penalty rate is one way to impose such costs (p.93)." Our results do not concur this view for several reasons. First, we show that if the expected return on the risky asset is sufficiently high, changing the lending rate has no influence on asset choice. Second, the penalty rate could reduce welfare largely. Finally, things are not so simple once the banks' asset choice is taken into account. For instance, a higher lending rate can lower the probability of liquidity crises but leads to a switch in banks' investment from the safe asset to the risky asset, leading to possible defaults. Hence, a penalty rate aimed at reducing the probability of a liquidity crisis can cause an adverse effect on risk-taking investments and defaults. The description of breadth (liquidity crises) and depth (banking defaults) of financial fragility is possible in the presence of the LLR only in a framework where investments in the risky asset are allowed. To

the best of our knowledge, our paper is the first to point out this possibility.

## 1.1 Related Literature

The LLR policy has a long history; its concept was elaborated in the 19th century by Thornton (1802) and Bagehot (1873). The classical doctrine has been criticized on two grounds. First, Goodfriend and King (1988), Kaufman (1991), and Schwartz (1992) argue that with efficient interbank markets, central banks should not lend to individual banks, but instead provide liquidity via open market operations. However, others argue that interbank markets may fail to allocate liquidity efficiently due to asymmetric information (Flannery, 1996; Freixas and Jorge, 2007; Heider, Hoerova, and Holthausen, 2015), free-riding (Bhattacharya and Gale, 1987), coordination failures (Freixas, Parigi, and Rochet, 2000), incomplete network (Allen and Gale, 2000), incomplete contracts (Allen, Carletti, and Gale, 2009), or market power (Acharya, Gromb, and Yorulmazer, 2010), which can justify the role of the LLR. In this paper, we do not model interbank markets explicitly, but consider a situation where a shortage of liquidity in a whole banking system occurs endogenously due to aggregate demand shocks that market capacity cannot satisfy.

Second, Goodhart (1987, 1999) argue that there is no clear-cut distinction between illiquidity and insolvency during a crisis and banks that require the LLR assistance are already under suspicions of insolvency. See also Solow (1982) and Schwartz (1992) for some more related issues. Our model captures Goodhart's emphasis well because in our setup the central bank must lend money to illiquid banks without knowing whether they would be insolvent.

In practice, central banks in many countries have expanded their LLR function beyond the classical Bagehot rule since the 1970s.[1] For example, Bordo (2014) points out that since the bailout of Franklin National Bank in 1974, the Fed's LLR policy has adopted the too-big-to-fail doctrine to prevent systemic risk and contagion irrespective of the classical doctrine. In addition, Giannini (1999) claims that most LLR policies adopted a non-penalty rate or even a

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[1]According to Bordo (1990), major central banks in European countries generally followed the classical doctrine between 1870 and 1970. In contrast, the Bank of Japan provided liquidity support to large illiquid and insolvent banks at a non-penalty rate based on the too-big-to fail doctrine in response to the financial panic of 1927 (Yokoyama, 2018).

subsidized rate without having stated it clearly in advance. There is no doubt that the moral hazard associated with these LLR policies is a serious concern. However, we do not have any theory taking into account monetary liquidity supports and moral hazard. We propose a new theory to fill this gap.

Our study is related to the following three strands of literature. The first strand focuses on financial crises and the role of the LLR in a standard non-monetary banking model, for example, Allen and Gale (1998), Freixas, Parigi, and Rochet (2000), Rochet and Vives (2004), Repullo (2005), Martin (2006, 2009), Allen, Carletti, and Gale (2009), and Acharya, Gromb, and Yorulmazer (2010). Some of them regard LLR policies as real tax-transfer schemes without monetary considerations, while others consider monetary transfers but treating nominal assets as an exogenous restriction. Furthermore, most models are based on partial equilibrium analysis. In contrast, our approach is to take monetary factors as the main driving force, because we believe that traditional banking crises should represent a widespread attempt by the public to convert their deposits into cash and a suspension of convertibility (Calomiris and Gorton, 1991; Champ, Smith, and Williamson, 1996), and that the abilities of creating high-powered money and distributing it quickly authorize a central bank to act as a lender of last resort (Schwartz, 2002).

The second strand examines monetary factors of the LLR in an overlapping-generations model with random relocation along the lines of Champ, Smith, and Williamson (1996) and Smith (2002). See also Antinolfi, Huybens and Keister (2001), Antinolfi and Keister (2006), and Matsuoka (2012). Unlike our model, these models do not consider risky financial technologies and moral hazard associated with the LLR. Williamson (1998) is an exception in the literature and develops a similar banking model with deposit insurance and discount window lending in order to study moral hazard associated with these policies. However, he abstracts monetary factors and does not endogenize the deposit level. Indeed, the distinction between monetary reserves and other real assets is crucial for moral hazard in our model. We can also analyze the influence of monetary factor, e.g., inflation, on the occurrence of moral hazard in monetary equilibrium. These are all new relative to the literature. In sum, we offer a new and

simple monetary general equilibrium approach to make the moral hazard problem, which is potentially very complicated, tractable.

The third strand considers banking with a New Monetarist approach along the lines of Lagos and Wright (2005) and Rocheteau and Wright (2005). This strand includes Andolfatto, Berentsen, and Martin (2019), Berentsen, Camera, and Waller (2007), Ferraris and Watanabe (2008, 2011), Bencivenga and Camera (2011), Williamson (2012, 2016), Gu et al. (2013), Gu et al. (2019), Sanches (2018), and Matsuoka and Watanabe (2019). None of them considers the economic role and consequences of the LLR and other related policies.

The rest of the paper is organized as follows. Section 2 describes the basic environment. Section 3 analyzes the case without the LLR. Section 4 analyzes the case with the LLR and explores the condition of moral hazard. Section 5 considers extensions. Section 6 concludes. All mathematical proofs are provided in the Appendix.

## 2   Environment

The model builds on a version of Lagos and Wright (2005). Time is discrete and continues forever. Each period is divided into two subperiods: day and night. A market is open in each subperiod. There are two types of $[0, 1]$ continuum of infinitely-lived agents. Agents of the same type are homogeneous. One type of agents, called sellers, have the production technology during the day, which allows them to produce perishable and divisible goods, referred to as special goods. The other type of agents, called buyers, do not have the production technology during the day but want to consume the special goods. Other divisible goods, referred to as general goods, are produced and consumed during the night. There is also an intrinsically worthless good, which is perfectly divisible and storable, called fiat money. Agents discount future payoffs at a rate $\beta \in (0, 1)$ across periods, but there is no discounting between the two subperiods.

The instantaneous utility functions for buyers and sellers are given by $u(q^b) + U(x) - h$ and $-q^s + U(x) - h$, respectively, where $q^b$ is the amount of the special goods consumed by the buyer, $q^s$ is the amount of the special goods produced by the seller, $x$ is the amount of

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the general goods consumed, and $h$ is the nighttime hours of work. Marginal production costs of both goods are constant, normalized to one. The utility function $u(q)$ is strictly increasing, strictly concave, and twice continuously differentiable with $u(0) = 0, u'(0) = \infty, u'(\infty) = 0$, and with the property that there exists some $\hat{q} > 0$ such that $u(\hat{q}) = \hat{q}$. Let $q^*$ denote the efficient quantity, which solves $u'(q^*) = 1$. For analytical tractability, we assume $\xi \equiv -\frac{qu''(q)}{u'(q)}$ is a positive constant. The utility function of general goods, $U(x)$, is also strictly increasing, concave, and twice continuously differentiable. We normalize $U(x^*) - x^* = 0$, where $x^*$ solves $U'(x^*) = 1$.

During the day, buyers and sellers can trade special goods in decentralized markets (DM), which involves bilateral random matching. Just like in Sanches and Williamson (2010) and Williamson (2012) (see also Section 5 of Williamson and Wright, 2010), we assume that in a DM, there are a fraction $\alpha \in (0, 1)$ of sellers who are engaged in a *non-monitored* exchange and a fraction $1 - \alpha$ of sellers who are engaged in a *monitored* exchange. At the beginning of the day, sellers meet with their counterparts and buyers learn whether they will trade with sellers in non-monitored or monitored meetings. In the non-monitored exchanges, exchanges are anonymous and trading histories are private knowledge, and thus, given the random meeting, sellers must receive money for immediate compensation of their products. In contrast, there is a record-keeping technology in the monitored exchange, and perfect commitment is possible so that buyers can promise credibly that they will make a payment to sellers later during the night. That individual buyers face randomness in different requirements of the medium of exchange, plays the role of a "liquidity preference shock." This is similar in the spirit to Diamond and Dybvig (1983) to motivate the banks' risk sharing role. In contrast to Diamond and Dybvig model, we assume that an individual buyer's type, non-monitored or monitored, is public information, implying that there is no self-fulfilling bank run in our model. In any meeting, we assume for simplicity that buyers make a take-it-or-leave-it offer to sellers.

The fraction $\alpha$ of monitored/non-monitored meetings is a random variable. It is publicly observable and identically distributed over time. Let $F = F(\alpha)$ represent the distribution function, which is assumed to be continuous, differentiable, and strictly increasing, and $f =$

$f(\alpha) > 0$ is the associated density function. As mentioned earlier, this randomness will play a key role in our model.

One can imagine several interpretations of the stochastic fluctuations of $\alpha$. First, it is typically thought of as a seasonal variation in the demand for money. Historically, large seasonal pressures, mostly in the spring planting season and fall crop moving season, have caused banking panics in agricultural economies (e.g., Sprague, 1910; Miron, 1986; Calomiris and Gorton, 1991). Second, small changes in the cost of information acquisition about counterparty or asset quality used as collateral in an imperfect credit system would have large effects on credit transactions (Lester, Postlewaite, and Wright, 2012). Finally, unexpected events such as large-scale natural disasters, blackouts, and September 11, 2001, would damage social communication tools necessary for credit transactions and suddenly increase the aggregate demand for money. As these seem potentially relevant, we are agnostic here about the exact nature of stochastic fluctuations.

During the night, general goods are traded in the centralized market (CM), which is Walrasian. Buyers have access to a storage technology which allows them to hold long-term assets across periods. We consider two types of assets. A safe asset transforms one unit of the general good into $R > 1$ units of the general good in the next period. We consider the case $\beta R < 1$ to avoid an explosive solution or indeterminacy. A risky asset yields a stochastic return, $\lambda R$, with probability $\eta$, or zero with probability $1 - \eta$, where $\lambda > 1$ and $\eta\lambda \leq 1$.[2] Thus, risk-averse and risk-neutral agents prefer the safe asset to the risky asset, and moral hazard is said to occur if the risky asset is selected. In addition, we assume that only a fraction $R$ of the successful return on the risky asset, $\lambda R$, is observable and verifiable, implying that only this fraction can be used as collateral when needed for discount window borrowing (see below). The remaining fraction $(\lambda - 1)R$ of the return is neither observable nor verifiable by outsiders (e.g., the central bank). This unobservable extra return can be interpreted as "private benefit" as in a standard moral hazard model (e.g., Holmstrom and Tirole, 1997). The asymmetric information implies that outsiders cannot observe whether the bank chooses the safe or risky asset until the risky

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[2] A similar assumption is made in Cooper and Ross (2002) and Martin (2006).

asset produces nothing. The term "investment in the risky assets" can be interpreted broadly to include low efforts for screening projects, monitoring projects, or management of financial risks. Typically, these efforts are costly and not observable by outsiders, implying that there is private benefit from shirking.

In a CM, buyers form private banks. A bank offers a contract to each of its depositors which stipulates a repayment plan as specified below. Depositors are buyers who deposit general goods in the CM and may need cash in the following DM. The bank stocks $m$ money (or $z \equiv \phi_{-1}m$ real money balances) and $k$ safe and $l$ risky assets in the CM. Any credit contracts in the DM are settled in the CM of the same period.
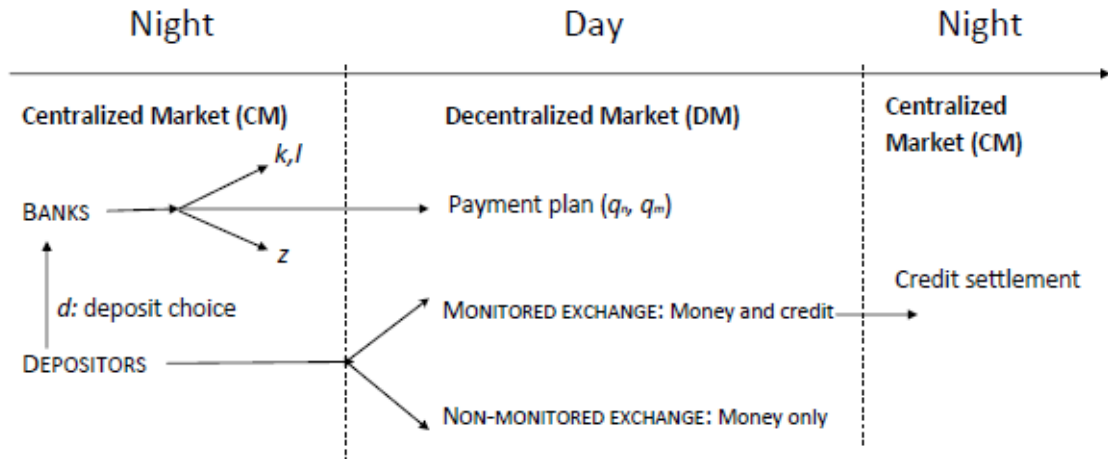


Fig 1: Timing of events

The timing of events is illustrated in Figure 1. At the end of a night, banks collect deposits and invest them in a portfolio of monetary reserves ($z$) and real assets ($k$ and $l$). At the beginning of the next day, agents observe the realized value of $\alpha$, and buyers learn their individual types of meeting. Then, the buyers who will trade with sellers in a non-monitored exchange, referred to as non-monitored buyers, receive money that their banks allocate and use it for their consumption, $q^n$, in the DM, while the buyers who will trade with credit in a monitored exchange, referred to as monitored buyers, consume, $q^m$. Finally, the banks distribute their wealth among their depositors and dissolve at the beginning of a night, and

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agents trade general goods and settle debts in the CM.

The stock of money available in a period is denoted by $M$. It grows (or shrinks) at a constant rate $\pi > \beta$, that is, $M_+ = \pi M$, through injection to (or withdrawing it from) buyers in a lump-sum manner in the CM st night, where the subscript "+" stands for the next period. Let $\phi$ denote the price of money in terms of general goods. In a stationary monetary equilibrium, where the real money balances are constant over time, the return on money must be equal to the inverse of the inflation rate, $\frac{\phi_+}{\phi} = \frac{1}{\pi}$.[3]

The first-best solution in our economy is straightforward. The socially optimal level of long-term assets, both safe and risky, equals 0 since $\beta \max\{\eta \lambda R, R\} = \beta R < 1$. The socially optimal levels of consumption equal $q^* \equiv u^{-1\prime}(1)$, that is, the marginal utility of special goods consumption $(= u'(q^*))$ is equal to the marginal cost $(= 1)$, and $x^* \equiv U^{-1\prime}(1)$, that is, the marginal utility of general goods consumption $(= U'(x^*))$ is equal to the marginal cost $(= 1)$.

## 3    Monetary Equilibrium without LLR

We first derive a stationary monetary equilibrium without an LLR. Given that perfect (unsecured) credit is available, a buyer in a monitored exchange can purchase by using credit (or, equivalently, issuing an IOU) and consume any quantity they wish to irrespective of their daytime money holdings, since the payment can be made later at night. Hence, in each period, a buyer in the monitored exchange consumes the first best quantity, that is, $q^m = q^*$. As the buyers in a monitored meeting do not need cash during the day, banks do not allocate cash to these buyers.

Banks' repayment schedules should determine how much money to allocate to each non-monitored buyer. At the beginning of a day, before buyers find out the type of their meeting, banks choose a payment schedule given their holdings of cash reserves, $m$, selected in the

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[3]In this paper, it is assumed that monetary reserves earn a zero nominal interest rate, that is, there is no deposit facility. One reason to focus on inflation rates rather than interest rates on reserves is that this model tries to capture a variety of banking economies where standing facilities are not established. Another reason is that it allows us to facilitate comparison with previous studies, including Smith (2002) and Antinolfi and Keister (2006). We conjecture, however, that changing interest rates on reserves would affect the real values in the same way as inflation does.

previous CM (see below). The payment can be contingent on the realized aggregate state. We assume competitive banks with free entry, so that each maximizes the expected value of its representative depositor (i.e., buyer). Denote by $q^n = q^n(\alpha)$ the consumption of a non-monitored buyer, and by $\theta = \theta(\alpha)$ the fraction of its cash reserves that a bank pays out to non-monitored buyers during a day. Without loss of generality, the remaining cash reserves are distributed uniformly among monitored buyers after the DM closes. For each realized value $\alpha \in (0,1)$, a bank's maximization problem in the DM can be written as:

$$\max_{\theta \in [0,1]} \alpha u(q^n) + (1 - \theta)\frac{z}{\pi},$$

subject to

$$\alpha q^n = \frac{\theta z}{\pi},$$

where $z \equiv \phi_{-1} m$ is the real reserve balances. The first term in the objective function represents the daytime utility of non-monitored buyers, who need cash for the daytime trade, and the second term, the nighttime real value of the remaining cash reserves. The constraint states that each individual non-monitored buyer receives $\frac{\theta m}{\alpha}$ units of cash from the bank and, given take-it-or-leave-it offers, exchanges it with the matched seller for $\frac{\theta m \phi}{\alpha} = \frac{\theta z}{\pi \alpha}$ units of special goods.

The first order condition is

$$\frac{z}{\pi}\left\{ u'(q^n) - 1 \right\} \geq 0 \quad ( = \text{if } \theta < 1). \tag{1}$$

This condition shows that two situations are possible in a monetary equilibrium (i.e., with $z > 0$). The first condition, $\theta < 1$, implies $q^n = q^m = q^*$, that is, consumption smoothing. The other, $\theta = 1$, implies that the bank exhausts all its cash reserves and fails to achieve consumption smoothing, that is, $q^n < q^m = q^*$. We refer to such an event as *a liquidity crisis*.[4] This notation captures the situation where a significant number of depositors suddenly demand to redeem bank debt for cash, leading to a shortage of the overall amount of reserves in the

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[4]Champ, Smith, and Williamson (1996) and many others call this situation as a banking panic or banking crisis.

banking system and a suspension of convertibility. Note that the suspension of convertibility is embedded in our setup in the sense that the bank refuses to liquidate real assets prematurely and only pay out reserves selected in the previous CM. It follows:

**Lemma 1 (Banks' Optimal Payment Plan)** *Given cash reserves $z > 0$, the optimal payment plan of banks is described by $q^m(\alpha) = q^*$,*

$$q^n(\alpha) = \begin{cases} q^* & \text{if } 0 < \alpha < \alpha^*, \\ \frac{z}{\alpha\pi} & \text{if } \alpha^* \leq \alpha < 1, \end{cases}$$

*and*

$$\theta(\alpha) = \begin{cases} \frac{\alpha}{\alpha^*} & \text{if } 0 < \alpha < \alpha^*, \\ 1 & \text{if } \alpha^* \leq \alpha < 1, \end{cases}$$

*where $\alpha^* \equiv \frac{z}{\pi q^*} > 0$.*

The lemma shows that the occurrence of a liquidity crisis can be stated in terms of realized values of $\alpha$. For low values of $\alpha < \alpha^*$, the realized aggregate demand for money in the DM is relatively low, so that banks' cash reserves are sufficient to cover the needs of the non-monitored buyers, leading to $q^n = q^*$ and $\theta < 1$. For high values of $\alpha \geq \alpha^*$, the realized aggregate demand for money is relatively high and, thus, banks' cash reserves are not enough to cover the needs of non-monitored buyers, leading to $q^n \leq q^*$ and $\theta = 1$. This results in a liquidity crisis. Figure 2 illustrates this lemma.

Given the total deposit, denoted by $d > 0$, and the repayment plan $(q^n(\alpha), q^m(\alpha))$ and $\theta = \theta(\alpha)$, described in Lemma 1, the banks' portfolio choice problem in the CM can be written as

$$V(d) = \max_{z,k,l \geq 0} \int_0^1 \left[ \alpha \left\{ u(q^n) + W(0,k,l,0) \right\} + (1-\alpha) \left\{ u(q^*) + W\left( \frac{(1-\theta)z}{1-\alpha}, k, l, q^* \right) \right\} \right] f(\alpha) d\alpha, \quad (2)$$

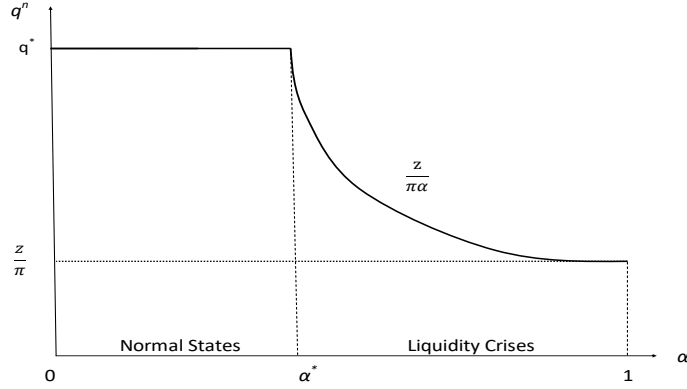subject to the balance sheet constraint,

$$d = z + k + l.$$

Fig 2: Consumption in a non-monitored exchange

In this expression, notice that a non-monitored (monitored) buyer consumes $q^n$ ($q^*$) in the DM, and the bank's long-term assets, $k$ and $l$, are distributed uniformly among all buyers. $W(z', k, l, c)$ represents the expected value of a buyer entering the next night market with holdings of $z'$ real cash balances, $k$ safe assets, $l$ risky assets, and $c$ debt from the DM and is given by

$$W(z', k, l, c) = \max_{x, h, d_+ \geq 0} U(x) - h + \beta V(d_+)$$

subject to

$$x + d_+ + c = h + R(k + \eta \lambda l) + \frac{z'}{\pi} + T,$$

and the usual non-negativity constraints, where $T$ denotes the real value of the monetary transfers (or taxes) from the central bank, that is, $T = \phi(M_+ - M) = \left(1 - \frac{1}{\pi}\right) \phi M_+$, and $V(d_+)$ is the expected value in the next DM with deposits $d_+$.

Substituting for $h$, we have

$$W(z', k, l, c) = R(k + \eta \lambda l) + \frac{z'}{\pi} + T - c + \max_{x \geq 0} \{U(x) - x\} + \max_{d_+ \geq 0} \{-d_+ + \beta V(d_+)\},$$

implying that the choice of $d_+$ is independent of the wealth $R(k + \eta \lambda l) + \frac{z'}{\pi} + T - c$ as in the Lagos and Wright model.

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Define

$$\Upsilon(z) \equiv F(\alpha^*) + \int_{\alpha^*}^{1} u'(q^n) f(\alpha) d\alpha.$$

Then, the marginal benefit from real cash balances holding is given by

$$\frac{1}{\pi} \Upsilon(z),$$

since when $\alpha \leq \alpha^*$, an additional unit of real balances is useful only in the CM, where it yields a real value $\frac{1}{\pi}$ (as captured in the first term in $\Upsilon$), while when $\alpha > \alpha^*$, it is useful in the DM, where it yields $\frac{u'(q^n)}{\pi}$ (as captured in the second term in $\Upsilon$). Thus, the liquidity factor is positive if $\alpha^* < 1$.

**Lemma 2 (Banks' Optimal Portfolio Choice)** *Given deposit $d > 0$, the optimal portfolio of banks is described by $k = d - z \geq 0$, $l = 0$, and*

$$z = \begin{cases} z(d) & \text{if } \Upsilon(d) \leq \pi R, \\ d & \text{if } \Upsilon(d) > \pi R, \end{cases}$$

*where $z(d) \in (0, d)$ is a solution to $\Upsilon(z(d)) = \pi R$.*

Note that since the risky asset is strictly dominated by the safe asset, the bank does not invest in the risky asset at all, that is, $l = 0$. Note also that the marginal benefit from cash holdings is positive and is monotonically decreasing in $z$ (see the proof of Lemma 2). Hence, if the marginal benefit from holding real balances, $\frac{1}{\pi}\Upsilon$, is greater than the marginal cost, $R$, at $z = d$, then $\Upsilon(z) > \pi R$ for all $z \leq d$ and the bank holds cash only, hence, $z = d$ and $k = 0$. Otherwise, the bank's portfolio includes not only cash $z = z(d) \in (0, d)$, but also assets $k > 0$.

The last step is to determine the optimal deposits of the buyers, given the deposit contract offered by the banks, $(q^n, q^m, z, k, l)$, as described above. The buyer's problem is

$$\max_{d \geq 0} \{-d + \beta V(d)\},$$

where $V(d)$ is the buyer's expected value entering the DM with deposit $d > 0$ as described above.

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For $d < \Upsilon^{-1}(\pi R)$, we have $z = d$ and the Euler equation is $1 = \beta V'(d)$ or

$$\frac{\pi}{\beta} = F\left(\frac{d}{\pi q^*}\right) + \int_{\frac{d}{\pi q^*}}^{1} u'\left(\frac{d}{\pi \alpha}\right) f(\alpha) d\alpha \quad (= \Upsilon(d)), \tag{3}$$

while for $d \geq \Upsilon^{-1}(\pi R)$, we have $z = z(d) \in (0, d)$ and $V'(d) = R$, that is, the marginal value of deposits hits the lower bound $R$ for high values of $d$. We therefore have the following result.

**Theorem 1 (Monetary Equilibrium without LLR)** *A stationary monetary equilibrium exists without an LLR, and is unique, where the deposit and cash holdings, denoted by $d_N$ and $z_N$, satisfy $d_N = z_N = \Upsilon^{-1}(\frac{\pi}{\beta}) \in (0, \pi q^*)$, and a liquidity crisis occurs with probability $1 - F(\alpha^*) \in (0, 1)$, where $\alpha^* = \frac{d_N}{\pi q^*} < 1$.*

Not surprisingly, without an LLR, banks hold deposits only as cash and no long-term assets, that is, $d_N = z_N = \Upsilon^{-1}(\frac{\pi}{\beta}) \in (0, \pi q^*)$, because both long-term assets, safe and risky, generate a real return less than the associated costs, $\beta \max\{\eta \lambda R, R\} = \beta R < 1$, and there is no use for them in the DM trade. With aggregate money demand uncertainty, the probability of a crisis is positive. This creates a situation in which in the absence of an LLR, non-monitored buyers are unable to consume the first best quantity because the bank runs out of cash reserves. The probability of a crisis goes to zero following the Friedman rule, that is, $\alpha^* = \frac{d_N}{\pi q^*} \to 1$ as $\pi \to \beta$.

The result on the effects of inflation is as follows:

**Corollary 1 (Effects of Inflation)** *For $\xi \in (0, \hat{\xi})$ with some $\hat{\xi} > 1$ (see the proof), the equilibrium deposit level, $d_N$, is monotonically decreasing in inflation, and the probability of a liquidity crisis, $1 - F(\alpha^*)$, is monotonically increasing in inflation.*

A higher inflation yields both a substitution effect and an income effect. The substitution effect occurs because higher inflation raises the cost of holding monetary reserves, and, therefore, banks will attempt to economize on such holdings, leading to a lower deposit level. Higher inflation also implies that banks can obtain the same amount of returns by increasing their reserves. In this manner, the income effect leads to an increase in the demand for cash reserves, leading to a higher deposit level. If $\xi$ is not too big, the substitution effect dominates

the income effect, which is assumed to be the case throughout this paper, implying that the bank's reserves are decreasing in inflation. As a result, the likelihood that the entire banking system will run out of liquidity increases. This result is consistent with empirical studies (e.g., Demirgüç-Kunt and Detragiache 1998, 2005).

So far, we have considered the baseline model with perfect (unsecured) credit and without the LLR. In the baseline model, because long-term assets are not used in the DM, banks never invest in them (100-percent-reserves banks or narrow banks) in equilibrium, and consumption levels of the two types of buyers in the DM are dichotomized. If perfect credit is not available, the bank's portfolio choice is not trivial. Our earlier paper, Matsuoka and Watanabe (2019), considers secured credit where these assets are used as a medium of exchange or collateral in the DM monitored meeting, and shows that banks actually invest in long-term assets even in the absence of the LLR. Introducing the LLR into our earlier model with secured credit would be very complicated, and a liquidity premium of the long-term asset would arise due to spending it for consumption in the DM as a medium of exchange and pledging it for a discount window loan in case of crises. For ease of exposition and to focus on the latter collateral role, we assume perfect credit in the monitored meeting in the baseline model. Note that the banks still play important roles of sharing risks and liquidity in our model because the opportunity cost of holding monetary reserves is reduced. We conjecture, however, that our main insight is not affected by this simplification.

## 4 Lender of Last Resort

In this section, we extend our baseline framework and allow the central bank to act as the LLR.

After the realization of $\alpha$, the central bank opens a discount window at the beginning of the DM, and offers private banks an intra-day cash loan with a real gross interest rate, $R^C > 1$. The central bank can limitlessly prepare cash for illiquid private banks at a rate of $R^C$. Clearly, private banks do not need additional cash if they have enough cash reserves

to cover their depositors' needs, and so the discount window will be activated only when a liquidity crisis occurs. Note that since private banks operate subject to limited liability, the central bank needs a guarantee for their loans' repayment given a possible default by the private banks. That is, the central bank's loans are fully collateralized as in current practice. We assume that the revenue (or loss) earned by the LLR through this lending policy is rebated to (or taxed on) the buyers in a lump sum manner.

In order to provide a role for the LLR, we now assume that the rate of return on money is low enough to satisfy

$$\frac{1}{\pi} < \min\{\eta \underline{R}^C, R\},$$

where $\underline{R}^C \equiv R/\{1 - \beta\eta(\lambda - 1)R\} > R$. If this condition were violated, money would dominate the safe asset or the risky asset or both, and the equilibrium allocation would be the same as in Theorem 1.

With the LLR, a private bank's deposit contract is modified to $(q^n, q^m, z, b, k, l)$, where $b$ is the amount of real balances a private bank borrows from the LLR in the event of a liquidity crisis. Note that safe and risky assets are substitutes. In what follows, we therefore consider two extreme cases, one with only the safe asset, that is, $k \geq 0 = l$, and the other with only the risky asset, that is, $l \geq 0 = k$. Banks compare the expected utilities of these two cases and choose the one with the higher utility in equilibrium.[5]

## 4.1 Safe Asset

We first consider the case where private banks invest only in the safe asset. It is now assumed that[6]

$$R^C > \pi R.$$

_____

[5]One can extend our model by assuming that the central bank can distinguish types of assets held by private banks and set appropriate discount rates and haircuts when they are used as collateral. For example, the model can allow only a fraction $\chi_i \in [0, 1]$ of asset $i$ to be used as collateral for a discount window loan. In this setup, the banks would have both assets to deal with a liquidity shortage appropriately.

[6]This assumption also captures the view of Bagehot (1873) that "[a very high interest rate] will operate as a heavy fine on unreasonable timidity, and will prevent the greatest number of applications by persons who don't require it. The rate should be raised early in the panic, so that the fine may be paid early; that no one may borrow out of idle precaution without paying well for it; that the banking reserve may be protected as far as possible"(p. 199).

If $R^C \leq \pi R$, banks would avoid the cost of holding cash reserves by not transferring money over time (i.e., by setting $z = 0$), and so a monetary equilibrium would not exist.[7] In addition, the assumption $R^C > 1$, which holds automatically under the above assumption, guarantees the "pecking order" that the banks first use their cash reserves and next borrow from the LLR. To derive the monetary equilibrium with the LLR, we follow the same steps as in the previous section.

As mentioned above, since private banks operate subject to limited liability, the central bank needs a guarantee of repayment. We assume that private banks can use their long-term assets as collateral.[8] Formally, if a private bank holds $k \geq 0$ safe assets and borrows $b \geq 0$ real cash balances, then it should satisfy the borrowing constraint,

$$R^C b \quad \leq \quad Rk. \tag{4}$$

The constraint states that a bank that demands and borrows $\frac{b}{\phi_+}$ yen (in nominal terms) from the central bank after the realization of $\alpha$, must pay back $\frac{R^C b}{\phi_+}$ yen ($\frac{R^C \phi_+ b}{\phi_+} = R^C b$ real balances) during the following night. This quantity must be no greater than the expected return on safe assets $\frac{Rk}{\phi_+}$ yen. Note that since the central bank loans are intra-day, its marginal cost is free from inflation.

Since the revenue earned by the LLR is rebated to the buyers, the government budget constraint in a given period is now modified as follows:

$$T = \phi(M_+ - M) + (R^C - 1)b,$$

where $b$ is determined by the banks in the previous DM. The first term on the right-hand side represents seigniorage, while the second is the profit on the discount window loans.

---

[7]The LLR would not like to support liquidity at a very low interest rate, which leads banks to hold zero monetary reserves, for several reasons. One possible reason is that if the LLR would do this, interbank markets, which are not modeled here, would collapse because no bank could supply liquidity. As a result, massively lending liquidity to all private banks without relying on the markets, which involves extensive evaluation and monitoring, would be very costly for the central bank. Another possible reason is that there would be large political pressure against lending zero-reserves banks because some taxpayers would refuse it. While we do not pursue further this problem, we just focus on a monetary equilibrium which is more relevant in practice.

[8]Equivalently, we can assume that the central bank makes a repurchase agreement with a troubled bank, whereby the bank sells some of its long-term assets to the central bank during the day in exchange for money and buys them back for a price of $R^C$ during the night.

Given values of $z > 0$ and $k \geq 0$, for each realization of $\alpha \in (0, 1)$, a private bank's problem of choosing the payment schedule in the DM is described as follows:

$$\max_{\theta \in [0,1], \ b \geq 0} \alpha u(q^n) + (1 - \theta)\frac{z}{\pi} - R^C b,$$

subject to the borrowing constraint (4) and

$$\alpha q^n = \frac{\theta z}{\pi} + b.$$

The first order conditions are (1) and

$$u'(q^n) - R^C + \mu_b - \mu_k R^C = 0, \tag{5}$$

where $\mu_b \geq 0$ is the Lagrange multiplier of the non-negativity constraint $b \geq 0$ and $\mu_k \geq 0$ is the Lagrange multiplier of the borrowing constraint (4). The occurrence of a liquidity crisis is described by the same critical value as before, $\alpha^* \equiv \frac{z}{\pi q^*}$, determined by the same condition (1). The condition (5) determines two more critical values (see below) that describe the borrowing quantity of private banks in the case of crises.

**Lemma 3 (Banks' Optimal Payment Plan with the LLR and the Safe Asset)** *Given* $z > 0$, $k \geq 0 = l$, *the optimal payment plan of banks with the safe asset in the presence of the LLR is described by* $q^m = q^*$ *and* $\theta = \theta(\alpha)$, *similar to that in Lemma 1, and*

$$b(\alpha) = \begin{cases} 0 & \text{if } 0 < \alpha \leq \alpha^{**}, \\ \alpha u^{-1\prime}(R^C) - \frac{z}{\pi} & \text{if } \alpha^{**} < \alpha < \alpha^{***}, \\ \frac{Rk}{R^C} & \text{if } \alpha^{***} \leq \alpha < 1, \end{cases} \quad \text{and} \quad q^n(\alpha) = \begin{cases} q^* & \text{if } 0 < \alpha < \alpha^*, \\ \frac{z}{\alpha \pi} & \text{if } \alpha^* \leq \alpha \leq \alpha^{**}, \\ u^{-1\prime}(R^C) & \text{if } \alpha^{**} < \alpha < \alpha^{***}, \\ \frac{\frac{R^C}{\pi} z + Rk}{R^C \alpha} & \text{if } \alpha^{***} \leq \alpha < 1, \end{cases}$$

*where*

$$\alpha^* \equiv \frac{z}{\pi q^*}, \quad \alpha^{**} \equiv \min\left\{\frac{z}{\pi u^{-1\prime}(R^C)}, 1\right\}, \quad \text{and} \quad \alpha^{***} \equiv \min\left\{\frac{\frac{R^C}{\pi} z + Rk}{R^C u^{-1\prime}(R^C)}, 1\right\}.$$

As in the model without the LLR, liquidity crises will never occur when $\alpha \in (0, \alpha^*)$ and private banks do not need to borrow from the central bank. A liquidity crisis occurs when $\alpha \in [\alpha^*, 1)$ in any one of the following three cases: for $\alpha \in [\alpha^*, \alpha^{**}]$, banks exhaust their cash reserves but do not borrow because the benefit of the LLR is relatively low; for $\alpha \in (\alpha^{**}, \alpha^{***})$, banks borrow cash as much as they want at a rate of $R^C$; for $\alpha \in [\alpha^{***}, 1)$, the borrowing constraint is binding and banks borrow the maximum level, $Rk$. Note that the assumption $R^C > 1$ guarantees the condition $\alpha^* < \alpha^{**}$ for any $z > 0$.

Given the repayment plan, we now consider the portfolio choice. A bank's problem is described by the same value function as in (2), but now the value function in the CM is given by

$$W(z', k, 0, c, b) = Rk + \frac{z'}{\pi} + T - c - R^C b + \max_{x \geq 0} \{U(x) - x\} + \max_{d_+ \geq 0} \{-d_+ + \beta V^s(d_+)\},$$

where the payment amount of $R^C b$ will be made to the LLR. Applying this, $\theta = \min\{1, \frac{\alpha}{\alpha^*}\}$, the result in Lemma 3 and the balance sheet constraint, $d = z + k$, to the value function (2) and rearranging, we have:

$$V^s(d) = \max_{0 \leq z \leq d} \int_0^{\alpha^*} \left[ \alpha u(q^*) + \left(1 - \frac{\alpha}{\alpha^*}\right) \frac{z}{\pi} \right] f(\alpha)d\alpha + \int_{\alpha^*}^{\alpha^{**}} \alpha u\left(\frac{z}{\alpha\pi}\right) f(\alpha)d\alpha$$

$$+ \int_{\alpha^{**}}^{\alpha^{***}} \left[ \alpha u\left(u^{-1\prime}(R^C)\right) - R^C \left(\alpha u^{-1\prime}(R^C) - \frac{z}{\pi}\right) \right] f(\alpha)d\alpha + \int_{\alpha^{***}}^1 \left[ \alpha u\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) - R(d-z) \right] f(\alpha)d\alpha$$

$$+ \{1 - E(\alpha)\} \{u(q^*) - q^*\} + R(d-z) + T + \max_{x \geq 0}\{U(x) - x\} + \max_{d_+ \geq 0}\{-d_+ + \beta V^s(d_+)\},$$

where $E(\alpha) \equiv \int_0^1 \alpha f(\alpha)d\alpha$.

Let us define

$$\Psi_z(z, d) \equiv F(\alpha^*) + \int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right) f(\alpha)d\alpha + [F(\alpha^{***}) - F(\alpha^{**})]R^C + \int_{\alpha^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha,$$

$$\Psi_k(z, d) \equiv F(\alpha^{***}) + \frac{1}{R^C} \int_{\alpha^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha.$$

Then, noting $\alpha^{***} \leq 1$, the first order condition of the optimal portfolio choice is:

$$\Psi(z, d) \equiv \frac{1}{\pi} \Psi_z(z, d) - R\Psi_k(z, d) \geq 0 \quad (= \text{ if } z < d). \tag{6}$$
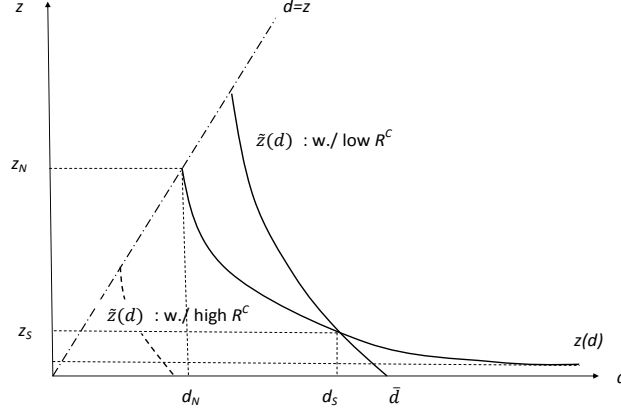
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Fig 3: Monetary equilibrium with the LLR and the safe asset

Denote by $z = z(d)$ a solution to (6) and $k(d) \equiv d - z(d)$. Then, given the optimal portfolio $(z(d), k(d))$, the deposit choice of depositors is described by the Euler equation, $-1 + \beta V^{s\prime}(d) = 0$, where

$$V^{s\prime}(d) = \frac{\partial z}{\partial d} \frac{\partial V^s(d)}{\partial z} + \frac{\partial V^s(d)}{\partial d}.$$

There are two cases. For $z = z(d) \in (0, d)$ (an interior solution), by applying the envelope condition, we have $\Psi(z, d) = \frac{\partial V^s(d)}{\partial z} = 0$ and $V^{s\prime}(d) = \frac{\partial V^s(d)}{\partial d} = R\Psi_k(z, d)$, and so the Euler equation is

$$\Psi_k(z, d) = \frac{1}{\beta R}. \tag{7}$$

Hence, an equilibrium, denoted by $(z_S, d_S) > 0$, is identified by (7) and

$$\Psi_z(z, d) = \frac{\pi}{\beta}. \tag{8}$$

For $z = d$ (the corner solution), we have $\alpha^{**} = \alpha^{***}$, $\Psi(z, d) = \frac{\partial V^s(d)}{\partial z} > 0$ and $V^{s\prime}(d) = \Psi(z, d) + R\Psi_k(z, d) = \frac{\Psi_z(z,d)}{\pi}$, and so the Euler equation is given by (3); since the bank does not hold any long-term assets, it cannot borrow from the LLR, that is, $b = 0$.

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**Proposition 1 (Monetary Equilibrium with the LLR and the Safe Asset)** *With the LLR and the safe asset, a stationary monetary equilibrium with bank deposits exists and is unique such that cash reserves and bank deposits satisfy*

$$z = \begin{cases} z_S \ (< z_N) & for \ \ R^C \in (\pi R, R^{C*}], \\ z_N \ (= d_N) & for \ \ R^C \in (R^{C*}, \infty), \end{cases}$$

*with some critical value $R^{C*} \in (\pi R, \infty)$, and $z_S \leq z_N$ and $d_S \geq d_N$. Further, whenever $R^C < R^{C*}$, it holds that $\alpha^{***} < 1$.*

As illustrated in Figure 3, a monetary equilibrium with a positive amount of the safe asset can be identified using two curves: $\tilde{z} = \tilde{z}(d)$, which is constructed using the Euler equation (7), and $z = z(d)$, which is constructed using the first order condition (8). Both these curves are downward sloping, because intuitively, cash and long-term assets are substitutes when the LLR lending is available during crises. As long as the loan rate from the LLR is not too high, that is, $R^C \in (\pi R, R^{C*}]$, the $\tilde{z}(d)$ curve intersects the $z(d)$ curve at $z = z_S < z_N$ and $d = d_S > d_N$ where the bank invests in a long-term safe asset $d_S - z_S > 0$, and the discount window lending is activated in case of crises. In contrast, if the rate is sufficiently high, that is, $R^C > R^{C*}$, banks will never borrow from the discount window, and the equilibrium allocation is identical to the one without the LLR as described in Theorem 1.

In the presence of the LLR, a long-term asset has a benefit in the DM that allows the bank to use the discount window lending during a liquidity crisis. To derive the liquidity premium on the safe asset, rewriting (7) yields

$$\frac{1 - \beta R}{\beta R} = \int_{\alpha^{***}}^{1} \left\{ \frac{u'(q^n)}{R^C} - 1 \right\} f(\alpha) d\alpha. \tag{9}$$

The left-hand side of this equation represents the cost of investing an additional unit of the safe asset, and the right-hand side represents the liquidity premium. When the borrowing constraint (4) is not binding, which occurs with probability $F(\alpha^{***})$, an additional unit of the safe asset has no effects in the DM, and thus the liquidity return is zero. When the borrowing constraint (4) is binding, which occurs with probability $1 - F(\alpha^{***})$, a bank can use $\frac{1}{R}$ unit
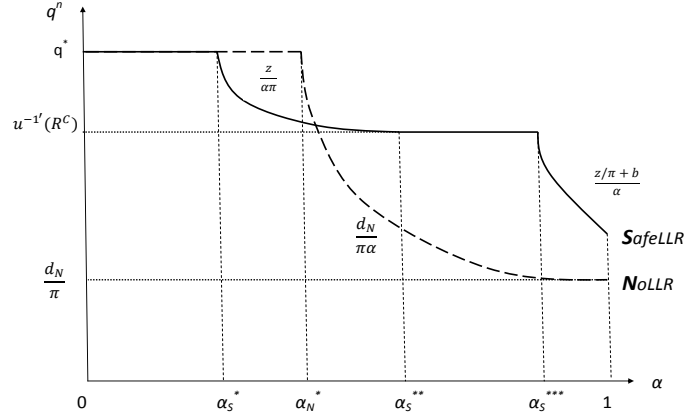
Fig 4: Monetary equilibrium with versus without the LLR

of the safe asset to borrow $\frac{1}{R^C}$ units of real cash balances, which allows each non-monitored buyer to acquire $\frac{1}{\alpha R^C}$ units of special goods with a total marginal utility of $\frac{\alpha u'(q^n)}{\alpha R^C}$.

The banks always have the option not to borrow from the discount window, and, thus, the LLR must be welfare improving if it is used. It mitigates the loss of liquidity crises by providing liquidity, leading to a higher level of deposit and consumption in the DM non-monitored meeting. The LLR will, however, reduce the incentive of banks to invest in liquid assets. As a result, banks hold a lower amount of cash reserves, and thus the LLR will increase the probability of liquidity crises. In Figure 4, the critical value without the LLR is represented by $\alpha_N^*$, and those with the LLR are represented by $\alpha_S^*, \alpha_S^{**}$, and $\alpha_S^{***}$.

**Corollary 2 (Implication of the LLR)** *The LLR is welfare improving, but increases the probability of a liquidity crisis.*

We now study the consequences of changing $\pi$ and $R^C$. With corner solutions, since all deposits are invested in cash reserves and banks do not borrow from the central bank, that is, $(z_S, d_S) = (z_N, d_N)$, the comparative static results are identical to those described in Proposition 1. Using interior solutions, we obtain:

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**Corollary 3 (Effects of Inflation and the Loan Rate)** *For $\xi \in (0, \check{\xi})$ with some $\check{\xi} > 0$ (see the proof), the interior solutions with the safe asset $(z_S, d_S)$ satisfy*

$$\frac{\partial z_S}{\partial \pi} < 0, \quad \frac{\partial d_S}{\partial \pi} > 0, \quad \frac{\partial z_S}{\partial R^C} > 0, \quad and \quad \frac{\partial d_S}{\partial R^C} < 0.$$

*Furthermore,*

$$\frac{\partial \alpha^*}{\partial \pi} < 0, \quad \frac{\partial \alpha^{**}}{\partial \pi} < 0, \quad \frac{\partial \alpha^{***}}{\partial \pi} = 0, \quad \frac{\partial \alpha^*}{\partial R^C} > 0, \quad \frac{\partial \alpha^{**}}{\partial R^C} > 0, \quad and \quad \frac{\partial \alpha^{***}}{\partial R^C} = 0.$$

If $\xi > 0$ is not too big (i.e., the substitution effect dominates the income effect), a higher inflation leads to a higher money holding cost, and a higher incentive for banks to economize their cash reserves, leading to a positive effect of inflation on the probability of a liquidity crisis and the likelihood that banks borrow from the central bank. Somewhat surprisingly, the effect of an increase in $\pi$ on $d$ differs depending on the presence of the LLR. In the case with the LLR, at a higher inflation, banks attempt to rely more on the central bank loans to avoid higher money holding costs. To do this, they have to collect more deposits and invest more in long-term assets as collateral for loans. In contrast, there is no such channel without the LLR, and, thus, the deposit level is simply decreasing in inflation, as most monetary general equilibrium models predict. Inflation has no effect on $\alpha^{***}$ because the negative impact on reserves and the positive impact on the safe asset cancel out. An increase in $R^C$ gives the bank an incentive to be self-guarding and increase their cash reserves, leading to positive impacts on $\alpha^*$ and $\alpha^{**}$, but since it makes the deposit contracts less attractive the deposit level will be lower.

## 4.2 Risky Asset

Consider next the case of the risky asset. An important difference is that given limited liability, banks must default when the risky asset produces nothing, which is referred to as a *banking default*. Since banks are able to honor their promise to repay the central bank loans only if the project is successful, the expected payment rate of the discount window loan with the risky asset is reduced to $\eta R^C$. In line with the previous analysis, we assume that

$$R^C > \pi \underline{R}^C,$$

where $\underline{R}^C \equiv R/\{1 - \beta\eta(\lambda-1)R\} > R$. If $R^C \leq \pi\underline{R}^C$, the demand for cash reserves would then be zero, and money would not circulate between periods. As before, the condition $R^C > 1/\eta$, which holds automatically since $\eta\underline{R}^C > 1/\pi$, guarantees the "pecking order" of the banks' use of cash reserves.

Note that given the information structure associated with the observability of the risky asset returns, the lending strategy of the LLR cannot be contingent on the type of assets banks hold, and the banks face the same borrowing constraint as described in (4) for the discount window borrowing.

With a risky asset, there is an issue of who should take on the financial burden of losses associated with banking defaults on the discount window loan. Ultimately, additional taxes levied on agents should compensate for these losses. This implies, together with the constant money supply rule, that the expected lump sum transfer (or tax) is given by

$$E(T) = \phi(M_+ - M) + (\eta R^C - 1)b.$$

With a risky asset, the optimal payment plan of private banks is modified as follows.

**Lemma 4 (Banks' Optimal Payment Plan with the LLR and the Risky Asset)** *Given* $z > 0$, $l \geq 0 = k$, *the optimal payment plan of banks with the risky asset in the presence of the LLR is described by* $q^m = q^*$ *and* $\theta = \theta(\alpha)$, *the same as in Lemma 1, and*

$$b(\alpha) = \begin{cases} 0 & \text{if } 0 < \alpha \leq \alpha_\eta^{**}, \\ \alpha u^{-1\prime}(\eta R^C) - \frac{z}{\pi} & \text{if } \alpha_\eta^{**} < \alpha < \alpha_\eta^{***}, \quad \text{and} \quad q^n(\alpha) = \begin{cases} q^* & \text{if } 0 < \alpha \leq \alpha^*, \\ \frac{z}{\alpha\pi} & \text{if } \alpha^* \leq \alpha \leq \alpha_\eta^{**}, \\ u^{-1\prime}(\eta R^C) & \text{if } \alpha_\eta^{**} < \alpha < \alpha_\eta^{***}, \\ \frac{\frac{R^C}{\pi}z+Rl}{R^C\alpha} & \text{if } \alpha_\eta^{***} \leq \alpha < 1, \end{cases}$$

*where*

$$\alpha^* \equiv \frac{z}{\pi q^*}, \quad \alpha_\eta^{**} \equiv \min\left\{\frac{z}{\pi u^{-1\prime}(\eta R^C)}, 1\right\}, \quad \text{and} \quad \alpha_\eta^{***} \equiv \min\left\{\frac{\frac{R^C}{\pi}z + Rl}{R^C u^{-1\prime}(\eta R^C)}, 1\right\}.$$

The solutions are quite similar to the ones of Lemma 3, except that, since the borrowing cost associated with the discount window is reduced, that is, $u^{-1\prime}(\eta R^C) > u^{-1\prime}(R^C)$, the bank increases the amount of borrowing compared to the case with a safe asset.

One of the important criticisms of Bagehot's rule is that there is no clear-cut distinction between illiquidity and insolvency during a crisis (e.g., Goodhart, 1987; 1999). We capture this point well because in our model the timings of bank's illiquidity and insolvency are different; illiquidity can occur during a day, while insolvency can occur at the beginning of a night. Thus, the LLR must decide whether to lend its funds to illiquid banks before their asset returns are realized. In such an environment, the banks have an incentive to borrow and default.

Given the repayment plan with a risky asset, the value function in the DM is modified to

$$
\begin{aligned}
V^r(d) = & \max_{0 \le z \le d} \int_0^{\alpha^*} \left[ \alpha u(q^*) + \left(1 - \frac{\alpha}{\alpha^*}\right) \frac{z}{\pi} \right] f(\alpha) d\alpha + \int_{\alpha^*}^{\alpha_\eta^{**}} \alpha u\left(\frac{z}{\alpha\pi}\right) f(\alpha) d\alpha \\
& + \int_{\alpha_\eta^{**}}^{\alpha_\eta^{***}} \left[ \alpha u\left(u^{-1\prime}(\eta R^C)\right) - \eta R^C \left(\alpha u^{-1\prime}(\eta R^C) - \frac{z}{\pi}\right) \right] f(\alpha) d\alpha \\
& + \int_{\alpha_\eta^{***}}^1 \left[ \alpha u\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha}\right) - \eta R(d-z) \right] f(\alpha) d\alpha \\
& + \{1 - E(\alpha)\}\{u(q^*) - q^*\} + \eta\lambda R(d-z) + T + \max_{x \ge 0}\{U(x) - x\} + \max_{d_+ \ge 0}\{-d_+ + \beta V^r(d_+)\}.
\end{aligned}
$$

Based on this value function, a monetary equilibrium, denoted by $(z_R, d_R)$, with $l_R = d_R - z_R > 0$ is characterized by the following two implicit equations:

$$
\Phi_z(z,d) \equiv F(\alpha^*) + \int_{\alpha^*}^{\alpha_\eta^{**}} u'\left(\frac{z}{\alpha\pi}\right) f(\alpha) d\alpha + [F(\alpha_\eta^{***}) - F(\alpha_\eta^{**})]\eta R^C + \int_{\alpha_\eta^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha}\right) f(\alpha) d\alpha = \frac{\pi}{\beta},
\tag{10}
$$

$$
\Phi_l(z,d) \equiv \eta F(\alpha_\eta^{***}) + \frac{1}{R^C} \int_{\alpha_\eta^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha}\right) f(\alpha) d\alpha + \eta(\lambda - 1) = \frac{1}{\beta R}.
\tag{11}
$$

According to the Euler equation (11), the liquidity premium on the risky asset should satisfy:

$$
\frac{1 - \beta\eta\lambda R}{\beta\eta\lambda R} = \frac{1}{\lambda} \int_{\alpha_\eta^{***}}^1 \left\{ \frac{u'(q^n)}{\eta R^C} - 1 \right\} f(\alpha) d\alpha.
\tag{12}
$$

The cost of investing an additional unit of the risky asset should equal the liquidity premium. When the borrowing constraint is not binding, which occurs with probability $F(\alpha_\eta^{***})$, the

liquidity return is zero since an additional unit of the risky asset has no effect in the DM. When the borrowing constraint is binding, which occurs with probability $1 - F(\alpha_\eta^{***})$, the liquidity return is positive since a bank can use $\frac{1}{\eta\lambda R}$ units of the risky asset to borrow $\frac{1}{\eta R^C}$ units of real cash balances, which allows each non-monitored buyer to acquire $\frac{1}{\alpha\eta R^C}$ units of special goods and its total marginal utility is $\frac{\alpha u'(q^n)}{\alpha\eta R^C}$. Finally, since only a fraction $\frac{R}{\lambda R}$ of the successful return of the risky asset is pledgeable, the liquidity return is simply discounted by $\frac{1}{\lambda}$.

Identifying a monetary equilibrium will follow the same steps as before.

**Proposition 2 (Monetary Equilibrium with the LLR and the Risky Asset)** *With the LLR and risky asset, a stationary monetary equilibrium with bank deposits exists and is unique so that the cash reserve balances and the bank's deposits satisfy*

$$z = \begin{cases} z_R \ (< z_N) & \text{for} \ \ R^C \in (\pi\underline{R}^C, \hat{R}^{C*}], \\ z_N \ (= d_N) & \text{for} \ \ R^C \in (\hat{R}^{C*}, \infty), \end{cases}$$

*with some critical value $\hat{R}^{C*} \in (\pi\underline{R}^C, \infty)$, and $z_R < z_N$ and $d_R > d_N$ for any $\eta \in (0,1)$. Further, whenever $R^C < \hat{R}^{C*}$, it holds that $\alpha_\eta^{***} < 1$.*

As in the case with a safe asset, a monetary equilibrium with a positive amount of the risky asset can be identified by using two curves: $\tilde{z} = \tilde{z}_\eta(d)$, which is constructed using (11), and $z = z_\eta(d)$, which is constructed using (10), as illustrated in Figure 5. The $z_\eta(d)$ curve with a risky asset pivots down, centered at the monetary equilibrium allocation with no LLR, $(z_N, d_N)$, relative to that with a safe asset. This occurs because the expected payment cost of the discount window lending is lower with the risky asset than the safe asset, and so given values of $d > 0$, the bank has a higher incentive to invest in long-term asset, leading to a higher crowding out of cash reserves. Whether the $\tilde{z} = \tilde{z}_\eta(d)$ curve with the risky asset is located below or above the one with the safe asset is determined by the relative size of the cost advantage versus the lower expected return of the risky asset. The figure illustrates a situation where the former benefit outweighs the latter cost.
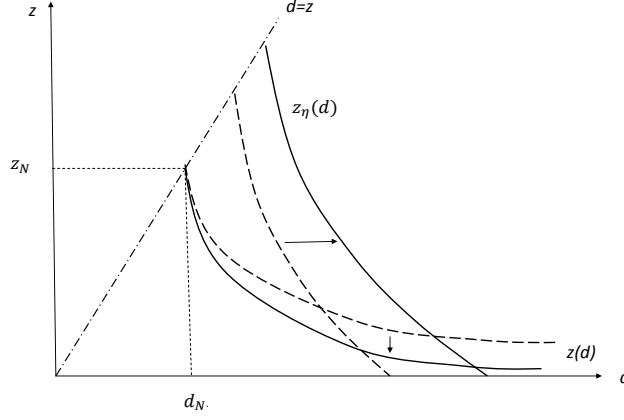
Fig 5: Monetary equilibrium with the LLR and the risky asset

As before, the LLR is beneficial since it mitigates the loss of liquidity crises, $d_R > d_N$ and $z_R < z_N$, but accompanies a higher probability of crises in the first place.

The following comparative static results and their intuitions are quite similar to the ones with a safe asset. The proof is lengthy but is quite similar to the one offered in Corollary 3. We therefore do not provide it here (available upon request).

**Corollary 4 (Effects of Inflation and the Loan Rate)** *For $\xi \in (0, \bar{\xi})$ with some $\bar{\xi} > 0$, the interior solutions with the risky asset satisfy*

$$\frac{\partial z_R}{\partial \pi} < 0, \quad \frac{\partial d_R}{\partial \pi} > 0, \quad \frac{\partial z_R}{\partial R^C} > 0, \quad and \quad \frac{\partial d_R}{\partial R^C} < 0.$$

*Furthermore,*

$$\frac{\partial \alpha^*}{\partial \pi} < 0, \quad \frac{\partial \alpha_\eta^{**}}{\partial \pi} < 0, \quad \frac{\partial \alpha_\eta^{***}}{\partial \pi} = 0, \quad \frac{\partial \alpha^*}{\partial R^C} > 0, \quad \frac{\partial \alpha_\eta^{**}}{\partial R^C} > 0, \quad and \quad \frac{\partial \alpha_\eta^{***}}{\partial R^C} = 0.$$

## 4.3 Asset Choice

We now study the optimal asset choice of private banks in the presence of the LLR. As mentioned before, safe and risky assets are substitutes, and so depositors choose the higher

expected utilities of these two assets. Hence, the problem is

$$\max\{-d_S + \beta V^s(d_S), -d_R + \beta V^r(d_R)\}. \tag{13}$$

Let us define the difference between the net expected values of the safe and risky assets as:

$$\Delta(R^C, \eta) \quad \equiv \quad (1-\beta)[\{-d_S + \beta V^s(d_S)\} - \{-d_R + \beta V^r(d_R)\}]. \tag{14}$$

Clearly, the safe asset is selected if $\Delta(R^C, \eta) > 0$, while the risky asset is selected if $\Delta(R^C, \eta) < 0$.

The next lemma establishes some properties of $\Delta(R^C, \eta)$.

**Lemma 5**     *1. If $R^C > \max\{R^{C*}, \hat{R}^{C*}\}$, then $\Delta(R^C, \eta) = 0$.*

  *2. If $R^C \in [R^{C*}, \hat{R}^{C*})$, then $\Delta(R^C, \eta) < 0$.*

  *3. If $R^C \in [\hat{R}^{C*}, R^{C*})$, then $\Delta(R^C, \eta) > 0$.*

  *4. As $\eta \to 1$ and $\lambda \to 1$, $\Delta(R^C, \eta) \to 0$.*

  *5. As $\eta \to 1$ and $\lambda \to 1$, $\frac{\partial \Delta}{\partial R^C}(R^C, \eta) \to 0$.*

  *6. Given $\lambda > 1$, if $\eta\lambda$ and $\beta R$ are sufficiently close to 1, then $\lim_{R^C \to \pi \underline{R}^C} \Delta(R^C, \eta) < 0$.*

The first property follows immediately from the above discussion and Propositions 1 and 2. If $R^C > \max\{R^{C*}, \hat{R}^{C*}\}$, it follows that $-d_S + \beta V^s(d_S) = -d_R + \beta V^r(d_R) = -d_N + \beta V(d_N)$, so that the discount window lending is not used, and asset choice becomes irrelevant. To examine the second property, suppose $R^C \in [R^{C*}, \hat{R}^{C*})$. Then, since the loan rate $R^C$ is greater than the threshold rate at which the safe asset is used as collateral $(R^{C*})$, but less than the threshold rate at which the risky asset is used $(\hat{R}^{C*})$, the bank is willing to invest in the risky asset. That is, moral hazard occurs. Banks can borrow from the discount window and default with probability $1-\eta$. To analyze the third property, suppose $R^C \in [\hat{R}^{C*}, R^{C*})$. Then, applying an argument to the second, $R^C$ is greater than the threshold rate of the risky asset but less than the rate of the safe asset, and so banks use the safe asset to borrow money from the discount window. Banks behave prudently and never default on the loans. The intuition for the fourth property is that if $\eta$ is very close to unity, the safe and risky assets are fundamentally the same and the net expected values are equalized. Note here that $\eta \to 1$ implies that the only

admissible parameter value is $\lambda \to 1$ given $\lambda > 1$ and $\eta\lambda \le 1$. The fifth property states that the effect of the lending rate on the choice of investment becomes negligible as $\eta$ gets closer to unity. This property is in sharp contrast to the conventional view that a higher lending rate has an impact on moral hazard. The final property states that if $R^C$ is close to $\pi\underline{R}^C$, banks with risky assets will hold almost no cash reserves and rely on the discount window because it is free from inflation costs. On the other hand, banks with the safe asset will hold a positive amount of cash reserves since $\underline{R}^C > R$ and will use the discount window loan only when they exhaust the reserves. Since the safe asset's advantage in returns will disappear if $\eta\lambda$ is sufficiently close to unity, and banks can avoid inflation costs by investing in the risky asset, the risky asset will dominate the safe asset as $R^C \to \pi\underline{R}^C$. This advantage of the risky asset becomes greater as the cost of holding collateral becomes small, that is, $R \to \frac{1}{\beta}$. The last two properties of Lemma 5 are the key building blocks for Proposition 3.

The next lemma provides a characterization of $R^{C*} \gtrless \hat{R}^{C*}$.

**Lemma 6** *There exists $(\eta\lambda)^* \in (\frac{1}{\pi R}, 1)$ such that $R^{C*} < \hat{R}^{C*}$ if $\eta\lambda \in ((\eta\lambda)^*, 1]$ and $R^{C*} > \hat{R}^{C*}$ if $\eta\lambda \in (0, (\eta\lambda)^*)$.*

Combining Lemmas 5 and 6, we can say that if the expected return on the risky asset, $\eta\lambda$, is sufficiently high, the risky asset is selected for $R^C \in [\pi\underline{R}^C, \hat{R}^{C*})$. In this case, since the risky asset is always selected whenever the discount window is activated, controlling the lending rate is not effective in preventing the banks from taking excessive risks.

To summarize, we provide the following proposition.

**Proposition 3 (Moral Hazard)** *In a stationary monetary equilibrium with the LLR, the discount window is activated if and only if the lending rate is low, $R^C < \max\{R^{C*}, \hat{R}^{C*}\}$. When LLR lending is used, private banks will invest in the risky asset rather than the safe asset if the expected return of the risky asset is sufficiently high and the cost of holding the collateral is sufficiently small.*

Relative to the safe asset, the risky asset has a cost advantage, but a relatively lower expected return. Thus, if the expected return of the risky asset is not too low, the risky asset

will be selected, and banking defaults can occur in equilibrium. Notice here that liquidity crises and banking defaults are correlated. The risky asset leads to a higher deposit level than the safe asset does, but to lower cash reserves, $z_R < z_S$, leading to a higher probability of liquidity crises. Thus, despite the higher probability of liquidity crises, the risky asset can better mitigate the loss of crises since the banks can borrow more during a crisis, that is, $u^{-1\prime}(\eta R^C) > u^{-1\prime}(R^C)$.

Historically, discount window loans made during banking crises are often defaulted partially (sometimes totally) or their payback dates are extended since it is difficult for the LLR to distinguish between an illiquid and an insolvent bank. For example, the Bank of Japan provided emergency special loans (called *toku-yu*) to 114 selected banks in response to the panic of 1927, but about half of the rescued banks had been insolvent and were overdue in their repayments in 1933. Furthermore, the Bank of Japan could not collect more than 52 million yen in loans even in 1952 (see Yokoyama, 2018). Since, in the model, the timings of illiquidity and insolvency are different and there is asymmetric information about the quality of a bank's portfolio, our model captures some of the important elements of the LLR policy.

To further explore the equilibrium outcomes, we use some numerical examples. We assume $u(q) = q^{1-\xi}/1 - \xi$ with $\xi > 0$, and $\alpha$ is uniformly distributed, with the following parameter values: $\beta = 0.96$, $R = 1.03$, $\lambda = 1.1$, $\xi = 0.8$, and $\pi = 1.03$. The top panel of Table 1 summarizes the result with $\eta = 0.895$ ($\eta\lambda = 0.9845$), where the risky asset is selected for any $R^C < \hat{R}^{C*} = 1.266$, that is, $\Delta(R^C, \eta) < 0$. The lower panel of Table 1 illustrates the result with $\eta = 0.88$ ($\eta\lambda = 0.968$) where the safe asset is selected for any $R^C < R^{C*} = 1.233$ and $\Delta(R^C, \eta) > 0$. Clearly, for values of $\eta\lambda < 0.968$, equilibrium allocations with the safe asset are not affected by $\eta$ and $\lambda$. In any case, an increase in $R^C$ raises the bank's cash reserves and reduces deposits, leading to higher cutoffs, $\alpha_R^*$ and $\alpha_R^{**}$, or $\alpha_S^*$ and $\alpha_S^{**}$, but $\alpha_R^{***}$ or $\alpha_S^{***}$ remains unchange, as shown in Corollaries 3 and 4. Once $R^C$ exceeds the threshold $\hat{R}^{C*}$ or $R^{C*}$, banks will no longer use the central bank loans.

| $R^C$ | $\eta\lambda$ | $(z_R, d_R)$ | $(\alpha_R^*, \alpha_R^{**}, \alpha_R^{***})$ | $\Delta(R^C, \eta)$ | Types of Equilibrium |
|---|---|---|---|---|---|
| 1.18 | 0.9845 | (0.2632, 0.7828) | (0.2555, 0.2736, 0.7591) | -0.0033 | Risky |
| 1.20 | 0.9845 | (0.4426, 0.7509) | (0.4297, 0.4698, 0.7591) | -0.0019 | Risky |
| 1.22 | 0.9845 | (0.5479, 0.7234) | (0.5319, 0.5937, 0.7591) | -0.0013 | Risky |
| 1.18 | 0.9680 | (0.6186, 0.7172) | (0.6006, 0.7387, 0.8445) | 0.0021 | Safe |
| 1.20 | 0.9680 | (0.6428, 0.6990) | (0.6241, 0.7839, 0.8445) | 0.0011 | Safe |
| 1.22 | 0.9680 | (0.6608, 0.6810) | (0.6416, 0.8226, 0.8445) | 0.0004 | Safe |

Table 1: Monetary equilibria with safe and risky assets

## 4.4 Discussions

From our analysis so far, we can summarize the main implications of the economic roles and consequences of the LLR as follows.

(i) The LLR reduces a bank's monetary reserves and increases deposists, which increase the likelihood of depletion of the reserves (a liquidity crisis). However, the magnitude of a crisis is mitigated.

(ii) The LLR may create moral hazard, that is, private banks may take more financial risks in terms of long-term assets.

(iii) The occurrence of moral hazard is determined mainly by the expected relative returns of the safe and risky assets and the cost of holding collateral given the asymmetric information about the quality of a bank's assets that can be used as collateral.

From (i) and (ii), we can conclude that the existence of a credible LLR can make a financial system unstable but it is potentially welfare improving. In this sense, there is a trade-off between efficiency and financial stability. Furthermore, in contrast to conventional wisdom, from (iii), a high (penalty) lending rate cannot have much impact on moral hazard when the expected return on the risky asset is sufficiently high, the cost of holding collateral is low, and the problem of asymmetric information is severe. However, a high lending rate has still large impacts on economic variables because it increases the bank's reserves and reduces deposits, which decrease the likelihood of a liquidity crisis.

One might think that if the LLR could commit in advance regarding the quality of acceptable collaterals during a crisis, based on its efforts to keep track of banks' portfolio strategies and eliminate asymmetric information about their assets (i.e., audit and stress test), it would increase the banks' incentives to behave prudently. While we admit the significance of efforts for overcoming the asymmetric information problem, there is an important time-inconsistency problem, as discussed in Kydland and Prescott (1977) and Ennis and Keister (2009). The central bank would like banks to believe that it will accept only safe assets (e.g., T-bills) as collateral for the discount window loans during a crisis. However, if a crisis actually occurred, the central bank would find it optimal to lend liquidity to troubled banks against even their risky assets (e.g., equity shares and mortgage-backed securities). If so, banks will be willing to hold risky assets in advance, and as a result, the lack of commitment leads to moral hazard and banking defaults.

Our results are related to some existing literature on the LLR policy and moral hazard. Martin (2006) shows that a liquidity provision policy by the central bank can prevent panics without creating moral hazard. In his model, which is based on a banking model of Cooper and Ross (1998), risk-averse depositors prefer the safe asset to the risky asset because borrowing strategies and portfolio choices are dichotomized. In contrast, in our model the risky asset has a comparative advantage in borrowing money from the central bank due to limited liability, which gives banks an incentive to choose the risky asset in advance. Repullo (2005) also shows that the existence of the LLR does not increase bank's risk-taking incentives, but simply reduces their liquid reserves. His paper and our paper share a common view that the expected return on the risky asset is the main factor in determining the risk of a bank's portfolio. However, in our model the existence of the LLR still matters since the collateralized LLR lending creates a liquidity premium on illiquid long-term assets.

Finally, we have abstracted from reputation effects by assuming that banks live only for one period. These assumptions make the analysis simple and highlight banks' risk-taking investment behavior in the presence of the LLR. If banks lived infinitely, the LLR could use history-dependent lending policies that have a positive effect on banks' incentives for

prudent behavior. However, once the LLR adopts the too-big-to-fail doctrine or fail to make a strong commitment, the reputation effect would be weakened so that our analysis still remains relevant.

## 5 Extensions

In this section, we examine some policies to curb moral hazard. As stated earlier, the presence of the LLR improves welfare but decreases financial stability. Though not modeled explicitly here, a government would place more value on financial stability than ex ante efficiency, because a crisis may have significant negative impacts on the real sector (e.g., increasing unemployment, decreasing output, etc.). Can the government eliminate banking defaults or moral hazard at the expense of welfare? To answer this question, we consider two policies that are often implemented in practice: a liquidity requirement and constructive ambiguity.

### 5.1 Liquidity Requirement

Under the liquidity requirement, all banks must hold a certain proportion of liquid reserves in their portfolio. The liquidity coverage ratio is envisioned by Basel III. Specifically, the government forces banks to invest at least a fraction $\kappa \in [0,1]$ of their deposits $d$ in liquid reserves in the CM, so it has to hold that

$$z \geq \kappa d.$$

The liquidity requirement imposes a lower bound of cash reserves $z$ banks should hold. A higher $\kappa$ implies a tighter constraint for banks.

Suppose now that the liquidity requirement is severe enough to be binding only with the risky asset, but not with the safe asset, that is, $\kappa$ satisfies

$$\frac{z_R}{d_R} < \kappa < \frac{z_S}{d_S},$$

where $(z_R, d_R)$ and $(z_S, d_S)$ are the unconstrained equilibrium allocation, obtained in Proposition 1 and 2, respectively, without imposing the liquidity requirement. Remember that the

point $(z_S, d_S)$ is in the northwestern area of $(z_R, d_R)$ in Figure 5, implying that $(z_R, d_R)$ is more likely to be restricted by the requirement. Then, the first order condition of the optimal portfolio choice with risky asset must be

$$\Phi(z, d) \equiv \frac{1}{\pi} \Phi_z(z, d) - R\Phi_l(z, d) < 0,$$

with $z = \kappa d$. The Euler equation now becomes, $-1 + \beta V'(d) = 0$ where

$$V'(d) = \frac{\partial z}{\partial d} \frac{\partial V(d)}{\partial z} + \frac{\partial V(d)}{\partial d} = \kappa \Phi(z, d) + R\Phi_l(z, d) = \frac{\kappa}{\pi} \Phi_z(z, d) + (1 - \kappa)R\Phi_l(z, d).$$

Thus, the solutions with a binding liquidity requirement, denoted by $(z_R^\kappa, d_R^\kappa)$, should satisfy $z_R^\kappa = \kappa d_R^\kappa$ where $d_R^\kappa$ is the unique solution to

$$\kappa \Phi_z(\kappa d, d) + (1 - \kappa)\pi R\Phi_l(\kappa d, d) = \frac{\pi}{\beta}. \tag{15}$$

The effects of the liquidity constraint on the optimal values of deposits and cash reserves are summarized as follows.

**Proposition 4** *The optimal value of $d_R^\kappa$ is strictly decreasing in $\kappa$. The effect of $\kappa$ on $z_R^\kappa$ is negative if $\xi$ is sufficiently small, and positive otherwise.*

A tighter liquidity requirement (i.e., a higher $\kappa$) makes the deposit contract less attractive so that the buyers reduce their deposits. The effect on the reserves is ambiguous because it increases the required reserves but discourages depositing. The proposition shows that the latter effect dominates the former if the degree of relative risk aversion is sufficiently small (the substitution effect is strong) and vice versa. The two opposite effects are missing in a partial equilibrium model of banking where deposit levels are exogenously fixed. Clearly, in the case of $\kappa > \frac{z_S}{d_S}$, where the liquidity requirement is also binding with the safe asset, the similar result of the effects of $\kappa$ on $z_S^\kappa$ and $d_S^\kappa$ can be obtained.

We now examine the effect of the liquidity requirement on the choice of asset investments. Suppose that $\xi$ is not so small that $z_R^\kappa$ is strictly increasing in $\kappa$. Then, a gradual increase in $\kappa$ decreases $d_R^\kappa$ and increases $z_R^\kappa$, reducing the probability of a crisis and welfare. If the expected return on the risky asset is sufficiently high (i.e., $\eta\lambda \approx 1$), the risky asset will be adopted even

when $\kappa$ is large. We demonstrate it using numerical examples in which the parameter values are the same as in the top panel of Table 1, that is, with $\eta = 0.895$ ($\eta\lambda = 0.9845$).

Table 2 compares two situations where the liquidity requirement is slack with $\kappa = 0$, and sufficiently tight with $\kappa = 0.85$. Note that the equilibrium outcomes with $\kappa = 0$ are the same as in the top panel of Table 1. Observe that the safe asset is selected when $R^C = 1.18$ under the liquidity requirement, while the risky asset continues to be selected even in the presence of the liquidity requirement when $R^C = 1.20$ and $1.22$ (but the net benefits of selecting it become very low). The liquidity requirement reduces the benefit of the risky asset used as collateral. This result suggests that a proper combination of the liquidity regulation and the lending rate can reduce not only the likelihood of a liquidity crisis but also control the moral hazard problem at the expense of ex-ante welfare.

| $R^C$ | $\kappa$ | $(z, d)$ | $(\alpha^*, \alpha^{**}, \alpha^{***})$ | $\Delta(R^C, \eta)$ | Types of Equilibrium |
|---|---|---|---|---|---|
| 1.18 | 0.00 | (0.2632, 0.7828) | (0.2555, 0.2736, 0.7591) | -0.0033 | Risky |
| 1.20 | 0.00 | (0.4426, 0.7509) | (0.4297, 0.4698, 0.7591) | -0.0019 | Risky |
| 1.22 | 0.00 | (0.5479, 0.7234) | (0.5319, 0.5937, 0.7591) | -0.0013 | Risky |
| 1.18 | 0.85 | (0.6186, 0.7172) | (0.6006, 0.7387, 0.8445) | 0.0002 | Safe |
| 1.20 | 0.85 | (0.6186, 0.7278) | (0.6006, 0.6567, 0.7591) | -0.0005 | Risky |
| 1.22 | 0.85 | (0.6073, 0.7145) | (0.5897, 0.6581, 0.7591) | -0.0009 | Risky |

Table 2: Monetary equilibria with liquidity requirement: $\kappa = 0.85$

## 5.2 Constructive Ambiguity

Constructive ambiguity is defined as not declaring in advance and being ambiguous about which banks would be regarded as large enough to fail and be rescued. Some economists and policymakers are positive about its effectiveness in controlling the moral hazard problem. For example, Giannini (1999, p.14) states that "the task of curbing moral hazard appears to have been performed largely by *constructive ambiguity*," while Schwartz (2002, p.452) puts "Constructive ambiguity supposedly constrains excessive risk taking by banks."[9]

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[9]See also Corrigan (1990) and Freixas et al. (2000).

To address this issue, we now consider the LLR policy with constructive ambiguity and assume that discount window lending is available with probability $\rho \in [0,1]$. This probability is known to depositors and banks. Clearly, when $\rho = 1$ the analysis is identical to the one given so far, while when $\rho = 0$ the model is identical to the basic model without the LLR. We can think of $1 - \rho$ as measuring the degree of policy ambiguity and controllable by the central bank.

Suppose that $\rho \in (0,1)$ and consider first the case of the safe asset. Note that the bank's solutions after the realization of $\alpha$ are the same as that in Lemma 2 when the bank has access to the LLR and that in Lemma 3 when the bank does not. Then, the value function at the beginning of a day can be modified as follows:

$$
\begin{aligned}
V_\rho(d) &= \max_{0 \le z \le d} \rho \Bigg\{ \int_0^{\alpha^*} \left[ \alpha u(q^*) + \left(1 - \frac{\alpha}{\alpha^*}\right) \frac{z}{\pi} \right] f(\alpha) d\alpha + \int_{\alpha^*}^{\alpha^{**}} \alpha u \left(\frac{z}{\alpha \pi}\right) f(\alpha) d\alpha \\
&+ \int_{\alpha^{**}}^{\alpha^{***}} \left[ \alpha u \left( u^{-1\prime}(R^C)\right) - R^C \left( \alpha u^{-1\prime}(R^C) - \frac{z}{\pi} \right) \right] f(\alpha) d\alpha \\
&+ \int_{\alpha^{***}}^{1} \left[ \alpha u \left( \frac{\frac{R^C}{\pi} z + R(d-z)}{R^C \alpha} \right) - R(d-z) \right] f(\alpha) d\alpha \Bigg\} \\
&+ (1-\rho) \Bigg\{ \int_0^{\alpha^*} \left[ \alpha u(q^*) + \left(1 - \frac{\alpha}{\alpha^*}\right) \frac{z}{\pi} \right] f(\alpha) d\alpha + \int_{\alpha^*}^{1} \alpha u \left( \frac{z}{\alpha \pi}\right) f(\alpha) d\alpha \Bigg\} \\
&+ \{1 - E(\alpha)\} \{u(q^*) - q^*\} + R(d-z) + T + \max_{x \ge 0} \{U(x) - x\} + \max_{d_+ \ge 0} \{-d_+ + \beta V(d_+)\}.
\end{aligned}
$$

Since a bank has an opportunity to borrow form the LLR with probability $\rho$, the value function is a convex combination of the expected utility with and without access to the LLR with weights $\rho$ and $1 - \rho$. The first order condition with respect to $z$ yields

$$
\Psi_\rho(z,d) \equiv \rho \left[ \frac{1}{\pi} \Psi_z(z,d) - R \Psi_k(z,d) \right] + (1-\rho) \left[ \frac{1}{\pi} \Upsilon(z) - R \right] \ge 0
$$

with equality if $z < d$.

The Euler equation becomes $\beta V_\rho'(d) = 1$ where

$$
V_\rho'(d) = \frac{\partial z}{\partial d} \frac{\partial V_\rho(d)}{\partial z} + \frac{\partial V_\rho(d)}{\partial d}.
$$

Thus, an interior solution, denoted by $(z_S^\rho, d_S^\rho)$, must satisfy the Euler equation, $\beta V_\rho'(d) =$

$\beta \frac{\partial V_\rho(d)}{\partial d}$ or

$$\rho \Psi_k(z, d) + (1 - \rho) = \frac{1}{\beta R}, \tag{16}$$

and the first order condition

$$\rho \Psi_z(z, d) + (1 - \rho)\Upsilon(z) = \frac{\pi}{\beta}. \tag{17}$$

The effects of $\rho$ on the equilibrium allocations are characterized in the following proposition.

**Proposition 5** *The optimal cash reserves $z_S^\rho$ are decreasing in $\rho$, and the optimal deposits $d_S^\rho$ are increasing in $\rho$.*

This result shows that constructive ambiguity gives the banks an incentive to insure themselves by increasing their monetary reserves, thus reducing the probability of a liquidity crisis, while the deposit level decreases because the deposit contract becomes less attractive.

We next examine the effect of $\rho$ on the threshold for the activated LLR. Let $R_\rho^{C*}$ be defined by $\tilde{\Psi}_\rho(R_\rho^{C*}) = 0$ where

$$\tilde{\Psi}_\rho(R^C) \equiv \pi \Psi_\rho(d, d) = \rho \{\Psi_z(d, d) - \pi R \Psi_k(d, d)\} + (1 - \rho)\{\Upsilon(d) - \pi R\}$$

$$= \frac{\pi}{\beta} - \pi R \left[\rho \left\{F(\alpha^{***}) + \frac{1}{R^C} \int_{\alpha^{***}}^1 u'\left(\frac{d}{\alpha \pi}\right) f(\alpha) d\alpha\right\} + (1 - \rho)\right].$$

From this it follows that $R_\rho^{C*}$ is increasing in $\rho$ and $R_\rho^{C*} \to R^{C*}$ as $\rho \to 1$. This result shows that the parameter set $(\pi R, R_\rho^{C*}]$, when the discount window is activated, becomes smaller as $\rho$ decreases, implying that constructive ambiguity (low $\rho$) will be inconsistent with a higher (penalty) lending rate. If both the policies were implemented rigorously, banks might stop borrowing from the LLR even when there is a liquidity crisis.

Not surprisingly, similar results are obtained in the case of the risky asset. The proof is quite similar to that of Proposition 5 and hence omitted here (available upon request).

**Proposition 6** *The optimal cash reserves $z_R^\rho$ are decreasing in $\rho$, and the optimal deposits $d_R^\rho$ are increasing in $\rho$. In addition, the threshold, $\hat{R}_\rho^{C*}$ is also increasing in $\rho$.*

To illustrate the effect of constructive ambiguity on the choice of investment, we use numerical examples again. We compare two regimes, $\rho = 0.85$ and $\rho = 1$. Notice that the case of $\rho = 1$ is identical to the upper panel of Table 1. Table 3 illustrates that constructive ambiguity gives the banks an incentive to insure themselves by increasing their cash reserves. Thus, by reducing the probability of a liquidity crisis, constructive ambiguity can prevent the banks from taking risks. This occurs when $R^C = 1.18$ and 1.2. Intuitively, since a bank relies more heavily on the discount window with the risky asset than with the safe asset, the ambiguity reduces the value of investing in the risky asset more drastically than in the safe asset. These numerical results imply that constructive ambiguity may be an effective policy tool to curb moral hazard, supporting the view of Giannini and Schwartz, but the depositors will bear the welfare cost of such a policy.

| $R^C$ | $\rho$ | $(z, d)$ | $(\alpha^*, \alpha^{**}, \alpha^{***})$ | $\Delta(R^C, \eta)$ | Types of Equilibrium |
|-------|--------|----------|------------------------------------------|---------------------|----------------------|
| 1.18  | 1.0    | (0.2632, 0.7828) | (0.2555, 0.2736, 0.7591) | -0.0033 | Risky |
| 1.20  | 1.0    | (0.4426, 0.7509) | (0.4297, 0.4698, 0.7591) | -0.0019 | Risky |
| 1.22  | 1.0    | (0.5479, 0.7234) | (0.5319, 0.5937, 0.7591) | -0.0013 | Risky |
| 1.18  | 0.85   | (0.6435, 0.7032) | (0.6248, 0.7684, 0.8324) | 0.0007 | Safe |
| 1.20  | 0.85   | (0.6585, 0.6858) | (0.6393, 0.8030, 0.8324) | 0.0001 | Safe |
| 1.22  | 0.85   | (0.6231, 0.7040) | (0.6050, 0.6753, 0.7515) | -0.0003 | Risky |

Table 3: Monetary equilibria with constructive ambiguity

Finally, it is worth noting that our results raise an important time-inconsistency problem again. If constructive ambiguity is effective, the central bank would like banks to believe that they cannot be certain if they will be rescued during a crisis. However, if a crisis actually occurred, the central bank would find it optimal to provide liquidity to troubled banks even in a state where it should not. If so, banks will believe that they will be rescued for sure during a crisis ex ante, ending up making this policy ineffective. Constructive ambiguity requires a strong commitment to future actions of the LLR.

# 6 Conclusion

We developed a monetary model of liquidity crises that allows us to investigate the economic role and consequence of the LLR. Given that private banks operate subject to limited liability, collateralized assets have liquidity values so that cash and long-term assets become substitutes during panics in the presence of the LLR. We showed that the LLR's liquidity provision will diminish banks' incentive to hold liquid assets, which in turn increases the probability of a liquidity crisis. Despite this unpleasant side effect, the LLR will mitigate the loss from liquidity crises and is beneficial. We also showed limited liability frees them from their payment responsibility in case of a default; hence, private banks can be induced to invest in risky assets rather than safe assets. That is, the LLR can create moral hazard in investment where private banks take more financial risk in terms of long-term assets.

Our results point to the public debate on the classical doctrine versus the moral hazard problem. To the best of our knowledge, our paper is the first to consider the effect of the LLR's liquidity provision to influence banks' portfolio decision, which eventually increases the ex ante probability of liquidity crises, and causes the moral hazard problem that increases financial risks in long-term assets in a general equilibrium model. We have done this in a monetary framework where the role of liquidity is made explicit in the occurrence of a liquidity crisis. Our results are negative about the conventional view that a high rate on the discount window can prevent banks from taking excessive risks. Additional policies such as liquidity requirements and constructive ambiguity can give banks self-guarding incentives and increase cash reserves, reducing the likelihood of a shortage of reserves, and in addition can influence the choice of investments.

There are several directions in which our model could be extended to address additional issues about the LLR. First, our model would be used to evaluate the impact of a capital requirement on the choice of investments as discussed in Repullo (2004, 2005). This issue could be addressed by adding risk-neutral investors who provide equity capital to the banks. Second, our model would be extended to consider moral hazard in a channel system which is widely used by many central banks in practice. This issue could be addressed by introducing

a deposit facility which pays banks the interest rate on monetary reserves. Third, our model would be extended to an open economy to assess the desirability and design of an international LLR. We leave these important issues for future research.

# Appendix

### Proof of Lemma 1

The first order condition with respect to $\theta$ is as follows:

$$\frac{z}{\pi}\left\{u'\left(\frac{\theta z}{\alpha\pi}\right) - 1\right\} \geq 0,$$

with equality if $\theta < 1$. There are two possible cases for the solution. If $\theta < 1$, then $u'(q^n) = 1 \Leftrightarrow q^n = q^*$. This case is true when $\theta = \frac{\alpha\pi q^*}{z} < 1 \Leftrightarrow \alpha < \frac{z}{\pi q^*} \equiv \alpha^*$. If $\theta = 1$, then $u'(q^n) \geq 1 \Leftrightarrow q^n = \frac{z}{\alpha\pi} \leq q^*$, leading to $\alpha \geq \alpha^*$. The lemma follows. $\square$

### Proof of Lemma 2

Applying the optimal payment plan $q^n = \min\{\frac{z}{\alpha\pi}, q^*\}$ and $\theta = \min\{\frac{\alpha}{\alpha^*}, 1\}$ with the critical value $\alpha^* = \frac{z}{\pi q^*}$ described in Lemma 1, the value function in the CM, $W(\cdot)$, and the balance sheet constraint, $d = z + k + l$, the banks' portfolio choice problem can be written as

$$
\begin{aligned}
V(d) &= \max_{z,k,l\geq 0} \int_0^1 \left[\alpha\{u(q^n) + W(0, k, l, 0)\} + (1-\alpha)\left\{u(q^*) + W\left(\frac{(1-\theta)z}{(1-\alpha)\pi}, k, l, q^*\right)\right\}\right] f(\alpha)d\alpha, \\
&= \max_{z,l\geq 0} \int_0^{\frac{z}{\pi q^*}} \left[u(q^*) - (1-\alpha)q^* + \left(1 - \frac{\alpha\pi q^*}{z}\right)\frac{z}{\pi}\right] f(\alpha)d\alpha \\
&\quad + \int_{\frac{z}{\pi q^*}}^1 \left[\alpha u\left(\frac{z}{\alpha\pi}\right) + (1-\alpha)\{u(q^*) - q^*\}\right] f(\alpha)d\alpha \\
&\quad + R[(d - z - l) + \eta\lambda l] + T + \max_{x\geq 0}\{U(x) - x\} + \max_{d_+\geq 0}\{-d_+ + \beta V(d_+)\}.
\end{aligned}
$$

The first order conditions yield $l = 0$ (since $\eta\lambda < 1$) and

$$\frac{1}{\pi}\Upsilon(z) \equiv \frac{1}{\pi}\left[F(\alpha^*) + \int_{\alpha^*}^1 u'(q^n) f(\alpha)d\alpha\right] \geq R,$$

with equality if $\Upsilon(d) < \pi R$. Since

$$\Upsilon'(z) = \int_{\alpha^*}^1 \frac{1}{\alpha\pi} u''(q^n) f(\alpha)d\alpha < 0$$

and $\Upsilon(0) = +\infty > \pi R$, there exists a unique solution $z = z(d) \in (0, d)$ to $\Upsilon(z(d)) = \pi R$, if $\Upsilon(d) < \pi R$. Otherwise, $\Upsilon(d) \geq \pi R$ for all $z \in (0, d]$ and so we must have a corner solution, $z = d$. The quantity of long-term safe assets is given by $k = d - z \geq 0$. $\square$

**Proof of Theorem 1**

The Euler equation is

$$-1 + \beta V'(d) = 0$$

where

$$V'(d) = \frac{\partial z}{\partial d}\frac{\partial V(d)}{\partial z} + \frac{\partial V(d)}{\partial d}.$$

There are two cases. For $d \geq \Upsilon^{-1}(\pi R)$ we have $z = z(d) \in (0, d)$ (an interior solution) and $\Upsilon(z(d)) = \pi R$ from Lemma 2, which implies

$$V'(d) = \frac{\partial V(d)}{\partial d} = R,$$

and, thus, the Euler equation is $-1 + \beta R = 0$, which is impossible under our assumption $\beta R < 1$. Hence, $d \geq \Upsilon^{-1}(\pi R)$ with $z = z(d) \in (0, d)$ (an interior solution) cannot be an equilibrium.

For $d < \Upsilon^{-1}(\pi R)$, we have $z = d$ (the corner solution) from Lemma 2, and so the Euler equation is $-1 + \beta V'(d) = -1 + \beta \left[ \{ \frac{1}{\pi}\Upsilon(d) - R \} + R \right] = 0$ or

$$\frac{\pi}{\beta} = F\left(\frac{d}{\pi q^*}\right) + \int_{\frac{d}{\pi q^*}}^{1} u'\left(\frac{d}{\pi\alpha}\right) f(\alpha)d\alpha \quad (= \Upsilon(d)), \tag{A.1}$$

where the R.H.S. is the marginal value of cash holdings at $z = d$. It satisfies: $\Upsilon(0) = \infty > \frac{\pi}{\beta} > 1 = \Upsilon(\pi q^*)$ and $\Upsilon'(\cdot) < 0$ (see above). Hence, there exists a unique solution $d_N \in (0, \pi q^*)$ to (A.1). Since $d_N < \pi q^*$ implies $\alpha^* < 1$, the probability of a liquidity crisis is positive, that is, $1 - F(\alpha^*) \in (0, 1)$, in equilibrium. $\square$

**Proof of Corollary 1**

Differentiating (A.1) with respect to $\pi$ and using (A.1) yield

$$\frac{\partial d_N}{\partial \pi} = \frac{\frac{1}{\beta} + \frac{1}{\pi}\int_{\frac{d_N}{\pi q^*}}^{1} u''(\frac{d_N}{\pi\alpha})\frac{d_N}{\alpha\pi}f(\alpha)d\alpha}{\int_{\frac{d_N}{\pi q^*}}^{1} u''(\frac{d_N}{\pi\alpha})\frac{f(\alpha)}{\alpha\pi}d\alpha} = -\frac{\frac{1}{\beta} - \frac{\xi}{\pi}\left\{\frac{\pi}{\beta} - F\left(\frac{d_N}{\pi q^*}\right)\right\}}{\frac{\xi}{d_N}\int_{\frac{d_N}{\pi q^*}}^{1} u'(\frac{d_N}{\pi\alpha})f(\alpha)d\alpha} < 0 \tag{A.2}$$

if $\xi < \hat{\xi} \equiv \frac{\frac{\pi}{\beta}}{\frac{\pi}{\beta} - F\left(\frac{d_N}{\pi q^*}\right)}$. Clearly, if $\xi < \hat{\xi}$, $\alpha^* = \frac{d_N}{\pi q^*}$ is decreasing in $\pi$, implying that $1 - F(\alpha^*)$ is increasing in $\pi$. $\square$

**Proof of Lemma 3**

We have the following system of equations induced by the first order conditions:

$$u'\left(\frac{\theta z}{\alpha\pi} + \frac{b}{\alpha}\right) - 1 = \frac{\mu\pi}{z}, \tag{A.3}$$

$$u'\left(\frac{\theta z}{\alpha\pi} + \frac{b}{\alpha}\right) - R^C(1 + \mu_k) = -\mu_b, \tag{A.4}$$

where $\mu \geq 0$ is the Lagrange multiplier of $\theta \leq 1$, $\mu_b \geq 0$ is the Lagrange multiplier of the non-negativity constraint $b \geq 0$, and $\mu_k \geq 0$ is the Lagrange multiplier of the borrowing constraint (4), $R^C b \leq Rk$.

There are four cases for the solution to the above equations (A.3) and (A.4).

**Case 1: $\theta < 1$ and $b = 0$.** Since $\theta < 1$ implies $\mu = 0$, we have in (A.3),

$$u'\left(\frac{\theta z}{\alpha\pi} + \frac{b}{\alpha}\right) = 1.$$

This implies in (A.4) that we must have $\mu_b > 0$, leading to $b = 0$. Note that $b = 0$ and $u'(q^n) = 1$ further lead to

$$\theta = \frac{\alpha\pi q^*}{z},$$

and $\theta < 1 \iff \alpha < \alpha^* \equiv \frac{z}{\pi q^*}$. Hence, for $\alpha \in (0, \alpha^*)$, the solution is $\theta < 1$ and $b = 0$.

**Case 2: $\theta = 1$ and $b = 0$.** $\theta = 1$ and $b = 0$ lead to $\mu \geq 0$, $\mu_k = 0$, and $\mu_b \geq 0$, which yield

$$u'\left(\frac{z}{\alpha\pi}\right) \geq 1 \iff \alpha \geq \alpha^*,$$

$$u'\left(\frac{z}{\alpha\pi}\right) \leq R^C \iff \alpha \leq \alpha^{**},$$

where $\alpha^{**} \equiv \frac{z}{\pi u^{-1\prime}(R^C)} > \frac{z}{\pi q^*} \equiv \alpha^*$ with $R^C > 1$. Hence, for $\alpha \in [\alpha^*, \alpha^{**}]$, the solution is $\theta = 1$ and $b = 0$.

**Case 3: $\theta = 1$ and $b \in \left(0, \frac{Rk}{R^C}\right)$.** $\mu_k = 0$ and $\mu_b = 0$ lead to

$$u'\left(\frac{z}{\alpha\pi} + \frac{b}{\alpha}\right) = R^C \iff b = \alpha u^{-1\prime}(R^C) - \frac{z}{\pi}.$$

This implies that $b > 0 \iff \alpha > \alpha^{**} \equiv \frac{z}{\pi u^{-1\prime}(R^C)}$ and $R^C b < Rk \iff \alpha < \alpha^{***} \equiv \frac{\frac{R^C}{\pi}z + Rk}{R^C u^{-1\prime}(R^C)}$. Hence, for $\alpha \in (\alpha^{**}, \alpha^{***})$, the solution is $\theta = 1$ and $b \in \left(0, \frac{Rk}{R^C}\right)$.

**Case 4: $\theta = 1$ and $b = \frac{Rk}{R^C}$.** $R^C b = Rk$ leads to $\mu_k \geq 0$ and $\mu_b = 0$, which yield

$$u'\left(\frac{z}{\alpha\pi} + \frac{Rk}{\alpha R^C}\right) \geq R^C \iff \alpha \geq \alpha^{***} \equiv \frac{\frac{R^C}{\pi}z + Rk}{R^C u^{-1\prime}(R^C)}.$$

Hence, for $\alpha \in [\alpha^{***}, 1)$, the solution is $\theta = 1$ and $b = \frac{Rk}{R^C}$.

The above covers all the possible cases and completes the proof. $\square$

**Proof of Proposition 1**

First, we identify the parameter space in which $k = d - z > 0$. In the first order condition (6), observe that

$$\Psi(0, d) = \left( \frac{R^C}{\pi} - R \right) \left[ F(\alpha^{***}|_{z=0}) + \frac{1}{R^C} \int_{\alpha^{***}|_{z=0}}^1 u'\left( \frac{Rd}{R^C \alpha} \right) f(\alpha)d\alpha \right] = \frac{1}{\beta R}\left( \frac{R^C}{\pi} - R \right) > 0,$$

where $\alpha^{***}|_{z=0} \equiv \frac{Rd}{R^C u^{-1\prime}(R^C)}$, and the last expression uses the Euler equation (7). Differentiating (6) with respect to $z$ yields

$$\pi \frac{\partial \Psi(z, d)}{\partial z} = \int_{\alpha^*}^{\alpha^{**}} \frac{1}{\alpha \pi} u''\left( \frac{z}{\alpha \pi} \right) f(\alpha)d\alpha + \left( 1 - \frac{\pi R}{R^C} \right) \int_{\alpha^{***}}^1 \frac{\frac{R^C}{\pi} - R}{R^C \alpha} u''\left( \frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha} \right) f(\alpha)d\alpha < 0.$$

Further, evaluating (6) at $z = d$ and $d > 0$ satisfying (3), we have:

$$\pi \Psi(d, d) = F(\alpha^*|_{z=d}) + \int_{\alpha^*|_{z=d}}^1 u'\left( \frac{d_N}{\pi \alpha} \right) f(\alpha)d\alpha - \pi R\left[ F(\alpha^{***}|_{z=d}) + \frac{1}{R^C} \int_{\alpha^{***}|_{z=d}}^1 u'\left( \frac{d_N}{\pi \alpha} \right) f(\alpha)d\alpha \right]$$

$$= \frac{\pi}{\beta} - \pi R\left[ F(\alpha^{***}|_{z=d}) + \frac{1}{R^C} \int_{\alpha^{***}|_{z=d}}^1 u'\left( \frac{d_N}{\pi \alpha} \right) f(\alpha)d\alpha \right]$$

$$\equiv \tilde{\Psi}(R^C),$$

where $\alpha^*|_{z=d} = \frac{d_N}{\pi q^*}$, $\alpha^{***}|_{z=d} = \frac{d_N}{\pi u^{-1\prime}(R^C)}$ and $d_N = z_N$ is the equilibrium deposit without the LLR as described in Theorem 1. In the last expression, we have:

$$\tilde{\Psi}(R^C) \to \frac{\pi}{\beta} - \pi R\left[ F(\alpha^{***}|_{z=d}) + \frac{1}{\pi R} \int_{\alpha^{***}|_{z=d}}^1 u'\left( \frac{d_N}{\pi \alpha} \right) f(\alpha)d\alpha \right] < 0$$

as $R^C \to \pi R$, since the above term goes to zero as $R \to \frac{1}{\pi}$ and is decreasing in $R$; $\tilde{\Psi}(R^C) = \frac{\pi}{\beta} - \pi R > 0$ for $R^C \geq u'(\frac{d_N}{\pi})$ satisfying $\alpha^{***}|_{z=d} = \min\left\{ \frac{d_N}{\pi u^{-1\prime}(R^C)}, 1 \right\} = 1$; and

$$\tilde{\Psi}'(R^C) = \frac{\pi R}{(R^C)^2} \int_{\alpha^{***}|_{z=d}}^1 u'\left( \frac{d_N}{\pi \alpha} \right) f(\alpha)d\alpha > 0. \tag{A.5}$$

Hence there exists a unique critical value, denoted by $R^{C*} \in (\pi R, u'(\frac{d_N}{\pi}))$ satisfying $\tilde{\Psi}(R^{C*}) = 0$, $\tilde{\Psi}(R^C) > 0$ for $R^C > R^{C*}$ and $\tilde{\Psi}(R^C) < 0$ for $R^C < R^{C*}$. We showed that $z = d$ for $R^C \geq R^{C*}$, hence, the solution has to be an interior one if $z < d$ for $R^C < R^{C*}$ (if it exists).

We now show the existence and uniqueness of an equilibrium solution $(z_S, d_S) > 0$ given $R^C \in (\pi R, R^{C*})$.

The solution has to satisfy the following system of equations:

$$\Psi_z(z,d) \equiv F(\alpha^*) + \int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right) f(\alpha)d\alpha + [F(\alpha^{***}) - F(\alpha^{**})]R^C$$

$$+ \int_{\alpha^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha = \frac{\pi}{\beta} \tag{A.6}$$

$$\Psi_k(z,d) \equiv F(\alpha^{***}) + \frac{1}{R^C}\int_{\alpha^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha = \frac{1}{\beta R}. \tag{A.7}$$

Observe that:

$$\frac{\partial \Psi_z(z,d)}{\partial z} = \int_{\alpha^*}^{\alpha^{**}} \frac{1}{\alpha\pi}u''\left(\frac{z}{\alpha\pi}\right) f(\alpha)d\alpha + \int_{\alpha^{***}}^1 \frac{\frac{R^C}{\pi} - R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha < 0; \tag{A.8}$$

$$\frac{\partial \Psi_z(z,d)}{\partial d} = \int_{\alpha^{***}}^1 \frac{R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha < 0; \tag{A.9}$$

$\Psi_z(z_N, d_N) = \frac{\pi}{\beta}$; $\Psi_z(\underline{z}, d) = \frac{\pi}{\beta}$ for sufficiently high values of $d$ such that $\alpha^{***} = \min\left\{\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C u^{-1\prime}(R^C)}, 1\right\} = 1$ with some lower bound $\underline{z} \in (0, z_N)$. Hence, the implicit equation (A.6) determines a function $z = z(d)$ that satisfies $z'(d) < 0$, $z(d_N) = z_N$ and $z(d) = \underline{z} > 0$ at some $d \in (d_N, \infty)$ that leads to $\alpha^{***} = 1$.

Observe also that

$$\frac{\partial \Psi_k(z,d)}{\partial z} = \frac{1}{R^C}\int_{\alpha^{***}}^1 \frac{\frac{R^C}{\pi} - R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha < 0; \tag{A.10}$$

$$\frac{\partial \Psi_k(z,d)}{\partial d} = \frac{1}{R^C}\int_{\alpha^{***}}^1 \frac{R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha < 0. \tag{A.11}$$

Further,

$$\Psi_k(d,d) = F(\alpha^{***}|_{z=d}) + \frac{1}{R^C}\int_{\alpha^{***}|_{z=d}}^1 u'\left(\frac{d_N}{\pi\alpha}\right) f(\alpha)d\alpha > \frac{1}{\beta R}$$

for $R^C < R^{C*}$ and $d = d_N$ (satisfying (3)). In this expression, since $\Psi_k(d,d)$ is decreasing in all $d > d_N$, we must have $\Psi_k(d,d) = \frac{1}{\beta R}$ (i.e., condition (A.7) holds true with $z = d$) for $R^C < R^{C*}$ only when $d > d_N$. Finally, since $\frac{1}{\beta R} > 1$, it is impossible to hold the condition (A.7) for values of $d > d_N$ that lead to $\alpha^{***} = \min\left\{\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C u^{-1\prime}(R^C)}, 1\right\} = 1$ and hence to $\Psi_k(z,d) = 1$. Therefore, we must necessarily have $z = 0$ at some upper bound $\bar{d} \in (d_N, \infty)$, and so condition (A.7) holds true only when $d \in (d_N, \bar{d}]$. Hence, the implicit equation (A.7) determines a function $\tilde{z} = \tilde{z}(d)$ that satisfies $\tilde{z}'(d) < 0$, $\tilde{z}(d) = d$ at some $d > d_N$ and $\tilde{z}(\bar{d}) = 0$.

Combining the above analysis, with some low $d \in (d_N, \bar{d})$, we have $\tilde{z}(d) = d > z(d)$, while with $d = \bar{d}$, we have $\tilde{z}(\bar{d}) = 0 < z(\bar{d})$. Further,

$$\frac{\partial}{\partial d}\left[\tilde{z}(d) - z(d)\right] = -\frac{\int_{\alpha^*}^{\alpha^{**}} \frac{1}{\alpha\pi} u''\left(\frac{z}{\alpha\pi}\right) f(\alpha)d\alpha \times \frac{\partial\Psi_z}{\partial d}}{\int_{\alpha^{***}}^{1} \frac{\frac{R^C}{\pi} - R}{R^C\alpha} u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha \times \frac{\partial\Psi_z}{\partial z}} < 0.$$

Therefore, the two curves, $z = z(d)$ and $\tilde{z} = \tilde{z}(d)$, must intersect once, at $d \in (d_N, \bar{d})$ and $z \in (\underline{z}, z_N)$, implying that there exists a unique solution to (A.6) and (A.7). $\square$

### Proof of Corollary 2

The claim follows from $z_S \leq z_N$ and $d_S \geq d_N$. $\square$

### Proof of Corollary 3

Remember that the solutions satisfy equations (A.6) and (A.7). Observe that

$$\frac{\partial\Psi_z(z, d; \pi, R^C)}{\partial\pi} = -\int_{\alpha^*}^{\alpha^{**}} \frac{z}{\alpha\pi^2} u''\left(\frac{z}{\alpha\pi}\right) f(\alpha)d\alpha - \int_{\alpha^{***}}^{1} \frac{z}{\alpha\pi^2} u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha > 0;$$
(A.12)

$$\frac{\partial\Psi_z(z, d; \pi, R^C)}{\partial R^C} = [F(\alpha^{***}) - F(\alpha^{**})] - \int_{\alpha^{***}}^{1} u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) \frac{R(d-z)}{(R^C)^2\alpha} f(\alpha)d\alpha > 0; \quad \text{(A.13)}$$

$$\frac{\partial\Psi_k(z, d; \pi, R^C)}{\partial\pi} = -\frac{1}{R^C}\int_{\alpha^{***}}^{1} \frac{z}{\alpha\pi^2} u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha > 0; \quad \text{(A.14)}$$

$$\frac{\partial\Psi_k(z, d; \pi, R^C)}{\partial R^C} = \frac{1}{(R^C)^2}\left(-1 + \frac{R(d-z)\xi}{\frac{R^C}{\pi}z + R(d-z)}\right)\int_{\alpha^{***}}^{1} u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right) f(\alpha)d\alpha. \quad \text{(A.15)}$$

These implicit equations determine two continuous and differentiable functions $d = d_S(\pi, R^C)$ and $z = z_S(\pi, R^C)$ which are characterized as follows:

$$\begin{pmatrix} \frac{\partial z_S}{\partial\pi} \\ \frac{\partial d_S}{\partial\pi} \end{pmatrix} = -\begin{pmatrix} \frac{\partial\Psi_k}{\partial z} & \frac{\partial\Psi_k}{\partial d} \\ \frac{\partial\Psi_z}{\partial z} & \frac{\partial\Psi_z}{\partial d} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial\Psi_k}{\partial\pi} \\ \frac{\partial\Psi_z}{\partial\pi} - \frac{1}{\beta} \end{pmatrix}$$

$$= -\frac{1}{\Lambda}\begin{pmatrix} \frac{\partial\Psi_z}{\partial d} & -\frac{\partial\Psi_k}{\partial d} \\ -\frac{\partial\Psi_z}{\partial z} & \frac{\partial\Psi_k}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial\Psi_k}{\partial\pi} \\ \frac{\partial\Psi_z}{\partial\pi} - \frac{1}{\beta} \end{pmatrix} \quad \text{(A.16)}$$

$$\begin{pmatrix} \frac{\partial z_S}{\partial R^C} \\ \frac{\partial d_S}{\partial R^C} \end{pmatrix} = -\frac{1}{\Lambda}\begin{pmatrix} \frac{\partial\Psi_z}{\partial d} & -\frac{\partial\Psi_k}{\partial d} \\ -\frac{\partial\Psi_z}{\partial z} & \frac{\partial\Psi_k}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial\Psi_k}{\partial R^C} \\ \frac{\partial\Psi_z}{\partial R^C} \end{pmatrix} \quad \text{(A.17)}$$

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where

$$\Lambda \equiv \det \begin{pmatrix} \frac{\partial \Psi_k}{\partial z} & \frac{\partial \Psi_k}{\partial d} \\ \frac{\partial \Psi_z}{\partial z} & \frac{\partial \Psi_z}{\partial d} \end{pmatrix} = \frac{\partial \Psi_k}{\partial z}\frac{\partial \Psi_z}{\partial d} - \frac{\partial \Psi_k}{\partial d}\frac{\partial \Psi_z}{\partial z}$$

$$= \frac{\xi}{R^C z}\left[\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha\right]\left[\int_{\alpha^{***}}^{1}\frac{R}{\alpha R^C}u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha\right] < 0,$$

using (A.8)–(A.11).

$\odot$ The effect of $\pi$ on $z_S$. Now, inserting (A.9), (A.11), (A.12), and (A.14) into (A.16) and rearranging, we get

$$\Lambda\frac{\partial z_S}{\partial \pi} = -\frac{\partial \Psi_z}{\partial d}\frac{\partial \Psi_k}{\partial \pi} + \frac{\partial \Psi_k}{\partial d}\left(\frac{\partial \Psi_z}{\partial \pi} - \frac{1}{\beta}\right)$$

$$= \frac{1}{\pi R^C}\left[\int_{\alpha^{***}}^{1}\frac{R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha\right]\left[\xi\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha - \frac{\pi}{\beta}\right] > 0,$$

if $\xi < \check{\xi} \equiv \frac{\frac{\pi}{\beta}}{\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha}$. Since $\Lambda < 0$, we have, if $\xi < \check{\xi}$,

$$\frac{\partial z_S}{\partial \pi} < 0. \tag{A.18}$$

In addition, we will make use of the following result.

$$\frac{\pi}{z}\frac{\partial z_S}{\partial \pi} - 1 = \frac{1}{\xi\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha}\left[(\xi - 1)\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha - F(\alpha^*)\right.$$

$$\left. - [F(\alpha^{***}) - F(\alpha^{**})]R^C - \int_{\alpha^{***}}^{1} u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha\right] - 1$$

$$= -\frac{\frac{\pi}{\beta}}{\xi\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha} < 0.$$

The last equality follows from the first order condition (8).

$\odot$ The effect of $\pi$ on $d_S$. Similarly, inserting (A.8), (A.10), (A.12), and (A.14) into (A.16), using the first oder condition (8), and rearranging terms, we have

$$\Lambda\frac{\partial d_S}{\partial \pi} = \frac{\partial \Psi_z}{\partial z}\frac{\partial \Psi_k}{\partial \pi} - \frac{\partial \Psi_k}{\partial z}\left(\frac{\partial \Psi_z}{\partial \pi} - \frac{1}{\beta}\right)$$

$$= \left[\int_{\alpha^{***}}^{1}\frac{1}{R^C\pi\alpha}u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha\right]\left[\frac{\xi R}{R^C}\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha + \frac{\frac{R^C}{\pi} - R}{R^C}\frac{\pi}{\beta}\right] < 0.$$

Since $\Lambda < 0$, we obtain

$$\frac{\partial d_S}{\partial \pi} > 0. \tag{A.19}$$

In addition, we will make use of the following result later.

$$\Lambda\left(\frac{\partial d_S}{\partial \pi} - \frac{\partial z_S}{\partial \pi}\right) = \frac{1}{\pi \beta R}\left[\int_{\alpha^{***}}^{1} \frac{R}{\alpha R^C} u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha}\right)f(\alpha)d\alpha\right] < 0,$$

which leads to

$$R\left(\frac{\partial d_S}{\partial \pi} - \frac{\partial z_S}{\partial \pi}\right) = \frac{\frac{R^C z}{\pi \beta}}{\xi \int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha \pi}\right)f(\alpha)d\alpha} > 0.$$

Thus, using the above results, we have

$$\frac{\partial \alpha^*}{\partial \pi} = \frac{\pi \frac{\partial z}{\partial \pi} - z}{(\pi)^2 q^*} < 0,$$

$$\frac{\partial \alpha^{**}}{\partial \pi} = \frac{\pi \frac{\partial z}{\partial \pi} - z}{(\pi)^2 u^{-1\prime}(R^C)} < 0,$$

$$\{R^C u^{-1\prime}(R^C)\}\frac{\partial \alpha^{***}}{\partial \pi} = \frac{R^C z}{\pi^2}\left(\frac{\pi}{z}\frac{\partial z}{\partial \pi} - 1\right) + R\left(\frac{\partial d}{\partial \pi} - \frac{\partial z}{\partial \pi}\right)$$

$$= -\frac{\frac{R^C z}{\pi^2}\frac{\pi}{\beta}}{\xi \int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha \pi}\right)f(\alpha)d\alpha} + \frac{\frac{R^C z}{\pi \beta}}{\xi \int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha \pi}\right)f(\alpha)d\alpha} = 0.$$

⊙ The effect of $R^C$ on $z_S$. Similarly, substituting (A.9), (A.11), (A.13), and (A.15) into (A.17), and rearranging, we get

$$\Lambda \frac{\partial z_S}{\partial R^C} = -\frac{\partial \Psi_z}{\partial d}\frac{\partial \Psi_k}{\partial R^C} + \frac{\partial \Psi_k}{\partial d}\frac{\partial \Psi_z}{\partial R^C}$$

$$= \left[\int_{\alpha^{***}}^{1} \frac{R}{R^C \alpha} u''\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha}\right)f(\alpha)d\alpha\right]$$

$$\times \left[\frac{1}{(R^C)^2}\int_{\alpha^{***}}^{1} u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha}\right)f(\alpha)d\alpha + \frac{F(\alpha^{***}) - F(\alpha^{**})}{R^C}\right] < 0.$$

Since $\Lambda < 0$, we have

$$\frac{\partial z_S}{\partial R^C} > 0. \tag{A.20}$$

In addition, we will make use of the following result.

$$\frac{R^C}{z_S}\frac{\partial z_S}{\partial R^C} = \frac{\int_{\alpha^{***}}^{1} u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C \alpha}\right)f(\alpha)d\alpha + R^C[F(\alpha^{***}) - F(\alpha^{**})]}{\xi \int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha \pi}\right)f(\alpha)d\alpha} > 0.$$

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$\odot$ The effect of $R^C$ on $d_S$. Similarly, inserting (A.8), (A.10), (A.13), and (A.15) into (A.17), we obtain

$$
\begin{aligned}
\Lambda \frac{\partial d_S}{\partial R^C} &= \frac{\partial \Psi_z}{\partial z}\frac{\partial \Psi_k}{\partial R^C} - \frac{\partial \Psi_k}{\partial z}\frac{\partial \Psi_z}{\partial R^C} \\
&= \frac{1}{(R^C)^2}\left(-1 + \frac{R(d-z)\xi}{\frac{R^C}{\pi}z + R(d-z)}\right)\left[\int_{\alpha^*}^{\alpha^{**}}\frac{1}{\alpha\pi}u''\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha\right]\left[\int_{\alpha^{***}}^{1}u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha\right] \\
&\quad + \frac{1}{(R^C)^2}\frac{\xi(\frac{R^C}{\pi}-R)}{\frac{R^C}{\pi}z + R(d-z)}\left[\int_{\alpha^{***}}^{1}u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha\right]^2 \\
&\quad + \left[\frac{\xi}{R^C}\frac{\frac{R^C}{\pi}-R}{\frac{R^C}{\pi}z + R(d-z)}\int_{\alpha^{***}}^{1}u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha\right][F(\alpha^{***}) - F(\alpha^{**})] > 0,
\end{aligned}
$$

if $\xi < 1 + \frac{R^C z}{\pi R(d-z)}$. Since $\Lambda < 0$, we have

$$\frac{\partial d_S}{\partial R^C} < 0, \tag{A.21}$$

if $\xi$ is not too big. Using the above results, some long but straightforward calculations show that

$$
\begin{aligned}
\frac{\partial k_S}{\partial R^C} &= \frac{\partial d_S}{\partial R^C} - \frac{\partial z_S}{\partial R^C} \\
&= -\frac{\frac{R^C}{\pi}z + (1-\xi)R(d-z)}{\xi R R^C} - \frac{z\int_{\alpha^{***}}^{1}u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha}{\xi R\pi\int_{\alpha^*}^{\alpha^{**}}u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha} - \frac{\frac{R^C}{\pi}z\{F(\alpha^{***}) - F(\alpha^{**})\}}{\xi R\int_{\alpha^*}^{\alpha^{**}}u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha} < 0.
\end{aligned}
$$

Further, we have

$$\frac{\partial \alpha^*}{\partial R^C} = \frac{1}{\pi q^*}\frac{\partial z}{\partial R^C} > 0,$$

$$\frac{\partial \alpha^{**}}{\partial R^C} = \frac{\frac{\partial z}{\partial R^C}u^{-1\prime}(R^C) - z\frac{\partial u^{-1\prime}(R^C)}{\partial R^C}}{\pi[u^{-1\prime}(R^C)]^2} > 0,$$

$$\frac{\partial \alpha^{***}}{\partial R^C} = \frac{\{\frac{R^C}{\pi}\frac{\partial z}{\partial R^C} + R(\frac{\partial d}{\partial R^C} - \frac{\partial z}{\partial R^C})\}R^C u^{-1\prime}(R^C)}{[R^C u^{-1\prime}(R^C)]^2} - \frac{R(d-z)u^{-1\prime}(R^C) + \{\frac{R^C}{\pi}z + R(d-z)\}R^C\frac{\partial u^{-1\prime}(R^C)}{\partial R^C}}{[R^C u^{-1\prime}(R^C)]^2}.$$

Since $\frac{R^C}{u^{-1\prime}(R^C)}\frac{\partial u^{-1\prime}(R^C)}{\partial R^C} = \frac{u'(u^{-1\prime}(R^C))}{u^{-1\prime}(R^C)u''(u^{-1\prime}(R^C))} = -\frac{1}{\xi}$, we have

$$\{R^C u^{-1\prime}(R^C)\}\frac{\partial \alpha^{***}}{\partial R^C} = \frac{R^C}{\pi}\frac{\partial z}{\partial R^C} + R\left(\frac{\partial d}{\partial R^C} - \frac{\partial z}{\partial R^C}\right) + \left(\frac{1}{\xi} - 1\right)\frac{R(d-z)}{R^C} + \frac{z}{\pi\xi},$$

$$= \frac{z}{\pi}\left[\frac{\int_{\alpha^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha}{\xi\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha} + \frac{R^C[F(\alpha^{***}) - F(\alpha^{**})]}{\xi\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha} + \frac{1}{\xi}\right]$$

$$+\left[-\frac{\frac{R^C}{\pi}z + (1-\xi)R(d-z)}{\xi R^C} - \frac{z\int_{\alpha^{***}}^1 u'\left(\frac{\frac{R^C}{\pi}z + R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha}{\xi\pi\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha}\right.$$

$$\left.-\frac{\frac{R^C}{\pi}z\{F(\alpha^{***}) - F(\alpha^{**})\}}{\xi\int_{\alpha^*}^{\alpha^{**}} u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha} + \frac{R(d-z)(1-\xi)}{\xi R^C}\right] = 0.$$

This completes the proof of the corollary. $\square$


**Proof of Lemma 4**

We have the following system of equations derived from the first order conditions:

$$u'\left(\frac{\theta z}{\alpha\pi} + \frac{b}{\alpha}\right) - 1 = \frac{\mu\pi}{z}, \tag{A.22}$$

$$u'\left(\frac{\theta z}{\alpha\pi} + \frac{b}{\alpha}\right) - R^C(\eta + \mu_z) = -\mu_b, \tag{A.23}$$

where $\mu \geq 0$ is the Lagrange multiplier of $\theta \leq 1$, $\mu_b \geq 0$ is the Lagrange multiplier of the non-negativity constraint $b \geq 0$, and $\mu_l \geq 0$ is the Lagrange multiplier of the borrowing constraint, $Rl \geq R^C b$.

There are four cases for the solution to the above equations (A.22) and (A.23).

**Case 1: $\theta < 1$ and $b = 0$.** Since $\theta < 1$ implies $\mu = 0$, we have in (A.22),

$$u'\left(\frac{\theta z}{\alpha\pi} + \frac{b}{\alpha}\right) = 1.$$

This implies in (A.23) that we must have $\mu_b > 0$, leading to $b = 0$. Note that $b = 0$ and $u'(q^n) = 1$ further lead to

$$\theta = \frac{\alpha\pi q^*}{z},$$

and $\theta < 1 \iff \alpha < \alpha^* \equiv \frac{z}{\pi q^*}$. Hence, for $\alpha \in (0, \alpha^*)$, the solution is $\theta < 1$ and $b = 0$.

**Case 2: $\theta = 1$ and $b = 0$.** $\theta = 1$ and $b = 0$ lead to $\mu \geq 0$, $\mu_z = 0$ and $\mu_b \geq 0$, which yield

$$u'\left(\frac{z}{\alpha\pi}\right) \geq 1 \iff \alpha \geq \alpha^*$$

$$u'\left(\frac{z}{\alpha\pi}\right) \leq \eta R^C \iff \alpha \leq \alpha^{**}_\eta$$

where $\alpha_\eta^{**} \equiv \frac{z}{\pi u^{-1\prime}(\eta R^C)} > \frac{z}{\pi q^*} \equiv \alpha^*$ with $\eta R^C > 1$. Hence, for $\alpha \in [\alpha^*, \alpha_\eta^{**}]$, the solution is $\theta = 1$ and $b = 0$.

**Case 3: $\theta = 1$ and $b \in \left(0, \frac{Rl}{R^C}\right)$.** $\mu_z = 0$ and $\mu_b = 0$ lead to

$$u'\left(\frac{z}{\pi\alpha} + \frac{b}{\alpha}\right) = \eta R^C \iff b = \alpha u^{-1\prime}(\eta R^C) - \frac{z}{\pi}.$$

This implies that $b > 0 \iff \alpha > \alpha_\eta^{**} \equiv \frac{z}{\pi u^{-1\prime}(\eta R^C)}$ and $R^C b < Rl \iff \alpha < \alpha_\eta^{***} \equiv \frac{Rl + \frac{R^C}{\pi}z}{R^C u^{-1\prime}(\eta R^C)}$. Hence, for $\alpha \in (\alpha_\eta^{**}, \alpha_\eta^{***})$, the solution is $\theta = 1$ and $b \in \left(0, \frac{Rl}{R^C}\right)$.

**Case 4: $\theta = 1$ and $b = \frac{Rl}{R^C}$.** $\mu_l \geq 0$ and $\mu_b = 0$ lead to $R^C b = Rl$ and

$$u'\left(\frac{z}{\alpha\pi} + \frac{Rl}{\alpha R^C}\right) \geq \eta R^C \iff \alpha \geq \alpha_\eta^{***} \equiv \frac{\frac{R^C}{\pi}z + Rl}{R^C u^{-1\prime}(\eta R^C)}.$$

Hence, for $\alpha \in [\alpha_\eta^{***}, 1)$, the solution is $\theta = 1$ and $b = \frac{Rl}{R^C}$.

The above covers all the possible cases and completes the proof. $\square$

**Proof of Proposition 2**

First of all, we shall identify the parameter space in which $l = d - z > 0$. In the first order condition,

$$\Phi(z, d) \equiv \frac{1}{\pi}\Phi_z(z, d) - R\Phi_l(z, d) \geq 0 \quad (= \text{ if } z < d),$$

where $\Phi_z, \Phi_l$ are given by (10) and (11), respectively. Observe that

$$
\begin{aligned}
\Phi(0, d) &= \left(\frac{\eta R^C}{\pi} - \eta R\right)\left[F\left(\frac{Rd}{R^C u^{-1\prime}(\eta R^C)}\right) + \frac{1}{\eta R^C}\int_{\frac{Rd}{R^C u^{-1\prime}(\eta R^C)}}^1 u'\left(\frac{Rd}{R^C \alpha}\right)f(\alpha)d\alpha\right] - \eta R(\lambda - 1) \\
&= \frac{1}{\beta}\left[\frac{R^C\{1 - \eta\beta R(\lambda - 1)\}}{\pi R} - 1\right] \\
&\equiv \bar{\Phi}(R^C),
\end{aligned}
$$

where the second expression uses the Euler equation (11). Clearly, $\bar{\Phi}(R^C)$ is linear and strictly increasing in $R^C$, and $\bar{\Phi}(R^C) = -\eta R(\lambda - 1) < 0$ as $R^C \to \pi R$. There exists a unique value such that $\bar{\Phi}(R^C) = 0$, or

$$R^C = \frac{\pi R}{1 - \eta\beta R(\lambda - 1)} \equiv \pi\underline{R}^C. \tag{A.24}$$

Then it follows that $\bar{\Phi}(R^C) < 0$ if $R^C < \pi\underline{R}^C$ and $\bar{\Phi}(R^C) > 0$ if $R^C > \pi\underline{R}^C$.

Further,

$$\pi\frac{\partial\Phi(z,d)}{\partial z} = \int_{\alpha^*}^{\alpha_\eta^{**}}\frac{1}{\alpha\pi}u''\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha + \left(1-\frac{\pi R}{R^C}\right)\int_{\alpha_\eta^{***}}^{1}\frac{\frac{R^C}{\pi}-R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z+R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha < 0,$$

and with $d > 0$ satisfying (3),

$$\begin{aligned}\pi\Phi(d,d) &= F(\alpha^*|_{z=d}) + \int_{\alpha^*|_{z=d}}^{1}u'\left(\frac{d}{\pi\alpha}\right)f(\alpha)d\alpha - \pi\eta R\left[F(\alpha_\eta^{***}|_{z=d}) + \frac{1}{\eta R^C}\int_{\alpha_\eta^{***}|_{z=d}}^{1}u'\left(\frac{d}{\pi\alpha}\right)f(\alpha)d\alpha + \lambda - 1\right] \\ &= \frac{\pi}{\beta} - \pi\eta R\left[F(\alpha_\eta^{***}|_{z=d}) + \frac{1}{\eta R^C}\int_{\alpha_\eta^{***}|_{z=d}}^{1}u'\left(\frac{d_N}{\pi\alpha}\right)f(\alpha)d\alpha + \lambda - 1\right] \\ &\equiv \tilde{\Phi}(R^C).\end{aligned} \tag{A.25}$$

In the last expression, we have: $\tilde{\Phi}'(R^C) > 0$; $\tilde{\Phi}(R^C) > 0$ for $R^C \geq \frac{1}{\eta}u'(\frac{d_N}{\pi})$ leading to $\alpha_\eta^{***}|_{z=d} = \min\left\{\frac{d_N}{\pi u^{-1\prime}(\eta R^C)},1\right\} = 1$. Since $\Phi(0,d) = 0$ at $R^C = \pi\underline{R}^C$ and $\Phi(z,d)$ is strictly decreasing in $z$, we must have $\tilde{\Phi}(\pi\underline{R}^C) < 0$. Thus, there exists a value $\hat{R}^{C*}(> \pi\underline{R}^C)$ such that $\tilde{\Phi}(R^C) = 0$. We have $\tilde{\Phi}(R^C) > 0$ if $R^C > \hat{R}^{C*}$ and $\tilde{\Phi}(R^C) < 0$ if $R^C < \hat{R}^{C*}$.

We now show the existence and uniqueness of an equilibrium solution $(z_R, d_R) > 0$ given $R^C \in (\max\{\pi\underline{R}^C, \hat{R}^{C*})$.

The solution has to satisfy the following system of equations:

$$\Phi_z(z,d) \equiv F(\alpha^*) + \int_{\alpha^*}^{\alpha_\eta^{**}}u'\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha + [F(\alpha_\eta^{***}) - F(\alpha_\eta^{**})]\eta R^C + \int_{\alpha_\eta^{***}}^{1}u'\left(\frac{\frac{R^C}{\pi}z+R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha = \frac{\pi}{\beta} \tag{A.26}$$

$$\Phi_l(z,d) \equiv \eta F(\alpha_\eta^{***}) + \frac{1}{R^C}\int_{\alpha_\eta^{***}}^{1}u'\left(\frac{\frac{R^C}{\pi}z+R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha + \eta(\lambda-1) = \frac{1}{\beta R}. \tag{A.27}$$

We first examine (A.26). Observe that:

$$\frac{\partial\Phi_z(z,d)}{\partial z} = \int_{\alpha^*}^{\alpha_\eta^{**}}\frac{1}{\alpha\pi}u''\left(\frac{z}{\alpha\pi}\right)f(\alpha)d\alpha + \int_{\alpha_\eta^{***}}^{1}\frac{\frac{R^C}{\pi}-R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z+R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha < 0;$$

$$\frac{\partial\Phi_z(z,d)}{\partial d} = \int_{\alpha_\eta^{***}}^{1}\frac{R}{R^C\alpha}u''\left(\frac{\frac{R^C}{\pi}z+R(d-z)}{R^C\alpha}\right)f(\alpha)d\alpha < 0;$$

$\Phi_z(z_N,d_N) = \frac{\pi}{\beta}$ where $d_N = z_N$; $\Phi_z(\underline{z}_\eta,d) = \frac{\pi}{\beta}$ for sufficiently high values of $d$ such that $\alpha_\eta^{***} = \min\{\frac{\frac{R^C}{\pi}z+R(d-z)}{R^C u^{-1\prime}(\eta R^C)},1\} = 1$ with some lower bound $\underline{z}_\eta \in (0,z_N)$. Hence, the implicit equation (A.26) determines a function $z = z_\eta(d)$ that satisfies $z_\eta'(d) < 0$, $z_\eta(d_N) = z_N$ and $z_\eta(d) = \underline{z}_\eta > 0$ at some $d \in (d_N, \infty)$ that leads to $\alpha_\eta^{***} = 1$. Notice that with $\eta \to 1$ and

$\lambda \to 1$, this curve becomes identical to the one with the safe asset, but when $\eta < 1$, it gives a lower $z$ with the risky asset than with the safe asset, for any given values of $d > d_N$.

Consider next (A.27). Observe also that

$$
\begin{aligned}
\frac{\partial \Phi_l(z,d)}{\partial z} &= \frac{1}{R^C} \int_{\alpha_\eta^{***}}^{1} \frac{\frac{R^C}{\pi} - R}{R^C \alpha} u'' \left( \frac{\frac{R^C}{\pi} z + R(d-z)}{R^C \alpha} \right) f(\alpha) d\alpha < 0; \\
\frac{\partial \Phi_l(z,d)}{\partial d} &= \frac{1}{R^C} \int_{\alpha_\eta^{***}}^{1} \frac{R}{R^C \alpha} u'' \left( \frac{\frac{R^C}{\pi} z + R(d-z)}{R^C \alpha} \right) f(\alpha) d\alpha < 0.
\end{aligned}
$$

Further,

$$
\Phi_l(d,d) = \eta F(\alpha_\eta^{***}|_{z=d}) + \frac{1}{R^C} \int_{\alpha_\eta^{***}|_{z=d}}^{1} u' \left( \frac{d}{\pi \alpha} \right) f(\alpha) d\alpha + \eta(\lambda - 1) > \frac{1}{\beta R}
$$

for $R^C < \hat{R}^{C*}$ and $d = d_N$. Just like in the previous analysis with the safe asset, since $\Phi_l(d,d)$ is strictly decreasing in $d > d_N$, we have $\Phi_l(d,d) = \frac{1}{\beta R}$ (i.e., the condition (A.27) holds true with $z = d$) for $R^C < \hat{R}^{C*}$ only if $d > d_N$. Finally, since $\frac{1}{\beta R} > 1$, it is impossible to hold the condition (A.27) for values of $d > d_N$ that lead to $\alpha_\eta^{***} = \min \left\{ \frac{\frac{R^C}{\pi} z + R(d-z)}{R^C u^{-1\prime}(\eta R^C)}, 1 \right\} = 1$ and hence to $\Phi_l(z,d) = 1$. Therefore, we must necessarily have $z = 0$ at some upper bound $\bar{d}_\eta \in (d_N, \infty)$ and so the condition (A.27) has a solution $(z,d) > 0$ only with $d \in (d_N, \bar{d}_\eta]$. Hence, the implicit equation (A.27) determines a function $\tilde{z} = \tilde{z}_\eta(d)$ that satisfies $\tilde{z}_\eta'(d) < 0$, $\tilde{z}_\eta(d) = d$ at some $d > d_N$ and $\tilde{z}_\eta(\bar{d}_\eta) = 0$.

Finally, combining the above analysis, with some low $d \in (d_N, \bar{d}_\eta)$, we have $\tilde{z}_\eta(d) = d > z_\eta(d)$, while with $d = \bar{d}$, we have $\tilde{z}_\eta(\bar{d}) = 0 < z_\eta(\bar{d})$. Further,

$$
\frac{\partial}{\partial d} [\tilde{z}_\eta(d) - z_\eta(d)] = -\frac{\int_{\alpha^*}^{\alpha_\eta^{**}} \frac{1}{\alpha \pi} u'' \left( \frac{z}{\alpha \pi} \right) f(\alpha) d\alpha \times \frac{\partial \Phi_z}{\partial d}}{\int_{\alpha_\eta^{***}}^{1} \frac{\frac{R^C}{\pi} - R\phi}{R^C \alpha} u'' \left( \frac{\frac{R^C}{\pi} z + R(d-z)}{R^C \alpha} \right) f(\alpha) d\alpha \times \frac{\partial \Phi_z}{\partial z}} < 0.
$$

Therefore, the two curves, $z = z_\eta(d)$ and $\tilde{z} = \tilde{z}_\eta(d)$, must intersect once, at $d \in (d_N, \bar{d}_\eta)$ and $z \in (\underline{z}_\eta, z_N)$, implying that there exists a unique solution to (A.26) and (A.27). $\square$

**Proof of Lemma 5**

Part 1 of the Lemma follows from Propositions 1 and 2. Part 2 of the Lemma follows since $-d^S + \beta V^S(d^S) = -d^N + \beta V(d^N) < -d^R + \beta V^R(d^R)$ for $R^C \in [R^{C*}, \hat{R}^{C*})$ from Proposition 1 and 2. Similarly, part 3 of the Lemma follows since $-d^S + \beta V^S(d^S) > -d^R + \beta V^R(d^R) = -d^N + \beta V(d^N)$ for $R^C \in [\hat{R}^{C*}, R^{C*})$ from Propositions 1 and 2.

To prove parts 4 and 5, it is necessary to derive $\Delta(R^C, \eta)$ as a function of $R^C$ and $\eta$. In a stationary monetary equilibrium, the net expected values of depositing in a bank with the safe and risky assets are given by

$$
\begin{aligned}
-d_S + \beta V^s(d_S) &= -\frac{d_S}{1-\beta} + \frac{\beta}{1-\beta}\left[ \int_0^{\alpha^*} \left\{ \alpha(u(q^*) - q^*) + \frac{z_S}{\pi} \right\} f(\alpha)d\alpha + \int_{\alpha^*}^{\alpha^{**}} \alpha u\left(\frac{z}{\alpha\pi}\right) f(\alpha)d\alpha \right. \\
&\quad + \int_{\alpha^{**}}^{\alpha^{***}} \left\{ \alpha u\left(u^{-1\prime}(R^C)\right) - \left(\alpha u^{-1\prime}(R^C) - \frac{z_S}{\pi}\right) \right\} f(\alpha)d\alpha \\
&\quad + \int_{\alpha^{***}}^1 \left\{ \alpha u\left(\frac{\frac{R^C}{\pi}z_S + R(d_S - z_S)}{R^C\alpha}\right) - \frac{R}{R^C}(d_S - z_S) \right\} f(\alpha)d\alpha \\
&\quad \left. + \{1 - E(\alpha)\}\{u(q^*) - q^*\} + R(d_S - z_S) + \left(1 - \frac{1}{\pi}\right)z_S \right],
\end{aligned}
$$

and

$$
\begin{aligned}
-d_R + \beta V^r(d_R) &= -\frac{d_R}{1-\beta} + \frac{\beta}{1-\beta}\left[ \int_0^{\alpha^*} \left\{ \alpha(u(q^*) - q*) - \frac{z_R}{\pi} \right\} f(\alpha)d\alpha + \int_{\alpha^*}^{\alpha_\eta^{**}} \alpha u\left(\frac{z_R}{\alpha\pi}\right) f(\alpha)d\alpha \right. \\
&\quad + \int_{\alpha_\eta^{**}}^{\alpha_\eta^{***}} \left\{ \alpha u\left(u^{-1\prime}(\eta R^C)\right) - \left(\alpha u^{-1\prime}(\eta R^C) - \frac{z_R}{\pi}\right) \right\} f(\alpha)d\alpha \\
&\quad + \int_{\alpha_\eta^{***}}^1 \left\{ \alpha u\left(\frac{\frac{R^C}{\pi}z_R + R(d_R - z_R)}{R^C\alpha}\right) - \frac{R}{R^C}(d_R - z_R) \right\} f(\alpha)d\alpha \\
&\quad \left. + \{1 - E(\alpha)\}\{u(q^*) - q^*\} + \eta\lambda R(d_R - z_R) + \left(1 - \frac{1}{\pi}\right)z_R \right],
\end{aligned}
$$

respectively. Using the two above equations, we have

$$
\begin{aligned}
\Delta(R^C, \eta) &= (1 - \beta\eta\lambda R)(d_R - d_S) + \beta R(1 - \eta\lambda)d_S - \beta(R-1)z_S + \beta(\eta\lambda R - 1)z_R \\
&\quad + \beta\left[ \int_{\alpha_\eta^*}^{\alpha^*} \alpha\{u(q^*) - q^*\} f(\alpha)d\alpha + \int_{\alpha^*}^{\alpha^{**}} \alpha\left\{ u\left(\frac{z_S}{\alpha\pi}\right) - \frac{z_S}{\alpha\pi} \right\} f(\alpha)d\alpha \right. \\
&\quad + \int_{\alpha^{**}}^{\alpha^{***}} \alpha\left\{ u\left(u^{-1\prime}(R^C)\right) - u^{-1\prime}(R^C) \right\} f(\alpha)d\alpha \\
&\quad + \int_{\alpha^{***}}^1 \alpha\left\{ u\left(\frac{\frac{R^C}{\pi}z_S + R(d_S - z_S)}{R^C\alpha}\right) - \frac{\frac{R^C}{\pi}z_S + R(d_S - z_S)}{R^C\alpha} \right\} f(\alpha)d\alpha \\
&\quad - \int_{\alpha_\eta^*}^{\alpha_\eta^{**}} \alpha\left\{ u\left(\frac{z_R}{\alpha\pi}\right) - \frac{z_R}{\alpha\pi} \right\} f(\alpha)d\alpha - \int_{\alpha_\eta^{**}}^{\alpha_\eta^{***}} \alpha\left\{ u\left(u^{-1\prime}(\eta R^C)\right) - u^{-1\prime}(\eta R^C) \right\} f(\alpha)d\alpha \\
&\quad \left. - \int_{\alpha_\eta^{***}}^1 \alpha\left\{ u\left(\frac{\frac{R^C}{\pi}z_R + R(d_R - z_R)}{R^C\alpha}\right) - \frac{\frac{R^C}{\pi}z_R + R(d_R - z_R)}{R^C\alpha} \right\} f(\alpha)d\alpha \right], \quad\quad (A.28)
\end{aligned}
$$

where $\alpha_\eta^* \equiv z_R/\pi q^* \leq \alpha^* = z_S/\pi q^*$. Since $d_R \to d_S$ and $z_R \to z_S$ as $\eta \to 1$ and $\lambda \to 1$ from Proposition 2, then as $\eta \to 1$ and $\lambda \to 1$, $\Delta(R^C, \eta) \to 0$, which proves part 4 of the Lemma.

To prove the fifth claim, differentiating $\Delta(R^C, \eta)$ with respect to $R^C$ and using the envelop

theorem imply that

$$\frac{\partial \Delta}{\partial R^C}(R^C, \eta) = (1-\beta)\beta \left\{ \frac{\partial V^s(d_S)}{\partial R^C} - \frac{\partial V^r(d_R)}{\partial R^C} \right\}$$

$$= \frac{R^C - 1}{u''(u^{-1'}(R^C))} \int_{\alpha^{**}}^{\alpha^{***}} \alpha f(\alpha)d\alpha - \frac{R(d_S - z_S)}{(R^C)^2} \int_{\alpha^{***}}^{1} \left\{ u'\left( \frac{\frac{R^C}{\pi} z_S + R(d_S - z_S)}{R^C \alpha} \right) - 1 \right\} f(\alpha)d\alpha$$

$$- \frac{\eta(\eta R^C - 1)}{u''(u^{-1'}(\eta R^C))} \int_{\alpha_\eta^{**}}^{\alpha_\eta^{***}} \alpha f(\alpha)d\alpha + \frac{R(d_R - z_R)}{(R^C)^2} \int_{\alpha_\eta^{***}}^{1} \left\{ u'\left( \frac{\frac{R^C}{\pi} z_R + R(d_R - z_R)}{R^C \alpha} \right) - 1 \right\} f(\alpha)d\alpha.$$

Since $d_R \to d_S$ and $z_R \to z_S$ as $\eta \to 1$ and $\lambda \to 1$, it follows that $\frac{\partial \Delta}{\partial R^C}(R^C, \eta) \to 0$ as $\eta \to 1$ and $\lambda \to 1$.

To prove the final claim, taking the limit as $R^C \to \pi \underline{R}^C$ yields $z_R \to 0$ by Proposition 2, which leads to $\alpha_\eta^* \to 0$, $\alpha_\eta^{**} \to 0$, $\alpha_\eta^{***} \to \frac{R d_R}{\pi \underline{R}^C u^{-1'}(\eta \pi \underline{R}^C)}$, and

$$\Delta(R^C, \eta) \to (1 - \beta \eta \lambda R)(d_R - d_S) + \beta R d_S (1 - \eta \lambda) - \beta(R-1)z_S$$

$$+ \beta \left[ \int_0^{\alpha^*} \alpha \left\{ u(q^*) - q^* \right\} f(\alpha)d\alpha + \int_{\alpha^*}^{\alpha^{**}} \alpha \left\{ u\left( \frac{z_S}{\alpha \pi} \right) - \frac{z_S}{\alpha \pi} \right\} f(\alpha)d\alpha \right.$$

$$+ \int_{\alpha^{**}}^{\alpha^{***}} \alpha \left\{ u\left( u^{-1'}(\pi \underline{R}^C) \right) - u^{-1'}(\pi \underline{R}^C) \right\} f(\alpha)d\alpha$$

$$+ \int_{\alpha^{***}}^{1} \alpha \left\{ u\left( \frac{\underline{R}^C z_S + R(d_S - z_S)}{\pi \underline{R}^C \alpha} \right) - \frac{\underline{R}^C z_S + R(d_S - z_S)}{\pi \underline{R}^C \alpha} \right\} f(\alpha)d\alpha$$

$$\left. - \int_0^{\alpha_\eta^{***}} \alpha \left\{ u\left( u^{-1'}(\eta \pi \underline{R}^C) \right) - u^{-1'}(\eta \pi \underline{R}^C) \right\} f(\alpha)d\alpha - \int_{\alpha_\eta^{***}}^{1} \alpha \left\{ u\left( \frac{R d_R}{\pi \underline{R}^C \alpha} \right) - \frac{R d_R}{\pi \underline{R}^C \alpha} \right\} f(\alpha)d\alpha \right].$$

(A.29)

The latter big blanket is negative, and if $\eta \lambda \approx 1$ and $\beta R \approx 1$, the first line of (A.29) becomes $-(1-\beta)z_S < 0$, leading to $\Delta(R^C, \eta) < 0$. This completes all the proofs of Lemma 5. $\square$

### Proof of Lemma 6

Denote the expected return of the risky asset by $R_E(\eta)R \equiv \eta \lambda R$ ($\leq R$). Then, equation (A.25), $\tilde{\Phi}(\hat{R}^{C*}) = 0$, which determines $\hat{R}^{C*} > \pi \underline{R}^C$, can be rewritten as

$$-\eta\{1 - F(\alpha_\eta^{***})\} + \frac{1}{\hat{R}^{C*}} \int_{\alpha_\eta^{***}}^{1} u'\left( \frac{d}{\pi \alpha} \right) f(\alpha)d\alpha + R_E(\eta) = \frac{1}{\beta R}. \qquad \text{(A.30)}$$

Note that, in the limit as $\eta \to 1$, $\hat{R}^{C*} \to R^{C*}$ (i.e., the critical value with the risky asset approaches to the one with the safe asset) and $R_E(\eta) \to 1$. For our purpose, we consider a fixed amount of decrease in $\eta$ away from $\eta = 1$ to $\eta = 1 - \epsilon$, with some admissible $\epsilon > 0$, and a varying decrease in $R_E(\eta)$ away from 1 to $1 - \epsilon_\eta$, with some arbitrary $\epsilon_\eta > 0$, and its influence on (A.30). Differentiating the L.H.S. of (A.30) with respect to $\eta$,

$$-\{1 - F(\alpha_\eta^{***})\} + R_E'(\eta),$$

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which is negative (positive) when $R'_E(\eta) > 0$ is small (large), that is, when $\epsilon_\eta$ is relatively large (small). This implies that given values of $\hat{R}^{C*}$ and $\eta < 1$, the L.H.S. of (A.30) is high (low) when $R_E(\eta)$ is high (low). Therefore, since the L.H.S. of (A.30) is decreasing in $\hat{R}^{C*}$, there exists a unique critical value denoted by $R_E^*(\eta) = (\eta\lambda)^* < 1$ such that $\hat{R}^{C*} > R^{C*}$ if $\eta\lambda > (\eta\lambda)^*$ and $\hat{R}^{C*} < R^{C*}$ if $\eta\lambda < (\eta\lambda)^*$. $\square$

**Proof of Proposition 3**

From Lemma 5, if the expected return on the risky asset, $\eta\lambda$, is sufficiently high or close to unity, the set of $(R^{C*}, \hat{R}^{C*})$ is nonempty, and the risky asset is selected for $R^C \in [R^{C*}, \hat{R}^{C*})$. In addition, from Lemma 5 and 6, since $\lim_{R^C \to \pi \underline{R}^C} \Delta(R^C, \eta) < 0$, $\Delta(R^{C*}, \eta) < 0$, and $\frac{\partial \Delta}{\partial R^C}(R^C, \eta) \approx 0$ if $\eta\lambda$ and $\beta R$ are sufficiently high or close to unity, by continuity the risky asset is also selected for $R^C \in [\pi \underline{R}^C, R^{C*})$. Thus, the risky asset is always selected when the bank uses the discount window, that is, when $R^C \in [\pi \underline{R}^C, \hat{R}^{C*})$. $\square$

**Proof of Proposition 4**

Differentiating (15) with respect to $\kappa$ yields

$$\frac{\partial z_R^\kappa}{\partial \kappa}\left\{\kappa \frac{\partial \Phi_z}{\partial z}(\kappa d, d) + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial z}(\kappa d, d)\right\} + \frac{\partial d_R^\kappa}{\partial \kappa}\left\{\kappa \frac{\partial \Phi_z}{\partial d}(\kappa d, d) + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial d}(\kappa d, d)\right\}$$
$$= \pi R \Phi_l(\kappa d, d) - \Phi_z(\kappa d, d) > 0. \quad \text{(A.31)}$$

The last inequality follows from $\Phi(\kappa d_R^\kappa, d_R^\kappa) < 0$. Applying $\frac{\partial z_R^\kappa}{\partial \kappa} = d_R^\kappa + \kappa \frac{\partial d_R^\kappa}{\partial \kappa}$ into (A.31) yields

$$\frac{\partial d_R^\kappa}{\partial \kappa}\left[\kappa\left\{\kappa \frac{\partial \Phi_z}{\partial z}(\kappa d, d) + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial z}(\kappa d, d)\right\} + \left\{\kappa \frac{\partial \Phi_z}{\partial d}(\kappa d, d) + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial d}(\kappa d, d)\right\}\right]$$
$$= \pi R \Phi_l(\kappa d, d) - \Phi_z(\kappa d, d) - d_R^\kappa\left\{\kappa \frac{\partial \Phi_z}{\partial z}(\kappa d, d) + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial z}(\kappa d, d)\right\}.$$

Since $\frac{\partial \Phi_z}{\partial z} < 0$, $\frac{\partial \Phi_z}{\partial d} < 0$, $\frac{\partial \Phi_l}{\partial z} < 0$, and $\frac{\partial \Phi_l}{\partial d} < 0$, we obtain

$$\frac{\partial d_R^\kappa}{\partial \kappa} < 0. \quad \text{(A.32)}$$

In addition, we have

$$\frac{\partial z_R^\kappa}{\partial \kappa} = d_R^\kappa + \frac{\kappa\left[\pi R \Phi_l - \Phi_z - d_R^\kappa\left\{\kappa \frac{\partial \Phi_z}{\partial z} + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial z}\right\}\right]}{\kappa\left\{\kappa \frac{\partial \Phi_z}{\partial z} + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial z}\right\} + \left\{\kappa \frac{\partial \Phi_z}{\partial d} + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial d}\right\}},$$
$$= \frac{d_R^\kappa\left\{\kappa \frac{\partial \Phi_z}{\partial d} + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial d}\right\} + \kappa(\pi R \Phi_l - \Phi_z)}{\kappa\left\{\kappa \frac{\partial \Phi_z}{\partial z} + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial z}\right\} + \left\{\kappa \frac{\partial \Phi_z}{\partial d} + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial d}\right\}}$$

The terms in the denominator are negative. The terms in the numerator can be rewritten as

$$d_R^\kappa \left\{ \kappa \frac{\partial \Phi_z}{\partial d} + (1-\kappa)\pi R \frac{\partial \Phi_l}{\partial d} \right\} + \kappa(\pi R \Phi_l - \Phi_z)$$

$$= \left[ \kappa \left( \frac{\pi R}{R^C} - 1 \right) - \xi \frac{\pi R}{R^C} \right] \int_{\alpha_\eta^{***}}^1 u' \left( \frac{\frac{R^C}{\pi} z_R^\kappa + R(d_R^\kappa - z_R^\kappa)}{R^C \alpha} \right) f(\alpha) d\alpha$$

$$+ \kappa \pi R [\eta F(\alpha_\eta^{***}) + \eta(\lambda - 1)] - \kappa \left[ F(\alpha^*) + \int_{\alpha^*}^{\alpha_\eta^{**}} u' \left( \frac{z_R^\kappa}{\alpha \pi} \right) + [F(\alpha_\eta^{***}) - F(\alpha_\eta^{**})]\eta R^C \right],$$

which is positive if $\xi$ is sufficiently small. Thus, we obtain

$$\frac{\partial z_R^\kappa}{\partial \kappa} < 0, \tag{A.33}$$

if $\xi$ is sufficiently small. Otherwise, it is positive. The proposition follows. $\square$


**Proof of Proposition 5**

The effects of a change in $\rho$ on the optimal values of $z_S^\rho$ and $d_S^\rho$ are determined by

$$\begin{pmatrix} \frac{\partial z_S^\rho}{\partial \rho} \\ \frac{\partial d_S^\rho}{\partial \rho} \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} \frac{\partial \Psi_z}{\partial z} + \frac{1-\rho}{\rho} \Upsilon'(z) & \frac{\partial \Psi_z}{\partial d} \\ \frac{\partial \Psi_k}{\partial z} & \frac{\partial \Psi_k}{\partial d} \end{pmatrix}^{-1} \begin{pmatrix} \Upsilon(z) - \Psi_z(z,d) \\ 1 - \Psi_k(z,d) \end{pmatrix}$$

$$= \frac{1}{\rho \Lambda_\rho} \begin{pmatrix} \frac{\partial \Psi_k}{\partial d} & -\frac{\partial \Psi_z}{\partial d} \\ -\frac{\partial \Psi_k}{\partial z} & \frac{\partial \Psi_z}{\partial z} + \frac{1-\rho}{\rho} \Upsilon'(z) \end{pmatrix} \begin{pmatrix} \Upsilon(z) - \Psi_z(z,d) \\ 1 - \Psi_k(z,d) \end{pmatrix} \tag{A.34}$$

where

$$\Lambda_\rho \equiv \det \begin{pmatrix} \frac{\partial \Psi_z}{\partial z} + \frac{1-\rho}{\rho} \Upsilon'(z) & \frac{\partial \Psi_z}{\partial d} \\ \frac{\partial \Psi_k}{\partial z} & \frac{\partial \Psi_k}{\partial d} \end{pmatrix}$$

$$= - \left( \frac{\partial \Psi_z}{\partial d} \frac{\partial \Psi_k}{\partial z} - \frac{\partial \Psi_z}{\partial z} \frac{\partial \Psi_k}{\partial d} \right) + \frac{1-\rho}{\rho} \Upsilon'(z) \frac{\partial \Psi_k}{\partial d} > 0.$$

The last inequality follows since $\Upsilon'(z) < 0$, $\frac{\partial \Psi_k}{\partial d} < 0$ and $\Lambda \equiv (\frac{\partial \Psi_z}{\partial d} \frac{\partial \Psi_k}{\partial z} - \frac{\partial \Psi_z}{\partial z} \frac{\partial \Psi_k}{\partial d}) < 0$ (see the proof of Corollary 3).

$\odot$ The effect of $\rho$ on $z_S^\rho$. From (A.34), using (16) and (17) we obtain

$$\rho \Lambda_\rho \frac{\partial z_S^\rho}{\partial \rho} = \frac{\partial \Psi_k}{\partial d} \left\{ \Upsilon(z_S^\rho) - \Psi_z(z_S^\rho, d_S^\rho) \right\} - \frac{\partial \Psi_z}{\partial d} \left\{ 1 - \Psi_k(z_S^\rho, d_S^\rho) \right\}$$

$$= \frac{1}{\rho} \left[ \frac{\partial \Psi_k}{\partial d} \left\{ \Upsilon(z_S^\rho) - \frac{\pi}{\beta} \right\} + \frac{\partial \Psi_z}{\partial d} \left( \frac{1}{\beta R} - 1 \right) \right] < 0.$$

The last inequality follows since $\frac{\partial \Psi_k}{\partial d} < 0$, $\frac{\partial \Psi_z}{\partial d} < 0$, and $\Upsilon(z_S^\rho) > \Upsilon(z_N) = \frac{\pi}{\beta}$.

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⊙ The effect of $\rho$ on $d_S^\rho$. Similarly, from (A.34), we obtain

$$
\begin{aligned}
\rho \Lambda_\rho \frac{\partial d_S^\rho}{\partial \rho} &= -\frac{\partial \Psi_k}{\partial z} \left\{ \Upsilon(z_S^\rho) - \Psi_z(z_S^\rho, d_S^\rho) \right\} + \left\{ \frac{\partial \Psi_z}{\partial z} + \frac{1-\rho}{\rho} \Upsilon'(z_S^\rho) \right\} \left\{ 1 - \Psi_k(z_S^\rho, d_S^\rho) \right\} \\
&= -\frac{1}{\rho} \left[ \frac{\partial \Psi_k}{\partial z} \left\{ \Upsilon(z_S^\rho) - \frac{\pi}{\beta} \right\} + \left\{ \frac{\partial \Psi_z}{\partial z} + \frac{1-\rho}{\rho} \Upsilon'(z_S^\rho) \right\} \left( \frac{1}{\beta R} - 1 \right) \right] > 0,
\end{aligned}
$$

since $\frac{\partial \Psi_k}{\partial z} < 0$, $\frac{\partial \Psi_z}{\partial z} < 0$, and $\Upsilon'(z_S^\rho) < 0$. The proposition follows. $\square$

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